Jackknife variance estimation corrections

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Contents

1 Jackknife variance correction

If we assume the S is a smooth functions of emperical CDF, especially a quadratic functions, then it can be shown the leading terms of $E(\tilde{Var}(S(X_1, \dots, S_{n-1}))) \geq Var(S(X_1, \dots, S_{n-1}))$ is a quadratic term in expecation. Therefore we could try to estimate the quadratic term and correct the bias for the jackknife variance estimation.

Define $Q_{ii'} \equiv nS - (n-1)(S_i + S_{i'}) + (n-2)S_{(ii')}$, then the correction will be

$$\hat{Var}^{corr}(S(X_1, \dots, X_n)) = \hat{Var}(S(X_1, \dots, X_n)) - \frac{1}{n(n-1)} \sum_{i<i'} (Q_{ii'} - \bar{Q})^2$$

where $\bar{Q} = \sum_{i<i'}(Q_{ii'})/(n(n-1)/2)$

2 Simulation study compare two GCTA and GCTA_rr

GCTA_rr is the `mixed.solve` function from `rrBLUP` r package.
Based on the following simulation results,

1. when $n < p$ case, those two methods' results are very closed to each other.
2. when $n > p$ case, in terms of effect estimation and jackknife variance estimation those two methods's reuslts are similar to each other. But for the variance corrections are quite different. That is the statistics Q of our method has a very large variance which leads to negative correction result.

2.0.1 setup

- Independent
- Normal
- $p = 100$
- $n = \{50, 75, 100, 150, 200\}$
- with interaction terms
- main effect: $Var(X^T\beta) = \{0, 8, 100\}$

2.0.2 Simulation result

2.0.3 $Var(X^T\beta) = \{0\}$

```
     n  MSE est_var est_mean NA_main GCTA_main_jack GCTA_v_jack_1
1:  50 3.40    1.83     1.32       0           0.59          9.55
2:  75 1.19    0.98     0.56       0           0.46          2.73
3: 100 1.08    0.84     0.57       0           0.44          1.35
4: 150 0.28    0.19     0.32       0          -1.09          0.86
5: 200 0.21    0.12     0.32       0          -1.60          0.78
   GCTA_v_jack_2 GCTA_v_corr
1:          9.62       -9.28
2:          2.68       -5.64
3:          1.35       -0.77
4:          0.67      -64.08
5:          0.69      -46.14
```

```
     n  MSE est_var est_mean NA_main GCTA_rr_main_jack GCTA_rr_v_jack_1
1:  50 3.40    1.83     1.32       0              0.60             9.55
2:  75 1.19    0.98     0.56       0              0.46             2.73
3: 100 1.08    0.84     0.57       0              0.44             1.35
4: 150 0.28    0.19     0.33       0             -0.17             0.62
5: 200 0.21    0.12     0.33       0              0.28             0.61
   GCTA_rr_v_jack_2 GCTA_rr_v_corr
1:             9.47         -3.560
2:             2.68         -5.643
3:             1.35         -0.770
4:             0.61         -1.204
5:             0.61         -0.041
```

2.0.4 $Var(X^T\beta) = \{100\}$

```
     n   MSE est_var est_mean NA_main GCTA_main_jack GCTA_v_jack_1
1:  50  9247    1784       87       0             66          8795
2:  75 10077    1863       92       0            103          5170
3: 100 11839    2142      100       0             84          2072
4: 150 10953     443      103       0             31          1280
5: 200  9778     245       98       0             30           725
   GCTA_v_jack_2 GCTA_v_corr
1:          8793       -3687
2:          5109       -3122
3:          2081         194
4:          1148      -80475
5:           673      -32124
```

```
     n   MSE est_var est_mean NA_main GCTA_rr_main_jack GCTA_rr_v_jack_1
1:  50  9247    1784       87       0                66             8795
2:  75 10077    1863       92       0               103             5170
3: 100 11839    2142      100       0                84             2072
4: 150 11194     414      104       0               103              969
5: 200  9854     238       98       0                98              616
   GCTA_rr_v_jack_2 GCTA_rr_v_corr
1:             8787          -3492
2:             5109          -3124
```

```
3:              2081            194
4:               970            158
5:               616            220
```

2.0.5 $Var(X^T\beta) = \{8\}$

```
     n MSE est_var est_mean NA_main GCTA_main_jack GCTA_v_jack_1
1:  50  90    25.8      8.0       0            8.5          74.1
2:  75  70    13.1      7.5       0            7.5          32.1
3: 100  68     6.3      7.8       0            7.5          13.7
4: 150  70     4.0      8.1       0            8.4           9.2
5: 200  65     2.5      7.9       0            7.6           4.6
   GCTA_v_jack_2 GCTA_v_corr
1:          73.8     -190.67
2:          31.9      -25.67
3:          13.8       -0.97
4:           8.1     -502.59
5:           4.3     -214.51
```

```
     n  MSE est_var est_mean NA_main
1:  50 24.0    24.0      8.0       0
2:  75 13.8    13.8      7.9       0
3: 100  8.6     8.6      8.1       0
4: 150  3.7     3.7      8.0       0
5: 200  2.7     2.7      8.0       0
```

```
     n MSE est_var est_mean NA_main GCTA_rr_main_jack GCTA_rr_v_jack_1
1:  50  90    25.8      8.0       0               8.5             74.1
2:  75  70    13.1      7.5       0               7.5             32.1
3: 100  68     6.3      7.8       0               7.5             13.7
4: 150  70     4.1      8.1       0               8.1              6.9
5: 200  65     2.5      7.9       0               7.9              3.9
   GCTA_rr_v_jack_2 GCTA_rr_v_corr
1:             73.6        -177.35
2:             31.8         -16.78
3:             13.8          -0.97
4:              6.9           1.49
5:              3.9           1.38
```

```
     n  MSE est_var est_mean NA_main
1:  50 23.8    23.9      8.0       0
2:  75 13.7    13.7      7.9       0
3: 100  8.6     8.6      8.1       0
4: 150  3.8     3.8      8.0       0
5: 200  2.7     2.7      8.1       0
```

2.0.6 correlation test $

```
     n    MSE est_var est_mean NA_main cor_main_jack cor_v_jack_1
1:  50 0.0131  0.0130     0.49       0          0.49       0.0127
2:  75 0.0083  0.0083     0.50       0          0.50       0.0079
3: 100 0.0057  0.0057     0.50       0          0.50       0.0059
4: 150 0.0038  0.0038     0.50       0          0.50       0.0039
5: 200 0.0030  0.0030     0.50       0          0.50       0.0029
```

```
   cor_v_jack_2 cor_v_corr
1:       0.0128     0.0120
2:       0.0079     0.0076
3:       0.0059     0.0057
4:       0.0039     0.0038
5:       0.0029     0.0029
```

2.1 compare the performance of delete 1 and delete d in variance estimation

The delete-d jackknife varinace estimator is

$$\sqsubseteq_{J(d)} = \frac{n-d}{d} \cdot \frac{1}{S} \sum_{S} (\hat{\theta}_s - \hat{\theta}_{s.})$$

, where $S = \binom{n}{d}$. Note that S could a very large value, so in the following simulation, only $S = 1000$ is used. In Jun Shao's another paper, he proposed an approximation of the deletel-d variance estimation. That is just select m from $S = \binom{n}{d}$ sub-samples and in that paper it recommended $m = n^{1.5}$.

2.1.1 setup

- Independent
- Normal
- $p = \{100, 1000\}$
- $n = \{50, 75, 100, 150, 200, 500, 750, 1000, 1500, 2000\}$
- $d = 0.5 \times n$
- $n_{repeat} = 1000$ for delete d jackknife
- main effect: $Var(X^T\beta) = 8$

2.1.2 GCTA with p = 100

n	MSE	est_var	est_mean	NA_main	GCTA_main_jack	GCTA_v_jack	GCTA_v_jack_var	d	n_sub
50	25.6	25.8	8.0	0	8.5	74.1	8383.8	1.0	NA
75	13.2	13.1	7.5	0	7.5	32.1	685.1	1.0	NA
100	6.2	6.3	7.8	0	7.5	13.7	102.2	1.0	NA
150	4.0	4.0	8.1	0	8.4	9.2	16.4	1.0	NA
200	2.5	2.5	7.9	0	7.6	4.6	2.1	1.0	NA
50	25.6	25.8	8.0	0	45.5	41.2	365.2	0.5	NA
75	13.2	13.1	7.5	0	-177.5	27.1	99.7	0.5	NA
100	6.2	6.3	7.8	0	-237.3	18.5	38.1	0.5	NA
150	4.0	4.0	8.1	0	-13.8	9.4	7.5	0.5	NA
200	2.5	2.5	7.9	0	17.3	5.0	1.4	0.5	NA
50	25.6	25.8	8.0	0	35.1	41.1	366.6	0.5	354
75	13.2	13.1	7.5	0	-107.6	27.0	100.1	0.5	650
100	6.2	6.3	7.8	0	-237.3	18.5	38.1	0.5	1000
150	4.0	4.0	8.1	0	-20.2	9.3	7.0	0.5	1837
200	2.5	2.5	7.9	0	53.4	5.1	1.3	0.5	2828

2.1.3 GCTA with p = 1000

n	MSE	est_var	est_mean	NA_main	GCTA_main_jack	GCTA_v_jack	GCTA_v_jack_var	d
500	2.88	2.91	8.0	0	7.8	4.65	1.08	1.0
750	1.29	1.30	8.0	0	8.0	2.26	0.15	1.0
1000	0.77	0.78	8.0	0	8.0	1.28	0.04	1.0
1500	0.47	0.48	7.9	0	6.7	0.80	0.01	1.0
500	2.88	2.91	8.0	0	-79.1	6.56	1.17	0.5
750	1.29	1.30	8.0	0	-5.9	3.04	0.13	0.5
1000	0.77	0.78	8.0	0	40.8	1.71	0.05	0.5
1500	0.41	0.41	8.0	0	9.9	0.80	0.01	0.5
2000	0.31	0.31	8.0	0	25.6	0.48	0.00	0.5

2.1.4 GCTA_rr_rr with p = 100

n	MSE	est_var	est_mean	NA_main	GCTA_rr_main_jack	GCTA_rr_v_jack	GCTA_rr_v_jack_var	d	n_sub
50	25.6	25.8	8.0	0	8.5	74.1	8378.6	1.0	NA
75	13.2	13.1	7.5	0	7.5	32.1	685.3	1.0	NA
100	6.2	6.3	7.8	0	7.5	13.7	102.2	1.0	NA
150	4.1	4.1	8.1	0	8.1	6.9	8.5	1.0	NA
200	2.5	2.5	7.9	0	7.9	3.9	1.3	1.0	NA
50	25.6	25.8	8.0	0	52.5	40.6	363.1	0.5	NA
75	13.2	13.1	7.5	0	-198.0	26.6	100.2	0.5	NA
100	6.2	6.3	7.8	0	-257.6	18.1	38.6	0.5	NA
150	4.1	4.1	8.1	0	-11.9	9.3	7.5	0.5	NA
200	2.5	2.5	7.9	0	25.4	5.0	1.4	0.5	NA
50	25.6	25.8	8.0	0	35.2	40.5	363.4	0.5	354
75	13.2	13.1	7.5	0	-120.5	26.6	100.8	0.5	650
100	6.2	6.3	7.8	0	-257.6	18.1	38.6	0.5	1000
150	4.1	4.1	8.1	0	-17.0	9.3	7.1	0.5	1837
200	2.5	2.5	7.9	0	76.2	5.1	1.3	0.5	2828

2.1.5 GCTA_rr with p = 1000

n	MSE	est_var	est_mean	NA_main	GCTA_rr_main_jack	GCTA_rr_v_jack	GCTA_rr_v_jack_var	d
500	2.88	2.91	8.0	0	7.8	4.65	1.08	1.0

(continued)

n	MSE	est_var	est_mean	NA_main	GCTA_rr_main_jack	GCTA_rr_v_jack	GCTA_rr_v_jack_var	d
750	1.29	1.30	8.0	0	8.0	2.26	0.15	1.0
1000	0.77	0.78	8.0	0	8.0	1.28	0.04	1.0
1500	0.48	0.48	7.9	0	8.0	0.62	0.00	1.0
500	2.88	2.91	8.0	0	-79.1	6.56	1.17	0.5
750	1.29	1.30	8.0	0	-5.9	3.04	0.13	0.5
1000	0.77	0.78	8.0	0	40.8	1.71	0.05	0.5
1500	0.41	0.41	8.0	0	11.8	0.80	0.01	0.5
2000	0.31	0.31	8.0	0	24.4	0.48	0.00	0.5

2.1.6 cor with n = 200

n	MSE	est_var	est_mean	NA_main	cor_main_jack	cor_v_jack	d
50	0.01252	0.01265	0.50001	0	0.50432	0.01229	1.0
75	0.00774	0.00782	0.50050	0	0.50323	0.00815	1.0
100	0.00607	0.00613	0.50148	0	0.50334	0.00582	1.0
150	0.00383	0.00385	0.49584	0	0.49709	0.00391	1.0
200	0.00281	0.00284	0.49930	0	0.50027	0.00288	1.0
50	0.01252	0.01265	0.50001	0	5.32154	0.01282	0.5
75	0.00774	0.00782	0.50050	0	3.26213	0.00844	0.5
100	0.00607	0.00613	0.50148	0	2.50378	0.00595	0.5
150	0.00383	0.00385	0.49584	0	1.59064	0.00396	0.5
200	0.00281	0.00284	0.49930	0	1.46439	0.00293	0.5

2.1.7 median with n = 200

n	MSE	est_var	est_mean	NA_main	median_main_jack	median_v_jack	d
50	0.03138	0.03135	-0.00775	0	-0.00775	0.06818	1.0
75	0.02211	0.02212	-0.00212	0	0.05228	0.03113	1.0
100	0.01523	0.01523	-0.00378	0	-0.00378	0.02720	1.0
150	0.01072	0.01072	-0.00279	0	-0.00279	0.01885	1.0
200	0.00804	0.00804	-0.00051	0	-0.00051	0.01614	1.0
50	0.03138	0.03135	-0.00775	0	5.04459	0.03477	0.5
75	0.02211	0.02212	-0.00212	0	8.44376	0.02248	0.5
100	0.01523	0.01523	-0.00378	0	2.68868	0.01587	0.5
150	0.01072	0.01072	-0.00279	0	-1.47581	0.01110	0.5
200	0.00804	0.00804	-0.00051	0	3.96797	0.00827	0.5

2.2 Jackknife variance estimation's bias and sample size n

2.2.1 setup

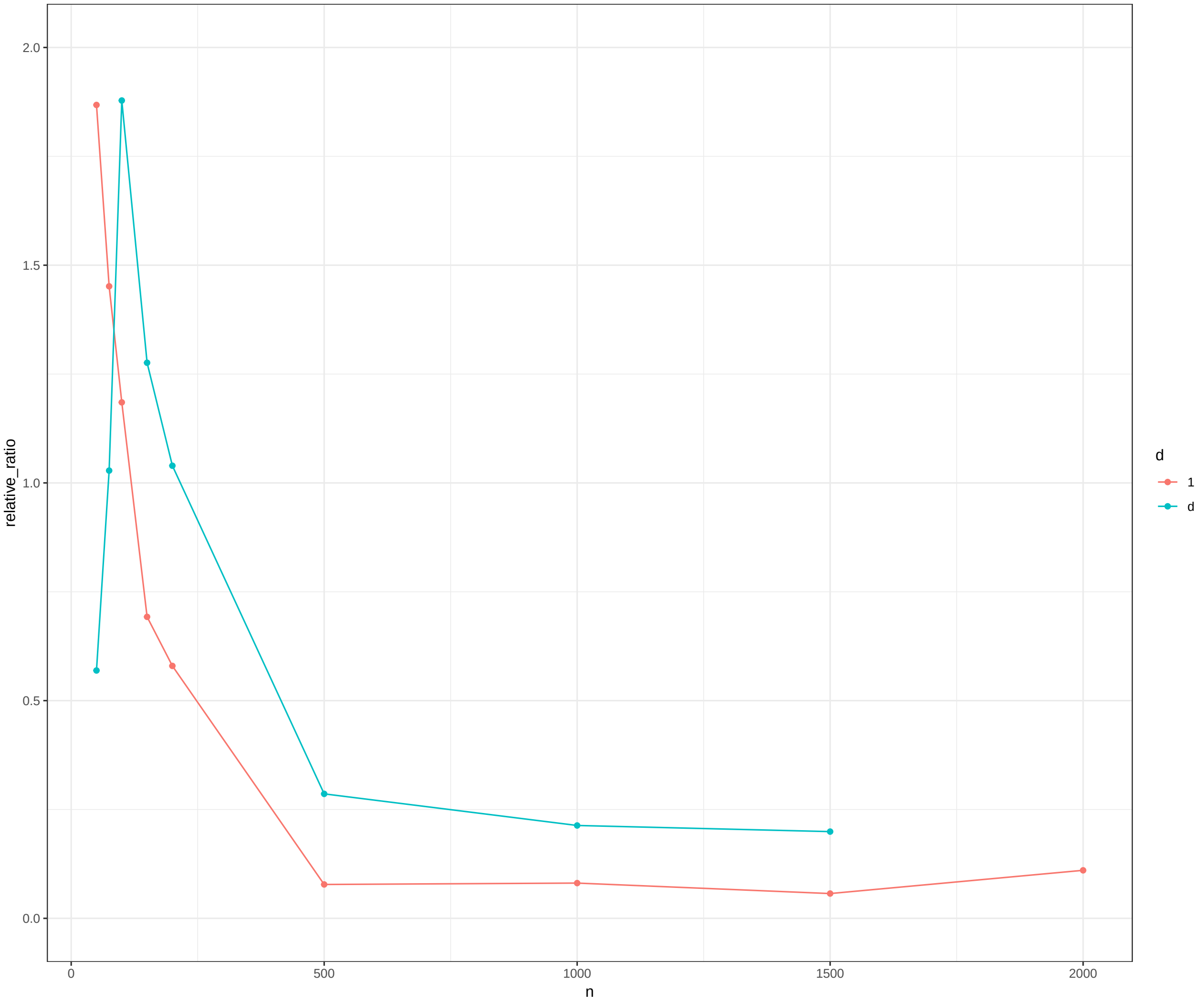
- Independent
- Normal
- $p = 100$
- $n = \{50, 75, 100, 150, 200, 500, 750, 1000, 1500, 2000\}$
- $d = 0.5 \times n$
- $n_{repeat} = n^{1.5}$ for delete d jackknife
- main effect: $Var(X^T\beta) = 8$

Based on the previous simulation results, we find there is a bias among all the jackknife variance estimation. Based on the Efron's result, the overestimation is because the statistics S is not a smooth function of the distribution function, so that the correct coefficient actually inflate the variance estimation.

The following reuslt is trying to see the relation between the bias and the sample size n

2.2.2 GCTA with p = 100

n	MSE	est_var	est_mean	NA_main	GCTA_rr_main_jack	GCTA_rr_v_jack	GCTA_rr_v_jack_var	d	relative_ratio
50	25.57707	25.83503	8.0196	0	8.4975	74.09557	8378.63491	1	1.86803
50	25.57707	25.83503	8.0196	0	35.1628	40.54145	363.39717	25	0.56924
75	13.17873	13.09873	7.5407	0	7.5224	32.11198	685.27217	1	1.45153
75	13.17873	13.09873	7.5407	0	-120.5219	26.56808	100.83487	38	1.02829
100	6.25254	6.28648	7.8299	0	7.4523	13.73592	102.19922	1	1.18499
100	6.25254	6.28648	7.8299	0	-257.5807	18.09388	38.62832	50	1.87822
150	4.06803	4.09156	8.1319	0	8.1135	6.92455	8.46409	1	0.69240
150	4.06803	4.09156	8.1319	0	-16.9623	9.31223	7.10912	75	1.27596
200	2.47559	2.48901	7.8929	0	7.9447	3.93194	1.32257	1	0.57972
200	2.47559	2.48901	7.8929	0	76.1900	5.07627	1.31029	100	1.03948
500	0.82596	0.82766	8.0811	0	8.0819	0.89206	0.02461	1	0.07782
500	0.82596	0.82766	8.0811	0	4.8643	1.06427	0.02959	250	0.28589
1000	0.33152	0.32460	8.1008	0	8.1036	0.35088	0.00145	1	0.08097
1000	0.33152	0.32460	8.1008	0	95.6021	0.39383	0.00168	500	0.21327
1500	0.21016	0.20455	8.0876	0	8.0888	0.21620	0.00050	1	0.05694
1500	0.19380	0.19373	8.0537	0	-100.0448	0.23232	0.00051	750	0.19923
2000	0.13784	0.13756	8.0407	0	8.0458	0.15273	0.00013	1	0.11029



2.2.3 Eg with p = 100

n	MSE	est_var	est_mean	NA_main	EigenPrism_main_jack	EigenPrism_v_jack	EigenPrism_v_jack_var	d	relative_ratio
50	21.6516	21.7303	8.3722	0	7.6266	48.0986	454.15195	1	1.21343
50	21.6516	21.7303	8.3722	0	-366.8806	48.3710	463.03961	25	1.22597
75	12.1880	12.2997	7.8937	0	7.3525	24.4991	94.99580	1	0.99184
75	12.1880	12.2997	7.8937	0	-463.3255	25.6137	77.40674	38	1.08246
100	7.1233	7.1933	7.9561	0	7.0599	15.1132	40.83974	1	1.10102
100	7.1233	7.1933	7.9561	0	-710.8797	16.1844	33.40698	50	1.24993
150	NaN	NA	NaN	100	NaN	NaN	NA	1	NaN
150	NaN	NA	NaN	100	NaN	8.4296	4.89393	75	NA
200	NaN	NA	NaN	100	NaN	NaN	NA	1	NaN
200	NaN	NA	NaN	100	NaN	5.0164	0.94967	100	NA
500	NaN	NA	NaN	100	NaN	NaN	NA	1	NaN
500	NaN	NA	NaN	100	NaN	NaN	NA	250	NaN
1000	NaN	NA	NaN	100	NaN	NaN	NA	1	NaN
1000	NaN	NA	NaN	100	NaN	NaN	NA	500	NaN
1500	NaN	NA	NaN	99	NaN	NaN	NA	1	NaN
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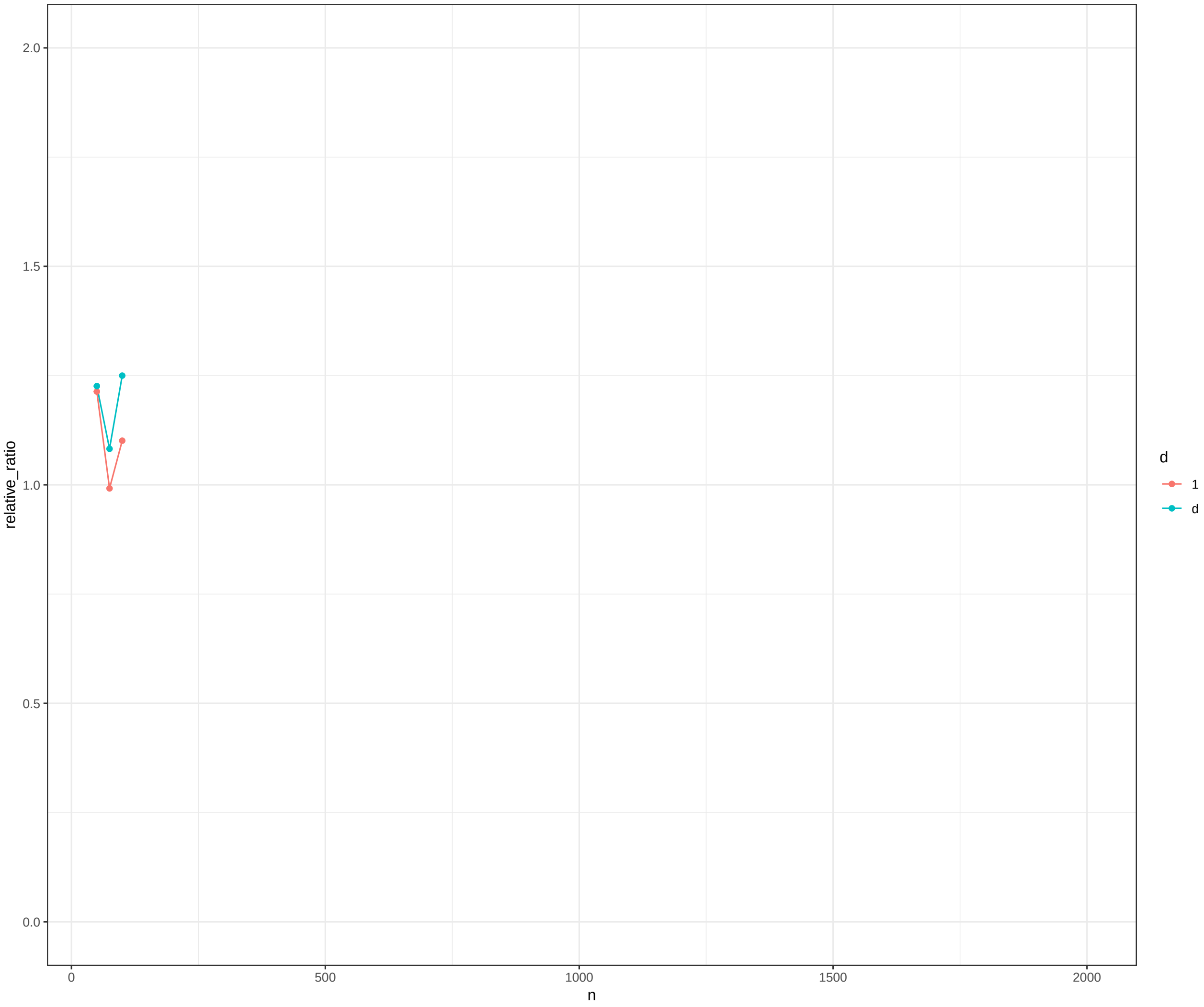
2000	NaN	NA	NaN	100	NaN	NaN	NA	1	NaN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