RMIT University, School of Science
Math2136 – Computational Mathematics
Assignment 3, Version 5

Submit by 11pm, October 23, 2016.

Do this assignment if the last digit of your student number is 1 or 6.

An explicit five stage Runge-Kutta scheme in vector form is:

$$\begin{aligned}
\mathbf{k}_1 &= h\mathbf{f}(\mathbf{y}_n) \\
\mathbf{k}_2 &= h\mathbf{f}(\mathbf{y}_n + \tfrac{1}{4}\mathbf{k}_1) \\
\mathbf{k}_3 &= h\mathbf{f}(\mathbf{y}_n + \tfrac{1}{2}\mathbf{k}_2) \\
\mathbf{k}_4 &= h\mathbf{f}(\mathbf{y}_n + \tfrac{1}{2}\mathbf{k}_1 - \mathbf{k}_2 + \tfrac{5}{4}\mathbf{k}_3) \\
\mathbf{k}_5 &= h\mathbf{f}(\mathbf{y}_n + \tfrac{2}{3}\mathbf{k}_2 + \tfrac{1}{3}\mathbf{k}_4) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \tfrac{1}{24}(3\mathbf{k}_1 + 4\mathbf{k}_2 + 10\mathbf{k}_3 + 4\mathbf{k}_4 + 3\mathbf{k}_5) \qquad \text{for } n = 0, 1, 2, \ldots.
\end{aligned}$$

This scheme can be used to compute approximate solutions for an autonomous system of n coupled first order differential equations. These equations will be of the form:

$$\frac{d\mathbf{y}}{dx} = \mathbf{f}(\mathbf{y}) \qquad \mathbf{y}(x_0) = \mathbf{y}_0,$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ and $\mathbf{f} = (f_1, f_2, \ldots, f_n)$.

1. (a)
   - **Maple:** Write a Maple procedure to implement this algorithm to solve an autonomous first order system of equations. The vectors $\mathbf{f}$ and $\mathbf{y}_0$, the step size h and the number of steps should be passed to the procedure.
   - **MatLab:** Write a MatLab .m file to implement this algorithm to solve an autonomous first order system of equations. The vectors $\mathbf{f}$ and $\mathbf{y}_0$, the step size h and the number of steps should be inputs to the function.
   - ***Mathematica*:** Write a *Mathematica* module to implement this algorithm to solve an autonomous first order system of equations. The vectors $\mathbf{f}$ and $\mathbf{y}_0$, the step size h and the number of steps should be inputs to the module.

   (b) Consider the initial value problem

   $$y' = \frac{4y}{y - 4x}, \qquad y(1) = 3.$$

   Given that the exact solution of Initial Value Problem, takes the value

   $$y(5) = 20 - \sqrt{385},$$

   using your Maple procedure, MatLab function or *Mathematica* module, numerically determine how the error at $x = 5$ depends on h. Choosing suitable values for step-size h, plot $\log|\text{error}|$ versus $\log h$. Use your plot to determine a relationship between the error and h. How accurate is the method? Is this what you would expect? Please discuss.

The Northern Tuli Game Reserve (part of the Tuli Block) is situated in the south eastern corner of Botswana, in a unique and historically significant location where the country meets its neighbours Zimbabwe and South Africa and at the confluence of two rivers, the Limpopo and the Shashe.

Elephant-Tree ecology in The Northern Tuli Game Reserve (area about 710 km^2) can be modeled by the following system of first order differential equations:

$$\frac{dT}{dt} = T(a - bT) - \frac{cT}{m+T}E,$$

$$\frac{dE}{dt} = E\left(-k + \frac{hT}{E+n}\right),$$

where;

- T is the density of trees [trees km^{-2}],
- E is the density of elephants [elephants km^{-2}],
- a [year^{-1}] is the natural rate of increase of trees per year,
- b [km^2 tree^{-1} year^{-1}] is the amount by which an additional unit of tree density depresses the rate of increase of trees,
- c [trees elephant^{-1} year^{-1}] is the rate of elimination of trees per elephant,
- m [trees km^{-2}] is a threshold density of trees,
- $-k$ [year^{-1}] is the rate of decrease of elephants (per year) in the absence of trees,
- h [elephants tree^{-1} year^{-1}] is the rate at which elephant decrease is ameliorated at a given ratio of trees to elephants and
- n [elephants km^{-2}] is the threshold density of elephants.

The system exhibits periodic behavior, which will appear as a closed curve in the phase plane (T-E plane), called a limit cycle. Large limit cycles are undesirable in game parks. If the system follows a large limit cycle, there will be periods in which the trees have been devastated by elephants and the elephants are dying out. This devastation can cause tourism to drop dramatically. The problem is solved by controlled culling by the game park managers. The increased death rate due to culling causes the spiral equilibrium point to change from repelling to attracting. The stable limit cycle disappears and the system tends to a stable equilibrium point. This transformation from stable limit cycle to stable equilibrium point is called Hopf bifurcation.

Based on observed data from Northern Tuli Game Reserve we know that

$$b = 4 \times 10^{-6}\left[\frac{\text{km}^2}{\text{tree year}}\right], \quad h = 2 \times 10^{-6}\left[\frac{\text{elephants}}{\text{tree year}}\right], \quad n = 0.1\left[\frac{\text{elephants}}{\text{km}^2}\right]$$

and based on the <u>SECOND last digit of your student number</u> the remaining parameters are given in the following Table.

Second Last digit	0	1	2	3	4	5	6	7	8	9
$a\ \left[\frac{1}{\text{year}}\right]$	.04	.05	.045	.035	.045	.04	.05	.055	.06	.03
$c\ \left[\frac{\text{trees}}{\text{elephant year}}\right]$	240	300	200	260	400	500	450	350	400	200
$m\ \left[\frac{\text{trees}}{\text{km}^2}\right]$	1000	800	1100	1150	1000	1200	800	1000	900	700
$k\ \left[\frac{1}{\text{year}}\right]$	.009	.012	.007	.009	.013	.014	.016	.012	.013	.009

2. With the given parameters use your MatLab function, Maple procedure or *Mathematica* module, from Question 1, to show that the system of first order differential equations, exhibits limit cycle behavior. That is integrate the system forward in time and plot the numerical solution in the phase plane (T-E plane).
   Try 10,000 years, with initial tree density a/b (the carrying capacity for trees in the absence of elephants) and elephant density 1, as a start!
   Also;

   - Does it matter what the starting values are?
   - Do you obtain limit cycle behavior for different starting values?
   - If starting values are chosen inside the limit cycle does the solution spiral outwards towards the limit cycle?

   Answer these question. Supply numerical evidence and a discussion.

3. Determine the period of the limit cycle. Use your plot from Question 2 to find a suitable starting point.
   Then, experiment with the number of time steps to find the period of the limit cycle.
   Considering the length of the limit cycle period (in years) is it feasible to confirm the limit cycle behavior by observation? Please comment.

4. Based upon the <u>SECOND LAST DIGIT OF YOUR STUDENT NUMBER</u>, the current number of elephants (E_0) and tree density (T_0) in the Northern Tuli Game Reserve is given in the following Table.

Second Last digit	0	1	2	3	4	5	6	7	8	9
E_0 [elephants]	452	488	712	391	402	249	328	488	577	337
$T_0\ \left[\frac{\text{trees}}{\text{km}^2}\right]$	5500	6900	6100	4900	6300	6500	6900	6900	8600	5100

The game park managers have determined that the current population of elephants is growing rapidly and they fear an environmental disaster.

- Your task is to determine an appropriate culling program so that the populations (elephants and trees) tend to a stable equilibrium.
- You have to achieve near equilibrium conditions within 50 years.
- After 50 years no more than 3 elephants are to be culled per year.

If $P(t)$ elephants are culled in year t, where $P(t)$ is an integer, then the governing first order system is now

$$\frac{dT}{d\tau} = T(a - bT) - \frac{cT}{m+T}E, \qquad T(\tau_0) = T_0,$$

$$\frac{dE}{d\tau} = E\Big(-k + \frac{hT}{E+n}\Big) - \frac{P(t)}{710}, \qquad E(\tau_0) = \frac{E_0}{710},$$

$$\frac{dt}{d\tau} = 1, \qquad t(\tau_0) = 0.$$

- By numerical experimentation find an appropriate culling program, where the system tends to a stable equilibrium point. That is define $P = P(t)$ so that within 50 years the populations are in near equilibrium state. After 50 years P should equal 3 or less.

  Note: You cannot cull half an elephant.
  - $P(t)$ must evaluate to an integer and be non-negative at all values of $t \geq 0$.
  - You can define $P(t)$ using a Matlab .m file. That is define a function and use an `if` statement to set $P(t)$ based on the year $t \geq 0$.
  - In Maple $P = P(t)$ could be defined using the `piecewise()` command.
  - The numerical solution should step in units of one year. (This avoids culling a fractional elephant.)
- Confirm your result with a phase plane plot (T-E plane), for $t \leq 1000$ years. Note that in Maple a three dimensional plot of a line through (T, E, t) points, can be rotated so that we see only the T-E plane.
- Estimate the number of elephants and the tree density when the system is in equilibrium.

Warning: If you recommend excessive culling of elephants this may result in culling of your assignment mark.