4. Minimization

Prove that for a symmetric matrix $A \in \mathbb{R}^{n,n}$, the following are equivalent:

1. $$Ax = b$$

2. $$v^T A x = v^T b, \quad \text{for all} \quad v \in \mathbb{R}^n$$

3. $$I[x] = \min_{y \in \mathbb{R}^n} I[y] \quad \text{where} \quad I[y] := \frac{1}{2} y^T A y - y^T b$$

Hint: x is a minimum of $I[x]$ if and only if the function $f_v(t) := I[x + tv]$ has a minimum at $t = 0$ for all v.

Remark: If A is the finite element matrix, the last minimization problem can be used to solve the linear system. An analogous statement for the Laplace equation is used to prove the Lax-Milgram theorem.