

5.1

$$U^{n+1} = U^n + h f\left(u^n + \frac{h}{2} f(u^n, t), t+h\right)$$

Waldmann

$$U^{n+1} = u(x+h) = u(x) + hu'(x) + \frac{h^2}{2} u''(x) + o(h^3).$$

$$U^n = u(x)$$

$$\begin{aligned} f\left(u^n + \frac{h}{2} f(u^n, t), t+h\right) &= f\left(u(x) + \frac{h}{2} f(u(x), t)\right) \\ &= f\left(u(x) + \frac{h}{2} \left(f(u(x) + hu'(x) + o(h^2))\right)\right) \\ &= f\left(u(x) + \frac{h}{2} \left(u'(x) + \cancel{hu'(x)} + hf(u'(x) + o(h^2))\right)\right) \\ &\xrightarrow{f(u(x)) = u'(x)} = f\left(u(x) + \frac{h}{2} u'(x) + o(h^2)\right) \\ &= u'(x) + \frac{h}{2} u''(x) + o(h^2) \end{aligned}$$

$$\cancel{u(x+h)} = u(x) + h(u'(x) + \frac{h}{2} u''(x) + o(h^2))$$

$$u(x) + hu'(x) + \frac{h^2}{2} u''(x) + o(h^3) = u(x) + hu'(x) + \frac{h^2}{2} u''(x) + o(h^3)$$

$$\begin{aligned} U^{n+1} - \left(u^n + hf\left(u^n + \frac{h}{2} f(u^n, t), t+h\right)\right) &= u(x) - u(x) + hu'(x) - hu'(x) + \frac{h^2}{2} u''(x) - \frac{h^2}{2} u''(x) + o(h^3), \\ \tau &= \cancel{o(h^3)}. \end{aligned}$$