4. Gradient Descent

Consider the PDE

$$-u'' + u^3 = 0, \qquad u(0) = 0, \qquad u(1) = 1$$

Note that this is nonlinear because of the u^3 term.

1. Find a discretization of this PDE with a standard three point finite difference stencil for the second derivative. This should give you a system of the form
$$AU + U^3 = 0 \tag{E}$$
for some matrix A and vector $U \in \mathbb{R}^m$, where $U^3 = [U_1^3, \ldots, U_m^3]$.

2. Show that if U is a minimum of the optimization problem
$$\min_{U \in \mathbb{R}^m} \frac{1}{2} U^T A U + \frac{1}{4} \sum_i U_i^4 \tag{M}$$
then U is a solution of (E).
Hint: Calculate the partial derivatives $\frac{\partial}{\partial U_i}$ for each component U_i.

3. Write down a gradient descent method to solve the minimization problem (M).