

# **Session 2: Linear and logistic regression as Generalized Linear Models**

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CUNY SPH Biostatistics 2

**Session 2:  
Linear and  
logistic  
regression as  
Generalized  
Linear Models**

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**Learning  
Objectives  
and Outline**

Review of  
multiple linear  
regression

Linear  
Regression as  
a GLM

Logistic  
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Residuals for  
logistic  
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Likelihood  
and  
hypothesis  
testing

Additive  
vs. Multiplica-  
tive models

# Learning Objectives and Outline

# Learning objectives

- 1 define generalized linear models (GLM)
- 2 define linear and logistic regression as special cases of GLMs
- 3 distinguish between additive and multiplicative models
- 4 define Pearson and deviance residuals
- 5 describe application of the Wald test

# Outline

- 1 Brief overview of multiple regression (Vittinghoff 4.1-4.3)
- 2 Linear Regression as a GLM (Vittinghoff 4.1-4.3)
- 3 Logistic Regression as a GLM (Vittinghoff 5.1-5.3)
- 4 Statistical inference for logistic regression (Vittinghoff 5.1-5.3)

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# Review of multiple linear regression

# Systematic component

$$E[y|x] = \beta_0 + \beta_1x_1 + \beta_2x_2 + \dots + \beta_px_p$$

- $x_p$  are the predictors or independent variables
- $y$  is the outcome, response, or dependent variable
- $E[y|x]$  is the expected value of  $y$  given  $x$
- $\beta_p$  are the regression coefficients

# Systematic plus random component

$$y_i = E[y|x] + \epsilon_i$$

$$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \dots + \beta_p x_p + \epsilon_i$$

Assumption:  $\epsilon_i \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2)$

- Normal distribution
- Mean zero at every value of predictors
- Constant variance at every value of predictors
- Values that are statistically independent

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# Linear Regression as a GLM



# Generalized Linear Models (GLM)

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- Linear regression is a special case of a broad family of models called “Generalized Linear Models” (GLM)
- This unifying approach allows to fit a large set of models using maximum likelihood estimation methods (MLE) (Nelder & Wedderburn, 1972)
- Can model many types of data directly using appropriate distributions, e.g. Poisson distribution for count data
- Transformations of  $Y$  not needed

# Components of GLM

- **Random component** specifies the conditional distribution for the response variable
  - doesn't have to be normal
  - can be any distribution in the “exponential” family of distributions
- **Systematic component** specifies linear function of predictors (linear predictor)
- **Link** [denoted by  $g(\cdot)$ ] specifies the relationship between the expected value of the random component and the systematic component
  - can be linear or nonlinear

# Linear Regression as GLM

- **The model:**  
$$y_i = E[y|x] + \epsilon_i = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi} + \epsilon_i$$
- **Random component** of  $y_i$  is normally distributed:  
$$\epsilon_i \stackrel{iid}{\sim} N(0, \sigma_\epsilon^2)$$
- **Systematic component** (linear predictor):  
$$\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi}$$
- **Link function** here is the *identity link*:  
$$g(E(y|x)) = E(y|x).$$
 We are modeling the mean directly, no transformation.

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# Logistic Regression as a GLM

# The logistic regression model

- **The model:**

$$\text{Logit}(P(x)) = \log \left( \frac{P(x)}{1 - P(x)} \right) = \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi}$$

- **Random component:**  $y_i$  follows a Binomial distribution (outcome is a binary variable)
- **Systematic component:** linear predictor

$$\beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi}$$

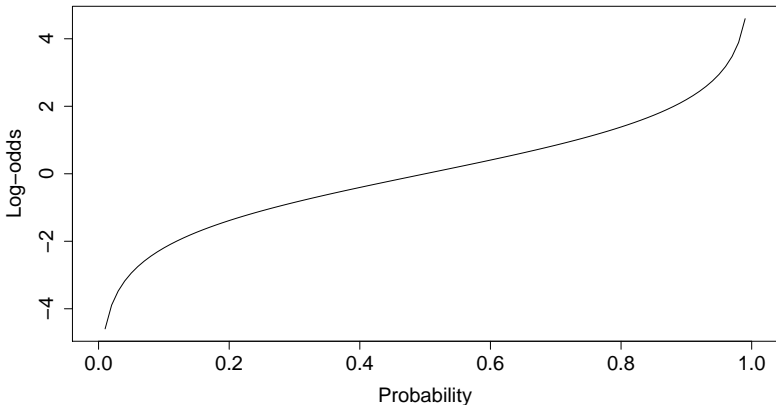
- **Link function:** *logit* (log of the odds that the event occurs)

$$g(P(x)) = \text{logit}(P(x)) = \log \left( \frac{P(x)}{1 - P(x)} \right)$$

$$P(x) = g^{-1} \left( \beta_0 + \beta_1 x_{1i} + \beta_2 x_{2i} + \dots + \beta_p x_{pi} \right)$$

# The logit function

```
logit <- function(P) log(P/(1-P))  
plot(logit, xlab="Probability", ylab="Log-odds",  
      cex.lab=1.5, cex.axis=1.5)
```



# Inverse logit function

```
invLogit <- function(x) 1/(1+exp(-x))
```



# Example: contraceptive use data

```
## [1] "<table class=\"Rtable1\">\n<thead>\n<tr>\n<th class='rowlabel firstrow lastrow'>Age (years)
```

Source: <http://data.princeton.edu/wws509/datasets/#cuse>



# Perform regression

- Outcome: whether using contraceptives or not
- Predictors: age, education level (high/low), whether wants more children or not

```
fit1 <- glm(cbind(using, notUsing) ~ age + education + wantsMore,  
            data=cuse, family=binomial("logit"))  
summary(fit1)
```

```
##  
## Call:  
## glm(formula = cbind(using, notUsing) ~ age + education + wantsMore,  
##      family = binomial("logit"), data = cuse)  
##  
## Deviance Residuals:  
##      Min       1Q   Median       3Q      Max   
## -2.5148  -0.9376   0.2408   0.9822   1.7333   
##  
## Coefficients:  
##              Estimate Std. Error z value Pr(>|z|)      
## (Intercept)   -0.8082    0.1590  -5.083 3.71e-07 ***  
## age25-29       0.3894    0.1759   2.214 0.02681 *    
## age30-39       0.9086    0.1646   5.519 3.40e-08 ***  
## age40-49       1.1892    0.2144   5.546 2.92e-08 ***  
## educationlow  -0.3250    0.1240  -2.620 0.00879 **    
## wantsMoreyes  -0.8330    0.1175  -7.091 1.33e-12 ***  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## (Dispersion parameter for binomial family taken to be 1)  
##  
##      Null deviance: 165.772  on 15  degrees of freedom  
##      Residual deviance: 15.144  on 14  degrees of freedom  
##      AIC: 175.91
```

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# Residuals for logistic regression

# Pearson residuals for logistic regression

- Traditional residuals  $y_i - E[y_i|x_i]$  don't make sense for binary  $y$ .
- One alternative is *Pearson residuals*
  - take the difference between observed and fitted values (on probability scale 0-1), and divide by the standard deviation of the observed value.
- Let  $\hat{y}_i$  be the best-fit predicted probability for each data point, i.e.  $g^{-1}(\beta_0 + \beta_1 x_{1i} + \dots)$
- $y_i$  is the observed value, either 0 or 1.

$$r_i = \frac{y_i - \hat{y}_i}{\sqrt{Var(\hat{y}_i)}}$$

Summing the squared Pearson residuals produces the *Pearson Chi-squared statistic*:

# Deviance residuals for logistic regression

- Deviance residuals and Pearson residuals converge for high degrees of freedom
- Deviance residuals indicate the contribution of each point to the model *likelihood*
- Definition of deviance residuals:

$$d_i = s_i \sqrt{-2(y_i \log \hat{y}_i + (1 - y_i) \log(1 - \hat{y}_i))}$$

Where  $s_i = 1$  if  $y_i = 1$  and  $s_i = -1$  if  $y_i = 0$ .

- Summing the deviances gives the overall deviance:  
$$D = \sum_i d_i^2$$

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# Likelihood and hypothesis testing

# What is likelihood?

- The *likelihood* of a model is the probability of the observed outcomes given the model, sometimes written as:
  - $L(\theta|data) = P(data|\theta)$ .
- Deviance residuals and the difference in log-likelihood between two models are related by:

$$\Delta(D) = -2 * \Delta(\log \text{ likelihood})$$

# Likelihood Ratio Test

- Use to assess whether the reduction in deviance provided by a more complicated model indicates a better fit
- It is equivalent of the nested Analysis of Variance is a nested Analysis of Deviance
- The difference in deviance under  $H_0$  is *chi-square distributed*, with df equal to the difference in df of the two models.

# Likelihood Ratio Test (cont'd)

```
fit0 <- glm(cbind(using, notUsing) ~ -1, data=cuse,  
            family=binomial("logit"))  
anova(fit0, fit1, test="LRT")
```



# Wald test for individual regression coefficients

- Can use partial Wald test for a single coefficient:
  - $\frac{\hat{\beta}}{\sqrt{\text{var}(\hat{\beta})}} \sim t_{n-1}$
  - $\frac{(\hat{\beta}-\beta_0)^2}{\text{var}(\hat{\beta})} \sim \chi^2_{df=1}$  (large sample)
- Wald CI for  $\beta$ :  $\hat{\beta} \pm t_{1-\alpha/2, n-1} \sqrt{\text{var}(\hat{\beta})}$
- Wald CI for odds-ratio:  $e^{\hat{\beta} \pm t_{1-\alpha/2, n-1} \sqrt{\text{var}(\hat{\beta})}}$

*Note:* Wald test confidence intervals on coefficients can provide poor coverage in some cases, even with relatively large samples

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# Additive vs. Multiplicative models

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- Linear regression is an *additive* model
  - e.g. for two binary variables  $\beta_1 = 1.5$ ,  $\beta_2 = 1.5$ .
  - If  $x_1 = 1$  and  $x_2 = 1$ , this adds 3.0 to  $E(y|x)$
- Logistic regression is a *multiplicative* model
  - If  $x_1 = 1$  and  $x_2 = 1$ , this adds 3.0 to  $\log(\frac{P}{1-P})$
  - Odds-ratio  $\frac{P}{1-P}$  increases 20-fold:  $\exp(1.5 + 1.5)$  or  $\exp(1.5) * \exp(1.5)$