FE630 Portfolio Theory and Applications - Assignment #1
Deadline for submission: June 27th 2021.

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1 Instructions

- **Please read carefully these instructions.**
- **Independence:** All students must work independently.
- **Submission:** Submit your answer document via Canvas. No handwritten answer document will be accepted.
- **Answer Document:** Your answer document MUST be in the form of a single pdf file that contains all of your answers including code printouts and graphs. Do not submit your answer document in any format other than pdf. An answer document that does not comprise a single pdf file complete with all answers will not be graded.
- **Cover Sheet:** When required, your answer document must include a cover sheet that states the course name, the homework number, the date, and your name.
- **Legibility and Logical Presentation:** Answer documents that are not easily legible, or not logically presented, or have a non-professional appearance will not be graded.
- **Source Code Requirement:** Your submission MUST also contain a separate set of source code files for all of your solutions. when programming is required. As part of the evaluation, the grader will run the source code to ensure that it provides results coherent with the answer document.
- **Permissible Computer Languages:** You can use any matrix-oriented computer programming language (Python, R or Matlab for example), but do not use any spreadsheets. Problems solved with spreadsheets will receive no credit.
- **Late Submission Policy:** If answer documents and source code files are not submitted by the due date and time, they will incur an immediate penalty of 20%. An additional 10% of penalty will be applied for each additional late day.

Topics for this assignment are the following:

- Utility functions,
- Basics of diversification

Please submit via Canvas a PDF file with your answers and also a zip file with the code used for computations and graphic display, if applicable. **Deadline for submission: June 27th 2021.**

2 Utility functions

1. **Log Utility** (15 points)

   (a) **Certainty Equivalent and Risk Premium**
   An investor whose initial wealth is \$1,000 is offered an opportunity to play a fair game with 2 possible outcomes: winning \$150 with a probability of 1/2 or losing \$150 with a probability of 1/2. The investor utility function is the natural logarithm of his wealth, $u(W) = \ln(W)$.

   - What is the Certainty Equivalent of this risky game?
   - What is the exact Risk Premium of this risky game?
   - How good is the approximation obtained by using a Taylor series expansion?

   (b) **Sensitivity to initial wealth**
   Assume that the initial wealth is \$2,000. What is the risk premium now?

   (c) **Sensitivity to volatility**
   Assume that the initial wealth is \$1,000 and the outcomes are winning \$300 with a probability of 1/2 or losing \$300 with a probability of 1/2. What is the risk premium now?

2. **Certainty Equivalent and Risk Premium for a Power Utility** (30 points)

   An investor whose initial wealth is \$1,000 is offered an opportunity to play a gamble with 2 possible outcomes: winning \$205 with a probability of 2/3 or losing \$400 with a probability of 1/3. The investor utility function is given by $u(W) = W^k$, where k is a real number.

   - Compute the absolute risk aversion coefficient and describe the risk attitude of the investor (Risk-averse or Risk-taker) as a function of k. Justify your answer.
   - Assume that $k = 1/2$. What is the Certainty equivalent of this risky game? What is the exact risk premium of this risky game? How good is the approximation obtained by using a Taylor series expansion and absolute Risk-aversion coefficient?
   - Assume that $k = 2$. What is the Certainty equivalent of this risky game? What is the exact risk premium of this risky game? How good is the approximation obtained by using a Taylor series expansion and absolute Risk-aversion coefficient?

3. **Exponential Utility** (Total 15 points)

   (a) **Setting and notations**
   Consider an investment universe with n securities, $S_1, S_2, \ldots, S_n$. Given a fixed investment horizon, we denote by $r = (r_1, r_2, \ldots, r_n)^T$, the vector of random returns of the securities, and by $\mu = E(r)$ and $\Sigma = cov(r)$ the Expected Return and Variance-covariance matrix respectively. We assume that an investor has a fully invested Portfolio defined by a vector of weights or holdings $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ with random return $r_p(\omega)$.

   (b) **Risk-aversion** (10 points)
   Prove that an investor with utility function $U(r_p) = -e^{-\lambda r_p}$ for $\lambda > 0$ is Risk-averse.

   (c) **Equivalent Quadratic Problem** (5 points)

   - Provide the expressions of $\mu_p(\omega)$, the Expected return of the portfolio as a function of μ. Provide the expression of $\sigma_p(\omega)^2$, the variance of the portfolio as a function of Σ.

- Now we assume that the distribution of r, random return of the securities, is Gaussian. Prove that maximizing the expected utility $E[U(r_P(\omega))]$ is equivalent to a linear quadratic minimization of an objective function depending solely on μ, Σ and ω.

4. **Numerical Application for Exponential Utility Under the constraint that the sum of weights equals to 1.** (20 points)

   (a) Download daily closing prices for the common stocks of Apple, Google, Facebook and Amazon from Jan 1st 2019 to December 31st 2019.

   (b) Use the entire data to compute the Expected Returns Vector and Covariance matrix.

   (c) Create a sequence $\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ containing numbers from 0 to 0.5 in steps of 0.001

   (d) Run through a loop for each value in $\lambda_\imath \in \Lambda$ to

   - Using a numerical solver in R, Matlab or Python, find the optimal portfolio corresponding to Item 3-(c) with the computed ρ, Σ and $\lambda_\imath$ selected from Λ
   - Compute the optimal portfolio's expected return and standard deviation of return;
   - Store the optimal's portfolio return and standard deviation.
   - After completing the loop, plot the efficient frontier.

3 Diversification

1. **Equally Weighted Portfolio** (Total 20 points)

   (a) **Setting and notations**

   Consider an investment universe with n securities, $S_1, S_2, \ldots, S_n$. Given a fixed investment horizon, we denote by $r = (r_1, r_2, \ldots, r_n)^T$, the vector of random returns of the securities, and by $\mu = E(r)$ and $\Sigma = cov(r)$ the Expected Return and Variance-covariance matrix respectively. We assume that an investor has a fully invested **equally weighted Portfolio** defined by a vector of weights or holdings $\omega = (\omega_1, \omega_2, \ldots, \omega_n)^T$ (meaning that $\omega_i = \frac{1}{n}$ for each $i \in \{1, \ldots, n\}$. We will denote by $r_p(\omega)$ the Portfolio's random return and by $\sigma_P(n)$ the Portfolio's risk.

   (b) **Diversifying the risk away** (10 points)

   - Give the expression of the Portfolio's variance $\sigma_P^2(n)$, as a function of Σ and n.
   - Prove that when goes to infinity, the variance of the Equally Weighted Portfolio's variance $\sigma_P^2(n)$ converges to the average covariance between all securities' returns.

   (c) **Case of two a portfolio with 2 securities: bringing the risk to zero** (10 points)

   We assume that $n = 2$ and denote by ρ the covariance between r_1 and r_2. Analyze the Portfolio's Risk σ_P and Expected return $\mu_p = E(r_p)$, as a function of ρ and write a computer program in Python, R or Matlab to plot the efficient frontier (graph of μ_p [y-axis] as a function σ_P [x-axis]) in the three following scenarios.

   - Perfect correlation: $\rho = 1$.
   - Perfect anticorrelation: $\rho = -1$.
   - No correlation: $\rho = 0$.

   For the graphical display of the efficient frontier curve, you may pick arbitrary values for the expected returns (μ_1, μ_2) and volatilities (σ_1, σ_2) with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.