

Estimating realized spot volatility with Gabor frames.

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Motivation: (Co)volatility (σ^2) is important

1. Risk management: VaR, CVaR, etc. . .
2. Portfolio optimization: Mean-variance optimization
3. Option pricing: Black-Scholes formula
4. etc. . .

Motivation: Semimartingales as prices

1. Only a security market with prices that evolve as semimartingales can be priced. Delbean & Schachermayer (1998)
2. X is a semimartingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ if there is a finite variation process A and a local martingale M s.t.

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$$\begin{aligned} X_t &= A_t + M_t \\ &= A_t^c + A_t^J + M_t^c + M_t^J \end{aligned} \tag{1}$$

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(2)

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$$\begin{aligned} X_t &= \int_0^t b_s ds + A_t^J + \int_0^t \sigma_s dW_s + M_t^J \\ &= \int_0^t b_s ds + \int_0^t \int_{\{|x|>1\}} x \mu(dt, dx) \\ &\quad + \int_0^t \sigma_s dW_s + \int_0^t \int_{\{|x|\leq 1\}} x [\mu(dt, dx) - dt\nu(dx)] \end{aligned}$$

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σ , b are stochastic processes. μ is a measure-valued random variable, i.e. $\mu : \Omega \times \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \rightarrow \mathbb{R}$, and ν is its Levy intensity measure.

The problem

1. Discrete observations X_1, \dots, X_n from

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + A_t^J + M_t^J, \quad \forall t \geq 0$$

over a fixed interval $[0, T]$

2. Estimate σ^2 over the interval $[0, T]$.

2.1 Nonparametrically: Arbitrary functional specification

2.2 Globally: find a sequence of random *functions*

$\hat{\sigma}_n^2 : \Omega \times [0, T] \rightarrow \mathbb{R}$ s.t. $\|\hat{\sigma}_n^2 - \sigma^2\|_{L^2[0, T]} \rightarrow 0$ in probability.
(possibly in L^p , $p > 1$).

General approach I

1. Assume $\sigma^2 \in L^2[0, T]$, a.s., where

$$L^2[0, T] = \left\{ f \in \mathbb{C}^{[0, T]} : \int_0^T f(s) \overline{f(s)} ds < \infty \right\} \quad (2)$$

is a Hilbert space, so it behaves just like \mathbb{R}^2 . In particular, the idea of an orthogonal basis (just like $\{(1, 0), (0, 1)\}$ in \mathbb{R}^2) makes sense in $L^2[0, T]$.

2. If $\{\psi_k\}_{k=1}^\infty$ is such an orthogonal basis and $f \in L^2[0, T]$, then there is a sequence of constants $c := \{c_k\}_{k=1}^\infty$ s.t.
 $f = \sum_k^\infty c_k \psi_k$. In particular,

$$\sigma^2(t) = \sum_k^\infty c_k \psi_k(t) \quad t \in [0, T] \quad (3)$$

General approach II

where

$$c_k = \langle \sigma^2, \psi_k \rangle := \int_0^T \sigma^2(s) \bar{\psi}_k(s) ds$$

3. Examples of $\{\psi_k\}_{k=1}^\infty$: Polynomials (Hermite), Fourier series (basis), wavelets, etc. . .
4. Given discrete X_i at times t_i for $i = 0, \dots, n-1$.

$$\begin{aligned} \hat{\sigma}_n^2 &= \sum_h^H \hat{c}_h \psi_h \\ \hat{c}_h &= \sum_{i=0}^n \bar{\psi}_h(t_i) (X_{i+1} - X_i)^2 \end{aligned} \tag{4}$$

Previous solutions

1. Fourier orthogonal basis (Malliavin et. al, 2007)
2. Wavelet orthogonal basis (Hoffmann et. al, 2012)

Consistent ONLY if X_t is a continuous Ito Semimartingale. That is

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s.$$

Our contribution

1. Consistent estimate for the general case in which you have jumps in addition to the continuous part.

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + A_t^J + M_t^J.$$

2. Generalization of the previous methods based on orthonormal basis. We use frames which generalize the concept of orthogonal basis.
3. The extension to frames is not just a technical extension, frames, Gabor frames in particular, possess coefficient noise reduction capabilities not found in orthonormal basis.

Frames

1. They come in pairs: $\{\psi_k\}$ and $\{\tilde{\psi}_k\}$.
2. Together, they possess the representation property: If $f \in L^2[0, T]$ then

$$f = \sum_k c_k \psi_k$$

where

$$c_k = \langle f, \tilde{\psi}_k \rangle$$

3. An orthonormal basis is a frame such that $\psi_k = \tilde{\psi}_k$.

Gabor frames I

1. They are frames constructed by translating a single function g in time and frequency:

$$\{g_{h,k}(t)\} := \{e^{2\pi i h a t} g(t - kb)\}, \quad h, k \in \mathbb{Z}$$

2. If g satisfies the partition of unity property then its dual \tilde{g} is a finite linear combination of translates of g , so it is easy to compute.

Gabor frames II

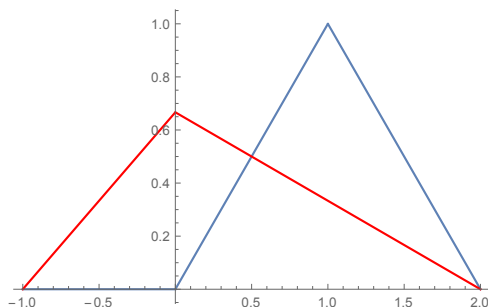


Figure: A dual pair of Gabor frame generators. In BLUE is the B_2 spline. The red line is the dual generator.

Nonparametric jump-robust global spot volatility estimator

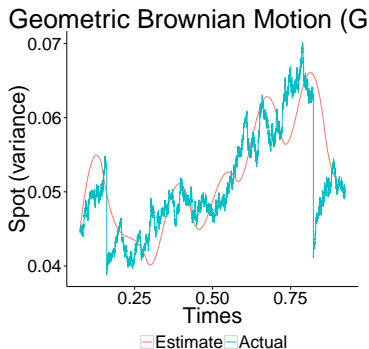
$$\begin{aligned}\hat{\sigma}_n^2 &= \sum_{|h| < H_n, |k| < K_0} \hat{c}_{h,k} g_{h,k} \\ \hat{c}_{h,k} &= \sum_{i=0}^n \bar{\bar{g}}_{h,k}(t_i) (X_{i+1} - X_i)^2 I_{\{|X_{i+1} - X_i| < u_n\}}\end{aligned}\tag{5}$$

where

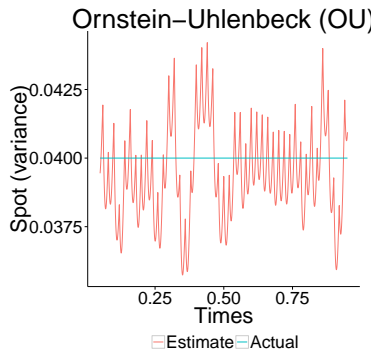
1. K_0 is a constant
2. $H_n = O(\sqrt{T/n})$
3. $\frac{\sqrt{(T/n) \log(n/T)}}{u_n} = o(1)$.

Performance I

Figure: Estimated vs. actual spot volatility of common price models



(a) GBM + JMP

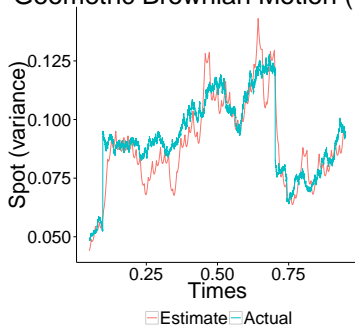


(b) OU + JMP

Performance II

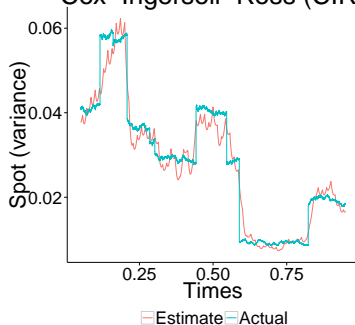
Figure: Estimated vs. actual spot volatility of common price models

Geometric Brownian Motion (GBM)



(a) GBM+JUMP

Cox–Ingersoll–Ross (CIR)



(b) CIR+JUMP

Performance III

Table: Mean integrated square error (MISE) of the frame-based estimator $\hat{\sigma}_n^2$ for popular price models.

n	FP			OU		
	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	1.53×10^{-4}	8.95×10^{-6}	1.44×10^{-4}	8.51×10^{-4}	1.31×10^{-4}	7.20×10^{-4}
5000	2.19×10^{-5}	2.27×10^{-6}	1.96×10^{-5}	5.48×10^{-5}	9.76×10^{-6}	4.50×10^{-5}
50000	2.13×10^{-6}	9.00×10^{-8}	2.04×10^{-6}	6.61×10^{-6}	2.65×10^{-6}	3.97×10^{-6}

n	GBM			CIR		
	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	6.13×10^{-3}	8.70×10^{-4}	5.26×10^{-3}	3.74×10^{-4}	2.32×10^{-4}	1.43×10^{-4}
5000	3.42×10^{-4}	4.07×10^{-5}	3.02×10^{-4}	1.12×10^{-5}	8.29×10^{-6}	2.95×10^{-6}
50000	7.11×10^{-5}	6.36×10^{-6}	6.47×10^{-5}	7.05×10^{-6}	5.64×10^{-6}	1.40×10^{-6}

Note: The mean of the integrated square errors are obtained by taking an average over 50 sample paths generated for each model/number of observations pair.

Conclusion

Theorem

$\hat{\sigma}_n^2$ converges in $L^2[0, T]$ in probability to σ^2 as $n \rightarrow \infty$.