Nonparametric estimation of multivariate volatility: A frame duality approach

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1 Model

Let $\{p_t\}_{0\leq t\leq T}$ be log prices with dynamics given by

$$\mathrm{d}p_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t, \qquad t \in [0, T], \tag{1.1}$$

where $\{W_t\}$ is a standard Brownian motion with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathrm{P})$ satisfying the usual conditions; $\{\mu_t\}$ is locally bounded and cadlag, whereas $\{\sigma_t\}$ is predictable and locally bounded; both $\{\mu_t\}$ and $\{\sigma_t\}$ satisfy the Lipschitz and growth conditions required for the existence of a strong solution. The obeservation horizon T is fixed and finite.

Suppose n log prices $p_{t_i} := p_i$, $i = 1, 2, \cdots, n$, are observed discretely at equidistant intervals $\Delta_n := T/n$. Given this data we wish to obtain an estimate of the spot volatility, σ, within the observation interval $[0, T]$. If all paths of σ are such that

$$\int_0^T \sigma_t^2 \mathrm{d}t < \infty,$$

then the spot volatility is a random element (function) in $L^2(0, T)$, the set of square integrable functions on $[0, T]$. Without loss of generality we set T equal to 1.

Now $L^2(0, 1)$ is a separable Hilbert space so that it admits a frame representation. That is, there is a sequence, $\{\phi_k\}_{k=1}^{\infty}$, of elements in $L^2(0, 1)$ such that for all $f \in L^2(0, 1)$, $f = \sum_{k=1}^{\infty} c_k \phi_k$, where $\{c_k\}_{k=1}^{\infty}$ is a sequence in $\ell^2(\mathbb{N})$, the set of square summable sequences. Frames generalize the notion of orthogonal basis: both share the representation property, but frames need not have elements that are mutually orthogonal. As a result, the representation in terms of the elements of a frame need not be unique. This is by no means a disadvantage; the redundancies in the frame yield computational stability and parsimony in the representation.

2 Frames and Riesz bases

We start by giving a general definition of frames. The specialization to the space of interest $L^2(0,1)$ is immediate.

2.1 Definition *Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ linear in the first argument. A sequence $\{\phi_k\}_{k\in\mathbb{N}}$, with $\phi_k \in \mathcal{H}$ for all k, is a frame if there exists positive constants c and C such that*

$$c\|f\|^2 \leq \sum_{k\in\mathbb{N}} |\langle f, \phi_k\rangle|^2 \leq C\|f\|^2, \tag{2.1}$$

for all $f \in \mathcal{H}$.

The constants c and C are the *frame bounds.* If $\{\phi_k\}$ is a frame then we may associate with it a bounded operator, $A : \ell^2(\mathbb{N}) \to \mathcal{H}$, known as the *synthesis operator* and given by $A\,\{c_k\} := \sum_{k\in\mathbb{N}} c_k\phi_k$. Its adjoint, $A^* : \mathcal{H} \to \ell^2(\mathbb{N})$, is known as the *analysis operator* and is given by $A^*f := \{\langle f, \phi_k\rangle\}$. By composing the analysis operator with the synthesis operator, we obtain the *frame operator*, $F : \mathcal{H} \to \mathcal{H}$, given by

$$Ff := AA^*f = \sum_{k\in\mathbb{N}} \langle f, \phi_k\rangle\phi_k.$$

The frame operator F is bounded, invertible, and self-adjoint[1]. This yields the reprensentation result

$$f = FF^{-1}f = \sum_{k\in\mathbb{N}} \langle f, F^{-1}\phi_k\rangle\phi_k. \tag{2.2}$$

The sequence $\{F^{-1}\phi_k\}_{k\in\mathbb{N}}$ is also a frame, and it is called the *canonical dual* of $\{\phi_k\}_{k\in\mathbb{N}}$. A frame will generally have other duals besides the canonical dual. Frames are quite general objects. What is needed is some control over the type of redundancies allowed in a frame. Without such a restriction results about the rate of convergence of the frame expansion would be impossible to come by. A Riesz basis provides just the type of control needed. Informally, a Riesz basis is a frame whose elements are all essential.

2.2 Definition *A sequence $\{\phi_k\}_{k\in\mathbb{N}}$, with $\phi_k \in \mathcal{H}$ for all k, is a Riesz basis if there exists an orthonormal basis $\{\xi_k\}_{k\in\mathbb{N}}$ of $\mathcal{H}$ and a bounded invertible operator $T : \mathcal{H} \to \mathcal{H}$ such that $\phi_k = T\xi_k$, for all k.*

[1] See Christensen (2001) and the references therein.

A frame is Riesz basis if it is *complete*; i.e. whenever $\langle f, \phi_k \rangle = 0$ for all k then $f = 0$; and there are positive cosntants c, C such that

$$c \sum_{k=1}^{N} |c_k|^2 \leq \left\| \sum_{k=1}^{N} c_k \phi_k \right\|^2 \leq C \sum_{k=1}^{N} |c_k|^2, \tag{2.3}$$

for all finite sequences $\{c_k\}_{1 \leq k \leq N}$. This is equivalent to the condition

$$c \leq \sum_{k \in \mathbb{Z}} |\hat{\phi}(\omega + 2\pi k)| \leq C, \qquad \forall \omega \in [0, 2\pi], \tag{2.4}$$

where $\hat{\phi}$ is the Fourier transform of ϕ. To proceed in our analysis, we specialize further the type of Riesz basis to those that may be generated by a single element (function), $\phi \in L^2(0,1)$. The entire Riesz basis is then generated by translating ϕ across the closed unit interval. By appropriately scaling the ϕ we end up with different levels of granularity in the representation. That is, we have in mind a collection[2] $\{\phi_{h,k}\}_{k,h \in \mathbb{Z}}$, where $\phi_{h,k} := \phi(x/h - k)$. We denote the function space generated by this basis as follows:

$$V_h(\phi) := \left\{ \sum_{k \in \mathbb{Z}} c_{h,k} \phi_{h,k} : \{c_{h,k}\} \in \ell^2(\mathbb{N}) \right\} \tag{2.5}$$

3 Volatility estimation by duality

To obtain estimates for $\sigma^2 \in L^2(0,1)$, we appeal to the duality theorem of Riesz basis. That is, there exists a Riesz basis $\{\psi_{h,k}\}_{k \in \mathbb{Z}}$ such that the projection of σ^2 onto $V_h(\phi)$ is given by

$$\sigma^2_{h,k}(t) = \sum_{k \in \mathbb{Z}} c_{h,k}\ \phi_{h,k}(t), \text{ where} \tag{3.1}$$

$$c_{h,k} = \langle \sigma^2, \psi_{h,k} \rangle \tag{3.2}$$

Now given n observations of the log price process p, we propose the following estimator of the volatility

$$\hat{\sigma}^2_n(t) = \sum_{k=-K}^{K} \hat{c}_{h,k}\ \phi_{h,k}(t) \tag{3.3}$$

$$\hat{c}_{h,k} = \sum_{i=2}^{n} \psi_{h,k}((i-1)/n)(p_{i\Delta_n} - p_{(i-1)\Delta_n}) \tag{3.4}$$

[2]See Unser & Daubechies (1997) for further elaboration on these ideas.

References

Christensen, Ole (2001) “Frames, Riesz bases, and discrete Gabor/wavelet expansions”, *Bull. Amer. Math. Soc*, pp. 273–291.

Unser, Michael and Daubechies, Ingrid (1997) “On the approximation power of convolution-based least squares versus interpolation”, *IEEE Transactions on Signal Processing*, Vol. 45, No. 7, pp. 1697–1711.