PhD thesis proposal: On nonparametric volatility estimation and bond market stability

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Contents

Chapter 1

Nonparametric spot volatility estimation by Gabor frames methods

Volatility estimation for discretely observed asset prices has received a great deal of attention recently, but much of that effort has been focused on the integrated volatility and, to a lesser extent, the spot volatility at a given point in time. In both cases, the object of interest is local: spot volatility at a given point in time or integrated volatility up to a given terminal time. Notable contributions to the spot and integrated volatility estimation literature include the papers by Foster & Nelson (1996), Fan & Wang (2008), Florens-Zmirou (1993), and Barndorff-Nielsen & Shephard (2004). On the other hand, global approaches, which aim to obtain volatility estimates not just at a point in time but for an entire time window, have received much less coverage. As a result, very little is known about the behaviour of these estimators; in fact, what is known is confined to the case where both the price and volatility processes have continuous paths.

So far, two types of global spot volatility estimators have been proposed. The first of these is the Fourier-based volatility estimator proposed by Malliavin & Mancino (2002) and Malliavin & Mancino (2009). This estimator is built up by first obtaining an estimate of the Fourier series expansion of the price process. The estimated Fourier coefficients of the price process are then used to obtain estimates for the Fourier coefficients of the volatility function. Even though Malliavin & Mancino show that estimates of the individual coefficients in the Fourier expansion of the volatility function converge in a mean square sense, they stopped short of provoding an explicit rate of convergence for the entire volatiltity function. The second class of global estimators is the wavelet-based estimators proposed by Genon-Catalot et al. (1992) and Hoffmann et al. (2012). A lot more is known about this class of estimators; for instance, uniform and integrated mean square rates are well known. In both the Fourier and the wavelet approaches, there is a reliance on orthonormal bases: the Fourier and wavelet orthonormal bases, respectively. Using orthonormal bases is optimal provided the individual coefficients in the orthonormal basis expansion of the volatility function can be estimated with good precision. This may not always be the case when using a finite number of data points to estimate the basis coefficients. The global spot volatility estimator we propose is aimed squarely at this problem by employing a Gabor frame methodology. Frames are very flexible and yield robust estimates when the individual coefficients cannot be estimated with high precision; which is probably the case when using a finite amount of data. Additionally, Gabor frames possess highly efficient noise reduction capabilities. This point becomes important when price measurements are subject to market microstructure noise. We elaborate on these points further below.

The rest of this paper is organized as follows: Section 1.1 gives a description of the price dynamics and observed prices; Section 1.2 reviews the part of the theory of Gabor frames that is pertinent to our study; Section 1.3 gives a specification of the Gabor frame based estimator; Section 1.4 contains a discussion of the convergence of the estimator in a MISE sense; Section 1.5 contains a simulation exercise providing further support for the estimator; in Section 1.6 we use the proposed estimator to gain insight into the diurnal pattern of intra day volatility in the bond market; Section 1.7 concludes and briefly discusses future work. All proofs are relegated to the Appendix.

1.1 Model

Let $\{X_t\}_{t\geq 0}$ be log prices with dynamics given by

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t, \qquad X_0 = x, \tag{1.1}$$

where $\{W_t\}$ is a standard Brownian motion with respect to the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathrm{P})$ satisfying the usual conditions; the initial price $x \in \mathbb{R}$ is known; the non-stochastic functions μ and σ are as yet unknown, but assumed to satisfy the Lipschitz and growth conditions sufficient for the existence of a strong solution. We assume prices are observed in the fixed time interval $[0, T]$ at discrete, equidistant times $t_i := i\Delta_n$, where $i = 0, 1, \cdots, n$ and $\Delta_n = T/n$. Given the finite sequence $\{X_{t_i}, i = 0, 1, 2, \cdots, n\}$, our aim is to estimate the spot variance σ^2 in the time interval $[0, T]$ by means of projection methods. We approach this task by estimating the projection of the spot variance in the finite dimensional subspace spanned by finite Gabor frame elements.

1.2 Gabor frames

Frames generalize the notion of orthonormal bases in Banach spaces. If $\{f_k\}_{k\in\mathbb{N}}$ is a frame for a separable Hilbert space $\mathcal{H}$ then every vector $f \in \mathcal{H}$ may be expressed as a linear combination of the frame elements, i.e.

$$f = \sum_{k\in\mathbb{N}} c_k f_k. \tag{1.1}$$

This is similar to how elements in a Hilbert space may be expressed in terms of orthonormal basis; but unlike orthonormal basis, the representation in (1.1) need not be unique, and the frame elements need not be orthogonal. Loosely speaking, frames contain redundant elements. The absence of uniqueness in the frame representation is by no means a shortcoming; on the contrary, we are afforded a great deal of flexibility and stability as a result. In fact, given a finite data sample, the estimated basis expansion coefficients are likely to be imprecise. This lack of precision can create significant distortions when using an orthonormal basis. These distortions are somewhat mitigated when using frames because of the built-in redundancies they contain. Of course, we end up computing more coefficients but there is no hard limit on the number of coefficients we should compute; we use the same n data points whether we compute k or $k + 10$ coefficients.

Furthermore, if $\{f_k\}_{k\in\mathbb{N}}$ is a frame for $\mathcal{H}$, then surjective, bounded transformations of $\{f_k\}_{k\in\mathbb{N}}$ also constitute frames for $\mathcal{H}$, e.g. $\{f_k + f_{k+1}\}_{k\in\mathbb{N}}$ is a frame. So, once we have a frame, we can generate an arbitrary number of them very easily. We may then obtain estimates using each frame and compare results. If our results using the different frames fall within a tight band, then we are afforded some indication of the robustness of our computations.

Another reason frames might be a good idea is that high-frequency financial data is seldom without market microstructure noise, while Fourier and wavelet methods have noise reduction capabilities, Gabor frames are particularly efficient in this regards. As a result, Gabor frames can potentially yield much sparser representations of the volatility process when working in a noisy environment. We will not deal explicitly with market microstructure noise here, we will do so in a second paper. Our discussion of frame theory is rather brief, we only mention concepts needed for our specification of the volatility estimator, for a detailed treatment see the book by Christensen (2008).

1.1 Definition *A sequence $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R})$ is a frame if there exists positive constants c and C such that*

$$c\|f\|^2 \leq \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2 \leq C\|f\|^2, \qquad \forall f \in L^2(\mathbb{R}),$$

where $\|\cdot\|$ is the norm on $L^2(\mathbb{R})$, and $\langle\cdot,\cdot\rangle$ is the inner product defined as:

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\mathrm{d}t, \qquad \forall f, g \in L^2(\mathbb{R}),$$

where $\overline{g(\cdot)}$ is the complex conjugate of $g(\cdot)$.

The constants c and C are called *frame bounds*. If $\{f_k\}$ is a frame then we may associate with it a bounded operator $A : \ell^2(\mathbb{N}) \to L^2(\mathbb{R})$ known as the *synthesis operator*, which, for all $c := \{c_k\} \in \ell^2(\mathbb{N})$, is definied as:

$$A\, c = \sum_{k\in\mathbb{N}} c_k f_k.$$

The adjoint of the synthesis operator $A^* : L^2(\mathbb{R}) \to \ell^2(\mathbb{N})$ is known as the *analysis* operator and is defined as:

$$A^* f := \{\langle f, f_k\rangle\},$$

$\forall f \in L^2(\mathbb{R})$. By composing the analysis and the synthesis operators, we obtain the *frame operator* $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined as:

$$Ff := AA^* f = \sum_{k\in\mathbb{N}} \langle f, f_k\rangle f_k,$$

for all $f \in L^2(\mathbb{R})$. The frame operator F is bounded, invertible, and self-adjoint[1]. This yields the representation result

$$f = FF^{-1} f = \sum_{k\in\mathbb{N}} \langle f, F^{-1} f_k\rangle f_k.$$

The sequence $\{F^{-1} f_k\}_{k\in\mathbb{N}}$ is also a frame, and it is called the *canonical dual* of $\{f_k\}_{k\in\mathbb{N}}$. A frame will generally have other duals besides the canonical dual. That is, there exists sequences $\{\tilde{f}_k\}_{k\in\mathbb{N}}$ besides the canonical sequence such that

$$f = \sum_{k\in\mathbb{N}} \langle f, \tilde{f}_k\rangle f_k \qquad \forall f \in L^2(\mathbb{R}). \tag{1.2}$$

[1]See Christensen (2001) and the references therein.

Next, we specialize the discussion to Gabor frames. The analysis of Gabor frames involves two operators T and M, called translation and modulation operators, respectively. (T as used here will not be confused with the upper bound of the observation interval $[0, T]$, as the meaning of T will be clear from the context). If $f \in L^2(\mathbb{R})$ then

$$T_b f(t) := f(t-b),$$
$$M_a f(t) := e^{2\pi iat} f(t),$$

for $a, b \in \mathbb{R}$, where $i^2 = -1$. Both T and M are shift operators: T is a shift or translation operator on the time axis, whereas M performs shifts on the frequency axis. A Gabor system is constructed by preforming time-frequency shifts on a single function $g \in L^2(\mathbb{R})$, i.e.

$$\{M_h T_k g\}_{h,k\in\mathbb{Z}}$$

is a Gabor system. A Gabor system need not be a frame.

1.2 Definition *Let $g \in L^2(\mathbb{R})$ and $a, b > 0$ and, for all $t \in \mathbb{R}$, define*

$$g_{h,k}(t) := e^{ihat} g(t-kb), \qquad \forall h, k \in \mathbb{Z},$$

where $i = \sqrt{-1}$. If the sequence $\{g_{h,k}\}_{h,k\in\mathbb{Z}}$ constitutes a frame for $L^2(\mathbb{R})$, then it is called a Gabor frame or a Weyl-Heisenberg frame.

The fixed function g is called the *generator* or the *window function*. In order to obtain sharp asymptotic rates, we require g and its dual $\tilde{g}$ to be continuous and compactly supported. The following Lemma taken from Christensen (2006) and Zhang (2008) tells us exactly how to construct such dual pairs.

1.1 Lemma *Let $[r, s]$ be a finite interval, a and b positive constants, and g a continuous function. If $g(t) \neq 0$ when $t \in (r, s)$; $g(t) = 0$ when $t \notin (r, s)$; $a < 2\pi/(s-r)$; and $0 < b < s-r$; then $\{g, \tilde{g}\}$ is a pair of dual Gabor frame generators with*

$$\tilde{g}(t) := g(t)/G(t), \text{ where} \tag{1.3}$$
$$G(t) := (2\pi/a) \sum_{k\in\mathbb{Z}} |g(t-kb)|^2. \tag{1.4}$$

Furthermore,

$$\tilde{g}_{h,k}(\cdot) := e^{ihat} \tilde{g}(\cdot - kb), \qquad \forall h, k \in \mathbb{Z} \tag{1.5}$$

is compactly supported.

Next, we establish that the dual generator $\tilde{g}$ also inherits the continuity properties of g.

1.2 Lemma *Let the dual Gabor frame generator $\tilde{g}$ be constructed as in (1.3). If $\omega_g(\delta)$ denotes the modulus of continuity of g, i.e. $\omega_g(\delta) := \sup\{|g(t) - g(t')| : t, t' \in \mathbb{R}$ and $|t - t'| < \delta\}$, then*

$$\omega_{\tilde{g}_{j,k}}(\delta) = C\omega_g(\delta) \qquad \forall h, k \in \mathbb{Z},$$

where C is a positive constant.

In the sequel, we assume the Gabor frame setup in Lemma (1.1).

1.3 Volatility estimation

We make the following assumptions about the drift and volatility coefficients explicit.

1.1 Assumption

1. The volatility function is strictly positive, bounded, and continuous.
2. The modulus of continuity of the volatility coefficient, $\omega_\sigma(\Delta_n)$, is equal to $o(1/\log(1/\Delta_n))$ as $n \to \infty$.
3. There is $0 < C_T < \infty$ such that $|\mu(t, x)| \leq C_T(1 + |x|)$, for all $t \in [0, T]$ and $x \in \mathbb{R}$.

Since σ^2 may not necessarily be defined on the entire real line, we proceed as in Genon-Catalot et al. (1992) by constructing an extension over the real line that has support in $[0, T]$; this way we are able to apply the Hilbert space machinery in $L^2(\mathbb{R})$. We let $\bar{\sigma}^2$ denote the extension of σ^2, i.e. $\bar{\sigma}^2$ coincides exactly with σ^2 on $[0, T]$ but vanishes outside of $[0, T]$. We summarize these notions as follows:

1.2 Assumption $\bar{\sigma}^2$ is in $L^2(\mathbb{R})$, has support in $[0, T]$, and coincides with σ^2 on $[0, T]$.

With this substitution, we end up with a new process $\bar{X}$ coinciding with X on $[0, T]$ such that

$$\mathrm{d}\bar{X} = \mu(t, \bar{X}_t)\mathrm{d}t + \bar{\sigma}(t)\mathrm{d}W_t, \qquad \bar{X}_0 = x.$$

Now we may avail ourselves of the Gabor frame representation on $L^2(\mathbb{R})$. Let $\{g, \tilde{g}\}$ be a dual Gabor pair constructed as in Lemma (1.1), then $\bar{\sigma}^2$ admits a Gabor frame expansion given by:

$$\bar{\sigma}^2(t) = \sum_{h,k\in\mathbb{Z}} c_{h,k}\ g_{h,k}(t), \text{ where} \tag{1.1}$$

$$c_{h,k} = \langle \bar{\sigma}^2, \tilde{g}_{h,k} \rangle. \tag{1.2}$$

Note that both $\bar{\sigma}^2$ and $\tilde{g}$ have compact support. Indeed $\bar{\sigma}^2$ has support in $[0, T]$, whereas $\tilde{g}$ has support in $[s, r]$. So, $c_{h,k} \neq 0$ only if the supports of $\bar{\sigma}^2$ and $\tilde{g}_{h,k}$ overlap. Furthermore, we note from (1.5) that $\tilde{g}_{h,k+1}$ is simply $\tilde{g}_{h,k}$ shifted by b units; so, $c_{h,k} = 0$ if $|k| > K_0$ with

$$K_0 := \lceil (T + |s| + |r|)/b \rceil, \tag{1.3}$$

where $\lceil x \rceil$ is the smallest positive integer larger than $x \in \mathbb{R}$. Thus $\bar{\sigma}^2$ admits a representation of the form:

$$\bar{\sigma}^2(t) = \sum_{\substack{(h,k)\in\mathbb{Z}^2 \\ |k|\le K_0}} c_{h,k}\ g_{h,k}(t). \tag{1.4}$$

Now, suppose n observations of the log price process are available, and let

$$\Theta_n := \{(h, k) \in \mathbb{Z}^2 : |h| \le H_n \text{ and } |k| \le K_0\},$$

where H_n is an increasing sequence in n. We propose the following estimator of the volatility coefficient in $[0, T]$:

$$\hat{\sigma}_n^2(t) = \sum_{(h,k)\in\Theta_n} \hat{c}_{h,k}\ g_{h,k}(t), \text{ where} \tag{1.5}$$

$$\hat{c}_{h,k} = \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(X_{t_{i+1}} - X_{t_i})^2. \tag{1.6}$$

In the next section we show that the estimator converges to σ^2 on $[0, T]$ in a mean integrated square error sense.

1.4 Asymptotic properties

Let R_n denote the average integrate square deviation of $\hat{\sigma}_n^2$ from $\bar{\sigma}^2$, i.e.

$$R_n = \mathrm{E} \int_{\mathbb{R}} \{\hat{\sigma}_n^2(t) - \bar{\sigma}^2(t)\}^2 \lambda(t) \mathrm{d}t, \tag{1.1}$$

where λ is a positive and continuous weight function with support in $(0, T)$. The weight function allows us to emphasis different time windows when estimating the volatility. For instance, we may wish to emphasize the recent past in real-time applications. We show that R_n tends to 0 as a function of the sample size, n. Note that R_n is the sum of a bias and a variance component, which we write as follows:

$$R_n = B_n^2 + V_n,$$

where

$$\begin{aligned} B_n^2 &:= \int_{\mathbb{R}} (\mathrm{E}[\hat{\sigma}_n^2(t) - \bar{\sigma}^2(t)])^2 \lambda(t) \mathrm{d}t \\ V_n &:= \int_{\mathbb{R}} \mathrm{E}[\{\hat{\sigma}_n^2(t) - \mathrm{E}[\hat{\sigma}_n^2(t)]\}^2] \lambda(t) \mathrm{d}t. \end{aligned}$$

1.1 Proposition *Let $\{g, \tilde{g}\}$ be pair of dual Gabor generators constructed as in Lemma* (1.1). *Suppose the conditions in Assumptions* (1.1) *and* (1.2) *hold. If $H_n^2 \Delta_n, H_n \omega_g(\Delta_n)$, and $\omega_{\sigma^2}(1/H_n) \log H_n \to 0$, then the mean integrated square error R_n tends to 0 as n tends to infinity, with*

$$\begin{aligned} B_n^2 &= O(H_n^2 \Delta_n + \{H_n \omega_g(\Delta_n)\}^2 + \{\omega_{\sigma^2}(1/H_n) \log H_n\}^2) \\ V_n &= O(H_n^2 \Delta_n), \end{aligned} \tag{1.2}$$

where $\omega_g(\delta) := \sup\{|g(t) - g(t')| : t, t' \in \mathbb{R}$ *and* $|t - t'| < \delta\}$.

1.1 Remark

1. First, the above bounds are remarkably similar to those achievable using an orthonormal basis such as wavelets (Genon-Catalot et al., 1992). The variance component is slower by a factor of H_n. This comes about because the vectors in a frame need not be orthogonal. The bias term is slower by a logarithmic factor. Intuitively, the logarithmic term shows up because we are expanding $\bar{\sigma}^2$ using a frame, which may be thought of as containing some redundant term. In practical implementations, this may be a small price to pay for the added flexibility and robustness gained by using frames.

2. Second, this result shows that the variance component of the MISE does not depend on the smoothness properties of either σ^2 and g.

1.5 Simulation study

1.6 Volatility in the bond market

1.7 Conclusion

We proposed an estimator for the spot volatility function using Gabor frame methods. We showed that the estimator converges in a MISE sense and obtained an explicit convergence rate. The evidence for the validity of the proposed estimator will be further reinforced in a simulation study. We will also take the estimator to task using data from the Forex and bond market.

Chapter 2

Market microstructure noise and spot volatility estimation

The need to estimate the volatility of an asset is pervasive in finance. Volatility is the key component in portfolio selection, option pricing, and risk management. Without further restriction on the price process the estimation of the volatility coefficient would be all but impossible. Thankfully financially theory provides some guidance in this regard. At relatively long observation intervals such as a week or a month, it is generally agreed that prices may be modeled as semimartigales. The theory of semimartingales provides a complete answer on what form we should expect the volatility coefficient of the price process to take. Unfortunately, general agreement breaks down when we consider prices sampled at much higher frequencies. High frequency asset price data is seldom without distortions resulting from the established processes in the market. Distortions may result from the so-called bid-ask bound, the release of asynchronous information, and rounding error resulting from discrete prices etc. We elaborate on this sources below. The term *market microstructure noise* was coined by Garman (1976) to describe this type of price contamination.

There is a large literature dealing with nonparametric estimation of the volatility process in the presence of market microstructure noise. Here too the focus of these efforts has been on obtaining estimates for the integrated volatility. Some of the proposed approaches such as the two-time scale estimator of Zhang et al. (2005) and the pre-averging estimator of Podolskij & Vetter (2007) have been extended to cover local spot volatility estimation; see for example Zu & Boswijk (2014). The study of market-microstructure-robust global volatility estimators has thus far been rather scant. A notable contribution, is the wavelet-based estimator proposed by Hoffmann et al. (2012), which works essentially by estimating the wavelet coefficient using pre-averaged market data. In this chapter, we propose an alternative estimator which combines the Gabor frame estimator from the previous chapter with the multi-time scale procedure popularized by Zhang et al. (2005). We believe that the Gabor frame type estimators truly come into their own when dealing with high frequency data that is corrupted with market microstructure noise. Their highly developed time-frequency localization properties ensures that market microstructure noise occurring as either an independent or dependent process can be effectively tracked and eliminated by a simple thresholding approach.

The rest of this chapter is organized as follows: In Section 2.1 we specify the price process and the structure of the noise process. In Section 2.2 we give a specification of the noise robust spot volatility estimator. In Section 2.3 we conduct a simulation exercise to verify the validity of the proposed estimator. In section 2.4 we use the estimator to gain insight about the diurnal pattern of volatility in the bond market; Section 2.5 concludes the paper. A rigorous proof of the consistency of the estimator will be given in the Appendix.

2.1 Model

We consider the problem of making inference on the spot volatility of a security price process using market data sampled discretely in the presence of market microstructure noise. The presence of the noise component implies that the price process of interest is unobservable directly; instead what we have are discrete transaction or bid and ask price data with market microstructure noise contamination. The usual way this market setup is modeled in the literature is via the Additive Market Microstructure (AMN) model, which as the name implies, states that prices are affected additively by microstructure noise. That is, for $i = 0, 1, \cdots, n$ and $0 = t_0 \leq t_i \leq t_n = T$, the i-th observed price at time t_i may be modeled as:

$$Y_{t_i} = X_{t_i} + \varepsilon_{t_i}, \tag{2.1}$$

where ε_{t_i} is the i-th coordinate in an i.i.d sequence of market microstructure noise. The noise component is assumed to be independent of the efficient price process. The unobserved efficient price of interest X is the unique solution to the following stochastic differential equation:

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t)\mathrm{d}W_t, \tag{2.2}$$

where W_t is a Brownian motion, σ is strictly positive, and both μ and σ are continuous and bounded from above. The functions μ and σ are referred to respectively as the drift and volatility functions. This setup is similar to the microstructure environment considered by Zhang et al. (2005) in their estimation of the integrated volatility.

2.1.1 Market microstructure noise

According to Garman (1976), market microstructure noise arises from the moment-to-moment aggregate exchange behavior. Some major sources of market microstructure noise include:

1. *The bid-ask spread.* The price at which an investor can buy an asset, at any fixed point in time, is almost always greater than the price at which he may sell the asset. The *real* or efficient price of the asset is somewhere in between (in some cases, it could be outside the range if there is private information not available to the other participants in the market)

2. *The price impact of trade.* The idea is that each transaction releases information about the underlying asset. For instance, a buyer-initiated transaction tells the market that the asset is more valuable than its current price to somebody. Now, a really big buyer-initiated transaction tells the market that someone with a lot of money and, with no doubt a sophisticated knowledge of the market, thinks the asset is more valuable in the future than its current price. This type of information release can lead to a domino effect where the market goes through several rounds overbidding the price of the asset even though the fundamentals of the asset may not have changed. A pioneering work in the theoretical study of the price impact of trades is the paper by Roll (1984); Hasbrouck (1991) provides an empirically-oriented treatment.

3. *Price round-off* Suppose the market valuation of IBM stock is CHF 19.95666, but because markets price are quoted up to a certain decimal place, the security may be exchanged at say CHF 19.95. Economically, this seems like a small matter, but implementation-wise this a problem for any statistical procedure relying on the assumption that prices satisfy some form of *recurrence or mixing* property. This is because with prices rounded at 2 decimal places, it is no longer the case that any possible value in the continuous range of the asset price will eventually show up in the data, given enough time. Thus the vast majority of price data in fact will *never* be reported.

4. *Human error* This is especially a problem for prices resulting from trading pits. The chaos of the trading pit almost surely guarantees

that data entry mistakes will occur through many times during the trading day.

2.2 Noise-robust estimator

We start by highlighting the difficulty with which we are faced when trying to estimate the volatility function in the presence of market microstructure noise. The main difficulty in Chapter 1 is that we are attempting to estimate an unobservable or latent volatility function on the basis of discretely observable price process; in the present setting, not even the price process X is observable. What we have a our disposal is data Y, which is part efficient price X and part noise ε. We have no idea how much of the observe price Y is noise and how much is efficient price. It is thus safe to assume that we need to do something different in order to take care of the contamination. Motivated by the ideas in Zhang et al. (2005), we propose to divide the sample n into R_n subsamples containing m_n data points so that $R_n m_n$ is approximately n or $R_n m_n \sim n$. We compute the coefficients of the Gabor frame expansion over these subsamples, average the coefficients out, and use the coefficient estimates over the entire sample to bias-correct the resulting coefficients. More rigorously, we propose the following:

$$\hat{\sigma}^2_{n,b}(t) = \sum_{(h,k)\in\Theta_n} \hat{c}^b_{h,k}\; g_{h,k}(t), \tag{2.1}$$

where

$$\hat{c}^b_{h,k} = \hat{c}^R_{h,k} - (m_n/n)\hat{c}_{h,k} \tag{2.2}$$

$$\hat{c}^R_{h,k} = (1/R_n)\sum_{i=0}^{n-R_n} \overline{\tilde{g}_{h,k}(t_i)}(Y_{t_{i+R_n}} - Y_{t_i})^2 \tag{2.3}$$

$$\hat{c}_{h,k} = \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(Y_{t_{i+1}} - Y_{t_i})^2. \tag{2.4}$$

Here as in Chapter 1, $\{g_{h,k}, \tilde{g}_{h,k}\}_{h,k\in\mathbb{Z}}$ is a pair of dual Gabor frames generated as in Lemma (1.1), and $\Theta_n := \{(h,k) \in \mathbb{Z}^2 : |h| \le H_n \text{ and } |k| \le K_0\}$ with H_n an increasing sequence n and $K_0 = \lceil (T + |s| + |r|)/b \rceil$. Any similarities between $\hat{\sigma}^2_n$ from Chapter 1 and $\hat{\sigma}^2_{n,b}$ are at best superficial. Note that $\hat{\sigma}^2_n$ is constructed using the actual efficient price process X, whereas in the present context we have to make do with corrupted market data Y. Also, note that the coefficients $\hat{c}_{h,k}$ play a secondary role here; they merely serve as a devise for bias correction. On the other hand there are strong similarities between the two time scale estimators of Zhang et al. (2005) and Zu & Boswijk (2014). To see this note that we may express (2.3) as follows:

$$\hat{c}^R_{h,k} = (1/R_n) \sum_{i=1}^{R_n - 1} \sum_{j=1}^{m_n} \overline{\tilde{g}_{h,k}(t_{i+(j-1)R_n\Delta_n})}(Y_{i+jR_n\Delta_n} - Y_{i+(j-1)R_n\Delta_n})^2.$$

Now it is clear that $\hat{c}^R_{h,k}$ is the average coefficient taken over the R_n coefficients estimates obtained using the R_n subsamples. In the sequel we show using a simulation study the validity of the proposed estimator. We also apply the estimator compute and study diurnal patterns in intra day volatility in the bond market. A rigorous proof of the consistency of the estimator will be given in the Appendix.

2.3 Simulation study

2.4 Diurnal pattern in the bond market revisited

2.5 Conclusion

Chapter 3

Detecting changes in bond market market stability

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Appendices

We now give the proof of Lemma (1.2).

Proof. Note that G is bounded away from zero. Indeed, since g has support in $[r,s]$, the series on the left hand side of (1.4) has finitely many terms for each t. In addition, it is straight forward to verify that $G(t) = G(t+b)$ for all t; so, G is periodic with period b. It is also clear that because g is continuous, so is G. It follows that G attains its min and max on any interval of length b. Let $I_b := [(s+r-b)/2, (s+r+b)/2]$, then

$$\begin{aligned}\min_{t\in\mathbb{R}} G(t) &= \min_{t\in I_b} G(t)\\ &\geq (2\pi/a)\min_{t\in I_b}|g(t)|^2.\end{aligned}$$

Because g is continuous and g doesn't vanish in (r,s), we conclude that $G_* := \min_{t\in\mathbb{R}} G(t) > 0$. It is also straight forward that $G^* := \max_{t\in\mathbb{R}} G(t) < \infty$. Now, let $t, t' \in \mathbb{R}$ such that $|t-t'| \leq \delta$, then

$$\begin{aligned}|\tilde{g}(t) - \tilde{g}(t')| &= |(G(t)G(t'))^{-1}(g(t)G(t') - g(t')G(t))|\\ &\leq (G_*^{-2})\{|g(t)||G(t)-G(t')| + |G(t)||g(t)-g(t')|\}. \qquad (.1)\end{aligned}$$

Let $\tau := r + (t \bmod b)$, and $\tau' := r + (t' \bmod b)$. It is straight forward to verify that if $|\tau - \tau'| \leq \delta$, then

$$\begin{aligned}|G(t) - G(t')| &\leq \sum_{j=0}^{\lfloor (s+r)/b\rfloor} |g(\tau+jb)^2 - g(\tau'+jb)^2|\\ &\leq \sum_{j=0}^{\lfloor (s+r)/b\rfloor} |g(\tau+jb) - g(\tau'+jb)||g(\tau+jb)+g(\tau'+jb)|\\ &\leq 2\lceil (s+r)/b\rceil g^*\omega_g(\delta), \qquad (.2)\end{aligned}$$

where $g^* := \max_{t\in\mathbb{R}}|g(t)|$. On the other hand, if $|\tau - \tau'| > \delta$, then

$$\begin{aligned}&|G(t) - G(t')|\\ &\quad\leq |g(\tau')^2 - g(r)^2| + |g(s)^2 - g(\tau+c)^2|\\ &\qquad+ \sum_{j=1}^{\lfloor (s+r)/b\rfloor} \{|g(\tau+(j-1)b)^2 - g(\tau'+jb)^2|\}.\end{aligned}$$

where $c = \lfloor (s+r)/b\rfloor b$. It follows as above that

$$|G(t) - G(t')| \leq 2(\lceil (s+r)/b\rceil + 1)g^*\omega_g(\delta, T). \qquad (.3)$$

Returning to (.1), we see that

$$|\tilde{g}(t) - \tilde{g}(t')| \leq C_{\tilde{g}} \omega_g(\delta),$$

where $C_{\tilde{g}} = G_*^2(2(\lceil (s+r)/b \rceil + 1)(g^*)^2 + G^*)$. Now let $h, k \in \mathbb{Z}$, then

$$\begin{aligned} |\tilde{g}_{h,k}(t) - \tilde{g}_{h,k}(t')| &= |e^{ihat}(\tilde{g}(t-kb) - \tilde{g}(t'-kb)| \\ &\leq |(\tilde{g}(t-kb) - \tilde{g}(t'-kb)| \leq C_{\tilde{g}} \omega_g(\delta). \end{aligned} \tag{.4}$$

The last inequality follows because translating a function leaves its modulus of continuity unchanged.

□

Next, we provide a proof of Proposition (1.1).

Proof. We begin with B_n^2, the bias component of the integrated mean square error. Using Itô's product formula (Applebaum, 2009, p. 257), we may express, using Itô's isometry theorem, the average deviation at time t as follows:

$$\mathrm{E}[\hat{\sigma}_n^2(t) - \bar{\sigma}^2(t)] = B_{1,n}(t) + B_{2,n}(t) + B_{3,n}(t),$$

where

$$\begin{aligned} B_{1,n}(t) &:= \sum_{(h,k)\in\Theta_n} g_{h,k}(t) \left\{ \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} \int_{t_i}^{t_{i+1}} \sigma^2(s) \mathrm{d}s - c_{h,k} \right\}, \\ B_{2,n}(t) &:= 2 \sum_{(h,k)\in\Theta_n} g_{h,k}(t) \left\{ \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} \mathrm{E}\left[\int_{t_i}^{t_{i+1}} (X_s - X_{t_i}) \mu(s, X_s) \mathrm{d}s \right] \right\}, \\ B_{3,n}(t) &:= - \sum_{(j,k)\notin\Theta_n} g_{h,k}(t) c_{h,k}. \end{aligned} \tag{.5}$$

The first two components, $B_{1,n}(t)$ and $B_{2,n}(t)$, result from the fact that X is being observed discretely; whereas $B_{3,n}$ results because only a finite number of the frame elements are being used in the approximation. We refer to Theorem 4.1 in Zhang (2008) for an estimate of $B_{3,n}$:

$$B_{3,n} = O(\omega_{\bar{\sigma}^2}(1/H_n) \log H_n).$$

So, the smoother the volatility coefficient the smaller the number of frame elements needed to obtain a decent approximation. We obtain a bound on

$B_{1,n}$ by noting that

$$\begin{aligned}\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}\int_{t_i}^{t_{i+1}}\sigma^2(s)\mathrm{d}s - c_{h,k} &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\sigma^2(s)\{\overline{\tilde{g}_{h,k}(t_i)}-\overline{\tilde{g}_{h,k}(s)}\}\mathrm{d}s\\ &\leq C_\beta \omega_{\tilde{g}_{h,k}}(\Delta_n),\end{aligned}$$

where $C_\beta = \int_0^T \sigma^2(s)\mathrm{d}s < \infty$. It follows from Lemma (1.2) that

$$B_{1,n}(t) \leq C_B H_n \omega_g(\Delta_n)$$

with $C_B = 2(2K_0+1)g^*\tilde{g}^* C_\beta C_{\tilde{g}}$. Next, we estimate $B_{2,n}(t)$. Note that

$$X_s - X_{t_i} = \int_{t_i}^{s}\mu(u,X_u)\mathrm{d}u + \int_{t_i}^{s}\sigma(u)\mathrm{d}W_u.$$

So, we may write

$$\begin{aligned}&\mathrm{E}\left[\int_{t_i}^{t_{i+1}}(X_s - X_{t_i})\mu(s,X_s)\mathrm{d}s\right]\\ &\quad= \mathrm{E}\left[\int_{t_i}^{t_{i+1}}\left(\int_{t_i}^{s}\mu(u,X_u)\mathrm{d}u\right)\mu(s,X_s)\mathrm{d}s\right]\\ &\qquad+ \mathrm{E}\left[\int_{t_i}^{t_{i+1}}\left(\int_{t_i}^{s}\sigma(u)\mathrm{d}W_u\right)\mu(s,X_s)\mathrm{d}s\right]\\ &\quad=: \beta_{2,i}+\beta_{3,i}.\end{aligned}$$

By Fubinni's theorem, the Cauchy-Schwarz inequality, Itô's isometry theorem, and the linear growth condition on the drift, we have

$$\begin{aligned}\beta_{3,i} &\leq C_T\int_{t_i}^{t_{i+1}}\left(\int_{t_i}^{s}\sigma^2(u)\mathrm{d}u\right)^{1/2}\mathrm{E}[(1+|X_s|)^2]^{1/2}\mathrm{d}s\\ &\leq C_3\Delta_n^{3/2}.\end{aligned}\tag{.6}$$

Similarly, it may be verified that $\beta_{2,i} \leq C_2\Delta_n^2$. Now we may write

$$\begin{aligned}B_{2,n}(t) &\leq \Delta_n^{1/2}\sum_{(h,k)\in\Theta_n} g_{h,k}(t)\left\{\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}C_4\Delta_n\right\}\\ &\leq ((2K_0+1)C_4 g^*\tilde{g}^* T)(2H_n+1)\Delta_n^{1/2}\\ &= O(H_n\Delta_n^{1/2}).\end{aligned}\tag{.7}$$

Thus the square bias is bounded as follows:

$$
\begin{aligned}
B_n^2 &= \int_R (B_{1,n}(t) + B_{2,n}(t) + B_{3,n}(t))^2 \lambda(t)\mathrm{d}t \\
&= O(H_n^2 \Delta_n + H_n^2 \omega_g^2(\Delta_n) + \omega_{\bar{\sigma}^2}^2(1/H_n)\log^2 H_n).
\end{aligned} \tag{.8}
$$

Next, we obtain a bound for the variance term. Note that V_n may be expressed as follows:

$$
V_n = V_{1,n} + V_{2,n}
$$

with

$$
\begin{aligned}
V_{1,n} &:= \sum_{(h,k)\in\Theta_n} \mathrm{var}[\hat{c}_{h,k}]\gamma_{h,k}^2, \text{ and} \\
V_{2,n} &:= \sum_{\substack{(h,k),(h',k')\in\Theta_n \\ (h,k)\neq(h',k')}} \mathrm{cov}[\hat{c}_{h,k}, \hat{c}_{h',k'}]\gamma_{h,k}\gamma_{h',k'},
\end{aligned}
$$

where $\gamma_{h,k} := \int_0^T g_{h,k}(t)\mathrm{d}t$. We start with $V_{1,n}$. If we set

$$
\begin{aligned}
Y_i &:= \left(\int_{t_i}^{t_{i+1}} \sigma(s)\mathrm{d}W_s \right)^2 \qquad \text{and} \\
Z_i &:= \left(\int_{t_i}^{t_{i+1}} \mu(s, X_s)\mathrm{d}s \right)^2 + 2\left(\int_{t_i}^{t_{i+1}} \mu(s, X_s)\mathrm{d}s \right)\left(\int_{t_i}^{t_{i+1}} \sigma(s)\mathrm{d}W_s \right),
\end{aligned}
$$

then we may write

$$
\hat{c}_{h,k} = \sum_{i=0}^{n-1} g_{h,k}(t_i)(Y_i + Z_i).
$$

Furthermore, setting

$$
\alpha_{1,i} := \sum_{(h,k)\in\Theta_n} g_{h,k}^2(t_i)\gamma_{h,k}^2,
$$

allows us to write

$$
V_{1,n} := \sum_{i=0}^{n-1} \alpha_{1,i}(\mathrm{var}[Y_i] + \mathrm{var}[Z_i] + 2\mathrm{cov}[Y_i, Z_i]).
$$

Using the Cauchy-Schwarz inequality, we obtain

$$\operatorname{var}[Y_i] = \left(\int_{t_i}^{t_{i+1}} \sigma^2(t_i) \mathrm{d}s \right)^2 = O(\Delta_n^2); \tag{.9}$$

$$\operatorname{var}[Z_i] \leq \mathrm{E}[Z_i^2] = O(\Delta_n^3); \tag{.10}$$

$$\operatorname{cov}[Y_i, Z_i] \leq (\operatorname{var}[Z_i]\operatorname{var}[Y_i])^{1/2} = O(\Delta_n^{5/2}). \tag{.11}$$

Since $\alpha_{1,i} = O(H_n)$, we conclude that $V_{1,n} = O(H_n \Delta_n)$. It is clear that $V_{2,n}$ may be bounded in a similar fashion. Indeed, let

$$\alpha_{2,i} := \sum_{\substack{(h,k)\in\Theta_n \\ (h,k)\neq(h',k')}} g_{h',k'}(t_i) g_{h,k}(t_i) \gamma_{h',k'} \gamma_{h,k},$$

then we may write

$$V_{2,n} := \sum_{i=0}^{n-1} \alpha_{2,i} (\operatorname{var}[Y_i] + \operatorname{var}[Z_i] + 2\operatorname{cov}[Y_i, Z_i]).$$

Since $\alpha_{2,i} = O(H_n^2)$, it follows that $V_{2,n} := O(H_n^2 \Delta_n)$.

□