On nonparametric spot volatility estimation, market microstructure noise, and fixed-income market stability.

Wale Dare

University of St. Gallen

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Overview

1. Nonparametric spot volatility estimation by Gabor frames methods
2. Market microstructure noise and spot volatility estimation
3. Tracking changes in bond market stability

Section 1

Nonparametric spot volatility estimation by Gabor frames methods

The problem

1. Discrete observations $X_1, \cdots, X_n$ from

$$X_t = X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s, \qquad \forall t \geq 0$$

   over a fixed interval $[0, T]$
2. Estimate σ^2 over the interval $[0, T]$.

Why might spot variance be valuable to have?

1. *Versatility*. Given $\hat{\sigma}^2$ over $[0, T]$, we get
   1.1 $\hat{\sigma}, \hat{\sigma}^3, \ldots, \hat{\sigma}^p$
   $f(\hat{\sigma})$, where f is continuous.
   1.2 $\int_0^T \hat{\sigma}^2(s)ds, \int_0^T \hat{\sigma}^4(s)ds, \ldots, \int_0^T \hat{\sigma}^p(s)ds$
   $\int_0^T f(\hat{\sigma}(s))ds$, where f is continuous.

Previous solutions

Orthonormal bases

$$\hat{\sigma}^2 = \sum_k \hat{c}_k \phi_k,$$

where $\{\phi_k\}$ is an orthonormal basis, and

$$\hat{c}_k = \sum_i (X_{i+1} - X_i)^2 \phi_k(t_i)$$

1. Fourier basis
2. Wavelet basis

Why frames might be a good idea

1. Generalization of the previous methods based on orthonomal basis.
2. Possesses coefficient noise reduction capabilities not found in orthonormal basis.

Gabor frames

In a system with a lot of noise, coefficient estimates may not be precise. In such cases it makes sense to use a frame instead of an orthonormal basis. We specialize to Gabor frames.

$$
\begin{aligned}
\hat{\sigma}^2 &= \sum_{h,k} \hat{c}_{h,k} \phi_{h,k} \\
\hat{c}_{h,k} &= \sum_{i} (X_{i+1} - X_i)^2 \tilde{\phi}_{h,k}(t_i),
\end{aligned}
$$

where $\{\phi_{h,k}\}$ and $\{\tilde{\phi}_{h,k}\}$ is a pair of dual Gabor frames.

Performance I

Table: Mean integrated square error (MISE) of the frame-based estimator $\hat{\sigma}_n^2$ for popular price models.

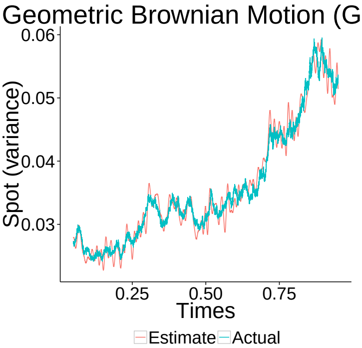
	ABM			OU		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	1.30×10^{-4}	2.86×10^{-6}	1.27×10^{-4}	1.43×10^{-4}	1.19×10^{-5}	1.31×10^{-4}
5000	1.41×10^{-5}	1.11×10^{-6}	1.30×10^{-5}	1.45×10^{-5}	1.62×10^{-6}	1.28×10^{-5}
50000	2.32×10^{-6}	1.02×10^{-6}	1.30×10^{-6}	2.36×10^{-6}	1.12×10^{-6}	1.23×10^{-6}

	GBM			CIR		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	2.18×10^{-4}	4.18×10^{-6}	2.14×10^{-4}	6.26×10^{-5}	8.51×10^{-7}	6.17×10^{-5}
5000	2.33×10^{-5}	1.58×10^{-6}	2.17×10^{-5}	6.82×10^{-6}	6.00×10^{-7}	6.22×10^{-6}
50000	4.66×10^{-6}	1.02×10^{-6}	3.64×10^{-6}	1.46×10^{-6}	6.06×10^{-7}	8.52×10^{-7}

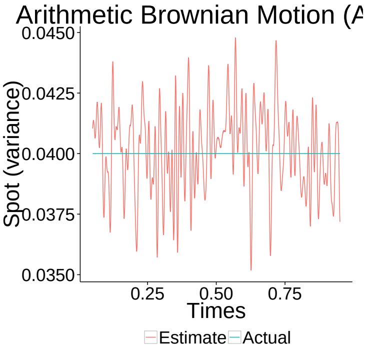
Note: The mean of the integrated square errors are obtained by taking an average over 100 sample paths generated for each model/number of observations pair.

Performance II

Figure: Estimated vs. actual spot volatility of common price models

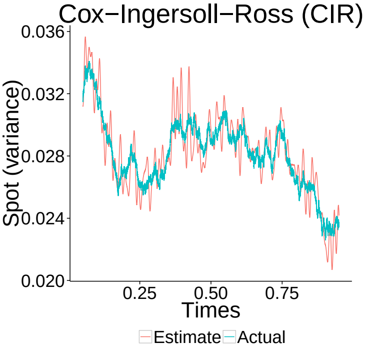


(a) GBM

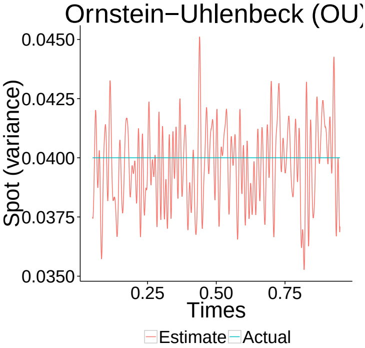


(b) ABM

Performance III



(c) CIR



(d) OU

Performance IV

Theorem

Let $\{g, \tilde{g}\}$ be pair of dual Gabor generators. If g is Lipschitz continuous and

$$H_n^2 \Delta_n + H_n^{-\alpha} \log H_n = o(1)$$

then $R_n(\alpha, c)$ converges to 0, with

$$B_n^2(\alpha, c) = O(H_n^2 \Delta_n + H_n^{-2\alpha} \log^2 H_n)$$
$$V_n(\alpha, c) = O(H_n^2 \Delta_n),$$

where $\Delta_n = 1/n$ is the step size, and H_n is the order of magnitude of the number of estimated frame coefficients.

Future work

1. cadlag volatility instead of just continuous volatility
2. Price processes with jumps

$$X_t = X_0 + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s + J_s, \qquad \forall t \geq 0$$

3. Multivariate extension
4. Emprirical study of spot volatility in fixed income markets

Section 2

Market microstructure noise and spot volatility estimation

Market microstructure noise I

1. Noise generated from the moment-to-moment aggregate exchange behavior. The main sources of noise are:
    1.1 *The bid-ask spread*. The price at which an investor can buy an asset, at any fixed point in time, is almost always greater than the price at which he may sell the asset. The *real* or efficient price of the asset is somewhere in between (in most cases, it could be outside the range if there is private information not available to the other participants in the market)
    1.2 *The price impact of trade*. The idea is that each transation releases information about the underlying asset. For instance, a buyer initiated transaction tells the market that the asset is more valuable than its current price to somebody. Now a really big buy transaction tells the market that someone with a lot of money and no doubt very sophisticated thinks the asset is more valuable than its currect. This can lead to the market overbidding the price of the asset even the foundamentals of the asset may not have changed.

Market microstructure noise II

1.3 *Price round-off* Say the market valuation of IBM stock is CHF 19.95666. Because markets prices are quoted up to a certain decimal place, the stock may be transacted at say CHF 19.95. In practice, this seems errelevant, but poses a big problem on any statistical procedure that assumes anything resembling *recurrence or mixing*. This means that any possible value in the continuous range of our price will eventually show up in the data, given enough time. With rounding only numbers up to 2 decimal places will ever show up in our data. The vast majority of data having more than two decimal places will *never* show up.

1.4 Human error Especially, in the days of physical trrading pits quote were frequently misentered.

Why care about microstructure noise I

Short answer: It makes our statistical tools *very, very inaccurate*. It also turns the world upside down. For example the more data you have the more inacurrate your estimate! Consider the following:

$$Y_{t_i} = X_{t_i} + \varepsilon_{t_i}, \tag{1}$$

where

$$X_t = x + \int_0^t \mu(s)ds + \int_0^t \sigma(s)dW_s \tag{2}$$

and W_s is a Brownian motion and ε is independent of X. The following is the coefficient associated with the Gabor frame expansion.

$$c_{h,k} = \int_0^1 \overline{g_{h,k}(s)}\sigma(s)^2 ds \tag{3}$$

Why care about microstructure noise II

A consiquence of the result from section 1 is that

$$\sum_{i=0}^{N} \overline{g_{h,k}(t_i)}(X_{i+1} - X_i)^2 = \int_0^1 \overline{g_{h,k}(s)}\sigma(s)^2 ds + o(1)$$
$$(MISE)$$

This works perfect without microstructure noise. With microstructure noise which in real markets is present, we have

$$\sum_{i=0}^{N} \overline{g_{h,k}(t_i)}(Y_{i+1} - Y_i)^2 = \int_0^1 \overline{g_{h,k}(s)}\sigma(s)^2 ds + 2N\omega_\varepsilon^2 + o(1)$$
$$(MISE)$$

Two-time scales/subsampling

We adapt the two timescales approach of Zhang et al. (2005) to obtain consistent coefficient estimates.

$$\hat{\sigma}_n^2(t) = \sum_{h,k} c_{h,k}^b \, g_{h,k}(t), \qquad \forall t \in [0, T], \tag{4}$$

where

$$c_{h,k}^b = c_{h,k}^R - (m_n/n) c_{h,k} \tag{5}$$

$$c_{h,k}^R = (1/R_n) \sum_{i=0}^{n-R_n} \overline{g_{h,k}(t_i)} (Y_{t_{i+R_n}} - Y_{t_i})^2 \tag{6}$$

$$c_{h,k} = \sum_{i=0}^{n-1} \overline{g_{h,k}(t_i)} (Y_{t_{i+1}} - Y_{t_i})^2. \tag{7}$$

Future work

1. Rigurous proof
2. Extension to Ito semimartingale. Jumps, cadlag volatility.
3. Simulation study
4. Empirical study

Section 3

Tracking changes in bond market stability

The empirical evidence

1. Litterman & Scheinkman (1991). 98% Volatility of spot short rate acconted for by three factors. 89% for the first component.
2. Bouchaud et al. (1999) report a decay rate faster than q^{-4}, where q is the rank of the component.

The HJM model of fixed income markets

Models the entire forward rate curve (FRC). The FRC is a stochastic "curve".For each maturity τ, we have

$$f(t,\tau) = f(0,\tau) + \int_0^t \mu(s,\tau)ds + \int_0^t \sum^m \sigma_j(s,\tau)dW_j(s),$$

which is the instantanious return on a loan contracted at time t for issue at time τ.

Realized Spectrum

The empirical evidence suggests there is $d \approx 3$ such that the co-volatility matrix of the foreward rates satisfies

$$\Sigma(t) \approx \sum_{i}^{d} \lambda_i(t)(v_i(t) \otimes v_i(t))$$

where $\lambda_i(t)$ and $v_i(t)$ are the i-th most principal eigen value and eigenvector of $\Sigma(t)$. We propose to estimate the spectrum of $\Sigma(t)$ via the spectrum of $\hat{\Sigma}(t)$, the Gabor frame estimate of the covolatility matrix. So that

$$\hat{\Sigma}(t) \approx \sum_{i}^{d} \hat{\lambda}_i(t)(\hat{v}_i(t) \otimes \hat{v}_i(t))$$

Tracking and predicting market swings

The first eigen vector has been found to be very stable, fluctuating only during periods of market turmoil. (Carmona & Tehranchi (2006)) It does makes sense to monitor changes in the first eigen vector. This can be summarized conveniently using consines:

$$\chi_1(t_i) = \frac{\langle \hat{v}_1(t_i), \hat{v}_1(t_{i-1}) \rangle}{\|\hat{v}_1(t_i)\| \|\hat{v}_1(t_{i-1})\|}$$

Does $\chi_1(t_i)$ contain leading information about foreward rates?

$$f(t_i, \tau) = \alpha_\tau + \sum_{j=1}^{q} \beta_{\tau,j} \hat{\chi}_{n,1}(t_{i-j}) + \sum_{j=1}^{p} \gamma_{\tau,j} f(t_{i-j}, \tau) + \eta_{\tau,i}, \tag{8}$$