Chapter 1

Estimating realized spot volatility with Gabor frames.

Volatility estimation using discretely observed asset prices has received a great deal of attention recently, however, much of that effort has been focused on estimating the *integrated* volatility and, to a lesser extent, the *spot* volatility at a given point in time. Notable contributions to this literature include the papers by Foster & Nelson (1996), Fan & Wang (2008), Florens-Zmirou (1993), and Barndorff-Nielsen & Shephard (2004). In these studies, the object of interest is local in nature: spot volatility at a given point in time or integrated volatility up to a terminal point in time. In contrast, estimators which aim to obtain volatility estimates for entire time windows have received much less coverage. These are the so-called global estimators; the objects of interest are global: random elements whose realizations are sample paths, i.e. functions defined on nontrivial time intervals.

In the global volatility estimation literature, two estimators stand out: the Fourier-based estimator proposed by Malliavin & Mancino (2002) and the wavelet-based estimator proposed by Genon-Catalot et al. (1992) and later developed by Hoffmann et al. (2012). The Fourier-based estimator is built up by first obtaining an estimate of the Fourier series expansion of the price process. The estimated Fourier coefficients of the price process are then used to obtain estimates for the Fourier coefficients of the volatility function. While there is no doubt that the Fourier-based estimator works (converges) both in theory and in practice (See Malliavin et al., 2007; Malliavin & Mancino, 2009), the theoretical investigation of the estimator seems somewhat incomplete. For instance, Malliavin & Mancino show that estimates of the individual coefficients in the Fourier expansion converge in a mean square sense but stopped short of providing an explicit rate of convergence for the entire volatility function. On the other hand, a lot more is known about the wavelet-based estimator; for instance, explicit rates of convergence for the entire volatility function are well known.

In both the Fourier and the wavelet approaches, there is a reliance on orthonormal bases: the Fourier and wavelet orthonormal bases, respectively. Now the use of orthonormal bases in *practical* work is optimal if the individual coefficients in the orthonormal basis expansion can be estimated with good precision. A coefficient with a large estimation error may be expected to cause a proportional distortion in the overall estimate of the volatility function. In practical work, where we must rely on a finite number of data points to obtain estimates for the bases coefficients, it is clear that coefficient error can easily become an issue. The global spot volatility estimator we propose is aimed squarely at this problem; it employs a Gabor frame methodology to mitigate the effects of bases coefficient error. Frames are very flexible and yield robust estimates in practical situations where coefficients lack precision or have been entirely *erased*. This robustness may be particularly pertinent in a high-frequency setting, where price measurements are subject to market microstructure noise. We elaborate on these points further below.

The rest of this paper is organized as follows: Section 1.1 gives a description of the dynamics of observed prices; Section 1.2 briefly reviews Gabor frames theory; Section 1.3 gives a specification of the Gabor frame based estimator; Section 1.3.1 discusses the asymptotic convergence of the frame-based estimator; Section 1.4.3 provides further support for the estimator via a simulation exercise; Section **??** provides a descriptive analysis of the diurnal pattern of intraday volatility in the bond markets; Section **??** proposes a multivariate extension; finally, Section **??** concludes and briefly discusses future work. The main technical arguments are contained in the Appendix.

1.1 Prices

We follow the literature by assuming that the asset price is a semimartingale[1]. This assumption is motivated in part by the fundamental theorem of asset pricing, which requires asset prices to be semimartingales as a neccessary and [SUFFICIENT?] condition for an arbitrage-free market. The class of semimartingales in its entirety is somewhat too broad and unwieldy. In fact, whether or not the notion of spot volatility makes sense for such a broad class is not immediately clear. We will instead confine our analysis to the class of stochastic processes known as *Itô semimartingales*: these are semimartingales

[1] Adapted processes which almost surely have right-continuous, left-limited paths and may be expressed as a sum of a local martingale and a finite variation process.

with absolutely continuous *characteristics*[2].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered probability space satifying the *usual conditions*. We consider prices that evolve over time according to:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t, \qquad t \geq 0 \tag{1.1}$$

where J is a pure jump process; W is a standard Brownian motion; X_0 is either known or observable at time 0; b and σ are suitably restricted processes. We follow the ususal practise by referring to either σ or σ^2 as *(spot) volatility*, with *(spot) variance* reserved exclusively for σ^2 when it is important to make a distinction between the two. In addition, to stress the connection with the simple Brownian motion with drift case, b and σ will occasionally be referred to, respectively, as the *drift coefficient* and the *diffusion coefficient*. The class of continuous Itô processes is large; it contains for example solutions of all stochastic differential equations.

We assume prices are observed in the fixed time interval $[0, 1]$ at discrete, equidistant times $t_i := i\Delta_n$, where $i = 0, 1, \cdots, n$ and $\Delta_n = 1/n$. Given the finite sequence $\{X_{t_i}, i = 0, 1, 2, \cdots, n\}$, our aim is to estimate the spot variance σ^2 in the time interval $[0, 1]$ by nonparametric methods. Note that our objective is not an approximation of a point but rather the approximation of an entire function. Thus an estimator of the spot variance may be viewed as a random element (function), as opposed to a random variable, that must converge in some sense to the spot variance, which itself is a random element. We approach this task by estimating the expansion of the spot variance using finite collections of Gabor frame elements.

1.2 Frames

Frames generalize the notion of orthonormal bases in Hilbert spaces. If $\{f_k\}_{k\in\mathbb{N}}$ is a frame for a separable Hilbert space $\mathcal{H}$ then every vector $f \in \mathcal{H}$ may be expressed as a linear combination of the frame elements, i.e.

$$f = \sum_{k\in\mathbb{N}} c_k f_k. \tag{1.2}$$

This is similar to how elements in a Hilbert space may be expressed in terms of orthonormal basis; but unlike orthonormal basis, the representation in (1.2) need not be unique, and the frame elements need not be orthogonal. Loosely speaking, frames contain redundant elements. The absence of uniqueness in

[2] See Jacod & Shiryaev (2003).

the frame representation is by no means a shortcoming; on the contrary, we are afforded a great deal of flexibility and stability as a result. In fact, given a finite data sample, the estimated basis expansion coefficients are likely to be imprecise. This lack of precision can create significant distortions when using an orthonormal basis. These distortions are somewhat mitigated when using frames because of the built-in redundancies they contain. Of course, we end up computing more coefficients but there is no hard limit on the number of coefficients we should compute; we use the same n data points whether we compute k or $k+10$ coefficients.

Furthermore, if $\{f_k\}_{k\in\mathbb{N}}$ is a frame for $\mathcal{H}$, then surjective, bounded transformations of $\{f_k\}_{k\in\mathbb{N}}$ also constitute frames for $\mathcal{H}$, e.g. $\{f_k + f_{k+1}\}_{k\in\mathbb{N}}$ is a frame. So, once we have a frame, we can generate an arbitrary number of them very easily. We may then obtain estimates using each frame and compare results. If our results using the different frames fall within a tight band, then we are afforded some indication of the robustness of our computations.

Our discussion of frame theory will be rather brief; we only mention concepts needed for our specification of the volatility estimator. For a more detailed treatment see the book by Christensen (2008). In the sequel if z is a complex number then we shall denote respectively by $\bar{z}$ and $|z|$ the complex conjugate and magnitude of z. Let $L^2(\mathbb{R})$ denote the space of complex-valued functions defined on the real line with finite norm given by

$$\|f\| := \left(\int_{\mathbb{R}} f(t)\overline{f(t)}dt \right)^{1/2} < \infty, \qquad \forall f \in L^2(\mathbb{R}).$$

Define the inner product of two elements f and g in $L^2(\mathbb{R})$ as $\langle f, g\rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}dt$.

Denote by $\ell^2(\mathbb{N})$ the set of complex-valued sequences defined on the set of natural numbers $\mathbb{N}$ with finite norm given by

$$\|c\| := \left(\sum_{k\in\mathbb{N}} c_k\overline{c_k} \right)^{1/2} < \infty, \qquad \forall c \in \ell^2(\mathbb{N}),$$

where c_k is the k-th component of c. The inner product of two sequences c and e in $\ell^2(\mathbb{N})$ is $\langle c, e\rangle := \sum_{k\in\mathbb{N}} c_k\overline{e_k}$. Now we may give a definition for frames:

1.1 Definition *A sequence $\{f_k\}_{k\in\mathbb{N}} \subset L^2(\mathbb{R})$ is a frame if there exists positive constants C_1 and C_2 such that*

$$C_1\|f\|^2 \le \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2 \le C_2\|f\|^2, \qquad \forall f \in L^2(\mathbb{R}).$$

The constants C_1 and C_2 are called *frame bounds.* If $C_1 = C_2$ then $\{f_k\}_{k\in\mathbb{N}}$ is said to be *tight.* Because an orthonormal basis satisfies Parceval's equality[3], it follows that an orthonormal basis is a tight frame with frame bounds identically equal to 1, i.e. $C_1 = C_2 = 1$. Now if $\{f_k\}$ is a frame, we may associate with it a bounded operator $\mathcal{A}$ that maps every function f in $L^2(\mathbb{R})$ to a sequence c in $\ell^2(\mathbb{N})$ in the following way:

$$\mathcal{A}f = c \qquad \text{where} \qquad c_k = \langle f, f_k\rangle, \qquad \forall k \in \mathbb{N}. \tag{1.3}$$

Because $\mathcal{A}$ takes a function defined on a continuum ($\mathbb{R}$) to a sequence, which is a function defined on the discrete set $\mathbb{N}$, $\mathcal{A}$ is known as the *analysis* operator associated with the frame $\{f_k\}_{k\in\mathbb{N}}$. The boundedness of the analysis operator follows from the frame bounds in Definition (1.1). Now $\mathcal{A}^*$, the adjoint[4] of $\mathcal{A}$, is well-defined and takes sequences in $\ell^2(\mathbb{N})$ to functions in $L^2(\mathbb{R})$. Using the fact that $\mathcal{A}^*$ must satisfy the equality $\langle \mathcal{A}f, c\rangle = \langle f, \mathcal{A}^*c\rangle$ for all $f \in L^2(\mathbb{R})$ and $c \in \ell^2(\mathbb{N})$, it may be deduced that

$$\mathcal{A}^*c = \sum_{k\in\mathbb{N}} c_k f_k, \qquad \forall c \in \ell^2(\mathbb{N}),$$

where c_k is the k-th component of the sequence c. The adjoint, $\mathcal{A}^*$, may be thought of as reversing the operation or effect of the analysis operator; for this reason it is known as the *synthesis* operator [5].

Now an application of the operator $(\mathcal{A}^*\mathcal{A})^{-1}$ to every frame element f_k yields a sequence $\{\tilde{f}_k := (\mathcal{A}^*\mathcal{A})^{-1}f_k\}_{k\in\mathbb{N}}$, which is yet another frame for $L^2(\mathbb{R})$. The frame $\{\tilde{f}_k\}_{k\in\mathbb{N}}$ is known as the *canonical dual* of $\{f_k\}_{k\in\mathbb{N}}$. Denoting the analysis operator associated with the canonical dual by $\tilde{\mathcal{A}}$, it may be shown[6] that

$$\mathcal{A}^*\tilde{\mathcal{A}} = \tilde{\mathcal{A}}^*\mathcal{A} = \mathcal{I}, \tag{1.4}$$

where $\mathcal{I}$ is the identity operator and $\tilde{\mathcal{A}}^*$ is the adjoint of the analysis operator of the canonical dual. Furthermore, Proposition 3.2.3 of Daubechies (1992)

[3]Parceval's equality states that if $\{f_k\}_{k\in\mathbb{N}}$ is an orthonormal basis for $\mathcal{H}$ a separable Hilbert space then

$$\|f\|^2 = \sum_{k\in\mathbb{N}} |\langle f, f_k\rangle|^2 = \|\hat{f}\|^2, \qquad \forall f \in \mathcal{H},$$

where $\hat{f}$ is the Fourier transform of f.

[4]The adjoint is the operator counterpart of the transpose of a real matrix.

[5]It is also occasionally referred to as the *reconstruction* operator

[6]See for example (Daubechies, 1992, Proposition 3.2.3)

shows that $\tilde{\mathcal{A}}$ satisfies

$$\tilde{\mathcal{A}} = \mathcal{A}(\mathcal{A}^*\mathcal{A})^{-1}, \tag{1.5}$$

so that the analysis operator of the canonical dual frame is fully characterized by $\mathcal{A}$ and its adjoint. It is easily seen that (1.4) yields a representation result since if $f \in L^2(\mathbb{R})$ then

$$f = \tilde{\mathcal{A}}^*\mathcal{A}f = \mathcal{A}^*\tilde{\mathcal{A}}f = \sum_{k\in\mathbb{N}} \langle f, \tilde{f}_k\rangle f_k. \tag{1.6}$$

Thus, in a manner reminiscent of orthonormal basis representations, every function in $L^2(\mathbb{R})$ is expressible as a linear combination of the frame elements, with the frame coefficients given by $\langle f, \tilde{f}_k\rangle$, the correlation between the function and the elements of the dual frame. It follows from the first equality in (1.4) and the commutativity of the duality relationship that functions in $L^2(\mathbb{R})$ may also be written as linear combinations of the elements in $\{\tilde{f}_k\}_{k\in\mathbb{N}}$, with coefficients given by $\langle f, f_k\rangle$, i.e. $f = \sum_{k\in\mathbb{N}}\langle f, f_k\rangle \tilde{f}_k$.

Now let $\mathcal{P} := \tilde{\mathcal{A}}\mathcal{A}^* = \mathcal{A}\tilde{\mathcal{A}}^*$. A consequence of the noncommutativity[7] of the composition operator is that even though $\mathcal{A}^*\tilde{\mathcal{A}} = \mathcal{I}$ (1.4), the operator $\mathcal{P}$ need not be equal to $\mathcal{I}$. In fact, Proposition 3.2.3 in (Daubechies, 1992) shows that in the general case $\mathcal{P}$ is the orthogonal projection operator of $\ell^2(\mathbb{N})$ onto $R(\mathcal{A})$, the range space of $\mathcal{A}$[8]. That $R(\mathcal{A})$ in general may not coincide with $\ell^2(\mathbb{N})$ is a consequence of the fact that frames may be redundant, i.e. that they may contain "more" elements than is required, and thus may be linearly dependent. The level of redundancy is inversely related to the "size" of $R(\mathcal{A})$. So that the more redundancy the frame contains the smaller the range space of $\mathcal{A}$. To see this, let $\{f_k\}_{k\in\mathbb{N}}$ be a frame such that $f_1 = f_2 + f_3$; so, $\{f_k\}_{k\in\mathbb{N}}$ is redundant and linearly dependent. Now if c is in the range space of $\mathcal{A}$, the associated analysis operator, then the k-th coordinate of c must satisfy $c_k = \langle f, f_k\rangle$ for some $f \in L^2(\mathbb{R})$. By the linearity of the inner product, $c_1 = c_2 + c_3$. Thus every element of $R(\mathcal{A})$ must satisfy the restriction that its first component must equal the sum of the second and the third. Since not every element of $\ell^2(\mathbb{N})$ is subject to this restriction, $R(\mathcal{A})$ must be a proper subset of $\ell^2(\mathbb{N})$. So, dependence relationships among the frame elements serve to restrict the range of $\mathcal{A}$; the greater the number of dependence relationships there are, the smaller $R(\mathcal{A})$ becomes. As we shall see shortly, this fact has important consequences for how coefficient error affects the precision of the synthesis operator. Of course if the frame has

[7] A binary operation $\star$ is noncommutative if $A \star B$ is generally not equal to $B \star A$. A classic example is matrix multiplication.

[8] $R(\mathcal{A}) := \{c \in \ell^2(\mathbb{N}) : c = \mathcal{A}f, \text{ for some } f \in L^2(\mathbb{R})\}$.

no redundancy, then $R(\mathcal{A})$ will coincide with $\ell^2(\mathbb{N})$ and $\mathcal{P}$ will just be the identity operator.

1.2.1 Why use frames?

The main reason we might be interested in frame methods for estimating volatility is robustness to coefficient noise. By this we mean the imprecision that may result by virtue of the fact that in practice the frame coefficients may not be known with precision and must be estimated. Coefficient error has many sources: error resulting from using a finite data sample, rounding or quantization error, and error arising from the use of data contaminated with market microstructure noise.

The robustness of redundant frames to coefficient error is well-documented. For instance, Munch (1992) report noise reduction that is directly proportional to the degree of redundancy of the frame. Cvetković & Vetterli (1998) consider coefficient error due to quantization, and report an even high degree of robustness to this type of coefficient errors. That redundant frames exhibit this kind of robustness is not entirely unexpected. Redundant frames in essence include near-duplicates of frame elements; so that, any error arising from a given frame coefficient is easily made up for by the presence of other frame elements with similar informational content.

Daubechies (1992) provides the following heuristic explanation in terms of the size of the range space of the analysis operator. Let $\{f_k\}_{k\in\mathbb{N}}$ be a redundant frame in $L^2(\mathbb{R})$, and $\mathcal{A} : L^2(\mathbb{R}) \to \ell^2(\mathbb{N})$ be the associated analysis operator defined as in (1.3). Now, it follows from (1.4) and (1.5) that

$$\mathcal{I} = \tilde{\mathcal{A}}^*\mathcal{A} = \tilde{\mathcal{A}}^*\tilde{\mathcal{A}}\mathcal{A}^*\mathcal{A}.$$

From the discussion in the previous section, the orthogonal projector of $\ell^2(\mathbb{N})$ onto $R(\mathcal{A})$ is $\tilde{\mathcal{A}}\mathcal{A}^* = \mathcal{P}$. Combining this with the equation above, we have $\mathcal{I} = \tilde{\mathcal{A}}^*\mathcal{P}\mathcal{A}$. So the representation result in (1.6) can be expressed as

$$f = \tilde{\mathcal{A}}^*\mathcal{A}f.$$

Now assuming the coefficients of f under the operation of the analysis operator were contaminated by white noise sequence ε, we would have at our disposal $\mathcal{A}f+\varepsilon$ instead of simply $\mathcal{A}f$. Further assume that ε is decomposable as follows: $\varepsilon = \varepsilon_{\mathcal{A}^\perp} + \varepsilon_A$, where ε_A resides in the range space of $\mathcal{A}$, and $\varepsilon_{\mathcal{A}^\perp}$ resides in the orthogonal complement of $R(\mathcal{A})$. So, by definition, $\mathcal{P}\varepsilon_{\mathcal{A}^\perp} = 0$. The operation of reconstructing a function from the noisy coefficients may now be expressed as

$$f_\varepsilon = \tilde{\mathcal{A}}^*\mathcal{P}(\mathcal{A}f + \varepsilon) = \tilde{\mathcal{A}}^*\mathcal{P}(\mathcal{A}f + \varepsilon_A + \varepsilon_{\mathcal{A}^\perp}) = \tilde{\mathcal{A}}^*\mathcal{P}(\mathcal{A}f + \varepsilon_A).$$

It is thus clear the deviation of the approximation $\|f - f_\varepsilon\| = \|\tilde{\mathcal{A}}^* \mathcal{P} \varepsilon_A\|$ should be lower than $\|\varepsilon\|$ to the extent that the range space of $\mathcal{A}$ is small, which is another way of saying that the approximation error is reduced to the extent that the frame is redundant. As was noted by Daubechies (1992), this explanation is heuristic and probably accounts for only a small portion of the noise reduction gain observed in practical work. Nevertheless, it provides a starting point for thinking about the source of the robustness of frames.

1.2.2 Gabor frames

Next, we specialize the discussion to Gabor frames. The analysis of Gabor frames involves two operators: the *translation* operator $\mathcal{T}$ and the *modulation* operator $\mathcal{M}$ defined as follows:

$$\mathcal{T}_b f(t) := f(t-b), \qquad b \in \mathbb{R}, f \in L^2(\mathbb{R}), \tag{1.7}$$

$$\mathcal{M}_a f(t) := e^{2\pi \mathrm{i} a t} f(t), \qquad a \in \mathbb{R}, f \in L^2(\mathbb{R}), \tag{1.8}$$

where i is the imaginary number, i.e. $\mathrm{i} = \sqrt{-1}$. Both $\mathcal{T}$ and $\mathcal{M}$ are shift operators: $\mathcal{T}$ is a shift or translation operator on the time axis, whereas $\mathcal{M}$ performs shifts on the frequency axis. A Gabor system is constructed by performing time-frequency shifts on a single function $g \in L^2(\mathbb{R})$. For example, if a and b are real numbers then the sequence of functions

$$\{\mathcal{M}_{ha} \mathcal{T}_{kb} g\}_{h,k \in \mathbb{Z}},$$

constitutes a Gabor system. A Gabor system need not be a frame.

1.2 Definition *Let $g \in L^2(\mathbb{R})$, and let $a > 0$, $b > 0$ be positive real numbers. Define for $t \in \mathbb{R}$*

$$g_{h,k}(t) := e^{2\pi \mathrm{i} h a t} g(t - kb), \qquad \forall h, k \in \mathbb{Z}.$$

If the sequence $\{g_{h,k}\}_{h,k \in \mathbb{Z}}$ constitutes a frame for $L^2(\mathbb{R})$, then it is called a Gabor frame[9].

The fixed function g is known as the *Gabor frame generator*[10]; a is known as the *modulation parameter*; and b is known as the *translation parameter*. In order to obtain sharp asymptotic rates, we require g and its dual $\tilde{g}$ (see (1.6)) to be continuous and compactly supported. The following Lemma taken from Christensen (2006) and Zhang (2008) tells us exactly how to construct such dual pairs. We restate the Lemma here without proof.

[9]It is also sometimes referred to as a *Weyl-Heisenberg* frame.
[10]It is referred to elsewhere as the *window function*.

1.1 Lemma *Let $[r, s]$ be a finite interval, let $a > 0$, $b > 0$ be positive constants, and let g be a continuous function. If $g(t) \neq 0$ when $t \in (r, s)$; $g(t) = 0$ when $t \notin (r, s)$; and a, b satisfy: $a < 1/(s - r)$, $0 < b < s - r$; then $\{g, \tilde{g}\}$ is a pair of dual Gabor frame generators, with the dual Gabor generator given by*

$$\tilde{g}(t) := g(t)/G(t), \text{ where} \tag{1.9}$$

$$G(t) := \sum_{k \in \mathbb{Z}} |g(t - kb)|^2/a. \tag{1.10}$$

Furthermore,

$$\tilde{g}_{h,k}(t) := e^{2\pi i hat} \tilde{g}(t - kb), \qquad \forall h, k \in \mathbb{Z} \tag{1.11}$$

is compactly supported.

Next, we state here and prove in the Appendix that the dual generator $\tilde{g}$ inherits the continuity properties of g.

1.2 Lemma *Let the dual Gabor frame generator $\tilde{g}$ be constructed as in (1.9). If $\bar{\omega}(g, \delta)$ denotes the modulus of continuity of g, i.e. $\bar{\omega}(g, \delta) := \sup\{|g(t) - g(t')| : t, t' \in \mathbb{R}$ and $|t - t'| < \delta\}$, then*

$$\bar{\omega}(\tilde{g}_{j,k}, \delta) = C\bar{\omega}(g, \delta) \qquad \forall h, k \in \mathbb{Z},$$

where C is a positive constant.

Proof. See Appendix A. □

In the sequel, we assume the Gabor frame setup in Lemma (1.1).

1.3 Volatility estimation: Continuous Itô semimartingale

In this section we specify a consistent estimator of the spot volatility within a framework of continuous prices. That is $J_t = 0$ and (1.1) reduces to:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s \qquad t \geq 0, \tag{1.12}$$

where both b and σ are adapted, b is càdlàg, and σ is continuous. This is similar to the framework considered by Hoffmann et al. (2012); the only difference is that spot volatility is càdlàg in their setting, whereas here it is continuous[11]. We label these assumptions for easy reference.

[11] The slight extension to the càdlàg spot volatility setting is underway.

1.1 Assumption

1. Spot volatility σ is a strictly positive and adapted stochastic process with continuous paths on $[0, 1]$.

2. The drift coefficient μ is adapted and càdlàg on $[0, 1]$.

The asymptotic results we shall establish rely heavily on the modulus of continuity of realized spot variance, so, it is not immediately clear that the continuity assumption can be further weakened. The right-continuity assumption (càdlàg is a French abbreviation for right continuous, with finite left limits) on the drift ensures that the drift coefficient is pathwise bounded on $[0, 1]$. Likewise, the spot variance is pathwise bounded as a result of the continuity assumption. Let $\{g, \tilde{g}\}$ be a pair of dual Gabor frame generators constructed as in Lemma (1.1), then σ^2 admits a Gabor frame expansion given by:

$$\sigma^2(t) = \sum_{h,k\in\mathbb{Z}} c_{h,k}\, g_{h,k}(t), \text{ where} \tag{1.13}$$

$$c_{h,k} = \langle \sigma^2, \tilde{g}_{h,k} \rangle. \tag{1.14}$$

Note that both σ^2 and $\tilde{g}$ have compact support. Indeed σ^2 has support in $[0, 1]$, whereas $\tilde{g}$ has support in $[s, r]$. So, $c_{h,k} \neq 0$ only if the supports of σ^2 and $\tilde{g}_{h,k}$ overlap. Furthermore, we note from (1.11) that $\tilde{g}_{h,k+1}$ is simply $\tilde{g}_{h,k}$ shifted by b units; so, $c_{h,k} = 0$ if $|k| > K_0$ with

$$K_0 := \lceil (1 + |s| + |r|)/b \rceil, \tag{1.15}$$

where $\lceil x \rceil$ is the smallest positive integer larger than $x \in \mathbb{R}$. Thus σ^2 admits a representation of the form:

$$\sigma^2(t) = \sum_{\substack{(h,k)\in\mathbb{Z}^2 \\ |k|\leq K_0}} c_{h,k}\, g_{h,k}(t),$$

and for sufficiently large positive integer H,

$$\sigma^2(t) \approx \sum_{\substack{|h|\leq H \\ |k|\leq K_0}} c_{h,k}\, g_{h,k}(t).$$

Now, suppose n observations of the price process are available, and let

$$\Theta_n := \{(h, k) \in \mathbb{Z}^2 : |h| \leq H_n \text{ and } |k| \leq K_0\}, \tag{1.16}$$

where H_n is an increasing sequence[12] in n. We propose the following estimator of the volatility coefficient:

$$v_n(X,t) := \sum_{(h,k)\in\Theta_n} \hat{c}_{h,k}\, g_{h,k}(t), \qquad \forall t \in [0,1], \text{ where} \tag{1.17}$$

$$\hat{c}_{h,k} := \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(X_{t_{i+1}} - X_{t_i})^2. \tag{1.18}$$

So Θ_n is the number of frame elements included in the expansion; it is also the number of coefficients to be estimated. Note that $|\Theta_n|$, the size or number of elements in Θ_n, is $(2K_0+1)(2H_n+1)$; and since K_0 is a finite quantity, it follows that $|\Theta_n| = O(H_n)$, i.e. the number of estimated coefficients is proportional to H_n, and therefore will grow with observation frequency. As will be seen shortly, the growth rate of H_n is asymptotically relevant. In the next section we show that the estimator converges to σ^2 on $[0,1]$ in a mean integrated square error sense.

1.3.1 Asymptotic properties

In this section we obtain an estimate of the rate of convergence of the Gabor frame estimator on compact intervals of the real line. We will take $[0,1]$ as the prototype for such intervals. The usual way to think about σ^2 is as a stochastic process, but it is just as natural to think of it as a random element. A random element is an extension of the familiar concept of a random variable to include situations where the state space can be any metric space E. Because our interest is in studying the convergence of the estimator on $[0,1]$, E will be set equal to $L^2[0,1]$, the set of real-valued, square integrable function on $[0,1]$. Now, due to the path continuity assumption on σ^2, we will restrict our attention to $C^0[0,1]$, the set of real-valued, continuous functions on $[0,1]$ equipped with the $L^2[0,1]$ norm. So, given an outcome ω in the sample space Ω, the restriction of realized volatility $\sigma^2(\omega)$ to $[0,1]$ is a continuous function defined on $[0,1]$. Now, with regards smoothness, $C^0[0,1]$ is a very diverse class, comprising functions that are infinitely differentiable, those that are nowhere differentiable, and everything in-between. For a random volatility coefficient it may very well be the case that for some outcome ω, realized volatility $\sigma^2(\omega)$ is very smooth with finite derivatives of all orders, whereas for outcome ω', $\sigma^2(\omega')$ is very rough with kinks everywhere. For instance, with probability one, the one-dimesional Brownian motion maps outcomes ω to continuous functions in $C^0[0,1]$, but it is almost surely the case that none of these functions will be differentiable.

[12]See Proposition (1.1) for restrictions on the growth of H_n.

Now, a good estimator should yield successively better approximations with increasing observation frequency, regarless of the degree of smoothness of the realized volatility function. But in all probability, the *rate* of convergence of the approximation would depend on the smoothness of the realized volatility, with faster rates achieved for smooth functions. That is, for outcomes ω, ω' in Ω, if $\sigma^2(\omega)$ is smoother as a function of time than $\sigma^2(\omega')$ then the number of observations required to achieve a given level of accuracy when $\sigma^2(\omega)$ is realized should not exceed the required number when $\sigma^2(\omega')$ is realized. So, while an estimator might eventually converge regardless of the regularity of volatility, the convergence rate may be outcome- or state-dependent.

To develop an asymptotic theory for the Gabor frame estimator which can account for state-dependency in convergence rates, we characterize the effective state space of the volatility coefficient, viewed as a random element in $C^0[0,1]$, according to a smoothness criterion. A simple way to achieve this characterization is via the Hölder continuity criterion. Let $0 < \alpha \leq 1$, a function f in $C^0[0,1]$ is said to be Hölder continuous with exponent α if there is a finite constant K such that whenever x and y are distinct numbers in $[0,1]$ then

$$|f|_\alpha := \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq K. \tag{1.19}$$

The set of Hölder continuous functions with exponent α is denoted by $C^{0,\alpha}[0,1]$. The Hölder class with $\alpha = 1$ is the familiar class of Lipschitz continuous functions. The Hölder classes admit a natural ordering relation whereby if α is larger than β then every function that is Hölder-continuous with exponent α is also Hölder-continuous with exponent β. Note that with regards regularity (smoothness), the ordering is reversed: the smoother the function the larger the Hölder exponent. Consequently, the Lipschitz class ($\alpha = 1$) is contained in every Hölder class, and it is also the class with the smoothest (most regular) functions. As mentioned earlier, Brownian paths are nowhere differentiable, but using Hölder classes the regularity of Brownian paths can be further qualified; this is a consequence of the well-known Lévy's Modulus of Continuity Theorem (Rogers & Williams, 2000, Theorem I.10.2), which states that, with probability one, no Brownian path is Hölder-continuous with exponent less than $1/2$ on $[0,1]$. Now for a fixed exponent α, the following norm may be defined on $C^{0,\alpha}[0,1]$:

$$\|f\|_\alpha := \sup_{t\in[0,1]} |f(t)| + |f|_\alpha,$$

where $|f|_\alpha$ is defined in (1.19). The norm is obviously well-defined since f is a continuous function defined on a compact set. Now using $\|\cdot\|_\alpha$ the state space may be characterized in a such a way that functions with similar regularity propoerties can be grouped together. We accomplish this via Hölder balls: A Hölder ball of radius $c > 0$ is given by:

$$\mathcal{H}(\alpha, c) := \{f \in C^{0,\alpha}[0,1] : \|f\|_\alpha \le c\}.$$

With this device it is possible to obtain convergence rates that take into account the regularity of realized volatility. While this is already quiet satisfying, it is also of some interest to achieve flexibility with regards the drift coefficient. Where the drift is concerned, regularity or even continuity for that matter is irrelevant; what is key is pathwise boundedness. The natural way to achieve flexibility in this respect is to group realized drift according to membership in balls of radius c in $L^\infty[0,1]$, that is,

$$\mathcal{U}(c) := \{f \in L^\infty[0,1] : \|f\|_\infty \le c\}. \tag{1.20}$$

In this way we are able to characterize the sample space according to the regularity of realized volatility and the boundedness of realized drift. This leads to the consideration of the asymptotic behavior of the estimator when $b \in \mathcal{U}(c)$ and $\sigma^2 \in \mathcal{H}(\alpha, c)$ for $c, \alpha > 0$[13]. We denote such events by $\mathcal{E}(\alpha, c)$, i.e.,

$$\mathcal{E}(\alpha, c) := \{\omega : b(\omega) \in \mathcal{U}(c)\} \cap \{\omega : \sigma^2(\omega) \in \mathcal{H}(\alpha, c)\}.$$

So, an outcome ω is in $\mathcal{E}(\alpha, c)$ if realized drift is caught between $-c$ and c on $[0,1]$; realized volatility $\sigma^2(\omega)$ is Hölder continuous with exponent α, and $\|\sigma^2(\omega)\|_\alpha \le c$. Note that the implication of the last statement is that there is some $c' \le c$ such that realized volatility is cought between 0 and c'.

We are now only left with the task of making the obvious modification to the usual integrated mean square error criterion :

$$R_n(\alpha, c) := E[\|v_n - \sigma^2\|^2 I_{\mathcal{E}(\alpha,c)}],$$

where $I_{\mathcal{E}(\alpha,c)}$ is the indicator function of $\mathcal{E}(\alpha, c)$, $\|\cdot\|$ is the $L^2[0,1]$ norm, and n is the observation frequency. Note that if $\mathcal{E}(\alpha, c) = \Omega$, the expression above will just be the usual integrated mean square error criterion. By restricting the volatility and the drift according to events $\mathcal{E}(\alpha, c)$, the asymptotic properties of the estimator may be studied with full flexibility. That is we may obtain results of the form $\limsup_{n\to\infty} \tilde{n}_{n,\alpha,c} R_n(\alpha, c) < \infty$, where the rate may vary for different values of α and c.

[13] It is not essential to use different c's for b and σ^2.

Much like the usual integrated mean square error risk, $R_n(\alpha, c)$ admits a decomposition in terms of an integrated square bias component and an integrated variance component. To see this, note the following:

$$R_n(\alpha, c) = E \int_0^1 \{(v_n(X,t) - \sigma^2(t)) I_{\mathcal{E}(\alpha,c)}\}^2 dt \tag{1.21}$$

$$= \int_0^1 E[\{(v_n(X,t) - \sigma^2(t)) I_{\mathcal{E}(\alpha,c)}\}^2] dt \tag{1.22}$$

$$= \int_0^1 E[(v_n(X,t) - \sigma^2(t)) I_{\mathcal{E}(\alpha,c)}]^2 dt + \int_0^1 \operatorname{var}[v_n(X,t) I_{\mathcal{E}(\alpha,c)}] dt. \tag{1.23}$$

The equality in (1.22) results from an interchange of the expectectation and integration operators justified by Fubini's theorem. The decomposition in line (1.23) results from the usual mean square error decomposition into a square bias and a variance component for each $t \in [0, 1]$. The two summands in the last line are the bias and variance components and will be denoted $B_n^2(\alpha, c)$ and $V_n(\alpha, c)$, respectively; we obtain estimates for their rates of convergence below.

1.1 Proposition *Let $\{g, \tilde{g}\}$ be pair of dual Gabor generators constructed as in Lemma* (1.1). *Suppose the conditions in Assumption* (1.1) *hold. If g is Lipschitz continuous and $H_n \uparrow \infty$ satisfies*

$$H_n^2 \Delta_n = o(1)$$

then $R_n(\alpha, c)$ converges to 0, with

$$B_n^2(\alpha, c) = O(H_n^2 \Delta_n + H_n^{-2\alpha} \log^2 H_n)$$
$$V_n(\alpha, c) = O(H_n^2 \Delta_n), \tag{1.24}$$

where $\Delta_n = 1/n$ is the step size, and H_n is the order of magnitude of the number of estimated frame coefficients.

Proof. See the appendix.

1.1 Remark

1. First, the above bounds are remarkably similar to those achievable using an orthonormal basis such as wavelets (Genon-Catalot et al., 1992). The variance component is slower by a factor of H_n. This comes about because the vectors in a frame need not be orthogonal. The bias term is

slower by a logarithmic factor. Intuitively, the logarithmic term shows up because we are expanding σ^2 using a frame, which may be thought of as containing some redundant term. Overall, the rate of convergence of the integrated mean square error differ only by logarithmic term. In practical implementations, this is an insignificant price to pay for the added flexibility and robustness gained by using frames.

2. Second, this result shows that the variance component of the MISE does not depend on the smoothness properties of either σ^2 and g.

1.2 Proposition *Suppose the price process is specified as in* (1.12). *Let* $\{g, \tilde{g}\}$ *be pair of dual Gabor generators satisfying the conditions of Lemma* (1.1) *such that* g *is Lipschitz continuous on the unit interval. If* $H_n \uparrow \infty$ *satisfies*

$$H_n(n^{-1}\log(n))^{1/2} = o(1),$$

then $v_n(X,t)$*, defined in* (1.17)*, converges in* $L^2[0,1]$ *to* σ^2 *in probability.*

Proof. We begin by noting that

$$\begin{aligned} v_n(X,t) - \sigma^2(t) = & \sum_{(h,k)\in\Theta_n} (\hat{c}_{h,k} \ - c_{h,k}) \ g_{h,k}(t) \\ & - \sum_{(h,k)\notin\Theta_n} c_{h,k} \ g_{h,k}(t), \end{aligned} \tag{1.25}$$

where

$$\begin{aligned} \hat{c}_{h,k} &= \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(X_{t_{i+1}} - X_{t_i})^2 \text{ and} \\ c_{h,k} &= \int_0^1 \overline{\tilde{g}_{h,k}(s)}\sigma^2(s)ds. \end{aligned}$$

We tackle the summands in (1.25) in turn starting with the first one. But first let

$$M_i := \int_{t_i}^{t_{i+1}} b(s)ds, \quad \text{and} \quad S_i := \int_{t_i}^{t_{i+1}} \sigma(s)dW_s,$$

and note that since $X_{t_{i+1}} - X_{t_i} = M_i + S_i$, it follows that

$$(X_{t_{i+1}} - X_{t_i})^2 = M_i^2 + 2M_iS_i + S_i^2.$$

So, (1.25) may be written as

$$v_n(X,t) - \sigma^2(t) = B_{1,n}(t) + B_{2,n}(t) + B_{3,n}(t) + B_{4,n}(t),$$

where

$$\begin{aligned}
B_{1,n}(t) &:= \sum_{(h,k)\in\Theta_n} g_{h,k}(t)\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}S_i^2 - c_{h,k}\right),\\
B_{2,n}(t) &:= 2\sum_{(h,k)\in\Theta_n} g_{h,k}(t)\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}S_iM_i\right),\\
B_{3,n}(t) &:= \sum_{(h,k)\in\Theta_n} g_{h,k}(t)\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}M_i^2\right),\\
B_{4,n}(t) &:= -\sum_{(h,k)\notin\Theta_n} g_{h,k}(t)c_{h,k}.
\end{aligned} \tag{1.26}$$

We will estimate the summands starting with $B_{4,n}(t)$. Note the following:

$$\begin{aligned}
\sum_{(h,k)\notin\Theta_n} g_{h,k}(t)c_{h,k} &= \sum_{(h,k)\notin\Theta_n} g_{h,k}(t)\int_0^1 \sigma^2(s)\overline{\tilde{g}_{h,k}(s)}ds\\
&\leq c\bar{\omega}(\tilde{g}_{h,k}, 1/H_n)\log H_n + c\bar{\omega}(\sigma^2, 1/H_n)\log H_n,
\end{aligned}$$

where the last line follows from Theorem (A.1). It from the Hölder continuity of σ that that $\bar{\omega}(\sigma^2, 1/H_n) \leq cH_n^{-\alpha}$. Furthermore, by Lemma (1.2) and the Lipschitz continuity of g we have $\bar{\omega}(\tilde{g}_{h,k}, 1/H_n) \leq cH^{-1}$. So,

$$B_{4,n}(t) = O(H_n^{-\alpha}\log H_n). \tag{1.27}$$

Note the generic use of the constant c. In the sequel, we will use c to denote the amalgamation of various constants resulting from multiple steps; this should be harmless since constants are not asymptotically relevant.

We now obtain an estimate for $B_{3,n}(t)$. Note the following:

$$\begin{aligned}
M_i^2 &= \left(\int_{t_i}^{t_{i+1}} b(s)ds\right)^2\\
&\leq \left(M^* n^{-1}\right)^2
\end{aligned} \tag{1.28}$$

Now since $g_{h,k}$ and $\tilde{g}_{h,k}$ are bounded independently of h and k, and $n\Delta_n = 1$, we have

$$B_{3,n}(t) = O(H_n\Delta_n). \tag{1.29}$$

Now consider the following,

$$\begin{aligned}S_i &= \int_{t_i}^{t_{i+1}} \sigma(s)dW_s \\ &= O(\sqrt{n^{-1}\log(n)}),\end{aligned} \tag{1.30}$$

for sufficiently large n by Lévy's modulus of continuity theorem. Hence

$$M_i S_i = O(\sqrt{n^{-3}\log(n)}). \tag{1.31}$$

Since $g_{h,k}$ and $\tilde{g}_{h,k}$ are bounded independently of h and k, and $n\Delta_n = 1$, we have

$$B_{2,n}(t) = O(H_n\sqrt{n^{-1}\log(n)}). \tag{1.32}$$

Now we tackle the final piece $B_{1,n}(t)$. Let

$$A := \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} S_i^2 - \int_0^1 \sigma^2(s)\overline{\tilde{g}_{h,k}(s)}ds. \tag{1.33}$$

We will first obtain an upper bound for A; we proceed by adding and subtracting $\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \overline{\tilde{g}_{h,k}(t_i)}\sigma^2(s)ds$ from A to yield:

$$\begin{aligned}A &= \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} \left(S_i^2 - \int_{t_i}^{t_{i+1}} \sigma^2(s)ds\right) \\ &\quad + \sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} \sigma^2(s)\{\overline{\tilde{g}_{h,k}(t_i)} - \overline{\tilde{g}_{h,k}(s)}\}ds\right) \\ &=: A_1 + A_2.\end{aligned}$$

We obtain estimates in turn for the summands. By linearity of expectation

$$\begin{aligned}A_2 &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \sigma^2(s)\{\overline{\tilde{g}_{h,k}(t_i)} - \overline{\tilde{g}_{h,k}(s)}\}ds \\ &\leq c\bar{\omega}(\tilde{g}_{h,k}, \Delta_n),\end{aligned}$$

where $\bar{\omega}(\tilde{g}_{h,k}, \Delta_n)$ is the modulus of continuity of $\tilde{g}_{h,k}$ on an interval of length Δ_n. By Lemma (1.2) and the Lipschitz continuity of g we have,

$$A_2 \leq c\bar{\omega}(g, \Delta_n) \leq c\Delta_n.$$

Now, we obtain an estimate for A_1. First, let

$$T_c = \inf\{t > 0 : \sigma^2(t) > c\}.$$

By the regularity assumption on σ, $P(T_c > t)$ can be made arbitrarily small by taking c sufficiently large. Hence, by the usual stopping time argument, it may be assumed that $\sigma^2(t)$ is bounded above by c almost surely. Now, let $D_i : \Omega \times [0,1] \to \mathbb{R}$ for $i = 0, \cdots, n-1$ be defined as follows:

$$D_i(t) := \overline{\tilde{g}_{h,k}(t_i)} \left(\int_{t_i}^{t} \sigma(u) dW_u \right) \mathbb{1}_{(t_i,t_{i+1}]}(t). \tag{1.34}$$

$$D_0(0) := 0. \tag{1.35}$$

So, $D_i(t)$ is 0 on $[0,1]$ except when t is in $(t_i, t_{i+1}]$.

Now, using the integration by parts formula for semimartingales, we may write

$$S_i^2 - \int_{t_i}^{t_{i+1}} \sigma^2(s) dW_s = 2 \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{s} \sigma(u) dW_u \right) \sigma(s) dW_s$$

so that

$$\begin{aligned} E(|A_1|) &= 2\, E \left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \overline{\tilde{g}_{h,k}(t_i)} \left(\int_{t_i}^{s} \sigma(u) dW_u \right) \sigma(s) dW_s \right] \\ &\le 2\, E \left[\left| \int_0^1 \sum_{i=0}^{n-1} D_i(s) \sigma(s) dW_s \right| \right]. \end{aligned}$$

Now using the fact that $\int_0^t \sum_{i=0}^{n-1} D_i(s)\sigma(s)dW_s$ is a martingale, we may make an appeal to the BDG inequality to yield:

$$\begin{aligned} E(|A_1|) &\le cE \left[\left| \int_0^1 \left(\sum_{i=0}^{n-1} D_i(s)\sigma(s) \right)^2 ds \right|^{1/2} \right] \\ &\le cE \left[\left| \int_0^1 \sum_{i=0}^{n-1} \{D_i(s)\sigma(s)\}^2 ds \right|^{1/2} \right], \end{aligned}$$

where the last line follows because $D_i(s)D_j(s) = 0$ whenever $i \neq j$. Now if we define $D_i^* := \sup_{t_i < s \le t_{i+1}} D_i(s)$, and use the fact that σ^2 is less than c

before T_c then

$$
\begin{aligned}
E(|A_1|) &\leq cE\left[\left|\sum_{i=0}^{n-1}\Delta_n(D_i^*)^2\right|^{1/2}\right] \\
&\leq cE\left[\left|\sum_{i=0}^{n-1}\Delta_n(D_i^*)^2\right|\right] \\
&\leq c\Delta_n\sum_{i=0}^{n-1}E[(D_i^*)^2]
\end{aligned}
\tag{1.36}
$$

where c is a generic constant representing the bound on σ^2 and the BDG constant. Note from the definition of D_i that it is itself a martingale, so we may bound D_i^* with yet another application of the BDG inequality. That is

$$
\begin{aligned}
E((D_i^*)^2) &\leq E\left(\int_{t_i}^{t_{i+1}}\sigma^2(s)ds\right) \\
&\leq c\Delta_n.
\end{aligned}
\tag{1.37}
$$

Hence, given any $\varepsilon > 0$, $P(\sup_{t\in[0,1]}|B_{1,n}(t)| > \varepsilon) = O(H_n n^{-1}) + P(T_c < 1)$.

Collecting the estimates for $B_{j,n}(t)$ for $j = 1, \cdots, 4$, it is easily seen that $|v_n(X,t) - \sigma^2(t)|$ tends to zero in probability uniformly for all $t \in [0,1]$. $\square$

1.4 Volatility estimation with jumps

In this section we specify a global spot volatility estimator within a framework of prices X evolving in time as Itô semimatingales with continuous diffusion coefficients. Let $\tau : \mathbb{R} \to \mathbb{R}$ be bounded and satisfy $\tau(x) = x$ in a neighborhood of 0. Let ι be the identity function on the real line, i.e. $\iota(x) = x$ for $x \in \mathbb{R}$. The price process X admits the following representation:

$$
X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \tau(x) * (\mu - \nu)_t + (\iota - \tau)(x) * \mu_t,
\tag{1.38}
$$

for $t \geq 0$ where W is a standard Brownian motion; X_0 is either known or observable at time 0; both b and σ are adapted; b is càdlàg, and σ is continuous; μ is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity ν, where ν is a σ-finite Lévy measure on $\mathbb{R}_+ \times \mathbb{R}$. Note that because μ is a

Poisson measure, if A and B are disjoint Borel sets on $\mathbb{R}_+ \times \mathbb{R}$, then the random measures $\mu(A)$ and $\mu(B)$ are Poisson distributed, independent, and and have intensity, $\nu(A)$ and $\nu(B)$, respectively. Moreover, because of the Lévy assumption on ν, it is the case that ν does not charge 0 and

$$(x^2 \wedge 1) * \nu_t < \infty, \qquad t \in [0,1].$$

The notation "$*$" denotes integration with respect to a random measure. So that

$$J_t^l := \tau * (\mu - \nu)_t = \int_0^t \int_{\mathbb{R}} \tau(x)[\mu(ds,dx) - \nu(ds,dx)],$$
$$J_t^s := (\iota - \tau) * \mu_t = \int_0^t \int_{\mathbb{R}} [\iota(x) - \tau(x)]\mu(ds,dx),$$

for $t \geq 0$. Both J^l and J^s are purely discontinuous in the sense that they are orthogonal to all continuous semimartingales. J^s accounts for small jumps; it is a square-integrable martingale with possibly infinite activity. J_2 accounts for large jumps, i.e. jumps with magnitude exceeding the bound on τ; it neccessarily has finite activity so it is a process with finite variation. In the sequel, we will specify τ as follows:

$$\tau(x) = xI_{\{|x|\leq 1\}}, \qquad x \in \mathbb{R}.$$

As in the preceeding section, we observe a realization of the price process at $n+1$ equidistant points t_i, $i = 0, 1, \cdots, n$. The observation interval is normalized to $[0,1]$ with practically no loss of generality. The estimator proposed in the previous section, where there is no jump activity, will not do here. It is inconsistent on account of the presence of jumps; its quality deteriorates as a function of how active the jumps of X are. We will counter this phenomenon with a modified spot variance estimator, but first we introduce the following notation. Let $\Delta^n X_i$ denote $X_{t_{i+1}} - X_{t_i}$ for $i = 0, 1, \cdots, n-1$, and let u_n be a positive decreasing sequence such that

$$u_n / \sqrt{\Delta_n \log(1/\Delta_n)} \tag{1.39}$$

diverges to infinity with n. We specify the jump-robust estimator of the spot volatility as follows:

$$V_n(X,t)(t) := \sum_{(h,k)\in\Theta_n} \hat{a}_{h,k}\, g_{h,k}(t), \qquad \forall t \in [0,1], \text{ where} \tag{1.40}$$

$$\hat{a}_{h,k} := \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} (\Delta_i X)^2 I_{\{(\Delta_i X)^2 \leq u_n\}}, \tag{1.41}$$

where $\{g_{h,k}, \tilde{g}_{h,k}\}$ is a pair of dual Gabor frames constructed as in Lemma (1.1); Θ_n retains its meaning from (1.16); and $I_{\{(\Delta_i X)^2 \leq u_n\}}$ is 1 if $|\Delta_i X|$ is less than or equal to u_n and 0 otherwise.

There are obvious similarities between $v_n(X,t)$ and $V_n(X,t)$ with the key difference being that $V_n(X,t)$ discards realized squared increments over intervals that likely contain jumps; u_n determines the threshold for what is included in the computation and what is not. Clearly it makes sense to use $v_n(X,t)$ if we have reason to believe that the price process is not subject to jumps since it uses all the data and therefore may be assumed to produce more accurate results.

1.4.1 Finite activity Lévy jumps

In order to demonstrate that the global estimator of spot volatility is consistent, we will proceed in stages. First suppose the price process specified in all generality in (1.38) experiences at most a finite number of jump in any finite time interval. That is we assume that X has finite activity jumps, which is equivalent to ν being finite on the complement of $\{0\}$. The finite activity assumption also implies that the price process may be expressed as

$$X_t := \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N_t} Y_i, \tag{1.42}$$

where jump sizes Y_i are i.i.d. with distribution F, and N is a Poisson process with intensity λ, independent of each Y_i. Under this conditions, we have the following:

1.3 Proposition *Suppose the price process is specified as in* (1.42). *Let* $\{g, \tilde{g}\}$ *be pair of dual Gabor generators satisfying the conditions of Lemma* (1.1) *such that* g *is Lipschitz continuous on the unit interval. Further assume the following conditions hold:*

(*i*) *the drift of* X *satisfies with probability 1:*

$$\limsup_{\Delta_n \to 0} \frac{M^*}{(\Delta_n \log(1/\Delta_n))^{1/2}} \leq C < \infty,$$

where $M^* := \sup_{1 \leq i < n} |\int_{t_i}^{t_{i+1}} b_s ds|$;

(*ii*) *the diffusion coefficient satisfies with probability 1:* $\int_0^T \sigma^2 ds < \infty$ *and*

$$\limsup_{\Delta_n \to 0} \frac{S^*}{\Delta_n} \leq B < \infty,$$

where $S^* := \sup_{1 \leq i < n} |\int_{t_i}^{t_{i+1}} \sigma_s^2 ds|$;

(iii) and $u_n \downarrow 0$ *is a sequence in* n *such that*

$$\lim_{\Delta_n \to 0} \frac{\Delta_n \log(1/\Delta_n)}{u_n} = 0.$$

(iv) If $H_n \uparrow \infty$ *satisfies*

$$H_n(n^{-1}\log(n))^{1/2} = o(1)$$

then $V_n(X,t)$ *as defined in* (1.40) *converges in* $L^2[0,1]$ *in probability to* σ^2.

Proof. First note that, there is a finite valued random variabe Λ such that $C + M + 1 \leq \Lambda$. Now, let X^c denote the continuous part of X so that $X = X^c + J$, and

$$X_t^c = \int_0^t b_s ds + \int_0^t \sigma_s dW_s, \tag{1.43}$$

for t in $[0,1]$. Denote

$$v_n(X^c, t) := \sum_{(h,k)\in\Theta_n} \hat{d}_{h,k}\, g_{h,k}(t), \qquad \forall t \in [0,1], \text{ where} \tag{1.44}$$

$$\hat{d}_{h,k} := \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(\Delta X_i^c)^2.$$

We have

$$\begin{aligned}&\int_0^1 (V_n(X,t) - \sigma^2(t))^2 dt \\ &\qquad \leq 2\int_0^1 (V_n(X,t) - v_n(X^c,t))^2 dt + 2\int_0^1 (v_n(X^c,t) - \sigma^2(t))^2 dt.\end{aligned} \tag{1.45}$$

That the second summand on the right converges to 0 in probability is a result of Proposition (1.1). Now note that

$$V_n(X,t)X - v_n(X^c,t)(t) = \sum_{(h,k)\in\Theta_n} (\hat{a}_{h,k} - \hat{d}_{h,k})\, g_{h,k}(t),$$

and

$$\hat{a}_{h,k} - \hat{d}_{h,k} = \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}\{(\Delta_i X)^2 I_{\{(\Delta_i X)^2 \leq u_n\}} - (\Delta X_i^c)^2\}.$$

By Theorem 3.1 of Mancini (2009), for almost all outcomes, there is n' such that for all $n \geq n'$

$$I_{\{(\Delta_i X)^2 \leq u_n\}} = I_{\{\Delta_i N = 0\}}.$$

Hence for almost all outcomes and sufficiently large n

$$\begin{aligned} \hat{a}_{h,k} - \hat{d}_{h,k} &\leq \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} (\Delta X_i^c)^2 I_{\{\Delta_i N \neq 0\}} \\ &\leq c \Lambda \Delta_n \log(1/\Delta_n) \sum_{i=0}^{n-1} I_{\{\Delta_i N \neq 0\}} \end{aligned} \tag{1.46}$$

Hence for almost all outcomes,

$$\begin{aligned} \int_0^1 \{V_n(X,t) - v_n(X^c,t)\}^2 dt &\leq c\Lambda^2 H_n^2 \Delta_n^2 \log^2(1/\Delta_n) N_1^2 \\ &= O(H_n^2 \Delta_n^2 \log^2(1/\Delta_n)) \\ &\to 0. \end{aligned} \tag{1.47}$$

□

1.4.2 Infinity activity Lévy jumps

We now turn to the case of a price process specified in full generality by (1.38), that is the price process is a sum of a continuous and a discontinuous process with possibly infinite activity. The infinite activity assumption is equivalent to the statement that ν assign infinite measure to the complement of the singleton containing zero. The following is the consistency Proposition in this more general framework:

1.4 Proposition *Let the price process X be specified as in* (1.38)*. Suppose the conditions of Proposition* (1.3) *hold. Suppose in addition that*

$$(x^2 \wedge u_n^{1/2}) * \nu_1 = o(H_n^{-2}), \tag{1.48}$$

as $n \to \infty$, then $V_n(X,t)$, defined in (1.40)*, converges in $L^2[0,1]$ in probability to σ^2.*

Proof. We wish to show that the random variable $\int_0^1 (V_n(X,t) - \sigma^2(t))^2 dt$ tends to zero in probability. The regularity conditions on X and σ^2 imply that $\sup_{t\in[0,1]}(V_n(X,t) - \sigma^2(t))^2$ is a random variable and that the previous claim would follow as soon as $\sup_{t\in[0,1]}(V_n(X,t) - \sigma^2(t))^2$ is shown to converge to zero in probability. To that end consider the following decomposition of the process X:

$$X = X^f + J^s, \tag{1.49}$$
$$X^f = X^c + J^l, \tag{1.50}$$

where $X^c = \int_0^t b_s ds + \int_0^t \sigma_s dW_s$, $J^l = xI_{|x|>1} * \mu$, and $J^s = xI_{|x|\le 1} * (\mu - \nu)$. Let t be a point in the unit interval, then

$$\begin{aligned} V_n(X,t) - \sigma^2(t) = &\sum_{(h,k)\in\Theta_n} (c_n(X,h,k) - c(X,h,k))g_{h,k}(t) \\ &- \sum_{(h,k)\notin\Theta_n} c(X,h,k)g_{h,k}(t). \end{aligned} \tag{1.51}$$

By Theorem 4.1 of Zhang (2008) (See Theorem A.1 of the Appendix), the last term on the right converges uniformly on the unit interval to zero, almost surely, as $n \to \infty$.

To obtain a bound on the first item on the right of (1.51), we may use (1.49) to write

$$\begin{aligned} &\sum_{(h,k)\in\Theta_n} (c_n(X,h,k) - c(X,h,k))g_{h,k}(t) \\ &\qquad = \sum_{(h,k)\in\Theta_n} (w_{h,k} + x_{h,k} + y_{h,k} + z_{h,k})g_{h,k}(t), \end{aligned} \tag{1.52}$$

where

$$\begin{aligned} w_{h,k} &:= \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(\Delta_i X^f)^2 I_{\{(\Delta_i X^f)^2 \le 4u_n\}} - \int_0^1 \sigma^2(s)\overline{\tilde{g}_{h,k}(s)}ds \\ x_{h,k} &:= \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(\Delta_i X^f)^2 (I_{\{(\Delta_i X)^2 \le u_n\}} - I_{\{(\Delta_i X^f)^2 \le 4u_n\}}) \\ y_{h,k} &:= 2\sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}\Delta_i X^f \Delta_i J^s I_{\{(\Delta_i X)^2 \le u_n\}} \\ z_{h,k} &:= \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(\Delta_i J^s)^2 I_{\{(\Delta_i X)^2 \le u_n\}}. \end{aligned} \tag{1.53}$$

By Proposition (1.3), if $\delta > 0$ then

$$P(\sup_{t\in[0,1]} |\sum_{(h,k)\in\Theta_n} w_{h,k}g_{h,k}(t)| > \delta) \leq c\delta^{-1}H_n n^{-1}\log(n), \tag{1.54}$$

for some $c \in \mathbb{R}$. It remains to show that the last three terms on the right converge to zero in probability. Starting with the second summand, denote $A_i := \{(\Delta_i X)^2 \leq u_n\}$, $B_i := \{(\Delta_i X^f)^2 \leq 4u_n\}$ and note that $I_{A_i} - I_{B_i} = I_{A_i\cap B_i^c} - I_{A_i^c\cap B_i}$. Now for each outcome in $A_i \cap B_i^c$, it is the case that $2u_n^{1/2} - |\Delta_i J^s| \leq |\Delta_i X^f| - |\Delta_i J^s| \leq |\Delta_i X^f + \Delta_i J^s| \leq u_n^{1/2}$, so that $|\Delta_i J^s| \geq u_n^{1/2}$ and

$$\begin{aligned}
&\sum_{(h,k)\in\Theta_n}\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}(\Delta_i X^f)^2 I_{A_i\cap B_i^c}\right)g_{h,k}(t)\\
&\quad\leq \sum_{(h,k)\in\Theta_n}\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}(\Delta_i X^f)^2 I_{\{(\Delta_i J^s)^2>u_n\}}\right)g_{h,k}(t)\\
&\quad\leq \sum_{(h,k)\in\Theta_n}\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}(\Delta_i X^c)^2 I_{\{(\Delta_i J^s)^2>u_n\}}\right)g_{h,k}(t)\\
&\qquad+\sum_{(h,k)\in\Theta_n}\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}(\Delta_i J^l)^2 I_{\{(\Delta_i J^s)^2>u_n\}}\right)g_{h,k}(t)\\
&\quad\leq v_n + w_n,
\end{aligned}\tag{1.55}$$

where

$$v_n := 2cH_n\Lambda n^{-1}\log(n)\sum_{i=0}^{n-1} I_{\{(\Delta_i J^s)^2>u_n\}}$$

$$w_n := cH_n\sum_{i=0}^{n-1}(\Delta_i J^l)^2 I_{\{(\Delta_i J^s)^2>u_n\}}$$

and c is a sufficienty large constant. Let $\delta > 0$ be given, put $x_n(t) := \sum_{(h,k)\in\Theta_n}\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}(\Delta_i X^f)^2 I_{A_i\cap B_i^c}\right)g_{h,k}(t)$ and note that

$$P(|x_n(t)| > \delta) \leq P(v_n > \delta/2) + P(w_n > \delta/2).$$

First note that the expression on the right is independent of t, so that the probabilities on the right bound $P(|x_n(t)| > \delta)$ uniformly on the unit interval. Now let $\varepsilon > 0$ be given and note that because Λ is almost surely finite, there is a sufficiently large $K > 0$ such that $P(\Lambda > K) \leq \varepsilon/2$. Hence,

$$\begin{aligned} P(v_n > \delta/2) &\leq 2cKH_n\delta^{-1}E(n^{-1}\log(n)\sum_{i=0}^{n-1} I_{\{(\Delta_i J^s)^2 > u_n\}}) + P(\Lambda > K) \\ &= 2cKH_n\delta^{-1}\log(n)P((\Delta_i J^s)^2 > u_n) + \varepsilon/2 \\ &\leq 2cKH_n\delta^{-1}\log(n)E((\Delta_1 J^s)^2))u_n^{-1} + \varepsilon/2 \\ &\leq 2cKH_n\delta^{-1}\log(n)n^{-1}\kappa u_n^{-1} + \varepsilon/2 \end{aligned} \tag{1.56}$$

where $\kappa := E((\Delta_1 J^s)^2)) < \infty$. Obviously there is a large enough n such that the first expression above is less than or equal to $\varepsilon/2$. Moreover, because $\delta > 0$,

$$\begin{aligned} P(w_n > \delta/2) &\leq P\left(\cup_i\{I_{\{|x|>1\}} * \mu((t_i, t_{i+1}] \times \mathbb{R}) > 0, (\Delta_i J^s)^2 > u_n\}\right) \\ &\leq nP(\mu([0, 1/n] \times \{|x| > 1\}) > 0)E((\Delta_1 J^s)^2)u_n^{-1} \\ &\leq cn^{-1}\kappa u_n^{-1}, \end{aligned}$$

which clearly tends to zero in n.

Now, by Theorem 3.1 of Mancini (2009), if an outcome is in $A_i^c \cap B_i$ then there is a sufficiently small u_n such that $I_{\{|x|>1\}} * \mu((t_i, t_{i+1}] \times \mathbb{R}) = 0$. Hence, for such an outcome, it is the case that $(\Delta_i X)^2 > u_n$ if and only if $(\Delta_i X^c + \Delta_i J^s)^2 > u_n$, which in turn would hold if either $(\Delta_i X^c)^2 > u_n/4$ or $(\Delta_i J^s)^2 > u_n/4$. However, by theorem 3.1 of Mancini (2009), by taking n large enough $\{(\Delta_i X^c)^2 > u_n/4\} = \emptyset$. Let $\delta > 0$ be a given positive number; put $y_n(t) := \sum_{(h,k)\in\Theta_n}\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}(\Delta_i X^f)^2 I_{A_i^c\cap B_i)}\right)g_{h,k}(t)$ and note that

$$P(|y_n(t)| > \delta) \leq P\left(cH_n\Lambda^2 n\left(I_{\{(\Delta_i J^s)^2 > u_n/4\}})\right) > \delta\right),$$

which tends to zero exactly as in (1.56). Hence,

$$P(\sup_{t\in[0,1]}|\sum_{(h,k)\in\Theta_n} x_{h,k}g_{h,k}(t)| > \delta) \to 0. \tag{1.57}$$

Now we obtain a bound for the third summand in (1.52). First, denote $C_i := \{(\Delta_i J^s)^2 \leq 4u_n\}$, $p_{h,k} := 2\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}\Delta_i X^f \Delta_i J^s I_{A_i\cap C_i}$, and $q_{h,k} := 2\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}\Delta_i X^f \Delta_i J^s I_{A_i\cap C_i^c}$. Clearly, $y_{h,k} = p_{h,k} + q_{h,k}$. Now note that on $A_i \cap C_i^c$, it is the case that $2u_n^{1/2} - |\Delta_i X^f| < |\Delta_i J^s| - |\Delta_i X^f| \leq |\Delta_i X| \leq u_n^{1/2}$, so that $u_n^{1/2} < |\Delta_i X^f| < |\Delta_i J^l| + |\Delta_i X^c|$. In turn, the last inequlity implies that either $|\Delta_i J^l| > u_n^{1/2}/2$ or $|\Delta_i X^c| > u_n^{1/2}/2$. Now, for sufficiently large n, it is almost surely never the case that $|\Delta_i X^c| > u_n^{1/2}/2$ for some i,

$0 \leq i \leq n-1$. Hence, for positive δ,

$$\begin{aligned} P(| & \sum_{(h,k)\in\Theta_n} q_{h,k} g_{h,k}(t)| > \delta/2) \\ & \leq P(\cup_i \{\mu((t_i, t_{i+1}] \times \{|x| > 1\}) > 0, (\Delta_i J^s)^2 > u_n\}) \\ & \leq cn^{-1}\kappa u_n^{-1}. \end{aligned} \tag{1.58}$$

Meanwhile, on $A_i \cap C_i$, it is easily seen that $|\Delta_i J^l| - |\Delta_i X^c + \Delta_i J^s| < |\Delta_i X| \leq u_n^{1/2}$, so that $|\Delta_i J^l| \leq u_n^{1/2} + |\Delta_i X^c| + |\Delta_i J^s|$. On the other hand, $|\Delta_i J^l| < u_n^{1/2} + \Lambda n^{-1/2} \log^{1/2}(n) + 2u_n^{1/2} = O(u_n^{1/2})$. Let $r_{h,k} := 2\sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} \Delta_i X^c \Delta_i J^s I_{A_i \cap C_i}$ and $s_{h,k} := 2cu_n^{1/2} \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} \Delta_i J^s I_{A_i \cap C_i}$. Then

$$\begin{aligned} P(| & \sum_{(h,k)\in\Theta_n} q_{h,k} g_{h,k}(t)| > \delta/2) \\ & \leq P(|\sum_{(h,k)\in\Theta_n} r_{h,k} g_{h,k}(t)| > \delta/4) + P(|\sum_{(h,k)\in\Theta_n} s_{h,k} g_{h,k}(t)| > \delta/4). \end{aligned}$$

Now consider that $\sum_{(h,k)\in\Theta_n} r_{h,k} g_{h,k}(t) \leq cH_n \sum_{i=0}^{n-1} \Delta_i X^c \Delta_i J^s I_{A_i \cap C_i}$, which implies that

$$\begin{aligned} P(|\sum_{(h,k)\in\Theta_n} r_{h,k} g_{h,k}(t)| > \delta/4) & \leq P(cH_n |\sum_{i=0}^{n-1} \Delta_i X^c \Delta_i J^s I_{A_i \cap C_i}| > \delta/4) \\ & \leq P\left(\left(\sum_{i=0}^{n-1} (\Delta_i X^c)^2\right)^{1/2} \left(\sum_{i=0}^{n-1} (\Delta_i J^s I_{A_i \cap C_i})^2\right)^{1/2} > \delta(4H_n c)^{-1}\right). \end{aligned}$$

It is a well known fact that $\sum_{i=0}^{n-1} (\Delta_i X^c)^2(t)$ converges to $\int_0^t \sigma^2(s)ds$ in probability uniformy on the unit interval. Hence, there is a sufficiently large N such that if $n > N$ then $P(|(\sum_{i=0}^{n-1} (\Delta_i X^c)^2)^{1/2} - (\int_0^1 \sigma^2(s)ds)^{1/2}| > \delta) \leq \varepsilon/4$, and because integrated volatility is almost surely finite, there is a sufficiently large K satisfying $K/2 > \delta$ such that $P(\int_0^1 \sigma^2(s)ds > K/2) \leq \varepsilon/4$. Hence, we may write

$$
\begin{aligned}
&P(|\sum_{(h,k)\in\Theta_n} r_{h,k}g_{h,k}(t)|>\delta/4)\\
&\quad\le P\left(\sum_{i=0}^{n-1}(\Delta_i J^s I_{A_i\cap C_i})^2>\delta^2(4KH_nc)^{-2}\right)+\varepsilon/2\\
&\quad\le P\left((x^2 I_{\{|x|\le 1\wedge 2u_n^{1/2}\}}*\mu)_1>\delta^2(4KH_nc)^{-2}\right)+\varepsilon/2\\
&\quad\le \delta^{-2}(4KH_nc)^2E\left((x^2 I_{\{|x|\le 1\wedge 2u_n^{1/2}\}}*\mu)_1\right)+\varepsilon/2\\
&\quad\le \delta^{-2}(4KH_nc)^2(x^2 I_{\{|x|\le 1\wedge 2u_n^{1/2}\}}*\nu)_1+\varepsilon/2
\end{aligned}
$$

which for sufficiently large n is less than ε by (1.48).

Now it is easily seen that for sufficiently large c

$$
\begin{aligned}
&P(|\sum_{(h,k)\in\Theta_n} s_{h,k}g_{h,k}(t)|>\delta/4)\le P(\sum_{i=0}^{n-1}\Delta_i J^s I_{A_i\cap C_i}>(8cu_n^{1/2})^{-1}\delta)\\
&\quad\le (64c^2u_n)\delta^{-2}E\left((x^2 I_{\{|x|\le 1\wedge 2u_n^{1/2}\}}*\mu)_1\right)\\
&\quad\le (64c^2u_n)\delta^{-2}(x^2 I_{\{|x|\le 1\wedge 2u_n^{1/2}\}}*\nu)_1 \qquad (1.59)
\end{aligned}
$$

which, as above, is less than $\varepsilon/4$ for sufficiently large n. Hence,

$$
P(\sup_{t\in[0,1]}|\sum_{(h,k)\in\Theta_n} y_{h,k}g_{h,k}(t)|>\delta)\to 0.
$$

Next, write $z_{h,k}=a_{h,k}+b_{h,k}$ where $a_{h,k}:=\sum_{i=0}^{n-1}\overline{\tilde g_{h,k}(t_i)}(\Delta_i J^s)^2 I_{A_i\cap C_i}$ and $b_{h,k}:=\sum_{i=0}^{n-1}\overline{\tilde g_{h,k}(t_i)}(\Delta_i J^s)^2 I_{A_i\cap C_i^c}$. Then

$$
\begin{aligned}
&P(|\sum_{(h,k)\in\Theta_n} z_{h,k}g_{h,k}(t)|>\delta)\\
&\quad\le P(|\sum_{(h,k)\in\Theta_n} a_{h,k}g_{h,k}(t)|>\delta/2)+P(|\sum_{(h,k)\in\Theta_n} b_{h,k}g_{h,k}(t)|>\delta/2).
\end{aligned}
$$

In the first instance,

$$
\begin{aligned}
&P(|\sum_{(h,k)\in\Theta_n} b_{h,k}g_{h,k}(t)|>\delta/2)\\
&\quad\le P\left(\cup_i\{I_{\{|x|>1\}}*\mu((t_i,t_{i+1}]\times\mathbb{R})>0,(\Delta_i J^s)^2>4u_n\}\right)\\
&\quad\le nP(I_{\{|x|>1\}}*\mu([0,1/n]\times\mathbb{R})>0)E((\Delta_i J^s)^2)(4u_n)^{-1}\\
&\quad\le cn^{-1}\kappa u_n^{-1}. \qquad (1.60)
\end{aligned}
$$

which can be made as small as desired. Now consider

$$
\begin{aligned}
&P(|\sum_{(h,k)\in\Theta_n} a_{h,k}g_{h,k}(t)| > \delta/2)\\
&\quad \le P(\sum_{i=0}^{n-1}(\Delta_i J^s)^2 I_{\{|\Delta_i J^s|\le 2u_n^{1/2}\}} > \delta(2cH_n)^{-1})\\
&\quad \le \delta^{-1}(2cH_n)E\left(x^2 I_{\{|x|\le 1\wedge 2u_n^{1/2}\}} * \mu_1\right)\\
&\quad \le \delta^{-1}(2cH_n)(x^2 I_{\{|x|\le 1\wedge 2u_n^{1/2}\}} * \nu)_1
\end{aligned}
$$

which can be made arbitrarily small. Hence,

$$
P(\sup_{t\in[0,1]} |\sum_{(h,k)\in\Theta_n} z_{h,k}g_{h,k}(t)| > \delta) \to 0. \tag{1.61}
$$

The result follows from (1.54),(1.57),(1.60), and (1.61). □

1.4.3 Simulation study

In this simulation study, we examine the impact of the observation frequency on the mean integrated square error (MISE), the square bias, and the variance of the frame-based estimator. We make this examination using prices generated by 4 commonly used models of asset prices, namely, the Arithmetic Brownian Motion (ABM), the Ornstein-Uhlenbeck process (OU), the Geometric Brownian Motion (GBM), and the Cox-Ingersoll-Ross (CIR) process.

Specifically, we simulate prices using the following stochastic differential equations:

$$
\begin{aligned}
dX &= 0.5dt + 0.2dW_t, && \text{(ABM)}\\
dX &= -4X_tdt + 0.2dW_t, && \text{(OU)}\\
dX &= 0.5X_tdt + 0.2X_tdW_t, && \text{(GBM)}\\
dX &= (0.1 - 0.5X_t)dt + 0.2\sqrt{X_t}dW_t, && \text{(CIR)}
\end{aligned}
$$

where W_t is a standard Brownian motion. For convenience, the observation interval is set to the unit interval $[0, 1]$. In all 4 cases, $X_0 = 0.8$. For each price model, we obtain estimates for the MISE, the square bias, and the variance of the estimator when the number of observations are 500, 5000, and 50000, respectively. In a high-frequency framework, 500 observations for an actively traded stock is likely too small; 5,000 is about right, but 50,000 is not entirely unheard of. At any rate, our objective is not to capture the average number of trades of any particular security, but rather, to obtain support for our asymptotic results by showing an inverse relationship between the number of observations and the MISE, and thereby gain a better understanding of the finite sample behavior of the estimator.

The starting point for constructing the estimator is to fix a generator for the Gabor frame. We have denoted the generator and its dual by g and $\tilde{g}$, respectively. For our purposes, any continuous and compactly supported function will work. In fact, part of the appeal of the frame method is this flexibility. We may chose the frame generator to match our prior assumptions about the smoothness of the latent volatility function. In this regard, a suitably *scaled*[14] member of the family of B-splines is particularly suited to the task of a Gabor frame *generator*. B-splines are piecewise polynomials, so, by varying their order or degree we may achieve any level of smoothness. Furthermore, the order of B-splines is directly related to the decay of their Fourier transforms. In fact the Fourier transform of a B-spline of order $p \geq 1$ decays like an $(p-1)$-th degree polynomial. This is important for the rate of decay of the MISE, and therefore, directly impacts the optimal choice of coefficients H_n to estimate. The upshot is, the higher the order of the B-spline, the smaller the number of coefficients needed to achieve a given level of accuracy.

Table 1.1: Mean integrated square error (MISE) of the frame-based estimator $\hat{\sigma}_n^2$ for popular price models.

	ABM			OU		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	1.30×10^{-4}	2.86×10^{-6}	1.27×10^{-4}	1.43×10^{-4}	1.19×10^{-5}	1.31×10^{-4}
5000	1.41×10^{-5}	1.11×10^{-6}	1.30×10^{-5}	1.45×10^{-5}	1.62×10^{-6}	1.28×10^{-5}
50000	2.32×10^{-6}	1.02×10^{-6}	1.30×10^{-6}	2.36×10^{-6}	1.12×10^{-6}	1.23×10^{-6}

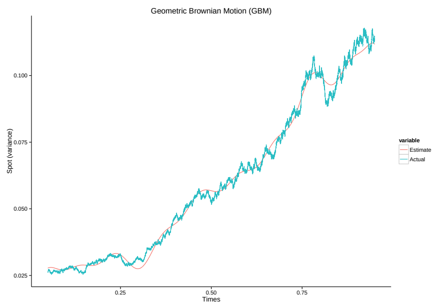
	GBM			CIR		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	2.18×10^{-4}	4.18×10^{-6}	2.14×10^{-4}	6.26×10^{-5}	8.51×10^{-7}	6.17×10^{-5}
5000	2.33×10^{-5}	1.58×10^{-6}	2.17×10^{-5}	6.82×10^{-6}	6.00×10^{-7}	6.22×10^{-6}
50000	4.66×10^{-6}	1.02×10^{-6}	3.64×10^{-6}	1.46×10^{-6}	6.06×10^{-7}	8.52×10^{-7}

Note: The mean of the integrated square errors are obtained by taking an average over 100 sample paths generated for each model/number of observations pair.

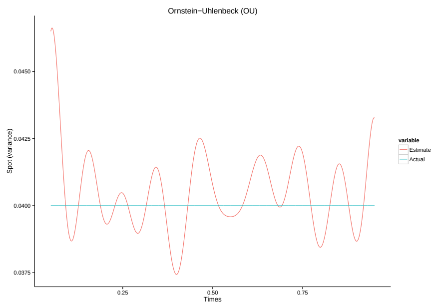
[14] More precisely *dilated*. See (1.62) for definition.

Figure 1.1: Estimated vs. actual spot volatility of common price models

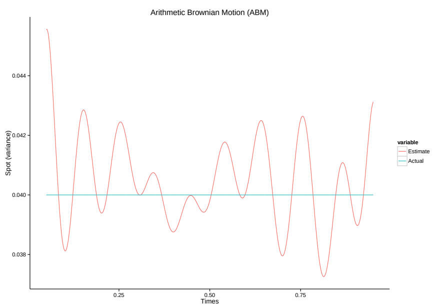
(a) GBM



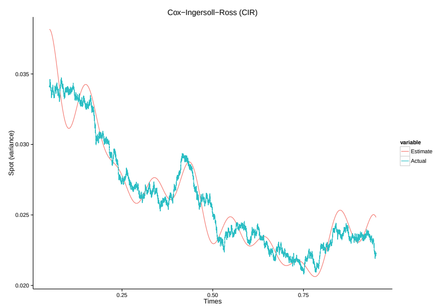
(b) OU



(c) ABM



(d) CIR



From an implementation perspective, using a B-spline makes the construction of a *dual* frame generator a trivial matter. This is a consequence of Theorems 2.2 and 2.7 in Christensen (2006), which together specify a very simple rule for constructing dual pairs: Let $a > 0$ and $b > 0$ denote translation and modulation parameters, and let h be a B-spline of order p. Define the dilation operator $\mathcal{D}_c$ as follows:

$$\mathcal{D}_c f(x) = c^{-1/2} f(x/c). \tag{1.62}$$

If $0 < ab \leq 1/(2p-1)$ then $\{\mathcal{D}_a h, \mathcal{D}_a \tilde{h}\}$, where

$$\tilde{h}(x) = abh(x) + 2ab \sum_{n=1}^{p-1} h(x+n), \qquad \forall x \in \mathbb{R}, \tag{1.63}$$

is a pair of dual Gabor frame generators. So if we start with a B-spline h then the dual generator will be a finite linear combination of scaled translates of h; consequently, the dual generator will be a spline, with similar regularity properties. For our simulation, we used a third-order B-spline. Our choice of the third order B-spline is motivated by a desire for a generator with a Fourier transform that decays like a quadratic polynomial. Specifically, we set

$$h(x) = \begin{cases} x^2/2 & x \in (1,0] \\ (-2x^2 + 6x - 3)/2 & x \in (2,1] \\ (3 - x^2)/2 & x \in (3,2] \\ 0 & x \notin (3,0] \end{cases},$$

with $\tilde{h}$ computed as in (1.63) above. Our choice of the modulation and translation parameters is rather arbitrary. The only constraint is that $0 < ab \leq 1/(2p-1) = 1/5$; from our experimentation with different values, performance seems to be about the same for different choices satisfying the inequality; we settled on $a = 1/5$ and $b = 1/3$. Ideally H_n, the order of the number of frequency domain shifts, would be selected optimally to minimize MISE while balancing integrated variance and integrated square bias; we will turn our attention to this problem in Section (TBA). For the time being we set H_n naively equal to 50.

The simulation results indicate that the Gabor frame estimator performs satisfactorily. Figure 1.2 displays, for each of the 4 price models (ABM, OU, GBM, and CIR), simulated spot variance sample paths plotted against spot variance paths produced by the Gabor frame estimator. A visual inspection shows that the estimator produces a relatively good fit even with the naive selection of H_n. This claim is further corroborated by the analysis of the

the integrated mean aquare error (MISE), the integrated square bias, and the integated variance summarized in Table 1.2. The figures in the table are arrived at in the following manner: first, 100 price histories are simulated for each observation frequency and model pair. So, each history is the result of sampling n price observations from distribution F, where n is the specified observation frequency and F is the distribution implied by the stochastic differential equation. The resulting data is a matrix with 100 rows and n columns. Each row represents a price history from which integrated quatities may be obatined, and each column indexes an orbservation time. Going down a column, average quantities may be computed. For instance, to arrive at the integrated square bias figures, average spot variances were computed for each observation times; the figures were then squared, weighted by Δ_n, and summed up. The integrated mean square error is computed similarly. We found that the variance, estimated in the foregoing manner, is only approximately the difference between the MISE and the integrated square bias. The reported figures for variance are in fact the difference between the MISE and the integrated square bias. The discrepancy is rather slight and does not materially change the result. In all 4 model, an inverse relation between MISE, square bias, and variance may be read off from the table. As was established mathematically, we expect MISE to vanish if the number of price observations were made to grow without bound.

Table 1.2: Mean integrated square error (MISE) of the frame-based estimator $\hat{\sigma}_n^2$ for popular price models.

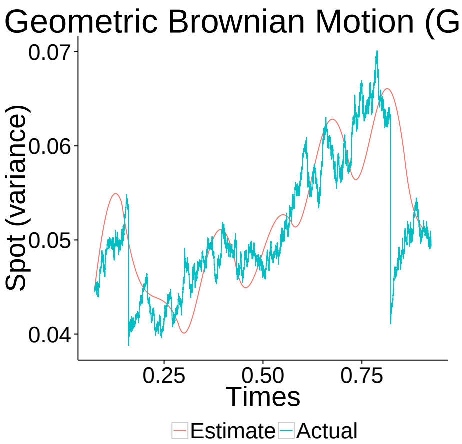
	FP			OU		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	1.53×10^{-4}	8.95×10^{-6}	1.44×10^{-4}	8.51×10^{-4}	1.31×10^{-4}	7.20×10^{-4}
5000	2.19×10^{-5}	2.27×10^{-6}	1.96×10^{-5}	5.48×10^{-5}	9.76×10^{-6}	4.50×10^{-5}
50000	2.13×10^{-6}	9.00×10^{-8}	2.04×10^{-6}	6.61×10^{-6}	2.65×10^{-6}	3.97×10^{-6}

	GBM			CIR		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	6.13×10^{-3}	8.70×10^{-4}	5.26×10^{-3}	3.74×10^{-4}	2.32×10^{-4}	1.43×10^{-4}
5000	3.42×10^{-4}	4.07×10^{-5}	3.02×10^{-4}	1.12×10^{-5}	8.29×10^{-6}	2.95×10^{-6}
50000	7.11×10^{-5}	6.36×10^{-6}	6.47×10^{-5}	7.05×10^{-6}	5.64×10^{-6}	1.40×10^{-6}

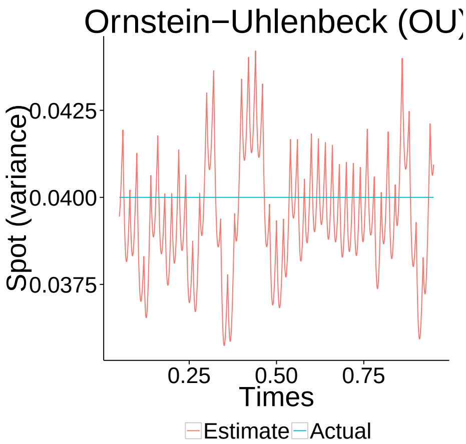
Note: The mean of the integrated square errors are obtained by taking an average over 50 sample paths generated for each model/number of observations pair.

Figure 1.2: Estimated vs. actual spot volatility of common price models

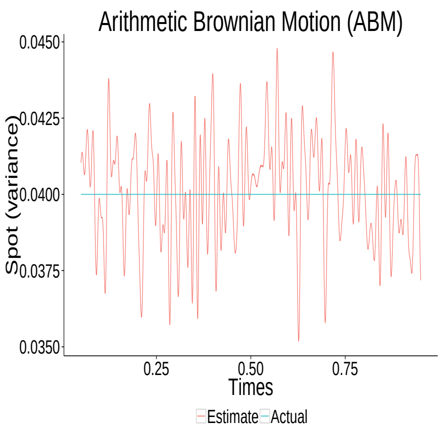
(a) GBM + JMP



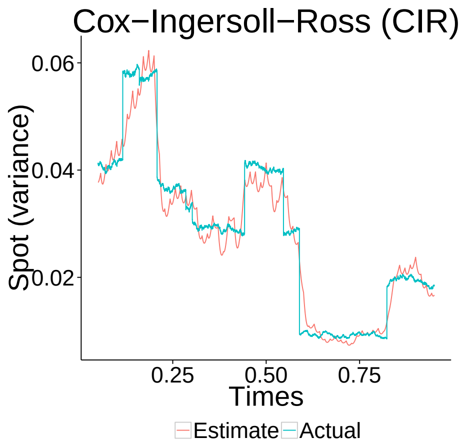
(b) OU + JMP



(c) ABM + JMP



(d) CIR + JUMP



Appendices

Appendix A

Proofs

We now give the proof of Lemma (1.2).

Proof. G is bounded away from zero. To see this, note that since g has support in $[r, s]$, the series on the left hand side of (1.10) has finitely many terms for each t. In addition, it is straight forward to verify that $G(t) = G(t+b)$ for all t; so, G is periodic with period b. It is also clear that because g is continuous, so is G. It follows that G attains its min and max on any interval of length b. Let $I_b := [(s+r-b)/2, (s+r+b)/2]$, then

$$\begin{aligned}\min_{t\in\mathbb{R}} G(t) &= \min_{t\in I_b} G(t)\\ &\geq (2\pi/a)\min_{t\in I_b}|g(t)|^2.\end{aligned}$$

Because g is continuous and g doesn't vanish in (r, s), we conclude that $G_* := \min_{t\in\mathbb{R}} G(t) > 0$. It is also straight forward that $G^* := \max_{t\in\mathbb{R}} G(t) < \infty$. Now, let $t, t' \in \mathbb{R}$ such that $|t - t'| \leq \delta$, then

$$\begin{aligned}|\tilde{g}(t) - \tilde{g}(t')| &= |(G(t)G(t'))^{-1}(g(t)G(t') - g(t')G(t))|\\ &\leq (G_*^{-2})\{|g(t)||G(t) - G(t')| + |G(t)||g(t) - g(t')|\}. \quad \text{(A.1)}\end{aligned}$$

Let $\tau := r + (t \bmod b)$, and $\tau' := r + (t' \bmod b)$. It is straight forward to verify that if $|\tau - \tau'| \leq \delta$, then

$$\begin{aligned}|G(t) - G(t')| &\leq \sum_{j=0}^{\lfloor (s+r)/b\rfloor} |g(\tau + jb)^2 - g(\tau' + jb)^2|\\ &\leq \sum_{j=0}^{\lfloor (s+r)/b\rfloor} |g(\tau + jb) - g(\tau' + jb)||g(\tau + jb) + g(\tau' + jb)|\\ &\leq 2\lceil (s+r)/b\rceil g^*\bar{\omega}(g, \delta), \quad \text{(A.2)}\end{aligned}$$

where $g^* := \max_{t\in\mathbb{R}} |g(t)|$. On the other hand, if $|\tau - \tau'| > \delta$, then

$$\begin{aligned}
&|G(t) - G(t')| \\
&\quad \le |g(\tau')^2 - g(r)^2| + |g(s)^2 - g(\tau + c)^2| \\
&\qquad + \sum_{j=1}^{\lfloor (s+r)/b \rfloor} \{|g(\tau + (j-1)b)^2 - g(\tau' + jb)^2|\}.
\end{aligned}$$

where $c = \lfloor (s+r)/b \rfloor b$. It follows as above that

$$|G(t) - G(t')| \le 2(\lceil (s+r)/b \rceil + 1) g^* \bar{\omega}(g, \delta). \tag{A.3}$$

Returning to (A.1), we see that

$$|\tilde{g}(t) - \tilde{g}(t')| \le C_{\tilde{g}} \bar{\omega}(g, \delta),$$

where $C_{\tilde{g}} = G_*^2(2(\lceil (s+r)/b \rceil + 1)(g^*)^2 + G^*)$. Now let $h, k \in \mathbb{Z}$, then

$$\begin{aligned}
|\tilde{g}_{h,k}(t) - \tilde{g}_{h,k}(t')| &= |e^{ihat}(\tilde{g}(t - kb) - \tilde{g}(t' - kb)| \\
&\le |(\tilde{g}(t - kb) - \tilde{g}(t' - kb)| \le C_{\tilde{g}} \bar{\omega}(g, \delta).
\end{aligned} \tag{A.4}$$

The last inequality follows because translating a function leaves its modulus of continuity unchanged.

□

The proof of Proposition (1.1) depends in a crucial way on Theorem 4.1 in Zhang (2008). We recall it here for convenience.

A.1 Theorem *Let $\{g_{h,k}, \tilde{g}_{h,k}\}$ be a pair of compactly supported dual Gabor frames satisfying the conditions of Lemma (1.1). If $f \in L^2(\mathbb{R})$ is bounded and continuous on $\mathbb{R}$ then for any positive integer l, if $K > (l + |s| + |r|)/b$ and H is a positive integer then the partial sums*

$$S_{H,K}(t) := \sum_{|h|\le H} \sum_{|k|\le K} \langle f, g_{h,k} \rangle \tilde{g}_{h,k}(t)$$

satisfies

$$S_{H,K}(t) - f(t) = O(\|g\|_\infty/ab)(\|g\|_\infty \bar{\omega}(f, 1/aH) + \|f\|_\infty \bar{\omega}(g, 1/aH)) \log H,$$

for all t such that $|t| \le l$.

Now we prove Proposition (1.1).

Proof. Without loss of generality let $X_0 = 0$, and take $\alpha \in (0,1]$ and $c > 0$ as given. We begin with $B_n^2(\alpha, c)$, the integrated square bias component of $R_n(\alpha, c)$, which is defined as:

$$B_n^2(\alpha, c) = \int_0^1 E[(v_n((,t)t) - \sigma^2(t))I_{\mathcal{E}(\alpha,c)}]^2 dt. \tag{A.5}$$

We make the following notational simplification:

$$E_{\mathcal{E}(\alpha,c)}[X] := E[X I_{\mathcal{E}(\alpha,c)}],$$

for all random variables X. We proceed by first obtaining an upper bound for the integrand in (A.5), i.e. the square bias at each fixed point t. To that end, let $t \in [0,1]$, and note that

$$\begin{aligned} E_{\mathcal{E}(\alpha,c)}[v_n((,t)t) - \sigma^2(t)] = &\sum_{(h,k)\in\Theta_n} E_{\mathcal{E}(\alpha,c)}[\hat{c}_{h,k} \ - c_{h,k}]\ g_{h,k}(t) \\ &- \sum_{(h,k)\notin\Theta_n} E_{\mathcal{E}(\alpha,c)}[c_{h,k}]\ g_{h,k}(t), \end{aligned} \tag{A.6}$$

where

$$\begin{aligned} \hat{c}_{h,k} &= \sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}(X_{t_{i+1}} - X_{t_i})^2 \text{ and} \\ c_{h,k} &= \int_0^1 \overline{\tilde{g}_{h,k}(s)}\sigma^2(s)ds. \end{aligned}$$

We tackle the summands in (A.6) in turn starting with the first one. But first let

$$M_i := \int_{t_i}^{t_{i+1}} \mu(s)ds, \quad \text{and} \quad S_i := \int_{t_i}^{t_{i+1}} \sigma(s)dW_s,$$

and note that since $X_{t_{i+1}} - X_{t_i} = M_i + S_i$, it follows that

$$\begin{aligned} E_{\mathcal{E}(\alpha,c)}[(X_{t_{i+1}} - X_{t_i})^2] = &E_{\mathcal{E}(\alpha,c)}[M_i^2] + 2E_{\mathcal{E}(\alpha,c)}[M_i S_i] \\ &+ E_{\mathcal{E}(\alpha,c)}[S_i^2]. \end{aligned}$$

So, (A.6) may be written as

$$E_{\mathcal{E}(\alpha,c)}[v_n((,t)t) - \sigma^2(t)] = B_{1,n}(t) + B_{2,n}(t) + B_{3,n}(t) + B_{4,n}(t),$$

where

$$B_{1,n}(t) := \sum_{(h,k)\in\Theta_n} g_{h,k}(t)\left(E_{\mathcal{E}(\alpha,c)}\left[\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}S_i^2 - c_{h,k}\right]\right),$$

$$B_{2,n}(t) := 2\sum_{(h,k)\in\Theta_n} g_{h,k}(t)\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}E_{\mathcal{E}(\alpha,c)}\left[S_iM_i\right]\right),$$

$$B_{3,n}(t) := \sum_{(h,k)\in\Theta_n} g_{h,k}(t)\left(\sum_{i=0}^{n-1}\overline{\tilde{g}_{h,k}(t_i)}E_{\mathcal{E}(\alpha,c)}\left[M_i^2\right]\right),$$

$$B_{4,n}(t) := -\sum_{(h,k)\notin\Theta_n} g_{h,k}(t)E_{\mathcal{E}(\alpha,c)}[c_{h,k}]. \tag{A.7}$$

We will estimate the summands starting with $B_{4,n}(t)$. Note the following:

$$\begin{aligned}\sum_{(h,k)\notin\Theta_n} g_{h,k}(t)E_{\mathcal{E}(\alpha,c)}[c_{h,k}] &= \sum_{(h,k)\notin\Theta_n} g_{h,k}(t)E[\langle\sigma^2,\tilde{g}_{h,k}\rangle I_{\mathcal{E}(\alpha,c)}]\\ &\leq E\left|\sum_{(h,k)\notin\Theta_n} g_{h,k}(t)\langle\sigma^2 I_{\mathcal{E}(\alpha,c)},\tilde{g}_{h,k}\rangle\right|\\ &\leq cE[\bar{\omega}(\tilde{g}_{h,k},1/H_n)\log H_n]\\ &\quad + cE[\bar{\omega}(\sigma^2 I_{\mathcal{E}(\alpha,c)},1/H_n)\log H_n],\end{aligned}$$

where the last line follows from Theorem (A.1). But since $\sigma^2 I_{\mathcal{E}(\alpha,c)}$ is bounded by c if it is in the Hölder ball $\mathcal{H}(\alpha,c)$ and 0 otherwise, it follows that $\bar{\omega}(\sigma^2 I_{\mathcal{E}(\alpha,c)},1/H_n) \leq cH_n^{-\alpha}$. Furthermore, by Lemma (1.2) and the Lipschitz continuity of g we have $\bar{\omega}(\tilde{g}_{h,k},1/H_n) \leq cH^{-1}$. So,

$$B_{4,n}(t) = O(H_n^{-\alpha}\log H_n). \tag{A.8}$$

Note the generic use of the constant c. In the sequel, we will use c to denote the amalgamation of various constants resulting from multiple steps; this should be harmless since constants are not asymptotically relevant.

We now obtain an estimate for $B_{3,n}(t)$. Note the following:

$$\begin{aligned}E_{\mathcal{E}(\alpha,c)}[M_i^2] &= E\left[\left(\int_{t_i}^{t_{i+1}}\mu(s)ds\right)^2 I_{\mathcal{E}(\alpha,c)}\right]\\ &= E\left[\left(\int_{t_i}^{t_{i+1}}\mu(s)I_{\mathcal{E}(\alpha,c)}ds\right)^2\right]\\ &\leq E\left[\left(\int_{t_i}^{t_{i+1}}|\mu(s)I_{\mathcal{E}(\alpha,c)}|ds\right)^2\right]\end{aligned}$$

Note that $|\mu(s)I_{\mathcal{E}(\alpha,c)}|$ is either 0 or less than c, so

$$E_{\mathcal{E}(\alpha,c)}[M_i^2] \leq c\Delta_n^2. \tag{A.9}$$

Now since $g_{h,k}$ and $\tilde{g}_{h,k}$ are bounded independently of h and k, and $n\Delta_n = 1$, we have

$$B_{3,n}(t) = O(H_n\Delta_n). \tag{A.10}$$

We obtain an estimate for $B_{2,n}(t)$ next, but first let

$$\begin{aligned} T_{\sigma^2}(c) &:= \inf\{t \in (0,1] : \sigma^2(t) > c\}, \\ T_\mu(c) &:= \inf\{t \in (0,1] : \mu(t) > c\}. \end{aligned} \tag{A.11}$$

So, $T_{\sigma^2}(c)$ and $T_\mu(c)$ are the hitting times of the open set (c,∞) by μ and σ^2, respectively; and they record the instant just before these coefficients exceed c. Because both processes are adapted and at least càdlàg, both hitting times are stopping times. Now, set

$$T_i(c) := T_{\sigma^2}(c) \wedge T_\mu(c) \wedge t_i, \qquad i = 1, \cdots, n.$$

Because $T_{\sigma^2}(c)$ and $T_\mu(c)$ are stopping times, so are the $T_i(c)$'s. The important thing to note is that at all times before $T_i(c)$, both μ and σ^2 are bounded by c.

Returning to $B_{2,n}(t)$ note that $I^2_{\mathcal{E}(\alpha,c)} = I_{\mathcal{E}(\alpha,c)}$ so that $E_{\mathcal{E}(\alpha,c)}[M_iS_i] = E[(M_iI_{\mathcal{E}(\alpha,c)})(S_iI_{\mathcal{E}(\alpha,c)})]$. By the Cauchy-Schwarz inequality,

$$E[(M_iI_{\mathcal{E}(\alpha,c)})(S_iI_{\mathcal{E}(\alpha,c)})] \leq E[(M_iI_{\mathcal{E}(\alpha,c)})^2]^{1/2}E[(S_iI_{\mathcal{E}(\alpha,c)})^2]^{1/2} \tag{A.12}$$

Now, by repeating the same steps as in the case of $B_{3,n}(t)$ above, we may conclude that

$$E[(M_iI_{\mathcal{E}(\alpha,c)})^2]^{1/2} \leq c\Delta_n. \tag{A.13}$$

Now consider the following: if $\omega \in \mathcal{E}(\alpha,c)$ then $\sigma^2(\omega) \in \mathcal{H}(\alpha,c)$ and $\|\sigma^2(\omega)\|_\infty \leq \|\sigma^2(\omega)\|_\alpha \leq c$ so that $T_{\sigma^2(\omega)}(c) = 1$. Similarly $T_{\mu(\omega)}(c) = 1$ so that $T_i(c) = t_i$. Hence,

$$\begin{aligned} E[(S_iI_{\mathcal{E}(\alpha,c)})^2] &= E\left[\left(\int_{t_i}^{t_{i+1}} \sigma(s)dW_s\right)^2 I_{\mathcal{E}(\alpha,c)}\right] \\ &= E\left[\left(\int_{t_i}^{T_{i+1}(c)} \sigma(s)dW_s\right)^2 I_{\mathcal{E}(\alpha,c)}\right] \\ &\leq E\left[\left(\int_{t_i}^{T_{i+1}(c)} \sigma(s)dW_s\right)^2\right], \end{aligned} \tag{A.14}$$

Note the role played by the $T_i(c)$'s; they serve to eliminate the factor $I_{\mathcal{E}(\alpha,c)}$ from the computations. An application of the Burkholder-Davis-Gundy (BDG) inequality now has the effect of eliminating the Wiener process W. So,

$$\begin{aligned} E[(S_i I_{\mathcal{E}(\alpha,c)})^2]^{1/2} &\leq cE\left[\left(\int_{t_i}^{T_{i+1}(c)} \sigma^2(s)ds\right)\right]^{1/2} \\ &\leq (c\Delta_n)^{1/2}. \end{aligned} \tag{A.15}$$

Now, substituting (A.13) and (A.15) into (A.12) yields the estimate

$$E_{\mathcal{E}(\alpha,c)}[M_i S_i] \leq (c\Delta_n)^{3/2}. \tag{A.16}$$

Since $g_{h,k}$ and $\tilde{g}_{h,k}$ are bounded independently of h and k, and $n\Delta_n = 1$, we have

$$B_{2,n}(t) = O(H_n \Delta_n^{1/2}). \tag{A.17}$$

Now we tackle the final piece $B_{1,n}(t)$. Let

$$A := E_{\mathcal{E}(\alpha,c)}\left[\sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)} S_i^2 - \int_0^1 \sigma^2(s)\overline{\tilde{g}_{h,k}(s)}ds\right]. \tag{A.18}$$

We will first obtain an upper bound for A; we proceed by adding and subtracting $E_{\mathcal{E}(\alpha,c)}[\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \overline{\tilde{g}_{h,k}(t_i)}\sigma^2(s)ds]$ from A to yield:

$$\begin{aligned} A &= E_{\mathcal{E}(\alpha,c)}\left[\sum_{i=0}^{n-1} \overline{\tilde{g}_{h,k}(t_i)}\left(S_i^2 - \int_{t_i}^{t_{i+1}} \sigma^2(s)ds\right)\right] \\ &\quad + E_{\mathcal{E}(\alpha,c)}\left[\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} \sigma^2(s)\{\overline{\tilde{g}_{h,k}(t_i)} - \overline{\tilde{g}_{h,k}(s)}\}ds\right)\right] \\ &=: A_1 + A_2. \end{aligned}$$

We obtain estimates in turn for the summands. By linearity of expectation

$$\begin{aligned} A_2 &= \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} E[\sigma^2(s) I_{\mathcal{E}(\alpha,c)}]\{\overline{\tilde{g}_{h,k}(t_i)} - \overline{\tilde{g}_{h,k}(s)}\}ds \\ &\leq c\bar{\omega}(\tilde{g}_{h,k}, \Delta_n), \end{aligned}$$

where $\bar{\omega}(\tilde{g}_{h,k}, \Delta_n)$ is the modulus of continuity of $\tilde{g}_{h,k}$ on an interval of length Δ_n. By Lemma (1.2) and the Lipschitz continuity of g we have,

$$A_2 \leq c\bar{\omega}(g, \Delta_n) \leq c\Delta_n. \tag{A.19}$$

Now, we obtain an estimate for A_1. First, let $D_i : \Omega \times [0,1] \to \mathbb{R}$ for $i = 0, \cdots, n-1$ be defined as follows:

$$D_i(t) := \overline{\tilde{g}_{h,k}(t_i)} \left(\int_{t_i}^{t} \sigma(u) dW_u \right) I_{(t_i, t_{i+1}]}(t). \tag{A.20}$$

$$D_0(0) := 0. \tag{A.21}$$

So, $D_i(t)$ is 0 on $[0,1]$ except when t is in $(t_i, t_{i+1}]$. Now, using the integration by parts formula for semimartingales, we may write

$$S_i^2 - \int_{t_i}^{t_{i+1}} \sigma^2(s) ds = 2 \int_{t_i}^{t_{i+1}} \left(\int_{t_i}^{s} \sigma(u) dW_u \right) \sigma(s) dW_s$$

so that

$$\begin{aligned} A_1 &= 2\, E_{\mathcal{E}(\alpha,c)} \left[\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \overline{\tilde{g}_{h,k}(t_i)} \left(\int_{t_i}^{s} \sigma(u) dW_u \right) \sigma(s) dW_s \right] \\ &= 2\, E_{\mathcal{E}(\alpha,c)} \left[\int_0^1 \sum_{i=0}^{n-1} D_i(s) \sigma(s) dW_s \right] \\ &= 2\, E \left[\left(\int_0^1 \sum_{i=0}^{n-1} D_i(s) \sigma(s) dW_s \right) I_{\mathcal{E}(\alpha,c)} \right]. \end{aligned}$$

Using the same stopping time argument as above, we may replace the upper limit of integration with $T_n(c)$ so that

$$\begin{aligned} A_1 &\le 2\, E \left[\left(\int_0^{T_n(c)} \sum_{i=0}^{n-1} D_i(s) \sigma(s) dW_s \right) I_{\mathcal{E}(\alpha,c)} \right] \\ &\le 2\, E \left[\left| \int_0^{T_n(c)} \sum_{i=0}^{n-1} D_i(s) \sigma(s) dW_s \right| \right]. \end{aligned}$$

Now using the fact that $\int_0^{T_n(c)} \sum_{i=0}^{n-1} D_i(s) \sigma(s) dW_s$ is a martingale, we may make another appeal to the BDG inequality to yield:

$$\begin{aligned} A_1 &\le cE \left[\left| \int_0^{T_n(c)} \left(\sum_{i=0}^{n-1} D_i(s) \sigma(s) \right)^2 ds \right|^{1/2} \right] \\ &\le cE \left[\left| \int_0^{T_n(c)} \sum_{i=0}^{n-1} \{ D_i(s) \sigma(s) \}^2 ds \right|^{1/2} \right], \end{aligned}$$

where the last line follows because $D_i(s)D_j(s) = 0$ whenever $i \neq j$. Now if we define $D_i^* := \sup_{t_i < s \leq T_n(c)} D_i(s)$, and use the fact that σ is less than c before $T_n(c)$ then

$$
\begin{aligned}
A_1 &\leq cE\left[\left|\sum_{i=0}^{n-1} \Delta_n (D_i^*)^2\right|^{1/2}\right] \\
&\leq cE\left[\left|\sum_{i=0}^{n-1} \Delta_n (D_i^*)^2\right|\right] \\
&\leq c\Delta_n \sum_{i=0}^{n-1} E[(D_i^*)^2] \qquad \text{(A.22)}
\end{aligned}
$$

where c is a generic constant representing the bound on σ^2 and the BDG constant. Note from the definition of D_i (A.20) that it is itself a martingale, so we may bound D_i^* with yet another application of the BDG inequality. That is

$$
\begin{aligned}
D_i^* &\leq \left(\int_{t_i}^{T_{i+1}(c)} \sigma^2(s) ds\right)^{1/2} \\
&\leq c\Delta_n^{1/2}. \qquad \text{(A.23)}
\end{aligned}
$$

Plugging the above into the estimate in (A.22) yields: $A_1 \leq c\Delta_n$. Combining the estimates for A_1 and A_2, it may be seen that

$$
B_{1,n}(t) = O(H_n \Delta_n). \qquad \text{(A.24)}
$$

Collecting the estimates for $B_{j,n}(t)$ for $j = 1, \cdots, 4$, it is easily seen that $E[v_n((,t)t) - \sigma^2(t)] = O(H_n \Delta_n^{1/2} + H_n^{-\alpha} \log H_n)$ for all $t \in [0,1]$. So that

$$
B^2(\alpha, c) = O(H_n^2 \Delta_n + H_n^{-2\alpha} \log^2 H_n).
$$

Next, we obtain a bound for the variance term $V_n(\alpha, c)$. Recall that

$$
V_n(\alpha, c) = \int_0^1 E_{\mathcal{E}(\alpha,c)}[\{v_n((,t)t) - E_{\mathcal{E}(\alpha,c)}[v_n((,t)t)]\}^2] dt.
$$

So that by the definition of the estimator, we may write

$$
\begin{aligned}
V_n(\alpha,c) &= \int_0^1 E_{\mathcal{E}(\alpha,c)}\left[\left\{\sum_{(h,k)\in\Theta_n}(c_{h,k}-E_{\mathcal{E}(\alpha,c)}[c_{h,k}])g_{h,k}(t)\right\}^2\right]dt \\
&= \sum_{(h,k)\in\Theta_n}\mathrm{var}_{\mathcal{E}(\alpha,c)}[\hat{c}_{h,k}]\left(\int_0^1 g_{h,k}^2(t)dt\right) \\
&\quad+\sum_{(h,k)\neq(h',k')\in\Theta_n}\mathrm{cov}_{\mathcal{E}(\alpha,c)}[\hat{c}_{h,k},\hat{c}_{h',k'}]\left(\int_0^1 g_{h,k}(t)g_{h',k'}(t)dt\right) \\
&=: V_1+V_2,
\end{aligned}
\tag{A.25}
$$

We will estimate these quantities in turn starting with V_1, but first let

$$
\begin{aligned}
Y_i &:= \left(\int_{t_i}^{t_{i+1}}\sigma(s)dW_s\right)^2, \\
Z_i &:= \left(\int_{t_i}^{t_{i+1}}\mu(s)ds\right)^2+2\left(\int_{t_i}^{t_{i+1}}\mu(s)ds\right)\left(\int_{t_i}^{t_{i+1}}\sigma(s)dW_s\right), \\
\beta_{1,i} &:= \sum_{(h,k)\in\Theta_n} g_{h,k}^2(t_i)\left(\int_0^1 g_{h,k}^2(t)dt\right),
\end{aligned}
$$

for $i=0,\cdots,n-1$. Now note that

$$
\hat{c}_{h,k}=\sum_{i=0}^{n-1}g_{h,k}(t_i)(X_{t_{i=1}}-X_{t_i})^2=\sum_{i=0}^{n-1}g_{h,k}(t_i)(Y_i+Z_i),
$$

and since increments of the Brownian motion are independent, we have

$$
V_1=\sum_{i=0}^{n-1}\beta_{1,i}(\mathrm{var}_{\mathcal{E}(\alpha,c)}[Y_i]+\mathrm{var}_{\mathcal{E}(\alpha,c)}[Z_i]+2\mathrm{cov}_{\mathcal{E}(\alpha,c)}[Y_i,Z_i]).
$$

We will estimate the first two moments of Y_i and Z_i in turn. Note that

$$\begin{aligned}
E_{\mathcal{E}(\alpha,c)}[Y_i] = E[Y_i I_{\mathcal{E}(\alpha,c)}] &= E\left[\left(\int_{t_i}^{t_{i+1}} \sigma(s)dW_s\right)^2 I_{\mathcal{E}(\alpha,c)}\right] \\
&= E\left[\left(\int_{t_i}^{T_i(c)} \sigma(s)dW_s\right)^2 I_{\mathcal{E}(\alpha,c)}\right] \\
&\le E\left[\left(\int_{t_i}^{T_i(c)} \sigma(s)dW_s\right)^2\right] \\
&\le cE\left[\left(\int_{t_i}^{T_i(c)} \sigma^2(s)ds\right)\right] \\
&\le c\Delta_n. \qquad \text{(A.26)}
\end{aligned}$$

where the fourth line results from an application of the BDG inequality. Repeating the exact same steps, it may be seen that $E_{\mathcal{E}(\alpha,c)}[Y^2] \le c\Delta_n^2$. Thus,

$$\mathrm{var}_{\mathcal{E}(\alpha,c)}[Y_i] = E_{\mathcal{E}(\alpha,c)}[Y_i^2] - E_{\mathcal{E}(\alpha,c)}[Y_i]^2 \le c\Delta_n^2. \qquad \text{(A.27)}$$

Next we obtain estimates for Z_i. From (A.9) and (A.16) we may conclude

$$E_{\mathcal{E}(\alpha,c)}[Z_i] = E_{\mathcal{E}(\alpha,c)}[M_i^2] + 2E_{\mathcal{E}(\alpha,c)}[M_iS_i] \le c\Delta_n^{3/2}.$$

Using similar computations as above, it may be seen that $E_{\mathcal{E}(\alpha,c)}[Z_i^2] \le c\Delta_n^3$ so that

$$\mathrm{var}_{\mathcal{E}(\alpha,c)}[Z_i] \le c\Delta_n^3.$$

Now by the Cauchy-Schwarz inequality we may write

$$\mathrm{cov}_{\mathcal{E}(\alpha,c)}[Y_i, Z_i] \le (\mathrm{var}_{\mathcal{E}(\alpha,c)}[Z_i]\mathrm{var}_{\mathcal{E}(\alpha,c)}[Y_i])^{1/2} \le c\Delta_n^{5/2}.$$

Now because $g_{h,k}$ is bounded, it follows that $\beta_{1,i} = O(H_n)$ so

$$V_1 = O(H_n\Delta_n)$$

It is straight forward to see that V_2 may be estimated in a similar fashion. Indeed, let

$$\beta_{2,i} := \sum_{(h,k)\neq(h',k')\in\Theta_n} g_{h',k'}(t_i)g_{h,k}(t_i)\left(\int_0^1 g_{h,k}(t)g_{h',k'}(t)dt\right),$$

then we may write

$$V_2 = \sum_{i=0}^{n-1} \beta_{2,i}(\mathrm{var}_{\mathcal{E}(\alpha,c)}[Y_i] + \mathrm{var}_{\mathcal{E}(\alpha,c)}[Z_i] + 2\mathrm{cov}_{\mathcal{E}(\alpha,c)}[Y_i, Z_i]),$$

the computations will then proceed identically as before from this point. Again, by the boundedness of $g_{h,k}$, it follows that $\beta_{2,i} = O(H_n^2)$ so that

$$V_2 = O(H_n^2 \Delta_n).$$

Therefore,

$$V(\alpha, c) = O(H_n^2 \Delta_n).$$

$\square$

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