Gabor series method for estimating multivariate volatilities

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1 Model

Let $\{p_t\}$ be a log prices process assumed to be a version of the strong solution of the stochastic differential equation

$$\mathrm{d}p_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t, \qquad t \in [0, T], \tag{1.1}$$

where $\{W_t\}$ is a standard Brownian motion with respect to the filtered probability basis $(\Omega, \mathcal{F}, \{\mathcal{F}\}_t, \mathrm{P})$ satisfying the usual conditions ; $\{\mu_t\}$ and $\{\sigma_t\}$ are the instantaneous drift and diffusion coefficients satisfying the Lipschitz and growth conditions for the existence of a strong solution; and T is a finite number. Note that the time horizon may be set to $[0, 1]$ without losing the generality of our analysis by using a change of variable argument.

We assume that n log prices p_i indexed by i from 1 to n are observed discreely at equidistant intervals $\Delta_n := 1/n$ between time zero and one. Given this data we wish to obtain an estimate of the spot volatility σ_t^2 during the observation interval $[0, 1]$. Note that the spot volatility is a random element (function) in $L^2(0, 1)$, the set of square integrable functions on $[0, 1]$. In other words, we wish to obtain a sequence of random elements $\hat{\sigma}_t^2$ whose approximation of σ_t^2 improves as the number of observations increases. To this end we note that since σ_t^2 is a random elements in $L^2(0, 1)$, it admits the following representation:

$$\sigma_t^2 = \sum_{j,k \in \mathbb{Z}} c_{j,k} e^{i2\pi jbt} g(t - ka), \tag{1.2}$$

where $i = \sqrt{-1}$; a and b are given real numbers; g is a suitably chosen function in $L^2(0, 1)$; $c_{j,k}$ for $j, k \in \mathbb{Z}$ are random coefficients given by

$$c_{j,k} = \int_{\mathbb{R}} \sigma_t^2 e^{i2\pi jbt} g(t - ka) \mathrm{d}t; \tag{1.3}$$

and $\{g_{j,k}(t) := e^{i2\pi jbt}g(t-ka)\}$ for $j,k \in \mathbb{Z}$ forms a Gabor frame[1] for $L^2(0,1)$. A frame generalizes the notion of a basis for a vector space by containing additional vectors beyond those absolutely neccessary to form a basis. The additional vectors offer flexibility and sparsity in the representation. Furthermore, a Gabor frame is localized in both time and frequency; so, we may expect it to be amenable to noise reduction applications where stationarity assumptions on the latent process may not be appropriate. So, given choices of a, b and g, the random element σ_t^2 may be estimated by obtaining approximations for the random coefficients $c_{j,k}$. To this end, we propose the following estimator

$$\hat{\sigma}_t^2 = \sum_{j,k\in[-N,N]} \hat{c}_{j,k} g_{j,k}(t), \tag{1.4}$$

where

$$\hat{c}_{j,k} := \sum_{i=1}^{n-1} g_{j,k}(i\Delta_n)(p_{i+1} - p_i)^2; \tag{1.5}$$

and N is some positive natural number. Note that the precision of $\hat{\sigma}_t^2$ improves as n and N increase, independently of each other.

1.1 Lemma *The estimator $\hat{c}_{j,k}$ converges in probability to $c_{j,k}$ as $n \to \infty$.*

Proof.

□

[1]See Christensen (2001) for a very asseccible introduction to Gabor frames