Estimating realized spot volatility with Gabor frames.

Wale Dare

University of St. Gallen

November 16, 2015

Motivation: (Co)volatility (σ^2) is important

1. Risk management: VaR, CVaR, etc…
2. Portfolio optimization: Mean-variance optimization
3. Option pricing: Black-Scholes formula
4. etc…

Motivation: Semimartingales as prices

1. Only a security market with prices that evolve as semimartingales can be priced. Delbean & Schachermayer (1998)
2. X is a semimartingale on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ if there is a finite variation process A and a local martingale M s.t.

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$$\begin{aligned} X_t &= A_t + M_t \\ &= A_t^c + A_t^J + M_t^c + M_t^J \end{aligned} \tag{1}$$

Ito Semimartingales

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$$ \tag{2} $$

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σ, b are stochastic processes. μ is a measure-valued random variable, i.e. $\mu : \Omega \times \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}) \to \mathbb{R}$, and ν is its Levy intensity measure.

The problem

1. Discrete observations $X_1, \cdots, X_n$ from

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + A_t^J + M_t^J, \qquad \forall t \geq 0$$

   over a fixed interval $[0, T]$
2. Estimate σ^2 over the interval $[0, T]$.
   2.1 Nonparametrically: Arbitrary functional specification
   2.2 Globally: find a sequence of random *functions* $\hat{\sigma}_n^2 : \Omega \times [0, T] \to \mathbb{R}$ s.t. $\|\hat{\sigma}_n^2 - \sigma^2\|_{L^2} \to 0$ in probability. (possibly in L^p, $p > 1$).

General approach I

1. Assume $\sigma^2 \in L^2[0, T]$, a.s., where

$$L^2[0, T] = \left\{ f \in \mathbb{C}^{[0,T]} : \int_0^T f(s)f\bar{(s)}ds < \infty \right\} \tag{2}$$

is a Hilbert space, so it behaves just like $\mathbb{R}^2$. In particular, the idea of an orhogonal basis (just like $\{(1,0),(0,1)\}$ in $\mathbb{R}^2$) makes sense in L^2.

2. If $\{\psi_k\}_{k=1}^{\infty}$ is such an orthogonal basis and $f \in L^2$, then there is a sequence of constants $c := \{c_k\}_{k=1}^{\infty}$ s.t. $f = \sum_k^{\infty} c_k \psi_k$. In particular,

$$\sigma^2(t) = \sum_k^{\infty} c_k \psi_k(t) \qquad t \in [0, T] \tag{3}$$

General approach II

where

$$c_k = \langle \sigma^2, \psi_k \rangle := \int_0^T \sigma^2(s) \bar{\psi}_k(s) ds$$

3. Examples of $\{\psi_k\}_{k=1}^{\infty}$: B-Splines, Fourier series (basis), wavelets, etc...
4. Given discrete X_i at times t_i for $i = 0, \ldots, n-1$.

$$\hat{\sigma}_n^2 = \sum_h^H \hat{c}_h \psi_h$$

$$\hat{c}_h = \sum_{i=0}^{n} \bar{\psi}_h(t_i)(X_{i+1} - X_i)^2 \qquad (4)$$

Previous solutions

1. Fourier orthogonal basis (Malliavin et. al 2007)
2. Wavelet orthogonal basis (Hoffmann et al 2012)

Consistent ONLY if X_t is a continuous Ito Semimartingale. That is

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s.$$

Our contribution

1. Consistent estimate for the general case in which you have jumps in addition to the continuous part.

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + A_t^J + M_t^J.$$

2. Generalization of the previous methods based on orthonomal basis. We use frames which generalize the concept of orthogonal basis.
3. The extension to frames is not just a technical extension, frames, Gabor frames in particular, possess coefficient noise reduction capabilities not found in orthonormal basis.

Frames

1. They come in pairs: $\{\psi_k\}$ and $\{\tilde{\psi}_k\}$.
2. Together, they possess the representation property: If $f \in L^2$ then

   $$f = \sum_k c_k \psi_k$$

   where

   $$c_k = \langle f, \tilde{\psi}_k \rangle$$

3. An orthonormal basis is a frame such that $\psi_k = \tilde{\psi}_k$.

Gabor frames

1. They are frames constructed by translating a single function g in time and frequency:

$$\{g_{h,k}(t)\} := \{e^{2\pi ihat} g(t - kb)\}, \qquad h, k \in \mathbb{Z}$$

2. If g satisfies the partition of unity property then its dual $\tilde{g}$ is a finite linear combination of translates of g, so it is easy to compute.

Nonparametric jump-robust global spot volatility estimator

$$\hat{\sigma}_n^2 = \sum_{|h|<H_n, |k|<K_0} \hat{c}_{h,k} g_{h,k}$$

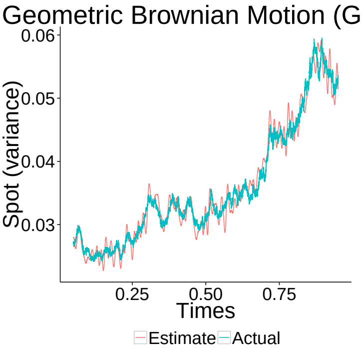
$$\hat{c}_{h,k} = \sum_{i=0}^{n} \bar{\bar{g}}_{h,k}(t_i)(X_{i+1} - X_i)^2 I_{\{|X_{i+1}-X_i|<u_n\}} \tag{5}$$

where

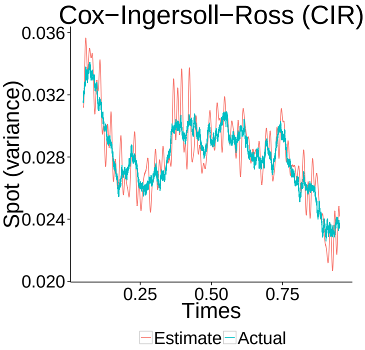
1. K_0 is a constant
2. $H_n = O(\sqrt{T/n})$
3. $\frac{\sqrt{(T/n)\log(n/T)}}{u_n} = o(1)$.

Performance I

Figure: Estimated vs. actual spot volatility of common price models



(a) GBM



(b) CIR

Performance II

Table: Mean integrated square error (MISE) of the frame-based estimator $\hat{\sigma}_n^2$ for popular price models.

	ABM			OU		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	1.30×10^{-4}	2.86×10^{-6}	1.27×10^{-4}	1.43×10^{-4}	1.19×10^{-5}	1.31×10^{-4}
5000	1.41×10^{-5}	1.11×10^{-6}	1.30×10^{-5}	1.45×10^{-5}	1.62×10^{-6}	1.28×10^{-5}
50000	2.32×10^{-6}	1.02×10^{-6}	1.30×10^{-6}	2.36×10^{-6}	1.12×10^{-6}	1.23×10^{-6}

	GBM			CIR		
n	MISE	Sq. Bias	Var	MISE	Sq. Bias	Var
500	2.18×10^{-4}	4.18×10^{-6}	2.14×10^{-4}	6.26×10^{-5}	8.51×10^{-7}	6.17×10^{-5}
5000	2.33×10^{-5}	1.58×10^{-6}	2.17×10^{-5}	6.82×10^{-6}	6.00×10^{-7}	6.22×10^{-6}
50000	4.66×10^{-6}	1.02×10^{-6}	3.64×10^{-6}	1.46×10^{-6}	6.06×10^{-7}	8.52×10^{-7}

Note: The mean of the integrated square errors are obtained by taking an average over 100 sample paths generated for each model/number of observations pair.

Performance III

Theorem

$\hat{\sigma}_n^2$ converges in $L^2[0, T]$ in probability to σ^2 as $n \to \infty$.