I think the source of my confusion is located in the previous line. It would appear that the middle estimate:

$$E\left[\left|\int_0^{T_c}(\sum_{i=2}^m H_{t,i})^2 dt\right|^{p/2}\right] \leq E\left[\left|\int_0^{T_c}\sum_{i=2}^m H_{t,i}^2 dt\right|^{p/2}\right]$$

is missing an m, according to the Cauchy inequality. Heuristically, when you square a sum you end up with cross terms that must be accounted for. So it would appear that the estimate would look like this instead (with an m inserted):

$$E\left[\left|\int_0^{T_c}\left(\sum_{i=2}^m H_{t,i}\right)^2 dt\right|^{p/2}\right] \leq E\left[\left|\int_0^{T_c}\sum_{i=2}^m \{m\} H_{t,i}^2 dt\right|^{p/2}\right]$$

Now on the next line instead of

$$E\left[\left|m^{-1}\sum_{i=2}^m (H_i^*)^2\right|^{p/2}\right] \leq m^{-1}\sum_{i=2}^m E[(H^*)^p]$$

we would have instead (by Hölder inequality):

$$E\left[\left|\sum_{i=2}^m (H_i^*)^2\right|^{p/2}\right] \leq m^{p/2-1}\sum_{i=2}^m E[(H^*)^p]$$

(I have left out the length of the support because it is asymptotically irrelevant). Combining the above with the last equation of the proof of Lemma 4.3, which estimates $E[(H^*)^p] = O(m^{-p/2})$, it would follow that

$$E_{\mathcal{D}_\infty(c)}\left[\left|\int_0^{T_c}\sum_{i=2}^m H_{t,i} dX_t\right|^p\right] = O(1).$$