Autonomous Robotics : Assignment 1

1. To combine two measurements y_1, y_2 of two sensors observing unknown x, for $\begin{cases} y_1 = h_1 x \\ y_2 = h_2 x \end{cases}$;

a. The state model equation is :

$$X_K = F_K x_{K-1} + W_K$$

X_K is the state vector at time K
W_K is the process time and
F_K is the state transition matrix from $K-1$ to next K

The measurement model equation is :

$$Y_K = H_K X_K + V_K$$

H_K is the measurement matrices of the two sensors.
V_K is the measurement noise terms
Y_K is the measurement vector of both sensors.

b. Show that if matrix R (measurement model covariance) the state uncertainty in Least square sense is :

$$\Sigma_X = (H^T H)^{-1} H R H^T ((H^T H)^{-1})^T$$

1. From $y = hx + v$ representing a simple linear regression model where y is dependent, x is independent and h is regression coefficient or slope. v is the error term of Y that cannot be explained by x.

if x is state vector to be optimized and y is measurement vector, to find best estimate of h;

~~the~~ minimize the sum square differences between y and predicted y values by ordinary least squares (OLS) so we minimize the cost function

$$J = (y - Hx)^T R^{-1} (y - Hx)$$

we take derivative of J w.r.t x and set to zero.

$$\frac{dJ}{dx} = -2H^T R^{-1}(y - Hx) = 0$$

$$x = (H^T R^{-1} H)^{-1} H^T R^{-1} y$$ represents ~~optimal~~ OLS estimator of x

The covariance of x is:

$$\Sigma x = E\left[(x - \hat{x})(x - \hat{x})^T\right]$$ $\hat{x}$ is the estimate of state x.

Expand term $(x - \hat{x})$ using x:

$$(x - \hat{x})(x - \hat{x})^T = (x - (H^T R^{-1} H)^{-1} H^T R^{-1} y)(x^T - y^T R^{-1} H (H^T R^{-1} H)^{-1})$$

Expanding the product:

$$(x - \hat{x})(x - \hat{x})^T = xx^T - x\hat{x}^T - \hat{x}x^T + \hat{x}\hat{x}^T$$

Taking Expectation of Both sides.

$$\Sigma x = E[xx^T] - E[x\hat{x}^T] - E[\hat{x}x^T] + E[\hat{x}\hat{x}^T] \quad \text{---(i)}$$

since true ~~vector~~ state vector x is a constant. & $E[\hat{x}] = x$, then

in ---(i), $E[x\hat{x}^T] = xE[\hat{x}^T] = xx^T$ and.

$$E[\hat{x}x^T] = E[(H^T R^{-1} H)^{-1} H^T R^{-1} yy^T R^{-1} H (H^T R^{-1} H)^{-1}$$

$$= \underbrace{H^T R^{-1} H}\,\underbrace{(H^T R^{-1} H)^{-1}} = 1$$

here R^{-1} is symmetric and $R^{-1}R = 1$ so the last term of ---(i)

$$E[\hat{x}\hat{x}^T] = E\left[((H^T R^{-1} H)^{-1} H^T R^{-1} y)((H^T R^{-1} H)^{-1} H^T R^{-1} y)^T\right]$$

$$= (\cancel{R}H^T R^{-1} H)^{-1} H^T R^{-1} R R^{-1} H (H^T R^{-1} H)^{-1}$$

$$= (H^T R^{-1} H)^{-1}$$

so putting everything together in ---(i) we get

$$\Sigma x = xx^T - xx^T + 1 + (H^T R^{-1} H)^{-1} = (H^T R^{-1} H)^{-1} H R^{-1} H^T ((H^T R^{-1} H)^{-1})^T$$

$$\text{So } \Sigma x = (H^T H)^{-1} H^T R H \, (H^T H)^{-1}$$ proved.

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c. Expression of uncertainty (variance) where variance of both sensors measurements is identical and not correlated?

R is diagonal and of form $\sigma^2 I_{(2\times2)}$

From $\Sigma_x = (H^T H)^{-1} H^T R H (H^T H)^{-1}$ and $R = \sigma^2 I_{(2\times2)}$

$I_{(2\times2)} H = H$ so

$$\Sigma_x = \sigma^2 \left[(H^T H)^{-1} \cancel{H^T H} \cancel{(H^T H)^{-1}} \right]$$

$$\Sigma_x = \sigma^2 (H^T H)^{-1}$$ representing uncertainty of the state x.

d. Considering an initial estimate of x, x_0 with variance P_0^2

Assuming Gaussian distribution and uncorrelated noise:

Kalman Filter with two y_1 and y_2 measurements:

State Prediction:

$\hat{X}_{K|K-1} = F_X \hat{X}_{K-1|K-1}$ Predicted state at time K, measured K-1

$P_{K|K-1} = F_X P_{K-1|K-1} F_X^T + Q_{K-1}$ Predicted state covariance

Q_{K-1} is process noise covariance matrix.

Measurement Update:

$$K_K = P_{K|K-1} H_K^T (H_K P_{K|K-1} H_K^T + R_K)^{-1}$$

$$\hat{X}_{K|K} = \hat{X}_{K|K-1} + K_K (y_K - H_K \hat{X}_{K|K-1})$$

$$P_{K|K} = (1 - K_K H_K) P_{K|K-1}$$

where $K = 1,2,3\ldots$ and $y_K = [y_{1K} ; y_{2K}]$, $H = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$

$R = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$

H_K is measurement model matrix

R_K is measurement noise covariance matrix

K_K is Kalman Gain matrix)

E. Considering both sensors are broken, third sensor $y_3 = h_3\sqrt{x}$ with variance σ_3^2.

For the EKF ;

- Time upade equation:

$$\hat{x}_{k|k-1} = F_{k-1}\hat{x}_{k-1|k-1} \quad \text{and}$$

$$P_{k|k-1} = F_{k-1}P_{k-1|k-1}F_{k-1}^T + Q_{k-1}$$

- Measurement update equation:

$$K_k = P_{k|k-1}H^T(HP_{k|k-1}H^T R)^{-1}$$

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k(Z_k - h\hat{x}_{k|k-1})$$

$$P_{k|k} = (1 - K_k H)P_{k|k-1}$$

F. Can we find an approximation using EKF for any values of state x ?

No, the Extended Kalman filter handles non linearities by approximating the linear functions using a first order taylor series expansion. so accuracy depends on how close the system and models are to linear. EKF also assumes a gaussian noise, measurement function is also aproximately linear.
We cannot assume linearity for all values of X.

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2.

Minimize Reprojection error ~~using~~ between views using warp function to find 3D and cam motion from correspondences.

$p' = w(p, Z, T)$ assumming 3D points can be tracked without occlusions.

a. Given multivariate random variable $D_1(m_p, \Sigma_p)$ and mapping function $p' = w(p)$. To approximate first two moments of p' show that approximation of w by its first two terms of Taylor series on m_p gives $p' \sim D_2(m_{p'}, \Sigma_{p'})$

$$m_{p'} = f(m_p) \text{ and } \Sigma_{p'} = J(m_p)\Sigma_p J(m_p)^T$$

Taylor series expansion of warping function w_p around m_p (mean point) is given by:

$$w_p = w m_p + J m_p (p - m_p) + o(\|p - m_p\|^2)$$

where norm $\|p - m_p\|^2$ is ignored since we are interested only in the first two moments and $J(m_p)$ is the Jacobian matrix of w at m_p

To find mean of p' we take Expected values of both sides.

- $E(p') = E[w_p] = w(m_p) + J m_p (E(p) - m_p)$

since $E[p - m_p] = 0$ p is centered around m_p,

$E(p') = w m_p$.

- Covariance of p':

$$\Sigma_{p'} = E[[p - w m_p][p' - w m_p]^T]$$

expanding terms using Taylor series of w_p,

$$\Sigma_{p'} = E[(w_p - w m_p - J m_p (p - m_p))(w_p - w m_p - J m_p (p - m_p))^T]$$

$$= E[(J(m_p) * (p - m_p))(J m_p * (p - m_p)^T)]$$

$$= J m_p * E[(p - m_p)(p - m_p)^T] J m_p^T$$

$$= J m_p \Sigma_p J m_p^T$$

$$p' \approx D_2(m_{p'}, \Sigma_{p'})$$

b. Show that Jacobian of Euclidean normalization of Homogeneous $p = [x \; w]^T$, $p = p/w$ and.

$$J = \begin{bmatrix} wI & -x \\ 0 & 0 \end{bmatrix} / w^2.$$

$$p' = (1/w) \begin{bmatrix} 1 & 0 \\ -x/w & 1 \end{bmatrix}_{2\times 2} [x \; w]^T$$

Jacobian, partial derivative of p' w.r.t p is given in matrix

First component $\frac{\delta p_1'}{\delta p_1} = \frac{\delta x/w}{\delta x} = \frac{1}{w}$ and $\frac{\delta p_1'}{\delta p_2} = \frac{\delta x/w}{\delta w} = \frac{-x}{w^2}$

Second component $\frac{\delta p_2'}{\delta p_1} = \frac{\delta(1)}{\delta x} = 0$ and $\frac{\delta p_2'}{\delta p_2} = \frac{\delta(1)}{\delta w} = 0$

so $$J = \begin{bmatrix} \frac{\delta p_1'}{\delta p_1} & \frac{\delta p_1'}{\delta p_2} \\ \frac{\delta p_2'}{\delta p_1} & \frac{\delta p_2'}{\delta p_2} \end{bmatrix} = \begin{bmatrix} 1/w & -x/w^2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{w}{w^2} & -\frac{x}{w^2} \\ 0 & 0 \end{bmatrix}.$$

or $$J = \begin{bmatrix} wI & -x \\ 0 & 0 \end{bmatrix} / w^2$$

C. Assuming Z is calculated within a disparity map.

$Z = bf/d$, $b \to$ baseline, $f \to$ focal length and $d \sim N(d, \sigma_d^2)$

show that depth error distribution variance can be approximated by $\sigma_z^2 = Z^4 \sigma_d^2 / (bf)^2 = KZ^4$

from disparity map

$$var(Z) = var(bf/d) = \left(\frac{bf}{d}\right)^2 var\left(\frac{d}{d}\right) + \left(\frac{f}{d}\right)^2 var(b) + \left(\frac{b}{d}\right)^2 var(f)$$

if $var(b) = var(f) = 0$ i.e assuming baseline & focal length are constant, $var(z) = \left(\frac{bf}{d}\right)^2 var(d/d) = \left(\frac{bf}{d}\right)^2 \frac{\sigma d^2}{\delta^2}$; $d = \frac{bf}{z}$

$var(z) = \frac{bf^2}{Z^2} \sigma d^2$ if $K = \sigma d^2 bf^2$ ∴

then variance of depth error is approximated $\sigma Z^2 = KZ^4$

d. Find uncertainty in 3d point $P = ZK^{-1}p$, $p \sim N(m_p, \Sigma_p)$
hint $p = [x \ 1]^T$ and $\Sigma_p = \begin{bmatrix} \Sigma x & 0 \\ 0 & 0 \end{bmatrix}$.

Σx = variance of pixel coordinates in x-direction

from $P = ZK^{-1}p$ and $Z = bf/d$.

$var(P)$ is the trace of Σ_P.

$$var(P) = tr(Z^2 K^{-1} \Sigma_p K^{-T})$$
$$= tr(K^{-1} \Sigma_p K^{-1} Z^2) \quad \text{cummulative property.}$$
$$= tr(K^{-1} \Sigma_p K^{-1} (bf/d^2))$$

$$K^{-1} = \begin{bmatrix} 1/f & 0 & -cx/f \\ 0 & 1/f & -cy/f \\ 0 & 0 & 1 \end{bmatrix}$$

f = focal length
cx, cy = principal point.

$$K^{-1} \Sigma_p K^{-1} = \begin{bmatrix} 1/f & 0 & -cx/f \\ 0 & 1/f & -cy/f \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Sigma x & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/f & 0 & -cx/f \\ 0 & 1/f & -cy/f \\ 0 & 0 & 1 \end{bmatrix}.$$

$$= \begin{bmatrix} \Sigma x/fx^2 & 0 & -cx\Sigma x/fx^2 \\ 0 & 0 & 0 \\ -\frac{cx\Sigma x}{fx^2} & 0 & \frac{cx^2 \Sigma x}{fx^2} \end{bmatrix}$$

$$tr(K^{-1}\Sigma_p K^{-1}) = \frac{\Sigma x}{fx^2} + \frac{cx^2 \Sigma x}{fx^2} = \frac{cx^2 + \Sigma x}{fx^2}$$

now $var(P) = \dfrac{cx^2 + \Sigma x}{fx}\left(\dfrac{bf}{d^2}\right)$.

if $d = bf/Z$ then:

$$var(P) = \frac{cx^2 + \Sigma x}{fx}\left(Z^2 \sigma d^2 / \cancel{f}^2\right)$$

so variance of uncertainty of 3D point $P = ZK^{-1}p$.
is $(cx^2 + \Sigma x) Z^2 \sigma d^2 / f^2$

f. Adopt w to be a warping function following perspective projection with K matrix.

3D $P = ZK^{-1}p$; after a motion $T = \{R, t\}$.

$$p' = w(p, Z, T) = [K \ 0] \, T \begin{bmatrix} ZK^{-1}p \\ 1 \end{bmatrix} = ZKRK^{-1}p + Kt.$$

find covariance of p' in de-homogeneous coordinates for w in 2D pixel following Gaussian distribution $p \sim N(m_p, \Sigma_p)$

- we find Jacobian of $w(p, Z, t)$ w.r.t p coordinates.

$$J = \partial w(p, Z, T) / \partial p = ZKRK^{-1}$$

Covariance $\Sigma_{p'} = J \Sigma_p J^T$

$$= ZKRK^{-1} \Sigma_p K^{-T} R^T K^T Z^T$$

$$= Z^2 KR \Sigma_p R^T K^T K^{-T} Z^T$$

Convert $\Sigma_{p'}$ to covariance matrix.

$$\Sigma_{p'} = Z^2 KR \, x_p x_p^T \, R^T K^T K^{-T} Z^T \quad \text{and.}$$

$$x_p = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}.$$