BIOT-SAVART LAW

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Introduction

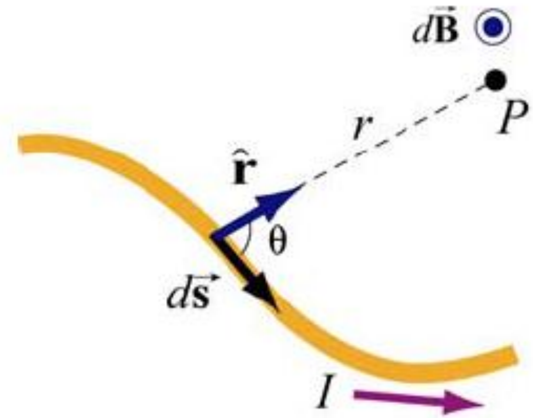
- A useful law that provides a method to calculate the magnetic field produced by an arbitrary current distribution.

- First discovered by Jean-Baptiste Biot and Félix Savart in the beginning of 19^{th} century

Definition

- The differential contribution d**B** to the magnetic field **B** from a length ds of a current I is given by the formula

$$d\vec{\mathbf{B}} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2}$$



with $\mu_0 = 4\pi\times 10^{-7}\,\mathrm{T\cdot m/A}$

the permeability of free space.

Thus the total magnetic field vector **B** is the sum of all of these small elements or, since they are differentially small, it is equivalent to the integral of d**B** over the current source.

$$\vec{\mathbf{B}} = \int_{\text{wire}} d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi} \int_{\text{wire}} \frac{d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}}{r^2}$$

Several key points to remember:

- **B** is a vector quantity which direction is determined by the cross product d**s** x **r** (and is perpendicular to both d**s** and **r**)
- The integration takes place over the entire current source (finite or infinite)
- Since the integral is a vector integral, the expression for **B** is really three integrals, one for each component of **B**.

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General Methodology To Follow

(1) Source point: Choose an appropriate coordinate system and write down an expression for the differential current element $I\,d\vec{\mathbf{s}}$, and the vector $\vec{\mathbf{r}}\,'$ describing the position of $I\,d\vec{\mathbf{s}}$. The magnitude $r' = |\vec{\mathbf{r}}\,'|$ is the distance between $I\,d\vec{\mathbf{s}}$ and the origin. Variables with a "prime" are used for the source point.

(2) Field point: The field point P is the point in space where the magnetic field due to the current distribution is to be calculated. Using the same coordinate system, write down the position vector $\vec{\mathbf{r}}_P$ for the field point P. The quantity $r_P = |\vec{\mathbf{r}}_P|$ is the distance between the origin and P.

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(3) Relative position vector: The relative position between the source point and the field point is characterized by the relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$. The corresponding unit vector is

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|}$$

where $r = |\vec{\mathbf{r}}| = |\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|$ is the distance between the source and the field point P.

(4) Calculate the cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$ or $d\vec{\mathbf{s}} \times \vec{\mathbf{r}}$. The resultant vector gives the direction of the magnetic field $\vec{\mathbf{B}}$, according to the Biot-Savart law.

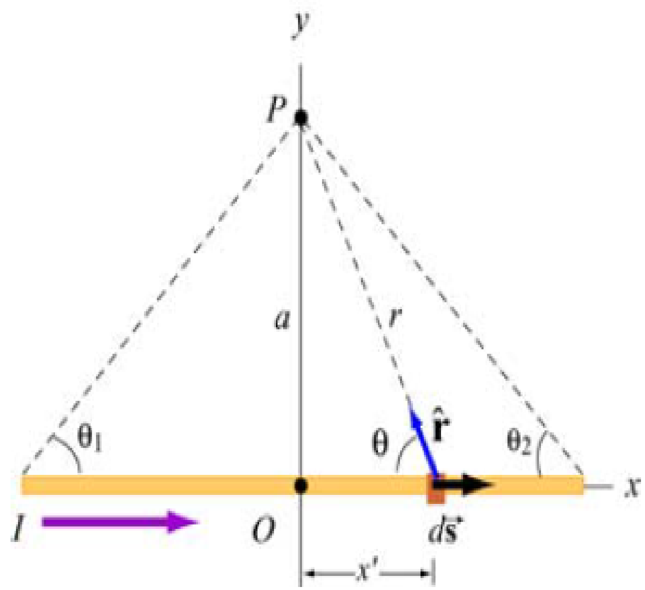
(5) Substitute the expressions obtained to $d\vec{\mathbf{B}}$ and simplify as much as possible.

(6) Complete the integration to obtain $\vec{B}$ if possible. The size or the geometry of the system is reflected in the integration limits. Change of variables sometimes may help to complete the integration.

Important Examples

I) Magnetic Field Due to a Finite Straight Wire

A thin, straight wire carrying a current I is placed along the x-axis. Evaluate the magnetic field at point P. Note that we have assumed that the leads to the ends of the wire make canceling contributions to the net magnetic field at the point P



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(1) Source point (coordinates denoted with a prime)

Consider a differential element $d\vec{\mathbf{s}} = dx'\hat{\mathbf{i}}$ carrying current I in the x-direction. The location of this source is represented by $\vec{\mathbf{r}}' = x'\hat{\mathbf{i}}$.

(2) Field point (coordinates denoted with a subscript "P")

Since the field point P is located at $(x, y) = (0, a)$, the position vector describing P is $\vec{\mathbf{r}}_P = a\hat{\mathbf{j}}$.

(3) Relative position vector

The vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$ is a "relative" position vector which points from the source point to the field point. In this case, $\vec{\mathbf{r}} = a\,\hat{\mathbf{j}} - x'\,\hat{\mathbf{i}}$, and the magnitude $r = |\vec{\mathbf{r}}| = \sqrt{a^2 + x'^2}$ is the distance from between the source and P. The corresponding unit vector is given by

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{a\,\hat{\mathbf{j}} - x'\,\hat{\mathbf{i}}}{\sqrt{a^2 + x'^2}} = \sin\theta\,\hat{\mathbf{j}} - \cos\theta\,\hat{\mathbf{i}}$$

(4) The cross product $d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}}$

The cross product is given by

$$d\,\vec{\mathbf{s}} \times \hat{\mathbf{r}} = (dx'\,\hat{\mathbf{i}}) \times (-\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) = (dx'\sin\theta)\,\hat{\mathbf{k}}$$

(5) Write down the contribution to the magnetic field due to $Id\vec{\mathbf{s}}$

The expression is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{dx\,\sin\theta}{r^2}\hat{\mathbf{k}}$$

which shows that the magnetic field at P will point in the $+\hat{\mathbf{k}}$ direction, or out of the page.

(6) Simplify and carry out the integration

The variables θ, x and r are not independent of each other. In order to complete the integration, let us rewrite the variables x and r in terms of θ.

$$\begin{cases} r = a/\sin\theta = a\csc\theta \\ x = a\cot\theta \;\;\Rightarrow\;\; dx = -a\csc^2\theta\,d\theta \end{cases}$$

Upon substituting the above expressions, the differential contribution to the magnetic field is obtained as

$$dB = \frac{\mu_0 I}{4\pi} \frac{(-a\csc^2\theta\, d\theta)\sin\theta}{(a\csc\theta)^2} = -\frac{\mu_0 I}{4\pi a}\sin\theta\, d\theta$$

Integrating over all angles subtended from $-\theta_1$ to θ_2 (a negative sign is needed for θ_1 in order to take into consideration the portion of the length extended in the negative x axis from the origin), we obtain

$$\boxed{B = -\frac{\mu_0 I}{4\pi a}\int_{-\theta_1}^{\theta_2} \sin\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_2 + \cos\theta_1)}$$

The first term involving θ_2 accounts for the contribution from the portion along the $+x$ axis, while the second term involving θ_1 contains the contribution from the portion along the $-x$ axis. The two terms add!

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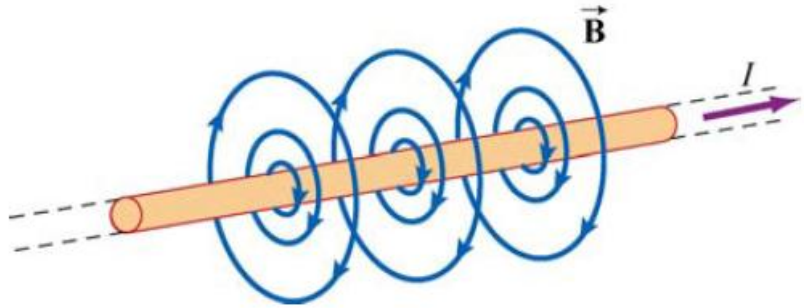
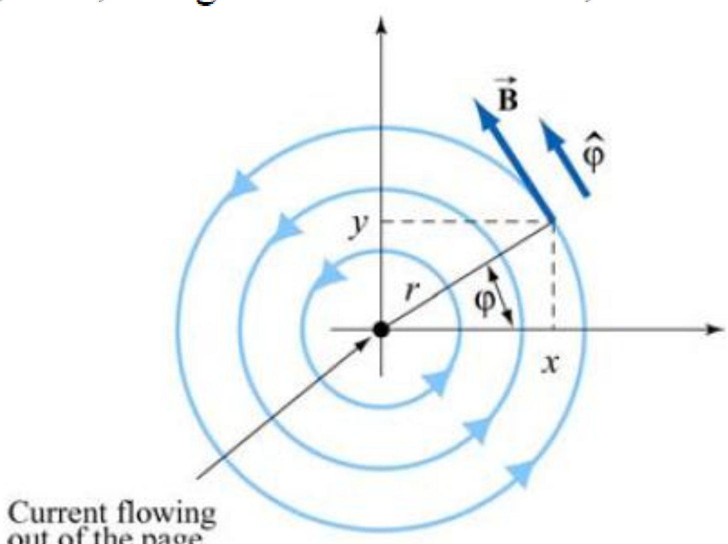
Note that in this limit, the system possesses cylindrical symmetry, and the magnetic field lines are circular



In fact, the direction of the magnetic field due to a long straight wire can be determined by the right-hand rule

If you direct your right thumb along the direction of the current in the wire, then the fingers of your right hand curl in the direction of the magnetic field. In cylindrical coordinates (r, φ, z) where the unit vectors are related by $\hat{\mathbf{r}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{z}}$, if the current flows in the $+z$-direction, then, using the Biot-Savart law, the magnetic field must point in the φ-direction.



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Let's examine the following cases:

(i) In the symmetric case where $\theta_2 = -\theta_1$, the field point P is located along the perpendicular bisector. If the length of the rod is $2L$, then $\cos\theta_1 = L/\sqrt{L^2+a^2}$ and the magnetic field is

$$B = \frac{\mu_0 I}{2\pi a}\cos\theta_1 = \frac{\mu_0 I}{2\pi a}\frac{L}{\sqrt{L^2+a^2}}$$

(ii) The infinite length limit $L \to \infty$

This limit is obtained by choosing $(\theta_1, \theta_2) = (0, 0)$. The magnetic field at a distance a away becomes

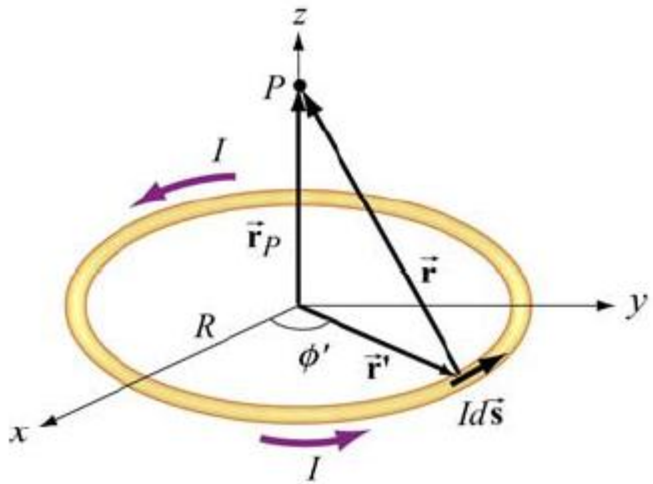
$$\boxed{B = \frac{\mu_0 I}{2\pi a}}$$

II) Magnetic Field Due to a Circular Current Loop

A circular loop of radius R in the xy plane carries a steady current I

(a) What is the magnetic field at a point P on the axis of the loop, at a distance z from the center?

(b) If we place a magnetic dipole $\vec{\boldsymbol{\mu}} = \mu_z \hat{\mathbf{k}}$ at P, find the magnetic force experienced by the dipole. Is the force attractive or repulsive? What happens if the direction of the dipole is reversed, i.e., $\vec{\boldsymbol{\mu}} = -\mu_z \hat{\mathbf{k}}$



(1) Source point

In Cartesian coordinates, the differential current element located at $\vec{\mathbf{r}}' = R(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$ can be written as $Id\vec{\mathbf{s}} = I(d\vec{\mathbf{r}}'/d\phi')d\phi' = IRd\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})$.

(2) Field point

Since the field point P is on the axis of the loop at a distance z from the center, its position vector is given by $\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$.

(3) Relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'$

The relative position vector is given by

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,' = -R\cos\phi'\,\hat{\mathbf{i}} - R\sin\phi'\,\hat{\mathbf{j}} + z\,\hat{\mathbf{k}}$$

and its magnitude

$$r = |\vec{\mathbf{r}}| = \sqrt{(-R\cos\phi')^2 + (-R\sin\phi')^2 + z^2} = \sqrt{R^2 + z^2}$$

is the distance between the differential current element and P. Thus, the corresponding unit vector from $Id\vec{\mathbf{s}}$ to P can be written as

$$\hat{\mathbf{r}} = \frac{\vec{\mathbf{r}}}{r} = \frac{\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'}{|\vec{\mathbf{r}}_P - \vec{\mathbf{r}}\,'|}$$

(4) Simplifying the cross product

The cross product $d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,')$ can be simplified as

$$d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,') = R\,d\phi'\left(-\sin\phi'\,\hat{\mathbf{i}}+\cos\phi'\,\hat{\mathbf{j}}\right)\times[-R\cos\phi'\,\hat{\mathbf{i}}-R\sin\phi'\,\hat{\mathbf{j}}+z\,\hat{\mathbf{k}}]$$
$$= R\,d\phi'[z\cos\phi'\,\hat{\mathbf{i}}+z\sin\phi'\,\hat{\mathbf{j}}+R\,\hat{\mathbf{k}}]$$

(5) Writing down $d\vec{\mathbf{B}}$

Using the Biot-Savart law, the contribution of the current element to the magnetic field at P is

$$d\vec{\mathbf{B}} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\hat{\mathbf{r}}}{r^2} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times\vec{\mathbf{r}}}{r^3} = \frac{\mu_0 I}{4\pi}\frac{d\vec{\mathbf{s}}\times(\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,')}{|\vec{\mathbf{r}}_P-\vec{\mathbf{r}}\,'|^3}$$
$$= \frac{\mu_0 IR}{4\pi}\frac{z\cos\phi'\,\hat{\mathbf{i}}+z\sin\phi'\,\hat{\mathbf{j}}+R\,\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}}\,d\phi'$$

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(6) Carrying out the integration

Using the result obtained above, the magnetic field at P is

$$\vec{\mathbf{B}} = \frac{\mu_0 IR}{4\pi} \int_0^{2\pi} \frac{z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}}}{(R^2+z^2)^{3/2}} d\phi'$$

The x and the y components of $\vec{\mathbf{B}}$ can be readily shown to be zero:

$$B_x = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \int_0^{2\pi} \cos\phi' d\phi' = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \sin\phi' \Big|_0^{2\pi} = 0$$

$$B_y = \frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \int_0^{2\pi} \sin\phi' d\phi' = -\frac{\mu_0 IRz}{4\pi(R^2+z^2)^{3/2}} \cos\phi' \Big|_0^{2\pi} = 0$$

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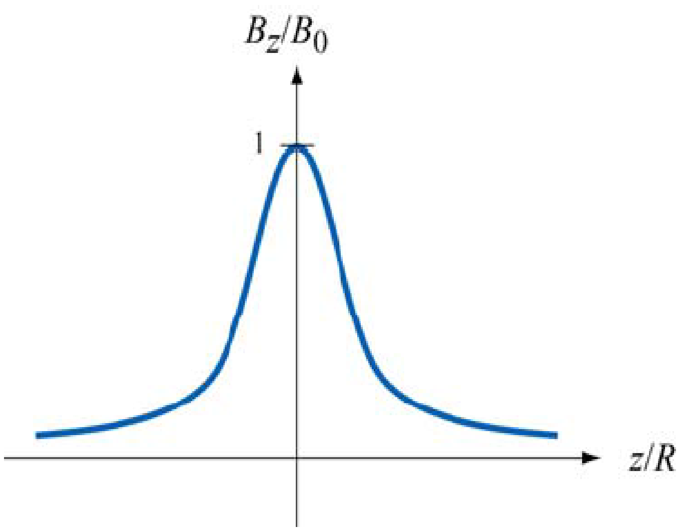
On the other hand, the z component is

$$B_z = \frac{\mu_0}{4\pi}\frac{IR^2}{(R^2+z^2)^{3/2}}\int_0^{2\pi} d\phi' = \frac{\mu_0}{4\pi}\frac{2\pi IR^2}{(R^2+z^2)^{3/2}} = \frac{\mu_0 IR^2}{2(R^2+z^2)^{3/2}}$$

Thus, we see that along the symmetric axis, B_z is the only non-vanishing component of the magnetic field. The conclusion can also be reached by using the symmetry arguments. The behavior of B_z / B_0 where $B_0 = \mu_0 I / 2R$ is the magnetic field strength at $z = 0$, as a function of z / R is shown



(b) If we place a magnetic dipole $\vec{\boldsymbol{\mu}} = \mu_z \hat{\mathbf{k}}$ at the point P due to the non-uniformity of the magnetic field, the dipole will experience a force given by

$$\vec{\mathbf{F}}_B = \nabla(\vec{\boldsymbol{\mu}} \cdot \vec{\mathbf{B}}) = \nabla(\mu_z B_z) = \mu_z \left(\frac{dB_z}{dz} \right) \hat{\mathbf{k}}$$

Upon differentiating and substituting, we obtain

$$\vec{\mathbf{F}}_B = -\frac{3\mu_z \mu_0 I R^2 z}{2(R^2 + z^2)^{5/2}} \hat{\mathbf{k}}$$

Thus, the dipole is attracted toward the current-carrying ring. On the other hand, if the direction of the dipole is reversed, $\vec{\boldsymbol{\mu}} = -\mu_z \hat{\mathbf{k}}$, the resulting force will be repulsive.

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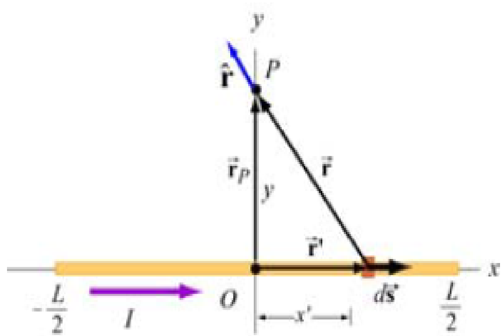
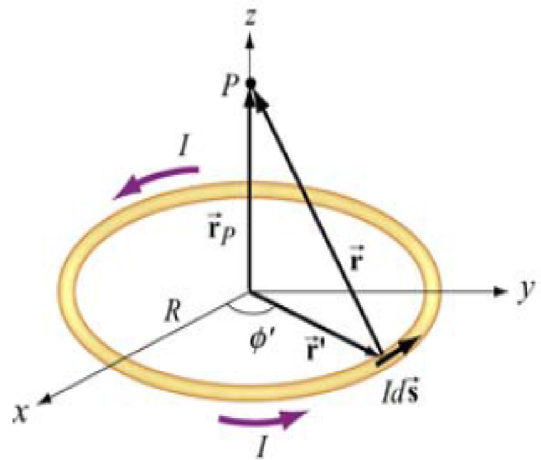
Summary of the two examples

Current distribution	Finite wire of length L	Circular loop of radius R
Figure		
(1) Source point	' $\vec{\mathbf{r}} = x'\hat{\mathbf{i}}$ $d\vec{\mathbf{s}} = (d\vec{\mathbf{r}}'/dx')dx' = dx'\hat{\mathbf{i}}$	$\vec{\mathbf{r}}' = R(\cos\phi'\hat{\mathbf{i}} + \sin\phi'\hat{\mathbf{j}})$ $d\vec{\mathbf{s}} = (d\vec{\mathbf{r}}'/d\phi')d\phi' = Rd\phi'(-\sin\phi'\hat{\mathbf{i}} + \cos\phi'\hat{\mathbf{j}})$
(2) Field point P	$\vec{\mathbf{r}}_P = y\hat{\mathbf{j}}$	$\vec{\mathbf{r}}_P = z\hat{\mathbf{k}}$

(3) Relative position vector $\vec{\mathbf{r}} = \vec{\mathbf{r}}_p - \vec{\mathbf{r}}\,'$	$\vec{\mathbf{r}} = y\hat{\mathbf{j}} - x'\hat{\mathbf{i}}$ $r = \lvert\vec{\mathbf{r}}\rvert = \sqrt{x'^2 + y^2}$ $\hat{\mathbf{r}} = \dfrac{y\hat{\mathbf{j}} - x'\hat{\mathbf{i}}}{\sqrt{x'^2 + y^2}}$	$\vec{\mathbf{r}} = -R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ $r = \lvert\vec{\mathbf{r}}\rvert = \sqrt{R^2 + z^2}$ $\hat{\mathbf{r}} = \dfrac{-R\cos\phi'\hat{\mathbf{i}} - R\sin\phi'\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{\sqrt{R^2 + z^2}}$
(4) The cross product $d\vec{\mathbf{s}} \times \hat{\mathbf{r}}$	$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = \dfrac{y\,dx'\hat{\mathbf{k}}}{\sqrt{y^2 + x'^2}}$	$d\vec{\mathbf{s}} \times \hat{\mathbf{r}} = \dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}})}{\sqrt{R^2 + z^2}}$
(5) Rewrite $d\vec{\mathbf{B}}$	$d\vec{\mathbf{B}} = \dfrac{\mu_0 I}{4\pi}\dfrac{y\,dx'\hat{\mathbf{k}}}{(y^2 + x'^2)^{3/2}}$	$d\vec{\mathbf{B}} = \dfrac{\mu_0 I}{4\pi}\dfrac{R\,d\phi'(z\cos\phi'\hat{\mathbf{i}} + z\sin\phi'\hat{\mathbf{j}} + R\hat{\mathbf{k}})}{(R^2 + z^2)^{3/2}}$

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(6) Integrate to get $\vec{\mathbf{B}}$	$B_x = 0$ $B_y = 0$ $B_z = \dfrac{\mu_0 I y}{4\pi}\displaystyle\int_{-L/2}^{L/2}\dfrac{dx'}{(y^2+x'^2)^{3/2}}$ $= \dfrac{\mu_0 I}{4\pi}\dfrac{L}{y\sqrt{y^2+(L/2)^2}}$	$B_x = \dfrac{\mu_0 I R z}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}\cos\phi' d\phi' = 0$ $B_y = \dfrac{\mu_0 I R z}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi}\sin\phi' d\phi' = 0$ $B_z = \dfrac{\mu_0 I R^2}{4\pi(R^2+z^2)^{3/2}}\displaystyle\int_0^{2\pi} d\phi' = \dfrac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}$

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