TABLE 1–5 Typical Order of Magnitude for Commonly Encountered Quantities

Quantity	U.S. Customary unit	SI unit
Dimensions of a wood standard 2 × 4	1.50 in × 3.50 in	38 mm × 89 mm
Moment of inertia of a 2 × 4 (3.50-in side vertical)	5.36 in^4	2.23×10^6 mm^4, or 2.23×10^{-6} m^4
Section modulus of a 2 × 4 (3.50-in side vertical)	3.06 in^3	5.02×10^4 mm^3, or 5.02×10^{-5} m^3
Force required to lift 1.0 gal of gasoline	6.01 lb	26.7 N
Density of water	1.94 $slugs/ft^3$	1000 kg/m^3, or 1.0 Mg/m^3
Compressed air pressure in a factory	100 psi	690 kPa
Yield point of SAE 1040 hot-rolled steel	42 000 psi, or 42 ksi	290 MPa
Modulus of elasticity of steel	30 000 000 psi, or 30×10^6 psi	207 GPa

Example Problem 1–1 Express the diameter of a shaft in millimeters if it is measured to be 2.755 in.

Solution Table A16 gives the conversion factor for length to be 1.00 in = 25.4 mm. Then

$$\text{Diameter} = 2.755\ \text{in}\ \frac{25.4\ \text{mm}}{1.00\ \text{in}} = 69.98\ \text{mm}$$

Example Problem 1–2 An electric motor is rotating at 1750 revolutions per minute (rpm). Express the speed in radians per second (rad/s).

Solution A series of conversions is required.

$$\text{Rotational speed} = \frac{1750\ \text{rev}}{\text{min}}\ \frac{2\pi\ \text{rad}}{\text{rev}}\ \frac{1\ \text{min}}{60\ \text{s}} = 183.3\ \text{rad/s}$$

1–10 DISTINCTION AMONG WEIGHT, FORCE, AND MASS

Distinction must be made among the terms *force*, *mass*, and *weight*. *Mass* is the quantity of matter in a body. A *force* is a push or pull applied to a body that results in a change in the body's motion or in some deformation of the body. Clearly these are two different physical phenomena, but the distinction is not always understood. The units for force and mass used in this text are listed in Table 1–2.

The term *weight*, as used in this book, refers to the amount of *force* required to support a body against the influence of gravity. Thus, in response to "What is the weight of 75 kg of steel?" we would use the relationship between force and mass from physics:

➩ Weight/Mass Relationship

$$F = ma \qquad \text{or} \qquad w = mg$$

where F = force
m = mass
a = acceleration
w = weight
g = acceleration due to gravity

We will use

$$g = 32.2\ \text{ft/s}^2 \qquad \text{or} \qquad g = 9.81\ \text{m/s}^2$$

Then, to compute the weight,

$$w = mg = 75\ \text{kg}\ (9.81\ \text{m/s}^2)$$
$$w = 736\ \text{kg}\cdot\text{m/s}^2 = 736\ \text{N}$$

Remember that, as shown in Table 1–2, the newton (N) is equivalent to 1.0 $kg \cdot m/s^2$. In fact, the newton is defined as the force required to give a mass of 1.0 kg an acceleration of 1.0 m/s^2. In our example, then, we would say that the 75-kg mass of steel has a weight of 736 N.

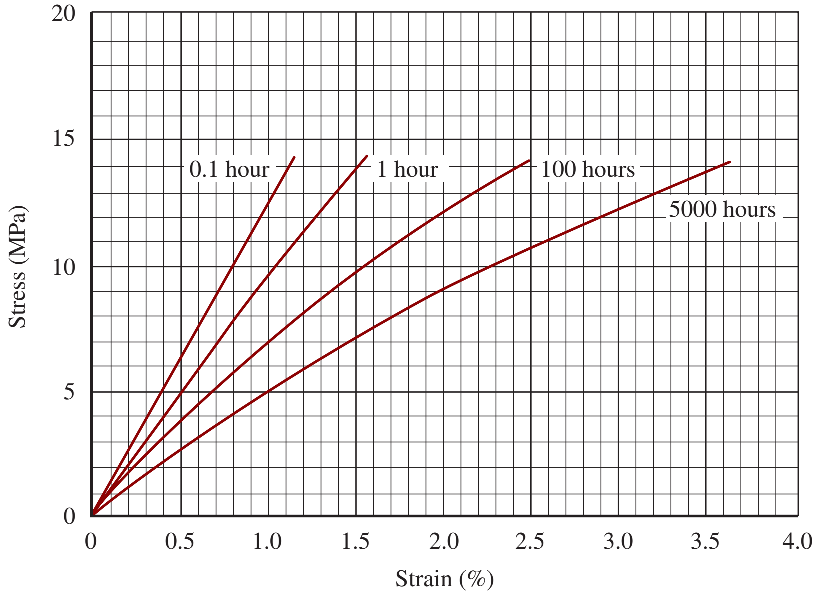


FIGURE 2–9 Example of stress versus strain as a function of time for nylon 66 plastic at 23°C (73°F)

dead weight, while the specimen is heated and maintained at a uniform temperature. Data for strain versus time are taken at least into the secondary creep stage and possibly all the way to fracture to determine the creep rupture strain. Testing over a range of temperatures gives a family of curves that are useful for design.

Creep can occur for many plastics even at or near room temperature. Figure 2–9 shows one way that creep data are displayed for plastic materials. (See References 14.) It is a graph of applied stress versus strain in the member with data shown for a specific temperature of the specimen. The curves show the amount of strain that would be developed within the specified times at increasing stress levels. For example, if this material were subjected to a constant stress of 5.0 MPa for 5000 hours, the total strain would be 1.0%. That is, the specimen would elongate by an amount 0.01 times the original length. If the stress were 10.0 MPa for 5000 hours, the total strain would be approximately 2.25%. The designer must take this creep strain into account to ensure that the product performs satisfactorily over time.

Relaxation

A phenomenon related to creep occurs when a member under stress is captured under load, giving it a certain fixed length and a fixed strain. Over time, the stress in the member would decrease, exhibiting a behavior called *relaxation*. This is important in such applications

Example Problem 2–1 A solid circular bar has a diameter of 5.0 mm and a length of 250 mm. It is made from nylon 66 plastic (30% Glass, 50% R.H.) and subjected to a steady tensile load of 240 N. Compute the elongation of the bar immediately after the load is applied and after 5000 hours (approximately seven months). See Appendix 13 and See Figure 2–9 for properties of the nylon.

Solution The stress and deflection immediately after loading will first be computed using fundamental equations of strength of materials:

$$\sigma = F/A \text{ and } \delta = FL/EA$$

See Chapter 3 for a review of strength of materials.
Then creep data from Figure 2–9 will be applied to determine the elongation after 5000 hours.

Results *Stress:*
The cross-sectional area of the bar is

$$A = \pi D^2/4 = \pi(5.0\text{ mm})^2/4 = 19.63\text{ mm}^2$$

$$\sigma = \frac{F}{A} = \frac{240\text{ N}}{19.63\text{ mm}^2} = 12.2\text{ N/mm}^2 = 12.2\text{ MPa}$$

Appendix 13 lists the tensile strength for nylon 66 to be 102 MPa. Therefore, the rod is safe from fracture.

Elongation:
The tensile modulus of elasticity for nylon 66 is found from Appendix 13 to be $E = 5500$ MPa. Then the initial elongation is,

$$\delta = \frac{FL}{EA} = \frac{(240\text{ N})(250\text{ mm})}{(5500\text{ N/mm}^2)(19.63\text{ mm}^2)} = 0.556\text{ mm}$$

Creep:
Referring to Figure 2–9 we find that when a tensile stress of 12.2 MPa is applied to the nylon 66 plastic for 5000 hours, a total strain of approximately 2.95% occurs. This can be expressed as

$$\epsilon = 2.95\% = 0.0295\text{ mm/mm} = \delta/L$$

Then,

$$\delta = \epsilon L = (0.0295\text{ mm/mm})(250\text{ mm}) = 7.375\text{ mm}$$

Comment This is approximately seven times as much deformation as originally experienced when the load was applied. So designing with the reported value of modulus of elasticity is not appropriate when stresses are applied continuously for a long time. We can now compute an apparent modulus of elasticity, E_{app}, for this material at the 5000 hours service life.

$$E_{app} = \sigma/\epsilon = (12.2\text{ MPa})/(0.0295\text{ mm/mm}) = 414\text{ MPa}$$

as clamped joints, press-fit parts, and springs installed with a fixed deflection. Figure 2–10 shows the comparison between creep and relaxation. For stresses below approximately 1/3 of the ultimate tensile strength of the material at any temperature, the apparent modulus in either creep or relaxation at any time of loading may be considered similar for engineering purposes. Furthermore, values for apparent modulus are the same for tension, compression, or flexure. (See Reference 14.) Analysis of relaxation is complicated by the fact that as the stress decreases, the rate of creep also decreases. Additional material data beyond that typically reported would be required to accurately predict the amount of relaxation at any given time. Testing under realistic conditions is recommended.

Physical Properties

Here we will discuss density, coefficient of thermal expansion, thermal conductivity, and electrical resistivity.

Density. *Density* is defined as the mass per unit volume of a material. Its usual units are kg/m^3 in the SI and lb/in^3 in the U.S. Customary Unit System, where the pound unit is taken to be pounds-mass. The Greek letter rho (ρ) is the symbol for density.

In some applications, the term *specific weight* or *weight density* is used to indicate the weight per unit volume of a material. Typical units are N/m^3 in the SI and lb/in^3 in the U.S. Customary Unit System, where the pound is taken to be pounds-force. The Greek letter gamma (γ) is the symbol for specific weight.

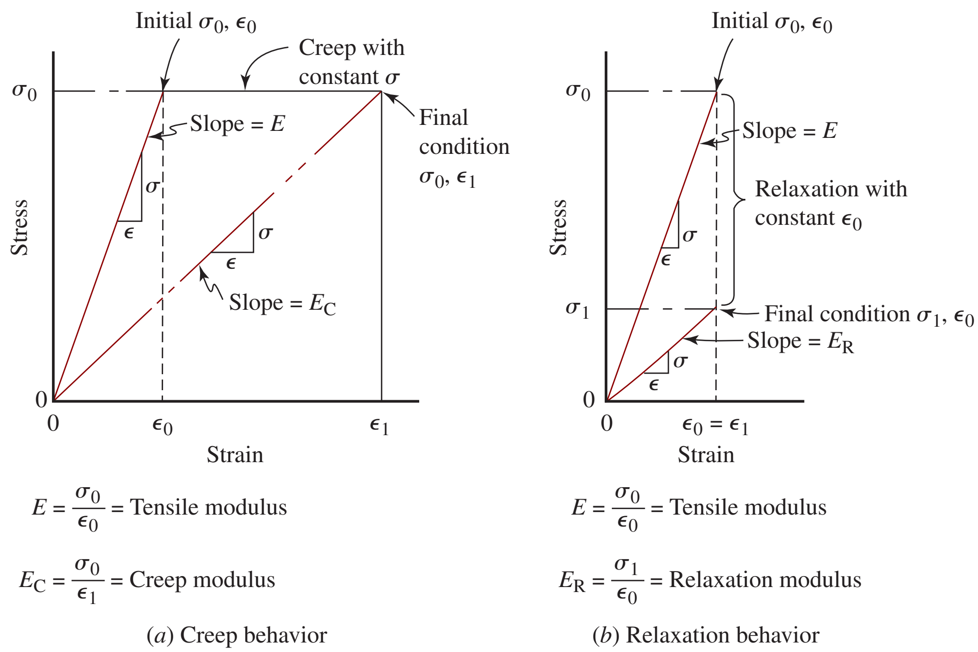


FIGURE 2–10 Comparison of creep and relaxation

TABLE 2–19 Example Properties of Matrix and Reinforcement Materials

	Tensile strength		Tensile modulus		Specific weight	
	ksi	MPa	10^6 psi	GPa	lb/in^3	kN/m^3
Matrix materials:						
Polyester	10	69	0.40	2.76	0.047	12.7
Epoxy	18	124	0.56	3.86	0.047	12.7
Aluminum	45	310	10.0	69	0.100	27.1
Titanium	170	1170	16.5	114	0.160	43.4
Reinforcement materials:						
S-glass	600	4140	12.5	86.2	0.09	24.4
Carbon-PAN	470	3240	33.5	231	0.064	17.4
Carbon-PAN (high-strength)	820	5650	40	276	0.065	17.7
Carbon (high-modulus)	325	2200	100	690	0.078	21.2
Aramid	500	3450	19.0	131	0.052	14.1

Example Problem 2–2 Compute the expected properties of ultimate tensile strength, modulus of elasticity, and specific weight of a composite made from unidirectional strands of carbon-PAN fibers in an epoxy matrix. The volume fraction of fibers is 30%. Use data from Table 2–19.

Solution

Objective: Compute the expected values of s_{uc}, E_c, and γ_c for the composite.

Given: Matrix-epoxy: $s_{um} = 18$ ksi; $E_m = 0.56 \times 10^6$ psi; $\gamma_m = 0.047$ lb/in^3.

Fiber-carbon-PAN: $s_{uf} = 470$ ksi; $E_f = 33.5 \times 10^6$ psi; $\gamma_f = 0.064$ lb/in^3.

Volume fraction of fiber: $V_f = 0.30$, and $V_m = 1.0 - 0.30 = 0.70$.

Analysis and Results: The ultimate tensile strength, s_{uc}, is computed from Equation (2–10):

$$s_{uc} = s_{uf}V_f + \sigma'_m V_m$$

To find σ'_m, we first find the strain at which the fibers would fail at s_{uf}. Assume that the fibers are linearly elastic to failure. Then

$$\epsilon_f = s_{uf}/E_f = (470 \times 10^3 \text{ psi})/(33.5 \times 10^6 \text{ psi}) = 0.014$$

At this same strain, the stress in the matrix is

$$\sigma'_m = E_m\epsilon = (0.56 \times 10^6 \text{ psi})(0.014) = 7840 \text{ psi}$$

Then, in Equation (2–10),

$$s_{uc} = (470\,000 \text{ psi})(0.30) + (7840 \text{ psi})(0.70) = 146\,500 \text{ psi}$$

The modulus of elasticity computed from Equation (2–12):

$$E_c = E_f V_f + E_m V_m = (33.5 \times 10^6)(0.30) + (0.56 \times 10^6)(0.70)$$
$$E_c = 10.4 \times 10^6 \text{ psi}$$

The specific weight is computed from Equation (2–14):

$$\gamma_c = \gamma_f V_f + \gamma_m V_m = (0.064)(0.30) + (0.047)(0.70) = 0.052 \text{ lb/in}^3$$

Summary of Results:

$$s_{uc} = 146\,500 \text{ psi}$$
$$E_c = 10.4 \times 10^6 \text{ psi}$$
$$\gamma_c = 0.052 \text{ lb/in}^3$$

Comment: Note that the resulting properties for the composite are intermediate between those for the fibers and the matrix.

Sign Convention for Shear Stresses

This book adopts the following convention:

> ***Positive shear stresses tend to rotate the element in a clockwise direction.***
>
> ***Negative shear stresses tend to rotate the element in a counterclockwise direction.***

A double subscript notation is used to denote shear stresses in a plane. For example, in Figure 3–5(c), drawn for the x–y plane, the pair of shear stresses, τ_{xy}, indicates a shear stress acting on the element face that is perpendicular to the x-axis and parallel to the y-axis. Then τ_{yx} acts on the face that is perpendicular to the y-axis and parallel to the x-axis. In this example, τ_{xy} is positive and τ_{yx} is negative.

3–4 NORMAL STRESSES DUE TO DIRECT AXIAL LOAD

Stress can be defined as the internal resistance offered by a unit area of a material to an externally applied load. *Normal stresses* (σ) are either *tensile* (positive) or *compressive* (negative).

For a load-carrying member in which the external load is uniformly distributed across the cross-sectional area of the member, the magnitude of the stress can be calculated from the direct stress formula:

⇨ Direct Tensile or Compressive Stress

$$\sigma = \text{force/area} = F/A \qquad \textbf{(3–1)}$$

The units for stress are always *force per unit area*, as is evident from Equation (3–1). Common units in the U.S. Customary system and the SI metric system follow.

U.S. Customary Units	*SI Metric Units*
$\text{lb/in}^2 = \text{psi}$	$\text{N/m}^2 = \text{pascal} = \text{Pa}$
$\text{kips/in}^2 = \text{ksi}$	$\text{N/mm}^2 = \text{megapascal}$
Note: 1.0 kip = 1000 lb	$= 10^6\ \text{Pa} = \text{MPa}$
1.0 ksi = 1000 psi	

The conditions on the use of Equation (3–1) are as follows:

1. The load-carrying member must be straight.
2. The line of action of the load must pass through the centroid of the cross section of the member.
3. The member must be of uniform cross section near where the stress is being computed.
4. The material must be homogeneous and isotropic.
5. In the case of compression members, the member must be short to prevent buckling. The conditions under which buckling is expected are discussed in Chapter 6.

Example Problem 3–1 A tensile force of 9500 N is applied to a 12-mm-diameter round bar, as shown in Figure 3–6. Compute the direct tensile stress in the bar.

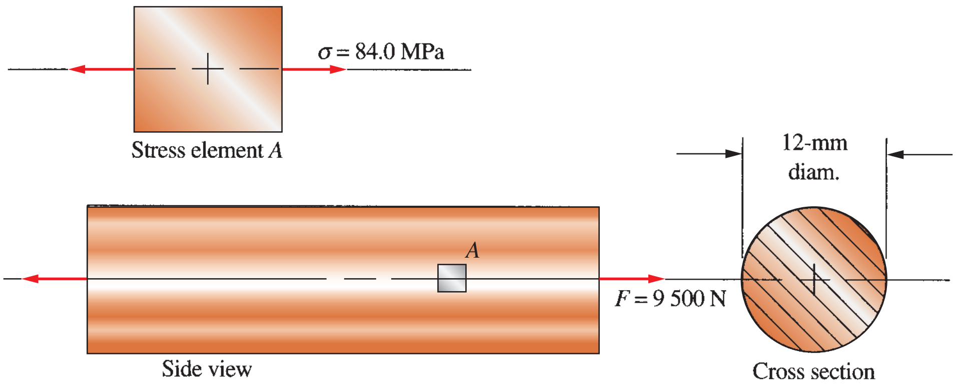


FIGURE 3–6 Tensile stress in a round bar

Solution

Objective: Compute the tensile stress in the round bar.

Given: Force $= F = 9500$ N; diameter $= D = 12$ mm.

Analysis: Use the direct tensile stress formula, Equation (3–1): $\sigma = F/A$. Compute the cross-sectional area from $A = \pi D^2/4$.

Results:

$$A = \pi D^2/4 = \pi(12\ \text{mm})^2/4 = 113\ \text{mm}^2$$

$$\sigma = F/A = (9500\ \text{N})/(113\ \text{mm}^2) = 84.0\ \text{N/mm}^2 = 84.0\ \text{MPa}$$

Comment: The results are shown on stress element A in Figure 3–6, which can be taken to be anywhere within the bar because, ideally, the stress is uniform on any cross section. The cube form of the element is as shown in Figure 3–5 (a).

Example Problem 3–2 For the round bar subjected to the tensile load shown in Figure 3–6, compute the total deformation if the original length of the bar is 3600 mm. The bar is made from a steel having a modulus of elasticity of 207 GPa.

Solution

Objective: Compute the deformation of the bar.

Given: Force = F = 9500 N; diameter = D = 12 mm.
Length = L = 3600 mm; E = 207 GPa

Analysis: From Example Problem 3–1, we found that σ = 84.0 MPa. Use Equation (3–3).

Results:

$$\delta = \frac{\sigma L}{E} = \frac{(84.0 \times 10^6\ \mathrm{N/m^2})(3600\ \mathrm{mm})}{(207 \times 10^9\ \mathrm{N/m^2})} = 1.46\ \mathrm{mm}$$

3–5 DEFORMATION UNDER DIRECT AXIAL LOAD

The following formula computes the stretch due to a direct axial tensile load or the shortening due to a direct axial compressive load:

$$\delta = FL/EA \tag{3–2}$$

➪ Deformation Due to Direct Axial Load

where δ = total deformation of the member carrying the axial load
F = direct axial load
L = original total length of the member
E = modulus of elasticity of the material
A = cross-sectional area of the member

Noting that $\sigma = F/A$, we can also compute the deformation from

$$\delta = \sigma L/E \tag{3–3}$$

3–6 SHEAR STRESS DUE TO DIRECT SHEAR LOAD

Direct shear stress occurs when the applied force tends to cut through the member as scissors or shears do or when a punch and a die are used to punch a slug of material from a sheet. Another important example of direct shear in machine design is the tendency for a key to be sheared off at the section between the shaft and the hub of a machine element when transmitting torque. Figure 3–7 shows the action.

The method of computing direct shear stress is similar to that used for computing direct tensile stress because the applied force is assumed to be uniformly distributed across the cross section of the part that is resisting the force. But the kind of stress is *shear stress* rather than *normal stress*. The symbol used for shear stress is the Greek letter tau (τ). The formula for direct shear stress can thus be written

➪ Direct Shear Stress

$$\tau = \text{shearing force/area in shear} = F/A_s \tag{3–4}$$

This stress is more properly called the *average shearing* stress, but we will make the simplifying assumption that the stress is uniformly distributed across the shear area.

3–7 TORSIONAL LOAD—TORQUE, ROTATIONAL SPEED, AND POWER

The relationship among the power (P), the rotational speed (n), and the torque (T) in a shaft is described by the equation

➪ Power–Torque–Speed Relationship

$$T = P/n \tag{3–5}$$

In SI units, power is expressed in the unit of *watt* (W) or its equivalent, *newton meter per second* (N·m/s), and the rotational speed is in *radians per second* (rad/s).

In the U.S. Customary Unit System, power is typically expressed as *horsepower*, equal to 550 ft·lb/s. The typical unit for rotational speed is rpm, or revolutions per minute. But the most convenient unit for torque is the pound-inch (lb·in). Considering all of these quantities and making the necessary conversions of units, we use the following formula to compute the torque (in lb·in) in a shaft carrying a certain power P (in hp) while rotating at a speed of n rpm.

➪ *P–T–n* Relationship for U.S. Customary Units

$$T = 63\,000\ P/n \tag{3–6}$$

The resulting torque will be in lb·in. You should verify the value of the constant, 63 000.

Example Problem 3–3 Figure 3–7 shows a shaft carrying two sprockets for synchronous belt drives that are keyed to the shaft. Figure 3–7 (b) shows that a force F is transmitted from the shaft to the hub of the sprocket through a square key. The shaft has a diameter of 2.25 in and transmits a torque of 14 063 lb·in. The key has a square cross section, 0.50 in on a side, and a length of 1.75 in. Compute the force on the key and the shear stress caused by this force.

Solution

Objective: Compute the force on the key and the shear stress.

Given: Layout of shaft, key, and hub shown in Figure 3–7.
Torque $= T = 14\,063$ lb·in; key dimensions $= 0.5$ in $\times$ 0.5 in $\times$ 1.75 in.
Shaft diameter $= D = 2.25$ in; radius $= R = D/2 = 1.125$ in.

Analysis: Torque $T =$ force $F \times$ radius R. Then $F = T/R$.
Use equation (3–4) to compute shearing stress: $\tau = F/A_s$.
Shear area is the cross section of the key at the interface between the shaft and the hub: $A_s = bL$.

Results: $F = T/R = (14\,063\ \text{lb·in})/(1.125\ \text{in}) = 12\,500$ lb
$A_s = bL = (0.50\ \text{in})(1.75\ \text{in}) = 0.875\ \text{in}^2$
$\tau = F/A = (12\,500\ \text{lb})/(0.875\ \text{in}^2) = 14\,300\ \text{lb/in}^2$

Comment: This level of shearing stress will be uniform on all parts of the cross section of the key.

Example Problem 3–4 Compute the torque on a shaft transmitting 750 W of power while rotating at 183 rad/s. (*Note:* This is equivalent to the output of a 1.0-hp, 4-pole electric motor, operating at its rated speed of 1750 rpm. See Chapter 21.)

Solution

Objective: Compute the torque T on the shaft.

Given: Power $= P = 750$ W $= 750$ N·m/s.
Rotational speed $= n = 183$ rad/s.

Analysis: Use Equation (3–5).

Results: $T = P/n = (750\ \text{N·m/s})/(183\ \text{rad/s})$
$T = 4.10$ N·m/rad $= 4.10$ N·m

Comments: In such calculations, the unit of N·m/rad is dimensionally correct, and some advocate its use. Most, however, consider the radian to be dimensionless, and thus torque is expressed in N·m or other familiar units of force times distance.

Example Problem 3–5 Compute the torque on a shaft transmitting 1.0 hp while rotating at 1750 rpm. Note that these conditions are approximately the same as those for which the torque was computed in Example Problem 3–4 using SI units.

Solution

Objective: Compute the torque on the shaft.

Given: $P = 1.0$ hp; $n = 1750$ rpm.

Analysis: Use Equation (3–6).

Results: $T = 63\,000\ P/n = [63\,000(1.0)]/1750 = 36.0$ lb·in

3–8 SHEAR STRESS DUE TO TORSIONAL LOAD

When a *torque*, or twisting moment, is applied to a member, it tends to deform by twisting, causing a rotation of one part of the member relative to another. Such twisting causes a shear stress in the member. For a small element of the member, the nature of the stress is the same as that experienced under direct shear stress. However, in *torsional shear*, the distribution of stress is not uniform across the cross section.

The most frequent case of torsional shear in machine design is that of a round circular shaft transmitting power. Chapter 12 covers shaft design.

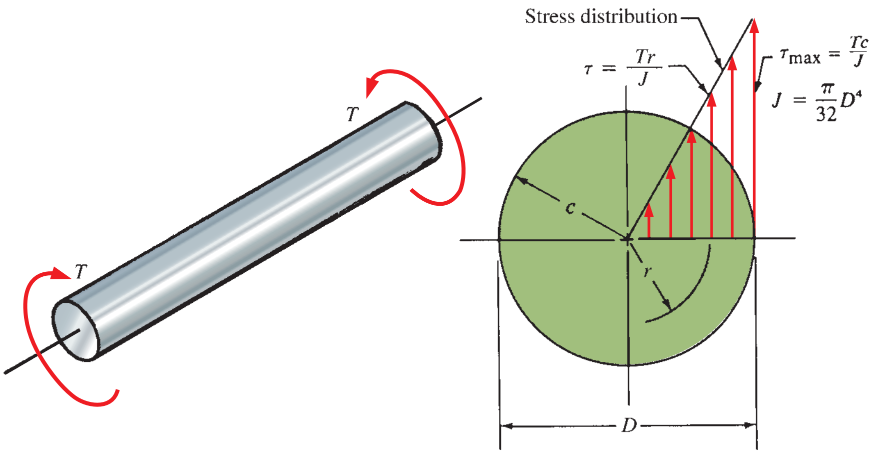


FIGURE 3–8 Stress distribution in a solid shaft

Torsional Shear Stress Formula

When subjected to a torque, the outer surface of a solid round shaft experiences the greatest shearing strain and therefore the largest torsional shear stress. See Figure 3–8. The value of the maximum torsional shear stress is found from

➪ Maximum Torsional Shear Stress in a Circular Shaft

$$\tau_{max} = Tc/J \qquad (3\text{–}7)$$

where c = radius of the shaft to its outside surface
J = polar moment of inertia

See Appendix 1 for formulas for J.

If it is desired to compute the torsional shear stress at some point inside the shaft, the more general formula is used:

➪ General Formula for Torsional Shear Stress

$$\tau = Tr/J \qquad (3\text{–}8)$$

where r = radial distance from the center of the shaft to the point of interest

Figure 3–8 shows graphically that this equation is based on the linear variation of the torsional shear stress from zero at the center of the shaft to the maximum value at the outer surface.

Equations (3–7) and (3–8) apply also to hollow shafts (Figure 3–9 shows the distribution of shear stress). Again note that the maximum shear stress occurs at the outer surface. Also note that the entire cross section carries a relatively high stress level. As a result, the hollow shaft is more efficient. Notice that the material near the center of the solid shaft is not highly stressed.

For design, it is convenient to define the *polar section modulus, Z_p*:

➪ Polar Section Modulus

$$Z_p = J/c \qquad (3\text{–}9)$$

Then the equation for the maximum torsional shear stress is

$$\tau_{max} = T/Z_p \qquad (3\text{–}10)$$

Formulas for the polar section modulus are also given in Appendix 1. This form of the torsional shear stress equation is useful for design problems because the polar section modulus is the only term related to the geometry of the cross section.

Example Problem 3–6 Compute the maximum torsional shear stress in a shaft having a diameter of 10 mm when it carries a torque of 4.10 N·m.

Solution

Objective: Compute the torsional shear stress in the shaft.

Given: Torque = T = 4.10 N·m; shaft diameter = D = 10 mm.
c = radius of the shaft = $D/2$ = 5.0 mm.

Analysiçs: Use Equation (3–7) to compute the torsional shear stress: $\tau_{max} = Tc/J$. J is the polar moment of inertia for the shaft: $J = \pi D^4/32$ (see Appendix 1).

Results: $J = \pi D^4/32 = [(\pi)(10\text{ mm})^4]/32 = 982\text{ mm}^4$

$$\tau_{max} = \frac{(4.10\text{ N}\cdot\text{m})(5.0\text{ mm})}{982\text{ mm}^4}\frac{10^3\text{ mm}}{\text{m}} = 20.9\text{ N/mm}^2 = 20.9\text{ MPa}$$

Comment: The maximum torsional shear stress occurs at the outside surface of the shaft around its entire circumference.

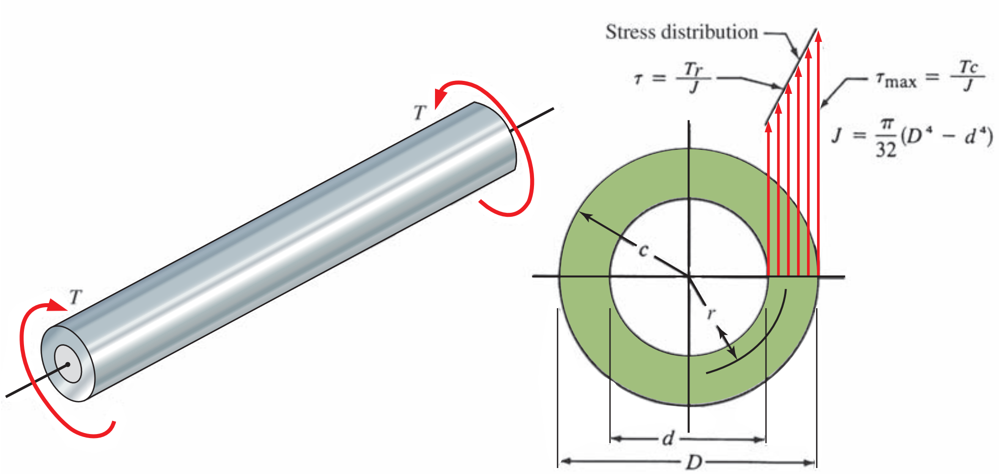


FIGURE 3–9 Stress distribution in a hollow shaft

3–9 TORSIONAL DEFORMATION

When a shaft is subjected to a torque, it undergoes a twisting in which one cross section is rotated relative to other cross sections in the shaft. The angle of twist is computed from

Torsional Deformation

$$\theta = TL/GJ \qquad (3\text{–}11)$$

where θ = angle of twist (radians)
L = length of the shaft over which the angle of twist is being computed
G = modulus of elasticity of the shaft material in *shear*

Example Problem 3–7 Compute the angle of twist of a 10-mm-diameter shaft carrying 4.10 N·m of torque if it is 250 mm long and made of steel with G = 80 GPa. Express the result in both radians and degrees.

Solution

Objective: Compute the angle of twist in the shaft.

Given: Torque = T = 4.10 N·m; length = L = 250 mm.
Shaft diameter = D = 10 mm; G = 80 GPa.

Analysis: Use Equation (3–11). For consistency, let $T = 4.10 \times 10^3$ N·mm and $G = 80 \times 10^3$ N/mm^2. From Example Problem 3–6, $J = 982$ mm^4.

Results:

$$\theta = \frac{TL}{GJ} = \frac{(4.10 \times 10^3 \text{ N·mm})(250 \text{ mm})}{(80 \times 10^3 \text{ N/mm}^2)(982 \text{ mm}^4)} = 0.013 \text{ rad}$$

Using π rad = 180°,

$$\theta = (0.013 \text{ rad})(180°/\pi \text{ rad}) = 0.75°$$

Comment: Over the length of 250 mm, the shaft twists 0.75°.

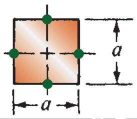
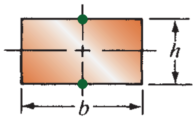
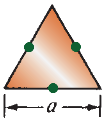
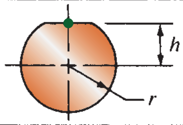
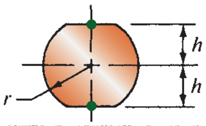
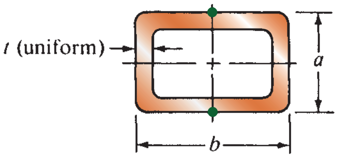
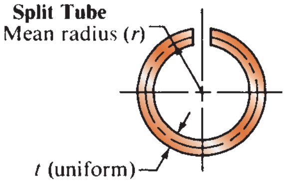
3–10 TORSION IN MEMBERS HAVING NON-CIRCULAR CROSS SECTIONS

The behavior of members having noncircular cross sections when subjected to torsion is radically different from that for members having circular cross sections. However, the factors of most use in machine design are the maximum stress and the total angle of twist for such members. The formulas for these factors can be expressed in similar forms to the formulas used for members of circular cross section (solid and hollow round shafts).

The following two formulas can be used:

Torsional Shear Stress

$$\tau_{max} = T/Q \qquad (3\text{–}12)$$

Cross-sectional shape	K = for use in $\theta = TL/GK$ Q = for use in $\tau = T/Q$	Colored dot (•) denotes location of τ_{max}
Square 	$K = 0.141a^4$ $Q = 0.208a^3$	τ_{max} at midpoint of each side
Rectangle 	$K = bh^3\left[\frac{1}{3} - 0.21\frac{h}{b}\left(1 - \frac{(h/b)^4}{12}\right)\right]$ $Q = \frac{bh^2}{[3 + 1.8(h/b)]}$	(Approximate; within ≈ 5%) τ_{max} at midpoint of long sides
Triangle (equilateral) 	$K = 0.0217a^4$ $Q = 0.050a^3$	
Shaft with One Flat 	$K = C_1 r^4$ $Q = C_2 r^3$	see table below
Shaft with Two Flats 	$K = C_3 r^4$ $Q = C_4 r^3$	see table below
Hollow Rectangle 	$K = \frac{2t(a-t)^2(b-t)^2}{(a+b-2t)}$ $Q = 2t(a-t)(b-t)$ Gives average stress; good approximation of maximum stress if t is small—thin-walled tube Inner corners should have generous fillets	
Split Tube 	$K = 2\pi r t^3/3$ $Q = \frac{4\pi^2 r^2 t^2}{(6\pi r + 1.8t)}$ t must be small—thin-walled tube	

Shaft with One Flat:

h/r	0	0.2	0.4	0.6	0.8	1.0
C_1	0.30	0.51	0.78	1.06	1.37	1.57
C_2	0.35	0.51	0.70	0.92	1.18	1.57

Shaft with Two Flats:

h/r	0.5	0.6	0.7	0.8	0.9	1.0
C_3	0.44	0.67	0.93	1.19	1.39	1.57
C_4	0.47	0.60	0.81	1.02	1.25	1.57

FIGURE 3–10 Methods for determining values for K and Q for several types of cross sections

Deflection for Noncircular Sections

$$\theta = TL/GK \tag{3–13}$$

Note that Equations (3–12) and (3–13) are similar to Equations (3–10) and (3–11), with the substitution of Q for Z_p and K for J. Refer Figure 3–10 for the methods of determining the values for K and Q for several types of cross sections useful in machine design. These values are appropriate only if the ends of the member are free to deform. If either end is fixed, as by welding to a solid structure, the resulting stress and angular twist are quite different. (See References 1–3, 6, and 7.)

Example Problem 3–8 A 2.50-in-diameter shaft carrying a chain sprocket has one end milled in the form of a square to permit the use of a hand crank. The square is 1.75 in on a side. Compute the maximum shear stress on the square part of the shaft when a torque of 15 000 lb·in is applied.

Also, if the length of the square part is 8.00 in, compute the angle of twist over this part. The shaft material is steel with $G = 11.5 \times 10^6$ psi.

Solution Objective: Compute the maximum shear stress and the angle of twist in the shaft.

Given: Torque = T = 15 000 lb·in; length = L = 8.00 in.
The shaft is square; thus, a = 1.75 in.
$G = 11.5 \times 10^6$ psi.

Analysis — Figure 3–10 shows the methods for calculating the values for Q and K for use in Equations (3–12) and (3–13).

Results
$Q = 0.208a^3 = (0.208)(1.75\text{ in})^3 = 1.115\text{ in}^3$
$K = 0.141a^4 = (0.141)(1.75\text{ in})^4 = 1.322\text{ in}^4$
Now the stress and the deflection can be computed.

$$\tau_{max} = \frac{T}{Q} = \frac{15\,000\text{ lb}\cdot\text{in}}{(1.115\text{ in}^3)} = 13\,460\text{ psi}$$

$$\theta = \frac{TL}{GK} = \frac{(15\,000\text{ lb}\cdot\text{in})(8.00\text{ in})}{(11.5\times 10^6\text{ lb/in}^2)(1.322\text{ in}^4)} = 0.0079\text{ rad}$$

Convert the angle of twist to degrees:

$$\theta = (0.0079\text{ rad})(180°/\pi\text{ rad}) = 0.452°$$

Comments — Over the length of 8.00 in, the square part of the shaft twists 0.452°. The maximum shear stress is 13 460 psi, and it occurs at the midpoint of each side as shown in Figure 3–10.

3–11 TORSION IN CLOSED, THIN-WALLED TUBES

A general approach for closed, thin-walled tubes of virtually any shape uses Equations (3–12) and (3–13) with special methods of evaluating K and Q. Figure 3–11 shows such a tube having a constant wall thickness. The values of K and Q are

$$K = 4A^2t/U \tag{3–14}$$

$$Q = 2tA \tag{3–15}$$

where A = area enclosed by the median boundary (indicated by the dashed line in Figure 3–11)
t = wall thickness (which must be uniform and thin)
U = length of the median boundary

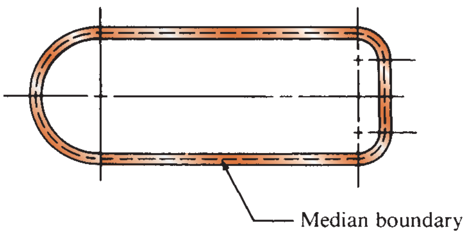


FIGURE 3–11 Closed, thin-walled tube with a constant wall thickness

The shear stress computed by this approach is the *average stress* in the tube wall. However, if the wall thickness t is small (a thin wall), the stress is nearly uniform throughout the wall, and this approach will yield a close approximation of the maximum stress. For the analysis of tubular sections having nonuniform wall thickness, see References 1–3, 6, and 7.

To design a member to resist torsion only, or torsion and bending combined, it is advisable to select hollow tubes, either round or rectangular, or some other closed shape. They possess good efficiency both in bending and in torsion.

3–12 TORSION IN OPEN, THIN-WALLED TUBES

The term *open tube* refers to a shape that appears to be tubular but is not completely closed. For example, some tubing is manufactured by starting with a thin, flat strip of steel that is roll-formed into the desired shape (circular, rectangular, square, and so on). Then the seam is welded along the entire length of the tube. It is interesting to compare the properties of the cross section of such a tube before and after it is welded. The following example problem illustrates the comparison for a particular size of circular tubing.

Example Problem 3–9 Figure 3–12 shows a tube before [Part (b)] and after [Part (a)] the seam is welded. Compare the stiffness and the strength of each shape.

Solution

Objective — Compare the torsional stiffness and the strength of the closed tube of Figure 3–12(a) with those of the open-seam (split) tube shown in Figure 3–12(b).

Given — The tube shapes are shown in Figure 3–12. Both have the same length, diameter, and wall thickness, and both are made from the same material.

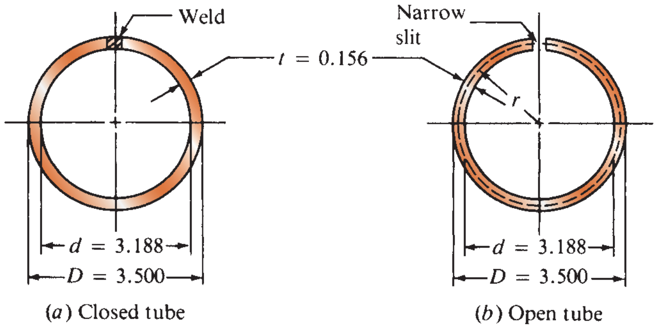


FIGURE 3–12 Comparison of closed and open tubes

Analysis Equation (3–13) gives the angle of twist for a noncircular member and shows that the angle is inversely proportional to the value of K. Similarly, Equation (3–11) shows that the angle of twist for a hollow circular tube is inversely proportional to the polar moment of inertia J. All other terms in the two equations are the same for each design. Therefore, the ratio of θ_{open} to θ_{closed} is equal to the ratio J/K. From Appendix 1, we find

$$J = \pi(D^4 - d^4)/32$$

From Figure 3–10, we find

$$K = 2\pi rt^3/3$$

Using similar logic, Equations (3–12) and (3–10) show that the maximum torsional shear stress is inversely proportional to Q and Z_p for the open and closed tubes, respectively. Then we can compare the strengths of the two forms by computing the ratio Z_p/Q. By Equation (3–9), we find that

$$Zp = J/c = J/(D/2)$$

The equation for Q for the split tube is listed in Figure 3–10.

Results We make the comparison of torsional stiffness by computing the ratio J/K. For the closed, hollow tube,

$$J = \pi(D^4 - d^4)/32$$
$$J = \pi(3.500^4 - 3.188^4)/32 = 4.592 \text{ in}^4$$

For the open tube before the slit is welded, from Figure 3–10,

$$K = 2\pi rt^3/3$$
$$K = [(2)(\pi)(1.672)(0.156)^3]/3 = 0.0133 \text{ in}^4$$
$$\text{Ratio} = J/K = 4.592/0.0133 = 345$$

Then we make the comparison of the strengths of the two forms by computing the ratio Z_p/Q.

The value of J has already been computed to be 4.592 in^4. Then

$$Z_p = J/c = J/(D/2) = (4.592 \text{ in}^4)/[(3.500 \text{ in})/2] = 2.624 \text{ in}^3$$

For the open tube,

$$Q = \frac{4\pi^2 r^2 t^2}{(6\pi r + 1.8t)} = \frac{4\pi^2(1.672 \text{ in})^2(0.156 \text{ in})^2}{[6\pi(1.672 \text{ in}) + 1.8(0.156 \text{ in})]} = 0.0845 \text{ in}^3$$

Then the strength comparison is

$$\text{Ratio} = Z_p/Q = 2.624/0.0845 = 31.1$$

Comments Thus, for a given applied torque, the slit tube would twist 345 times as much as the closed tube. The stress in the slit tube would be 31.1 times higher than in the closed tube. Also note that if the material for the tube is thin, it will likely buckle at a relatively low stress level, and the tube will collapse suddenly. This comparison shows the dramatic superiority of the closed form of a hollow section to an open form. A similar comparison could be made for shapes other than circular.

Example Problem 3–11 Compute the maximum shearing stress in the beam described in Example Problem 3–10 using the special shearing stress formula for a rectangular section.

Solution

Objective: Compute the maximum shearing stress τ in the beam in Figure 3–14.

Given: The data are the same as stated in Example Problem 3–10 and as shown in Figure 3–14.

Analysis: Use Equation (3–18) to compute $\tau = 3V/2A$. For the rectangle, $A = th$.

Results:

$$\tau_{max} = \frac{3V}{2A} = \frac{3(1000\ \text{lb})}{2[(2.0\ \text{in})(8.0\ \text{in})]} = 93.8\ \text{psi}$$

Comment: This result is the same as that obtained for Example Problem 3–10, as expected.

3–15 NORMAL STRESS DUE TO BENDING

A *beam* is a member that carries loads transverse to its axis. Such loads produce bending moments in the beam, which result in the development of bending stresses. Bending stresses are *normal stresses*, that is, either tensile or compressive. The maximum bending stress in a beam cross section will occur in the part farthest from the neutral axis of the section. At that point, the *flexure formula* gives the stress:

➭ Flexure Formula for Maximum Bending Stress

$$\sigma = Mc/I \qquad \textbf{(3–22)}$$

where M = magnitude of the bending moment at the section

I = moment of inertia of the cross section with respect to its neutral axis

c = distance from the neutral axis to the outermost fiber of the beam cross section

The magnitude of the bending stress varies linearly within the cross section from a value of zero at the neutral axis, to the maximum tensile stress on one side of the neutral axis, and to the maximum compressive stress on the other side. Figure 3–16 shows a typical stress distribution in a beam cross section. Note that the stress distribution is independent of the shape of the cross section.

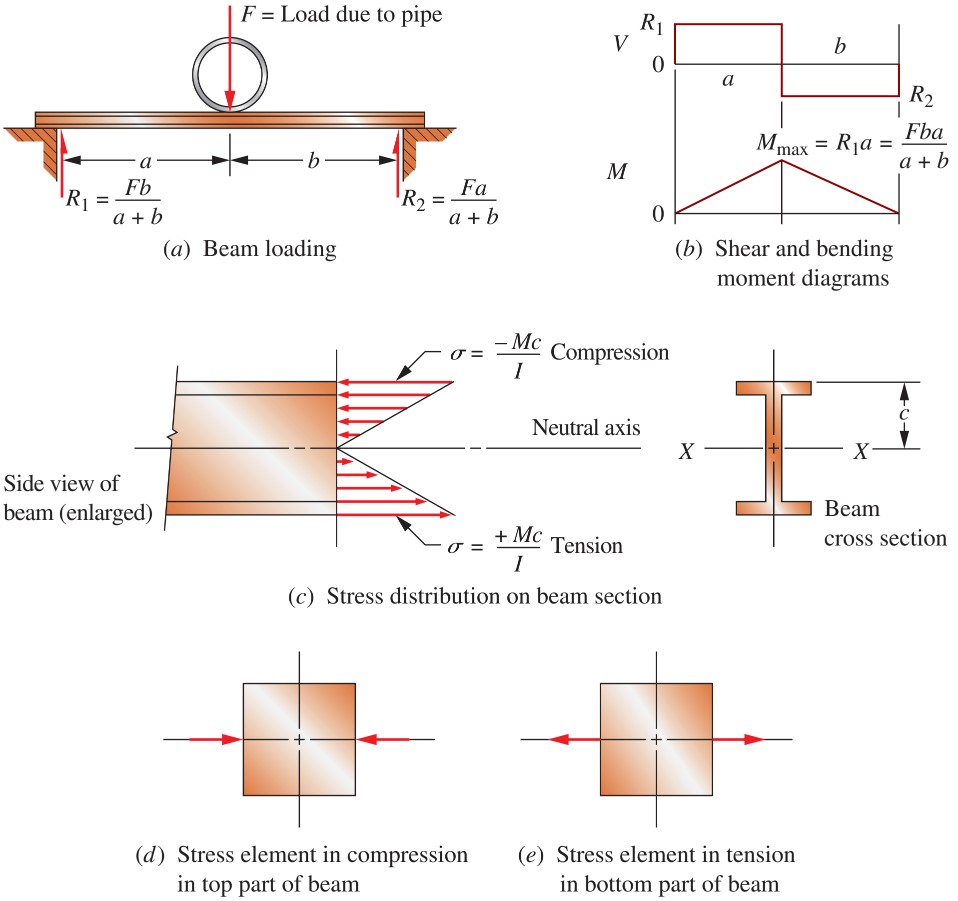


FIGURE 3–16 Typical bending stress distribution in a beam cross section

Note that *positive bending* occurs when the deflected shape of the beam is concave upward, resulting in compression on the upper part of the cross section and tension on the lower part. Conversely, *negative bending* causes the beam to be concave downward.

The flexure formula was developed subject to the following conditions:

1. The beam must be in pure bending. Shearing stresses must be zero or negligible. No axial loads are present.
2. The beam must not twist or be subjected to a torsional load.
3. The material of the beam must obey Hooke's law.
4. The modulus of elasticity of the material must be the same in both tension and compression.
5. The beam is initially straight and has a constant cross section.
6. Any plane cross section of the beam remains plane during bending.
7. No part of the beam shape fails because of local buckling or wrinkling.

If condition 1 is not strictly met, you can continue the analysis by using the method of combined stresses presented in Chapter 4. In most practical beams, which are long relative to their height, shear stresses are sufficiently small as to be negligible. Furthermore, the maximum bending stress occurs at the outermost fibers of the beam section, where the shear stress is in fact zero. A beam with varying cross section, which would violate condition 5, can be analyzed by the use of stress concentration factors discussed later in this chapter.

For design, it is convenient to define the term *section modulus*, *S*, as

$$S = I/c \tag{3–23}$$

The flexure formula then becomes

➪ Flexure Formula

$$\sigma = M/S \tag{3–24}$$

Since I and c are geometrical properties of the cross section of the beam, S is also. Then, in design, it is usual to define a design stress, σ_d, and, with the bending moment known, solve for S:

➪ Required Section Modulus

$$S = M/\sigma_d \tag{3–25}$$

This results in the required value of the section modulus. From this, the required dimensions of the beam cross section can be determined.

Example Problem 3–12

For the beam shown in Figure 3–16, the load F due to the pipe is 12 000 lb. The distances are $a = 4$ ft and $b = 6$ ft. Determine the required section modulus for the beam to limit the stress due to bending to 30 000 psi, the recommended design stress for a typical structural steel in static bending. Then specify the lightest suitable steel beam.

Solution

Objective: Compute the required section modulus S for the beam in Figure 3–16.

Given: The layout and the loading pattern are shown in Figure 3–16.

Lengths: Overall length $= L = 10$ ft; $a = 4$ ft; $b = 6$ ft.
Load $= F = 12\,000$ lb.
Design stress $= \sigma_d = 30\,000$ psi.

Analysis: Use Equation (3–25) to compute the required section modulus S. Compute the maximum bending moment that occurs at the point of application of the load using the formula shown in Part (b) of Figure 3–16.

Results:

$$M_{max} = R_1\,a = \frac{Fba}{a+b} = \frac{(12\,000\text{ lb})(6\text{ ft})(4\text{ ft})}{(6\text{ ft} + 4\text{ ft})} = 28\,800\text{ lb}\cdot\text{ft}$$

$$S = \frac{M}{\sigma_d} = \frac{28\,800\text{ lb}\cdot\text{ft}}{30\,000\text{ lb/in}^2}\,\frac{12\text{ in}}{\text{ft}} = 11.5\text{ in}^3$$

Comments: A steel beam section can now be selected from Tables A15–9 and A15–10 that has at least this value for the section modulus. The lightest section, typically preferred, is the W8×15 wide-flange shape with $S = 11.8\text{ in}^3$.

3–16 BEAMS WITH CONCENTRATED BENDING MOMENTS

Figures 3–16 and 3–17 show beams loaded only with concentrated forces or distributed loads. For such loading in any combination, the moment diagram is continuous. That is, there are no points of abrupt change in the value of the bending moment. Many machine elements such as cranks, levers, helical gears, and brackets carry loads whose line of action is offset from the centroidal axis of the beam in such a way that a concentrated moment is exerted on the beam.

Example Problem 3–13

The bell crank shown in Figure 3–18 is part of a linkage in which the 80-lb horizontal force is transferred to F_2 acting vertically. The crank can pivot about the pin at O. Draw a free-body diagram of the horizontal part of the crank from O to A. Then draw the shearing force and bending moment diagrams that are necessary to complete the design of the horizontal arm of the crank.

Solution

Objective: Draw the free-body diagram of the horizontal part of the crank in Figure 3–18. Draw the shearing force and bending moment diagrams for that part.

Given: The layout from Figure 3–18.

Analysis: Use the entire crank first as a free body to determine the downward force F_2 that reacts to the applied horizontal force F_1 of 80 lb by summing moments about the pin at O.

Then create the free-body diagram for the horizontal part by breaking it through the pin and replacing the removed part with the internal force and moment acting at the break.

Results: We can first find the value of F_2 by summing moments about the pin at O using the entire crank:

$$F_1 \cdot a = F_2 \cdot b$$
$$F_2 = F_1(a/b) = 80\ \text{lb}(1.50/2.00) = 60\ \text{lb}$$

Below the drawing of the complete crank, we have drawn a sketch of the horizontal part, isolating it from the vertical part. The internal force and moment at the cut section are shown. The externally applied downward force F_2 is reacted by the upward reaction at the pin. Also, because F_2 causes a moment with respect to the section at the pin, an internal reaction moment exists, where

$$M = F_2 \cdot b = (60\ \text{lb})(2.00\ \text{in}) = 120\ \text{lb} \cdot \text{in}$$

The shear and moment diagrams can then be shown in the conventional manner. The result looks much like a cantilever that is built into a rigid support. The difference here is that the reaction moment at the section through the pin is developed in the vertical arm of the crank.

Comments: Note that the shape of the moment diagram for the horizontal part shows that the maximum moment occurs at the section through the pin and that the moment decreases linearly as we move out toward point A. As a result, the shape of the crank is optimized, having its largest cross section (and section modulus) at the section of highest bending moment. You could complete the design of the crank using the techniques reviewed in Section 3–15.

Example Problem 3–14

Figure 3–19 represents a print head for a computer printer. The force F moves the print head toward the left against the ribbon, imprinting the character on the paper that is backed up by the platen. Draw the free-body diagram for the horizontal portion of the print head, along with the shearing force and bending moment diagrams.

Solution

Objective: Draw the free-body diagram of the horizontal part of the print head in Figure 3–19. Draw the shearing force and bending moment diagrams for that part.

Given: The layout from Figure 3–19.

Analysis: The horizontal force of 35 N acting to the left is reacted by an equal 35 N horizontal force produced by the platen pushing back to the right on the print head. The guides provide simple supports in the vertical direction. The applied force also produces a moment at the base of the vertical arm where it joins the horizontal part of the print head.

We create the free-body diagram for the horizontal part by breaking it at its right end and replacing the removed part with the internal force and moment acting at the break. The shearing force and bending moment diagrams can then be drawn.

Results: The free-body diagram for the horizontal portion is shown below the complete sketch. Note that at the right end (section D) of the print head, the vertical arm has been removed and replaced with the internal horizontal force of 35.0 N and a moment of 875 N·mm caused by the 35.0 N force acting 25 mm above it. Also note that the 25 mm-moment arm for the force is taken from the line of action of the force *to the neutral axis of the horizontal part*. The 35.0 N reaction of the platen on the print head tends to place the head in compression over the entire length. The rotational tendency of the moment is reacted by the couple created by R_1 and R_2 acting 45 mm apart at B and C.

Below the free-body diagram is the vertical shearing force diagram in which a constant shear of 19.4 N occurs only between the two supports.

The bending moment diagram can be derived from either the left end or the right end. If we choose to start at the left end at *A*, there is no shearing force from *A* to *B*, and therefore there is no change in bending moment. From *B* to *C*, the positive shear causes an increase in bending moment from 0 to 875 N·mm. Because there is no shear from *C* to *D*, there is no change in bending moment, and the value remains at 875 N·mm. The counterclockwise-directed concentrated moment at *D* causes the moment diagram to drop abruptly, closing the diagram.

Example Problem 3–15 Figure 3–20 shows a crank in which it is necessary to visualize the three-dimensional arrangement. The 60-lb downward force tends to rotate the shaft *ABC* around the *x*-axis. The reaction torque acts only at the end of the shaft outboard of the bearing support at *A*. Bearings *A* and *C* provide simple supports. Draw the complete free-body diagram for the shaft *ABC*, along with the shearing force and bending moment diagrams.

Solution

Objective: Draw the free-body diagram of the shaft *ABC* in Figure 3–20. Draw the shearing force and bending moment diagrams for that part.

Given: The layout from Figure 3–20.

Analysis: The analysis will take the following steps:

1. Determine the magnitude of the torque in the shaft between the left end and point *B* where the crank arm is attached.
2. Analyze the connection of the crank at point *B* to determine the force and moment transferred to the shaft *ABC* by the crank.
3. Compute the vertical reactions at supports *A* and *C*.
4. Draw the shearing force and bending moment diagrams considering the concentrated moment applied at point *B*, along with the familiar relationships between shearing force and bending moments.

Results: The free-body diagram is shown as viewed looking at the *x–z* plane. Note that the free body must be in equilibrium in all force and moment directions. Considering first the torque (rotating moment) about the *x*-axis, note that the crank force of 60 lb acts 5.0 in from the axis. The torque, then, is

$$T = (60\text{ lb})(5.0\text{ in}) = 300\text{ lb}\cdot\text{in}$$

This level of torque acts from the left end of the shaft to section *B*, where the crank is attached to the shaft.

Now the loading at *B* should be described. One way to do so is to visualize that the crank itself is separated from the shaft and is replaced with a force and moment caused by the crank. First, the downward force of 60 lb pulls down at *B*. Also, because the 60-lb applied force acts 3.0 in to the left of *B*, it causes a concentrated moment in the *x–z plane* of 180 lb·in to be applied at *B*.

Both the downward force and the moment at *B* affect the magnitude and direction of the reaction forces at *A* and *C*. First, summing moments about *A*,

$$(60\text{ lb})(6.0\text{ in}) - 180\text{ lb}\cdot\text{in} - R_C(10.0\text{ in}) = 0$$
$$R_C = [(360 - 180)\text{ lb}\cdot\text{in}]/(10.0\text{ in}) = 18.0\text{ lb upward}$$

Now, summing moments about C,

$$(60\text{ lb})(4.0\text{ in}) + 180\text{ lb}\cdot\text{in} - R_A(10.0\text{ in}) = 0$$
$$R_A = [(240 + 180)\text{ lb}\cdot\text{in}]/(10.0\text{ in}) = 42.0\text{ lb upward}$$

Now the shear and bending moment diagrams can be completed. The moment starts at zero at the simple support at *A*, rises to 252 lb·in at *B* under the influence of the 42-lb shear force, then drops by 180 lb·in due to the counterclockwise concentrated moment at *B*, and finally returns to zero at the simple support at *C*.

Comments: In summary, shaft *ABC* carries a torque of 300 lb·in from point *B* to its left end. The maximum bending moment of 252 lb·in occurs at point *B* where the crank is attached. The bending moment then suddenly drops to 72 lb·in under the influence of the concentrated moment of 180 lb·in applied by the crank.

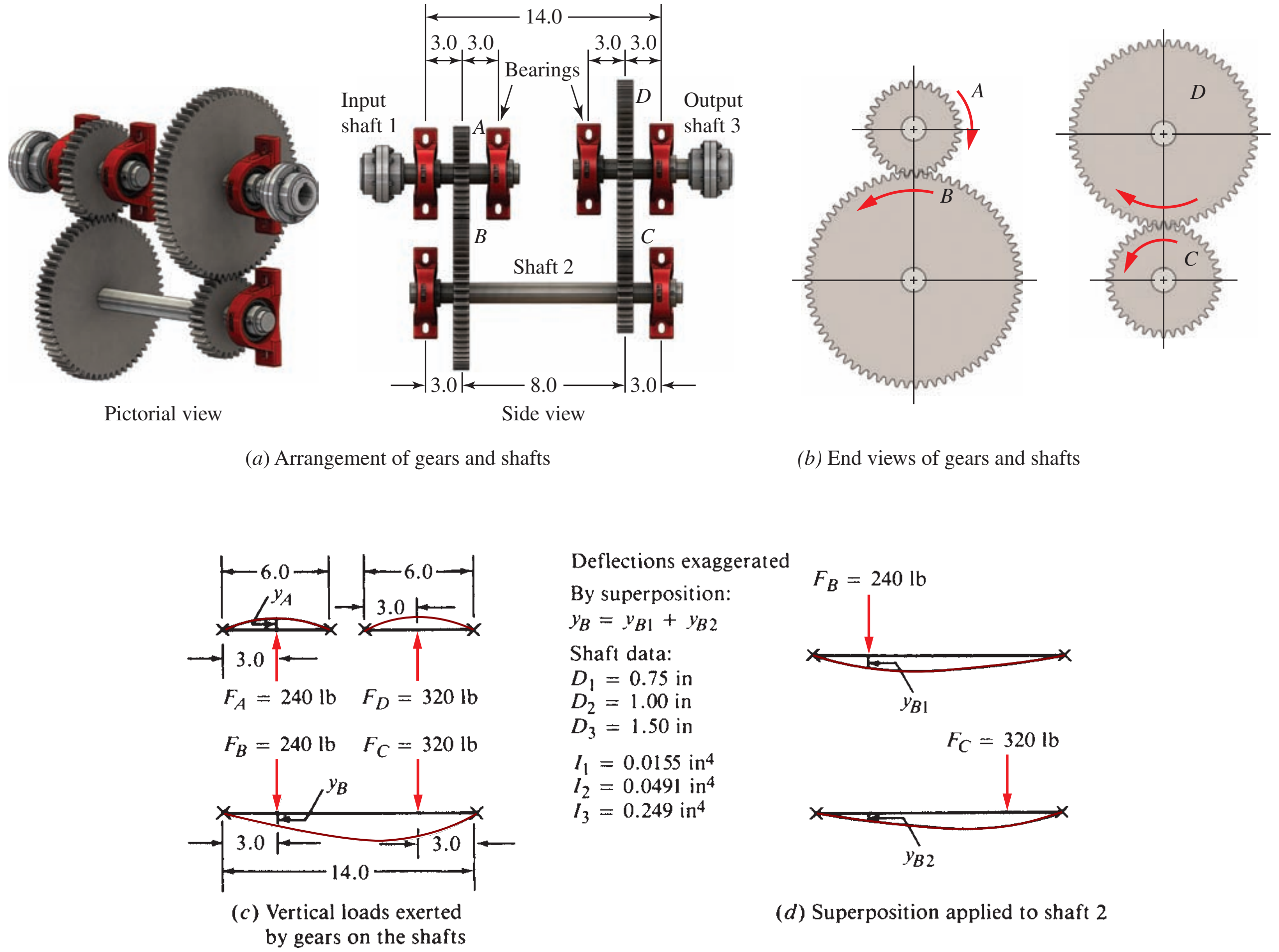


FIGURE 3–23 Shaft deflection analysis for a double-reduction speed reducer

For many additional cases, superposition is useful if the actual loading can be divided into parts that can be computed by available formulas. The deflection for each loading is computed separately, and then the individual deflections are summed at the points of interest.

Many commercially available computer software programs allow the modeling of beams having rather complex loading patterns and varying geometry. The results include reaction forces, shearing force and bending moment diagrams, and deflections at any point. It is important that you understand the principles of beam deflection, studied in strength of materials and reviewed here, so that you can apply such programs accurately and interpret the results carefully.

Example Problem 3–16

For the two gears, *A* and *B*, in Figure 3–23, compute the relative deflection between them in the plane of the paper that is due to the forces shown in Figure 3–23 (c). These *separating forces*, or *normal forces*, are discussed in Chapters 9 and 10. It is customary to consider the loads at the gears and the reactions at the bearings to be concentrated. The shafts carrying the gears are steel and have uniform diameters as listed in the figure.

Solution

Objective: Compute the relative deflection between gears *A* and *B* in Figure 3–23.

Given: The layout and loading pattern are shown in Figure 3–23. The separating force between gears *A* and *B* is 240 lb. Gear *A* pushes downward on gear *B*, and the reaction force of gear *B* pushes upward on gear *A*. Shaft 1 has a diameter of 0.75 in and a moment of inertia of 0.0155 in^4. Shaft 2 has a diameter of 1.00 in and a moment of inertia of 0.0491 in^4. Both shafts are steel. Use $E = 30 \times 10^6$ psi.

Analysis: Use the deflection formulas from Appendix 14 to compute the upward deflection of shaft 1 at gear *A* and the downward deflection of shaft 2 at gear *B*. The sum of the two deflections is the total deflection of gear *A* with respect to gear *B*.

Case (a) from Table A14–1 applies to Shaft 1 because there is a single concentrated force acting at the midpoint of the shaft between the supporting bearings. We will call that deflection y_A.

Shaft 2 is a simply supported beam carrying two nonsymmetrical loads. No single formula from Appendix 14 matches that loading pattern. But we can use superposition to compute the deflection of the shaft at gear B by considering the two forces separately as shown in Part (d) of Figure 3–23. Case (b) from Table A14–1 is used for each load.

We first compute the deflection at B due only to the 240-lb force, calling it y_{B1}. Then we compute the deflection at B due to the 320-lb force, calling it y_{B2}. The total deflection yat B is $y_B = y_{B1} + y_{B2}$.

Results The deflection of shaft 1 at gear A is

$$y_A = \frac{F_A L_1^3}{48\ EI} = \frac{(240)(6.0)^3}{48(30 \times 10^6)(0.0155)} = 0.0023 \text{ in}$$

The deflection of shaft 2 at B due only to the 240-lb force is

$$y_{B1} = -\frac{F_B a^2 b^2}{3\ EI_2 L_2} = -\frac{(240)(3.0)^2(11.0)^2}{3(30 \times 10^6)(0.0491)(14)} = -0.0042 \text{ in}$$

The deflection of shaft 2 at B due only to the 320-lb force at C is

$$y_{B2} = -\frac{F_c bx}{6\ EI_2 L_2}(L_2^2 - b^2 - x^2)$$

$$y_{B2} = -\frac{(320)(3.0)(3.0)}{6(30 \times 10^6)(0.0491)(14)}[(14)^2 - (3.0)^2 - (3.0)^2]$$

$$y_{B2} = -0.0041 \text{ in}$$

Then the total deflection at gear B is

$$y_B = y_{B1} + y_{B2} = -0.0042 - 0.0041 = -0.0083 \text{ in}$$

Because shaft 1 deflects upward and shaft 2 deflects downward, the total relative deflection is the sum of y_A and y_B:

$$y_{\text{total}} = y_A + y_B = 0.0023 + 0.0083 = 0.0106 \text{ in}$$

Comment This deflection is very large for this application. How could the deflection be reduced?

3–19 EQUATIONS FOR DEFLECTED BEAM SHAPE

The general principles relating the deflection of a beam to the loading on the beam and its manner of support are presented here. The result will be a set of relationships among the load, the vertical shearing force, the bending moment, the slope of the deflected beam shape, and the actual deflection curve for the beam. Figure 3–17 shows diagrams for these five factors, with θ as the slope and y indicating deflection of the beam from its initial straight position. The product of modulus of elasticity and the moment of inertia, EI, for the beam is a measure of its stiffness or resistance to bending deflection. It is convenient to combine EI with the slope and deflection values to maintain a proper relationship, as discussed next.

One fundamental concept for beams in bending is

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

where M = bending moment

x = position on the beam measured along its length

y = deflection

Thus, if it is desired to create an equation of the form $y = f(x)$ (i.e., y as a function of x), it would be related to the other factors as follows:

$$y = f(x)$$

$$\theta = \frac{dy}{dx}$$

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

$$\frac{V}{EI} = \frac{d^3y}{dx^3}$$

$$\frac{w}{EI} = \frac{d^4y}{dx^4}$$

where w = general term for the load distribution on the beam

The last two equations follow from the observation that there is a derivative (slope) relationship between shear and bending moment and between load and shear.

In practice, the fundamental equations just given are used in reverse. That is, the load distribution as a function of x is known, and the equations for the other

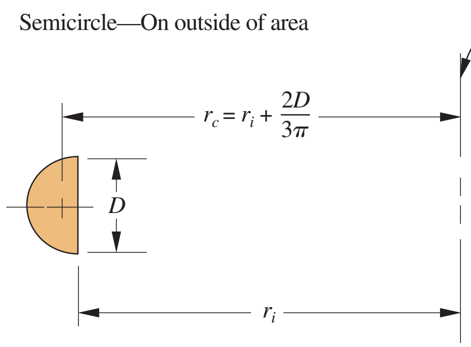
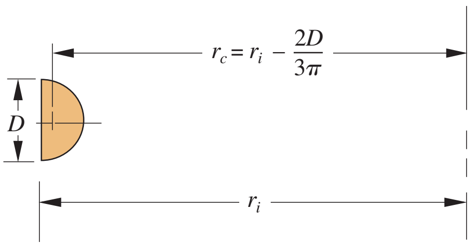
Shape of cross section	Area, A	Area shape factor, ASF
Semicircle—On outside of area 	$\pi D^2/8$	For $D/2 > r_i$ $r_i\pi - D + 2\sqrt{D^2/4 - r_i^2}\ \ln\left[\frac{D/2+\sqrt{D^2/4-r_i^2}}{r_i}\right]$ For $r_i > D/2$ $r_i\pi - D - \pi\sqrt{r_i^2-D^2/4} + 2\sqrt{r_i^2-D^2/4}\,\sin^{-1}(D/2r_i)$ [Note: Argument of inverse sine is in radians]
Semicircle—On inside of area 	$\pi D^2/8$	$r_i\pi + D - \pi\sqrt{r_i^2-D^2/4} - 2\sqrt{r_i^2-D^2/4}\,\sin^{-1}(D/r_i)$ [Note: Argument of inverse sine is in radians]

FIGURE 3–25 Area shape factors for selected cross sections of curved bars (*Continued*)

Example Problem 3–17

A curved bar has a rectangular cross section 15.0 mm thick by 25.0 mm deep as shown in Figure 3–26. The bar is bent into a circular arc producing an inside radius of 25.0 mm. For an applied bending moment of +400 N·m, compute the maximum tensile and compressive stresses in the bar.

Solution

Objective Compute the maximum tensile and compressive stresses.

Given $M = +400$ N·m that tends to straighten the bar. $r_i = 25.0$ mm.

Analysis Apply Equations (3–28) and (3–29).

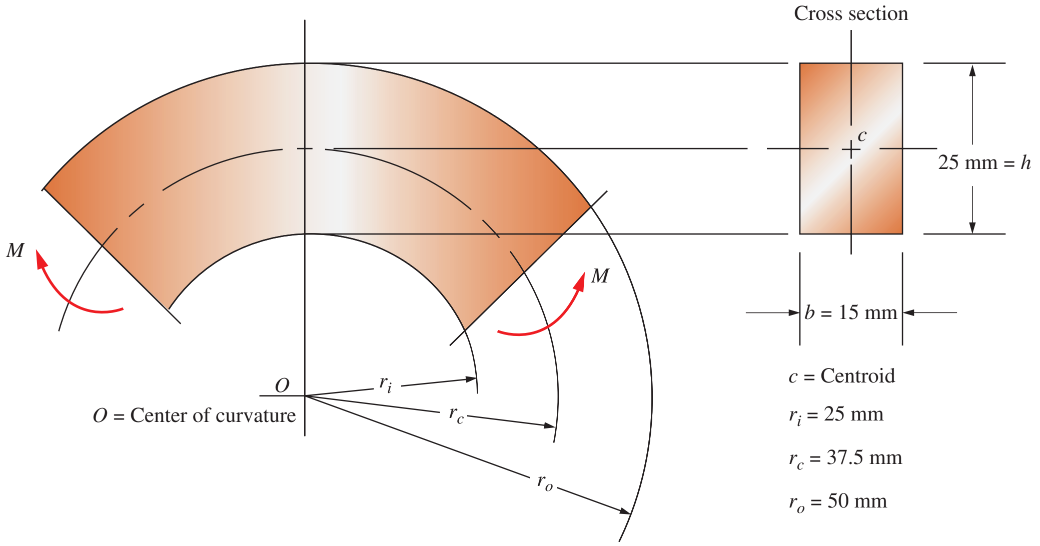


FIGURE 3–26 Curved bar with a rectangular cross section for Example Problem 3–17

Results First compute the cross-sectional area:

$$A = bh = (15.0\text{ mm})(25.0\text{ mm}) = 375.0\text{ mm}^2$$

Now compute the quantities involving radii. See Figure 3–26 for related dimensions:

$$r_o = r_i + h = 25.0\text{ mm} + 25.0\text{ mm} = 50.0\text{ mm}$$
$$r_c = r_i + h/2 = 25.0\text{ mm} + (25.0/2)\text{ mm} = 37.5\text{ mm}$$
$$R = A/ASF$$

From Figure 3–25, for a rectangular cross section:

$$ASF = b \cdot \ln(r_o/r_i) = (15\text{ mm})[\ln(50.0/25.0)] = 10.3972\text{ mm}$$

Then, $R = A/ASF = (375\text{ mm}^2)/10.3972\text{ mm} = 36.0674\text{ mm}$
Quantities needed in the stress equations include:

$$r_c - R = 37.5\text{ mm} - 36.0674\text{ mm} = 1.4326\text{ mm}$$

This is the distance e as shown in Figure 3–24.

$$R - r_o = 36.0674\text{ mm} - 50.0\text{ mm} = -13.9326\text{ mm}$$
$$R - r_i = 36.0674\text{ mm} - 25.0\text{ mm} = 11.0674\text{ mm}$$

This is the distance from the inside surface to the neutral axis.
Stress at outer surface, using Equation (3–28):

$$\sigma_o = \frac{M(R - r_o)}{Ar_o(r_c - R)} = \frac{(400\text{ N}\cdot\text{m})(-13.9326\text{ mm})[1000\text{ mm/m}]}{(375\text{ mm}^2)(50.0\text{ mm})(1.4326\text{ mm})}$$

$$\sigma_o = -207.5\text{ N/mm}^2 = -207.5\text{ MPa}$$

This is the maximum compressive stress in the bar.
Stress at inner surface, using Equation (3–29):

$$\sigma_i = \frac{M(R - r_i)}{Ar_i(r_c - R)} = \frac{(400\text{ N}\cdot\text{m})(11.0674\text{ mm})[1000\text{ mm/m}]}{(375\text{ mm}^2)(25.0\text{ mm})(1.4326\text{ mm})}$$

$$\sigma_i = 329.6\text{ N/mm}^2 = 329.6\text{ MPa}$$

This is the maximum tensile stress in the bar.

Comment The stress distribution between the outside and the inside is similar to that shown in Figure 3–24.

The analysis process introduced earlier can be modified to consider a composite shape for the cross section of the curved beam.

Procedure for Analyzing Curved Beams with Composite Cross-sectional Shapes Carrying a Pure Bending Moment

This procedure is used to compute the maximum tensile and compressive stresses for a curved beam at the inside (at r_i) and outside (at r_o) surfaces. The results are then compared to determine which value is the true maximum stress.

1. Determine the value of the applied bending moment, M, including its sign.
2. For the composite cross-sectional area of the beam:
   - a. Determine the inside radius, r_i, and the outside radius, r_o.
   - b. Separate the composite area into two or more parts that are shapes from Figure 3–25.
   - c. Compute the area of each component part, A_i, and the total area, A.
   - d. Locate of the centroid of each component area.
   - e. Compute the radius of the centroid of the composite area, r_c.
   - f. Compute the value of the area shape factor, ASF_i, for each component area using the equations in Figure 3–25.
   - g. Compute the radius, R, from the center of curvature to the neutral axis from:

$$R = A/\Sigma(ASF_i)$$

3. Compute the stress at the outside surface from:

$$\sigma_o = \frac{M(R - r_o)}{Ar_o(r_c - R)} \tag{3–30}$$

4. Compute the stress at the inside surface from:

$$\sigma_i = \frac{M(R - r_i)}{Ar_i(r_c - R)} \tag{3–31}$$

5. Compare σ_o and σ_i to determine the maximum value.