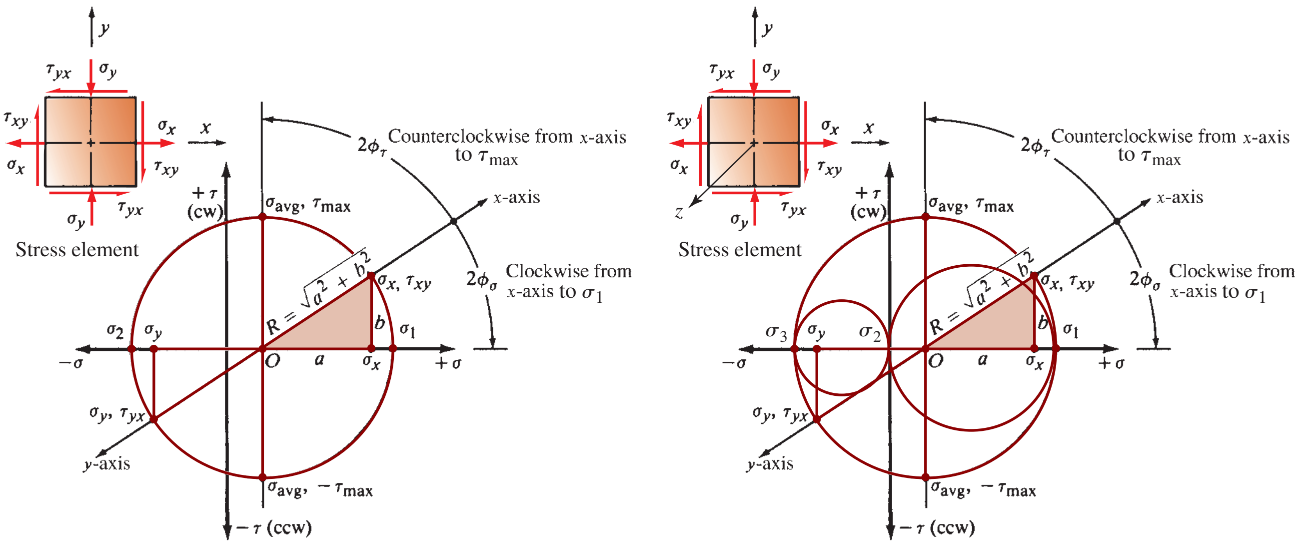


(*a*) Completed 2D Mohr's circle (*b*) Completed 3D Mohr's circle (Discussed after Example Problem 4-2)

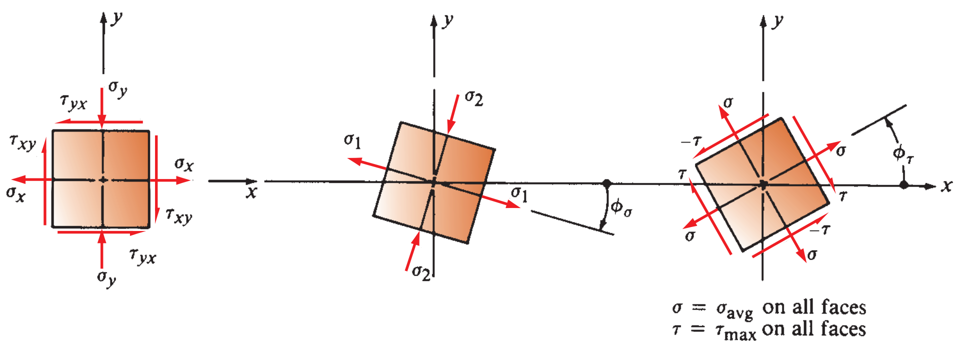
FIGURE 4–13 General form for a completed 2D Mohr's circle, Steps 8–14; Complete 3D Mohr's circle (discussed later)

9. Find σ_1, the maximum principal stress, at the point where the circle crosses the σ-axis at the right. Note that $\sigma_1 = \sigma_{avg} + R$. Because the Mohr's circle represents a plot of all possible combinations of normal and shear stresses on the element for any angle of orientation, it stands to reason that σ_1 lies at the right end of the horizontal diameter.
10. Find σ_2, the minimum principal stress, at the left end of the horizontal diameter. Note that $\sigma_2 = \sigma_{avg} - R$.
11. Find τ_{max}, the maximum shear stress, at the top end of the vertical diameter of the circle. By observation, $\tau_{max} = R$, the radius of the circle. Note also that the coordinates for the point at the top of the circle are (σ_{avg}, τ_{max}).

    At this point in the process, it is important to realize that angles on Mohr's circle are actually double the true angles.

The following steps define the general method for finding the angles of orientation of the principal stress element and the maximum shear stress element in relation to the original *x*-axis, located in Step 6 and shown in Figure 4–13(a). Now we can observe that the line from the center of the circle to the second plotted point, **(σ_y, τ_{yx})**, represents the *y*-axis from the original stress element. Of course, the *x*-axis and the *y*-axis are truly 90° apart, whereas they are 180° apart on Mohr's circle, illustrating the double-angle phenomenon.

12. Find the angle called $2\varphi_\sigma$, always measured ***from the x-axis to the σ-axis***, and ***note the direction—either clockwise or counterclockwise***. In the ***current problem***, we can observe that $2\varphi_\sigma = \alpha$, the angle found in Step 7, and the rotation is clockwise. ***However, in other problems, the angle must be found from the geometry of the circle.*** (This is illustrated in example practice problems that follow.) Now compute $\varphi_\sigma = 2\varphi_\sigma/2$.
13. Draw the *principal stress element* as shown in Figure 4–14(b). In Part (a) of the figure, we have reproduced the *original stress element* to indicate the direction of the *x*-axis and we draw a new element in relation to that axis at an angle of φ_σ. The new element must be rotated in the same direction, clockwise or counterclockwise, as observed in Step 12.
    a. On the face of the element found from the rotation, draw the vector σ_1.



(*a*) Original stress element **(*b*) Principal stress element** **(*c*) Maximum shear stress element**

FIGURE 4–14 Display of results from 2D Mohr's circle

b. Then show that the vector σ_1 is acting on the opposite parallel face in the opposite sense to indicate the tension or compression stress.

c. Draw two σ_2 vectors on the other perpendicular faces in the proper sense.

d. Note that ***on the principal stress element, the shear stress is always zero***. This is evident from the Mohr's circle where the coordinates of the principal stresses are always (σ_1, 0) and (σ_2, 0), indicating zero shear stress.

14. Find the angle called $2\varphi_\tau$, always measured ***from the x-axis to the τ_{max}-axis,*** and ***note the direction—either clockwise or counterclockwise***. In the ***current problem***, we can observe that $2\varphi_\tau = 90° - \alpha$, the angle found in Step 7, and the rotation is counterclockwise. ***However, in other problems, the angle must be found from the geometry of the circle.*** Now compute $\varphi_\tau = 2\varphi_\tau/2$.

15. Draw the *maximum shear stress element* as shown in Figure 4–14(c). Here, we draw a new element in relation to the *x*–axis at an angle of φ_τ. The new element must be rotated in the direction of either clockwise or counterclockwise, as observed in Step 14.

a. On the face of the element found from the rotation, draw the vector τ_{max}.

b. Then show that the vector τ_{max} is acting on the opposite parallel face in the opposite sense to indicate the shearing action.

c. Draw two $-\tau_{max}$ vectors on the other perpendicular faces. The combination of all four shearing vectors must indicate equilibrium of the element.

d. Note that ***on the maximum shear stress element, the average normal stress acts on all four faces.*** This is evident from the Mohr's circle where the coordinates of the maximum shear stress is always (σ_{avg}, τ_{max}) and for the minimum shear stress it is always (σ_{avg}, $-\tau_{max}$).

16. The final step in the process of preparing the 2D Mohr's circle is to summarize the primary results, typically: σ_1, σ_2, τ_{max}, σ_{avg}, φ_σ, and φ_τ.

We will now illustrate the construction of a 2D Mohr's circle by using the same data as in Example Problem 4–1, in which the principal stresses and the maximum shear stress were computed directly from the equations.

Example Problem 4–2

The shaft shown in Figure 4–7 is supported by two bearings and carries two V-belt sheaves. The tensions in the belts exert horizontal forces on the shaft, tending to bend it in the *x*–*z* plane. Sheave *B* exerts a clockwise torque on the shaft when viewed toward the origin of the coordinate system along the *x*-axis. Sheave *C* exerts an equal but opposite torque on the shaft. For the loading condition shown, determine the principal stresses and the maximum shear stress on element *K* on the front surface of the shaft (on the positive *z*-side) just to the right of sheave *B*. Use the procedure for constructing the 2D Mohr's circle in this section.

Solution

Objective: Determine the principal stresses and the maximum shear stresses on element *K*.

Given: Shaft and loading pattern shown in Figure 4–7.

Analysis: Use the *Procedure for Constructing a 2D Mohr's Circle*. Some intermediate results will be taken from the solution to Example Problem 4–1 and from Figures 4–7 to 4–9.

Results: ***Steps 1 and 2.*** The stress analysis for the given loading was completed in Example Problem 4–1. Figure 4–15 is identical to Figure 4–9 and represents the results of Step 2 of the 2D Mohr's circle procedure, the original stress element.

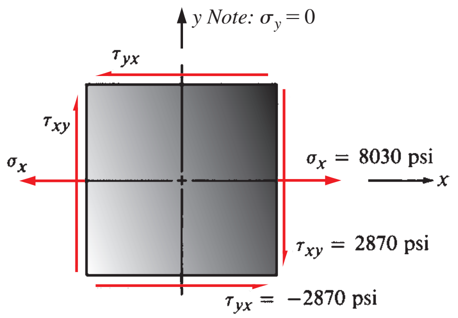


FIGURE 4–15 Stresses on element *K*

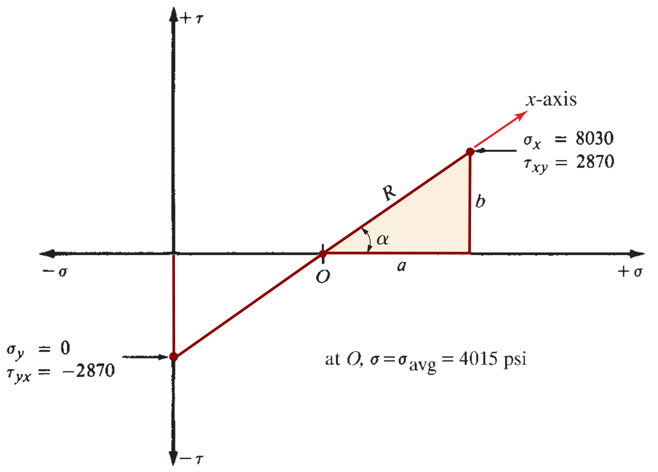


FIGURE 4–16 Partially completed 2D Mohr's circle

Steps 3–6. Figure 4–16 shows the results. The first point plotted was

$$\sigma_x = 8030 \text{ psi}, \tau_{xy} = 2870 \text{ psi}$$

The second point was plotted at

$$\sigma_y = 0 \text{ psi}, \tau_{yx} = -2870 \text{ psi}$$

Then a line was drawn between them, crossing the σ-axis at O. The value of the stress at O is

$$\sigma_{avg} = (\sigma_x + \sigma_y)/2 = (8030 \text{ psi} + 0 \text{ psi})/2 = 4015 \text{ psi}$$

Extend the line through the point (σ_x, τ_{xy}) and label it, x-axis.

Step 7. We compute the values for a, b, and R from

$$a = (\sigma_x - \sigma_{avg}) = (8030 \text{ psi} - 4018 \text{ psi}) = 4015 \text{ psi}$$

$$b = \tau_{xy} = 2870 \text{ psi}$$

$$R = \sqrt{a^2 + b^2} = \sqrt{(4015 \text{ psi})^2 + (2870 \text{ psi})^2} = 4935 \text{ psi}$$

Step 8. Figure 4–17(a) shows the completed 2D Mohr's circle. The circle has its center at O and the radius R. Note that the circle passes through the two points originally plotted. It must do so because the circle represents all possible states of stress on the element K.

Step 9. The maximum principal stress is at the right side of the circle.

$$\sigma_1 = \sigma_{avg} + R$$

$$\sigma_1 = 4015 + 4935 = 8950 \text{ psi}$$

Step 10. The minimum principal stress is at the left side of the circle.

$$\sigma_2 = \sigma_{avg} - R$$

$$\sigma_2 = 4015 - 4935 = -920 \text{ psi}$$

Step 11. At the top of the circle,

$$\sigma = \sigma_{avg} = 4015 \text{ psi}$$

$$\tau = \tau_{max} = R = 4935 \text{ psi}$$

The value of the normal stress on the element that carries the maximum shear stress is the same as the coordinate of O, the center of the circle.

Step 12. Compute the angles α, $2\phi_\sigma$, and then ϕ_σ. Use the circle as a guide.

$$\alpha = 2\phi_\sigma = \arctan(b/a) = \arctan(2870 \text{ psi})/(4015 \text{ psi}) = 35.6°$$

$$\phi_\sigma = 35.6°/2 = 17.8°$$

Note that ϕ_σ must be measured *clockwise* from the original x-axis to the direction of the line of action of σ_1 for this set of data. The principal stress element will be rotated in the same direction as part of step 13.

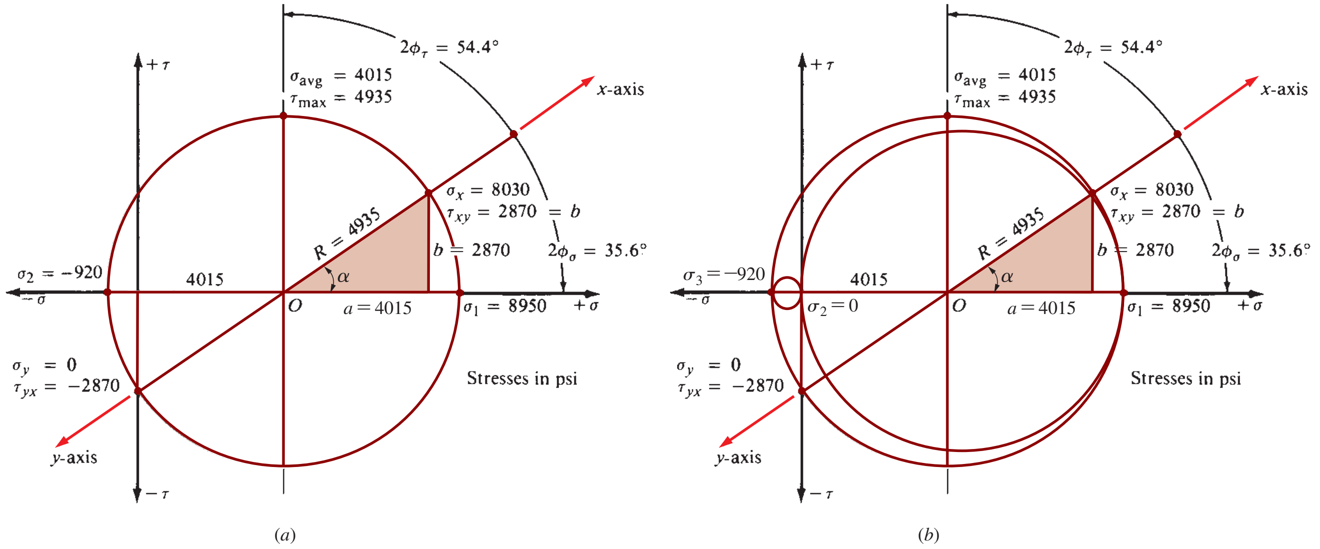
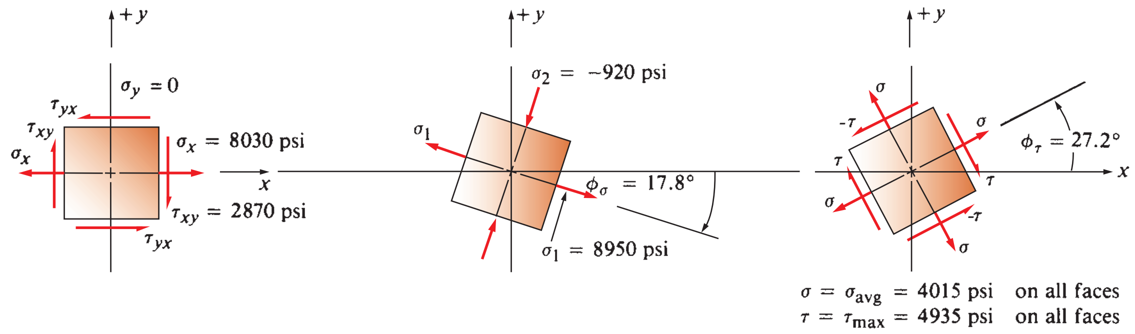


FIGURE 4–17 Completed Mohr's circles for (a) 2D result and (b) 3D result



(*a*) Original stress element, *K*, from Figure 4–7

(*b*) Principal stress element

(*c*) Maximum shear stress element

FIGURE 4–18 Results from 2D Mohr's circle analysis

Step 13. See Figure 4–18(a) where the original stress element has been reproduced. Now draw the principal stress element to the right (Part (b) of the figure). Rotate the element 17.8° *clockwise* from the *x*-axis as shown. Then draw the vectors for the maximum principal stress, $\sigma_1 = 8950$ psi (tensile), on that face and the one opposite. Complete the element by drawing vectors for $\sigma_2 = -920$ psi (compressive) on the other two faces. Label the element as shown in the figure.

Step 14. Compute the angle $2\phi_\tau$ and then ϕ_τ. From the circle we see that

$$2\phi_\tau = 90° - \alpha = 90° - 35.6° = 54.4°$$

$$\phi_\tau = 54.4°/2 = 27.2°$$

Note that the stress element on which the maximum shear stress acts must be rotated *counterclockwise* from the orientation of the original element for this set of data.

Step 15. In Figure 4–18(c), we draw the maximum shear stress element to the right of the principal stress element. It is rotated *counterclockwise* 27.2° from the *x*-axis.

a. On that face we show the positive maximum shearing stress, $\tau_{max} = 4935$ psi.

b. Then we complete τ_{max} by drawing an equal shearing stress vector on the opposite face acting in the opposite direction so that the two vectors tend to rotate the element clockwise, indicating a positive shearing stress.

c. Now we can draw the two vectors that make up $-\tau_{max}$ on the other two faces. The set of four vectors now put the element in rotational equilibrium.

d. The element is completed by drawing four equal vectors, σ_{avg}, on all four faces of the element.

Step 16. Summary of results:

- Maximum principal stress $= \sigma_1 = 8950$ psi—Tension
- Minimum principal stress $= \sigma_2 = -920$ psi—Compression
- Maximum shear stress $= \tau_{max} = 4935$ psi
- Angle of rotation of principal stress element $= \varphi_\sigma = 17.8°$ clockwise below the *x*-axis
- Angle of rotation of the maximum shear stress element $= \sigma_\tau = 27.2°$ counterclockwise above the *x*-axis
- Figure 4–18 shows the resulting stress elements. These are identical to those shown in Figure 4–11 from the hand calculations performed in Example Problem 4–1.

Mohr's Circles for Three-Dimensional Stresses

Obviously, real load-carrying members are three-dimensional objects, and all stress elements should be three-dimensional elements. In Example Problem 4–1, Figure 4–13(a), the Mohr's circle represents the stress transformation of a two-dimensional, plane-stress element. The maximum and minimum principal stresses are identified as σ_1 and σ_2. It is clear that the z direction is one of the principal directions, as the stress components involving the z direction are all zero, $\sigma_z = \tau_{yz} = \tau_{zx} = 0$. Rearranging/ranking the three principle stresses, $\sigma_1 > \sigma_2 > \sigma_3$, the Mohr's circles for the three-dimensional stress element in plane stress state is shown in Figure 4–13(b), where $\sigma_2 = \sigma_z = 0$. Note that the additional two circles are drawn from the stress states in the 1-2 plane and the 2-3 plane.

The largest circle is the same as that developed for the 2D example with the maximum stress labeled as σ_1 as before at the right end. The smallest stress is at the left end of the largest circle and it is now labeled as σ_3. The second circle is drawn with σ_1 as before at the right end and $\sigma_2 = 0$ at the left end. Finally, the third circle is drawn with $\sigma_2 = 0$ at the right end and σ_3 at the left end.

In Example Problem 4–2, the element at the location of interest is subjected to plane stress. In the full three-dimensional model of the stress element, the principal stresses are again rearranged/reordered:

$$\sigma_1 = 8950 \text{ psi}$$
$$\sigma_2 = 0 \text{ psi}$$
$$\sigma_3 = -920 \text{ psi}$$

The corresponding Mohr's circles for three-dimensional stress state is shown in Figure 4–17(b).

In three-dimensional stress transformation, it is possible that all of the roots in Equation (4–12) are non-zero. With three non-zero principal stresses, the resulting Mohr's circles are shown in Figure 4–19. Note that the three principal stresses are again ordered such that $\sigma_1 > \sigma_2 > \sigma_3$.

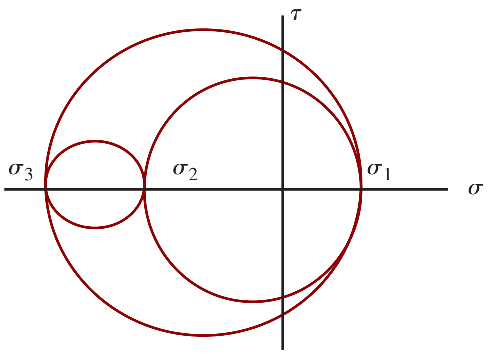


FIGURE 4–19 General example of Mohr's circles for three-dimensional stress state for which all three principal stresses are non-zero

Tresca Stress

Based on the three principal stresses obtained from stress transformation, a special form of stress, called the Tresca stress (after Henri Tresca), can be calculated for design decision making. The Tresca stress is defined as

$$\sigma' = (\sigma_1 - \sigma_3)/2 \qquad \textbf{(4–14)}$$

Observing from the Mohr's circles (three-dimensional), the Tresca stress is the maximum shear stress. In design, a failure criterion can be established to evaluate if yielding will occur. For a ductile material under static loading, a stress element is considered to have failed when $\sigma' > s_{sy}$, where s_{sy} is the shear yield strength of the material. The failure theory involving the Tresca stress is therefore called the Maximum Shear Stress Theory (MSST).

von Mises Stress

A more accurate assessment of the stress state for design with ductile material under static loading is the von Mises stress. It is defined as

$$\sigma_e = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}} \qquad \textbf{(4–15)}$$

The failure criterion based on the von Mises stress postulates that material will yield when $\sigma_e > s_y$, where s_y is the yield strength of the material. The application of the Tresca and the von Mises stresses in design will be further discussed in Chapter 5.

TABLE 4–1 Practice Problems for Mohr's Circle

Example Problem	x-axis	σ_x	σ_y	τ_{xy}	Fig. No.
4–3	1st quadrant	+10.0 ksi	+4.0 ksi	+5.0 ksi	4–20
4–4	2nd quadrant	−80 MPa	+20 MPa	+50 MPa	4–21
4–5	3rd quadrant	−80 MPa	+20 MPa	−50 MPa	4–22
4–6	4th quadrant	+10.0 ksi	−2.0 ksi	−4.0 ksi	4–23

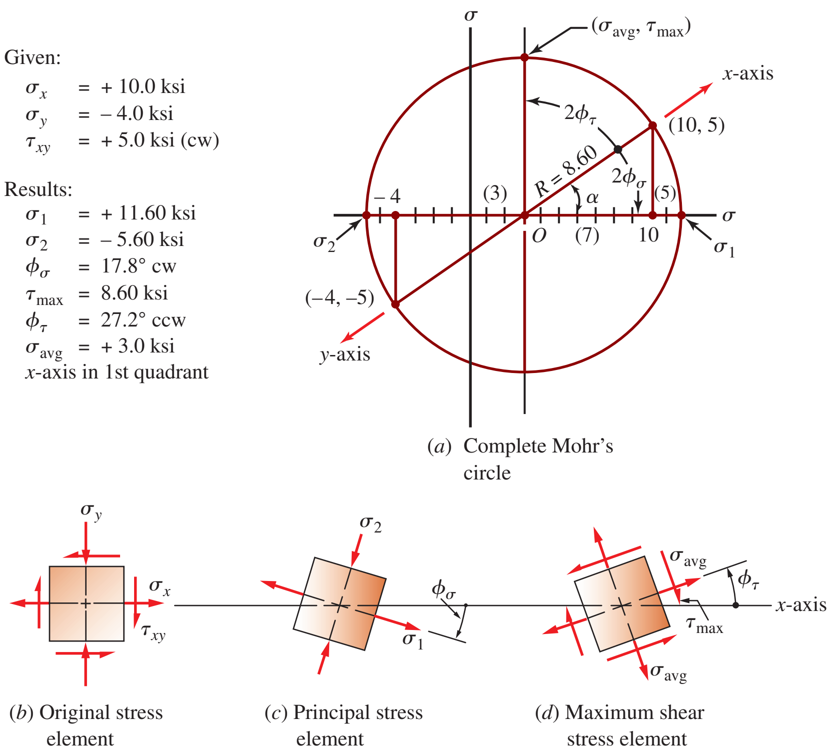
4–5 MOHR'S CIRCLE PRACTICE PROBLEMS

To a person seeing the construction of Mohr's circle for the first time, it may seem long and involved. But with practice under a variety of combinations of normal and shear stresses, you should be able to execute the 16 steps quickly and accurately.

Table 4–1 gives four sets of data (Example Problems 4–3 through 4–6) for normal and shear stresses in the x–y plane. You are advised to complete the Mohr's circle for each before looking at the solutions in Figures 4–20 through 4–23. From the circle, determine the two principal stresses, the maximum shear stress, and the planes on which these stresses act. Then draw the given stress element, the principal stress element, and the maximum shear stress element, all oriented properly with respect to the x- and y-directions. Note that each problem results in the x-axis being in a different quadrant.

After completing the solution for two-dimensional data, values are given for "3D stresses" considering that the z-axis stress is zero. The new values are the three principal stresses, σ_1, σ_2, and σ_3 as used in Sections 4–3 and 4–4.

Example Problem 4–3



3D stresses: $\sigma_1 = +11.60$ ksi, $\sigma_2 = 0$ ksi, $\sigma_3 = -5.60$ ksi

FIGURE 4–20 Solution for Example Problem 4–3, x-axis in 1st quadrant

Example Problem 4–4

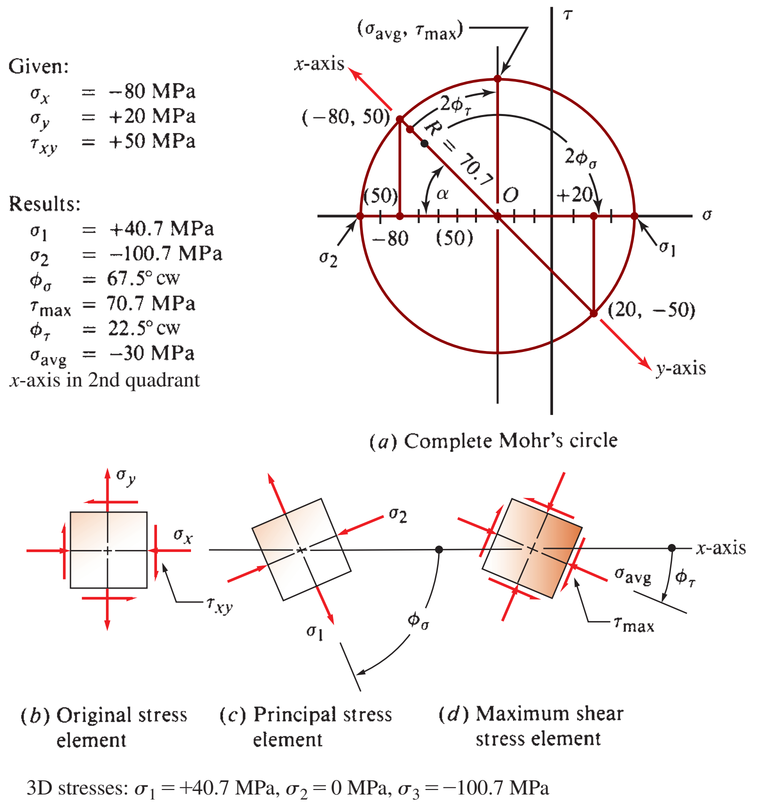


FIGURE 4–21 Solution for Example Problem 4–4 x-axis in 2nd quadrant

Example Problem 4–5

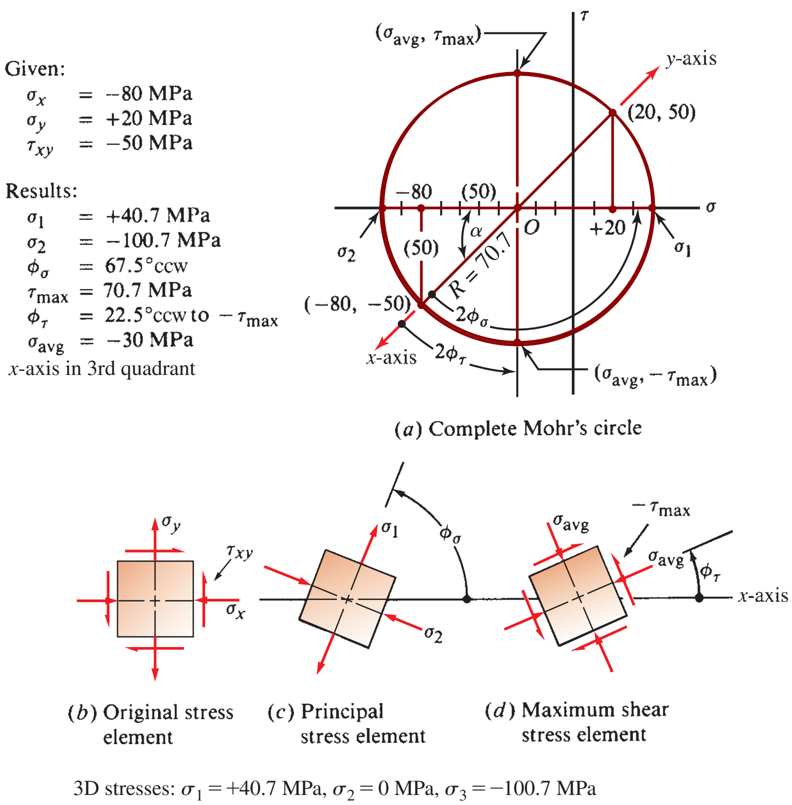


FIGURE 4–22 Solution for Example Problem 4–5, x-axis in 3rd quadrant

Example Problem 4–6

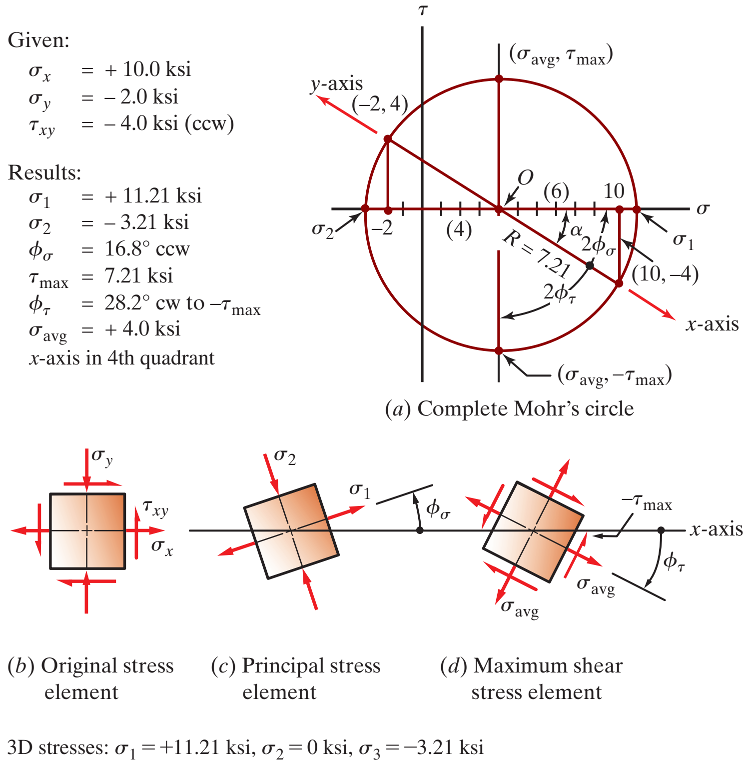


FIGURE 4–23 Solution for Example Problem 4–6, x-axis in 4th quadrant

4–6 MOHR'S CIRCLE FOR SPECIAL STRESS CONDITIONS

Mohr's circle is used here to demonstrate the relationship among the applied stresses, the principal stresses, and the maximum shear stress for the following special cases:

- Uniaxial tension
- Uniaxial compression
- Biaxial tension
- Biaxial tension and compression
- Pure shear
- Combined tension and shear

These are important, frequently encountered stress conditions, and they will be used in later chapters to illustrate failure theories and design methods. These failure theories are based on the values of the principal stresses and the maximum shear stress.

Uniaxial Tension

The stress condition produced in all parts of a standard tensile test specimen is pure uniaxial tension. Figure 4–24 shows the stress element and the corresponding Mohr's circle. Note that the maximum principal stress, σ_1, is equal to the applied stress, σ_x; the minimum principal stress, σ_3, is zero; and the maximum shear stress, τ_{max}, is equal to $\sigma_x/2$. Note that in three-dimensional stress state, the Mohr's circle drawn from the 1-2 plane coincides with that from the 1-3 plane, and the Mohr's circle drawn from the 2-3 plane is a point.

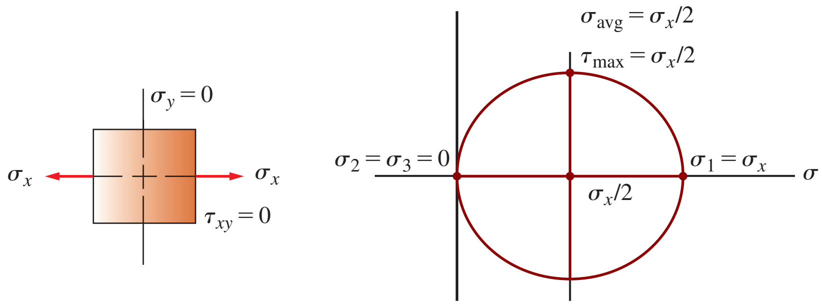


FIGURE 4–24 Mohr's circle for pure uniaxial tension

Uniaxial Compression

Figure 4–25 shows pure uniaxial compression as it would be produced by a standard compression test. Mohr's circle shows that $\sigma_1 = \sigma_2 = 0$ and $\sigma_3 = \sigma_x$ (a negative value); and the magnitude of the maximum shear stress is $\tau_{max} = \sigma_x/2$. Note that in a three-dimensional stress state, the Mohr's circle drawn from the 1-2 plane is a point, and the Mohr's circle drawn from the 1–3 plane coincides with that from the 2-3 plane.

Biaxial Tension

The stress condition produced in a spherical thin-walled pressure vessel or a filled balloon is biaxial tension. Figure 4–26 shows the stress element, where $\sigma_x = \sigma_y$, and the corresponding Mohr's circle. Note that the Mohr's circle drawn from the original two-dimensional stress element in the 1-2 plane is a point. Considering the third principal direction, two more circles are added. The Mohr's circle drawn from the 1-3 plane coincides with that from the 2-3 plane. The magnitude of the maximum shear stress is $\tau_{max} = \sigma_x/2$.

Biaxial Tension and Compression

Figure 4–27 shows that the element is subjected to biaxial tension and compression where σ_x is positive (tension) and σ_y is negative (compression) with the same magnitude. The resulting Mohr's circles show that $\sigma_1 = \sigma_x$, $\sigma_2 = 0$, and $\sigma_3 = \sigma_y$. The magnitude of the maximum shear stress is $\tau_{max} = \sigma_x$.

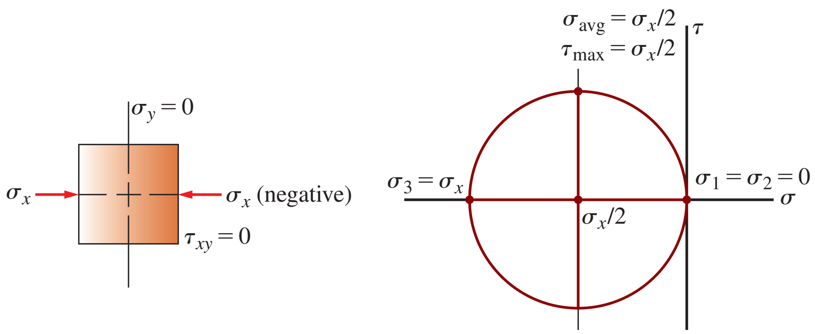


FIGURE 4–25 Mohr's circle for pure uniaxial compression

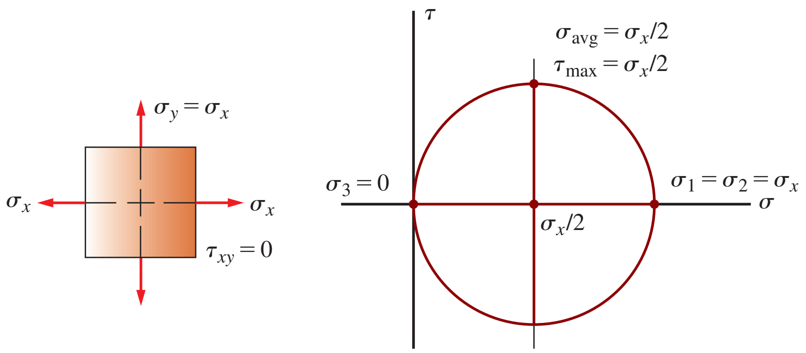


FIGURE 4–26 Mohr's circle for biaxial tension

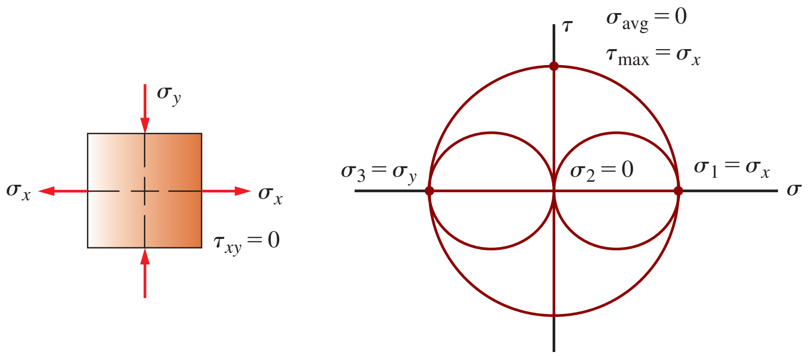


FIGURE 4–27 Mohr's circle for biaxial tension and compression

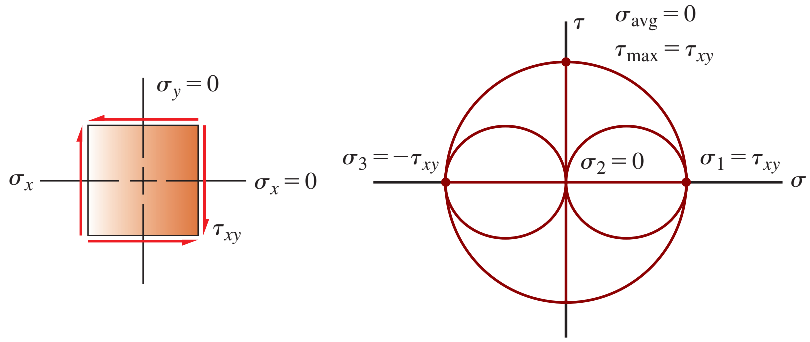


FIGURE 4–28 Mohr's circle for pure shear

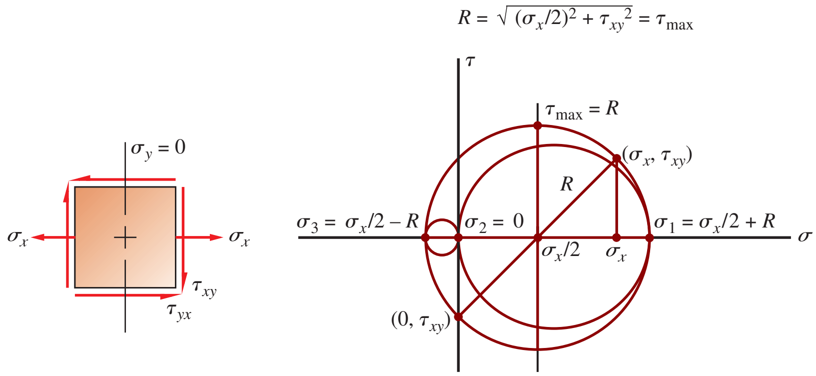


FIGURE 4–29 Mohr's circles for uniaxial tension and torsional shear

Pure Shear

Figure 4–28 shows the Mohr's circles for the case of pure shear that can result from a torsional load. The plane stress circle has its center at the origin of the $\sigma - \tau$ axes and that the radius of the circle is equal to the value of the applied shear stress, τ_{xy}. Considering the third principal direction, we have $\sigma_1 = \tau_{xy}$, $\sigma_2 = 0$, and $\sigma_3 = -\tau_{xy}$. The maximum shear $\tau_{max} = \tau_{xy}$. From Figures 4–27 and 4–28, it can be observed that if the magnitude of tensile stress σ_x and compressive stress σ_y in Figure 4–27 is the same as that of the shear stress τ_{xy} in Figure 4–28, the Mohr's circles in the two figures are identical. The two stress elements are identified to be in the *equivalent stress state*. That is, the two different stress states have the same effect on the material.

Combined Tension and Shear

This is an important special case because it describes the stress condition in a rotating shaft carrying bending loads while simultaneously transmitting torque. This is the type of stress condition on which the procedure for designing shafts, presented in Chapter 12, is based. If the applied stresses are called σ_x and τ_{xy}, the Mohr's circle in Figure 4–29 shows that

$$\tau_{max} = R = \text{radius of circle} = \sqrt{(\sigma_x/2)^2 + \tau_{xy}^2} \quad \textbf{(4–16)}$$

$$\sigma_1 = \sigma_x/2 + R = \sigma_x/2 + \sqrt{(\sigma_x/2)^2 + \tau_{xy}^2} \quad \textbf{(4–17)}$$

$$\sigma_2 = \sigma_x/2 - R = \sigma_x/2 - \sqrt{(\sigma_x/2)^2 + \tau_{xy}^2} \quad \textbf{(4–18)}$$

A convenient and useful concept called *equivalent torque* can be developed from Equation (4–16) for the special case of a body subjected to only bending and torsion.

An example is shown in Figure 4–30, where a circular bar is loaded at one end by a downward force and a torsional moment. The force causes bending in the bar with the maximum moment at the point where the bar is attached to the support. The moment causes a tensile stress on the top of the bar in the x-direction at the point called A, where the magnitude of the stress is

$$\sigma_x = M/S \quad \textbf{(4–19)}$$

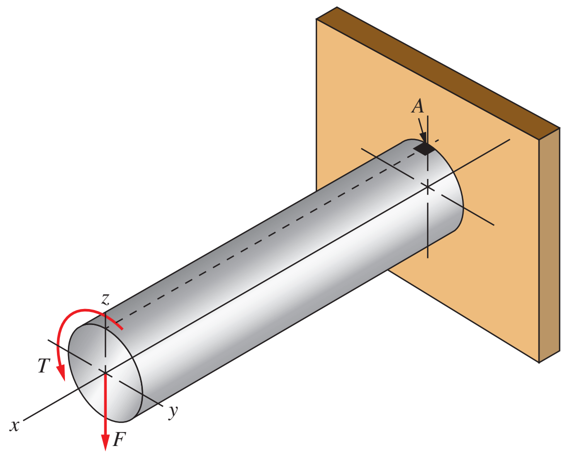


FIGURE 4–30 Circular bar in bending and torsion

where S = section modulus of the round bar.

Now the torsional moment causes torsional shear stress in the x–y plane at point A having a magnitude of

$$\tau_{xy} = T/Z_p \tag{4–20}$$

where Z_p = polar section modulus of the bar.

Point A then is subjected to a tensile stress combined with shear, the special case shown in the Mohr's circle of Figure 4–29. The maximum shear stress can be computed from Equation (4–16). If we substitute Equations (4–19) and (4–20) in to Equation (4–16), we get

$$\tau_{max} = \sqrt{(M/2S)^2 + (T/Z_p)^2} \tag{4–21}$$

Note from Appendix 1 that $Z_p = 2S$. Equation (4–21) can then be written as

$$\tau_{max} = \frac{\sqrt{M^2 + T^2}}{Z_p} \tag{4–22}$$

It is convenient to define the quantity in the numerator of this equation to be the *equivalent torque*, T_e. Then the equation becomes

$$\tau_{max} = T_e/Z_p \tag{4–23}$$

Stresses in a Cylinder with Internal Pressure

Another important special case of stress state is the set of stresses developed in a closed-end cylinder with internal pressure. In strength of materials, you learned that the outer surfaces of the walls of such cylinders are subjected to tensile stresses in two directions: (1) tangent to its circumference and (2) axially, parallel to the axis of the cylinder. The stress perpendicular to the wall at the outer surface is zero.

Figure 4–31 shows the stress condition on an element of the surface of the cylinder. The tangential stress, also called *hoop stress*, is aligned with the x-direction and is labeled σ_x. The axially directed stress, also called *longitudinal stress*, acts in line with the y-direction and is labeled σ_y.

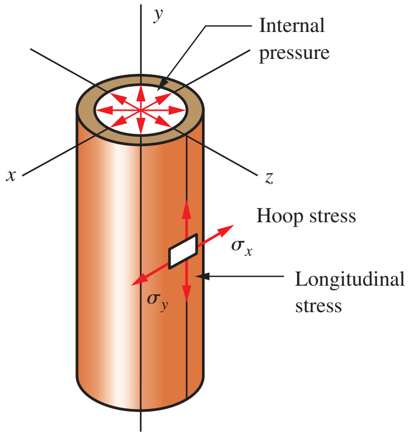


FIGURE 4–31 Thin-walled cylinder subjected to pressure with its ends closed

In strength of materials, you learned that if the wall of the cylinder is relatively thin, the maximum hoop stress is

$$\sigma_x = pD/2t$$

where p = internal pressure in the cylinder
D = mean diameter of the cylinder
t = thickness of the cylinder wall

Also, the longitudinal stress is

$$\sigma_y = pD/4t$$

Both stresses are tensile, and the hoop stress is twice as large as the longitudinal stress.

The analysis would be similar for any kind of thin-walled cylindrical vessel carrying an internal pressure. Examples are storage tanks for compressed gases, pipes carrying moving fluids under pressure, and the familiar beverage can that releases internal pressure when the top is popped open.

Let's use the following example to demonstrate the use of Mohr's circles for stress analysis.

Example Problem 4–7

As shown in Figure 4–32, a hydraulic motor drives a speed increaser gearbox which in turn drives the air screw compressor at 8000 rpm. The air screw compressor draws in outside air, compresses it, and discharges it into the air receiver tank. The receiver tank supplies service air to power air tools. A cut section of a receiver tank is shown below. The maximum design pressure of the system is set at 175 psi. The air receiver tank is made from aluminum tubing and has a 6 in outside diameter with a wall thickness of 1/8 in.

a. Can the receiver tank be considered a thin-walled pressure vessel?
b. Calculate the longitudinal and circumferential stresses on the receiver tank.
c. Use Mohr's circle to calculate the principal stresses and the maximum shear stress.

Solution

Given: The outside diameter of the tank, $D = 6.0$ in and the wall thickness, $t = 0.125$ in.
The internal pressure in the tank is 175 psi. [Note: Mean diameter $D_m = D_o - t = 5.875$ in]

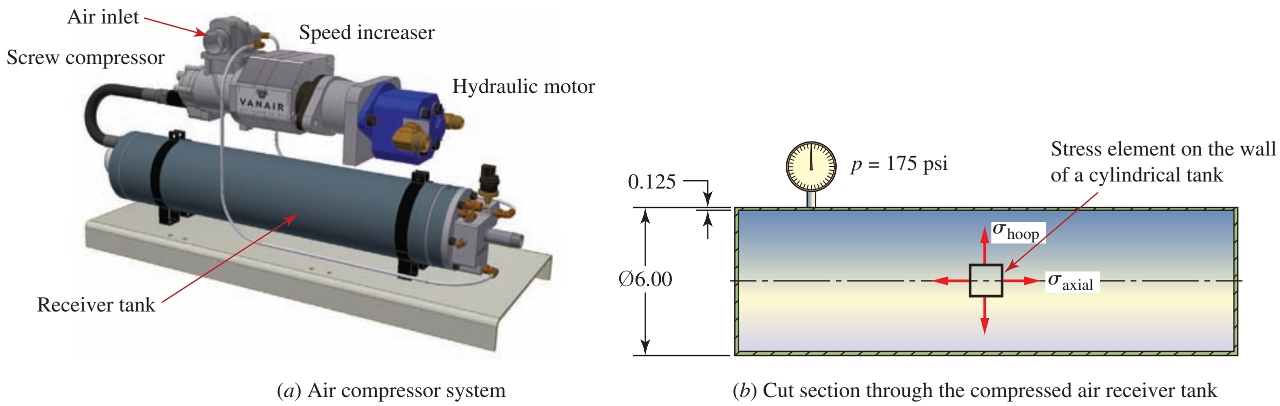


FIGURE 4–32 Screw-type air compressor system with a receiver tank (Courtesy of Vanair, Inc., Michigan City, IN)

Results

First check to determine if the pressure vessel is to be considered a thin-walled or thick-walled cylinder:

$$\frac{D_m}{t} = \frac{5.875}{0.125} = 47 > 20$$

Therefore, the vessel is considered to be a thin-walled cylinder.

The stress element obtained from the cylindrical wall has the following stress components:

Longitudinal/axial stress: $\dfrac{pD_m}{4t} = \dfrac{175 \times 5.875}{4 \times 0.125} = 2056 \text{ psi}$

Circumferential/hoop stress: $\sigma_{\text{hoop}} = \dfrac{pD_m}{2t} = \dfrac{175 \times 5.875}{2 \times 0.125} = 4113 \text{ psi}$

Radial stress: $\sigma_{\text{radial}} = 0$

Since there is no shear stress, the stress components are already in principal directions. Ordering the three principal stresses, we have

$$\sigma_1 = \sigma_{\text{hoop}} = 4113 \text{ psi}$$
$$\sigma_2 = \sigma_{\text{axial}} = 2056 \text{ psi}$$
$$\sigma_3 = \sigma_{\text{radial}} = 0$$

The corresponding three-dimensional stress element and Mohr's circles are shown in Figure 4–33. Note that the maximum shear stress is based on the consideration of the largest circle where

$$\tau_{\max} = 2056 \text{ psi}$$

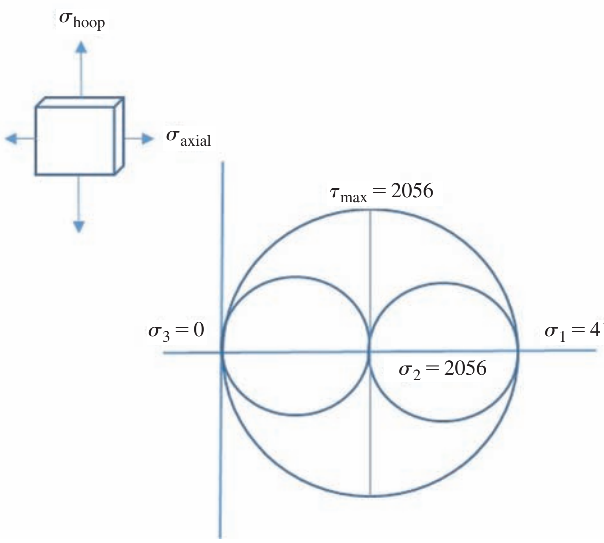


FIGURE 4–33 Stress element on the surface of the cylindrical tank and the resulting Mohr's circles

Example Problem 4–7 demonstrates the procedure of analyzing a stress element, a plane stress element in this case, and converting the stress components to three ranked principal stresses. In this case, one of the principal stresses is zero. In a design problem, we will need to further reduce the three principal stresses to an "effective stress" and make sure the yield strength of the selected material is greater than the effective stress to prevent yielding. While more is said in Chapter 5 about the concept of design stress, we now follow with the type of analysis necessary to judge the suitability of the tank design in Example Problem 4–7. Two approaches are demonstrated: the Maximum Shear Stress Theory (MSST), also called the *Tresca criterion*; and the Distortion Energy Theory, also called the *von Mises criterion*.

Using the Tresca criterion, the effective stress is

$$\sigma' = (\sigma_1 - \sigma_3)/2 = 4113/2 = 2056 > s_y \text{ (psi)}$$

Then the selected material should have a shear yield strength greater than 2013 psi. Since $s_{sy} = s_y/2$, the yield strength of the material should be greater than 4113 psi.

Using the von Mises criterion, the effective stress is

$$\sigma_{\text{eff}} = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}}$$

$$\sigma_{\text{eff}} = \sqrt{\frac{(4113 - 2056)^2 + (2056 - 0)^2 + (0 - 4113)^2}{2}} = 3562 \text{ psi}$$

Therefore, the yield strength of the selected material needs to be greater than 3562 psi.

Before we introduce the failure theories in Chapter 5, it is tempting to use the maximum principal stress to select the material. While considering only the maximum principle stress also results in $s_y \geq 4113$ psi, it should be noted that the solution $s_y \geq 4113$ psi is based on MSST with $\sigma_1 = 4113$ psi and $\sigma_3 = 0$. The above example also shows that when all three principal stresses are considered in the von Mises criterion, it yields a different result.

4–7 ANALYSIS OF COMPLEX LOADING CONDITIONS

The examples shown in this chapter involve relatively simple part geometries and loading conditions for which the necessary stress analysis can be performed using familiar methods of statics and strength of materials. If more complex geometries or loading conditions are involved, you may not be able to complete the required analysis to create the original stress element from which the Mohr's circle is constructed, as discussed in Section 4–4.

Consider, for example, a cast wheel for a high-performance racing car. The geometry would likely involve webs or spokes of a unique design connecting the hub to the rim to create a lightweight wheel. The loading would be a complex combination of torsion, bending, and compression generated by the cornering action of the wheel.

One method of analysis of such a load-carrying member would be accomplished by experimental stress analysis using strain gages or photoelastic techniques. The results would identify the stress levels at selected points in certain specified directions that could be used as the input to the construction of the Mohr's circle for critical points in the structure.

Another method of analysis would involve the modeling of the geometry of the wheel as a *finite-element model*. The three-dimensional model would be divided into several hundred small-volume elements. Points of support and restraint would be defined on the model, and then external loads would be applied at appropriate points. The complete data set would be input to a special type of computer analysis program called *finite-element analysis* (FEA). The output from the program lists the stresses and the deflection for each of the elements. These data can be plotted on the computer model so that the designer can visualize the stress distribution within the model. Most such programs list the principal stresses and the von Mises stress, eliminating the need to actually draw the Mohr's circle. (The application of von Mises stress is introduced in Chapter 5.) Several different finite-element analysis programs are commercially available for use on personal computers, on engineering work stations, or on mainframe computers. See Internet sites 1–5.

REFERENCE

1. Mott, Robert L. *Applied Strength of Materials*. 6th ed. Boca Raton, FL: CRC Press, 2017.

INTERNET SITES RELATED TO STRESS TRANSFORMATION

The following is a short list of the numerous companies that develop and provide finite-element analysis software for a wide variety of applications including static and dynamic structural analysis, thermal analysis, dynamic performance of mechanical systems, vibration analysis, computational fluid dynamics analysis, and other computer-aided engineering (CAE) capabilities.

1. ADINA R&D, Inc.
2. Autodesk Algor Simulation
3. ANSYS, Inc.
4. MSC Software, Inc.
5. NEi Software, Inc.

PROBLEMS

For the sets of given stresses on an element given in Table 4–2, draw a complete Mohr's circle, find the principal stresses and the maximum shear stress, and draw the principal stress element and the maximum shear stress element. Any stress components not shown are assumed to be zero.

31. Refer to Figure 3–20. For the shaft aligned with the *x*-axis, create a stress element on the bottom of the shaft just to the left of section *B*. Then draw the Mohr's circle for that element. Use $D = 0.50$ in.
32. Refer to Figure P3–48. For the shaft *ABC*, create a stress element on the bottom of the shaft just to the right of section *B*. The torque applied to the shaft at *B* is resisted at support *C* only. Draw the Mohr's circle for the stress element. Use $D = 1.50$ in.
33. Repeat Problem 32 for the shaft in Figure P3–49. Use $D = 0.50$ in.
34. Refer to Figure P3–50. For the shaft *ABC*, create a stress element on the bottom of the shaft just to the left of section *B*. The torque applied to the shaft by the crank is resisted at support *B* only. Draw the Mohr's circle for the stress element. Use $D = 50$ mm.
35. A short cylindrical bar having a diameter of 4.00 in is subjected to an axial compressive force of 75 000 lb and a torsional moment of 20 000 lb·in. Draw a stress element on the surface of the bar. Then draw the Mohr's circle for the element.
36. A torsion bar is used as a suspension element for a vehicle. The bar has a diameter of 20 mm. It is subjected to a torsional moment of 450 N·m and an axial tensile force of 36.0 kN. Draw a stress element on the surface of the bar, and then draw the Mohr's circle for the element.

TABLE 4–2 Given Stresses for Problems 1–30

Problem	σ_x	σ_y	τ_{xy}
1	20 ksi	0 ksi	10 ksi
2	−85 ksi	40 ksi	30 ksi
3	40 ksi	−40 ksi	−30 ksi
4	−80 ksi	40 ksi	−30 ksi
5	−120 ksi	40 ksi	−20 ksi
6	−120 ksi	40 ksi	20 ksi
7	60 ksi	−40 ksi	−35 ksi
8	120 ksi	−40 ksi	100 ksi
9	−100 MPa	0 MPa	80 MPa
10	−250 MPa	80 MPa	−110 MPa
11	50 MPa	−80 MPa	40 MPa
12	150 MPa	−80 MPa	−40 MPa
13	−150 MPa	80 MPa	−40 MPa
14	0 MPa	0 MPa	40 MPa
15	250 MPa	−80 MPa	0 MPa
16	50 MPa	−80 MPa	−30 MPa
17	400 MPa	−300 MPa	200 MPa
18	−120 MPa	180 MPa	−80 MPa
19	−30 MPa	20 MPa	40 MPa
20	220 MPa	−120 MPa	0 MPa
21	40 ksi	0 ksi	0 ksi
22	0 ksi	0 ksi	40 ksi
23	38 ksi	−25 ksi	−18 ksi
24	55 ksi	0 ksi	0 ksi
25	22 ksi	0 ksi	6.8 ksi
26	−4250 psi	3250 psi	2800 psi
27	300 MPa	100 MPa	80 MPa
28	250 MPa	150 MPa	40 MPa
29	−840 kPa	−335 kPa	−120 kPa
30	−325 kPa	−50 kPa	−60 kPa

CHAPTER
FIVE

DESIGN FOR DIFFERENT TYPES OF LOADING

The Big Picture
You Are the Designer
5–1 Objectives of This Chapter
5–2 Types of Loading and Stress Ratio
5–3 Failure Theories
5–4 Design for Static Loading
5–5 Endurance Limit and Mechanisms of Fatigue Failure
5–6 Estimated Actual Endurance Limit, s'_n
5–7 Design for Cyclic Loading
5–8 Recommended Design and Processing for Fatigue Loading
5–9 Design Factors
5–10 Design Philosophy
5–11 General Design Procedure
5–12 Design Examples
5–13 Statistical Approaches to Design
5–14 Finite Life and Damage Accumulation Method

THE BIG PICTURE — Design for Different Types of Loading

Discussion Map

- This chapter provides additional tools you can use to design load-carrying components that are safe and reasonably efficient in their use of materials.
- You must learn how to classify the kind of loading the component is subjected to: *static*, *repeated and reversed*, *fluctuating*, *shock*, or *impact*.
- You will learn to identify the appropriate analysis techniques based on the type of load and the type of material.

Discover

Identify components of real products or structures that are subjected to static loads.

Identify components that are subjected to equal, repeated loads that reverse directions.

Identify components that experience fluctuating loads that vary with time.

Identify components that are loaded with shock or impact, such as being struck by a hammer or dropped onto a hard surface.

Using the techniques you learn in this chapter will help you to complete a wide variety of design tasks.

For the concepts considered in this chapter, the big picture encompasses a huge array of examples in which you will build on the principles of strength of materials that you reviewed in Chapters 3 and 4 and extend them from the analysis mode to the design mode. Several steps are involved, and you must learn to make rational judgments about the appropriate method to apply to complete the design.

In this chapter, you will learn how to do the following:

1. Recognize the manner of loading for a part: Is it static, repeated and reversed, fluctuating, shock, or impact?
2. Select the appropriate method to analyze the stresses produced.

3. Determine the strength property for the material that is appropriate to the kind of loading and to the kind of material: Is the material a metal or a nonmetal? Is it brittle or ductile? Should the design be based on the yield strength, ultimate tensile strength, compressive strength, endurance limit, or some other material property?
4. Specify a suitable *design factor*, often called a *factor of safety*.
5. Design a wide variety of load-carrying members to be safe under their particular expected loading patterns.

The following paragraphs show by example some of the situations to be studied in this chapter.

An ideal *static load* is one that is applied slowly and is never removed. Some loads that are applied slowly and removed and replaced very infrequently can also be considered to be static. What examples can you think of for products or their components that are subjected to static loads? Consider load-carrying members of structures, parts of furniture pieces, and brackets or support rods holding equipment in your home or in a business or factory. Try to identify specific examples, and describe them to your colleagues. Discuss how the load is applied and which parts of the load-carrying member are subjected to the higher stress levels. Some of the examples that you discovered during **The Big Picture** discussion for Chapter 3 could be used again here.

Fluctuating loads are those that vary during the normal service of the product. They typically are applied for quite a long time so the part experiences many thousands or millions of cycles of stress during its expected life. There are many examples in consumer products around your home, in your car, in commercial buildings, and in manufacturing facilities. Consider virtually anything that has moving parts. Again, try to identify specific examples, and describe them to your colleagues. How does the load fluctuate? Is it applied and then completely removed each cycle? Or is there always some level of mean or average load with an alternating load superimposed on it? Does the load swing from a positive maximum value to a negative minimum value of equal magnitude during each cycle of loading? Consider parts with rotating shafts, such as engines or agricultural, production, and construction machinery.

Consider products that have failed. You may have identified some from **The Big Picture** discussion for Chapter 3. Did they fail the first time they were used? Or did they fail after some fairly long service? Why do you think they were able to operate for some time before failure?

Can you find components that failed suddenly because the material was brittle, such as cast iron, some ceramics, or some plastics? Can you find others that failed only after some considerable deformation? Such failures are called *ductile fractures*.

What were the consequences of the failures that you have found? Was anyone hurt? Was there damage to some other valuable component or property? Or was the failure simply an inconvenience? What was the order of magnitude of cost related to the failure? The answer to some of these questions can help you make rational decisions about design factors to be used in your designs.

It is the designer's responsibility to ensure that a machine part is safe for operation under reasonably foreseeable conditions. This requires that a stress analysis be performed in which the predicted stress levels in the part are compared with the *design stress*, or that level of stress permitted under the operating conditions.

The stress analysis can be performed either analytically or experimentally, depending on the degree of complexity of the part, the knowledge about the loading conditions, and the material properties. The designer must be able to verify that the stress to which a part is subjected is safe.

The manner of computing the design stress depends on the manner of loading and on the type of material. Loading types include the following:

Static
Repeated and reversed
Fluctuating
Shock or impact
Random

Material types are many and varied. Among the metallic materials, the chief classification is between *ductile* and *brittle* materials. Other considerations include the manner of forming the material (casting, forging, rolling, machining, and so on), the type of heat treatment, the surface finish, the physical size, the environment in which it is to operate, and the geometry of the part. Different factors must be considered for plastics, composites, ceramics, wood, and others.

This chapter outlines methods of analyzing load-carrying machine parts to ensure that they are safe. Several different cases are described in which knowledge of the combinations of material types and loading patterns leads to the determination of the appropriate method of analysis. It will then be your job to apply these tools correctly and judiciously as you continue your career.

YOU ARE THE DESIGNER

Recall the task presented at the start of Chapter 4, in which you were the designer of a bracket to hold a fabric sample during a test to determine its long-term stretch characteristics. Figure 4–2 showed a proposed design.

Now you are asked to continue this design exercise by selecting a material from which to make the two bent circular bars that are welded to the rigid support. Also, you must specify a suitable diameter for the bars when a certain load is applied to the test material. ■

5–1 OBJECTIVES OF THIS CHAPTER

After completing this chapter, you will be able to:

1. Identify various kinds of loading commonly encountered by machine parts, including *static*, *repeated and reversed*, *fluctuating*, *shock or impact*, and *random*.
2. Define the term *stress ratio* and compute its value for the various kinds of loading.
3. Describe the stress analysis process and the concept of failure theories for mechanical design.
4. Define the *maximum shear stress theory of failure* for design with ductile materials.
5. Define the *distortion energy theory*, also called the *von Mises theory* or the *Mises-Hencky theory* for design with ductile materials.
6. Define the *maximum normal stress theory*, the *Coulomb-Mohr Theory* and the *modified Mohr theory* for design with brittle materials under static loading.
7. Define the concept of *fatigue*.
8. Define the material property of *endurance limit* and determine estimates of its magnitude for different materials.
9. Recognize the factors that affect the magnitude of endurance limit.
10. Describe the *Soderberg* and the *Goodman methods* and apply them to the design of parts subjected to fluctuating stresses.
11. Define the term *design factor.*
12. Specify a suitable value for the design factor.
13. Consider *statistical approaches*, *finite life*, *fracture mechanics*, and *damage accumulation methods* for design.

5–2 TYPES OF LOADING AND STRESS RATIO

The primary factors to consider when specifying the type of loading to which a machine part is subjected are the manner of variation of the load and the resulting variation of stress with time. Stress variations are characterized by four key values, expressed here as normal stresses:

1. Maximum stress, σ_{max}
2. Minimum stress, σ_{min}
3. Mean (average) stress, σ_m
4. Alternating stress, σ_a (*stress amplitude*)

The maximum and minimum stresses are usually computed from known information by stress analysis or finite-element methods, or they are measured using experimental stress analysis techniques. Then the mean and alternating stresses can be computed from

$$\sigma_m = (\sigma_{max} + \sigma_{min})/2 \tag{5–1}$$

$$\sigma_a = (\sigma_{max} - \sigma_{min})/2 \tag{5–2a}$$

$$\sigma_a = (\sigma_{max} - \sigma_m) \tag{5–2b}$$

The behavior of a material under varying stresses is dependent on the manner of the variation. One method used to characterize the variation is called *stress ratio*. Two types of stress ratios that are commonly used are defined as follows:

$$\begin{aligned} \text{Stress ratio } R &= \frac{\text{minimum stress}}{\text{maximum stress}} = \frac{\sigma_{min}}{\sigma_{max}} \\ \text{Stress ratio } A &= \frac{\text{alternating stress}}{\text{mean stress}} = \frac{\sigma_a}{\sigma_m} \end{aligned} \tag{5–3}$$

Stress ratio R is used in this book.

Static Stress

When a part is subjected to a load that is applied slowly, without shock, and is held at a constant value, the resulting stress in the part is called *static stress.* An example is the load on a structure due to the dead weight of the building materials. Figure 5–1 shows a diagram of stress versus time for static loading. Because $\sigma_{max} = \sigma_{min}$, the stress ratio for static stress is $R = 1.0$.

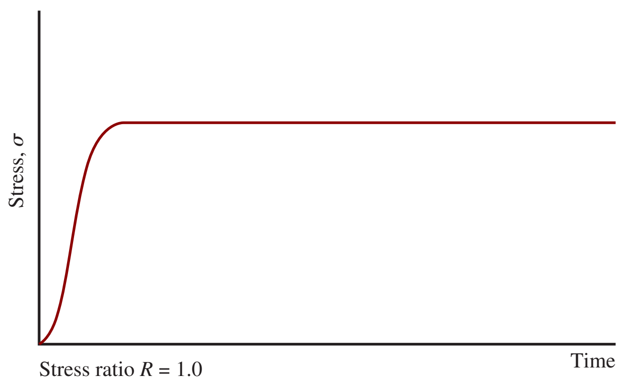


FIGURE 5–1 Static stress

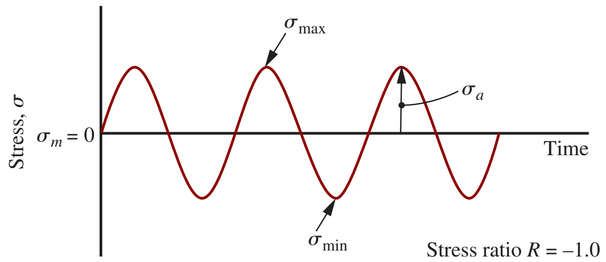


FIGURE 5–2 Repeated and reversed stress or pure oscillation

Static loading can also be assumed when a load is applied and is removed slowly and then reapplied, if the number of load applications is small, that is, under a few thousand cycles of loading.

Repeated and Reversed Stress—Pure Oscillation

A stress reversal occurs when a given element of a load-carrying member is subjected to a certain level of tensile stress followed by the *same level* of compressive stress. If this stress cycle is repeated many thousands of times, the stress is called *repeated and reversed* or *pure oscillation*. Figure 5–2 shows the diagram of stress versus time for repeated and reversed stress. Because $\sigma_{min} = -\sigma_{max}$, the stress ratio is $R = -1.0$, and the mean stress is zero.

An important example in machine design is a rotating circular shaft loaded in bending such as that shown in Figure 5–3. In the position shown, an element on the bottom of the shaft experiences tensile stress while an element on the top of the shaft sees a compressive stress of equal magnitude. As the shaft is rotated 180° from the given position, these two elements experience a complete reversal of stress. Now if the shaft continues to rotate, all parts of the shaft that are in bending see repeated, reversed stress. This is a description of the classic loading case of *repeated and reversed bending*.

This type of loading is often called *fatigue loading*, and a machine of the type shown in Figure 5–3 is often used to test the fatigue behavior of materials. The device shown is called the *R. R. Moore fatigue test device* and the material property thus measured is called *endurance limit* and this property is discussed in detail in Section 5–5. The shaft is supported by a bearing at each end while a yoke is supported on bearings. A known loading is applied to the yoke resulting in two concentrated loads being applied; one at each bearing that supports the yoke. Note from the shearing force and bending moment diagrams that this type of loading provides uniform bending moment between the yoke arms while the shearing force is zero. Thus pure bending occurs in the test section. The shaft is machined to precise dimensions with the middle portion having a very gradual taper down to a small diameter. That diameter is typically 0.300 in. With the gradual taper, the stress concentration factor is virtually 1.0. Furthermore, the shaft is polished to a fine surface finish so that machining marks do not affect the stress

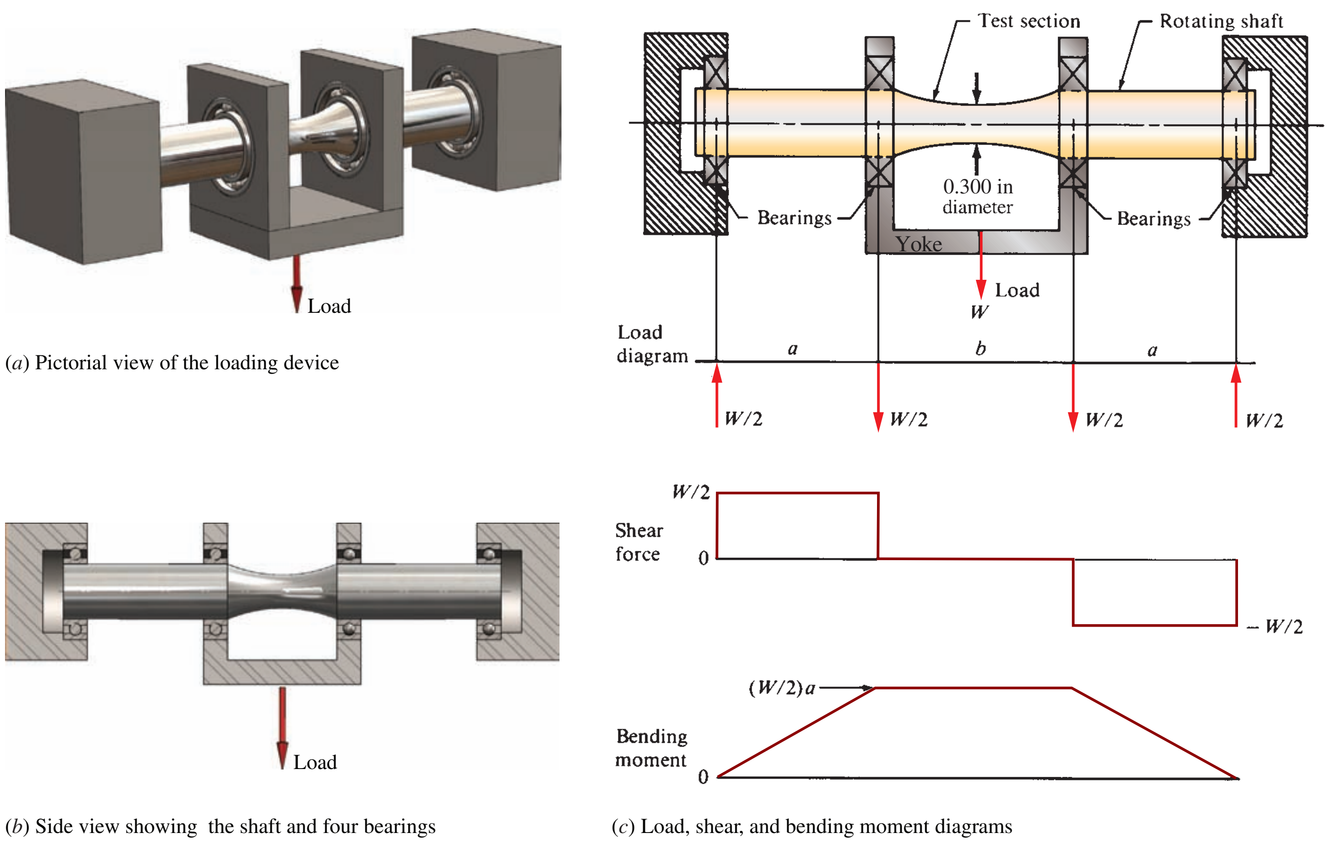


FIGURE 5–3 R. R. Moore fatigue test device; example of reversed bending

levels in the bar. The shaft is rotated by an electric motor while the system counts the number of revolutions. It also has a device to sense when the specimen breaks so that there is a known relationship between the stress level and the number of cycles to failure.

Actually, reversed bending is only a special case of fatigue loading, since any stress that varies with time can lead to fatigue failure of a part. Many materials test laboratories are using computer-controlled, repeated and reversed axial loading instead of rotating bending to acquire fatigue strength data. It is described later that there are differences between these two methods in regard to the strength values obtained. It is essential that care be exercised to determine what type of stress is used to measure the fatigue strength when using published data.

Fluctuating Stress—Pulsating Stress

When a load-carrying member is subjected to an alternating stress with a nonzero mean, the loading produces *fluctuating stress*, sometimes called *pulsating stress*. Figure 5–4 shows four diagrams of stress versus time for this type of stress. Differences among the four diagrams occur in whether the various stress levels are positive (tensile) or negative (compressive). ***Any varying stress with a nonzero mean is considered a fluctuating stress***. Figure 5–4 also shows the possible ranges of values for the stress ratio R for the given loading patterns.

A special, frequently encountered case of fluctuating stress is *repeated*, *one-direction stress*, or pure pulsating stress, in which the load is applied and removed many times. As shown in Figure 5–5, the stress varies from zero to a maximum with each cycle. Then, by observation,

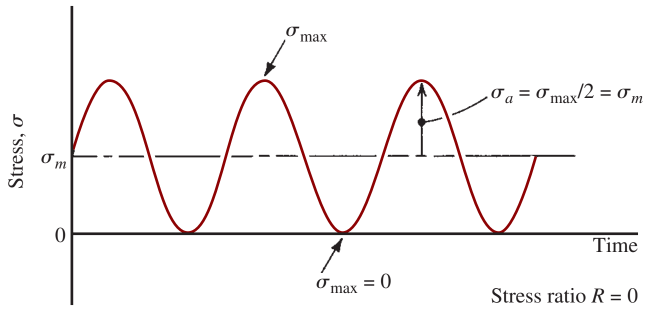


FIGURE 5–5 Repeated, one-direction stress, a special case of fluctuating stress or pure pulsating stress

$$\sigma_{min} = 0$$
$$\sigma_m = \sigma_a = \sigma_{max}/2$$
$$R = \sigma_{min}/\sigma_{max} = 0$$

An example of a machine part subjected to the more general nature of fluctuating stress of the type shown in Figure 5–4(a) is shown in Figure 5–6, in which a reciprocating cam follower feeds spherical balls one at a time from a chute. The follower is held against the rotating eccentric cam by a flat spring loaded as a cantilever. Part (a) of the figure shows the entire layout of the ball feed device and part (b) shows the cross section of the flat spring. Parts (c) and (d) show two views of just the cam, follower, and flat spring. When the follower is farthest

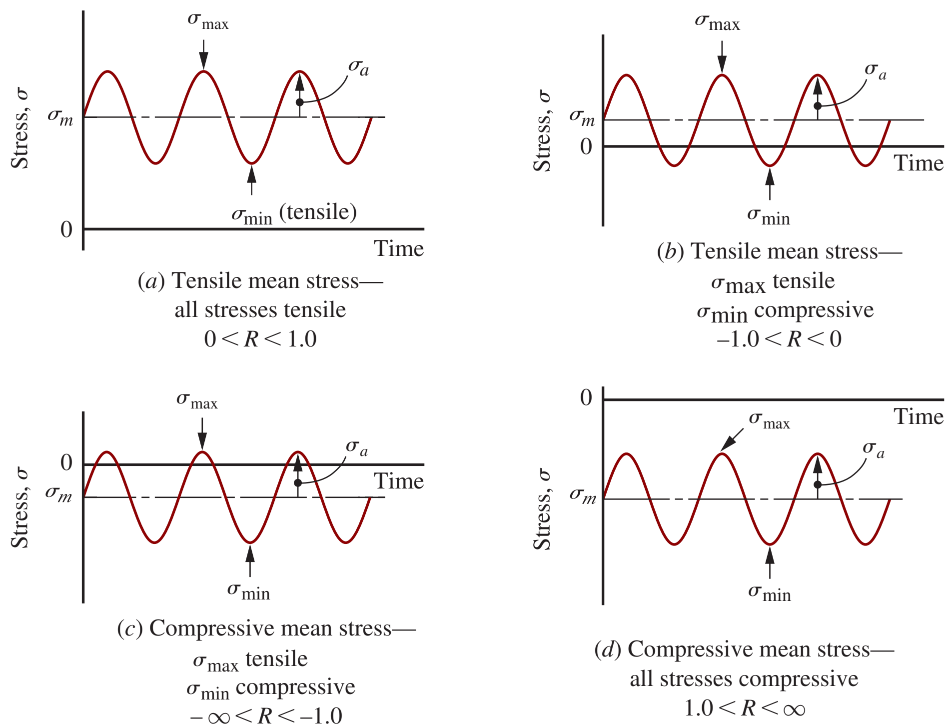


FIGURE 5–4 Fluctuating stresses—pulsating stresses

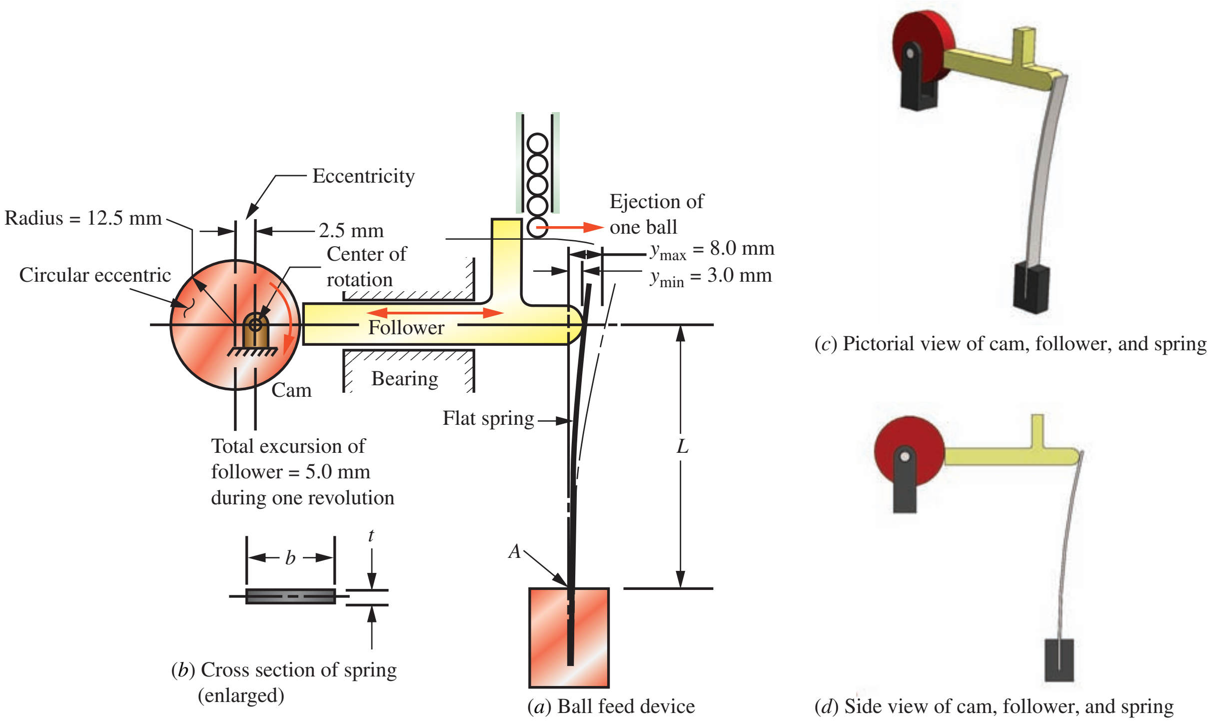


FIGURE 5–6 Example of cyclic loading in which the flat spring is subjected to fluctuating stress

to the left, the spring is deflected from its free (straight) position by an amount $y_{min} = 3.0$ mm. When the follower is farthest to the right, the spring is deflected to $y_{max} = 8.0$ mm. Then, as the cam continues to rotate, the spring sees the cyclic loading between the minimum and maximum values. Point A at the base of the spring on the convex side experiences the varying tensile stresses of the type shown in Figure 5–4(a). Example Problem 5–1 completes the analysis of the stress in the spring at point A.

Example Problem 5–1 For the flat steel spring shown in Figure 5–6, compute the maximum stress, the minimum stress, the mean stress, and the alternating stress. Also compute the stress ratio, R. The length L is 65 mm. The dimensions of the spring cross section are $t = 0.80$ mm and $b = 6.0$ mm.

Solution

Objective: Compute the maximum, minimum, mean, and alternating tensile stresses in the flat spring. Compute the stress ratio, R.

Given: Layout shown in Figure 5–6. The spring is steel: $L = 65$ mm.

Spring cross-sectional dimensions: $t = 0.80$ mm and $b = 6.0$ mm.

Maximum deflection of the spring at the follower $= 8.0$ mm.

Minimum deflection of the spring at the follower $= 3.0$ mm.

Analysis: Point A at the base of the spring experiences the maximum tensile stress. Determine the force exerted on the spring by the follower for each level of deflection using the formulas from Table A14–2, Case (a). Compute the bending moment at the base of the spring for each deflection. Then compute the stresses at point A using the bending stress formula, $\sigma = Mc/I$. Use Equations (5–1), to (5–3) for computing the mean, alternating stresses, and R.

Results: Case (a) of Table A14–2 gives the following formula for the amount of deflection of a cantilever for a given applied force:

$$y = PL^3/3EI$$

Solve for the force as a function of deflection:

$$P = 3EIy/L^3$$