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Lab Course
Scientific Computing

Worksheet 2

distributed: Tue., 24.11.2020
due: Sun., 06.12.2020, midnight (submission on the Moodle page)
oral examination: Tue., 08.12.2020 (exact time slots announced on the Moodle page)

We examine the following ordinary differential equation describing the dynamics of the population of a certain species:

$$\dot{p} = \left(1 - \frac{p}{10}\right) \cdot p \tag{1}$$

with initial condition

$$p(0) = 1. \tag{2}$$

The analytical solution is given by

$$p(t) = \frac{10}{1 + 9e^{-t}}.$$

We use this rather simple equation with a known exact solution to examine the properties of different numerical methods.

a) Use `matlab` to plot the function $p(t)$ in a graph.

b) Consider a general initial value problem

$$\dot{y} = f(y), \quad y(0) = y_0.$$

Implement the following explicit numerical methods

1) explicit Euler method,
2) method of Heun,
3) Runge-Kutta method (fourth order)

as a `matlab` function depending on the right hand side $f(y)^1$, the initial value y_0, the timestep size δt and the end time t_{end}. The output of the function shall be a vector containing all computed approximate values for y.

c) For each of the three methods implemented, compute approximate solutions p_k for equation (1) with initial conditions (2), end time $t_{end} = 5$. We want to investigate the behavior of the approximate solutions for different time step sizes $\delta t = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$.

i) For each of the three methods, create a separate figure. Each figure contains the four computed solutions for the four different timesteps δt. Add also the analytical solution for reference in each figure.

ii) For each case, compute the approximation error

$$E = \sqrt{\frac{\delta t}{t_{end}} \sum_k \left(p_k - p_{k,exact}\right)^2},$$

where p_k denotes the approximation, $p_{exact,k}$ the exact solution at $t = \delta t \cdot k$. Write down the errors in the tables below.

iii) For each of the three methods, determine the factor by which the error E is reduced if the step size δt is halved. Write down the results in the tables below.

iv) In general, we do not know the exact solution of an equation we have to solve numerically (otherwise, we would not have to use a numerical method, in fact). To anyhow guess the accuracy of a method, we can use the difference between our best approximation (the one with the smallest time step δt) and the other approximations:

$$\tilde{E} = \sqrt{\frac{\delta t}{t_{end}} \sum_k \left(p_k - p_{k,best}\right)^2},$$

[1] Use a *MATLAB Function Handle* as argument.

where p_k denotes the approximation with time step δt and $p_{best,k}$ the best approximation at $t = \delta t \cdot k$.

Compute $\tilde{E}$ for all time steps and methods used. Write down the results in the tables below and compare them to the exact error E.

explicit Euler method ($q = 1$)				
δt	1	0.5	0.25	0.125
error	0.7783	0.3827	0.1898	0.0945
error red.	0.4917	0.4960	0.4978	–
error app.	0.6868	0.2888	0.0954	0

method of Heun ($q = 2$)				
δt	1	0.5	0.25	0.125
error	0.2749	0.0717	0.0189	0.0049
error red.	0.2609	0.2633	0.2585	–
error app.	0.2701	0.0668	0.0140	0

Runge-Kutta method ($q = 4$)				
δt	1	0.5	0.25	0.125
error	0.0079	5.2249×10^{-4}	3.4554×10^{-5}	2.2331×10^{-6}
error red.	0.0662	0.0661	0.0646	–
error app.	0.0079	5.2024×10^{-4}	3.2313×10^{-5}	0

Questions:

Q1 By which factor is the error reduced for each halfing of δt if you apply a

1) first order ($O(\delta t)$),

2) second order ($O\left(\delta t^2\right)$),

3) third order ($O\left(\delta t^3\right)$),

4) fourth order ($O\left(\delta t^4\right)$)

method.

Q2 For which integer q can you conclude that the error of the

1) explicit Euler method,

2) method of Heun,

3) Runge-Kutta method (fourth order)

behaves like $O\left(\delta t^q\right)$?

Q3 Is a higher order method always more accurate than a lower order method (for the same stepsize δt)?

Q4 Assume you have to compute the solution up to a certain prescribed accuracy limit and that you see that you can do with less time steps if you use the Runge-Kutta-method than if you use Euler or the method of Heun. Can you conclude in this case that the Runge-Kutta method is the most efficient one of the three alternatives?