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The structure of paradoxes in a logic of sentential operators

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Abstract Any language $\mathcal{L}$ of classical logic, of first or higher order, is extended with sentential quantifiers and operators. The resulting language $\mathcal{L}^+$, capable of self-reference without arithmetic or syntax coding, can serve as its own metalanguage. The syntax of $\mathcal{L}^+$ is represented by directed graphs and its semantics, coinciding with the classical one on $\mathcal{L}$, utilizes the graph-theoretic concepts of kernels and semikernels. The former provide explosive semantics, generalized by the latter to the situations where expressed paradoxes do not cause explosion. This distinguishes paradoxes from contradictions, which do yield explosion. Paradoxes arise only at the metalevel due to specific definitions of the operators but can be avoided, so that $\mathcal{L}^+$, although capable of expressing paradoxes, is free from them. For such an extension $\mathcal{L}^+$ of any FOL language $\mathcal{L}$ and its non-explosive semantics, a complete reasoning is obtained by extending classical Gentzen's system LK with two rules for sentential quantifiers. Adding (cut) yields a complete system for the explosive semantics. The novel semantics and self-referential capabilities seem promising for further extension of FOL towards languages capable also of expressing consistently own syntax and truth theory.

Keywords sentential operators, semantic and intensional paradoxes, non-explosive logic, (semi)kernels of digraphs

1 Introduction

Arithmetic with the arithmetized syntax can serve as a base syntax theory over which truth theories can be formulated. We let AST stand for first-order arithmetic with an effective coding of expressions of its language augmented by a truth predicate, $\mathcal{T}$, on natural numbers. For each natural number n, if n is the code of an expression ϕ, let $\ulcorner\phi\urcorner$ be the numeral of n in the language. Convention T – instances of the schema $\mathcal{T}(\ulcorner\phi\urcorner) \leftrightarrow \phi$, for all sentences ϕ – provides *prima facie* desirable truth-theoretic principles. However, as Tarski has shown, it cannot hold unrestrictedly in a classical theory, on pain of inconsistency [43]. Since one can only approximate it, while even maximal consistency is not a reliable guide [23], search for adequate restrictions on Convention T has attracted a lot of attention, producing various proposals and impressive results.

This paper, however, lets the problems of syntax and truth theory loom in the background, at most hinting towards a different approach which might possibly arise from the presented model of self-reference and paradox analysis. Its main novelty is replacing predicates on coded syntax by sentential operators, which appear to be less paradox prone. Convention T, for instance, reflects the intended application of predicate $\mathcal{T}$ as a meta-predicate on sentences, identifying the (codes of the) true ones. Unrestricted, it gives only one form of paradoxes arising with such meta-predicates modelled by predicates on the arithmetized syntax. When the latter express basic modal notions, elements of temporality, or just negation, the diagonalization lemma yields paradoxes also without

Convention T [11, 18, 24, 36], while the corresponding paradoxes do not arise with the operator versions of such predicates. Certainly, operators handling enough of syntax, especially substitution, bring paradoxes back [15,29], but they act then on the open formulas, which must be left for future work. Here, operators act only on sentences (closed formulas). The operator approach has proven fruitful in modal logics, with its philosophical advantages over predicates reviewed in [17]. The paper gathers such indications into a general logic of sentential operators, LSO, neither restricted to modalities nor relying on Kripke semantics. Unlike more philosophical discussions, e.g., [25,41], it provides a fully formal account of general operators. Its language model differs from AST's in several ways, which are reviewed below along the general introduction of LSO's central features.

The main difference concerns the relation between the object-language and the metalanguage. In AST, coding of the syntax turns the latter into a subset of the former. In LSO, by an object-language we mean any classical propositional, first or higher order language, and by its interpretation any standard classical interpretation. Any such language $\mathcal{L}$ can be extended with sentential quantifiers and operators to the full language $\mathcal{L}^+$. Occurrences of the operators mark then what we view as the metalanguage of $\mathcal{L}^+$ (not only of $\mathcal{L}$). For an operator K and sentence S, a sentential atom, *s-atom*, $K(S)$ is an atomic sentence of the metalanguage about S, $K(K(S))$ is such an atomic sentence about $K(S)$, etc. (The paper is thus writen in the meta-metalanguage. S-atoms and atoms of $\mathcal{L}$, *o-atoms*, form the set of all atoms of $\mathcal{L}^+$.)

The only requirement on the operators is that they are boolean-valued functions on sentences. If $K(S)$ is true, we can read it as K being true about S, as S being said or known, and such choices may require additional axioms, which only specialize the general features of LSO. The operators need not be truth-functional and can treat arguments purely syntactically, e.g., $K(S)$ may be true exactly when S is a universal sentence in prenex normal form. This opens a way towards a theory of syntax, as well as to modelling intensionality and modalities, but these issues are not pursued here. LSO is an hyperintensional logic only in so far as the operators can be opaque, failing to preserve logical equivalence of arguments, but it neither provides any intensional semantics nor considers the status of propositions or propositional attitudes. Propositions appear at most as mere sentences, while examples blur easily the borders between "says ϕ", "assumes ϕ", "claims ϕ", etc. The significant distinction is that between the statements with and without the operators, between the metalevel and the object-level. Modalities, attitudes or intensions can be handled by further axiomatizations of operators. Nonetheless, paradoxes of the semantic and intensional character arise and are treated in LSO in the same way. The liar, saying only "Every sentence I am saying now is false", is significantly different neither from one not believing any of one's beliefs nor from a club whose members are all people not belonging to any club. Semantic and intensional paradoxes are caused by the same patterns, which will be noted in the text.

One of the consequences of the formal distinction between the object-language and the metalanguage is the difference between contradictions and paradoxes. Informally, we may distinguish sentences which always evaluate to false from ones which look plausibly but lead to contradictions. Formalized in classical logic, also the latter are just contradictions, but LSO retains this distinction. Unlike contradictions, paradoxes possess some models. Karen can claim John to say only true things while John is accusing Karen of always lying. The implied incompatibility does not preclude the event. In this sense, paradoxes can occur in the world – perhaps, at the linguistic metalevel, but that is part of the world, too. This also distinguishes them from contradictions, which cannot occur empirically. One can say S and not-S, but simultaneously saying S and not saying S is hardly acceptable. Even the dialetheists embrace only some contradictions, like the liar lying and not lying, which occurs at the metalevel, predicating contrary properties to the liar's statement. Opting thus for classical logic, with no contradictions in the world of objects, LSO admits them as consequences of paradoxes, arising only in the metalanguage.

In LSO, as for Tarski, *"[t]he appearance of an antinomy is [...] a symptom of disease"* [44]. It is caused by contagious valuations of s-atoms, i.e., definitions of the operators.[1] Reading $K(S)$ as Karen saying sentence S, let Kl abbreviate $\forall\phi(K\phi \to \neg\phi)$, stating falsehood of everything

[1] S-atoms express applications of operators to arguments, so their valuations define the operators. 'Definitions of operators' refer to such valuations, even if they are undefinable in the language of LSO.

Karen says. Karen saying this, $K(Kl)$, is intuitively not paradoxical, if she says also something true. However, Karen saying only Kl (and nothing true) implies Kl and $\neg Kl$, witnessing to the paradox which might disappear if Karen said other things. In LSO, this paradox arises when K is true about Kl and false about all other sentences.

As the example suggests, besides certain definitions of operators, paradox requires also sentential quantification. Stern observes that *"in the predicate setting the liar-like paradoxes depend crucially on the properties of the truth predicate or the modal predicates, whereas in the operator setting it is quantification simpliciter that leads to paradox"* [41]. We prove that extending classical logic with sentential quantifiers alone, or with operators alone, does not lead to paradoxes. Only adding both does and then it is not quantification simpliciter which leads to paradox, but only in combinations with certain definitions of the operators.

Hearing a paradox, one recognizes the implied contradiction but continues normal reasoning unaffected. This natural non-explosiveness is present in LSO in a specific way.

Karen's saying above implies that she is lying and not lying but neither snow ceases to be white nor anything about what John is saying follows. Paradoxes can be seen as such claims in the metalanguage causing the impossibility of evaluating consistently some sentences, but not leading to any explosion. In LSO, the possibility of Karen saying only Kl is reflected by the models realizing such a situation, where only some contradictions follow, e.g., $Kl \wedge \neg Kl$, but not others. This element of relevance reflects the informal way of identifying a paradox by showing such a 'right contradiction'. We do not explain the liar by deriving from it that snow is not white. The 'right contradiction' is that he is lying and not lying. In LSO, with the valuation of s-atoms where only $K(Kl)$ is true, basically only sentence Kl cannot be assigned any coherent value, while most others can. (Coherence will be defined as a local consistency, which may fail to extend to a consistent valuation of all sentences of the language.) This gives a precise hint which s-atoms need changes to resolve the paradox, here, atoms K affecting truth of Kl. Otherwise, the model can be extended by John saying (or not) whatever he likes, while all usual intepretations of the object-language remain unchanged. Formally, a paradox in LSO consists of a set of sentences possessing a model, namely, a valuation of atoms satisfying these sentences, which cannot be extended to any consistent valuation of all sentences of the metalanguage, due to the implied contradictions.

LSO models of theories involving paradoxes are in this way partial. We say that a model of the liar Karen, which does not provide a value for Kl, does not *cover* Kl. (A model can not be arbitrarily extended to cover desired sentences. In this case, Karen's model does not have any extension covering Kl.) The notion has no counterpart in classical logic, but could be compared to a colloquial argument which is classically consistent, provided that one ignores questions about Kl. Total models, covering the whole language, i.e., interpreting coherently all sentences, are special cases when the theory, with all its metalevel claims, is consistent. At this point, we can only signal that both are formalized using digraphs with sentences as vertices. So called *semikernels* [27] represent partial coherent valuations and non-explosive semantics. Simplifying, we can think here of a semikernel as a subset of sentences satisfying all its members and (vacuously) all sentences it does not cover. A contradiction can be satisfied only vacuously, by a model not covering it. This gives a weak form of paraconsistency. For instance, sound reasoning derives both Kl and $\neg Kl$ from Karen saying only Kl, without deriving anything about John's statements or any object-level contradictions. (Karen saying only Kl has namely partial models where John makes arbitrary or no statements, while all models covering object-language are classical.) A paradox in LSO possesses thus models which require satisfaction of some contradiction and, therefore, can not be extended to valuations of the whole language. Such total valuations are captured by *kernels*, i.e., semikernels covering all sentences of the language. (Kernel is the original concept from [46].) Requiring a consistent valuation of all sentences, they yield the explosive semantics. Valid sentences must be satisfied also by such models, so the logic remains two-valued and retains all classical tautologies. Classical contradictions can be satisfied vacuously but have no models covering them.

The close connections between the non-explosive (partial) and the explosive (total) semantics are reflected in the reasoning. Surprisingly, reasoning about the partial models allows all classical rules of the sequent calculus. The reasoning system, denoted (with the sans-serif font) LSO, extends Gentzen's classical sequent system LK, [14, 45], with two rules for s-quantifiers. Sentential operators and s-quantification bring a flavour of higher order, but the operator form of the former and the substitutional interpretation of the latter allow LSO to be sound and complete for the non-explosive semantics of the extension $\mathcal{L}^+$ of any first-order language $\mathcal{L}$ (denoted by FOL^+). The fact that LSO contains classical LK, adding only two rules for s-quantification, may justify calling it "classical". However, it works for the non-explosive semantics and LSO can be seen as classical logic exploding from contradictions, but not from paradoxes. Derivation of all sentences from a paradox, via the contradictions it implies, is prevented by the inadmissibility of (cut) $\frac{\Gamma\vdash\Delta,A \quad A,\Gamma\vdash\Delta}{\Gamma\vdash\Delta}$ at the metalevel. The explosive, "fully classical", semantics is captured by reasoning in LSO with unrestricted (cut). It allows to conjoin a contradiction, implied by a paradox, with all statements, implied in LSO by any contradiction. In this way also (cut) reflects the difference between contradictions and paradoxes, or between the object-level, where it is admissible, and the metalevel, where it makes paradoxes explode. LSO exemplifies thus non-transitive systems, e.g., [9,38,39] but unlike there, (cut) here merely turns a non-explosive logic into an explosive one.

Unlike valuations of o-atoms, which determine unique values of all object-language sentences, some valuations of s-atoms yield paradoxes which, implying contradictions, preclude any (consistent) valuation of all sentences of the language. This is one of the features distinguishing the object-language from the metalanguage, another being that some valuations of all atoms may still leave some sentences of the metalanguage, like the truth-teller, without any definite value. Viewing such phenomena as defects of the model seems to reflect the conflation of the metalanguage and the object-language, inherent in coding the former in the latter. In LSO, just as in natural language, they only manifest differences between the two language levels. Moreover, they can be avoided by using the language (formally, valuating s-atoms) in a cautious way.

This poses a specific challenge. In the language of arithmetic with truth predicate, self-reference arises in a specific theory (e.g., Robinson arithmetic) via the diagonalization lemma. In LSO it is a feature of the language. An s-quantifier in a sentence ranges over all sentences, including this very sentence, e.g., $\forall\phi(K\phi \to \neg\phi)$ has instance $K(\forall\phi(K\phi \to \neg\phi)) \to \neg\forall\phi(K\phi \to \neg\phi)$. Self-reference, enabling an expression of the liar, threatens consistency in both cases. AST retains it by various restrictions on Convention T, while LSO by the adequate valuations of atoms. The question thus arises whether the mere language of LSO, not only some theory over it, is consistent, whether any valuation of atoms extends to a valuation of all sentences of the language respecting its semantic restrictions. The proof that this is the case establishes non-trivial Theorem 3.20.

Consistency of the language modelled in LSO reflects the informal intuition that paradoxes do not reside in the language as such, but in the ways we use it and talk about it. Our model of paradoxes complies with the diagnosis from [8], according to which they arise from taking for granted some assumptions that, on a closer analysis, display a contradiction. LSO embodies this idea by paradoxes arising exactly due to the way operators are defined by valuations of s-atoms. So understood definitions are far from logically innocent. Unlike nominal definitions or definitional extensions, specific definitions of operators amount to specific axioms – assumptions or claims (the difference between them being here inconsequential).

This view opposes the one originating with Tarski's diagnosis and voiced occasionally in recent years, according to which natural language simply is inconsistent [3,5,13,28]. That opinion seems to arise from perceiving predicates on arithmetized syntax as an adequate model of the (natural) metalanguage, to which the inconsistency of unrestricted Convention T is then ascribed. In LSO paradox relies on an assumption or a claim, represented by the definition of the involved operators, and can be avoided by avoiding the unfortunate claims. If nobody claims to be (always) lying, no liar paradox results and the language remains consistent.

To this, however, one wants to object! Statements like L "This sentence is false" do not have to be claimed, they cause trouble by simply being there. Well, by simply being, the liar L makes a claim. Usually, it is represented by $L \leftrightarrow \neg\mathcal{T}(L)$ or simply $L \leftrightarrow \neg L$, expressing the pretence to truth of this usatisfiable equivalence, the liar's semantic claim. (We might say "truth condition" if this expression did not carry too heavy connotations.) The liar L and its claim "L is true iff it is false" are distinct sentences and the paradox amounts to the nonexistence of a boolean value for the former satisfying the latter. Casual interaction dismisses the liar so easily, in spite of its intriguing challenge, because no matter the truth-value of the pronounced sentence, the claim hidden behind it is false. Its analysis, starting from the assumption that L must be true or false, arrives in each case at a contradiction. In a way, LSO turns such an analysis around, starting with the inadmissibility of the claim, which renders the truth-value of the liar insignificant. Formally, the liar in LSO is Karen claiming only to be always lying, which makes it quite explicit that the only cause of trouble is an implausible claim – a valuation of s-atoms which can not be extended consistently to the full language.

The view of self-reference and paradoxes differs thus significantly from that arising in AST but a comparison to the truth theories over AST must wait for extensions of LSO with syntax and truth theory. The differences, concerning the present elements, can be summarized as follows:

1. Self-reference in LSO is a feature of the language; in AST it is obtained by specific theories.
2. In AST, coding of syntax turns metalanguage into a subset of the object-language. In LSO, operators distinguish syntactically the two without any coding. Semantically, they differ, e.g., in that a valuation of all atoms determines a valuation of all sentences of the object-language, but may yield paradoxes or fail to assign any truth-value to some sentences of the metalanguage.
3. Paradoxes in AST arise from extending Convention T to too many sentences. In LSO, they arise only in metalanguage due to some valuations of s-atoms, that is, definitions of operators.
4. Paradoxes in AST are only special cases of contradictions, distinguished at most from outside of the system. In LSO contradictions have no (covering) models, while paradoxes have some, though only partial ones, which can not be extended to the full language.
5. This enables non-explosive LSO reasoning in the presence of paradoxes, incorporating classical system LK. Limitations of (cut) reflect here the unwillingness of informal reasoning to conclude from the liar anything except that he is lying and not lying. Unrestricted (cut) extends this to the explosive reasoning, with the liar entailing everything, as he does in AST.
6. The explosive semantics of LSO is a special case of the non-explosive one, and the two coincide for consistent theories. There is, of course, no such distinction in AST.
7. LSO appears to be capable of including modal operators, though this requires further work. In AST, modalities can be represented only by predicates.
8. Natural language, seen through the glasses of LSO, is consistent and capable of expressing paradoxes. In AST, consistency requires limitations of Convention T that avoid paradoxes by preventing their expression and seem to have no counterpart in natural language.[2]

Sections 2-6 contain the main presentation, while appendix in Sections 7-8 all proofs and some lesser technicalities. Section 2 introduces an extension of any classical, first or higher order, language with sentential quantifiers and operators, as well as the reasoning system LSO for such an extension of any first-order language. Section 3 presents the semantics based on the language graphs, their kernels and semikernels, and discusses briefly relations to the classical semantics. It contains the main results of the paper: consistency of the full language $\mathcal{L}^+$ with sentential quantifiers and operators, as well as soundness and completeness of LSO for the non-explosive semantics of FOL$^+$. Section 4 relates the explosive and the non-explosive semantics and shows that adding (cut) to LSO yields sound and complete system LSO^c for the explosive semantics. Moreover, if a contradiction follows from a theory in LSO^c, some follows also in LSO: reasoning with (cut) does not introduce any new paradoxes but only makes paradoxes explode like usual contradictions. We comment the apparent similarity and the actual differences of (reasoning in) LSO and the non-transitive systems, e.g., [9, 38, 39]. Section 5 analyzes a series of examples from the literature and signals some extensions of LSO. Section 6 summarizes the encountered features of paradoxes.

2 Reasoning about sentences

A classical (propositional, first or higher order) language $\mathcal{L}$ is extended first to $\mathcal{L}^\Phi$ by adding sentential variables, *s-variables* Φ, which can be *s-quantified*, so that, e.g., $\forall\phi\forall x(A(x) \vee \phi)$ is a sentence. To this we add *operators*, applicable to sentences, so that also $\forall x\forall\phi(A(x) \vee \phi \vee P(\phi))$

[2] Rejecting $L \leftrightarrow \mathcal{T}(\ulcorner L \urcorner)$ for the sentence L, satisfying $L \leftrightarrow \neg\mathcal{T}(\ulcorner L \urcorner)$ by the diagonalization lemma, makes $\ulcorner L \urcorner$ denote *in the resulting theory* not L, which it is coding, but rather $\neg L$, by the latter equivalence and reading of $\mathcal{T}$ as the truth predicate. The liar disappears from the sentences which the theory can express or address.

is a sentence, for an operator P. (Below, we ignore the propositional case which can be obtained by obvious simplifications.) With X denoting a set of *o-variables* (used by $\mathcal{L}$) and $\mathbf{C}$ a set of s(entential)-constants, formulas of $\mathcal{L}^+$ are defined inductively:

1. Every $\mathcal{L}$ formula is an $\mathcal{L}^+$ formula.
2. All s-constant and s-variables, $\mathbf{C} \cup \Phi$, are $\mathcal{L}^+$ formulas.
3. If K is an n-ary operator and $\phi_1...\phi_n$ are $\mathcal{L}^+$ formulas, then so is $K(\phi_1...\phi_n)$.
4. If ϕ, ψ are $\mathcal{L}^+$ formulas, then so are $\neg\phi$ and $\phi \wedge \psi$.
5. If $x \in X$, $\phi \in \Phi$ and F is an $\mathcal{L}^+$ formula, then $\forall x F$ and $\forall \phi F$ are $\mathcal{L}^+$ formulas.

A *sentence* is a formula without any free (object or sentential) variables. S-constants $\mathbf{C}$ can serve as sentence names. In point 3, the arguments of operators could also be $\mathcal{L}$ terms, but we simplify the presentation ignoring their standard treatment. This point admits arbitrary formulas, but we require arguments to the operators to be sentences. (More general cases, admitted by the semantic definitions, can be ignored at first reading.) Atoms are divided into

(a) o-atoms $\mathbf{A}_X$, the atoms of $\mathcal{L}$, and

(b) s-atoms, $\mathbf{A}^\circ_X = \mathbf{A}^+_X \setminus \mathbf{A}_X$, s-constants $\mathbf{C}$ or formulas of type 3 above (not s-variables Φ).

For a set M, by $\mathbf{T}_M$ we denote the free algebra of $\mathcal{L}$ terms over M, by $\mathbf{S}_M/\mathbf{S}^+_M$ all $\mathcal{L}/\mathcal{L}^+$ sentences over $\mathbf{T}_M$, and by $\mathbf{S}/\mathbf{S}^+$ all $\mathcal{L}/\mathcal{L}^+$ sentences. Superscript $\_^\circ$ marks the metalevel, added to the object-level $\mathcal{L}$ and yielding the resulting extension $\_^+$, e.g., $\mathcal{L}^\circ = \mathcal{L}^+ \setminus \mathcal{L}$, $\mathbf{S}^\circ_M = \mathbf{S}^+_M \setminus \mathbf{S}_M$, etc.

By FOL$^+$ we denote any language of FOL extended as above. Reasoning system LSO for FOL$^+$, given below, extends LK with two rules for s-quantifiers. The basic syntax uses only $\{\wedge, \neg, \forall\}$, with other connectives and $\exists$, and rules for them, defined in the classical way. Sequents, written $\Gamma \Rightarrow \Delta$, are formed from countable sets $\Gamma \cup \Delta$ of $\mathcal{L}^+$ formulas. $\Gamma \vdash \Delta$ denotes provability of $\Gamma \Rightarrow \Delta$ in LSO.

(Ax) $\quad \Gamma \vdash \Delta \quad$ for $\quad \Gamma \cap \Delta \neq \emptyset$

$(\neg_L)\ \dfrac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta} \qquad (\neg_R)\ \dfrac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}$

$(\wedge_L)\ \dfrac{A, B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \qquad (\wedge_R)\ \dfrac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}$

$(\forall_L)\ \dfrac{F(t), \Gamma, \forall x F(x) \vdash \Delta}{\Gamma, \forall x F(x) \vdash \Delta}$ legal t/x in $F \qquad (\forall_R)\ \dfrac{\Gamma \vdash \Delta, F(y)}{\Gamma \vdash \Delta, \forall x F(x)}$ fresh y

$(\forall^+_L)\ \dfrac{F(S), \Gamma, \forall \phi F(\phi) \vdash \Delta}{\Gamma, \forall \phi F(\phi) \vdash \Delta}$ any $S \in \mathbf{S}^+ \qquad (\forall^+_R)\ \dfrac{\Gamma \vdash \Delta, F(S) \quad - \text{for all } S \in \mathbf{S}^+}{\Gamma \vdash \Delta, \forall \phi F(\phi)}$

The form of (Ax) makes weakening admissible, while using sets gives implicit rules of contraction and permutation, which can be added if one wants to use sequences or multisets. The infinitary rule $(\forall^+_R)$ signals the missing compactness, but should not present any challenge for the intuition.[3] It reflects the unrestricted substitution class for the interpretation of s-quantifiers, comprising all sentences of $\mathcal{L}^+$, unlike in other substitutional approaches to sentential quantification (e.g., [21], or more recent [2]), restricting this class to avoid problematic self-reference.

Infinite sequents allow to handle some cases of infinite axiomatizations. For instance, making only finitely many statements requires infinitely many premisses excluding all other sentences. This case has a finite representation using operator $\doteq$ for syntactic equality of sentences, *s-equality*. We consider practically only sequents with no free s-variables, but they can be useful for handling s-equalities. For instance, 'Karen saying only S' is expressible as $K(S) \wedge \forall\phi(K\phi \to \phi \doteq S)$, or $\forall\phi(K\phi \leftrightarrow \phi \doteq S)$, abbreviated by $K!\phi$. The following rules suffice.

(ref) $\dfrac{S \doteq S, \Gamma \vdash \Delta}{\Gamma \vdash \Delta} \qquad$ (rep) $\dfrac{A(S), A(Q), S \doteq Q, \Gamma \vdash \Delta}{A(Q), S \doteq Q, \Gamma \vdash \Delta} \qquad$ (neq) $\dfrac{\Gamma \vdash \Delta, Q \doteq S}{\Gamma \vdash \Delta}\ Q \not\doteq S.$

Claims like 'property D holds for each sentence, possibly except $S_1, ..., S_n$' are now finitely expressible as $\forall\sigma(\sigma \doteq S_1 \vee \ldots \vee \sigma \doteq S_n \vee D(\sigma))$, allowing sometimes to establish $\forall\phi D(\phi)$ by a finite case analysis, instead of $(\forall^+_R)$:

[3] Instead of this infinitary rule we can use the counterpart of $(\forall_R)$, with a fresh s-variable and s-quantifer replacing the object-level items, provided that semantics addresses not only the actual language, but all its extensions.

$$(\forall^{\doteq}_R)\ \frac{\Gamma \vdash \Delta, D(S_1)\ ...\ \Gamma \vdash \Delta, D(S_n) \qquad \Gamma \vdash \Delta, D(\sigma), \sigma \doteq S_1, ..., \sigma \doteq S_n}{\Gamma \vdash \Delta, \forall\phi D(\phi)}\ \text{fresh } \sigma \in \Phi.$$

Properties of LSO, like soundness/completeness, the interaction of its object-level and metalevel, the role of (cut), will be discussed along the corresponding semantic notions.

3 Semantics

We keep the presentation focused on FOL, but the semantic definitions and results of this section work with minimal adjustments for propositional or higher order classical logics. Roughly, any classical intepretation of $\mathcal{L}$ in a structure M is extended to an intepretation of $\mathcal{L}^+$ by providing a valuation of s-atoms and intepreting s-quantifiers substitutionally, e.g.:

$$M \models \forall\phi(P(\phi) \vee \phi) \Leftrightarrow \text{for all } S \in \mathbf{S}^+ : M \models P(S) \vee S. \tag{3.1}$$

The right side has instances like $P(\forall\phi P(\phi) \vee \phi) \vee (\forall\phi(P(\phi) \vee \phi)$, involving the definiendum. Such circularities are handled formally using graphs and their (semi)kernels.

3.1 Kernels and semikernels

By "graph" we mean a directed graph $G = (\mathbf{V}_G, \mathbf{E}_G)$, with $\mathbf{E}_G \subseteq \mathbf{V}_G \times \mathbf{V}_G$, dropping subscript $_{-G}$ when an arbitrary or fixed graph is addressed. For a binary relation R, we let $R(x) = \{y \mid R(x, y)\}$ and extend this notation pointwise to sets, $R(X) = \bigcup_{x\in X} R(x)$. The converse $\mathbf{E}^-$ of an edge relation $\mathbf{E}$ is obtained by flipping directions of all edges. A *neighbour* y of a vertex x is either its *out-neighbour*, $y \in \mathbf{E}(x)$, or its *in-neighbour*, $y \in \mathbf{E}^-(x)$. A *path* is a (typically finite) sequence of vertices $x_1x_2...x_n$, where each x_{i+1} is an out-neighbour of the preceding x_i. A path is *acylic* if each vertex occurs at most once. A *kernel* (or *solution* [46]) of a graph is a subset $K \subseteq \mathbf{V}$ which is

(a) independent, i.e., $\mathbf{E}^-(K) \subseteq \mathbf{V} \setminus K$ (no edges between vertices in K), and

(b) absorbing, i.e., $\mathbf{E}^-(K) \supseteq \mathbf{V} \setminus K$ (each vertex outside K has an edge to some vertex in K),

in short, such that $\mathbf{E}^-(K) = \mathbf{V} \setminus K$. Equivalently, it is an assignment $\kappa \in \mathbf{2}^{\mathbf{V}}$, with $\mathbf{2} = \{\mathbf{0}, \mathbf{1}\}$, such that

$$\forall x \in \mathbf{V} : \kappa(x) = \mathbf{1} \Leftrightarrow \forall y \in \mathbf{E}(x) : \kappa(y) = \mathbf{0}, \tag{3.2}$$

i.e., a vertex is $\mathbf{1}$ if all its out-neighbours are $\mathbf{0}$, and is $\mathbf{0}$ if at least one of its out-neighbours is $\mathbf{1}$. This allows to view each edge as the negation of its target, and branching as the conjunction of such negations. Given (3.2), the set $\{x \in \mathbf{V} \mid \kappa(x) = \mathbf{1}\}$ satisfies (a) and (b), while if K satisfies (a), (b) then $\kappa \in \mathbf{2}^{\mathbf{V}}$ given by $\kappa(x) = \mathbf{1} \Leftrightarrow x \in K$ satisfies (3.2). We therefore do not distinguish the two and by $sol(G)$ denote the set of kernels or such assignments. Graph G is *solvable* if $sol(G) \neq \emptyset$.

By Richardson's theorem from the 1950-ties, the absence of odd cycles ensures solvability of graphs having no infinite branchings or no infinite acyclic paths [37]. In particular, a finite graph wihout odd cycles is solvable. Our graphs, introduced in the next subsection, are solvable, but model paradoxes by unsolvabale subgraps. This is mostly due to the odd cycles, but also acyclic Yablo-like paradoxes are captured and commented in Section 5.5.

A *sink* is a vertex with no outgoing edges and sinks belong to every kernel by (b). One can think of them as the basic truths independent from the rest of the graph.

A subset $L \subseteq \mathbf{V}$ *covers* vertices in L and those pointing to L, that is, $L \cup \mathbf{E}^-(L)$, denoted by $\mathbf{E}^-[L]$. The equation $\mathbf{E}^-(K) = \mathbf{V} \setminus K$ means that kernel K covers the whole graph. A valuation is *coherent* on vertices for which it satisfies (3.2), so a kernel represents a coherent valuation of all sentences. Our semantics allows also more general situations which are only locally coherent and can not be extended to the whole graph. In the absence of a kernel, a relevant part of the graph may still be covered coherently by a *semikernel*, [27], namely, a subset $L \subseteq \mathbf{V}$ such that

$$\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L. \tag{3.3}$$

The set of semikernels of G is denoted by $SK(G)$. In terms of assignments, a semikernel L determines a coherent valuation of vertices it covers, satisfying (3.2) for every $x \in \mathbf{E}^-[L]$: each statement denied by any true one (in L) is false (in $\mathbf{E}^-(L)$), while every false statement (in $\mathbf{E}^-(L)$) denies some true one (in L). Although locally consistent, a coherent situation represented by a semikernel can entail inconsistency.[4]

Every graph possesses a semikernel, since $\emptyset$ satisfies trivially (3.3). But semikernels of interest are nontrivial, also in graphs not possessing any kernel.

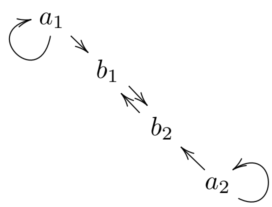
Example 3.4 The propositions to the left can be represented by the graph to the right.

a_1 : This and the next sentence are false.

b_1 : The next sentence is false.

b_2 : The previous sentence is false.

a_2 : This and the previous sentence are false.



The graph has no kernel, witnessing to the involved paradox, but has two semikernels: $\{b_1\}$ and $\{b_2\}$ covering, respectively, $\{a_1, b_1, b_2\}$ and $\{b_1, b_2, a_2\}$, which can be seen as coherent subdiscourses, where truth-values can be assigned coherently, i.e., respecting (3.2). □

3.2 Language graphs

The graph in Example 3.4 represents a specific theory, while a language graph represents all sentences of $\mathcal{L}^+$, relatively to a given $\mathcal{L}$-*domain*, that is, a nonempty set M with a standard classical interpretation of $\mathcal{L}$-terms $\mathbf{T}_M$, but not of the predicate symbols. For each such a domain, there is one language graph $\mathcal{G}_M(\mathcal{L}^+)$ representing the syntax of $\mathcal{L}^+$ over M, with sentences $\mathbf{S}^+_M$ as vertices. A (semi)kernel represents then a subset of sentences true in M under the valuation assigning $\mathbf{1}$ exactly to the *literals* (atoms or negated atoms) belonging to the subset. The structure of the graph, combined with the semikernel condition (3.3), ensures that the evaluation of the object-level sentences covered by L coincides with their classical semantics, generalizing this to self-referential sentences. We start by sketching the main ideas.

Each $S \in \mathbf{S}^+_M$ is the *source* (vertex reaching all others) of the subgraph $\mathcal{G}_M(S)$ of $\mathcal{G}_M(\mathcal{L}^+)$.

1. Dual literals form 2-cycles, e.g., $P(m) \leftrightarrows \neg P(m)$ or $K(R, Q) \leftrightarrows \neg K(R, Q)$, and so do s-constants $\mathbf{C}$. Consequently, in each kernel exactly one of the literals is $\mathbf{1}$ and the other $\mathbf{0}$.

2. Out-branching represents conjunction (or universal quantification), and each edge negation of its target. For instance, the source of $\mathcal{G}_M(\neg A)$ has a single outgoing edge $\boxed{\neg A} \to A...$, while the source of $\mathcal{G}_M(B \wedge C)$ has two: $...B \leftarrow \boxed{\neg B} \leftarrow \boxed{B \wedge C} \to \boxed{\neg C} \to C...$

Subgraph $\mathcal{G}_M(S)$ of each sentence S without s-quantifiers, in particular of each object-level sentence, can be seen as a tree (the source vertex S with no incoming edges and a unique path to every other vertex), except that instead of leaves (with no outgoing edges) there are atomic 2-cycles. It reminds of S's parse tree but, primarily, reflects the semantics of the formula constructors $(\neg, \wedge, \forall)$ in terms of kernels. Using (3.2), one checks easily that the source $B \wedge C$ above is $\mathbf{1}$ exactly when $B = \mathbf{1} = C$. The graph for $A \to B$, obtained from $A \to B \Leftrightarrow \neg(\neg\neg A \wedge \neg B)$, is $\boxed{A \to B} \to \boxed{\neg\neg A \wedge \neg B} \to B...\ \ \boxed{\neg A} \to A...$ By (3.2) its source is $\mathbf{1}$ exactly when $A = \mathbf{0}$ or $B = \mathbf{1}$.

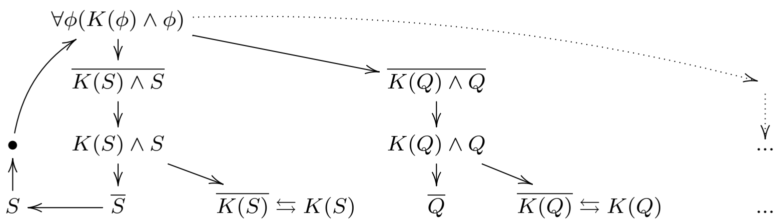
3. The source of the subgraph for a sentence with the universal o-quantifier has an edge to the negation of every instance, e.g., vertex $\forall x P(x)$ has an edge to $\neg P(m)$ for every $m \in M$.

The collection $\{\mathcal{G}_M(S) \mid S \in \mathbf{S}_M\}$ forms the subgraph $\mathcal{G}_M(\mathcal{L})$ of $\mathcal{G}_M(\mathcal{L}^+)$.

4. Similarly to sentences with o-quantifiers, a universally s-quantified sentence is the source with an edge to the negation of every instance. This requires more care and details are explained further down. The crucial point is that starting from the source, e.g., of $S = \forall\phi(K(\phi) \wedge \phi)$, also $K(S) \wedge S$ is an instance of S. The tree is formed as for the sentences without s-quantifiers, but the instantiations are, so to say, suspended until we reach the leaves of the tree. If sentence Q,

[4] The branch of argumentation theory arising from [12] shares only its origins in a similar reading of digraph (semi)kernels. Links to reference graphs, used in [6,33] for paradox analysis, although closer, are not essential either.

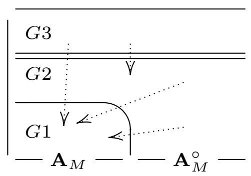
instantiating an s-variable quantified in the original sentence, becomes eventually such a leaf, it acquires a double edge (path of length 2) to the source of the subgraph $\mathcal{G}_M(Q)$, as shown in (3.5) below for the leaf S. ($\overline{X}$ stands for $\neg X$. Vertex $\bullet$ is commented below.) The literals have only 2-cycles to their duals, as shown for $K(S)$ and $K(Q)$.



(3.5)

By (3.2), the source $\forall\phi(K\phi\wedge\phi)$ is $\mathbf{1}$ iff, for each $X \in \mathbf{S}^+_M$, $\overline{K(X)\wedge X} = \mathbf{0}$ iff $K(X)\wedge X = \mathbf{1}$. This is, of course, falsified by any false sentence X, e.g., from the object-level $\mathcal{L}_M$. If Q in the drawing is an $\mathcal{L}_M$ sentence, vertex $\overline{Q}$ belongs to the subgraph $\mathcal{G}_M(\mathcal{L})$ and has an edge to the source of the tree $\mathcal{G}_M(Q)$ there. The whole graph $\mathcal{G}_M(\mathcal{L}^+)$ has the three main subgraphs:

- G1 is $\mathcal{G}_M(\mathcal{L})$ for the object-language,
- G2 extends G1 with 2-cycles for s-atoms $\mathbf{A}^\circ_M$, their propositional combinations, also with sentences from G1, and with the o-quantification of such combinations,
- G3 contains $\mathcal{G}_M(A)$ for each sentence A with s-quantifiers.



(3.6)

Subgraph G2, containing G1, is a collection of trees, with the atomic 2-cycles at all leaves. These 2-cycles provide only the possibility of different valuations of atoms, so subgraph G2 is essentially acyclic, following the inductive definition of the language. In subgraph G3, however, there is a path between each pair of source vertices, forming multiple cycles, as will be explained below. Dotted arrows indicate edges between these subgraphs, going only from G3 to G2/G1 and from G2 to G1, but never in the opposite directions. By our notational convention $\mathcal{G}_M(\mathcal{L}^\circ) = \mathcal{G}_M(\mathcal{L}^+) \setminus G1$.

The description suggests some redundancies. In (3.5), the leaf S represents actually the source, while $\bullet$ its negation $\neg S$. Such auxiliary vertices, serving only the presentation and denoted by AUX, help presenting the subgraphs for all sentences as trees of a similar type, but can be eliminated without changing any relevant properties of the graphs, in particular, their (semi)kernels. We can identify S with the source vertex, because any *double edge* $x \to \bullet \to z$, where x has no other out-neighbours and $\bullet$ no other neighbours, can be contracted by removing $\bullet$ and identifying $x = z$, virtually without changing (semi)kernels, Fact 7.2. Likewise, identifying vertices with identical out-neighbourhood does not change (semi)kernels in any essential way, as intuition suggests and Fact 7.3 shows. Such simplifications are often performed implicitly on the example graphs.

Definition 3.7 *The* language graph $\mathcal{G}_M(\mathcal{L}^+)$, *for a language* $\mathcal{L}^+$ *and* $\mathcal{L}$ *domain* M, *is given by:*

1. *Vertices* $\mathbf{V} = \mathbf{S}^+_M$.
2. *Each atomic sentence* $A \in \mathbf{A}^+_M$, *except s-equality, has a 2-cycle to its negation:* $A \leftrightarrows \overline{A}$.
3. *For each* $S \in \mathbf{S}^+_M$, *s-equality atom* $S \doteq S$ *is a sink; for each syntactically distinct* $S, Q \in \mathbf{S}^+_M$, *vertex* $Q \doteq S$ *has an edge to the sink* $\overline{Q \doteq S}$.
4. *Each nonatomic sentence* $S \in \mathbf{S}^+_M$ *is the source of the subgraph* $\mathcal{G}_M(S)$:

$\mathcal{G}_M(S)$: *source*		*with edges to the source of*:
(a) $\neg F$	$\to$	$\mathcal{G}_M(F)$,
(b) $F_1 \wedge F_2$	$\to$	$\mathcal{G}_M(\neg F_i)$, *for* $i \in \{1,2\}$,
(c) $\forall x F x$	$\to$	$\mathcal{G}_M(\neg F(m))$, *for each* $m \in M$,
(d) $\forall\phi F\phi$	$\to$	$\mathcal{G}_M(\neg F(S))$, *for each* $S \in \mathbf{S}^+$.

When $\mathcal{L}$ is higher order, the only difference is the domain M, containing required sets, with object quantifier(s) in point 4.(c) being those from $\mathcal{L}$. When it is inessential, we often drop M and write $\mathcal{G}(\mathcal{L}^+)$ for $\mathcal{G}_M(\mathcal{L}^+)$. By $\mathcal{LG}r(\mathcal{L}^+)$ we denote the class of all language graphs for language $\mathcal{L}^+$.

As can be expected, every $\mathcal{L}^+$ sentence has a prenex normal form, and its quantifier-free matrix a disjunctive normal form, yielding normal form PDNF (Section 7.2), used for graph construction. The universal and existential quantifiers give rise to the following branchings to instantiations of the quantified o-variables by all elements $a, b, c...$ of the domain, and of s-variables by all $\mathbf{S}^+$.

$$\begin{tikzcd}[column sep=small]
 & \forall\phi D(\phi) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, dotted] & & & & \exists x D(x) \arrow[d] & & \\
\overline{D(S_1)} \arrow[d] & \overline{D(S_2)} \arrow[d] & \overline{D(\forall\phi D(\phi))} \arrow[d] & ... \arrow[d, dotted] & & \forall x \neg D(x) \arrow[dl] \arrow[d] \arrow[dr] \arrow[drr, dotted] & & \\
D(S_1) & D(S_2) & D(\forall\phi D(\phi)) & ... & D(a) & D(b) & D(c) & ...
\end{tikzcd} \tag{3.8}$$

Quantifier prefix is converted to the graph by successively performing such instantiations and branchings, until no quantified variables remain. At the end of each branch of instantiations of all variables quantified in the original sentence, there remains the subgraph for its instantiated DNF matrix, a *DNF-foot*. For example, DNF matrix $D(\phi, x) = (\neg\phi \wedge \neg Q(x)) \vee (\neg P(\phi) \wedge R)$, with a ground atom R, instantiated by $S \in \mathbf{S}^+$ and $a \in M$, yields DNF-foot $\mathcal{G}_M(D(S,a))$:

$$\begin{tikzcd}
D(S,a) \arrow[rrr] & & & \circ_{\overline{D(S,a)}} \arrow[dll] \arrow[drr] & & & \\
 & C_1(S,a) \arrow[d] \arrow[dr] & & & & C_2(S,a) \arrow[d] \arrow[dr] & \\
 & S & Q(a) \arrow[d, shift left] & & & P(S) \arrow[d, shift left] & \overline{R} \arrow[d, shift left] \\
 & \neg S \arrow[u] & \overline{Q(a)} \arrow[u, shift left] & & & \overline{P(S)} \arrow[u, shift left] & R \arrow[u, shift left]
\end{tikzcd} \tag{3.9}$$

The auxiliary vertex $\circ_{\overline{D(S,a)}}$ is the sentence $\neg C_1(S,a) \wedge \neg C_2(S,a) \in \mathbf{S}^+_M$, while the auxiliary $C_1(S,a)$ is the sentence $\neg S \wedge \neg Q(a)$. For $L \in SK(\mathcal{G}_M(D(S,a))) : D(S,a) \in L \Leftrightarrow \circ_{\overline{D(S,a)}} \notin L \Leftrightarrow C_1(S,a) \in L \vee C_2(S,a) \in L \Leftrightarrow \{\neg S, \neg Q(a)\} \subseteq L \vee \{\neg P(S), R\} \subseteq L$, reflecting the expected $D(S,a) = \mathbf{1} \Leftrightarrow (S = \mathbf{0} = Q(a)) \vee (P(S) = \mathbf{0} \wedge R = \mathbf{1})$.

The full graph $\mathcal{G}_M(\mathcal{L}^+)$ has, besides the essentially acyclic $\mathcal{G}_M(\mathcal{L})$ described above, also subgraph $\mathcal{G}_M(\mathcal{L}^\circ)$ containing subgraphs $\mathcal{G}_M(S)$ for sentences $S \in \mathbf{S}^\circ$ with s-atoms and/or s-quantifiers. If S contains no s-quantifiers, its subgraph $\mathcal{G}_M(S)$ is a tree like those in $\mathcal{G}_M(\mathcal{L})$, only with 2-cycles for s-atoms $\mathbf{A}^\circ_M$ instead of $\mathbf{A}_M$ as some of the leaves.

Complexity comes with the s-quantified sentences. If S is such, each $A \in \mathbf{S}^+$ instantiating, according to point 4.(d) of Definition 3.7, an s-variable ϕ in a *sentential position* in S, i.e., not in the scope of any operator, like ϕ in $C_1 = \neg\phi \wedge \neg R_1$, becomes a leaf of $\mathcal{G}_M(S)$, that is, of its DNF-foot (3.9), with a double edge to the source of the subgraph $\mathcal{G}_M(A)$. In particular, sentence S also instantiates ϕ, and the resulting leaf has a double edge to the source of this very $\mathcal{G}_M(S)$, as in (3.5). Every $A \in \mathbf{S}^+$, instantiating ϕ in a sentential position in S, either occurs as an internal node, i.e., on some path from the source S to some source of a DNF-foot, or not. In the former case, the leaf A is called an *internal leaf* of $\mathcal{G}_M(S)$, and has a double edge back to its occurrence in $\mathcal{G}_M(S)$, possibly forming a cycle. In the latter case, when A occurs in $\mathcal{G}_M(S)$ only as a leaf, it is its *external leaf*, $ext(\mathcal{G}_M(S))$, and has a double edge to the source of its separate $\mathcal{G}_M(A)$. In this case, if A itself is s-quantified, its subgraph instantiates its s-variables by all sentences, in particular by S, giving paths back to the source of $\mathcal{G}_M(S)$. The subgraph $\mathcal{G}_M(S)$ is given by

- the source S,
- all paths to all the leaves (at all its DNF-feet)
- 2-cycles at the atomic sentences (occurring in S or resulting from the instantiations of atomic subexpressions of S), and
- cycles to the internal leaves of $\mathcal{G}_M(S)$.
- the external leaves, without (the double edges to the sources of) their subgraphs.

Sources of all s-quantified sentences among $\mathbf{S}^\circ_M$ belong to one strongly connected component of $\mathcal{G}_M(\mathcal{L}^+)$, subgraph G3 from (3.6). Their leaves instantiated with sentences $\mathbf{S}_M$ belong (have double edges) to the subgraph $\mathcal{G}_M(\mathcal{L})$, but there are no edges returning thence to $\mathcal{G}_M(\mathcal{L}^\circ)$.

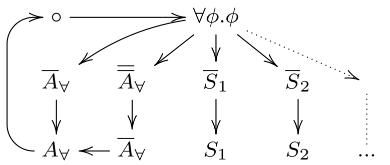
Such double edges, yielding cycles and connecting subgraphs of distinct sentences, arise only from sentences substituted for s-variables in sentential positions. Sentences substituted into *nominal positions*, i.e., into the scope of some operator, contribute only to the resulting s-atoms with 2-cycles to their duals, like $P(S) \leftrightarrows \overline{P(S)}$ in (3.9), arising from substituting S into $P(\phi)$.[5] This difference, as well as the overall effect, are well visible in drawing (3.5).

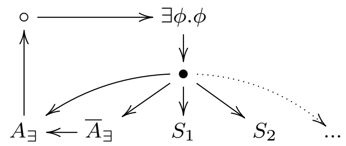
This completes our description of language graphs which represent the syntax of $\mathcal{L}^+$, so that the semantics of the involved self-reference is captured by their (semi)kernels, as we show now.

3.3 Satisfiaction relation and the (semi)kernel models

A (semi)kernel L of a language graph represents sentences satisfied under valuation of atoms determined by the literals contained in L, as indicated by the paragraph under (3.9). Here is a more intricate example involving s-quantified sentences.

Example 3.10 Let $S_1, S_2, ...$ stand for all $\mathbf{S}^+$, except the leftmost two in each graph sketched below: $\mathcal{G}(A_\forall)$, for the sentence $A_\forall = \forall\phi.\phi$, and $\mathcal{G}(A_\exists)$, for the sentence $A_\exists = \neg\forall\phi.\neg\phi$:





The drawings indicate only the essential aspects, ignoring other edges and cycles.

In the left graph $\mathcal{G}(A_\forall)$, $A_\forall$ is an internal leaf, while all S_i are external ones. The two vertices $\overline{A}_\forall$ (as well as $\forall\phi.\phi$ and $A_\forall$) could be identified. Any $S_i \in \mathbf{S}^+$ valuated to $\mathbf{0}$ yields $\overline{S}_i = \mathbf{1}$ and $\forall\phi.\phi = \mathbf{0}$, but even if all $S_i = \mathbf{1}$, the mere cycles involving $A_\forall$ and $\overline{A}_\forall$ force $\forall\phi.\phi = \mathbf{0}$. To obtain a kernel, the odd cycle via $\overline{\overline{A}}_\forall$ must namely be broken, i.e., some of its vertices must have an out-neighbour $= \mathbf{1}$. If all $\overline{S}_i = \mathbf{0}$, this still happens when both $\overline{A}_\forall = \mathbf{1} = \circ$, making $A_\forall = \overline{\overline{A}}_\forall = \mathbf{0} = \forall\phi.\phi$. We might thus say that $\forall\phi.\phi$ is a counterexample to its own truth.

A dual situation occurs in $\mathcal{G}(A_\exists)$, where kernel requires breaking the odd cycle via $\bullet$ and $\overline{A}_\exists$. This happens if any $S_i = \mathbf{1}$, making $\exists\phi.\phi = \mathbf{1}$, but even if all S_i were $\mathbf{0}$, the only way of breaking this odd cycle is with $\exists\phi.\phi = \mathbf{1}$, which provides thus a witness to its own truth. □

The truth-value of $\forall\phi.\phi$ depends on the value of this very sentence. Such circular dependencies can hardly be captured by any inductive semantic definitions, while semikernels covering this sentence determine this value uniquely in accordance with the informal understanding.

We define an $\mathcal{L}^+$-sequent $\Gamma \Rightarrow \Delta$ to be *valid*, $\Gamma \models \Delta$, iff in every language graph $\mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^+)$ every relevant situation satisfies it. A *situation* is a semikernel L, it is *relevant* if it covers $\Gamma \cup \Delta$, i.e., $\Gamma \cup \Delta \subseteq \mathbf{E}^-[L] = \mathbf{E}^-(L) \cup L$, and it *satisfies* the sequent if some $D \in \Delta$ is true, i.e., $\Delta \cap L \neq \emptyset$, or some $G \in \Gamma$ is false, i.e., $\Gamma \cap \mathbf{E}^-(L) \neq \emptyset$. (This is generalized to free object-variables $\mathcal{V}(\Gamma, \Delta)$, by considering all assignments $\alpha \in M^{\mathcal{V}(\Gamma,\Delta)}$, with $\alpha(S)$ denoting insertion of $\alpha(v)$ for each free object-variable v in formula S and $\alpha(\Gamma) = \{\alpha(S) \mid S \in \Gamma\}$. For free s-variables, all substitution instances with sentences from $\mathbf{S}^+$ must hold.)

$$\begin{aligned} \Gamma \models \Delta &\iff \forall \mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^+)\ \forall L \in SK(\mathcal{G}_M) : L \models \Gamma \Rightarrow \Delta, \text{ where} \\ L \models \Gamma \Rightarrow \Delta &\iff \forall \alpha \in M^{\mathcal{V}(\Gamma,\Delta)} : \\ &\qquad \alpha(\Gamma \cup \Delta) \subseteq \mathbf{E}^-[L] \to (\alpha(\Gamma) \cap \mathbf{E}^-(L) \neq \emptyset) \vee (\alpha(\Delta) \cap L \neq \emptyset). \end{aligned} \tag{3.11}$$

Semikernel models of $\Gamma \subseteq \mathbf{S}^+$ are pairs $(\mathcal{G}, L) \in \mathcal{LG}r(\mathcal{L}^+) \times SK(\mathcal{G})$ satisfying $\Rightarrow F$, for all $F \in \Gamma$.

The covering condition $\alpha(\Gamma \cup \Delta) \subseteq \mathbf{E}^-[L]$ brings the important aspect of (ir)relevance. According to it, a semikernel satisfies every sequent/sentence that it does not cover. For instance, the empty semikernel satisfies every sequent and every contradiction $A \wedge \neg A$, but only vacuously, since

[5] These 2-cycles are formed only for atoms with the outermost operator. Substituting S into $P(\phi, Q(\phi))$ yields atom $P(S, Q(S))$ with 2-cycle to its dual $\overline{P(S, Q(S))}$. The inner $Q(S)$ does not obtain any edges to its dual $\overline{Q(S)}$ here, but only when atom $Q(S)$ occurs in sentential position.

it does not cover anything.[6] A nonempty semikernel also satisfies A and $\neg A$ for every sentence it does not cover. This makes LSO paraconsistent, but in a degenerate way, as contradictions are satisfied only vacuously, in situations ignoring them. Semikernels covering $A \wedge \neg A$ contain either A or $\neg A$, but never both, so a contradiction is unsatisfiable by any semikernel covering it.

In spite of the vacuous satisfiability of contradictions, even possible validity $\Gamma \models A \wedge \neg A$ for some Γ and A, contradictions entail everything and $\Gamma, A \wedge \neg A \models \Delta$ holds for every A, Γ, Δ. By the previous paragraph, every semikernel satisfying $A \wedge \neg A$ does so only vacuously, not covering $A \wedge \neg A$, so no Γ or Δ falsify any such validity.

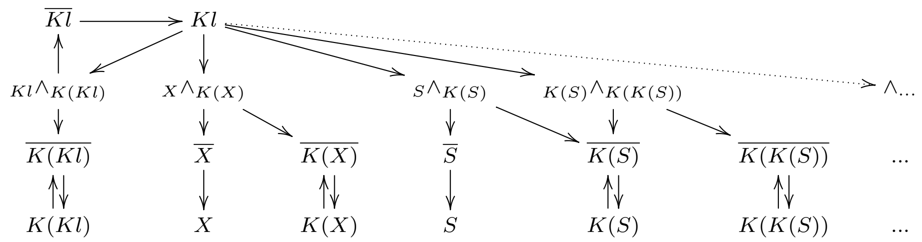
Satisfiability of a contradiction is an oxymoron, so we should clarify this notion in the present context. An $S \subseteq \mathbf{S}^+$ is a *contradiction*, $S \in \mathbb{C}$, if it is not contained in any semikernel, i.e., $S \not\subseteq L$, for every language graph $\mathcal{G}$ and $L \in SK(\mathcal{G})$. Each classical contradiction is thus an LSO contradiction and it can be satisfied only vacuously. As a typical example, $A \wedge \neg A$ has the graph $A \wedge \neg A \to \neg A \to A$... No semikernel L contains $A \wedge \neg A$, since then $\neg A \in \mathbf{E}(A \wedge \neg A) \subseteq \mathbf{E}^-(L)$ would force also $A \in L$, contradicting independence of L.

Dually, $S \subseteq \mathbf{S}^+$ is a *tautology* if it is contained in every covering semikernel, $S \subseteq \mathbf{E}^-[L] \Rightarrow S \subseteq L$. Semikernels not covering S satisfy it vacuosuly, so a tautology is satisfied by every semikernel. Classical tautologies are LSO tautologies, so these definitions just extend the classical notions to our generalized semantics (e.g., $\models \forall\phi(\neg(\phi \wedge \neg\phi))$, $\models \forall\phi, \psi((\neg\phi \wedge (\phi \vee \psi)) \to \psi)$, etc.). But as satisfiable contradictions indicate, their status and role are a bit more complex than in classical logic.

It might seem that a semikernel, not covering the whole graph, assigns a third value to all uncovered sentences. But this impression is more wrong than right. In spite of the partiality of the semikernel models and the involved paraconsistency, the semantics is two-valued, since by (3.11) each semikernel determines a unique boolean value of each sentence, perhaps vacuously by not covering it. If $\neg A$ is not covered by a semikernel L, then neither is A, and L satisfies both, so to say, independently of each other. Satisfaction of both A and $\neg A$ reflects the non-inductive, holistic character of the semantics. A semikernel satisfies simply all sentences it does not cover, in addition to the ones it contains. Although both uncovered A and $\neg A$ obtain thus value $\mathbf{1}$, the logic is not dialetheic, since neither A nor $\neg A$ obtain both values $\mathbf{1}$ and $\mathbf{0}$.

This way of satisfying contradictions underlies the use of semikernels to define models also for theories which, although locally coherent, perhaps even apparently plausible, lead nevertheless to contradictions, as illustrated by the following example.

Example 3.12 The liar Karen says only that everything she is saying is false, $Kl = \forall\phi(K\phi \to \neg\phi)$. Semikernel $L = \{K(Kl)\} \cup \{\overline{K(S)} \mid S \not\equiv Kl\}$, capturing this situation, cannot be extended to any kernel, because $K(Kl) = \mathbf{1}$ makes $\overline{K(Kl)} = \mathbf{0}$, while ${}_{X}\wedge_{K(X)} = \mathbf{0}$, for $X \not\equiv Kl$, due to $\overline{K(X)} = \mathbf{1}$. The remaining unresolved odd cycle $\overline{Kl} - Kl - {}_{Kl}\wedge_{K(Kl)}$ prevents any extension of L to a kernel.



□

Semikernels provide thus models for some situations that are merely contradictions in classicasl logic, yielding non-explosive semantics. In the example above, semikernel L can be extended with many other sentences, for instance, with $J(A)$ or with $\neg J(A)$ but not with both. The paradox captured by L implies that Karen lies and does not lie but, like in an informal analysis of the liar, virtually nothing beyond that. John may still say A and may not say A, but neither follows in LSO from Karen's statement. This suggests to define an *s-contradiction* (or a paradox) as a theory possessing semikernel models but no kernel models. Less formally, a paradox is an apparently

[6] Its existence might be philosophically pleasing, e.g., as a representation of the total ignorance, accepting both poles of each contradiction. Formally, it plays no role, as most theories have also non-empty models.

meaningful set of statements (possessing a limited, semikernel model) which at a closer analysis displays a contradiction (hence possesses no kernel model). A ‘closer analysis’ means here not so much deriving all consequences (which would unveil the paradox, if it were manageable), as rather expanding the attention to the relevant statements not included in the original context, but demonstrating the implied contradiction. E.g., trying to expand L with Kl or $\neg Kl$ fails, displaying only the contradiction $Kl \wedge \neg Kl$ which is, so to say, relevant for this paradox. A similar expansion with $J(A)$ would succeed, yielding a semikernel showing $K!(Kl) \not\models \neg J(A)$ and hence $K!(Kl) \not\models J(A) \wedge \neg J(A)$. This contradiction is not relevant for this paradox, while the relevant $Kl \wedge \neg Kl$ tells how to resolve it: valuate s-atoms so that Karen always lies or not, but not both.

Kernels are semikernels that cover the whole graph, providing a special case of (3.11) which misses the elements of relevance and non-explosiveness. They yield the explosive semantics, denoted by $\models_c$, with the subscript suggesting its classical explosive character, which later will also be related to (cut). An $\mathcal{L}^+$-sequent $\Gamma \Rightarrow \Delta$ is *c-valid*, $\Gamma \models_c \Delta$, iff in every language graph $\mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^+)$ every kernel satisfies it. This only specializes condition (3.11) replacing semikernels by kernels:

$$\Gamma \models_c \Delta \;\Leftrightarrow\; \forall \mathcal{G}_M \in \mathcal{LG}r(\mathcal{L}^+)\; \forall K \in sol(\mathcal{G}_M)\; \forall \alpha \in M^{\mathcal{V}(\Gamma,\Delta)} : \\ \alpha(\Gamma) \cap \mathbf{E}^-(K) \neq \emptyset \vee \alpha(\Delta) \cap K \neq \emptyset. \tag{3.13}$$

A *kernel model* of $\Gamma \subseteq \mathbf{S}^+$ is its semikernel model $(\mathcal{G}, K)$ with $K \in sol(\mathcal{G})$.

3.4 The graph versus classical semantics

By Definition 3.7, point 4, each complex sentence is a vertex with edges (or paths) to the sentence’s components (or instances for quantified sentences), while by semikernel condition (3.3), in terms of assignments (3.2), value of each vertex depends solely on the values of its out-neighbours. This is the compositional aspect of the semikernel semantics, determining value of a complex sentence from the values of its out-neighbours, eventually, its components (or instances). This local compositionality interacts, however, with the circular dependencies and evades any usual inductive definition. The holistic element of condition (3.3) lies in the requirement it puts on *the whole* set of vertices (sentences), like the holistic element of consistency requiring simultaneous satisfaction of *all* sentences. Consequently, non-vacuous satisfaction in (3.11) is given by membership of the satisfied formulas in the satisfying (semi)kernel, and not by an inductive definition (e.g., on the complexity of formulas), which seems rather unlikely to be possible for such circular phenomena.[7]

The restriction of kernel semantics (3.13) to the trees of subgraph G2 (sentences without s-quantifiers) coincides with the classical inductively defined satisfaction. For each sentence A without s-quantifers, its subgraph $\mathcal{G}_M(A)$ is a tree except that, instead of leaves, there are literals with 2-cycles. Exactly one element from each such cycle is in each kernel, capturing thus a valuation of atoms. Every such tree has exactly one kernel for every valuation of its 2-cycles, obtained by inducing values from such a valuation upwards,[8] reflecting thus the inductive definition of satisfaction. Kernel K of subgraph G2 (in particular, of $\mathcal{G}_M(\mathcal{L})$) represents exactly the formulas satisfied (in M) under valuation ρ of atoms given by intersecting each atomic 2-cycle with K, i.e., by setting $\rho(x) = \mathbf{1} \Leftrightarrow x \in K$, for $x \in \mathbf{A}^+_M$ ($x \in \mathbf{A}_M$ for $\mathcal{G}_M(\mathcal{L})$).

Example 3.14 Inclusion of $\overline{P(S)}$ and R from (3.9) in a kernel K forces, by independence, $P(S)$ and $\overline{R}$ out of it. This, in turn, forces $C_2(S,a) \in K$ by absorption, so $\circ_{\overline{D(S,a)}} \notin K$ and $D(S,a) \in K$. This implication from $\{\overline{P(S)}, R\} \subseteq K$ to $D(S,a) \in K$ reflects that from $\neg P(S) \wedge R$ to $D(S,a)$. □

[7] An inductive definition of a semantics based on (another use of) graphs in [40] assigns to sentences with circular dependencies sets of equations, specifying their semantic restrictions and formed from such sets assigned to the sentence’s components. Although the definition is apparently compositional, it merely relocates circularity and non-compositionality to solving sets of equations. The reader may judge whether such sets constitute sufficient and useful semantic values, exemplifying what the sentence carrying this footnote deems unlikely.

[8] Incidentally, this is the first result about kernels from the work which introduced the concept [46].

There is thus a bijection mapping a FOL structure (M, ρ), with an $\mathcal{L}$ domain M and $\rho \in \mathbf{2}^{\mathbf{A}_M}$, to the language graph with its kernel $(\mathcal{G}_M(\mathcal{L}), K_\rho)$, where $A \in K_\rho \Leftrightarrow \rho(A) = \mathbf{1}$ for $\mathcal{L}_M$-atoms A. Then also, for all $S \in \mathbf{S}_M$ (with $\models$ denoting here the standard FOL satisfaction),

$$(M, \rho) \models S \Leftrightarrow S \in K_\rho. \tag{3.15}$$

Unlike kernels, containing *exactly* one element from every atomic 2-cycle, each semikernel contains *at most* one such element, representing thus a partial valuation of atoms. Since atomic 2-cycles have no outgoing edges, every such partial valuation forms a semikernel of the whole graph.

An analogous observation applies to subgraph $\mathcal{G}(\mathcal{L})$. No edges go out of it to its complement $\mathcal{G}_M(\mathcal{L}^\circ)$, so every kernel of $\mathcal{G}(\mathcal{L})$ is a semikernel of the whole graph $\mathcal{G}(\mathcal{L}^+)$. All trees of $\mathcal{G}(\mathcal{L})$ have the atomic 2-cycles at their leaves, so this subgraph has a unique solution for every choice from these 2-cycles. This reflects the elementary fact that every valuation of atoms determines, by the inductive definition, semantic values of all sentences in classical and most other logics.

The same holds for subgraph G2. It, too, is a collection of trees with the atomic 2-cycles, hence is uniquely solvable for every choice from these cycles (valuation of atoms). It has no edges to its complement G3, hence each kernel of G2 is a semikernel of $\mathcal{G}(\mathcal{L}^+)$. This shows the unsurprising fact that adding operators to a classical language does not create any paradoxes (allows to evaluate inductively all sentences under every valuation of all atoms, including the new s-atoms).

What remains is subgraph G3 containing multiple cycles, many of which are odd. The reader may, even should, wonder if this subgraph hides perhaps some unavoidable paradoxes making the whole language graph $\mathcal{G}(\mathcal{L}^+)$ unsolvable. Its solvability is far from obvious. The undefinability of truth underlies various forms of the claim, originating with Tarski's [43], that natural languages actually are inconsistent [3,5,8,13,28]. The semantic elements needed for such a claim may require specific axioms in some formalizations, e.g., in AST, but in most cited works are intended as the fundamental principles of the language itself. Also in LSO, the presence of self-reference in the very language, not only of its model in some theory, leads to the possibility of inconsistency of this very language, and not only of various theories over it.

The following Section 3.5 resolves this worry showing that $\mathcal{G}(\mathcal{L}^+)$ does have a kernel, that is, all sentences of $\mathcal{L}^+$ can be evaluated respecting the classical semantics. First, Theorem 3.16 states that, for a language $\mathcal{L}^\Phi$ with s-quantifiers but no operators, language graphs not only have kernels, but have unique one for every valuation of $\mathcal{L}$ atoms. (Thus, adding such quantifiers to any classical language does not create any paradoxes, just as adding only sentential operators does not. Paradoxes require both operators and s-quantifiers.) Solvability of graphs for the full language $\mathcal{L}^+$ follows, Theorem 3.20, but paradoxes become possible and if they occur then semikernel semantics enables more detailed analysis than kernel semantics.

Section 3.5 is rather technical, referring to even more technical proofs in the appendix, so one can safely skip it on a casual reading, taking further only the two mentioned theorems. The reader satisfied with this can continue now with Section 4.

3.5 Solvability of $\mathcal{G}(\mathcal{L}^\Phi)$ and $\mathcal{G}(\mathcal{L}^+)$

In $\mathcal{L}^\Phi$, extending the object-language $\mathcal{L}$ with s-quantifiers but no operators, s-variables occur only in sentential positions. The only atoms are $\mathcal{L}$-atoms $\mathbf{A}$ (and possibly $\mathbf{C}$. When $\mathbf{A} = \emptyset$, the language $\emptyset^\Phi$ is that of quantified boolean sentences, QBS.) Given a domain M and $\rho \in \mathbf{2}^{\mathbf{A}_M}$, all $\mathcal{L}^\Phi$ sentences obtain unique values by a unique extension to a kernel $\hat{\rho}$ of the corresponding graph $\mathcal{G}_M(\mathcal{L}^\Phi)$. (Numbers in parentheses refer to the same statements with proofs in the appendix.)

Theorem 3.16 (7.6) *For each $\mathcal{G}_M(\mathcal{L}^\Phi)$ and $\rho \in \mathbf{2}^{\mathbf{A}_M}$ there is a unique $\hat{\rho} \in sol(\mathcal{G}_M(\mathcal{L}^\Phi))$ with $\hat{\rho}|_{\mathbf{A}_M} = \rho$.*

The proof relies on the lemma below, showing that for any solution of $\mathcal{G}^-_M(S)$ – denoting, for $S \in \mathbf{S}^\Phi_M \setminus \mathbf{S}_M$, vertices of $\mathcal{G}_M(S)$ without those in its DNF-feet – depends on the valuation of $\mathbf{A}_M$, but not of external leaves $ext(\mathcal{G}_M(S))$, as the second part of the lemma states. In a way, DNF-foot determines a boolean function, and the value of S depends on this function (and valuation of $\mathbf{A}_M$), rather than on the values of external leaves, which span all possibilities. Valuation of $ext(\mathcal{G}_M(S))$ affects, of course, values in DNF-feet, where they occur. For either A from Example 3.10, the lemma implies that the value of $\mathcal{G}^-(A)$, i.e., the source vertex with its marked cycles, is independent from the values of all external leaves among S_1, S_2, etc.

Lemma 3.17 (7.7) *For every $\mathcal{G}_M(\mathcal{L}^\Phi)$ and sentence $A \in \mathbf{S}_M^\Phi$, each valuation ρ of atoms $\mathbf{A}_M$ and external leaves of $\mathcal{G}_M(A)$, $\rho \in \mathbf{2}^{\mathbf{S}_M \cup ext(\mathcal{G}_M(A))}$, has a unique extension to $\rho_A \in sol(\mathcal{G}_M(A))$. Valuation of atoms, $\rho|_{\mathbf{A}_M}$, determines restriction of ρ_A to $\mathcal{G}_M^-(A)$.*

Valuation of sentences $\mathbf{S}_M^\Phi \setminus \mathbf{S}_M$ does not have any standard definition, which is merely suggested by (3.1). By Theorem 3.16, such a valuation $\hat{\rho}$ is determined by $\rho \in \mathbf{2}^{\mathbf{A}_M}$, just as is valuation of $\mathbf{S}_M$. Existence and uniqueness of $\hat{\rho}$ ensure well-definedness of (3.1), given by the following.

Definition 3.18 *An $\mathcal{L}_M^\Phi$-sentence A is true in an $\mathcal{L}$ domain M under valuation $\rho \in \mathbf{2}^{\mathbf{A}_M}$, $(M, \rho) \models A$, iff $\hat{\rho}(A) = \mathbf{1}$ for the unique solution $\hat{\rho} \in sol(\mathcal{G}_M(\mathcal{L}^\Phi))$ with $\hat{\rho}|_{\mathbf{S}_M} = \rho$.*

This gives a well-defined class $Mod(\Gamma) = \{(M, \rho) \mid \forall A \in \Gamma : (M, \rho) \models A\} = \bigcap_{A \in \Gamma} Mod(A)$ of $\mathcal{L}$-structures modelling any theory $\Gamma \subseteq \mathbf{S}^\Phi$. The bijection (3.15) between FOL structures and graphs with kernels, mapping (M, ρ) to $(\mathcal{G}_M, K_\rho)$, extends to FOL$^\Phi$ by mapping (M, ρ) to $(\mathcal{G}_M, \hat{\rho})$.

The hardly unexpected but significant Theorem 3.16 implies that a classical language $\mathcal{L}$ remains free from paradoxes, under every valuation of atoms, when extended with quantification over all sentences to $\mathcal{L}^\Phi$. In fact, by the following theorem, $\mathcal{L}^\Phi$ has the same expressive power as $\mathcal{L}$.

Theorem 3.19 (7.11) *For every $\Gamma \subseteq \mathcal{L}^\Phi$ there is a $\Gamma^- \subseteq \mathcal{L}$ with $Mod(\Gamma) = Mod(\Gamma^-)$.*

In particular, quantification over all sentences in FOL$^\Phi$, extending FOL apparently as far as possibility of self-reference, reduces to propositional quantification.

Solvability of $\mathcal{G}(\mathcal{L}^+)$

Operators applied to sentences provide only fresh atoms, so one might think that everything works unchanged. It does, if only such predicates are introduced without s-quantifiers. The language graph which is then, as for the object-language, a collection of trees with new s-atoms, is uniquely solvable for every valuation of atoms. However, combination of operators with s-quantifiers changes things dramatically. For instance, blind ascriptions of truth, called also infinitary conjunctions, namely claims like "All Ks are true", for an operator K, become expressible as $\forall\phi(K\phi \to \phi)$.[9] Technically, a more significant novelty is the dependence of the operators on their argument *sentences*, not only boolean values of these sentences, and the possibility of violating semantic equivalence of arguments. Consequently, only even cycles can be broken, without breaking the corresponding odd ones, leading to paradoxes. Unlike valuations of $\mathcal{L}$-atoms in a domain M, determining a unique solution of the graph $\mathcal{G}_M(\mathcal{L}^\Phi)$, some valuations of s-atoms can make language graph $\mathcal{G}_M(\mathcal{L}^+)$ unsolvable, as illustrated by Example 3.12. Although $\mathcal{L}^+$ can thus express some paradoxes, none are implied. They appear, as in the example, only due to unfortunate valuations of s-atoms. An extension $\mathcal{L}^+$ of a classical language $\mathcal{L}$ remains consistent.

Theorem 3.20 *Every language graph $\mathcal{G}_M(\mathcal{L}^+)$ has a kernel.*

A proof can be gathered from Theorem 5.7 stating preservation of solvability by *definitional extension*, that is, introduction of a fresh operator, say P, to a language $\mathcal{L}^+$, by a sentence of the form $\forall\phi(P(\phi) \leftrightarrow \exists\!\!\!\forall\psi F(\phi, \psi))$, where $F(\phi, \psi)$ is an $\mathcal{L}^+$-formula (not containing P), with free variables ϕ among those of the left side $P(\phi)$. Starting now with, e.g., $\mathcal{L}^\Phi$ without any operators, having solvable graphs by Theorem 3.16, yields solvability of graphs for its extension with any, also infinite, number of operators.

9 Their role for truth-theory has been discussed at least since Quine's [32]. When syntax is arithmetized, they become problematic due to the complications in controlling interaction with the restrictions on Convention T, e.g., [16, 30]. A paradox in LSO, in turn, requires a sentence or s-variable to occur in both a sentential and a nominal position, exemplified also by such blind ascriptions.

4 Reasoning, (non)explosiveness and (cut)

LSO provides sound and complete reasoning for the semikernel semantics (3.11).

Theorem 4.1 (8.1,8.3) *$\Gamma \vdash \Delta \Leftrightarrow \Gamma \models \Delta$, for countable $\Gamma \cup \Delta \subseteq FOL^+$.*

This fact may seem surprising given, on the one hand, the non-explosiveness of the semikernel semantics and, on the other hand, the classical character of LSO, so we comment it a bit closer.

Derivability $\Gamma \vdash A \wedge \neg A$ signals the impossibility of combining Γ with any coherent valuation of A. By soudness, $\Gamma \models A \wedge \neg A$, so every semikernel L satisfying Γ satisfies $A \wedge \neg A$. But L can do it only vacuously, by not covering this contradiction, so if Γ has any models, they are only semikernels not covering $A \wedge \neg A$.

Such Γs do not exist at the object-level, where entailing a contradiction amounts to being one. Paradoxes occur only at the metalevel – with the help of operators and s-quantifiers – and contradictions they imply involve also these operators.

For an illustration, we continue Example 3.12 with Karen claiming to be always lying. We can then prove that she does not, $K(Kl) \vdash \neg Kl$, but no paradox follows yet. If this is everything she says, then we can also prove that she is always lying, and $K!Kl \vdash Kl \wedge \neg Kl$ witnesses to a paradox, where $K!Kl$ abbreviates $\{K(Kl), \forall\phi(K\phi \to \phi \doteq Kl)\}$. (Section 5.1 contains more details.)

Provability $K!Kl \vdash Kl \wedge \neg Kl$ does not imply nonexistence of a semikernel containing $K!Kl$, as $K!Kl \vdash \emptyset$ would do, but nonexistence of such a semikernel covering also $Kl \wedge \neg Kl$. Most other contradictions and sentences are still not derivable from $K!Kl$. As we saw under Example 3.12, $K!Kl \not\models J(A)$, which follows from the infinite branch of the attempted derivation ($X = J(A)$):

$$(\forall^+_L)\ [Kl/\phi]\ \dfrac{\dfrac{(\forall^+_L)\ [X/\phi]\ \dfrac{\dfrac{(\forall^+_L)\ [S_1/\phi]\ \dfrac{\dfrac{\begin{matrix}\vdots\\ K!Kl \vdash X, K(X), K(S_1)\end{matrix} \qquad \dfrac{S_1 \doteq Kl, K!Kl \vdash X, K(X), S_1 \doteq Kl}{S_1 \doteq Kl, K!Kl \vdash X, K(X)}\ (\mathrm{neq})\ S_1 \not\doteq Kl}{K(S_1) \to S_1 \doteq Kl, K!Kl \vdash X, K(X)}}{K!Kl \vdash X, K(X)} \qquad (\mathrm{neq})\ X \not\doteq Kl\ \dfrac{X \doteq Kl, K!Kl \vdash X, X \doteq Kl}{X \doteq Kl, K!Kl \vdash X}}{K(X) \to X \doteq Kl, K!Kl \vdash X}}{Kl \doteq Kl, K!Kl \vdash X} \qquad K!Kl \vdash X, K(Kl)}{K(Kl) \to Kl \doteq Kl, K!Kl \vdash X}}{K!Kl \vdash X}$$

The branch keeps instantiating $\forall\phi(K\phi \to \phi \doteq Kl)$ by all sentences. In the limit, $K(S_i)$ for each $S_i \not\doteq Kl$ appears to the right of $\vdash$. Since no disjunction of X with $K(X)$ or other $K(S_i)$ is provable, the following countermodel results, reflecting $K!Kl \not\vdash J(A)$ (extending L from Example 3.12):

$$Z = \{\overline{J(A)}, K(Kl), \forall\phi(K\phi \to \phi \doteq Kl)\} \cup \{\overline{K(S_i)} \mid S_i \not\doteq Kl\}.\ ^{10} \tag{4.2}$$

In the same way, $K!Kl \not\vdash \emptyset$. Its derivation would copy the attempted $K!Kl \vdash X$ above, removing all Xs and yielding the infinite branch with the countermodel $Z_1 = Z \setminus \{\overline{J(A)}\}$. The possible situation where Karen claims to be lying, L, can be extended to one where Karen claims nothing else, Z_1, and then to Z where John is not saying A, so $K!Kl \not\vdash J(A)$.

Now, a contradiction entails every sentence S, e.g., $Kl \wedge \neg Kl \vdash S$, reflecting the fact that it does not belong to any semikernel. Since a paradox entails some contradiction, like $K!Kl \vdash \neg Kl \wedge Kl$, using (cut) $\frac{\Gamma \vdash \Delta, A \quad A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta}$ would yield $K!Kl \vdash S$. But semikernel (4.2) gives a countermodel $Z \not\models K!Kl \Rightarrow J(A)$, so (cut) is not sound. It is trivially admissible for the object-language, as long as only LK is used, but it changes the semantics for the whole LSO. The contradiction $Kl \wedge \neg Kl$, following from Karen's statement, is not 'discovered' in Z. A semikernel that is not a kernel represents a limited context which is coherent, that is, only locally consistent, without taking into account the whole language. Z provides a model for Karen saying only Kl, but inquiry into the truth of what she is saying, Kl or $\neg Kl$, expands this context to the point where the paradox – the impossibility of a valuation of Kl coherent with Z – is discovered.

To become "fully classical", that is, explosive, and to derive everything from $K!Kl$ via the contradiction it entails, it suffices to add unrestricted (cut), yielding LSO^c, with $\vdash_c$ denoting provability. This brings us to the kernel semantics (3.13), for which LSO^c is sound and complete.

[10] This shows only the crucial elements of Z, which contains also others implied by these ones: $K(X) \to X \doteq Kl$ for every $X \in \mathbf{S}^+$, the sinks $\overline{Y \doteq Kl}$ for all $Y \not\doteq Kl$, and $Kl \doteq Kl$.

Theorem 4.3 (8.4) *For a countable $\Gamma \cup \Delta \subseteq FOL^{+}$, $\Gamma \models_c \Delta$ iff $\Gamma \vdash_c \Delta$.*

Making thus contradictions explode seems the only contribution of (cut) to LSO. Paradoxes discovered using (cut) can be diagnosed also without it, since if LSO^c derives some contradiction from a theory, so does LSO. By the following theorem, LSO derives then a contradiction of the specific form $\bot_Q = \bigvee_{S\in Q}(S \wedge \neg S)$, for a finite set of sentences Q, denoted by $Q \Subset \mathbf{S}^{+}$.

Theorem 4.4 (7.20) *For a countable $\Gamma \subset FOL^{+}$: $(\forall Q \Subset \mathbf{S}^{+} : \Gamma \nvdash \bot_Q) \Rightarrow (\forall \bot \in \mathbb{C} : \Gamma \nvdash_c \bot)$.*

4.1 Non-transitivity

Unsoundness of (cut) for semikernel semantics (3.11) arises from the subtle element of (ir)relevance: vacuous satisfaction of a formula by a semikernel not covering it. If $\{X\} \in \mathbb{C}$ then $\Gamma \vdash X$ either if $\Gamma \vdash \emptyset$, i.e., $\Gamma \in \mathbb{C}$, or if $\Gamma \notin \mathbb{C}$ but every semikernel containing Γ satisfies $\Gamma \Rightarrow X$ vacuously, by not covering X. In the latter case, semikernels containing Γ may cover other sentences, enabling $\Gamma \not\models \Delta$. Unsoundness of (cut) is limited to such cases, when countermodels to the conclusion satisfy premise(s) only vacuously.

Fact 4.5 *(a1) $\Gamma \models \Delta, A$ and (a2) $\Gamma, A \models \Delta$ and (c) $\Gamma \not\models \Delta$ iff there is a semikernel $L \not\models \Gamma \Rightarrow \Delta$ but none such can be extended to a semikernel $L' \supseteq L$ with $A \in \mathbf{E}^{-}[L']$.*

Proof. $\Rightarrow$) (c) $\Gamma \not\models \Delta$ means that there is a semikernel $L \not\models \Gamma \Rightarrow \Delta$, i.e., $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^{-}(L)$. Let L be any such. If L can be extended to L' covering A, then $\Gamma \subseteq L'$ and $\Delta \subseteq \mathbf{E}^{-}(L')$, while either (i1) $A \in \mathbf{E}^{-}(L')$, contradicting (a1) since $L' \not\models \Gamma \Rightarrow \Delta, A$, or (i2) $A \in L'$, contradicting (a2) as $L' \not\models \Gamma, A \Rightarrow \Delta$. Thus, L cannot be extended to cover A.
$\Leftarrow$) A semikernel $L \not\models \Gamma \Rightarrow \Delta$ establishes (c). Let L' be any semikernel covering Γ, Δ and A. By assumption $L' \models \Gamma \Rightarrow \Delta$ (otherwise it could not cover A), establishing both (a1) and (a2). □

The impossibility to extend a semikernel $L \not\models \Gamma \Rightarrow \Delta$ to a semikernel covering A means that L harbours a paradox: excluding both A and $\neg A$, it cannot be extended to a kernel, precluding a valuation of the whole language. In Fact 4.5, this applies to all semikernels falsifying the conclusion $\Gamma \Rightarrow \Delta$ of (cut), meaning that its negation is paradoxical but not contradictory. The conclusion itself need not be paradoxical, as semikernels satisfying it may admit extensions to kernels, and (cut) will be sound for them.

This inadmissibility of (cut) at the metalevel of LSO reminds of the non-transitive consequence studied, e.g., in [9,38,39]. Although fundamentally different from LSO, addressing only transparent truth predicate in the manner of AST, the role of (cut), or rather of its absence, in the non-transitive systems seems sufficiently similar to warrant a closer comment.

Relation $\models^{st}_{+}$ from [38], appearing most relevant for the comparison, the external or meta-inferential level, as some call it: $A, \neg A \Rightarrow B$ is valid (vacuously), but (cut) in the form $\frac{\Gamma \vdash A \ ; \ \Gamma \vdash \neg A}{\Gamma \vdash B}$ is not admissible. Looking more specifically, relation $\models^{st}_{+}$ holds for $\mathbf{1} \models^{st}_{+} \frac{\mathbf{1}}{\mathbf{2}}$ and $\frac{\mathbf{1}}{\mathbf{2}} \models^{st}_{+} \mathbf{0}$, with $\frac{\mathbf{1}}{\mathbf{2}}$ representing paradox. To prevent $\mathbf{1} \models^{st}_{+} \mathbf{0}$, transitivity is blocked precisely when the cut formula is a paradox (relatively to the context). Fact 3.17 in [38] excludes (cut) between $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ when A evaluates to $\frac{\mathbf{1}}{\mathbf{2}}$ in the models falsifying $\Gamma \Rightarrow \Delta$. This is reflected, if not exactly repeated, by our Fact 4.5, barring transitivity to conclusions falsifiable only by models which display their latent paradox under extensions covering A. Conveying very similar informal messages, the two facts differ due to the different models of paradox.

This, however, is a fundamental difference between 3-valued semantics in [38, 39] and our 2-valued semantics. It might seem that a semikernel, not covering the whole graph, assigns a third value to all uncovered sentences. This impression is more wrong than right. Semikernels do handle inconsistency by leaving it out, but in each relevant semikernel L, covering the actual sequent, every sentence of the sequent is either true (in L) or false (in $\mathbf{E}^{-}(L)$). Kernels are only special cases that leave nothing out. Since they exist for language graphs, all sentences can obtain truth values. Paradox is not any third value but a failure to assign any of the two. Atomic claims are either true or false, while the unfortunate third value of paradox, or rather the unfortunate fact of not being amenable to evaluation, arises only from confused (compounds of) sentences. Semikernels allow Karen to say that she is always lying and even to say only that. That a paradox results is as unfortunate a consequenece as is the contradiciton that she is both telling the truth and not, which falls out of the reasonable discourse, out of any model. It could be seen as a third value, but it seems more adequate to see it as the impossibility of extending a bivalent valuation, given by a semikernel containing Karen's statements, to one determining also their truth.

These differences in the semantics come clearly forth in the reasoning systems. 3-sided sequent systems in [38], reflecting 3-valued semantics, come in two variants, disjunctive and conjunctive, which can be related to the 2-sided ones in the expected ways, but extend considerably their expressiveness. We do not dispute their merits, but limit the comparision to the two-sided system for $\models^{st}_{+}$. First, LSO restricted to the mere truth predicate is a trivial extension of LK admitting, besides unrestricted (cut), insertion of $\mathcal{T}$ around any sentence. ST reasoning from [39] almost coincides with the so restricted LSO except that, using $\models^{st}_{+}$ with 3-valued models, ST does not admit (cut). Restrictions on (cut) are very similar in ST and LSO, guarding against applications over paradoxes. However, while ST needs such restrictions for reasoning with truth predicate, responsible for paradoxes in AST, the restrictions on (cut) in LSO are used for reasoning with arbitrary operators. Limited to the mere object-language, LSO is just LK admitting (cut).

Perhaps the most significant difference emerges as the result of admitting unrestricted (cut). As noted above, since $\mathbf{1} \models^{st}_{+} \frac{\mathbf{1}}{\mathbf{2}}$ and $\frac{\mathbf{1}}{\mathbf{2}} \models^{st}_{+} \mathbf{0}$, (cut) trivializes $\models^{st}_{+}$ yielding $\mathbf{1} \models^{st}_{+} \mathbf{0}$. In LSO, (cut) does not trivialize the logic but only inconsistent theories, turning the non-explosive logic of semikernels into the explosive one of kernels. Paradoxical statements of Karen imply specific contradictions, but do not affect John or the object-level. Unrestricted (cut), letting paradox entail everything, destroys this bond of relevance, bringing LSO back to classical, explosive logic.

Given so diverging technical contexts, restricted transitvity via paradox is indeed a striking similarity between the two approaches. Still, differences in the scope of (cut)'s applicability and dramatically different consequences of lifting its restrictions suggest that the significance of this similarity may be smaller than its apparent appeal.

5 Some examples

This section gives some examples of paradox analysis in LSO. Even when approaches based on AST provide similar diagnoses, the analyses differ. Cases with the intensional or modal character may be problematic in AST due to paradoxes arising easier with modalities rendered by predicates than by operators. The finer distinctions of analysis in Section 5.2 are hardly expressible in AST. In general, non-explosiveness of LSO results in the derivability of only relevant consequences of paradoxes, which suggest specific changes for avoiding them.

5.1 Stating a paradox is possible, even if not evaluating its truth-value

Karen saying that she always lies, $K(Kl)$ with $Kl = \forall\phi(K\phi \to \neg\phi)$, tells sometimes truth:

$$\dfrac{\dfrac{\dfrac{\dfrac{\forall\phi(K\phi \to \neg\phi), K(Kl) \vdash Kl}{\neg Kl, \forall\phi(K\phi \to \neg\phi), K(Kl) \vdash} \qquad \forall\phi(K\phi \to \neg\phi), K(Kl) \vdash K(Kl)}{K(Kl) \to \neg Kl, \forall\phi(K\phi \to \neg\phi), K(Kl) \vdash}}{\forall\phi(K\phi \to \neg\phi), K(Kl) \vdash}\;(\forall^{+}_{L})\ Kl[Kl/\phi]}{K(Kl) \vdash \neg\forall\phi(K\phi \to \neg\phi)} \tag{5.1}$$

As noted by Prior, [31], Karen must then also sometimes lie:

$$\dfrac{\dfrac{\dfrac{Kl, K(Kl) \vdash K(Kl) \qquad \begin{array}{c}(5.1)\\ \vdots \\ Kl, K(Kl) \vdash\end{array}}{\forall\phi(K\phi \to \phi), K(Kl) \to Kl, K(Kl) \vdash}}{\forall\phi(K\phi \to \phi), K(Kl) \vdash}\;(\forall^{+}_{L})\ Kl[Kl/\phi]}{K(Kl) \vdash \neg\forall\phi(K\phi \to \phi)}$$

The resulting Prior's theorem

$$K(\forall\phi(K\phi \to \neg\phi)) \to (\exists\phi(K\phi \wedge \phi) \wedge \exists\phi(K\phi \wedge \neg\phi)), \tag{5.2}$$

is so far no paradox, as Karen can also say other things. If she does not, what follows is not that she is saying two things, one true and one false, but a contradiction, signalling a paradox seen in Example 3.12. On the one hand, (5.1) gives $K(Kl) \vdash \neg Kl$. To obtain Kl requires capturing that she says nothing else, which amounts to the infinite set of negated atoms $\overline{L} = \{\overline{K(S)} \mid S \mathrel{\dot{\neq}} Kl\}$.

$$\dfrac{\dfrac{\begin{array}{c}\overset{(5.1)}{\vdots}\\ \overline{L}, K(Kl) \vdash \neg Kl\end{array}}{\overline{L}, K(Kl) \vdash K(Kl) \to \neg Kl} \qquad \dfrac{\dfrac{\overline{L} \setminus \{K(S)\}, K(Kl), K(S), S \vdash K(S)}{\overline{L}, K(Kl), K(S), S \vdash}\ \neg K(S) \in \overline{L}}{\overline{L}, K(Kl) \vdash K(S) \to \neg S}\ \text{– for each } S \mathrel{\dot{\neq}} Kl}{\overline{L}, K(Kl) \vdash \forall\phi(K\phi \to \neg\phi)}\ (\forall^{+}_{R})$$

S-equality allows a finite expression and proof of this fact, using $(\forall^{\doteq}_{R})$ instead of $(\forall^{+}_{R})$:

$$\dfrac{\dfrac{\begin{array}{c}\overset{(5.1)}{\vdots}\\ K!Kl, K(Kl), Kl \vdash\end{array}}{K!Kl \vdash K(Kl) \to \neg Kl} \qquad \dfrac{\dfrac{\dfrac{..., K\sigma \vdash ..., K\sigma \qquad ..., \sigma \doteq Kl, ... \vdash \sigma \doteq Kl}{..., K\sigma \to \sigma \doteq Kl, K!Kl, K\sigma, \sigma \vdash \sigma \doteq Kl}}{K!Kl, K\sigma, \sigma \vdash \sigma \doteq Kl}\ (\forall^{\sigma}_{L})\ K!Kl[\sigma/\phi]}{K!Kl \vdash K\sigma \to \neg\sigma, \sigma \doteq Kl}}{K!Kl \vdash \forall\phi(K\phi \to \neg\phi)}\ (\forall^{\doteq}_{R})$$

The semikernel $L = \overline{L} \cup \{K(Kl)\}$ models Karen saying only that she is lying. The paradox amounts to the provability of contradiction, $K!Kl \vdash Kl \wedge \neg Kl$, and the impossibility of extending semikernel L to any covering Kl. Karen's paradoxical statement makes evaluation of its truth-value impossible. Semikernel L does not, however, validate any other statements. Most other facts – and contradictions – remain unprovable. Snow does not become not-white, while for any sentence S distinct from Kl, we have $K!Kl \vdash \neg K(S)$ but $K!Kl \not\vdash K(S)$. Karen saying only Kl does not become Karen saying everything, as she does in AST (or LSO^{c}).

Note also that this subsection's title does not hold for truth theories over AST, where ensuring concistency by restricting Convention T makes paradoxes inexpressible in the theory (footnote 2).

The diagonalization lemma

This beautiful and powerful result ensures for every formula with one free variable, $F(x)$, a sentence S such that $S \leftrightarrow F(\ulcorner S \urcorner)$ is provable (in a sufficiently strong arithmetic). It gives then the first equivalence for the liar $\neg\mathcal{T}(\ulcorner L \urcorner) \leftrightarrow L \leftrightarrow \mathcal{T}(\ulcorner L \urcorner)$, with Convention T yielding the second. The situation in LSO is different, although variants of the lemma hold here, too. For instance:

Fact 5.3 *For every formula $F\phi$ with one free variable, s-variable ϕ, there is a sentence S, with a fresh operator K, such that $K_1 \models S \to F(S)$, for valuations K_1 making $K(S) = \mathbf{1}$.*

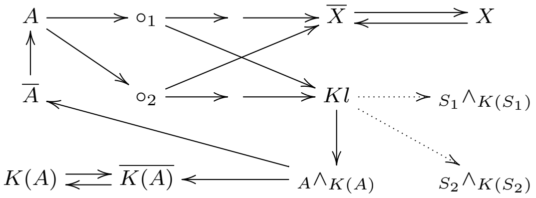
For $S = \forall\phi(K\phi \to F\phi)$, we have $K(S) \models S \to F(S)$, so S says (or implies) then "S is F". Just as the liar L is the paradigmatic paradoxical instance of the diagonalization lemma in AST, Karen is such an instance of this fact in LSO, with $F\phi = \neg\phi$ and Kl for S. Proof (5.1) gives then $K(S) \models F(S)$, hence $K(S) \models S \to F(S)$, and paradox results if K is true only about Kl. With the chosen S, no paradox arises unless ϕ has a sentential occurrence in $F\phi$, as S has only nominal ϕ under K, while a paradox requires both. Paradoxes arising potentially from self-reference in Fact 5.3 exemplify again dependence on the definition of operators.

5.2 Statements do not affect facts, but tautologies can make a difference

Just like the fact of making a statement should not be confused with the statement made, thinking a thought should be distinguished from the thought's content, not to mention its truth. When thoughts are confused with their truth-values, thinking may seem to force something outside one's thoughts. The following is quoted with inessential modifications after [1]:

"If K is thinking only 'Everything I am thinking now is false iff X is false', then X is true." We take X as an arbitrary atom, while Asher, reading it as "Everything Tarski is thinking is false", notes: *"By reasoning that is valid in the simple theory of types, we conclude that Tarski was not able to think that snow is white, a bizarre and unwanted consequence of a logic for belief"*.

The relevant part of the graph is shown below. K is thinking $A = Kl \leftrightarrow \neg X$, where $Kl = \forall\phi(K\phi \to \neg\phi)$ is as in Example 3.12; each ${}_S\wedge_{K(S)}$ has edges to $\overline{K(S)}$ and $\overline{S}$, as shown for ${}_A\wedge_{K(A)}$.



Each combination $\mathbf{2}^{\{K(A),X\}}$, of literals over $K(A)$ and X, forms a semikernel so $K(A) \not\models X$, since e.g. $\{K(A), \overline{X}\} \not\models X$. LSO proves something about the relations between truth-values of A and X, e.g., $A, K(A) \vdash X$, but this isn't as exciting as K's thought limiting Tarski's.

If A is the only thought of K, $K!A = \{K(A), \forall\psi(K\psi \to \psi \doteq A)\}$, then $K(S) = \mathbf{0} =_S \wedge_{K(S)}$ for each $S \not\equiv A$, leaving ${}_A\wedge_{K(A)}$ undetermined. We still have $K!A \not\vdash X$, as $\{K(A), \overline{X}\}$ can be extended to a semikernel containing $K!A$. Something new follows now about the relation between truth of A and X. Of the two cycles via $A - Kl -{}_A\wedge_{K(A)} - \overline{A}$, the one via $\circ_2$ is even and the one via $\circ_1$, odd. Having ${}_S\wedge_{K(S)} = \mathbf{0}$, for all $S \not\equiv A$, the only way to break the latter is by $\overline{X} = \mathbf{0}$, i.e., $X = \mathbf{1}$. Now, this follows also assuming $\neg A$, i.e., $\neg A, K!(A) \vdash X$.

The two proofs give $A \vee \neg A, K!A \vdash X$. Having also $\vdash A \vee \neg A$, (cut) would yield $K!A \vdash X$. But semikernel $\{K!A, \overline{X}\}$ shows $K!A \not\models X$. To obtain X requires an additional assumption. The two proofs show that A's value does not matter. What matters is that it has one, $A \vee \neg A = \mathbf{1}$. This is guaranteed by $\vdash A \vee \neg A$ only in situations covering $A \vee \neg A$. The needed assumption is that in situations containing $K!A$ and covering X, also A can be evaluated, so that $A \vee \neg A = \mathbf{1}$. No semikernel extension of $\{K!A, \overline{X}\}$ allows this, while $K!A \not\models X$ makes this (cut) unsound.

K's thought A, "This thought is false iff X is", is true iff the equivalence it states is. X is not entailed by K thinking A but by this equivalence, $A \leftrightarrow (\neg A \leftrightarrow \neg X) \models X$, and vacuously so when $X = \mathbf{0}$. To this propositional essence, our semantics adds the possibility of non-satisfaction of X when K thinks only A, so that $K!A \not\models X$. This thought of K, being a contingent paradox, yields inconsistency when X is false, $K!A, \neg X \vdash A \wedge \neg A$. Hence, X must be true if K thinking only A is to form a non-paradoxical situation with A's truth or falsity, $A \vee \neg A, K!A \models X$, not merely because K is thinking A and nothing else.

Although tautological, $A \vee \neg A$ adds thus the nonempty assumption that A is not part of a paradox. (Most other tautologies are here irrelevant and adding them will not entail X.) In classical logic, this tacit assumption of non-paradoxicality coincides with consistency. Distinguishing the two and admitting the plausible event $\{K!A, \neg X\}$, LSO notices also that A becomes then paradoxical.

5.3 Metalanguage: paradoxes and indeterminate statements

Non-explosive paradoxes are one feature distinguishing metalanguage from the object-language. Another is sentences that remain indeterminate under some valuations of all atoms. John saying only that he always tells the truth, the framed $J(Jt)$ with $Jt = \forall\phi(J\phi \to \phi)$ in Figure 5.4 below, is the truth-teller. Each ${}_{\overline{X}}\wedge_{J(X)}$, for $X \not\equiv Jt$, is false due to John not saying X, while $\overline{J(Jt)} = \mathbf{0}$ leaves 2-cycle ${}_{\overline{Jt}}\wedge_{J(Jt)} \leftrightarrows Jt$ with one solution $Jt = \mathbf{1}$ and the other $Jt = \mathbf{0}$.

LSO admits semikernel models with sentences like Jt left without any value. This does not happen at the object-level $\mathcal{L}$, nor $\mathcal{L}^{\Phi}$, where each valuation of o-atoms determines unique values of all sentences, but only at the metalevel. Considering this a flaw seems due to merging the metalanguage with the object-language by coding the former as a subset of the latter. In LSO such an

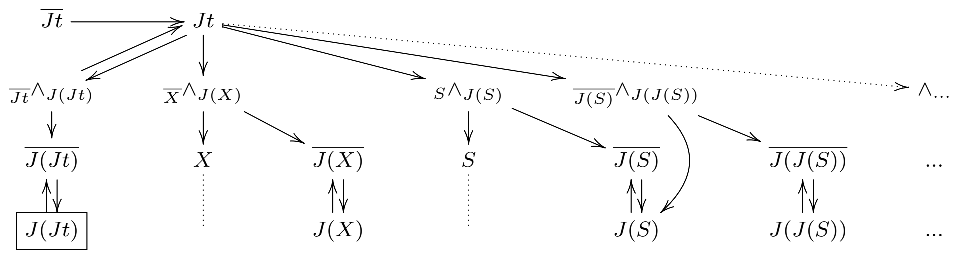


Fig. 5.4 A part of $\mathcal{G}(Jt)$, for $Jt = \forall\phi(J\phi \to \phi)$.

indeterminateness is simply another, besides paradoxes, feature distinguishing the two. Moreover, the difference between the innocent self-reference of the truth-teller and vicious circularity of paradoxes, reflecting the indeterminacy of the former versus the impossibility of valuating the latter, is captured in language graphs by even versus (unresolvable) odd cycles.

The unproblematic status of the truth-teller amounts to the informal observation that it says nothing. Making no real claim, its truth or falsity makes no difference. A difference appears if he says also something else, because then, no matter what else it is, the 2-cycle $Jt \leftrightarrows {}_{\overline{Jt}}{\wedge}_{J(Jt)}$ can always be solved by ${}_{\overline{Jt}}{\wedge}_{J(Jt)} = \mathbf{1}$ and $Jt = \mathbf{0}$. The mere claim of telling only the truth implies consistency of this claim being false.[11]

5.4 Modal logics

The authors of [7] consider the following situation, with a peculiar anaphoric self-reference.

(1) Ann believes that Bob assumes that (2) Ann believes that Bob's assumption is wrong.

The question whether (*) *Ann believes Bob's assumption to be wrong*, is answered by the following informal reasoning. (We insert (a), (b), (c) to mark the assumptions used later.)

"If so, then in Ann's view, Bob's assumption, namely 'Ann believes that Bob's assumption is wrong', is right. But then Ann does not believe that Bob's assumption is wrong (a), which contradicts our starting supposition. This leaves the other possibility (b), that Ann does not believe that Bob's assumption is wrong. If this is so, then in Ann's view, Bob's assumption, namely 'Ann believes that Bob's assumption is wrong', is wrong (c). But then Ann does believe that Bob's assumption is wrong, so we again get a contradiction."

Whether statements form a paradox or not depends often on representation, and authors build an impressive machinery to ensure that these do. The following is only one possible way of capturing the situation in LSO. We do not worry about distinction between 'believes' vs. 'assumes', central in [7], but denote Ann's thoughts by A, Bob's by B and let σ be what Ann believes to be Bob's assumption (2). This yields the following representation:

(1) $AB\sigma$

(2) $\sigma \leftrightarrow A(B\sigma \wedge \neg\sigma) \overset{(d)}{\leftrightarrow} AB\sigma \wedge A\neg\sigma$.

As in normal modal logic, A distributes over conjunction and implication, (d). Equivalence (2), available to the agents, has any number of As (or Bs), in particular, $A(\sigma \leftrightarrow A(B\sigma \wedge \neg\sigma))$. Adding the assumptions from the informal arguments yields, for all ϕ, ψ:

(a) $AA\phi \leftrightarrow A\phi$ (c) $\neg A\phi \to A\neg\phi$, for relevant ϕ

(b) $A\phi \vee \neg A\phi$ (d) $A(\phi \wedge \psi) \leftrightarrow (A\phi \wedge A\psi)$ and $A(\phi \to \psi) \to (A\phi \to A\psi)$

The question whether (*) *Ann believes Bob's assumption to be wrong*, asking apparently whether $A\sigma$ or $\neg A\sigma$ (or $A\neg\sigma$), since σ is what B assumes, asks equally whether $\neg\sigma$ or σ, as σ states exactly that Ann believes Bob's assumption to be wrong. Taking the former, LSO proves $Ax \vdash A\sigma \wedge \neg A\sigma$

[11] Buridan's early proposal, that each statement claims its own truth in addition to whatever it may be saying, provides thus a 'solution' to the liar and similar paradoxes by making them false – but for the price of consistency of all statements being false. Earlier, Bradwardine maintained falsity of paradoxes without this overgeneralization, taking only some – self-negating – statements as claiming also their truth, [34, 35]. However, which statements are so self-negating is contingent and hardly tractable.

from $Ax = \{(1), (2), (a), (b), (c), (d)\}$, but we spare the reader the involved intricacies. In spite of appearances, this paradox concerns only a single complex belief of Ann. No axioms about Bob are needed and their irrelevance becomes apparent when we note that (1) simplifies (2) to a version of knower's paradox, $\sigma \leftrightarrow A\neg\sigma$. Subtleties of the analyses in [7], intended for applications to multiagent games, result in a particular modal logic, while LSO is a general schema admitting various specializations and helping to unveil uniform patterns. Granting the ingenuity of the scenario, there seems to be nothing specifically intensional or modal about the paradox, at least, when represented as above. Reading $A\phi$ as 'A claims ϕ', the scenario becomes

(1) Ann claims that Bob claims that (2) Ann claims that Bob's claim is false,

yielding a paradox of semantic character, represented and analysed in exactly the same way.

In general, modal paradoxes have natural representation in LSO, modalities being specific operators with axiomatic theories specializing LSO. (LSO might need adjustments to handle such axiomatizations. To some extent this can be done in a structural way, e.g., as in [26], but one can also simply add modal axioms to the antecedents of sequents, as we did in the example above. Necessitation rule requires then infinity of such premisses, with all nestings of (operators corresponding to) box around axioms, but LSO admits that.) Relating modal logic to LSO would take at least another paper, so we only comment one example, utilizing the difference between language graphs and Kripke frames, reflected here by that between sentences and propositions.

Karen saying only ϕ, that is $K\phi \wedge \forall\psi(K\psi \rightarrow \psi \doteq \phi)$, enters Kaplan's formula from [19], $\forall\phi\Diamond\forall\psi(K\psi \leftrightarrow \psi \doteq \phi)$, written in the form

(A) $\forall\phi\Diamond(K\phi \wedge \forall\psi(K\psi \rightarrow \psi \doteq \phi))$.

The important difference is that Kaplan uses $=$ as equality of propositions, instead of our syntactic equality $\doteq$. With s-quantifiers ranging also over propositions, viewed as arbitrary subsets of possible worlds, a cardinality argument excludes an operator K satisfying (A), making it false rather than paradoxical. However, (A) says that for every ϕ it is possible for K to say ϕ and nothing else, which seems quite plausible in limited situations. According to [19], logic should not rule it out. Attempts to save (A) (e.g., by taking as propositions only some subsets, [22], or by restricting principle of universal instantiation, [4]) leave the issue open.

The cardinality argument does not affect language graphs, where $\Diamond$ and K act on sentences and not any subsets. (A) is trivially satisfied by semikernel L containing $\forall\phi\Diamond(K!\phi)$ and atom $\Diamond(K!S_i)$, for each $S_i \in \mathbf{S}^+$. Ensuring modal content of $\Diamond$ and K, by closing L under appropriate modal axioms, does not change the situation and keeps (A) satsifiable, as the problem is due to the model of propositions and not modalities.

5.5 Yablo paradox

All our examples so far have involved circular paradoxes because they are most common and natural. Yablo paradox, on the other hand, appears noncircular, unless one applies some esoteric notions of circularity. In LSO it means just graph cycles and Yablo graph $Y = (\omega, <)$ has none. The essential aspects of this paradox can be captured, e.g., by the following theory Y from [20]:

(a) a transitive binary relation R on a nonempty set of sentences,

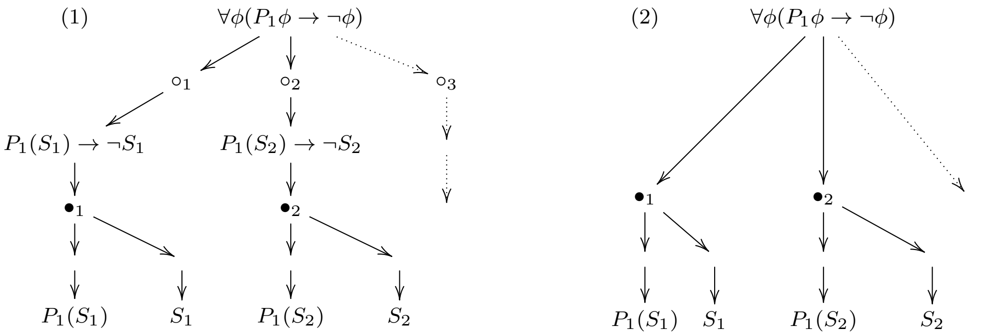
(b) that has no endpoints, $\forall\alpha\exists\beta R(\alpha, \beta)$, and where

(B) operator P satisfies the formula: $\forall\alpha(P(\alpha) \leftrightarrow \forall\beta(R(\alpha, \beta) \rightarrow \neg P(\beta)))$.

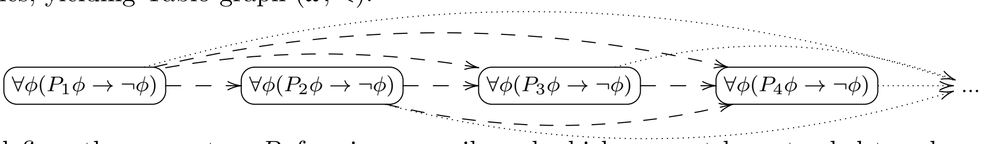
A single sentence with a loop provides a model of R, and so does ω ordering, but no semikernel contains Y. The author observes that its "inconsistency [...] has nothing to do with truth, for it [...] arises irrespective of what P means: other than the Yablo scheme itself (B) and the auxiliary axioms (a), (b), no specific axioms for P are used in the deduction of the inconsistency." Indeed, Y with variables ranging over objects rather than sentences is a contradiction and inconsistency of Y has nothing to do with the truth predicate. By our definition of paradox, as the inclusion in a semikernel which cannot be extended to a kernel, Y is not a paradox but a contradiction.

A *paradox*, according to this definition, can be obtained using a different formulation, with ω operators (persons) $P_1, P_2, ...$, each P_i holding for (saying) exactly $\forall\phi(P_j\phi \rightarrow \neg\phi)$ for all $j > i$. Theory Γ, containing for each P_i s-atoms $\{P_i(\forall\phi(P_j\phi \rightarrow \neg\phi)) \mid j > i\}$ and negations of P_i applied to all other sentences, captures the Yablo-like situation, where everybody makes infinitely many claims, amounting to: everything everybody after me is saying is false.

This Γ, composed of s-literals, is a semikernel of any language graph. Schema (1) below, for subgraph $\mathcal{G}(\forall\phi(P_1\phi \to \neg\phi))$, has equivalent (2), with contracted double edges from $\circ_i$ to $\bullet_i$.



When $P_1(S_k) = \mathbf{0}$, the respective $\bullet_k = \mathbf{0}$. With the valuation of s-atoms given by Γ, this leaves only subgraph (2) with S_{n_i} identical to $\forall\phi(P_i\phi \to \neg\phi)$ for $i > 1$. The same happens for every P_i, leaving pattern (3) below, where dashed arrows mark the triple arrows which can be contracted to single ones, yielding Yablo graph $(\omega, <)$:

(3) 

Theory Γ defines thus operators P_i forming a semikernel which can not be extended to a kernel due to the unsolvability of subgraph (3).

The infinitary character of the argument is captured in LSO^c, formalizing the following proof:

(a) P_1 says something true if either what he says about P_2 is true, i.e., everything P_2 says is false, or else if P_2 says something true: $\Gamma \models (\exists\phi(P_2\phi \wedge \phi) \vee \forall\phi(P_2\phi \to \neg\phi)) \to \exists\phi(P_1\phi \wedge \phi)$
(b) The antecedent of (a) is a tautology, so P_1 says somehting true: $\Gamma \models \exists\phi(P_1\phi \wedge \phi)$.
(c) By the analogous argument, $\Gamma \models \exists\phi(P_i\phi \wedge \phi)$ for every $i \geq 1$, i.e., $\Gamma \models \neg\forall\phi(P_i\phi \to \neg\phi)$.
(d) Collecting all formulas from point (c) for $i \geq 2$, gives actually $\Gamma \models \forall\phi(P_1\phi \to \neg\phi)$.
(e) Contradiction of (b) and (d) shows paradoxicality of Γ.

We abbreviate $L_i = \forall\phi(P_i\phi \to \neg\phi)$, so that $\neg L_i \leftrightarrow \exists\phi(P_i\phi \wedge \phi)$, and let $L_i[\phi\backslash S]$ denote instantiating $\forall\phi$ of L_i by S in the following proof of (a):

$$\dfrac{\dfrac{\dfrac{L_1[\phi\backslash S]\ (\forall^+_L)\ \dfrac{\dfrac{\dfrac{\dfrac{(\mathrm{x})}{\Gamma, L_1, P_2S \vdash P_1S} \quad \Gamma, L_1, P_2S, \neg S \vdash \neg S}{\Gamma, L_1, P_2S, P_1S \to \neg S \vdash \neg S}\ (\to_L)}{\Gamma, L_1, P_1S \to \neg S \vdash P_2S \to \neg S}}{\Gamma, L_1 \vdash P_2S \to \neg S} \quad \ldots \vdash \ldots}{\Gamma, L_1 \vdash L_2}\ (\forall^+_R)}{\Gamma, \neg L_2 \vdash \neg L_1} \quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma \ni P_1(L_2)}{\Gamma, L_2, L_1 \vdash P_1(L_2)} \quad \dfrac{\Gamma, L_2, L_1 \vdash L_2}{\Gamma, L_2, L_1, \neg L_2 \vdash}}{\Gamma, L_2, L_1, P_1(L_2) \to \neg L_2 \vdash}}{\Gamma, L_2, L_1 \vdash}\ (\forall^+_L)\ L_1[\phi\backslash L_2]}{\Gamma, L_2 \vdash \neg L_1}}{\Gamma, \neg L_2 \vee L_2 \vdash \neg L_1}$$

Branch (x), of the same kind for every S instantiating L_2 in the premises of $(\forall^+_R)$, has two cases. If $S = L_i$ for some i such that $P_2S \in \Gamma$, then also $P_1S \in \Gamma$, yielding an axiom. Otherwise, $\neg P_2S \in \Gamma$ yielding also an axiom.

(b) The extra assumption, $\neg L_2 \vee L_2$, of the resulting sequent is a tautology, hence $\Gamma \vdash \neg L_2 \vee L_2$. Applying (cut) to this and the result of (a) $\Gamma, \neg L_2 \vee L_2 \vdash \neg L_1$, yields $\Gamma \vdash_c \neg L_1$.

(c) In the same way, $\Gamma \vdash_c \neg L_i$ follows for each P_i, giving premises in the indicated branches of the following proof of point (d):

$$(\forall^+_R)\ \dfrac{(\to_R)\ \dfrac{\vdots \atop \Gamma \vdash_c \neg L_2}{\Gamma \vdash P_1(L_2) \to \neg L_2} \quad \dfrac{\vdots \atop \Gamma \vdash_c \neg L_3}{\Gamma \vdash P_1(L_3) \to \neg L_3} \quad \cdots \quad \text{for each S} \in \mathbf{S}^+ \quad \dfrac{\dfrac{\Gamma \ni \neg P_1(S)}{\Gamma, P_1(S) \vdash \neg S}}{\Gamma \vdash P_1(S) \to \neg S}}{\Gamma \vdash \forall\phi(P_1\phi \to \neg\phi)} \qquad (5.5)$$

Dots $\cdots$ stand also for the branches instantiating $\forall\phi$ of the conclusion with sentences S other than $\forall\phi(P_i\phi \to \neg\phi)$, which terminate with axioms as shown to the righ. Combined with (b), this yields (e) $\Gamma \vdash_c \forall\phi(P_1\phi \to \neg\phi) \wedge \neg\forall\phi(P_1\phi \to \neg\phi)$.

By Theorem 4.4, a proof of a contradiction from Γ in LSO^c ensures also one without (cut). (Cut) was used only in the proof of (b)/(c), which is here proven without it:

$$\dfrac{\dfrac{\dfrac{\Gamma \ni P_1(L_2)}{\Gamma \vdash P_1(L_2)} \qquad \dfrac{\dfrac{\dfrac{L_3[\phi\backslash L_1]\ (\forall^+_L)\ \dfrac{\dfrac{\dfrac{\Gamma \ni P_1(L_3)}{\Gamma, L_1, L_3 \vdash P_1(L_3)} \qquad \dfrac{\Gamma, L_1, L_3 \vdash L_3}{\Gamma, L_1, \neg L_3, L_3 \vdash}}{\Gamma, L_1, P_1(L_3) \to \neg L_3, L_3 \vdash}}{\Gamma, L_1, L_3 \vdash}}{\Gamma, L_1 \vdash \neg L_3} \qquad \dfrac{\dfrac{\vdots}{\Gamma, L_1, L_4 \vdash}}{\Gamma, L_1 \vdash \neg L_4} \qquad \cdots \qquad \forall i > 2}{\Gamma, L_1 \vdash L_2}\ (\forall^+_R)}{\Gamma, L_1, \neg L_2 \vdash}}{\dfrac{\Gamma, L_1, P_1(L_2) \to \neg L_2 \vdash}{\Gamma, \forall\phi(P_1\phi \to \neg\phi) \vdash}\ (\forall^+_L)\ L_1[\phi\backslash L_2]}}{\Gamma \vdash \neg\forall\phi(P_1\phi \to \neg\phi)}$$

In the same way $\neg L_i$ follows for every $i > 1$, and then L_1 by (5.5) with $\vdash_c$ replaced by $\vdash$.

5.6 Definitional extensions and "Convention T"

As every kernel is a semikernel, the explosive kernel semantics is a special case of the non-explosive semantics of semikernels, and the two coincide on consistent theories/languages. Theorem 4.1 can help checking whether we are in such a desirable situation, but some syntactic conditions ensuring this would be desirable. A modest example can be *definitional extension*, extending any language $\mathcal{L}^+$ to $\mathcal{L}^P$ with a fresh operator symbol P, defined by a sentence of the form ($\exists\!\!\!\forall \in \{\forall, \exists\}$)

$$\forall\phi(P(\phi) \leftrightarrow \exists\!\!\!\forall\psi F(\phi, \psi)), \tag{5.6}$$

where $\exists\!\!\!\forall\psi F(\phi, \psi)$ is an $\mathcal{L}^+$-formula, with free variables ϕ among those of the left side $P(\phi)$.

Definitional extension preserves solvability by the following theorem, according to which any solution of any graph $\mathcal{G}_M(\mathcal{L}^+)$ can be expanded to a solution of $\mathcal{G}_M(\mathcal{L}^P)$.

Theorem 5.7 (7.17) *For every $\Gamma \subseteq \mathcal{L}^+$ and its definitional extension F, every kernel model of Γ can be extended to a kernel model of $\Gamma \cup F$.*

The proof amounts to elimination of symbol(s) P, replacing each $P(S)$ by its definiens $\exists\!\!\!\forall\psi F[S, \psi]$. This operation, trivial in FOL, has to be performed recursively (e.g., $P(P(S))$ needs repeated replacements) on a cyclic graph and is given in Appendix 7.5.

As a special case, formula reminding of Convention T, $\forall\phi(\mathcal{T}\phi \leftrightarrow \phi)$, satisfies trivially (5.6).

Corollary 5.8 *Each kernel model of any $\Gamma \subseteq \mathcal{L}^+$ can be extended to a kernel model of $\Gamma \cup \{(\mathrm{T})\}$.*

Holding for FOL and higher order classical logics, this does not contradict Tarski's undefinability theorem. On the one hand, this $\mathcal{T}$ is just the identity operator, not any truth-predicate 'decoding' numbers (or names) as formulas. More siginificantly, LSO does not internalize substitution which underlies the diagonalization lemma and Tarski's theorem. Its treatment in LSO must be left for further work since it requires an extension to open formulas.

6 Summary

Like in natural reasoning, a paradox in LSO is unveiled by deriving from it a particular contradiction (possibly from a limited set of such), relevantly related to the specific statement of the paradox. From the liar we deduce that he lies and does not lie, not that snow is white and not white. As in natural reasoning, most other contradictions do not follow from the liar in LSO with the non-explosive semantics. If explosiveness is desired not only from contradictions but also from paradoxes, the non-explosive semantics can be specialized to the explosive one (replacing all semikernels by kernels), with the only difference that reasoning can then apply unrestricted (cut).

Paradoxes in LSO aris from unfortunate definitions of operators. One can hardly expect any simple, sufficient and necessary conditions on such definitions ensuring the absence of paradoxes. We can, however, look forward to results narrowing conditions of either kind and thus providing some general guidelines for paradox-free valuations of s-atoms.

From the purely syntactic perspective, a paradox occurs only at the metalevel (again, as in natural reasoning). It requires namely a sentence (or sentences) using both an operator and an s-quantifier and having both nominal and sentential occurrences of some variable (in one scope).

In terms of language graphs, most paradoxes we have seen involve unresolvable odd cycles, representing indirect liars. By Richardson's theorem, [37], every graph without infinite acyclic paths or infinite branchings, in particualr, every finite graph is solvable when it has no odd cycles. Thus, every finite paradox involves the liar pattern. We can therefore conclude non-paradoxicality if such a graph is left after inducing values from a given valuation of atoms, but cases like that are rare. Still, the crucial analysis can often be limited to finitely representable schemas, as was done in the examples.

The other pattern of paradox is acyclic Yablo-like, which was not defined here but merely exemplified by the graph $(\omega, <)$, with each edge subdivided into three edges. These two patterns are, in fact, the only ones, according to the recent theorem from [47], showing that every paradox involves either an indirect liar (odd cycle) or a Yablo-like pattern, defined there in terms of graphs.

The next stage is to extend LSO to operators acting on open formulas. This will require some form of quotation mechanism, needed for syntax theory and for handling substitution, and might lead to some finer syntactic conditions necessary for the occurrence of paradoxes. In a still further perspective, one should not exclude the possibility of developing in LSO a truth theory for LSO.

7 Appendix: Language graphs and (semi)kernels

7.1 Some facts about (semi)kernels

The following equivalent semikernel condition is used in some proofs.

Fact 7.1 *For any* $L \subseteq \mathbf{V} : \mathbf{E}(L) \subseteq \mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L \iff \mathbf{E}(L) \subseteq \mathbf{E}^-(L) \cap \mathbf{V} \setminus L$.

Proof. If $\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L$ then $\mathbf{E}(L) \subseteq \mathbf{E}^-(L) = \mathbf{E}^-(L) \cap \mathbf{V} \setminus L$. If $\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \cap \mathbf{V} \setminus L$ then $\mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L$, for if some $x \in \mathbf{E}^-(L) \cap L$ then $\mathbf{E}(x) \cap L \neq \emptyset$, i.e., $\mathbf{E}(L) \not\subseteq \mathbf{V} \setminus L$. □

The two facts below imply equisolvability of graphs, showing actually that the two have *essentially the same solutions*: each solution of one can be expanded to a solution of the other, and each solution of the other, restricted to the first, is its solution. These facts, applied implicitly on the drawings, justify also duplication of vertices $\mathbf{S}_M$ as Aux, without affecting solutions.

A path $a_0...a_k$ is *isolated* if $\mathbf{E}_G(a_i) = \{a_{i+1}\}$ for $0 \leq i < k$ and $\mathbf{E}^-_G(a_i) = \{a_{i-1}\}$ for $0 < i < k$. A *double edge*, introduced earlier, is an isolated path of length 2. *Contraction* of such an isolated path amounts to identifying the first and the last vertex, joining their neighbourhoods and removing the intermediate vertices, i.e., obtaining graph G' where $\mathbf{V}_{G'} = \mathbf{V}_G \setminus \{a_1...a_k\}$, $\mathbf{E}_{G'}(a_0) = \mathbf{E}_G(a_k)$ and $\mathbf{E}^-_{G'}(a_0) = \mathbf{E}^-_G(a_0) \cup \mathbf{E}^-_G(a_k) \setminus \{a_{k-1}\}$. The first fact is a trivial observation.

Fact 7.2 *If G' results from G by contracting an isolated path of even length, then* $\forall K' \in sol(G') \exists! K \in sol(G) : K' \subseteq K$, *and* $\forall K \in sol(G) : K \cap \mathbf{V}_{G'} \in sol(G')$.

The same holds if G' results from a transfinite number of such contractions, provided that no *ray*, i.e., an infinite outgoing path with no repeated vertex, is contracted to a finite path.

The second fact shows that identifying vertices with identical out-neighbourhoods preserves and reflects (semi)kernels. To define this operation, let $R_G \subseteq \mathbf{V}_G \times \mathbf{V}_G$ relate two vertices in G with identical out-neighbourhoods, i.e., $R_G(a, b) \Leftrightarrow \mathbf{E}_G(a) = \mathbf{E}_G(b)$. It is an equivalence, so let

$G^{\downarrow}$ denote the quotient graph over equivalence classes, $[v] = \{u \in \mathbf{V}_G \mid R_G(v,u)\}$, with edges $\mathbf{E}_{G^{\downarrow}}([v],[u]) \Leftrightarrow \exists v \in [v], u \in [u] : \mathbf{E}_G(v,u)$. The operation can be iterated any number n of times, denoted by $G^{\downarrow n}$ and defined by: $G^{\downarrow 1} = G^{\downarrow}$ and $G^{\downarrow(n+1)} = (G^{\downarrow n})^{\downarrow}$. Vertices of $G^{\downarrow n}$ are taken as subsets of $\mathbf{V}_G$, $[u]^n = \{v \in \mathbf{V}_G \mid \exists i \leq n : R_{G^{\downarrow i}}([v]^i,[u]^i)\}$. For limit ordinals λ, $G^{\downarrow\lambda}$ is given by

$\mathbf{V}_{G^{\downarrow\lambda}} = \{[u]^\lambda \mid u \in \mathbf{V}_G\}$ where $[u]^\lambda = \bigcup_{i<\lambda}[u]^i = \{v \in \mathbf{V}_G \mid \exists i < \lambda : R_{G^{\downarrow i}}([u]^i,[v]^i)\}$ and

$\mathbf{E}_{G^{\downarrow\lambda}}([v]^\lambda,[u]^\lambda) \Leftrightarrow \exists n \in \lambda : \mathbf{E}_{G^{\downarrow n}}([v]^n,[u]^n)$.

Fact 7.3 *For every ordinal n, and SKr denoting either kernels or semikernels (sol or SK):*

(a) $K \in SKr(G) \Rightarrow \{[v]^n \mid v \in K\} \in SKr(G^{\downarrow n})$, and

(b) $K^{\downarrow n} \in SKr(G^{\downarrow n}) \Rightarrow \bigcup K^{\downarrow n} \in SKr(G)$.

Proof. (1) The proof for $n = 1$ shows the claim also for every successor n.
(a) $K^{\downarrow} = \{[v] \mid v \in K\}$ is independent, for if $\mathbf{E}_{G^{\downarrow}}([v],[w])$ for some $[v],[w] \in K^{\downarrow}$, then $\mathbf{E}_G(v,w)$ for some $v \in [v], w \in [w]$. But then $v,w \in K$ contradicting independence of K – if $x \in K$ then $[x] \subseteq K$, since $\forall x,y \in [v] : \mathbf{E}_G(x) = \mathbf{E}_G(y)$, so $\mathbf{E}_G(x) \cap K = \emptyset \Leftrightarrow \mathbf{E}_G(y) \cap K = \emptyset$.

If $[v] \in \mathbf{V}_{G^{\downarrow}} \setminus K^{\downarrow}$, then $[v] \subseteq \mathbf{V}_G \setminus K \subseteq \mathbf{E}^-_G(K)$, so $\forall v \in [v]\ \exists w \in K : \mathbf{E}_G(v,w)$. Then $[w] \in K^{\downarrow}$ and $[v] \in \mathbf{E}^-_{G^{\downarrow}}([w]) \subseteq \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$. Thus $\mathbf{V}_{G^{\downarrow}} \setminus K^{\downarrow} \subseteq \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$, so $K^{\downarrow} \in sol(G^{\downarrow})$.

If $K \in SK(G)$ and $[v] \in \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$, i.e., for some $v \in [v], w \in K : v \in \mathbf{E}^-_G(w)$, then $[v] \subseteq \mathbf{E}^-_G(w)$ and $[w] \in K^{\downarrow}$, so $[v] \in \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$, i.e., $\mathbf{E}_{G^{\downarrow}}(K^{\downarrow}) \subseteq \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$, so $K^{\downarrow} \in SK(G^{\downarrow})$.
(b) $K = \bigcup K^{\downarrow} = \{v \in \mathbf{V}_G \mid [v] \in K^{\downarrow}\}$ is independent, for if $\mathbf{E}_G(v,u)$ for some $v,u \in K$, then also $\mathbf{E}_{G^{\downarrow}}([v],[u])$ contradicting independence of $K^{\downarrow}$. If $x \notin K$ then $[x] \notin K^{\downarrow}$, and since $\mathbf{E}_{G^{\downarrow}}([x],[v])$ for some $[v] \in K^{\downarrow}$, so for some $y \in [x]$ and $v \in [v] \subseteq K$, $\mathbf{E}_G(y,v)$. But since $\mathbf{E}_G(y) = \mathbf{E}_G(x)$, so also $\mathbf{E}_G(x,v)$, i.e., $x \in \mathbf{E}^-_G(K)$. Thus $\mathbf{V}_G \setminus K \subseteq \mathbf{E}^-_G(K)$, and $K \in sol(G)$.

If $K^{\downarrow} \in SK(G^{\downarrow})$ and $v \in \mathbf{E}_G(K)$, then $[v] \in \mathbf{E}_{G^{\downarrow}}(K^{\downarrow}) \subseteq \mathbf{E}^-_{G^{\downarrow}}(K^{\downarrow})$, i.e., $[v] \in \mathbf{E}^-_{G^{\downarrow}}([w])$ for some $[w] \in K^{\downarrow}$. Then $[w] \subseteq K$ and $[v] \subseteq \mathbf{E}^-_G([w])$, so that $\mathbf{E}_G(K) \subseteq \mathbf{E}^-_G(K)$.
(2) We show the claim for limit λ.
(a) If $K \in sol(G)$, let $K^{\downarrow\lambda} = \{[v]^\lambda \mid v \in K\}$. If $\mathbf{E}_{G^\lambda}([v]^\lambda,[u]^\lambda)$ for some $[v]^\lambda,[u]^\lambda \in K^{\downarrow\lambda}$, i.e., $v,u \in K$, then for some $n \in \lambda : \mathbf{E}_{G^n}([v]^n,[u]^n)$, which means that $K^{\downarrow n} = \{[x]^n \mid x \in K\}$ is not a kernel of G^n, contrary to point (1). Hence $K^{\downarrow\lambda}$ is independent. If $[v]^\lambda \in \mathbf{V}_{G^\lambda} \setminus K^{\downarrow\lambda}$, then $[v]^\lambda \subseteq \mathbf{V}_G \setminus K$, so for any $v \in [v]$, there is a $u \in \mathbf{E}_G(v) \cap K$. Then also $[u]^\lambda \in \mathbf{E}_{G^\lambda}([v]^\lambda) \cap K^{\downarrow\lambda}$, hence $\mathbf{V}_{G^\lambda} \setminus K^{\downarrow\lambda} \subseteq \mathbf{E}^-_{G^\lambda}(K^{\downarrow\lambda})$, and $K^{\downarrow\lambda} \in sol(G^{\downarrow\lambda})$.

If $K \in SK(G)$, i.e., $\mathbf{E}_G(K) \subseteq \mathbf{E}^-_G(K)$ and $[v]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}(K^{\downarrow\lambda})$, then for some $n \in \lambda : [v]^n \in \mathbf{E}_{G^{\downarrow}n}([w]^n)$ for some $[w]^n \in K^{\downarrow n}$. Then also $[w]^n \in \mathbf{E}^-_{G^{\downarrow n}}(K^{\downarrow n})$, as $K^{\downarrow n} \in SK(G^{\downarrow n})$ by IH, but then also $[w]^\lambda \in \mathbf{E}^-_{G^{\downarrow\lambda}}(K^{\downarrow\lambda})$. Thus $\mathbf{E}_{G^{\downarrow\lambda}}(K^{\downarrow\lambda}) \subseteq \mathbf{E}^-_{\downarrow\lambda}(K^{\downarrow\lambda})$.
(b) For a kernel $K^{\downarrow\lambda}$ of $G^{\downarrow\lambda}$, let $K = \bigcup K^{\downarrow\lambda} = \{v \in \mathbf{V}_G \mid [v]^\lambda \in K^{\downarrow\lambda}\}$. If $v \in \mathbf{E}^-_G(x)$ for some $x \in K$, then $v \notin K$ for if $v \in K$, i.e., $[v]^\lambda \subseteq K$, then $[v]^\lambda \in \mathbf{E}^-_{G^{\downarrow\lambda}}([x]^\lambda) \cap K^{\downarrow\lambda} \subseteq \mathbf{E}^-_{G^{\downarrow\lambda}}(K^{\downarrow\lambda}) \cap K^{\downarrow\lambda}$ contradicting independence of $K^{\downarrow\lambda}$. If $v \in \mathbf{V}_G \setminus K$, i.e., $[v]^\lambda \notin K^{\downarrow\lambda}$, then there is some $[u]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}([v]^\lambda) \cap K^{\downarrow\lambda}$. Since $[u]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}([v]^\lambda)$, so for some $n < \lambda, [u]^n \in \mathbf{E}_{G^{\downarrow n}}([v]^n)$, that is, for some $u' \in [u]^n, u' \in \mathbf{E}_G(v)$. Since $[u]^\lambda \in K^{\downarrow\lambda}$, so $[u]^n \subseteq [u]^\lambda \subseteq K$, hence $v \in \mathbf{E}^-_G(K)$ and $K \in sol(G)$.

If $K^{\downarrow\lambda} \in SK(G^{\downarrow\lambda})$, independence of K follows as above. If $v \in \mathbf{E}_G(K)$, then $[v]^\lambda \in \mathbf{E}_{G^{\downarrow\lambda}}(K^{\downarrow\lambda}) \subseteq \mathbf{E}^-_{G^{\downarrow\lambda}}(K^{\downarrow\lambda})$, i.e., for some $n \in \lambda : [v]^n \in \mathbf{E}^-_{G^{\downarrow n}}(K^{\downarrow n})$. By IH, $[v]^n \subseteq \mathbf{E}^-_G(K^{\downarrow n}) \subseteq \mathbf{E}^-_G(K)$. Hence $\mathbf{E}_G(K) \subseteq \mathbf{E}^-_G(K)$. □

7.2 Logical and graph equivalences

We formulate logical and some other notions of equivalence in terms of graphs. Two $\mathcal{L}^+_M$ sentences are equivalent, in $\mathcal{G}_M(\mathcal{L}^+)$, if they belong to the same kernels. $\mathcal{L}^+$ sentences are (logically) equivalent if they are so in every language graph:

$$\begin{array}{lll} \text{for a graph } G \text{ and } A,B\in\mathbf{V}_G: & A \overset{G}{\Longleftrightarrow} B \text{ iff } \forall K\in sol(G): A\in K \Leftrightarrow B\in K & \\ \text{for } A,B\in\mathbf{S}^+_M: & A \overset{\mathcal{L}^+_M}{\Longleftrightarrow} B \text{ iff } A \overset{\mathcal{G}_M(\mathcal{L}^+)}{\Longleftrightarrow} B & (7.4)\\ \text{for } A,B\in\mathbf{S}^+: & A \overset{\mathcal{L}^+}{\Longleftrightarrow} B \text{ iff } \forall M: A \overset{\mathcal{L}^+_M}{\Longleftrightarrow} B. & \end{array}$$

A more specific equivalence will be used, corresponding to prenex operations. Each sentence can be written in PDNF, that is, prenex normal form with matrix in DNF. Two $\mathcal{L}^+_M$ sentences are *PDNF equivalent*, denoted by $A \overset{P}{\Leftrightarrow} B$, if they have (also) identical PDNFs. To show that PDNF equivalence implies $\mathcal{L}^+$ equivalence, we use a more structural notion of equivalence in a graph.

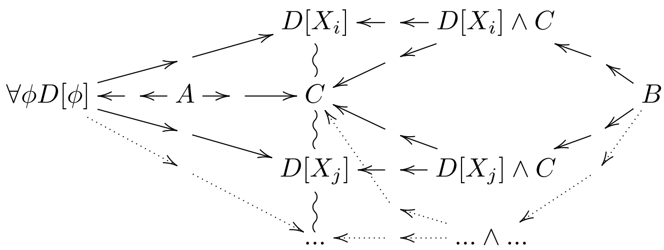
By $\mathbf{E}^*_G$ we denote the reflexive and transitive closure of $\mathbf{E}_G$ and by $\mathbf{E}^*_G(S)$, for $S\in\mathbf{V}_G$, the subgraph of G induced by all vertices reachable from S. A *common cut* of $A,B\in\mathbf{V}_G$ is a set of vertices $C\subseteq\mathbf{E}^*_G(A)\cap\mathbf{E}^*_G(B)$, such that every path leaving A and prolonged sufficiently far crosses C and so does every path leaving B. (C may intersect A and B and contain vertices on various cycles intersecting A and B.) We say that A and B are *cut equivalent*, $A \overset{c}{\Leftrightarrow} B$, if there is a common cut C such that for every correct (not falsifying (3.2)) valuation of C, every correct extension to $\{A,B\}$ forces identical value of A and B. Obviously, if $A \overset{c}{\Leftrightarrow} B$ in a graph G, then also $A \overset{G}{\Leftrightarrow} B$, as each $K\in sol(G)$ determines a correct valuation of every common cut of A and B. FOL tautologies/contradictions are in all/none kernels, implying graph equivalence. Contingent propositional/FOL equivalence $P(\overline{\phi})\Leftrightarrow Q(\overline{\phi})$ implies cut equivalence, $P(\overline{\phi}) \overset{c}{\Leftrightarrow} Q(\overline{\phi})$, with shared atoms giving possible common cut in graphs (trees) of each instance. We show this for PDNF.

Fact 7.5 *For $A,B\in\mathbf{S}^+_M$ in $\mathcal{G}_M(\mathcal{L}^+)$, if $A \overset{P}{\Leftrightarrow} B$ then $A \overset{c}{\Leftrightarrow} B$, hence $A \overset{\mathcal{L}^+_M}{\Longleftrightarrow} B$.*

Proof. Letting $\mathcal{G}=\mathcal{G}_M(\mathcal{L}^+)$ and assuming $sol(\mathcal{G})\neq\emptyset$, we verify standard prenex transformations, considering only s-quantifiers, as the object-quantifiers can be treated in the same way.

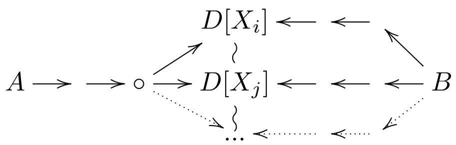
1. The claim holds trivially for B obtained by renaming bound s-variables (avoiding name clashes) in A, as the two have the same subgraph. This is also the case for the subgraphs of $A=\neg\forall\phi D[\phi]$ and $B=\exists\phi\neg D[\phi]$.

2. $A=(\forall\phi D[\phi])\wedge C \overset{\mathcal{G}}{\Leftrightarrow} \forall\phi(D[\phi]\wedge C)=B$, with no free occurrences of ϕ in C. On the schematic subgraph below, X_i,X_j... stand for all $\mathbf{S}^+_M$ and common cut is marked by the waved line.



Inspecting the graph, we see that, for any kernel K:
$B\in K\Leftrightarrow\big((D[X_i]\wedge C)\in K \text{ for all } X_i\big)\Leftrightarrow\big(C\in K\wedge(D[X_i]\in K \text{ for all } X_i)\big)\Leftrightarrow A\in K.$

3. For $A=\neg\exists\phi D[\phi] \overset{\mathcal{G}}{\Leftrightarrow} \forall\phi\neg D[\phi]=B$ the schematic subgraph is as follows:



Obviously, for any kernel $K: A\in K\Leftrightarrow\circ\in K\Leftrightarrow(D[X_i]\notin K \text{ for all } X_i)\Leftrightarrow B\in K.$ □

Thus, every sentence in $\mathcal{L}^+$ has an $\overset{\mathcal{L}^+}{\Longleftrightarrow}$-equivalent PDNF sentence. A useful consequence is that, considering below solvability of $\mathcal{G}_M(\mathcal{L}^\Phi)$ or $\mathcal{G}_M(\mathcal{L}^+)$, we can limit attention to sentences in PDNF.

7.3 No paradoxes in $\mathcal{L}^{\Phi}$ – solvability of $\mathcal{G}(\mathcal{L}^{\Phi})$

Extending any classical language $\mathcal{L}$ with s-quantifiers to $\mathcal{L}^{\Phi}$ does not introduce any paradoxes. The following theorem shows a stronger claim that, in a domain M, all $\mathcal{L}^{\Phi}_M$ sentences obtain unique values under every valuation of $\mathcal{L}_M$ sentences, which is determined by a valuation of atoms $\mathbf{A}_M$.

Theorem 7.6 (3.16) *In any $\mathcal{G}_M(\mathcal{L}^{\Phi})$, each $\rho \in \mathbf{2}^{\mathbf{S}_M}$ has a unique extension $\hat{\rho} \in sol(\mathcal{G}_M(\mathcal{L}^{\Phi}))$ with $\hat{\rho}|_{\mathbf{S}_M} = \rho$.*

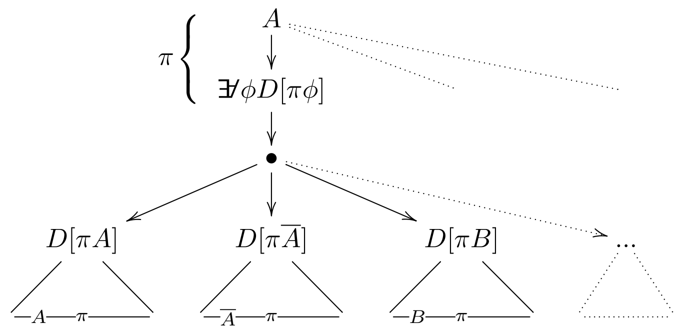
Proof. Graph $\mathcal{G}_M(\mathcal{L}^{\Phi})$ consists of two subgraphs, the strong component with all s-quantified sentences, $\mathcal{G}_M(\mathcal{L}^{\Phi} \setminus \mathcal{L}) = \bigcup_{A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M} \mathcal{G}_M(A)$, and the collection $\mathcal{G}_M(\mathcal{L}) = \bigcup_{B \in \mathbf{S}_M} T_M(B)$ of trees for object-language sentences, with no edges from the latter to the former. For each $A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M$ in the former, there are (single or double) edges from external leaves $V \in ext(\mathcal{G}_M(A))$, to the sources of $\mathcal{G}_M(V)$, that are trees $T_M(V)$ when $V \in \mathbf{S}_M$. By Lemma 3.17 below, valuation ρ of $\mathbf{S}_M = \mathbf{V}_{\mathcal{G}_M(\mathcal{L})}$, determines a solution ρ^-_A of each $\mathcal{G}^-_M(A)$ (subgraph of $\mathcal{G}_M(A)$ without its DNF-feet), compatible with every valuation of $ext(\mathcal{G}_M(A))$. Hence, these can be combined into $\rho \cup \bigcup_{A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M} \rho^-_A$ forcing value $\rho^-_V(V)$ at each $V \in ext(\mathcal{G}_M(A))$, and thus determining solutions of all DNF-feet. Each $\mathcal{G}_M(A)$ obtains thus a solution $\rho_A \supset \rho^-_A$, yielding a unique $\hat{\rho} = (\rho \cup \bigcup_{A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M} \rho_A) \in sol(\mathcal{G}_M(\mathcal{L}))$, extending ρ. $\square$

The missing lemma shows that for each sentence $A \in \mathbf{S}^{\Phi}_M \setminus \mathbf{S}_M$, solution of the subgraph of $\mathcal{G}_M(A)$ without its DNF-feet, denoted by $\mathcal{G}^-_M(A)$, depends on the valuation of $\mathbf{S}_M$, but not of external leaves $ext(\mathcal{G}_M(A))$, as the second part of the lemma states. Valuation of $ext(\mathcal{G}_M(A))$ affects, of course, values in DNF-feet in which they occur.

Lemma 7.7 (3.17) *For every graph $\mathcal{G}_M(\mathcal{L}^{\Phi})$ and sentence $A \in \mathbf{S}^{\Phi}_M$, each valuation ρ of $\mathbf{S}_M$ and of external leaves of $\mathcal{G}_M(A)$, $\rho \in \mathbf{2}^{\mathbf{S}_M \cup ext(\mathcal{G}_M(A))}$, has a unique extension to $\rho_A \in sol(\mathcal{G}_M(A))$. Restriction $\rho|_{\mathbf{S}_M}$ determines restriction of ρ_A to $\mathcal{G}^-_M(A)$: if $\rho|_{\mathbf{S}_M} = \sigma|_{\mathbf{S}_M}$ then $\rho_A|_{\mathcal{G}^-_M(A)} = \sigma_A|_{\mathcal{G}^-_M(A)}$.*

Proof. By Fact 7.5, we can limit attention to sentences in PDNF.

For $A \in \mathbf{S}^{\Phi}_M$, with the number $q(A) = n + 1 \geq 1$ of s-quantifiers and s-variables, and for n-sequence of sentences $\pi \in (\mathbf{S}^{\Phi}_M)^n$ substituted for the n s-variables of A bound by its first n quantifers, the sources of all feet, $A(\pi S) = D[\pi S]$, $S \in \mathbf{S}^{\Phi}_M$, are grandchildren of vertex $A(\pi) = \exists\!\!\!/ \phi D[\pi\phi]$. (In the drawing, $\exists\!\!\!/ = \exists$ and all feet have the common parent $\bullet$; when $\exists\!\!\!/ = \forall$, their distinct parents are children of $A(\pi)$.) Each foot $A(\pi S)$ represents an application of the same boolean function $d^{\pi}(\phi) = D[\pi\phi]$, evaluating $D[\pi\phi]$ given valuation of its parameters π, ϕ and, possibly, some atoms $L_A \subset \mathbf{S}_M$ occurring in the original matrix $D[...]$. For any $\rho \in \mathbf{2}^{\mathbf{S}_M}$, L_A obtain fixed values so, considering d^{π}, we assume the effects of $\rho(L_A)$ taken into account.



i. The *internal* vertices of π, $int(\pi)$ are sentences occurring on the path after substitutions, and *external* ones are those which do not, $ext(\pi) = \mathbf{S}^{\Phi}_M \setminus int(\pi)$. Some 'sinks' of the feet have single or double edges to vertices from π, which are $int(\pi)$, including $\pi_0 = A$ and $\exists\!\!\!/ \phi D[\pi\phi]$ (when this is substituted for ϕ in $D[\pi\phi]$.) As branches from $\bullet$ instantiate ϕ with every sentence $S \in \mathbf{S}^{\Phi}_M$, all sentences from $int(\pi)$ do occur in some feet.

ii. Depending on whether $\exists\!\!\!\forall$ is $\forall$ or $\exists$, the value at vertex $\exists\!\!\!\forall\phi D[\pi\phi]$, as a function of values of its grandchildren, is either

(*) $\exists\phi D[\pi\phi] = \bigvee_{S\in\mathbf{S}_M} d^\pi(S)$ or $\forall\phi D[\pi\phi] = \bigwedge_{S\in\mathbf{S}_M} d^\pi(S)$.

We consider first the case when $|\pi| = q(A) - 1$, i.e., $A(\pi) = \exists\!\!\!\forall\phi D[\pi\phi]$ is the grandparent of the completely substituted (sources of) DNF-feet ($D[\pi A], D[\pi B]$, etc., in the drawing).

Every valuation of sentences from π, abbreviated as $\alpha \in \mathbf{2}^\pi$, specializes function $d^\pi(\phi)$ to a unary boolean function $d^{\alpha(\pi)}(\phi) = D[\alpha(\pi)\phi]$, and (*) to

(**) $\exists\phi D[\alpha(\pi)\phi] = \bigvee_{S\in\mathbf{S}_M} d^{\alpha(\pi)}(S)$ or $\forall\phi D[\alpha(\pi)\phi] = \bigwedge_{S\in\mathbf{S}_M} d^{\alpha(\pi)}(S)$.

iii. As a boolean function of one variable, $d^{\alpha(\pi)}(\phi)$ is either constant or not. If it is constant, i.e., $d^{\alpha(\pi)}(\phi) = d^{\alpha(\pi)}(\neg\phi)$, then $\exists\!\!\!\forall\phi D[\alpha(\pi)\phi]$ obtains the same value in either case of (**). Otherwise, $d^{\alpha(\pi)}(\neg\phi) = \neg d^{\alpha(\pi)}(\phi)$ and, since for each $S \in \mathbf{S}_M^\Phi$ both $d^{\alpha(\pi)}(S)$ and $d^{\alpha(\pi)}(\neg S)$ enter evaluation of (**), this yields constant $\mathbf{0}$ at their least common predecessor ($\bullet$ when $\exists\!\!\!\forall = \exists$ and $A(\pi)$ when $\exists\!\!\!\forall = \forall$). In this way, for every $\alpha \in \mathbf{2}^\pi$, $A(\pi)$ obtains a unique value $\alpha^\uparrow(A(\pi))$, induced from all $D[\alpha(\pi)S]$ by (**), but determined already by $d^{\alpha(\pi)}(\phi)$, independently from

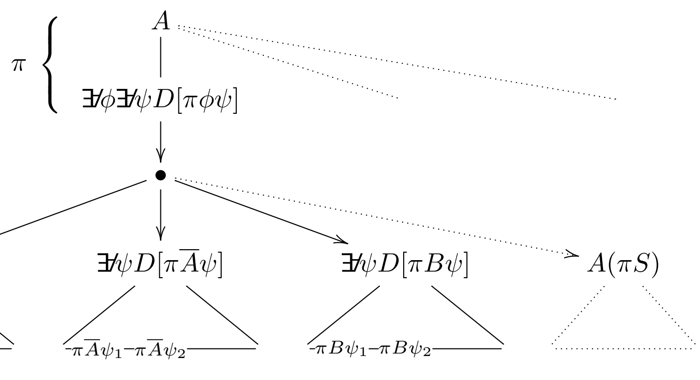
(i) valuation $\alpha(A(\pi))$, i.e., if $\alpha_\mathbf{0}, \alpha_\mathbf{1} \in \mathbf{2}^\pi$ differ only at $A(\pi)$, then $\alpha_\mathbf{0}^\uparrow(A(\pi)) = \alpha_\mathbf{1}^\uparrow(A(\pi))$, and
(ii) independently from valuation of $ext(\pi)$, since each external vertex S enters both evaluation of $d^{\alpha(\pi)}(S)$ and of $d^{\alpha(\pi)}(\neg S)$, with jointly constant contribution to (**) as just explained.

Point (i) means that cycles from the feet to $A(\pi)$ admit a unique solution $\rho_{A(\pi),\alpha}$ to the subgraph $\mathcal{G}_M(A(\pi))$ of $\mathcal{G}_M(A)$, given any $\rho \in \mathbf{2}^{\mathbf{S}_M\cup ext(\pi)}$ and $\alpha \in \mathbf{2}^{\pi^-}$, where π^- is π without its last element $A(\pi)$. By point (ii), $\rho|_{ext(\pi)}$ is inessential, so if $\rho|_{\mathbf{S}_M^-} = \sigma|_{\mathbf{S}_M^-}$ then $\rho_{A(\pi),\alpha}(A) = \sigma_{A(\pi),\alpha}(A)$.

iv. This is the basis for the claim that for each A with $q(A) \geq 1$ and each path π from the source A with $|\pi| < q(A)$, each valuation of π^- and $\mathbf{S}_M$ determines a unique value of $A(\pi)$. We use its formulation above, i.e., for each $\rho \in \mathbf{2}^{\mathbf{S}_M\cup ext(\pi)}$ and each $\alpha \in \mathbf{2}^\pi$, vertex $V = A(\pi)$ (above the sources of the feet) obtains a unique value $\alpha^\uparrow(V)$, which depends at most on valuation of vertices on π^- (above V), but neither on the value (i) of $\alpha(V)$ nor (ii) of $\rho(X)$, for any $X \in ext(\pi)$.

The claim is shown by induction on $h - l$, where $h \geq 1$ is the distance of the source A from the sources of the feet and l is the distance of V from the source A, $h > l \geq 0$. The basis $h - l = 1$ is **iii**.

v. The argument from **iii** works also in the induction step. For $0 \leq |\pi| = l < h - 1$, we have the following counterpart of the drawing from **iii**, with $A(\pi) = \exists\!\!\!\forall\phi\exists\!\!\!\forall\psi D[\pi\phi\psi]$, where $\exists\!\!\!\forall\psi$ is the sequence of remaining quantifiers, and ψ_1, ψ_2 at the bottom signal various substitutions for ψ.



Given $\alpha \in \mathbf{2}^\pi$, IH applied to the lowest triangles in the drawing, i.e., subgraphs $\mathcal{G}_M(A(\pi S))$ with sources $A(\pi S)$ for $S \in \mathbf{S}_M^\Phi$, gives to each $A(\pi S)$ a unique value, independent of valuation of $ext(\pi S)$. Consequently $A(\pi\phi)$ is a function of only π and ϕ, so that for any $\alpha \in \mathbf{2}^\pi$, it represents a function $d^{\alpha(\pi)}$ of ϕ. The same argument and cases for $d^{\alpha(\pi)}$ as in **iii** show that the value $\alpha^\uparrow(A(\pi))$, induced to the common grandparent of all $A(\pi S)$ under valuation $\alpha \in \mathbf{2}^\pi$, is equal whether $\alpha(A(\pi)) = \mathbf{1}$ or $\alpha(A(\pi)) = \mathbf{0}$, giving point (i) of induction. As for each $A(\pi S)$ its value under $\alpha^\uparrow$ is independent from valuation of $ext(\pi S)$ by IH, the induced value $\alpha^\uparrow(A(\pi))$ is independent from

$ext(\pi) = \bigcap_{S\in\mathbf{S}_M} ext(\pi S)$, giving point (ii) of induction. Consequently, $\alpha^{\uparrow}(A(\pi))$ is unique and independent of valuations of $ext(\pi)$ and of $A(\pi)$, which establishes the induction step.

vi. Thus, the value of the source A is determined, for each $\rho \in \mathbf{2}^{\mathbf{S}_M}$, independently from valuation of $ext(\mathcal{G}_M(A))$. Starting now from A and using claim **iv** downwards, the value of $A(S)$, for each $S \in \mathbf{S}^{\Phi}_M$, is determined by ρ and value of A (independently from valuation of $ext(\mathcal{G}_M(A))$). Since A is determined by ρ, so is the value of $A(S)$. Proceeding inductively down the tree $T_M(A)$, valuation ρ^-_A of $T_M(A)^-$ is seen determined by ρ, independently from valuation of $ext(\mathcal{G}_M(A))$. The latter determines then values in all feet of $\mathcal{G}_M(A)$, yielding a unique solution ρ_A of $\mathcal{G}_M(A)$, with $\rho^-_A \subset \rho_A$ and $\rho_A|_{\mathbf{S}_M\cup ext(\mathcal{G}_M(A))} = \rho$. $\square$

7.4 Expressive power of $\mathcal{L}^{\Phi}$

By Theorem 3.19 below, extending $\mathcal{L}$ to $\mathcal{L}^{\Phi}$ does not increase the expressive power, as the introduced s-quantification amounts to a complex form of quantification over boolean values. In models of $A = \forall\phi F[\phi]$, F is true for all sentences ϕ, including A itself. Guaranteeing a well-defined boolean value for each sentence (in each structure), the theorem makes this "including itself" harmless, reducing $\forall\phi$ to propositional quantifier. To verify A it suffices to verify $F[\phi]$ for $\phi = \mathbf{1}$ and $\phi = \mathbf{0}$. This follows provided that every sentential context $F[\phi]$ (having only ϕ free), is a congruence preserving equivalence of sentences, i.e., such that for each pair of $\mathcal{L}^{\Phi}$ sentences A, B,

$$A \overset{\mathcal{L}^{\Phi}}{\Longleftrightarrow} B \text{ implies } F[A] \overset{\mathcal{L}^{\Phi}}{\Longleftrightarrow} F[B]. \tag{7.8}$$

Given an internal equivalence $A \leftrightarrow B \overset{\mathcal{L}^{\Phi}}{\Leftrightarrow} (A \wedge B) \vee (\neg A \wedge \neg B)$, it suffices that for every structure M (abbreviating (M, ρ)), if $M \models A \leftrightarrow B$ then $M \models F[A] \leftrightarrow [B]$. These assumptions are satisfied for classical logic. Let $\top/\bot$ stand for an arbitrary tautology/contradiction in $\mathcal{L}$.

Fact 7.9 *For every $\mathcal{L}^{\Phi}$ formula $F[\phi]$ with only ϕ free and for every $\mathcal{L}$-structure M:*
$M \models \forall\phi F[\phi]$ iff $M \models F[\top] \wedge F[\bot]$, and $M \models \exists\phi F[\phi]$ iff $M \models F[\top] \vee F[\bot]$.

Proof. If $M \models \forall\phi F[\phi]$ then, in particular, $M \models F[\top]$ and $M \models F[\bot]$, so $M \models F[\top] \wedge F[\bot]$. Conversely, assuming $M \models F[\top] \wedge F[\bot]$, let S be an arbitrary $\mathcal{L}$-sentence. If $M \models S$ then also $M \models S \leftrightarrow \top$, hence $M \models F[S]$ by (7.8), since $M \models F[\top]$. If $M \not\models S$ then also $M \models S \leftrightarrow \bot$, hence $M \models F[S]$, since $M \models F[\bot]$. In either case $M \models F[S]$, and since S was arbitrary, $M \models \forall\phi F[\phi]$.

If $M \models \exists\phi F[\phi]$, let S be a sentence for which $M \models F[S]$. Either $M \models S$ or $M \not\models S$, i.e, $M \models \neg S$. In the first case $M \models S \leftrightarrow \top$ and in the latter $M \models S \leftrightarrow \bot$. Thus either $M \models F[\top]$ or $M \models F[\bot]$, hence $M \models F[\top] \vee F[\bot]$. Conversely, if $M \models F[\top] \vee F[\bot]$ then either $M \models F[\top]$ or $M \models F[\bot]$. In either case $M \models \exists\phi F[\phi]$. $\square$

In particular, the unique solution of $\mathcal{G}(\emptyset^{\Phi})$ contains exactly true quantified boolean sentences, QBS. The right sides of the equivalences in Fact 7.9 give their standard semantics.

By Theorem 3.16, values of $\mathcal{L}$ sentences determine values of all $\mathcal{L}^{\Phi}$ sentences. Consequently, if structures M, N are elementarily equivalent in $\mathcal{L}$, $M \overset{\mathcal{L}}{\equiv} N$, they are so also in $\mathcal{L}^{\Phi}$, $M \overset{\mathcal{L}^{\Phi}}{\equiv} N$.

Fact 7.10 *For any $\mathcal{L}$ structures M and N, $M \overset{\mathcal{L}}{\equiv} N$ iff $M \overset{\mathcal{L}^{\Phi}}{\equiv} N$.*

Proof. The non-obvious implication to the right follows by induction on the number of s-quantifiers. Let $M \overset{k}{\equiv} N$ denote that M and N model the same $\mathcal{L}^{\Phi}$ sentences with up to k s-quantifiers, so that $M \overset{\mathcal{L}}{\equiv} N$ corresponds to $M \overset{0}{\equiv} N$, giving the induction basis. For a PDNF sentence $A = \forall\phi\overline{\exists\!\!\!\forall\psi} D[\phi, \overline{\psi}]$, where $|\overline{\psi}| = k \geq 0$, suppose that

(m) $M \models A$, i.e., for every $F \in \mathbf{S}^{\Phi} : M \models \overline{\exists\!\!\!\forall\psi} D[F, \overline{\psi}]$, while

(n) $N \not\models A$, i.e., for some $F_0 \in \mathbf{S}^{\Phi} : N \not\models \overline{\exists\!\!\!\forall\psi} D[F_0, \overline{\psi}]$.

F_0 has some s-quantifiers, as otherwise (m), (n) contradict IH, $M \overset{k}{\equiv} N$. Now $N \not\models \overline{\exists\!\!\!\forall\psi} D[F_0, \overline{\psi}]$ implies $N \not\models \forall\phi\overline{\exists\!\!\!\forall\psi} D[\phi, \overline{\psi}]$ yielding, by Fact 7.9, either $N \not\models \overline{\exists\!\!\!\forall\psi} D[\top, \overline{\psi}]$ or $N \not\models \overline{\exists\!\!\!\forall\psi} D[\bot, \overline{\psi}]$. For any $\mathcal{L}$ sentence $P_0 \Leftrightarrow \top$ in the former case, and $P_0 \Leftrightarrow \bot$ in the latter, $N \not\models \overline{\exists\forall\psi}D[P_0,\overline{\psi}]$. This last sentence has k s-quantifiers so, by IH, $M \not\models \overline{\exists\forall\psi}D[P_0,\overline{\psi}]$, which contradicts (m). An analogical argument shows the induction step for $A = \exists\phi\overline{\exists\forall\psi}D[\phi,\overline{\psi}]$. □

For any theory in $\mathcal{L}^{\Phi}$, Fact 7.9 makes it straightforward to construct a theory in $\mathcal{L}$ with the same model class. For any $\mathcal{L}^{\Phi}$ sentence A in PDNF, an $\mathcal{L}$ sentence A^-, with $Mod(A) = Mod(A^-)$, is obtained replacing $\forall\phi F[\phi]$ by $F[\top] \wedge F[\bot]$ and $\exists\phi F[\phi]$ by $F[\top] \vee F[\bot]$. E.g., starting with $A = \forall\phi\exists\psi(C \wedge \phi) \vee (D \wedge \psi) \vee (\phi \wedge \psi)$, with $C, D \in \mathbf{S}$, one application of Fact 7.9 yields

$$\exists\psi\big((C \wedge \top) \vee (D \wedge \psi) \vee (\top \wedge \psi)\big) \wedge \exists\psi\big((C \wedge \bot) \vee (D \wedge \psi) \vee (\bot \wedge \psi)\big),$$

which simplifies to: $\exists\psi(C \vee (D \wedge \psi) \vee \psi) \wedge \exists\psi(D \wedge \psi) \iff \exists\psi(D \wedge \psi) \overset{7.9}{\iff} (D \wedge \top) \vee (D \wedge \bot) \iff D$, so $Mod(A) = Mod(D)$. Proceeding thus by induction on the number of s-quantifiers (in PDNF of $\mathcal{L}^{\Phi}$ sentences), Fact 7.9 yields $\forall A \in \mathbf{S}^{\Phi}\exists A^- \in \mathbf{S} : Mod(A) = Mod(A^-)$, establishing

Theorem 7.11 (3.19) *For every $\Gamma \subseteq \mathcal{L}^{\Phi}$ there is a $\Gamma^- \subseteq \mathcal{L}$ with $Mod(\Gamma) = Mod(\Gamma^-)$.*

7.5 Solvability of $\mathcal{G}(\mathcal{L}^+)$ and of definitional extensions

Proof of Lemma 3.17 relies on each DNF-foot being a boolean function, **ii-iii**. It can be repeated, ensuring absence of paradoxes in $\mathcal{L}^+$, if each operator respects boolean equivalence, for instance, is constant, so each language graph $\mathcal{G}_M(\mathcal{L}^+)$ is solvable, although such interpretations are hardly sufficient. We show non-paradoxicality of a definitional extension $\mathcal{L}^P$ of a non-paradoxical language $\mathcal{L}^+$, namely, extension with a fresh predicate P defined by a sentence

$$\forall\phi(P(\phi) \leftrightarrow \exists\forall\psi F[\phi,\psi]), \tag{5.6}$$

where F is an $\mathcal{L}^+$-formula (with free variables ϕ among those of the left side $P(\phi)$). Now, each extension of $\mathcal{L}^+$ to $\mathcal{L}^P$ introduces into each graph $\mathcal{G}(\mathcal{L}^+)$ all complex sentences with P and atomic 2-cycles $P(S) \leftrightarrows \overline{P(S)}$, for every sentence S of $\mathcal{L}^P$. For the special extension (5.6), $\mathcal{G}(\mathcal{L}^+)$ can be extended to $\mathcal{G}(\mathcal{L}^P)$ by drawing a double edge from $P(S)$ to its defining sentence $\exists\forall\psi F(S,\psi)$, instead of an edge to $\overline{P(S)}$. Every kernel of such a graph determines a kernel of the graph with atomic 2-cycles where each $P(S)$ and $\exists\forall\psi F(S,\psi)$ obtain the same values, and vice versa. Lemma 7.16 below, giving immediately Theorem 5.7, shows that for any language graph G for $\mathcal{L}^+$, such an extension G^P, for P axiomatized by (5.6), preserves solutions of G. Its proof amounts to elimination of symbol P, replacing each $P(S)$ by its definiens $\exists\forall\psi F[S,\psi]$. Such a replacement, trivial in FOL, must proceed recursively on a cyclic graph (e.g., $P(P(S))$ needs repeated replacements) and involves some technicalities. These end with the paragraph before Lemma 7.16.

The proof assumes a language graph G over some domain M, in which no two vertices have equal out-neighbourhoods. (If G contains such vertices, as language graphs typically do, their identification preserves essentially the solutions by Fact 7.3, and we apply the construction and fact below to the so quotiented G.) The graph $G^P = \mathcal{G}_M(\mathcal{L}^P)$ contains G as an induced subgraph.

As the first step, we quotient atoms of G^P containing P. Let $\simeq$ be congruence on $\mathcal{L}^P_M$-sentences induced by the basic reflexive relation $P(S) \simeq_0 \exists\forall\psi F[S,\psi]$, for every $\mathcal{L}^P_M$-sentence S. For every operator Q distinct from P, we identify every two atoms $Q(A_1...A_n) \simeq Q(B_1...B_n)$ when $A_i \simeq B_i$ for $1 \leq i \leq n$. Each equivalence class contains an atom $Q(S_1...S_n)$ for some $S_i \in \mathbf{S}^+_M$, not containing any P, so in the following we can assume only such atoms present. It is a simple observation that quotient $q : G^P \to H$, where $\mathbf{E}_H(q(x)) = \{q(y) \mid y \in \mathbf{E}_{G^P}(x)\}$ in the resulting graph H, reflects kernels, so the preimage of any kernel of H is a kernel of G^P.

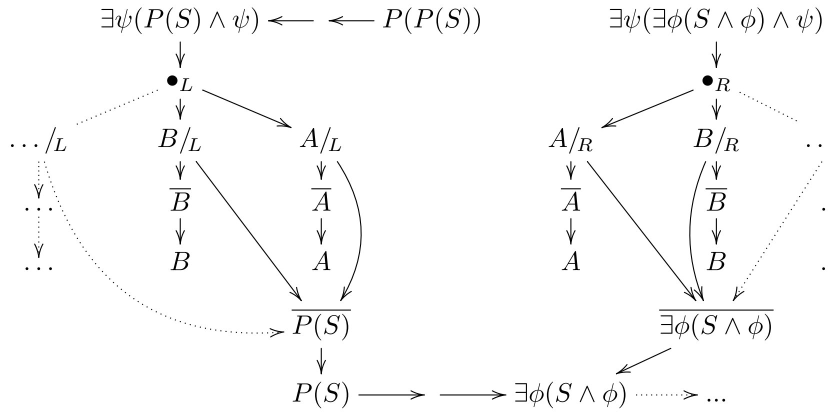
We now map $\gamma : H \to G$, performing a sequence of identifications $\gamma_i : H_{i-1} \to H_i$, for $0 < i \in \omega$ and $H_0 = H$. Each γ_i is identity on the subgraph G of H_i, identifying some vertices from $\mathbf{V}_i \setminus \mathbf{V}_G$ with some in $\mathbf{V}_G$. First, we identify $\gamma_1(P(S)) = \exists\forall\psi F[S,\psi]$, removing the double edge and the intermediate vertex $\bullet_{P(S)}$ between $P(S)$ and its definiens $\exists\forall\psi F[S,\psi]$, for $S \in \mathbf{S}^P_M$. Then $\gamma_{i+1}(v) = w$ when vertices $v \in \mathbf{V}_i \setminus \mathbf{V}_G$ and $w \in \mathbf{V}_G$ have the same out-neighbourhood. More precisely, (7.12) defines γ inductively, starting with $\mathbf{V}_0 = \mathbf{V}_H, \mathbf{E}_0 = \mathbf{E}_H$.

$$
\begin{array}{ll}
i = 1, & \text{letting } Re_0 = \bigcup\{\{P(S), \bullet_{P(S)}\} \mid S \in \mathbf{S}^P_M, \{\bullet_{P(S)}\} = \mathbf{E}_0(P(S))\} \text{ define:} \\
& \gamma_1(v) = \begin{cases} \nexists\psi F[S,\psi], & \text{if } v = P(S) \text{ for any } S \in \mathbf{S}^P_M \\ v & \text{if } v \notin Re_0 \end{cases} \\
& \text{The resulting graph } H_1 \text{ is given by :} \\
& \mathbf{V}_1 = \mathbf{V}_0 \setminus Re_0, \text{ and } \mathbf{E}_1(v) = \{\gamma_1(w) \mid w \in \mathbf{E}_0(v)\} \setminus Re_0 \\
i+1, & \text{letting } Re_i = \{v \in \mathbf{V}_i \setminus \mathbf{V}_G \mid \exists w \in \mathbf{V}_G : \mathbf{E}_i(v) = \mathbf{E}_i(w)\} \text{ define:} \\
& \gamma_{i+1}(v) = \begin{cases} w \in \mathbf{V}_G \text{ such that } \mathbf{E}_i(v) = \mathbf{E}_i(w) & \text{if } v \in Re_i \\ v & \text{if } v \notin Re_i \end{cases} \\
& \text{The resulting graph } H_{i+1} \text{ is given by :} \\
& \mathbf{V}_{i+1} = \mathbf{V}_i \setminus Re_i \text{ and } \mathbf{E}_{i+1}(v) = \mathbf{E}_i(\gamma_{i+1}(v)) \setminus Re_i
\end{array}
\tag{7.12}
$$

Let $\gamma(v) = \gamma_n(v)$, for $v \in \mathbf{V}_H$, where $n \in \omega$ is the least such that $\forall m > n : \gamma_m(v) = \gamma_n(v)$.

Function γ is well-defined by the assumption that G has no pair of vertices with identical out-neighbourhoods. For $A, B \in \mathbf{V}_H$ and $n \in \omega$, we denote by $A \sim_n B$ that $\gamma_n(A) = \gamma_n(B)$, and by $A \sim B$ that $\gamma(A) = \gamma(B)$, i.e., $\exists n \in \omega : A \sim_n B$.

Example 7.13 Let $P(\phi) \leftrightarrow \exists\psi(\phi \wedge \psi)$ and, for some $S \in \mathbf{S}_M$, consider vertex $P(P(S)) \in \mathbf{V}_H$. The relevant parts of the graph H are sketched below, with $A/_X, B/_X, ...$ denoting vertices with X substituted for the $\exists$-quantified ψ. The subscripts ${}_{L,R}$ mark these instantiations in the respective subgraphs, e.g., $A/_L = P(S) \wedge A$ and $A/_R = \exists\phi(S \wedge \phi) \wedge A$. Sentences $A, B, ...$ (and $\overline{A}, \overline{B}, ...$) are duplicated in both subgraphs to increase readability, but they are actually the same vertices.



1. $P(P(S)) \sim_1 \exists\psi(P(S) \wedge \psi)$ and $P(S) \sim_1 \exists\psi(S \wedge \psi)$, hence
   $\mathbf{E}_1(\overline{P(S)}) = \{\gamma_1(P(S))\} = \{\exists\psi(S \wedge \psi)\} = \mathbf{E}_1(\overline{\exists\psi(S \wedge \psi)})$ and, consequently,
2. $\overline{P(S)} \sim_2 \overline{\exists\psi(S \wedge \psi)}$. Then, for each $A \in \mathbf{S}^P_M$, $\mathbf{E}_2(A/_L) = \{\overline{\exists\psi(S \wedge \psi)}, \overline{A}\} = \mathbf{E}_2(A/_R)$, so
3. $A/_L \sim_3 A/_R$, for every $A \in \mathbf{S}^P_M$.
4. Consequently, $\bullet_L \sim_4 \bullet_R$ and then
5. $\exists\psi(\exists\phi(S \wedge \phi) \wedge \psi)) \sim_5 \exists\psi(P(S) \wedge \psi) \sim_1 P(P(S))$, leaving only G's subgraph to the right. $\Box$

The equivalence $\sim$ is a congruence on $\mathbf{V}_H$ in the sense that if all out-neighbours of A and B are $\sim$-equivalent then also $A \sim B$, i.e., for $\mathbf{E}_H(A) = \{A_i \mid i \in I\}$ and $\mathbf{E}_H(B) = \{B_i \mid i \in I\}$:

$$\text{if } (\forall i \in I : A_i \sim B_i) \text{ then } A \sim B. \tag{7.14}$$

This holds since each sentence subgraph $\mathcal{G}_M(A)$ (tree $\mathrm{T}_M(A)$) has finite height $h(A)$, in particular distance from the source A to atoms $P(S)$ of $\mathcal{G}_M(A)$ is at most $h(A)$. Hence, if $\forall i \in I : A_i \sim B_i$ then $\exists n \leq \max\{h(A), h(B)\} \forall i \in I : A_i \sim_n B_i$.[12] The equality $\gamma_n(A_i) = \gamma_n(B_i)$ implies, in turn, that $\mathbf{E}_n(A) = \{\gamma_n(A_i) \mid i \in I\} = \{\gamma_n(B_i) \mid i \in I\} = \mathbf{E}_n(B)$, which yields $A \sim_{n+1} B$.

[12] This implication fails in general graphs for $\sim$ defined by (7.12) from some basis $\sim_1$, when I is infinite and distance from $A_i, B_i, i \in I$, to relevant pairs $X \sim_1 Y$ is unbounded.

Fact 7.15 *(a)* $\forall S \in \mathbf{S}^P_M \setminus \mathbf{S}_M\ \exists Q \in \mathbf{S}_M : Q \sim S$*, hence* $\gamma(H) = G$*,*
(b) H *and* G *have essentially the same solutions,*
(c) Every solution of G *extends to a unique solution of* G^P*.*

Proof. Point (a) is shown by induction on the number p of Ps in a sentence $S \in \mathbf{S}^P_M \setminus \mathbf{S}_M$.

1. If $p = 1$ and S is atomic, then $S = P(R)$ for some $R \in \mathbf{S}_M$, so $S \sim_1 \exists\!\!\!/\psi F[R, \psi] \in \mathbf{S}_M$.

2. If $p = 1$ and S is not atomic, we proceed by structural induction on S, with point **1** providing the basis and induction hypothesis IH_2:

i. $\bigwedge_{i\in I} S_i$, for finite I. By IH_2, for each S_i there is $Q_i \in \mathbf{S}_M$ with $S_i \sim Q_i$, so $\bigwedge_{i\in I} S_i \sim \bigwedge_{i\in I} Q_i$ by (7.14), and $\bigwedge_{i\in I} Q_i \in \mathbf{S}_M$.

ii. $\neg A$. By IH_2, $A \sim Q$ for some $Q \in \mathbf{S}_M$, so $\neg A \sim \neg Q$ by (7.14), while $\neg Q \in \mathbf{S}_M$.

iii. $S = \exists\!\!\!/\phi A[\phi]$, where ϕ does not occur under P, so that $S = \exists\!\!\!/\phi A[\phi, P(R)]$, for some $R \in \mathbf{S}_M$ and context $A[\phi, \_]$ with no P. Since $P(R) \sim_1 \exists\!\!\!/\psi F[R] \in \mathbf{S}_M$, taking $Q = \exists\!\!\!/\phi A[\phi, \exists\!\!\!/\psi F[R]] \in \mathbf{S}_M$, we obtain $A[T, P(R)] \sim A[T, \exists\!\!\!/\psi F[R]]$ for every $T \in \mathbf{S}^P_M$ by (7.14), i.e., for all grandchildren of S and Q. By (7.14), this yields $S \sim Q$.

iv. $S = \exists\!\!\!/\phi A[P(C[\phi])]$, i.e., S contains quantification into P, for some contexts $A[\_], C[\_]$ without any P, as $p = 1$. For grandchildren of S, namely, $A[P(C[T])]$ for all $T \in \mathbf{S}^P_M$, the equivalence $P(C[T]) \sim_1 \exists\!\!\!/\psi F[C[T], \psi]$ gives $A[P(C[T])] \sim A[\exists\!\!\!/\psi F[C[T], \psi]]$ by (7.14). Sentences on the left, for all $T \in \mathbf{S}^P_M$, comprise all grandchildren of S, and those on the right all grandchildren of $Q = \exists\!\!\!/\phi A[\exists\!\!\!/\psi F[C[\phi], \psi]] \in \mathbf{S}_M$, so $S \sim Q$ by (7.14).

3. For the induction step for $p > 1$, the two cases depend on whether P is nested or not.

i. If the number of Ps not nested under others is $n > 1$, consider all these highest Ps in $T_M(S)$, i.e., $S = C[P(A_1), ..., P(A_n)]$, where $C[\_]$ contains no Ps. For $R = C[\exists\!\!\!/\psi F[A_1, \psi], ..., \exists\!\!\!/\psi F[A_n, \psi]]$, $S \sim R$ by (7.14). R has $p - n < p$ Ps so, by IH, $R \sim Q$ for some $Q \in \mathbf{S}_M$. Hence $S \sim Q$.

ii. If all Ps are nested under each other, then $S = C[P(A)]$ for some context $C[\_]$ without any Ps, and with $p - 1$ occurrences of P in A. $P(A) \sim_1 \exists\!\!\!/\psi F[A, \psi]$ and, by IH, $\exists\!\!\!/\psi F[A, \psi] \sim R$ for some $R \in \mathbf{S}_M$, so that also $P(A) \sim R$. Then $C[P(A)] \sim C[R]$, by (7.14) and $C[R] \in \mathbf{S}_M$, as required.

The equality $\gamma(H) = G$ follows since each $S \in \mathbf{V}_H \setminus \mathbf{V}_G$ represents a sentence in $\mathbf{S}^P_M \setminus \mathbf{S}_M$.

(b) For $i \geq 0$, H_i is the quotient of H by $\sim_1$,...,$\sim_i$. By Fact 7.2, H_1 has essentially the same solutions as H. (No ray is contracted to a finite path, because the case $P(S) \sim_1 \exists\!\!\!/\psi F[S, \psi]$ is applied at most finitely many times along each path under each sentence Q, since Q contains at most finitely many nested Ps.) By Fact 7.3, the same holds for H_1 and every H_i, $i > 1$, including limits H_λ. Thus, H and $\gamma(H) = G$ have essentially the same solutions.

(c) By the observation before this fact, quotient $G^P \to H$ reflects solutions, so that the preimage of every solution of H is a solution of G^P. Using (b), each solution of G extends to one for G^P. □

Let *definitional extension* refer to any well-ordered chain starting with any theory $\mathit{\Gamma}_0 \subseteq \mathcal{L}_0 \subseteq \mathcal{L}^+$ and adding, at step $i+1$, axiom (5.6) with a fresh predicate $P \notin \mathcal{L}_i$ and $F[\phi, \psi] \in \mathcal{L}_i$, for language $\mathcal{L}_i$ of theory $\mathit{\Gamma}_i$ obtained at step i. In the limits, the language and theory are unions of all steps. The following counterpart of model theoretic conservativity of usual definitional extensions holds.

Lemma 7.16 *Each solution of a language graph* $G_0 = \mathcal{G}_M(\mathcal{L}_0)$ *extends to a solution of the graph of its definitional extension.*

Proof. Fact 7.15.(c) gives the claim for an extension with a single predicate. By IH, definitional extension G_i of G_0 with $P_1, ..., P_i$, preserves all solutions of G. Graph G_{i+1}, obtained now by adding P_{i+1}, whose definiens F_{i+1} can utilize $P_j, j \leq i$, preserves by Fact 7.15 solutions of G_i, and hence of G. This establishes successor step.

For any limit, the language $\mathcal{L}^\omega_M = \bigcup_{i\in\omega} \mathcal{L}^i_M$ extends the initial language $\mathcal{L}^0_M$ with all ω predicates $P_1, P_2, ...$ introduced on the way. Its graph $G_\omega = \bigcup_{i\in\omega} G_i$, with unions taken on vertices and on edges, contains all double edges from the new predicate's instances to their definienses. We repeat the proof with the unions of all equivalences used along the way. As the first step, let $\simeq^\omega$

be a congruence on $\mathcal{L}^{\omega}_M$-sentences induced from the relation $A \simeq^{\omega} B \Leftrightarrow \exists n \in \omega : A \simeq^n B$, where $\simeq^n$ is the congruence $\simeq$ on $\mathcal{L}^n_M$-sentences from step n. Identification of all atoms $Q(A_1...A_k) \simeq^{\omega} Q(B_1...B_k)$ when $A_i \simeq^{\omega} B_i$ for $1 \leq i \leq k$ gives a quotient H reflecting kernels as before. Each equivalence class contains an atom from $\mathcal{L}^0_M$. Let H denote the resulting graph, and H_i its restriction to the subgraph induced by vertices of G_i (with the atoms identified as just described), so that $H = \bigcup_{i \in \omega} H_i$.

In the chain $G_0 = H_0 \subseteq H_1 \subseteq H_2 \subseteq ...$, for each pair of subsequent $H_{i-1} \subseteq H_i$, the construction (7.12) yields $\gamma^i : H_i \to H_{i-1}$ satisfying Fact 7.15. Composing $\gamma^1(\gamma^2(...(\gamma^{i-1}(\gamma^i(H_i)))...))$ gives surjective $\overline{\gamma}^i : H_i \to G_0$, where $\overline{\gamma}^j(H_i) = \overline{\gamma}^i(H_i)$ for any $j \geq i$. Hence, the union $\gamma^{\omega} = \bigcup_{i \in \omega} \gamma^i$ gives a surjective quotient $\gamma^{\omega} : H \to G_0$, reflecting solutions. □

A non-paradoxical language $\mathcal{L}^+$ is one having a solvable graph $\mathcal{G}_M(\mathcal{L}^+)$ so, by this lemma, its definitional extension remains non-paradoxical.

Theorem 7.17 (5.7) *For every $\Gamma \subseteq \mathcal{L}^+$ and its definitional extension F, every kernel model of Γ can be extended to a kernel model of $\Gamma \cup F$.*

7.6 (Cut) preserves consistency

An important theorem from [27] states that if every induced proper subgraph has a semikernel then the graph has a kernel. For language graphs we would rather ask more specifically about a kernel containing a given theory, depending on the existence of semikernels extending the theory to some (finite) parts of the language. The following fact gives such a compactness-like claim for any language graph. (Recall the notation $S \Subset X$ for S being a finite subset of X.)

Fact 7.18 *For $\Gamma \subseteq \mathcal{L}^+$ and any language graph $G = \mathcal{G}_M(\mathcal{L}^+)$, if for each $S \Subset \mathbf{S}^+_M$ there is a semikernel of G containing Γ and covering S, then G has a kernel containing Γ.*

Proof. Let SK_{Γ} denote all semikernels of G containing Γ. For a finite set $X \Subset \mathbf{S}^+_M$, denote by
$SK_X = \{L \in SK_{\Gamma} \mid X \subseteq L\}$ – semikernels of G containing Γ and X, and
$SK^c_X = \{L \in SK_{\Gamma} \mid X \subseteq \mathbf{E}^-_G[L]\}$ – semikernels of G containing Γ and covering X.
The set $F = \{SK^c_X \mid X \Subset \mathbf{S}^+\}$ has finite intersection property by the main assumption. We show that, for an ultrafilter $U \supseteq F$ on $\mathcal{P}(SK_{\Gamma})$, existing by the ultrafilter lemma, a kernel of G can be given by $K = \{S \in \mathbf{S}^+_M \mid SK_S \in U\}$ (subscript $_{-S}$ abbreviates now $_{-\{S\}}$ for a single sentence S).

1. K covers G because if $S \in \mathbf{S}^+_M \setminus K$, then $SK_S \notin U$ so $\overline{SK_S} = \{L \in SK_{\Gamma} \mid S \notin L\} \in U$. If also $SK_{\neg S} \notin U$, then $\overline{SK_{\neg S}} = \{L \in SK_{\Gamma} \mid \neg S \notin L\} \in U$. Hence, if both $S \notin K$ and $\neg S \notin K$, then $nS = \overline{SK_S} \cap \overline{SK_{\neg S}} = \{L \in SK_{\Gamma} \mid S \notin L \wedge \neg S \notin L\} \in U$. As $SK_S \cup SK_{\neg S} = SK^c_{\neg S} \in F \subseteq U$, so $nS \cap SK^c_{\neg S} = \emptyset \in U$ contradicts U being an ultrafilter. Hence $S \in K$ or $\neg S \in K$ for each $S \in \mathbf{S}^+_M$. Since $SK_S \cap SK_{\neg S} = \emptyset$, also $S \notin K$ or $\neg S \notin K$, so $S \in K \Leftrightarrow \neg S \notin K$ for every sentence S.

2. Independence of K is shown for each kind of vertex $S \in K$ by showing $\mathbf{E}_G(S) \subseteq \mathbf{V}_G \setminus K$.
(i) If $\neg S \in K$ then $\mathbf{E}_G(\neg S) = \{S\} \subset \mathbf{V}_G \setminus K$, by **1**, while for literals $S \in K \Leftrightarrow \mathbf{E}_G(S) = \{\overline{S}\} \subset \mathbf{V}_G \setminus K$ ($\overline{S}$ is dual literal to S). For nonatomic S with main connective other than $\neg$, one of the following cases applies.
(ii) For each conjunction $\mathbf{E}_G(S_1 \wedge S_2) = \{\neg S_1, \neg S_2\}$, hence $SK_{S_1 \wedge S_2} \cap SK_{\neg S_i} = \emptyset$, for $i \in \{1, 2\}$, so either both $\neg S_i \notin K$ or else $S_1 \wedge S_2 \notin K$, i.e., $S_1 \wedge S_2 \in K \Rightarrow \mathbf{E}_G(S_1 \wedge S_2) \subset \mathbf{V}_G \setminus K$.
(iii) For each $\forall$-quantified sentence $\mathbf{E}_G(\forall x F(x)) = \{\neg F(m) \mid m \in M\}$, so $SK_{\forall x F(x)} \cap SK_{\neg F(m)} = \emptyset$ for each $m \in M$, hence if $\forall x F(x) \in K$ then $\mathbf{E}_G(\forall x F(x)) \subset \mathbf{V}_G \setminus K$. (The same works for $\forall \phi F(\phi)$.)

3. We show $\mathbf{E}_G(K) \subseteq \mathbf{E}^-_G(K)$ by similar case analysis.
(i) For literals $S \in K \overset{\mathbf{1}}{\Leftrightarrow} \overline{S} \notin K$, so $S \in K \Leftrightarrow \mathbf{E}_G(S) = \{\overline{S}\} \subseteq \mathbf{E}^-_G(S) \subseteq \mathbf{E}^-_G(K)$.
(ii) $\mathbf{E}_G(S_1 \wedge S_2) = \{\neg S_1, \neg S_2\}$ and $(SK_{S_1} \cap SK_{S_2}) \cap SK^c_{S_1 \wedge S_2} = SK_{S_1 \wedge S_2}$, so

(*) $(SK_{\neg S_1} \notin U \wedge SK_{\neg S_2} \notin U) \overset{\mathbf{1}}{\Leftrightarrow} (SK_{S_1} \in U \wedge SK_{S_2} \in U) \Leftrightarrow SK_{S_1 \wedge S_2} \in U \overset{\mathbf{1}}{\Leftrightarrow} SK_{\neg(S_1 \wedge S_2)} \notin U$
Thus, if $(S_1 \wedge S_2) \in K$ then each $\neg S_i \notin K$ so $S_i \in K$ and $\mathbf{E}_G(S_1 \wedge S) \subseteq \mathbf{E}^-_G(K)$. For negated conjuction: $\neg(S_1 \wedge S_2) \in K \overset{\mathbf{1}}{\Leftrightarrow} (S_1 \wedge S_2) \notin K \overset{*}{\Leftrightarrow} \neg S_1 \in K \vee \neg S_2 \in K \Rightarrow \mathbf{E}_G(\neg(S_1 \wedge S_2)) =$

$\{S_1 \wedge S_2\} \subseteq \mathbf{E}^-_G(K)$.
(iii) $\mathbf{E}_G(\forall x F(x)) = \{\neg F(m) \mid m \in M\} \subseteq \mathbf{V}_G \setminus K \Leftrightarrow \{F(m) \mid m \in M\} \subseteq K$. (The proof for s-quantified $\forall \phi F(\phi)$ is identical, so we write only $\forall x F(x)$.) For each $L \in SK^c_{\forall x F(x)}$:

$\forall x F(x) \in L \Leftrightarrow \{F(m) \mid m \in M\} \subseteq L \Leftrightarrow \{\neg F(m) \mid m \in M\} \subseteq \mathbf{V}_G \setminus L$.

Thus $SK_{\forall x F(x)} \cap SK_{\neg F(m)} = \emptyset$, for every $m \in M$, so

(*) $SK_{\forall x F(x)} \in U \Leftrightarrow \forall m \in M : SK_{\neg F(m)} \notin U \overset{\mathbf{1}}{\Leftrightarrow} \forall m \in M : SK_{F(m)} \in U$

which yields:

$\forall x F(x) \in K \Leftrightarrow SK_{\forall x F(x)} \in U \overset{*}{\Leftrightarrow} \forall m \in M : SK_{F(m)} \in U \Rightarrow \mathbf{E}_G(\forall x F(x)) \subseteq \mathbf{E}^-_G(K)$

and for negated quantifier:

$\neg(\forall x F(x) \in K) \overset{\mathbf{1}}{\Leftrightarrow} (\forall x F(x)) \notin K \Leftrightarrow SK_{\forall x F(x)} \notin U \overset{*}{\Leftrightarrow} \exists m \in M : SK_{\neg F(m)} \in U \Rightarrow$
$\Rightarrow \mathbf{E}_G(\neg \forall x F(x)) = \{\forall x F(x)\} \subseteq \mathbf{E}^-_G(\neg F(m)) \subseteq \mathbf{E}^-_G(K)$. □

From a syntactic perspective, a shortcoming of this fact is that it concerns a single graph $\mathcal{G}_M(\mathcal{L}^+)$, rather than a theory, and involves all $\mathbf{S}^+_M$ sentences, not only $\mathbf{S}^+$. The idea of a theory having a consistent extension to the whole language, provided that it has such extensions to its finite parts, is better captured by the next lemma, leading to the theorem that LSO-unprovability of any (specific) contradiction implies existence of a graph with a kernel, and hence unprovability of any contradiction also when using (cut). For a finite set $Q \Subset \mathbf{S}^+$, we let $\bot_Q = \bigvee_{S \in Q}(S \wedge \neg S)$.

Lemma 7.19 $(\forall Q \Subset \mathbf{S}^+ : \mathit{\Gamma} \not\vdash \bot_Q) \Rightarrow \exists G, K \in sol(G) : \mathit{\Gamma} \subseteq K$.

Proof. To use Fact 7.18, we construct a graph G over domain M with semikernels containing $\mathit{\Gamma}$ and covering every finite subset of $\mathbf{S}^+_M$. Letting I index finite subsets of $\mathbf{S}^+$, the assumption gives a semikernel L_i of a graph G_i containing $\mathit{\Gamma}$ and covering $\bot_i$, for every $i \in I$. Let G be the language graph over $M = \prod M_i$ with $f^M(\prod m_i) = \prod f^{M_i}(m_i)$ and constants $c^M = \prod c^{M_i}$ Define inductively the operation $\_^{\uparrow M}$, lifting terms $\mathbf{T}^+_{M_i} \to \mathcal{P}(\mathbf{T}^+_M)$ and sentences $\mathbf{S}^+_{M_i} \to \mathcal{P}(\mathbf{S}^+_M)$:

$m_i^{\uparrow M} = (m_i)^{\uparrow M} = \{n \in M \mid n_i = m_i\} = \{m_i\} \times \prod_{j \neq i} M_j$, for $m_i \in M_i$, and likewise
$(f(m_i))^{\uparrow M} = f^M(m_i^{\uparrow M}) = \{f^M(n) \in \mathbf{T}^+_M \mid n \in m_i^{\uparrow M}\}$ and
$F(m_i)^{\uparrow M} = F(m_i^{\uparrow M}) = \{F(n) \in \mathbf{S}^+_M \mid n \in m_i^{\uparrow M}\}$, for a formula $F(\phi_1 ... \phi_k)$ and $m_i \in M_i^k$.

Notation $F(m_i)$ implies that the only M_i elements are among m_i. In general, S_i denotes an $\mathcal{L}^+_{M_i}$ sentence with possibly some elements from M_i, that equals $S \in \mathbf{S}^+$ if no such elements occur. Then $S_i^{\uparrow M}$ is the set of sentences obtained by replacing each $m_i \in M_i$ by all $m_i^{\uparrow M}$. (S_i denotes also an $\mathcal{L}^+/\mathcal{L}^+_M$ sentence S with all terms interpreted in/projected onto M_i.) Some observations:

(a) $\bigcup\{m_i^{\uparrow M} \mid m_i \in M_i\} = M$ and $\bigcup\{S_i^{\uparrow M} \mid S_i \in \mathbf{S}^+_{M_i}\} = \mathbf{S}^+_M$, and
$\bigcup\{F(S_i)^{\uparrow M} \mid S_i \in \mathbf{S}^+_{M_i}\} = \{F(S) \mid S \in \mathbf{S}^+_M\}$.

(b) $\forall S_i, R_i \in \mathbf{S}^+_{M_i} : S_i \not\doteq R_i \Rightarrow S_i^{\uparrow M} \cap R_i^{\uparrow M} = \emptyset$ ($\doteq$ modulo renaming of bound variables)

(c) For $S \in \mathbf{S}^+$, $S^{\uparrow M} = \{S\}$, e.g., $(\forall x P(x))^{\uparrow M} = \{\forall x P(x)\}$, so for $S \in \mathbf{S}^+$ we identify $S = S^{\uparrow M}$.

Setting $L_i^{\uparrow M} = \bigcup\{S_i^{\uparrow M} \subseteq \mathbf{S}^+_M \mid S_i \in L_i\}$, for each $i \in I$, we also have:

(d) $S_i \notin L_i \Leftrightarrow \forall X \in L_i : S_i \not\doteq X \overset{\mathbf{(b)}}{\Rightarrow} \forall X \in L_i : S_i^{\uparrow M} \cap X^{\uparrow M} = \emptyset \Rightarrow S_i^{\uparrow M} \cap L_i^{\uparrow M} = \emptyset$. This implies: $S(m) \in L_i^{\uparrow M} \Rightarrow S(m_i) \in L_i$ and, as a special case

(e) $S \notin L_i \Rightarrow S^{\uparrow M} = S \notin L_i^{\uparrow M}$, for $S \in \mathbf{S}^+$.

By the main assumption $\mathit{\Gamma} \subseteq L_i$ for each i, hence $\mathit{\Gamma} \subset L_i^{\uparrow M}$. We show that $L_i^{\uparrow M} \in SK(G)$.

1. That $L_i^{\uparrow M}$ is absorbing, $\mathbf{E}_G(L_i^{\uparrow M}) \subseteq \mathbf{E}^-_G(L_i^{\uparrow M})$, follows by considering cases of its vertices:

1.i. If $P(m) \in L_i^{\uparrow M}$, then $\mathbf{E}_G(P(m)) = \{\overline{P(m)}\} \subseteq \mathbf{E}^-_G(P(m)) \subseteq \mathbf{E}^-_G(L_i^{\uparrow M})$.

1.ii. For a negated $\neg S \in L_i^{\uparrow M}$, we have $\neg S \in L_i^{\uparrow M} \overset{\mathbf{(e)}}{\Rightarrow} \neg S_i \in L_i \Rightarrow S_i \in \mathbf{E}^-_{G_i}(L_i)$, and show $S \in \mathbf{E}^-_G(L_i^{\uparrow M})$ for $\mathbf{E}_G(\neg S) = \{S\}$ by cases of S. Where relevant, we mark possible $n \in M$ occurring in the considered sentences as extra parameters.

1. Negated atom $\neg P(n) \in L_i^{\uparrow M} \Rightarrow \mathbf{E}_G(\neg P(n)) = \{P(n)\} \subset \mathbf{E}^-_G(\neg P(n)) \subset \mathbf{E}^-_G(L_i^{\uparrow M})$.

2. $\neg\neg S \in L_i^{\uparrow M} : \neg S_i \in \mathbf{E}^-_{G_i}(L_i) \Rightarrow S_i \in L_i \overset{\textbf{(d)}}{\Rightarrow} S \in L_i^{\uparrow M} \Rightarrow \neg S \in \mathbf{E}^-_G(L_i^{\uparrow M})$.
3. $\neg(S \wedge R) \in L_i^{\uparrow M} : S_i \wedge R_i \in \mathbf{E}^-_{G_i}(L_i) \Rightarrow \neg S_i \in L_i \vee \neg R_i \in L_i \overset{\textbf{(d)}}{\Rightarrow} \neg S \in L_i^{\uparrow M} \vee \neg R \in L_i^{\uparrow M}$
$\Rightarrow S \wedge R \in \mathbf{E}^-_G(L_i^{\uparrow M})$.
4. $\neg\forall x F(x,n) \in L_i^{\uparrow M} : \forall x F(x,n_i) \in \mathbf{E}^-_{G_i}(L_i) \Rightarrow \exists m_i \in M_i : \neg F(m_i,n_i) \in L_i \Rightarrow$
$\overset{\textbf{(d)}}{\Rightarrow} \neg F(m_i^{\uparrow M}, n_i^{\uparrow M}) \subseteq L_i^{\uparrow M} \Rightarrow \neg F(m,n) \in L_i^{\uparrow M} \Rightarrow \forall x F(x,n) \in \mathbf{E}^-_G(L_i^{\uparrow M})$.
5. $\neg\forall \phi F(\phi,n) \in L_i^{\uparrow M} : \forall \phi F(\phi,n_i) \in \mathbf{E}^-_{G_i}(L_i) \Rightarrow \exists S_i \in \mathbf{S}^+_{M_i} : \neg F(S_i,n_i) \in L_i \Rightarrow$
$\overset{\textbf{(d)}}{\Rightarrow} \neg F(S_i^{\uparrow M}, n_i^{\uparrow M}) \subseteq L_i^{\uparrow M} \Rightarrow \neg F(S,n) \in L_i^{\uparrow M} \Rightarrow \forall \phi F(x,n) \in \mathbf{E}^-_G(L_i^{\uparrow M})$.

1.iii. $S_1 \wedge S_2 \in L_i^{\uparrow M} \overset{\textbf{(e)}}{\Rightarrow} (S_1 \wedge S_2)_i \in L_i \Rightarrow \{(\neg S_1)_i, (\neg S_2)_i\} \subset \mathbf{E}^-_{G_i}(L_i) \Rightarrow \{(S_1)_i, (S_2)_i\} \subseteq L_i \Rightarrow$
$\Rightarrow (S_1)_i^{\uparrow M} \cup (S_2)_i^{\uparrow M} \subset L_i^{\uparrow M} \Rightarrow \{S_1, S_2\} \subset L_i^{\uparrow M} \Rightarrow \{\neg S_1, \neg S_2\} \subset \mathbf{E}^-_G(L_i^{\uparrow M})$.

1.iv. $\forall x F(x,n) \in L_i^{\uparrow M} \overset{\textbf{(e)}}{\Rightarrow} \forall x F(x,n_i) \in L_i \Leftrightarrow \{\neg F(m_i,n_i) \mid m_i \in M_i\} \subseteq \mathbf{E}^-_{G_i}(L_i) \Leftrightarrow$
$\Leftrightarrow \{F(m_i,n_i) \mid m_i \in M_i\} \subseteq L_i$
$\Rightarrow \bigcup\{(F(m_i^{\uparrow M}, n_i^{\uparrow M})) \mid m_i \in M_i\} \overset{\textbf{(a)}}{=} \{F(m, n_i^{\uparrow M}) \mid m \in M\} \subseteq L_i^{\uparrow M} \Rightarrow$
$\Rightarrow \{F(m,n) \mid m \in M\} \subseteq L_i^{\uparrow M} \Rightarrow \mathbf{E}_G(\forall x F(x,n)) = \{\neg F(m,n) \mid m \in M\} \subseteq \mathbf{E}^-_G(L_i^{\uparrow M})$

1.v. $\forall \phi F(\phi,n) \in L_i^{\uparrow M} \overset{\textbf{(e)}}{\Rightarrow} \forall \phi F(\phi,n_i) \in L_i$
$\Leftrightarrow \{\neg F(S_i,n_i) \mid S_i \in \mathbf{S}^+_{M_i}\} \subseteq \mathbf{E}^-_{G_i}(L_i) \Leftrightarrow \{F(S_i,n_i) \mid S_i \in \mathbf{S}^+_{M_i}\} \subseteq L_i$
$\Rightarrow \bigcup\{F(S_i^{\uparrow M}, n_i^{\uparrow M}) \mid S_i \in \mathbf{S}^+_{M_i}\} = \{F(S, n_i^{\uparrow M}) \mid S \in \mathbf{S}^+_M\} \subseteq L_i^{\uparrow M}$
$\Rightarrow \{F(S,n) \mid S \in \mathbf{S}^+_M\} \subseteq L_i^{\uparrow M} \Rightarrow \mathbf{E}_G(\forall \phi F(\phi,n)) = \{\neg F(S,n) \mid S \in \mathbf{S}^+_M\} \subseteq \mathbf{E}^-_G(L_i^{\uparrow M})$

2. Also independence of $L_i^{\uparrow M}$, $\mathbf{E}_G(L_i^{\uparrow M}) \subseteq \mathbf{V}_G \setminus L_i^{\uparrow M}$, follows by considering its possible vertices:

2.i. $P(m) \in (L_i)^{\uparrow M} \overset{\textbf{(d)}}{\Rightarrow} P(m_i) \in L_i \Rightarrow \overline{P(m_i)} \notin L_i \overset{\textbf{(d)}}{\Rightarrow} \overline{P(m)} \notin L_i^{\uparrow M}$

2.ii. $\neg S(m) \in L_i^{\uparrow M} \overset{\textbf{(d)}}{\Rightarrow} \neg S_i(m_i) \in L_i \Rightarrow S_i(m_i) \in \mathbf{E}^-_{G_i}(L_i) \Rightarrow S_i(m_i) \notin L_i$
$\overset{\textbf{(d)}}{\Rightarrow} S_i(m_i)^{\uparrow M} \cap L_i^{\uparrow M} = \emptyset \overset{S(m) \in S(m_i)^{\uparrow M}}{\Longrightarrow} S(m) \in \mathbf{V}_G \setminus L_i^{\uparrow M}$.

2.iii. $(S^1(m) \wedge S^2(n)) \in L_i^{\uparrow M} \overset{\textbf{(d)}}{\Rightarrow} (S^1(m_i) \wedge S^2(n_i)) \in L_i \Rightarrow$
$\Rightarrow \{\neg S^1(m_i), \neg S^2(n_i)\} \subset \mathbf{E}^-_{G_i}(L_i)$
$\Rightarrow \{\neg S^1(m_i), \neg S^2(n_i)\} \cap L_i = \emptyset$
$\Rightarrow (\neg S^1(m_i^{\uparrow M}) \cup \neg S^2(n_i^{\uparrow M})) \cap L_i^{\uparrow M} = \emptyset \Rightarrow \mathbf{E}_G(S^1(m) \wedge S^2(n)) \cap L_i^{\uparrow M} = \emptyset$
The last implication holds by $(\neg S^1(m_i^{\uparrow M}) \cup \neg S^2(n_i^{\uparrow M})) \supset \{\neg S^1(m), \neg S^2(n)\} = \mathbf{E}_G(S^1(m) \wedge S^2(n))$.

2.iv. $\forall x F(x,n) \in L_i^{\uparrow M} \overset{\textbf{(d)}}{\Rightarrow} \forall x F(x,n_i) \in L_i \Rightarrow$
$\Rightarrow \{\neg F(m_j,n_i) \mid m_j \in M_i\} \cap L_i = \emptyset$
$\overset{\textbf{(d)}}{\Rightarrow} \bigcup\{\neg F(m_j^{\uparrow M}, n_i^{\uparrow M}) \mid m_j \in M_i\} \cap L_i^{\uparrow M} = \emptyset$
$\overset{\textbf{(a)}}{\Rightarrow} \bigcup\{(\neg F(m, n_i^{\uparrow M})) \mid m \in M\} \cap L_i^{\uparrow M} = \emptyset$
$\overset{n \in n_i^{\uparrow M}}{\Longrightarrow} \{\neg F(m,n) \mid m \in M\} \cap L_i^{\uparrow M} = \emptyset \Rightarrow \mathbf{E}_G(\forall x F(x,n)) \cap L_i^{\uparrow M} = \emptyset$

2.v. $\forall \phi F(\phi,n) \in L_i^{\uparrow M} \overset{\textbf{(d)}}{\Rightarrow} \forall \phi F(\phi,n_i) \in L_i \Rightarrow$
$\Rightarrow \{\neg F(S_j,n_i) \mid S_j \in \mathbf{S}^+_{M_i}\} \subseteq \mathbf{E}^-_{G_i}(L_i)$
$\Rightarrow \{\neg F(S_j,n_i) \mid S_j \in \mathbf{S}^+_{M_i}\} \cap L_i = \emptyset$
$\overset{\textbf{(d)}}{\Rightarrow} \bigcup\{(\neg F(S_j,n_i))^{\uparrow M} \subset \mathbf{S}^+_M \mid S_j \in \mathbf{S}^+_{M_i}\} \cap L_i^{\uparrow M} = \emptyset$
$\overset{\textbf{(a)}}{\Rightarrow} \bigcup\{\neg F(S,n) \subset \mathbf{S}^+_M \mid S \in \mathbf{S}^+_M\} \cap L_i^{\uparrow M} = \emptyset \Rightarrow \mathbf{E}_G(\forall \phi F(\phi,n)) \cap L_i^{\uparrow M} = \emptyset$.

Thus, for every L_i, its lifiting $L_i^{\uparrow M}$ is a semikernel of G (by Fact 7.1), containing Γ and covering $\bot_i$. To use Fact 7.18, we only have to show that every finite $Q \Subset \mathbf{S}^+_M$ is covered by some $L_i^{\uparrow M}$.

3. For an arbitrary $S \in \mathbf{S}^+$, $\bot_{\{S\}} = S \wedge \neg S$ is covered by $L_{\{S\}}$ which does not contain it. Hence $(S \wedge \neg S) \in \mathbf{E}^-_{G_{\{S\}}}(L_{\{S\}})$, so either $\neg S \in L_{\{S\}}$ or $\neg\neg S \in L_{\{S\}}$. In the former case $S \in \mathbf{E}_{G_{\{S\}}}(L_{\{S\}}) \subseteq \mathbf{E}^-_{G_{\{S\}}}(L_{\{S\}})$ while in the latter $S \in L_{\{S\}}$. In each case, $L_{\{S\}}^{\uparrow M}$ covers $S = S^{\uparrow M}$.

For an arbitrary $S \in \mathbf{S}^+_M \setminus \mathbf{S}^+$, i.e., $S = Fm$, where $m \in M$ are all M elements occurring in S, contradiction $N(F) = \forall x \neg(Fx \wedge \neg Fx) \wedge \neg\forall x \neg(Fx \wedge \neg Fx)$ has, in some graph G_i, a countermodel L_i covering it by $N(F) \in \mathbf{E}^-_{G_i}(L_i)$. Thus either $\neg\forall x\neg(Fx \wedge \neg Fx) \in L_i$ or $\neg\neg\forall x\neg(Fx \wedge \neg Fx) \in L_i$, but since the former sentence is a contradiction, the latter is the case. Then $\forall x\neg(Fx \wedge \neg Fx) \in L_i$, hence also $\neg(Fm_i \wedge \neg Fm_i) \in L_i$ for every $m_i \in M_i$. Since then $(Fm_i \wedge \neg Fm_i) \in \mathbf{E}^-_{G_i}(L_i)$, for every $m_i \in M_i$, either $\neg\neg Fm_i \in L_i$ or $\neg Fm_i \in L_i$. In the former case $Fm_i \in L_i$, while in the latter, $Fm_i \in \mathbf{E}^-_{G_i}(L_i)$. Thus, L_i covers Fm_i, containing either Fm_i or $\neg Fm_i$, for every $m_i \in M_i$. By **(c)**, $Fm \in L_i^{\uparrow M}$ or $\neg Fm \in L_i^{\uparrow M}$, which means that $L_i^{\uparrow M}$ covers Fm, for every $m \in M$.

These arguments for single sentences are extended to an arbitrary $Q = \{F^1m^1, ..., F^km^k\} \Subset \mathbf{S}^+_M$, by considering semikernel L_i which covers, without containing, $\bot_Q = N(F^1) \vee ... \vee N(F^k)$, and existing by the assumption that $\Gamma \not\vdash \bot_Q$. The subgraph of $\bot_Q$ has the form

$$\begin{tikzcd}
 & \bot_Q \arrow[d] & \\
 & \bullet \arrow[dl] \arrow[d, dotted] \arrow[dr] & \\
N(F^1) & ... & N(F^k)
\end{tikzcd}$$

hence $\bot_Q \in \mathbf{E}^-_{G_Q}(L_Q)$ implies $\bullet \in L_Q$, so $N(F^j) \in \mathbf{E}^-_{G_Q}(L_Q)$, for $1 \leq j \leq k$. By the argument for a single $N(F)$, this implies that L_Q covers every F^jm^j. $\square$

If LSO does not derive from Γ contradiction $\bot_Q = \bigvee_{S \Subset Q}(S \wedge \neg S)$, for any $Q \Subset \mathbf{S}^+$, there is thus a graph with a kernel containing Γ. Soundness, Theorem 4.3, implies then $\Gamma \not\vdash_c \bot$ for each $\bot \in \mathbb{C}$. Thus, if LSO^c proves a contradiction from a theory Γ, then so does LSO.

Theorem 7.20 (4.4) *For a countable $\Gamma \subset FOL^+$:* $(\forall Q \Subset \mathbf{S}^+ : \Gamma \not\vdash \bot_Q) \Rightarrow (\forall \bot \in \mathbb{C} : \Gamma \not\vdash_c \bot)$.

8 Appendix: Soundness and completeness

Facts 8.1 and 8.3 below show soundness and completeness of LSO for semikernel semantics from (3.11), establishing Theorem 4.1. Theorem 4.3 shows these properties for LSO with (cut) for kernel semantics (3.13).

Fact 8.1 (4.1) *The rules of* LSO *are sound and invertible for (3.11).*

Proof. Given an arbitrary language graph $\mathcal{G}^+$ (over an arbitrary domain M), soundness for each rule follows by showing that any semikernel L covering the conclusion satisfies it, assuming validity of the premise(s), while invertibility by showing that any semikernel L covering (each) premise satisfies it, assuming validity of the rule's conclusion.

1. $(\wedge_R)$. For soundness, assume $\Gamma \models \Delta, A_1$ and $\Gamma \models \Delta, A_2$, and let semikernel L cover the rule's conclusion, under a given $\alpha \in M^{\mathcal{V}(\Gamma,\Delta)}$. Assume that $\alpha(\Gamma) \subseteq L$, $\alpha(\Delta) \subseteq \mathbf{E}^-(L)$ and $\alpha(A_1 \wedge A_2) \in \mathbf{E}^-(L)$ – if not, then $L \models_\alpha \Gamma \Rightarrow \Delta, A_1 \wedge A_2$, as desired. Since $\alpha(A_1 \wedge A_2) \in \mathbf{E}^-(L)$ and $\mathbf{E}(\alpha(A_1 \wedge A_2)) = \{\neg\alpha(A_1), \neg\alpha(A_2)\}$ so, for some $i \in \{1,2\}$, $\neg\alpha(A_i) \in L$, and then $\alpha(A_i) \in \mathbf{E}^-(L)$, contradicting the assumption $\Gamma \models \Delta, A_i$.

For invertibility, let $\Gamma \models \Delta, A_1 \wedge A_2$ and L cover A_1 (or A_2) under α. If (*) $\alpha(\Gamma) \subseteq L$ and $\alpha(\Delta \cup \{A_1\}) \subseteq \mathbf{E}^-(L)$, then $L' = L \cup \{\neg\alpha(A_1)\}$ is a semikernel, since $\mathbf{E}(\neg\alpha(A_1)) = \{\alpha(A_1)\} \subseteq \mathbf{E}^-(L) \subseteq \mathbf{E}^-(L')$. L' covers also $\alpha(A_1 \wedge A_2) \in \mathbf{E}^-(\neg\alpha(A_1))$. Thus L' covers the conclusion, while $\alpha(\Gamma) \cap \mathbf{E}^-(L') = \emptyset$ and $\alpha(\Delta \cup \{A_1 \wedge A_2\}) \cap L' = \emptyset$, so $L' \not\models \Gamma \Rightarrow \Delta, A_1 \wedge A_2$, contrary to $\Gamma \models \Delta, A_1 \wedge A_2$. Hence (*) fails, so $\alpha(\Gamma) \cap \mathbf{E}^-(L) \neq \emptyset$ or $\alpha(\Delta \cup \{A_1\}) \cap L \neq \emptyset$, yielding the claim.

Assignments to free FOL-variables do not affect the argument, so covering by L below is to be taken relatively to a given α, which we do not mention, except for $(\forall_R)$.

2. $(\wedge_L)$. For soundness, assume $\Gamma, A_1, A_2 \models \Delta$, let semikernel L cover the rule's conclusion, $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$. If $A_1 \wedge A_2 \in L$, then $\mathbf{E}(A_1 \wedge A_2) = \{\neg A_1, \neg A_2\} \subseteq \mathbf{E}^-(L)$, so $\mathbf{E}(\{\neg A_1, \neg A_2\}) = \{A_1, A_2\} \subseteq L$, contradicting $\Gamma, A_1, A_2 \models \Delta$. Thus $A_1 \wedge A_2 \in \mathbf{E}^-(L)$ and $L \models \Gamma, A_1 \wedge A_2 \Rightarrow \Delta$.

For invertibility, assume $\Gamma, A_1 \wedge A_2 \models \Delta$, let semikernel L cover the rule's premise, and assume $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$. If $A_1, A_2 \in L$, which is the only way L can contradict $\Gamma, A_1, A_2 \models \Delta$, then $\{\neg A_1, \neg A_2\} \subseteq \mathbf{E}^-(L)$, and $L' = L \cup \{A_1 \wedge A_2\}$ is also a semikernel:

$$\mathbf{E}(L') = \mathbf{E}(L \cup \{A_1 \wedge A_2\}) = \mathbf{E}(L) \cup \mathbf{E}(\{A_1 \wedge A_2\}) \subseteq \mathbf{E}^-(L) \cup \{\neg A_1, \neg A_2\} \subseteq$$
$$\subseteq \mathbf{E}^-(L) \subseteq \mathbf{V} \setminus (L \cup \{A_1 \wedge A_2\}).$$

The last inclusion follows because $\mathbf{E}^-(L) \subseteq \mathbf{V} \setminus L$ and $A_1 \wedge A_2 \notin \mathbf{E}^-[L]$, since $A_1 \wedge A_2 \in L$ contradicts $\Gamma, A_1 \wedge A_2 \models \Delta$ (as $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$), while $A_1 \wedge A_2 \in \mathbf{E}^-(L)$ contradicts independence of L, implying $\neg A_i \in L$ (for $i = 1$ or $i = 2$), while $\neg A_i \in \mathbf{E}^-(L)$ since $A_i \in L$.

Since $L' \not\models \Gamma, A_1 \wedge A_2 \Rightarrow \Delta$ contradicts the assumption, either $A_1 \notin L$ or $A_2 \notin L$, and $L \models \Gamma, A_1 \wedge A_2 \Rightarrow \Delta$ as desired.

3. $(\neg_R)$. For soundness, assume $\Gamma, A \models \Delta$, let semikernel L cover the rule's conclusion, and assume $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$. If $\neg A \in L$, we are done, while if $\neg A \in \mathbf{E}^-(L)$ then $A \in L$, which contradicts the assumption, since now $\Gamma \cup \{A\} \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$.

For invertibility, assuming $\Gamma \models \Delta, \neg A$, let L cover the rule's premise, $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$. If $A \in L$ then $\neg A \in \mathbf{E}^-(L)$ and $L \not\models \Gamma \Rightarrow \Delta, \neg A$, contradicting the assumption. Hence $A \in \mathbf{E}^-(L)$, as required for $L \models \Gamma, A \Rightarrow \Delta$.

4. $(\neg_L)$. For soundness, assume $\Gamma \models \Delta, A$, let L cover the rule's conclusion, $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$. If $\neg A \in \mathbf{E}^-(L)$, we are done, while if $\neg A \in L$ then $A \in \mathbf{E}(\neg A) \subseteq \mathbf{E}^-(L)$, contradicting the assumption, since now $\Gamma \cup \{A\} \subseteq L$ and $(\Delta \cup \{A\}) \subseteq \mathbf{E}^-(L)$.

For invertibility, assume $\Gamma, \neg A \models \Delta$, let L cover the rule's premise, $\Gamma \subseteq L$ and $\Delta \subseteq \mathbf{E}^-(L)$. If $A \in \mathbf{E}^-(L)$ then $L' = L \cup \{\neg A\}$ is a semikernel, because L is and $\mathbf{E}(\neg A) = \{A\} \subseteq \mathbf{E}^-(L)$. But L' contradicts the assumption, so $A \in L$, as required for $L \models \Gamma \Rightarrow \Delta, A$.

5. $(\forall_L)$. For soundness, assume $F(t), \Gamma, \forall x F(x) \models \Delta$ and let L cover the rule's conclusion. If $\forall x F(x) \notin L$, i.e., $\forall x F(x) \in \mathbf{E}^-(L)$, then $(\Gamma \cup \{\forall x F(x)\}) \cap \mathbf{E}^-(L) \neq \emptyset$, so $L \models \Gamma, \forall x F(x) \Rightarrow \Delta$. If $\forall x F(x) \in L$ then also $F(t) \in L$, since $\neg F(t) \in \mathbf{E}(\forall x F(x)) \subseteq \mathbf{E}^-(L)$ and $\mathbf{E}(\neg F(t)) = \{F(t)\}$. As L covers the premise, either $\Gamma \cap \mathbf{E}^-(L) \neq \emptyset$, since $F(t) \notin \mathbf{E}^-(L)$, or $\Delta \cap L \neq \emptyset$. Either case yields the claim for L, so $\Gamma, \forall x F(x) \models \Delta$.

For invertibility, assuming $\Gamma, \forall x F(x) \models \Delta$, any L covering the premise of the rule covers also its conclusion, yielding the claim.

6. $(\forall_R)$. For soundness, let (*) $\Gamma \models \Delta, F(y)$, with eigenvariable y, and L cover the rule's conclusion, under a given assignment α to $\mathcal{V}(\Gamma, \Delta, \forall x F(x)) \not\ni y$. Assume also $\alpha(\Gamma) \subseteq L$ and $\alpha(\Delta) \subseteq \mathbf{E}^-(L)$. If $\alpha(\forall x F(x)) \notin L$ then $\alpha(\forall x F(x)) \in \mathbf{E}^-(L)$ and some $\alpha(\neg F(m)) \in L$, since $\mathbf{E}(\alpha(\forall x F(x))) = \{\alpha(\neg F(m)) \mid m \in M\}$. Extending α with $\alpha(y) = m$, we obtain $L \not\models_\alpha \Gamma \Rightarrow \Delta, F(y)$, contrary to (*). Thus, $\alpha(\forall x F(x)) \in L$ and $L \models_\alpha \Gamma \Rightarrow \Delta, \forall x F(x)$.

For invertibility, if $L \not\models_\alpha \Gamma \Rightarrow \Delta, F(y)$, for $\alpha(y) = m$, i.e., $\alpha(\Gamma) \subseteq L$, $\alpha(\Delta) \subseteq \mathbf{E}^-(L)$ and $\alpha(F(m)) \in \mathbf{E}^-(L)$, then $L' = L \cup \{\alpha(\neg F(m))\}$ is a semikernel, because L is and $\mathbf{E}(\alpha(\neg F(m))) = \{\alpha(F(m))\} \subseteq \mathbf{E}^-(L) \subseteq \mathbf{E}^-(L')$. L' covers the conclusion since $\alpha(\forall x F(x)) \in \mathbf{E}^-(\alpha(\neg F(m)))$, but $L' \not\models_\alpha \Gamma \Rightarrow \Delta, \forall x F(x)$.

7. $(\forall_L^+)$. The argument repeats that for $(\forall_L)$. For soundness, assume $\Gamma, F(S), \forall \phi F(\phi) \models \Delta$ and let L cover the rule's conclusion. If $\forall \phi F(\phi) \notin L$ then $\forall \phi F(\phi) \in \mathbf{E}^-(L)$, yielding $L \models \Gamma, \forall \phi F(\phi) \Rightarrow \Delta$. If $\forall \phi F(\phi) \in L$ then also $F(S) \in L$, since $\neg F(S) \in \mathbf{E}(\forall \phi F(\phi)) \subseteq \mathbf{E}^-(L)$ and $\mathbf{E}(\neg F(S)) = \{F(S)\}$. Thus L covers also the premise, hence, either $\Gamma \cap \mathbf{E}^-(L) \neq \emptyset$, since $F(S) \notin \mathbf{E}^-(L)$, or $\Delta \cap L \neq \emptyset$. Either case yields the claim for L, so $\Gamma, \forall \phi F(\phi) \models \Delta$.

For invertibility, assuming $\Gamma, \forall \phi F(\phi) \models \Delta$, any L covering the premise of the rule covers also its conclusion, yielding the claim.

8. $(\forall_R^+)$. For soundness, let $\Gamma \models \Delta, F(S)$ for every S, and L cover the rule's conclusion. If $\forall \phi F(\phi) \in L$ then L satisfies the rule's conclusion. If $\forall \phi F(\phi) \notin L$ then $\forall \phi F(\phi) \in \mathbf{E}^-(L)$ and some $\neg F(S) \in L$, since $\mathbf{E}(\forall \phi F(\phi)) = \{\neg F(S) \mid S \in \mathbf{S}^+\}$. Now L covers also the premise $\Gamma \Rightarrow \Delta, F(S)$ and $F(S) \notin L$, hence either $\Gamma \cap \mathbf{E}^-(L) \neq \emptyset$ or $\Delta \cap L \neq \emptyset$. Each case yields the claim that L satisfies the conclusion.

For invertibility, assume $\Gamma \models \Delta, \forall \phi F(\phi)$, and let L cover a premise $\Gamma \Rightarrow \Delta, F(S)$. If (*) $\Gamma \subseteq L$ and $\Delta \cup \{F(S)\} \subseteq \mathbf{E}^-(L)$ then $L' = L \cup \{\neg F(S)\}$ is a semikernel, since $\mathbf{E}(\neg F(S)) = \{F(S)\} \subseteq$

$\mathbf{E}^-(L) \subseteq \mathbf{E}^-(L')$. L' covers also $\forall\phi F(\phi)$, since $\forall\phi F(\phi) \in \mathbf{E}^-(\neg F(Q))$, for every $Q \in \mathbf{S}^+$, in particular, $\forall\phi F(\phi) \in \mathbf{E}^-(\neg F(S))$. Thus $\Gamma \cup \Delta \cup \{F(S), \forall\phi F(\phi)\} \subseteq \mathbf{E}^-[L']$, while $\Gamma \cap \mathbf{E}^-(L') = \emptyset$, $\Delta \cap L' = \emptyset$ and $\forall\phi F(\phi) \notin L'$, contrary to $\Gamma \models \Delta, \forall\phi F(\phi)$. Hence (*) fails, so either $\Gamma \cap \mathbf{E}^-(L) \neq \emptyset$ or $(\Delta \cup \{F(S)\}) \cap L \neq \emptyset$, yielding the claim.

9. S-equality rules are sound and invertible, because atoms occurring in premises but not conclusionsare redundant (due to point 3 of Definition 3.7 of language graph). E.g., $S \doteq S$ in the premise of (ref) is always satisfied (being a sink), hence satisfaction of the premise implies satisfaction of the conclusion. Conversely, satisfaction of the conclusion by any semikernel allows its extension with any sink, in particular, with $S \doteq S$. Analogous argument works for (rep) and (neq). ☐

The following simple consequence of Definition 3.7 is used in the completeness proof below.

Fact 8.2 *In any graph $\mathcal{G}^+_M$, the following relations hold between the form of a nonatomic sentence $X \in \mathbf{S}^+_M$ and forms of its out- and in-neighbours:*

1. $\mathbf{E}^-(X) = \{\neg X\}$ *– when X does not start with $\neg$,*
2. $\mathbf{E}^-(\neg X) = \{\neg\neg X\} \cup \{X \wedge S \mid S \in \mathbf{S}^+_M\} \cup \{\forall(\phi.D(\phi) \mid \exists S \in \mathbf{S}^+_M : D(S) = X\} \cup ...$
   $... \cup \{\forall x.D(x) \mid \exists t \in \mathbf{T}_M : D(t) = X\}$
3. *when X does not start with $\neg$, then each out-neighbour of X does,*
4. $\mathbf{E}(\neg X) = \{X\}$.

For atomic X, $\mathbf{E}^-(X) = \{\neg X\} = \mathbf{E}(X)$ and $\mathbf{E}^-(\neg X) = \{X\} = \mathbf{E}(\neg X)$.

The proof of completeness can apply the standard techniques because proofs in LSO, even if infinite, are well-founded trees with axioms as leaves. A few adjustments are needed for handling deviations from LK. One is the infintiary rule $(\forall^+_R)$, needed because substitution of fresh eigenvariables for s-variables, although sound, does not necessarily lead to a countermodel in an unsuccessful derivation, since s-variables are not sentences. (Replacing $(\forall^+_R)$ by a usual $\forall_R$-rule using eigen-variables would yield a complete system for a modified notion of $\models$, admitting extensions of the language with new s-constants.) In the proof, we ensure not only that all formulas are processed and all terms are substituted by $(\forall_L)$, but also that all sentences are substituted by $(\forall^+_L)$. Missing subformula property, due to substitution of all sentences for s-variables, is handled by retaining the principal formula from the conclusion in all its premisses, in a bottom-up construction of a derivation tree. As a special case of violation of this property, a branch can be cyclic, with the same sequent appearing infinitely often. Any nonaxiomatic (e.g., cyclic) branch provides a countermodel.

Fact 8.3 (4.1) *For a countable $\Gamma \cup \Delta \subseteq FOL^+$: $\Gamma \not\vdash \Delta \Rightarrow \exists \mathcal{G}^+ \exists L \in SK(\mathcal{G}^+) : L \not\models \Gamma \Rightarrow \Delta$.*

Proof. We fix an enumeration E^+ of all sentences $\mathbf{S}^+$ so that each occurs infinitely often, an enumeration $E_T = t_1, t_2...$ of terms $\mathbf{T}_X$ so that each occurs infinitely often, and an enumeration $E_S = S_1, S_2...$ of FOL$^+$ formulas without free s-variables and with operators applied only to sentences $\mathbf{S}^+$, so that each occurs infinitely often. (FOL variables, requiring special care, are treated in the standard way and ignored below, e.g., we keep also an enumeration of eigenvariables). We enumerate all triples $\langle S_i, t_j, S_k\rangle \in E_S \times E_T \times E^+$, with each $\langle S_i, t_j, \_\rangle$ and $\langle S_i, \_, S_k\rangle$ occurring infinitely often. This is interleaved with an enumeration of all pairs $E_S \times E_S$, with each pair occurring infinitely often.

1. We construct a derivation tree, starting with the root $\Gamma \vdash \Delta$, which is to be proven. An *active* sequent – intially, only the root – is a nonaxiomatic leaf of the tree constructed bottom-up so far. We proceed along the enumeration of the triples and pairs considering, for each $\langle S_i, t_j, S_k\rangle$, the cases of active occurrences (in the active sequents) of S_i. Pairs $\langle S_i, S_j\rangle$ are considered in point **v**.

i. If $S_i \in \mathbf{A}^+$, or S_i has no active occurrrences, proceed to the next triple.

ii. Otherwise, proceed retaining S_i from the active sequent, which instantiates the conlusion of the relevant rule, in the new leaves obtained from the rule's premisses. E.g., if $S_i = A \wedge B$ then every active sequent of the form $\Gamma', A \wedge B, \Gamma'' \vdash \Delta$ is replaced by

$$\frac{A, B, \Gamma', A\wedge B, \Gamma'' \vdash \Delta}{\Gamma', A\wedge B, \Gamma'' \vdash \Delta}$$

while every active sequent of the form $\Gamma \vdash \Delta', A\wedge B, \Delta''$ by

$$\frac{\Gamma \vdash A, \Delta', A\wedge B, \Delta'' \qquad \Gamma \vdash B, \Delta', A\wedge B, \Delta''}{\Gamma \vdash \Delta', A\wedge B, \Delta''}$$

In the same way, for other connectives.

iii. If $S_i = \forall x D(x)$, every active sequent of the form $\Gamma', \forall x D(x), \Gamma'' \vdash \Delta$, is replaced by the derivation with a new leaf adding $D(t_j)$ to its antencedent

$$\frac{D(t_j), \Gamma', \forall x D(x), \Gamma'' \vdash \Delta}{\Gamma', \forall x D(x), \Gamma'' \vdash \Delta}.$$

Every active sequent of the form $\Gamma \vdash \Delta', \forall x D(x), \Delta''$ is replaced by

$$\frac{\Gamma \vdash D(c), \Delta', \forall x D(x), \Delta''}{\Gamma \vdash \Delta', \forall x D(x), \Delta''}$$

where c is a fresh eigenvariable c.

iv. If $S_i = \forall \phi D(\phi)$ then replace every active sequent of the form $\Gamma', \forall\phi D(\phi), \Gamma'' \vdash \Delta$ by

$$\frac{D(S_k), \Gamma', \forall\phi D(\phi), \Gamma'' \vdash \Delta}{\Gamma', \forall\phi D(\phi), \Gamma'' \vdash \Delta}$$

while every active sequent of the form $\Gamma \vdash \Delta', \forall\phi D(\phi), \Delta''$ by the infinitely branching derivation with a new leaf sequent for each $S_n \in \mathbf{S}^+$:

$$\frac{\Gamma \vdash D(S_1), \Delta', \forall\phi D(\phi), \Delta'' \qquad \Gamma \vdash D(S_2), \Delta', \forall\phi D(\phi), \Delta'' \qquad \text{... – for all } S_n \in \mathbf{S}^+}{\Gamma \vdash \Delta', \forall\phi D(\phi), \Delta''}$$

v. For a pair $\langle S_i, S_j\rangle$, we apply rules for $\doteq$. If $S_i \not\equiv S_j$, we add atom $S_i \doteq S_j$ to the consequent of every active sequent. Otherwise, we add it to the antencedent. Finally, for each active sequent containing $S_i \doteq Q$ in its antecedent, along with any formula $A(S_i)$, we add to it $A(Q)$.

2. A branch gets closed when its leaf is an axiom, and the tree is obtained as the ω-limit of this process. If all branches are closed (finite), the derivation yields a proof of the root.

Otherwise, an infinite branch allows us to construct a countermodel of all sequents on this branch, including the root sequent. (Such an infinite branch can represent a finite process of derivation terminating with a nonaxiomatic sequent, which remains unchanged in an infinite tail of the branch. It can also be cyclic. These special cases are treated uniformly with an infinite branch without any repeated sequents.)

3. The claim is that if β is an infinite branch, with β'_L/β'_R all formulas occurring in β on the left/right of $\vdash$, then there is a language graph $\mathcal{G}$ with a semikernel L' such that $\beta'_L \subseteq L'$ and $\beta'_R \subseteq \mathbf{E}^-(L')$. The rest of the proof establishes this claim.

Absence of any axiom in β implies that $\beta'_L \cap \beta'_R = \emptyset$, which is often applied implicitly. $\beta'_L = \beta_L \cup Eq_L$, where Eq_L are $\doteq$-atoms $S \doteq S$ occurring on the left. $\beta'_R = \beta_R \cup Eq_R$, where Eq_R are $\doteq$-atoms occurring on the right, with $\overline{Eq}_R$ denoting the set of their negations.

If β contains any FOL-atoms, construct first a FOL-structure M, giving a countermodel to $(\beta_L \cap \mathbf{S}_M) \Rightarrow (\beta_R \cap \mathbf{S}_M)$, in the standard way. Otherwise, set $M = \emptyset$. Let $\mathcal{G} = \mathcal{G}_M$ (when $M = \emptyset$, this is the graph for QBS). We show that (def) $L = \beta_L \cup (\mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R))$ is a semikernel of $\mathcal{G}$, with $\beta_R \subseteq \mathbf{E}^-(L)$ and $\beta_L \subseteq L$. Then $L' = Eq_L \cup \overline{Eq}_R \cup L$ is a required semikernel of $\mathcal{G}$.

4. First, $\doteq$-atoms can be treated separately. Since $\beta'_L \cap \beta'_R = \emptyset$, each $\doteq$-atom $A \in Eq_R$ has the form $S \doteq T$ for syntactically distinct sentences, while each such atom in Eq_L has the form $S \doteq S$. Any semikernel of $\mathcal{G}$, in particular L, can be extended to semikernel $L' = L \cup Eq_L \cup \overline{Eq}_R$, as the added vertices are sinks of $\mathcal{G}$, by Definition 3.7. Thus $\mathbf{E}(Eq_L \cup \overline{Eq}_R) = \emptyset$, while $Eq_R \subseteq \mathbf{E}^-(L') \cap (\mathbf{V} \setminus L')$.

5. To show $L \in SK(\mathcal{G})$, we show first $\beta_R \subseteq \mathbf{E}^-(L)$, which follows from definitions of L and $\mathcal{G}$ by considering the cases for $A \in \beta_R$. Use of Fact 8.2/Definition 3.7 is marked by superscript $\_^{8.2}$.

i. If $A \in \mathbf{A}^+$ then $\mathbf{E}(A) \overset{8.2}{=} \{\neg A\} \overset{8.2}{=} \mathbf{E}^-(A)$, so $\neg A \in L$ by (def) and $A \overset{8.2}{\in} \mathbf{E}^-(L)$.

ii. If $A = \neg C$ then $C \in \beta_L \subseteq L$, so $A \overset{8.2}{\in} \mathbf{E}^-(L)$.

iii. If $A = C \wedge D$ then $C \in \beta_R$ (or $D \in \beta_R$), so $\neg C \overset{8.2}{\in} \mathbf{E}^-(C) \cap \mathbf{E}(C \wedge D) \subseteq \mathbf{E}^-(\beta_R) \cap \mathbf{E}(\beta_R) \subseteq L$, and thus $A = C \wedge D \overset{8.2}{\in} \mathbf{E}^-(\neg C) \subseteq \mathbf{E}^-(L)$. (The case of $D \in \beta_R$ is analogous.)

iv. If $A = \forall x.D(x)$ then $D(c) \in \beta_R$, for some $c \in M$, so $\neg D(c) \overset{8.2}{\in} \mathbf{E}^-(D(c)) \cap \mathbf{E}(\forall x.D(x)) \subseteq L$, and $A \overset{8.2}{\in} \mathbf{E}^-(\neg D(c)) \subseteq \mathbf{E}^-(L)$.

v. If $A = \forall \phi.D(\phi)$ then $D(S) \in \beta_R$ for some $S \in \mathbf{S}^+$, so $\neg D(S) \overset{8.2}{\in} \mathbf{E}^-(D(S)) \cap \mathbf{E}(\forall \phi.D(\phi)) \subseteq L$, and $A \in \mathbf{E}^-(\neg D(S)) \subseteq \mathbf{E}^-(L)$.

6. We show $\mathbf{E}(L) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$, partitioning $L = \beta_L \cup Z$, where $Z = (\mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)) \setminus \beta_L$, and establish first $\mathbf{E}(\beta_L) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$, considering cases of $A \in \beta_L$.

i. For atoms $A \in \mathbf{A}^+$, $A \in \beta_L \subseteq L$ and $A \notin \beta_R$ imply $\neg A \notin \beta_L$ and, since $\mathbf{E}(\neg A) \overset{8.2}{=} \{A\}$, $\neg A \notin \mathbf{E}^-(\beta_R)$. Thus $\mathbf{E}(A) \overset{8.2}{=} \{\neg A\} \subseteq \mathbf{E}^-(A) \cap \mathbf{V} \setminus L \subseteq \mathbf{E}^-(L) \cap \mathbf{V} \setminus L$.

ii. $A = \neg C \in \beta_L$ implies $C \in \beta_R$, so $\mathbf{E}(A) \overset{8.2}{=} \{C\} \subseteq \beta_R \subseteq \mathbf{E}^-(L)$ by **5**.

We show $\mathbf{E}(A) \subset \mathbf{V} \setminus L$. $C \notin \beta_L$ since $\beta_L \cap \beta_R = \emptyset$. Suppose $C \in \mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)$. If $C = \neg D$ then $\neg D \in \mathbf{E}^-(\beta_R)$, i.e., $\mathbf{E}(\neg D) \overset{8.2}{=} \{D\} \subset \beta_R$, while $A = \neg C = \neg\neg D \in \beta_L$ implies also $\neg D \in \beta_R$ and $D \in \beta_L$, contradicting $\beta_L \cap \beta_R = \emptyset$.

Otherwise, i.e., if C does not start with $\neg$, then for any $F \in \beta_R$ for which $C \in \mathbf{E}(F)$, Fact 8.2.(3-4) forces $F = \neg C = A$, contradicting $\beta_R \cap \beta_L = \emptyset$.

iii. $A = B \wedge C \in \beta_L$ implies $\{B, C\} \subset \beta_L$ and $\{\neg B, \neg C\} \cap \beta_L = \emptyset$, so $\mathbf{E}(B \wedge C) \overset{8.2}{=} \{\neg B, \neg C\} \subseteq \mathbf{V} \setminus \beta_L$ and $\mathbf{E}(B \wedge C) = \{\neg B, \neg C\} \overset{8.2}{\subseteq} \mathbf{E}^-(\{B, C\}) \subseteq \mathbf{E}^-(\beta_L)$. If, say, $\neg B \in \mathbf{E}^-(\beta_R)$, then $B \in \beta_R$ would contradict $\beta_L \cap \beta_R = \emptyset$. The same if $\neg C \in \mathbf{E}^-(\beta_R)$. Thus, $\mathbf{E}(B \wedge C) \subseteq \mathbf{E}^-(L) \cap \mathbf{V} \setminus L$.

iv. $A = \forall \phi D(\phi) \in \beta_L \Rightarrow \{D(S) \mid S \in \mathbf{S}^+\} \subseteq \beta_L$, so $\mathbf{E}(\forall \phi D(\phi)) \overset{8.2}{=} \{\neg D(S) \mid S \in \mathbf{S}^+\} \overset{8.2}{\subseteq} \mathbf{E}^-(\{D(S) \mid S \in \mathbf{S}^+\}) \subseteq \mathbf{E}^-(\beta_L) \subseteq \mathbf{E}^-(L)$.

If any $\neg D(S) \in L$ then either $\neg D(S) \in \beta_L$, so $D(S) \in \beta_R$, or $\neg D(S) \in \mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)$, which implies $D(S) \in \beta_R$, since $\mathbf{E}(\neg D(S)) \overset{8.2}{=} \{D(S)\}$. In either case, $D(S) \in \beta_R$ contradicts $\beta_L \cap \beta_R = \emptyset$. Thus $\mathbf{E}(\forall \phi.D(\phi)) \subseteq \mathbf{V} \setminus L$.

v. For $A = \forall x.D(x)$, the argument is as in **iv**. $\forall x.D(x) \in \beta_L$ implies $\{D(t) \mid t \in \mathbf{T}_M\} \subseteq \beta_L$, so $\mathbf{E}(\forall x D(x)) \overset{8.2}{=} \{\neg D(t) \mid t \in \mathbf{T}_M\} \overset{8.2}{\subseteq} \mathbf{E}^-(\{D(t) \mid t \in \mathbf{T}_M\}) \subseteq \mathbf{E}^-(\beta_L) \subseteq \mathbf{E}^-(L)$.

If any $\neg D(t) \in L$, then either $\neg D(t) \in \beta_L$, so $D(t) \in \beta_R$, or $\neg D(t) \in \mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)$, which implies $D(t) \in \beta_R$, since $\mathbf{E}(\neg D(t)) \overset{8.2}{=} \{D(t)\}$. In either case, $D(t) \in \beta_R$ contradicts $\beta_L \cap \beta_R = \emptyset$. Thus $\mathbf{E}(\forall x.D(x)) \subseteq \mathbf{V} \setminus L$.

7. Also each sentence $S \in Z = (\mathbf{E}(\beta_R) \cap \mathbf{E}^-(\beta_R)) \setminus \beta_L$ satisfies $\mathbf{E}(S) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$:

i. If $S \in Z$ does not start with $\neg$, then $\mathbf{E}^-(S) \overset{8.2}{=} \{\neg S\}$, so $\neg S \in \beta_R$, implying $S \in \beta_L$, so $S \notin Z$.

ii. If $S = \neg A \in Z \subseteq \mathbf{E}^-(\beta_R)$ then $\mathbf{E}(\neg A) \overset{8.2}{=} \{A\} \subset \beta_R \overset{\mathbf{5}}{\subseteq} \mathbf{E}^-(L)$. If $A \in Z$, then it starts with $\neg$ by **7.i**, i.e., $A = \neg B$ and $\mathbf{E}(\neg B) \overset{8.2}{=} \{B\} \subset \beta_R$. Since also $A \in \beta_R$ so $B \in \beta_L$, contradicting $\beta_L \cap \beta_R = \emptyset$. Hence $A \notin Z$ and $A \notin \beta_L$ (since $A \in \beta_R$), i.e., $A \notin L = Z \cup \beta_L$, so that $\mathbf{E}(\neg A) = \{A\} \subseteq \mathbf{V} \setminus L$.

By **6** and **7**, $\mathbf{E}(L) = \mathbf{E}(\beta_L) \cup \mathbf{E}(Z) \subseteq \mathbf{E}^-(L) \cap (\mathbf{V} \setminus L)$, so $L \in SK(\mathcal{G})$ by Fact 7.1. $\square$

Unlike in variants of circular proof theory, an infinite branch gives always a rise to a countermodel. A paradigmatic example of a cyclic proof, with the same sequent reappearing infinitely often in a branch, can be the attempted derivation of $\forall \phi.\phi$:

$$(\forall_L^+)\ \dfrac{\dfrac{\vdots}{\vdash \forall \phi.\phi} \qquad \dfrac{A \vdash \qquad \vdash A}{\vdash \neg A \wedge A} \qquad \dfrac{\vdots}{\vdash S} \quad \text{... for all } S \in \mathbf{S}^+}{\vdash \forall \phi.\phi}$$

Any sentence $\neg A \wedge A$ gives a counterexample when any of its branches does not terminate with an axiom, providing a countermodel. Such is, in particular, the leftmost branch where $\forall \phi.\phi$ instantiates ϕ in the root sentence and is expanded further, giving a copy of the whole tree and, eventually, a special branch β with $\beta_L = \emptyset$ and $\beta_R = \{\forall\phi.\phi\}$. This infinite branch provides also a countermodel, with $\forall\phi.\phi = \mathbf{0}$. This looks strange, but is verified by inspecting graph $\mathcal{G}_M(A_\forall)$ in Example 3.10, according to which $\forall\phi.\phi$ does act as a witness to its own falsity.

A branch with a repeated sequent can be terminated, with the conclusion of unprovability, if one can verify that subsequent substitutions, higher up in the derivation, will also yield the same sequent. A single repetition is not enough, as it may be due to a specific substitution, while subsequent ones might yield new sequents.

The remaining fact is soundness and completeness with (cut) for kernel semantics (3.13).

Theorem 8.4 (4.3) *For a countable $\Gamma \cup \Delta \subseteq FOL^+$, $\Gamma \models_c \Delta$ iff $\Gamma \vdash_c \Delta$.*

Proof. Soundness and invertibility follow by essentially the same argument as in Fact 8.1, with some simplifications due to each kernel $K \in sol(G)$ covering the whole graph, $\mathbf{E}^-_G(K) = \mathbf{V}_G \setminus K$. We fix an arbitrary graph G and show each case for an arbitrary fixed $K \in sol(G)$.

1. $(\wedge_R)$. For soundness, let $\Gamma \models_c \Delta, A_1$, $\Gamma \models_c \Delta, A_2$, and $\alpha \in M^{\mathcal{V}(\Gamma,\Delta)}$. Then $\alpha(\Gamma) \cap \mathbf{V} \setminus K \neq \emptyset$ or $\alpha(\Delta) \cap K \neq \emptyset$, in which case also conclusion is satisfied under α, or else $\{\alpha(A_1), \alpha(A_2)\} \subseteq K$. Then $\{\neg\alpha(A_1), \neg\alpha(A_2)\} \subseteq \mathbf{V} \setminus K$, and hence $\alpha(A_1 \wedge A_2) \in K$ since $\mathbf{E}(\alpha(A_1 \wedge A_2)) = \{\neg\alpha(A_1), \neg\alpha(A_2)\}$.

For invertibility, let $\Gamma \models_c \Delta, A_1 \wedge A_2$. If $\alpha(\Gamma) \cap \mathbf{V} \setminus K \neq \emptyset$ or $\alpha(\Delta) \cap K \neq \emptyset$, then K satisfies also both premises under α. If neither is the case, then $\alpha(A_1 \wedge A_2) \in K$, hence $\mathbf{E}(\alpha(A_1 \wedge A_2)) = \{\neg\alpha(A_1), \neg\alpha(A_2)\} \subseteq \mathbf{V} \setminus K$ and, $\mathbf{E}(\neg\alpha(A_i)) = \alpha(A_i) \in K$, for $i \in \{1, 2\}$, hence $K \models_\alpha \Gamma \Rightarrow \Delta, A_i$.

Assignments to free FOL-variables do not affect the arguments below. They are relative to a given α, which we do not mention, except for $(\forall_R)$. In each case, we assume that $\Gamma \subseteq K$ and $\Delta \subseteq \mathbf{V} \setminus K$ focusing on the active/principal formulas.

2. $(\wedge_L)$. For soundness, assuming $\Gamma, A_1, A_2 \models_c \Delta$ (and $\Gamma \subseteq K$ and $\Delta \subseteq \mathbf{V} \setminus K$), $A_i \in \mathbf{V} \setminus K$, for $i = 1$ or $i = 2$. Then $\neg A_i \in K$, since $\mathbf{E}(\neg A_i) = A_i$, and $A_1 \wedge A_2 \in \mathbf{E}^-(\neg A_i) \subseteq \mathbf{E}^-(K) \subseteq \mathbf{V} \setminus K$. Thus $K \models \Gamma, A_1 \wedge A_2 \Rightarrow \Delta$.

For invertibility, assume $\Gamma, A_1 \wedge A_2 \models_c \Delta$. If $A_1, A_2 \in K$, which is the only way K can contradict $\Gamma, A_1, A_2 \models_c \Delta$, then $\mathbf{E}(A_1 \wedge A_2) = \{\neg A_1, \neg A_2\} \subseteq \mathbf{E}^-(\{A_1, A_2\}) \subseteq \mathbf{E}^-(K) = \mathbf{V} \setminus K$, and hence $A_1 \wedge A_2 \in K$, contradicting $K \models \Gamma, A_1 \wedge A_2 \Rightarrow \Delta$.

3. $(\neg_R)$. For soundness, $\Gamma, A \models_c \Delta$ implies $A \in \mathbf{V} \setminus K$, so $\neg A \in K$, since $\mathbf{E}(\neg A) = A$.

For invertibility, $\Gamma \models_c \Delta, \neg A$ implies $\neg A \in K$, so $\mathbf{E}(\neg A) = \{A\} \subseteq \mathbf{V} \setminus K$ and $K \models \Gamma, A \Rightarrow \Delta$.

4. $(\neg_L)$. For soundness, $\Gamma \models_c \Delta, A$, implies $A \in K$ hence $\neg A \in \mathbf{V} \setminus K$ and $K \models \Gamma, \neg A \Rightarrow \Delta$.

For invertibility, $\Gamma, \neg A \models_c \Delta$ implies $\neg A \in \mathbf{V} \setminus K$, hence $\mathbf{E}(\neg A) = A \in K$ and $K \models \Gamma \Rightarrow \Delta, A$.

5. $(\forall_L)$. For soundness, assume $F(t), \Gamma, \forall x F(x) \models_c \Delta$. If $\forall x F(x) \notin K$, i.e., $\forall x F(x) \in \mathbf{V} \setminus K$, then $(\Gamma \cup \{\forall x F(x)\}) \cap \mathbf{E}^-(K) \neq \emptyset$, so $K \models \Gamma, \forall x F(x) \Rightarrow \Delta$. If $\forall x F(x) \in K$ then also $F(t) \in K$, since $\neg F(t) \in \mathbf{E}(\forall x F(x)) \subseteq \mathbf{V} \setminus K$, so $\mathbf{E}(\neg F(t)) \cap K \neq \emptyset$ while $\mathbf{E}(\neg F(t)) = \{F(t)\}$. Thus either $\Gamma \cap (\mathbf{V} \setminus K) \neq \emptyset$ or $\Delta \cap K \neq \emptyset$, yielding $K \models \Gamma, \forall x F(x) \Rightarrow \Delta$.

Invertibility follows by weakening since $\Gamma, \forall x F(x) \models_c \Delta$ implies $F(t), \Gamma, \forall x F(x) \models_c \Delta$.

6. $(\forall_R)$. For soundness, let (*) $\Gamma \models_c \Delta, F(y)$, with eigenvariable $y \notin \mathcal{V}(\Gamma, \Delta)$, and $\alpha(\Gamma) \subseteq K$ and $\alpha(\Delta) \subseteq \mathbf{V} \setminus K$. If $\alpha(\forall x F(x)) \notin K$ then $\alpha(\forall x F(x)) \in \mathbf{E}^-(K)$ and some $\alpha(\neg F(m)) \in K$, since $\mathbf{E}(\alpha(\forall x F(x))) = \{\alpha(\neg F(m)) \mid m \in M\}$. Extending α with $\alpha(y) = m$ yields $\alpha(\Gamma) \subseteq K$ and $\alpha(\Delta, F(y)) \subseteq \mathbf{V} \setminus K$, contrary to (*). Hence, $\alpha(\forall x F(x)) \in K$.

For invertibility, if $\alpha(\Gamma) \subseteq K$ and $\alpha(\Delta, F(y)) \subseteq \mathbf{V} \setminus K$, for $\alpha(y) = m$, then $\neg\alpha(F(m)) \in K$, since $\mathbf{E}(\neg\alpha(F(m))) = \alpha(F(m)) \in \mathbf{V} \setminus K$. Then $\alpha(\forall x F(x)) \in \mathbf{E}^-(\neg\alpha(F(m))) \subseteq \mathbf{E}^-(K) = \mathbf{V} \setminus K$, giving $\alpha(\Gamma) \subseteq K$ and $\alpha(\Delta, \forall x F x) \subseteq \mathbf{V} \setminus K$, which contradicts $K \models \Gamma \Rightarrow \Delta, \forall x F x$.

7. $(\forall_L^+)$. The argument repeats that for $(\forall_L)$. For soundness, assume $\Gamma, F(S), \forall\phi F(\phi) \models_c \Delta$. If $\forall\phi F(\phi) \in K$ then also $F(S) \in K$, since $\neg F(S) \in \mathbf{E}(\forall\phi F(\phi)) \subseteq \mathbf{E}^-(K)$ and $\mathbf{E}(\neg F(S)) = \{F(S)\}$. Hence, as $\Gamma \subseteq K$ and $\Delta \subseteq \mathbf{V} \setminus K$, it holds $\forall\phi F(\phi) \in \mathbf{V} \setminus K$ and $K \models \Gamma, \forall\phi F(\phi) \Rightarrow \Delta$.

For invertibility, assuming $\Gamma, \forall\phi F(\phi) \models_c \Delta$, weakening yields $F(S), \Gamma, \forall\phi F(\phi) \models_c \Delta$.

8. $(\forall_R^+)$. For soundness, let $\Gamma \models_c \Delta, F(S)$ for every $S \in \mathbf{S}^+$. If $\forall\phi F(\phi) \notin K$ then $\forall\phi F(\phi) \in \mathbf{E}^-(K)$ and some $\neg F(S) \in K$, since $\mathbf{E}(\forall\phi F(\phi)) = \{\neg F(S) \mid S \in \mathbf{S}^+\}$. Since $F(S) \notin K$, either

$\Gamma \cap \mathbf{E}^-(K) \neq \emptyset$ or $\Delta \cap K \neq \emptyset$, contradicting the assumption $\Gamma \subseteq K$ and $\Delta \subseteq \mathbf{V} \setminus K$. Hence $\forall\phi F(\phi) \in K$ and K satisfies the rule's conclusion.

For invertibility, assume $\Gamma \models_c \Delta, \forall\phi F(\phi)$, and $\forall\phi F(\phi) \in K$. If for some $S \in \mathbf{S}^+$, $F(S) \in \mathbf{V}\setminus K$, then $\neg F(S) \in K$ since $\mathbf{E}(\neg F(S)) = \{F(S)\} \subseteq \mathbf{V} \setminus K$. But $\neg F(S) \in \mathbf{E}(\forall\phi F(\phi))$, contradicting independence of K. Hence, $F(S) \in K$ for all $S \in \mathbf{S}^+$.

9. The rules for $\doteq$ are sound and invertible by the same argument as in case of semikernels, point **1.v** in proof of Fact 8.3. Every kernel contains all sinks, so $\overline{S \doteq T}$ for syntactically distinct sentences, and all $S \doteq S$ belong to every kernel.

This concludes the proof of soudnenss. For completeness, we modify the construction from the proof of Fact 8.3, by interleaving the enumeration of all triples $E_S \times E_T \times E^+$ and pairs $E_S \times E_S$ with enumeration E'_S of all FOL$^+$ formulas without free s-variables, where each such formula occurs only once. Following this interleaved enumeration yields now a new case, **1.5** of an $A \in E'_S$, in which we expand each active sequent $\Gamma \Rightarrow \Delta$ with the premises of (cut) over A, i.e., with $\Gamma \Rightarrow \Delta, A$ and $\Gamma, A \Rightarrow \Delta$. A semikernel falsifying any one of them, falsifies the conclusion. Given an infinite nonaxiomatic branch β, a language graph $\mathcal{G}_M$ is obtained as in the proof of Fact 8.3, over domain M consisting of free variables and ground terms used in the standard construction of a FOL countermodel for $\beta \cap \mathbf{S}_M$. Point **3** of the proof of Fact 8.3 shows β to determine a semikernel K of $\mathcal{G}_M$, falsifying each sequent on β. Now, β contains one of the premises of an application of (cut) for each $A \in E'_S = \mathbf{S}^+_M$. As every sentence $\mathbf{S}^+_M$ occurs thus in β_L or β_R, semikernel K covers all $\mathbf{S}^+_M$, so it is a kernel of $\mathcal{G}_M$. □

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