# The holism of truth and paradox

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May 10, 2013

**Abstract**

We argue that discourses, understood as relative and bounded totalities of statements, provide an ontological grounding for truth and paradox. Single statements may be the carriers of truth-values but their truth-claims become justifed or invalid only relatively to the actual discourse. Invalidity of such claims, that is, the situation when no coherent assignment of truth-values to all involved statements is possible, amounts to a semantic paradox. Typical examples of paradoxes involve discourses consisting of a single statement but, in general, only the totality of the actual discourse can justify the conclusion of paradoxicality. The absence of paradox amounts exactly to the applicability of the truth-concept, which allows to distribute some truth-values among all statements of the discourse. Appendix shows that diagnosis of paradoxes and analysis of truth-claims can be conducted in classical logic.

## 1 Introduction

It is hard to make up one's mind about the accusation "the next man is a liar" without clarifying first who the next man is. Similarly, it is unclear what to make of a claim like "the next statement is false", without first gathering some information about the next statement. Is this merely a question of geography? Are we merely trying to locate the object we have in mind? The answer, it seems to us, is that there is more to the story of the next man and the next statement than the possible ambiguity of pointing at something. Who the next man and the next statement are, *in themselves*, might depend on what others say about them or, more generally, how others depend on them. If you call a man a liar, it witnesses to a relation between you and him, and this relation might very well be a critical part of what determines his nature, as well as yours. Thus, it is not possible to completely seperate the question of who the next man is from the question of what we are to make of you calling him a liar. In the case of men, this is a fundamental insight of social psychology, going back at least to the influential work of Mead and the Chicago school [Mea67]. For the case of statements, the same insight, perhaps equally fundamental, has been overlooked by the branch of analytic philosophy investigating the workings of truth and paradox. Truth does not pertain to statements as isolated entities, but requires their anchoring in an appropriate context, a discourse, which is the source of possible legitimacy or failure of truth-claims.[1]

[1] Although we grant that the *discursive* context matters for truth and may even be crucial, the affinity with contextualism ends approximately at this point. We give a uniform schema for evaluating truth-claims in all discourses and we do not "resolve" paradoxes by indexing truth-predicates, but simply diagnose them as such.

The paper argues for this claim and, through a series of examples, shows how it suggests itself already from a more careful consideration of the semantic paradoxes. We take statements to be some minimal units susceptible of carrying a truth-value. Their existence should be acceptable even if their precise nature is not clear. We do not ask if they are propositions, beliefs, types, tokens, instances of linguistic expressions, or something else. Rather, we argue that regardless of what notion of statement one adheres to, one needs a corresponding notion of discourse in order to make sense of truth.

A discourse itself is not to be understood as The Language nor any Universal Whole, but simply as a limited list of statements, typically but not necessarily, interconnected by some cross-references, for instance:

(x) The last statement on this list is true.

(y) Snow is white and the first statement on this list is false. (1)

A moment's thought is enough to see that this is a paradox. Starting from the first statement, since snow is in fact white, one is forced to conclude that the statement is true if and only if it is false. Notice, however, that what it says is completely harmless. It leads to a paradox only when taken together with the statement following it, which, in addition to making a truthful claim about snow, also ensures that the first statement indirectly negates its own truth. Examples such as this suggest that a correct account of truth needs to take discourses into account. Truth arises only in the absence of paradox, but paradox, as the example illustrates, is a property of the referential structure of the discourse, not a property of any particular among its statements.

Therefore, we need to include discourses in our ontology, and not merely as syntactic objects, but also as entities independent from syntax. That said, the distinction between syntax and semantics will not play any significant role in our argument.[2] Importantly, we will not view language as some other-worldly dual of the world, but as its part. This statement, this very statement you are now reading, is an element of our ontology, and so is this article, not only as a list of sentences, but also as a discourse, a collection of interrelated statements.

Occasionally, it will be useful to distinguish two aspects of a statement's meaning: what the statement says, its *literal* meaning, emerging in a compositional manner from the literal meanings of the words used to express it, and the discursive, *actual* meaning, including the possible truth-value, which depends also on the referential interactions of the statement and, therefore, according to the ontology we propose, on the discursive context where it occurs. For instance, all occurrences of "The last statement on this list is false." say the same, have the same literal meaning. However, the nature of such statements, their actual meaning, and whether they are true or not, depends on the actual discourses where they appear.

The paper does not present a complete theory of truth but only some observations which such a theory should take into account. The main claims are as follows:

1. Although bearers of truth-values are single statements, discourses are necessary for truth-claims. The concept of truth applies only to the totality of a discourse.[3]

[2] If it is of relevance, we will speak of sentences as purely syntactic expressions of statements.

[3] The concept of truth encompasses also falsehood. It is, perhaps, just the concept of the distinction between these two truth-values. The ambiguities between these uses of 'truth', in particular truth-concept versus truth-values, are hopefully resolved in each case by the context. Truth-claims pertain to statements which appear to possess a truth-value, not necessarily the value 'true'.

2. The concept of truth arises in opposition to (semantic) paradox, which is the limit suspending legitimacy of truth-claims. As such, paradox is a property of discourses, not of single statements.

3. More specifically, paradox amounts to the impossibility of assigning truth-values to the discourse's statements in a manner that reflects the correspondence between what the statement says and its position in the discursive network, and that respects the workings of truth. This is not to be confused with logical inconsistency of the discourse itself, but as we show in the appendix, it can be represented formally as the logical inconsistency of its truth-theory.

   (a) Paradox is not "resolved" by introducing any new semantic values, but describes a *real property of discourses*, namely, the unavoidable failure of any coherent attribution of truth-values to its constituents. Paradox is the end of truth-claims, not the beginning of a search for new truth-values. Thus, the presented view is not threatened by any revenge. It supports the idea of truth-gaps, but such gaps arise from properties of discourses, not from properties of the statements that happen to be contained in them.

   (b) Often, paradox can be localized, allowing us to extract coherent subdiscourses with truth-bearing elements. An appropriate definition of a coherent subdiscourse allows us to apply classical logic in the presence of paradox, without any deductive explosion.

   (c) Paradox arises only in a purely discursive context, as the result of an unfortunate interaction of statements, detached from non-discursive roots, and contingent paradoxes also conform to this claim.

4. It follows that the nature of truth has a *holistic* character. For instance, in an ungrounded discourse, apparently undetermined statements can obtain truth-values, not on the basis of what they say, but as a result of the way in which other statements depend on them. The meaning of a statement is not exhausted by what it says about the world, but also depends on what other statements say about it.

5. The holism of truth and paradox is of a limited kind, not to be taken as an argument for a theory of truth based on a notion of ultimate coherence. It indicates only that truth has an essentially contextual, and hence local and modal character, with a mere possibility of truth being the primary modus, and necessary truths being special cases.

6. This limited holism does not interefere with the basic view of a true statement as corresponding with some aspect of reality, its truthmaker. The truthmaker theory is not our subject, however, and we neither elaborate nor depend on it. Our claim is only that discourses, in addition to providing the necessary grounds for the legitimacy of truth-claims, act often also as truthmakers for many statements.

These abstract claims are elaborated and argued for in the following section. The technical appendix presents a formalization, which enables analysis of truth-claims and diagnosis of paradoxes, using only classical logic.

## 2 Discourses as truthmakers

### 2.1 The missing link – discourse as truthmaker for cases of failed denotation

Consider the following discourse, consisting of a single statement:

- The next statement on this list is false. (2)

When presented with a peculiar looking statement such as this, the natural philosophical approach is to determine its exact nature and then analyze the object so obtained, asking whether it is true or false or, more generally, what it can tell us about the nature of statements or of truth. However, with (2) already the first step seems doomed to fail. For no matter how meticulous we are in studying the statement, no matter how good we are at breaking it apart and looking at its constituents, one question will remain forever unanswered: what is the 'next statement'? In the search for an answer we look *beyond* the statement itself and in the particular case of (2) we have a problem since, as far as we can tell, there is no next statement.

This is not just another common case of reference gone wrong. We are not purporting to refer to an object in the non-linguistic world, like the king of France. We are referring quite explicitly to an entity in a list of statements, a collection of linguistic objects. We are purporting to point out an element in a discourse, and not just any discourse, but *the* discourse of which the statement itself forms part. No doubt, this discourse exists – it is there, on the page! – yet the 'next statement' does not. The point is not that it is difficult to arrive at this conclusion; ostensibly, the crucial next statement is missing. Rather, the point is that in order to establish this simple fact, we have to take the discourse into account, either explicitly or at least implicitly. It is not enough to say simply that there is no next statement, in the world, as it were. Saying that there is no next statement, we mean that there is no next statement *in the actual discourse*, and the distinction is crucial. For it is the discourse that must exist, in order to witness to the claim that it, indeed, contains no next statement. The discourse itself is the only plausible truthmaker for the true claim that our statement fails to denote.

### 2.2 No no-no: discourse as truthmaker for distinctness of opposites

Consider the following discourse:

- The next statement on this list is false.
- The previous statement on this list is false. (3)

In this case, we do not have a case of failed reference, but a discourse with two statements, both of which seem perfectly meaningful. In fact, discourses with the same basic form as this appear in real life all the time, whenever two people disagree by negating each others claims. What makes this discourse special is that it points to nothing whatsoever outside of itself. Making a further analogy with real life situations, it is like two people disagreeing as a matter of principle, without making reference to any particular subject matter. However, just as this is a relatively common occurrence in life, so does it seem intuitively clear that while discourse

(3), referring only to its own statements, is ungrounded, it is still perfectly meaningful and can be analyzed further with respect to truth.

But doing so presents us with a problem, since there seems to be nothing in the non-discursive world on the basis of which we can judge the truth of the involved statements. Whatever their truth might be, there seems to be an acute shortage of plausible truthmakers. Based on this, a prevailing metaphysical intuition is that the semantic status of the two statements, being symmetric, should be the same. This creates problems, since by classical logic alone, it follows that if one statement is true, then the other must be false. This conundrum is known as the "no-no paradox" in the literature, e.g., [Coo11].

The standard account of the no-no paradox is, however, flawed since it fails to consider one important object present in the scenario, the discourse itself. While the statements are symmetric and do not say anything about the rest of the world that could help to distinguish between them, they appear in the world at different places in the discourse, and as such are in fact distinguishable. This is not a conclusion we can arrive at by breaking them down and analyzing their parts. It is only attainable if we take broader look, and note that they occupy different positions in the discourse. In this way, we also come to accept that they say different things about the discourse, and hence also about the world. Their different position in the discourse is their essential feature, it is a part of their identity. Therefore, while we reject the view that there is a paradox, we do not follow [Sor03] in suggesting that these statements lack truthmakers. The discourse is their truthmaker, and without it the statements could not even exist.[4]

This might be quite a radical account of the nature of statements, but the discourses presented so far suggest that it is both attractive and plausible. For it could not have escaped the reader that the statement of discourse (2), modulo the discursive context, is identical to the first statement of discourse (3). Disregarding the discourses, there is nothing that could possibly distinguish between them, their literal meaning being the same. Intuitively, however, they are clearly distinct; the first statement of (2) fails to denote, but the first statement of (3) does express a meaningful claim, to which a notion of truth should apply.

So, if someone who does not believe in discourses can arrive at a no-no paradox by noting that the first and second statement of (3) appear equivalent, does it not then follow, by exactly the same line of thought, that the first statement of (2) is equivalent to the first statement of (3)? This, however, is not just a logically problematic conclusion – it also appears completely counterintuitive and, indeed, metaphysically unreasonable. However, as soon as we admit discourses into our ontology of truthmakers, the necessary distinction between these statements, and between the first and second statement of (3), is no longer just an artifact of logical representation, but becomes also a consequence of our metaphysical stipulations. The no-no paradox disappears; classical logic is correct when it designates the first and second statement of (3) as contrary to each other, and there *is* an object in the world that bears witness to this, namely the discourse in which they appear.

Now, while we conclude that the two statements of (3) are contrary, we cannot say more than that. In particular, we cannot say that one is true or the other is false, for there is no information, neither in the discourse nor elsewhere to indicate an answer. The truth-claims of the involved statements are legitimate but not settled, and one might pose this as an

[4]We use the term "truthmaker" rather liberally, as will be seen in a moment, since the discourse does not make any of the statements actually true. In this case, it is only the 'corresponding aspect of reality' which witnesses to their contrariety.

accusation of insufficiency. While we provide some metaphysical sugarcoating for the logical analysis by which they are opposites, this does not really change the fact that the truth-values of both statements are undetermined and, as such, are in some sense the same.

If this is true, then it is true in much the same way as both 'it will rain tomorrow' and its negation are contingent (on what happens tomorrow). In such cases, there is little doubt that insufficient is only the notion that they are the same. It will rain or it will not, and this recognition of mutual exclusiveness represents a more adequate analysis than the mere recognition that, as of yet, we do not know which one it will be.

With the discourse being part of the ontology, the position in the discourse becomes part of the statement. It can be irrelevant, but sometimes it is crucial. So, while the choice between the first and second statement of (3) still appears unmotivated, it has now become metaphysically clear that they are different and, in fact, contrary. Without the discourse, on the other hand, their indistinctness seems inescapable, turning the logically unproblematic situation (3) into a genuine paradox.

Let us remark that ungrounded discourses are by no means the only circumstances in which the truth-values of contingent statements appear fundamentally so. While there is certainly a sense of arbitrariness involved in discourses such as (3), this aspect, being more about epistemic access to truth-values than about truth's nature, should not be overemphasized. Most counterfactual claims, for instance, are of this sort. Still, not many people follow Quine in claiming that this makes them meaningless or incapable of having definite truth-values. In fact, the inherently contingent, hence modal, character of discourses like (3) is not merely a pragmatic annoyance, but is the basis of most, if not all, legitimate truth-claims which a theory of truth should capture. We believe, moreover, that our approach succeeds in doing so, as it admits here only what classical logic admits and nothing more: it establishes the possibility of specific truth-values and the necessity of these values being opposite.

### 2.3 Grounding the ungrounded – discourse as truthmaker for the definite truth of its own statements

Contingency of discourses is the most common situation, but it is not the only one. Non-discursive facts determine sometimes unique truth-values and in Section 2.5 we show how our proposal accommodates the definite nature of such truths. However, even ungrounded discourses can give rise to unique truth-values of their constituents. For instance, in the discourse "The next statement is false. The previous statement is false. The previous statement is true or false.", indeterminate truth-values of the first two statements do not prevent us from seeing that the last statement is true. But beware! It is not enough to notice that it is a tautology. If the last statement referred to the liar, for instance, then its truth would be questionable, to put it mildly. It is true because, in addition to being a tautology, it resides in a non-paradoxical discourse.

To illustrate the general point, let us consider the following list of statements:

(u1) This statement is false and so is the next one on this list.

(u2) The next statement on this list is false. (4)

(u3) The previous statement on this list is false.

The statement (u1) is a contingent liar. If (u2) is false, then (u1) is true iff it is false, and

semantic paradox ensues. But if (u2) is true then (u1) is simply false because, regardless of its negating self-reference, it wrongly claims falsehood of (u2).

Paradox will be considered in more detail in the following sections, while for our current purpose we only postulate that avoiding it is the fundamental attitude. We will not consider a discourse paradoxical if we can make sense of it. Sense, however, must be made both with respect to our ontology and our logic, and while the harmless nature of (4) is a logical triviality, it is ontologically subtle. For it is only by accepting that the discourse itself exists, that we obtain an ontological grounding for its analysis. We proceed as follows: Since paradox ensues unless (u2) is true and there is no other information present that can help us determine the truth of any involved statements, we conclude, on the basis of this *discursive* fact, that (u2) is indeed true. This entails then that (u1) and (u3) are both false, so that every statement of this discourse obtains a definite truth-value even if, as in the two previous discourses, neither of them concern any object in the world besides the discourse itself.

This conclusion is a truth of classical logic: the discourse is inconsistent unless (u2) is true, from which it follows by classical logical consequence that it *is* true.[5] The philosophical challenge is to connect this to metaphysical truth, but our account does exactly this. It is the discourse itself that acts as a truthmaker for this distribution of truth-values among its statements, and the distribution is as it is because it must be, to prevent paradox.

This introduces a highly specific notion of *holism* into the account of truth, paradox and, indeed, into the very nature of statements. A statement is not an isolated entity, but is always anchored in a specific discourse.[6] The discourse is part of the statement's essence, providing, one might say, an account of its origin, analogously to the Kripkean account of identity of objects in general [Kri81]. Moreover, as illustrated by examples considered so far, the discourse to which a statement belongs can exercise a crucial influence on its meaning and truth. Example (4) shows a situation where, in spite of ungroundedness, (u2) is *necessarily* true, not because it refers to something making it true, but because it is part of something in the world – the discourse – that contains statements referring to it in such a way that it has to be.

### 2.4 Discourse as truthmaker for the claims of lacking truth-value

Consider the following variation, from [Gai92], of a scholastic puzzle:

(5)
(a) The first statement on this list is not true.
(b) The first statement on this list is not true.

Literally, these two statements say exactly the same thing, yet they do it in different ways, and have different truth status. While the first statement is obviously paradoxical, the second one seems to be true.

Acknowledging this difference, one can follow some scholastics in viewing them as two propositions with the same formal signification but distinct virtual implications, as two tokens of the same type or, as we do, as two statements with the same literal but different actual meaning. Here, differences between such alternatives are not as important as their common

[5]This might require qualification, in particular, logical formalization of discourses. Formalization may be possible in various ways, so we give one example in the appendix.

[6]Isolated statements can be seen as limiting cases of discourses consisting of a single statement.

element, namely the contextual nature of truth-carriers. The crucial insight concerns the reason for the semantic difference of (a) and (b). It does not arise from any property of their parts, but appears only when we take a step back and look at them jointly, in the context of their discourse. Indeed, the whole (5) seems to be the only candidate for explaining the difference between (a) and (b), and it is by taking the totality of the discourse into account that we can argue for it, by pointing to the difference in their referential structure. It is precisely *not* possible to distinguish between (a) and (b) if one insists on breaking the discourse apart in order to analyze its constituents separately.

While the difference between (a) and (b) is witnessed by the discourse, the conclusion that (b) is true might be hasty. Certainly, if we agree on (a)'s paradoxicality, (b)'s truth seems indisputable. However, paradox is a property of the discourse as a whole, and since (a) and (b) are part of the same discourse, perhaps paradox affects (b) just as much as it affects (a)?

Suspending for the moment a judgment about the truth of (b), let us consider another well-known phenomenon, the liar cycles such as the following.

(c) The next statement on this list is not true.

(d) The next statement on this list is not true. (6)

(e) The first statement on this list is not true.

The same reasoning as just suggested for (b) in (5) can be applied to any of the three statements involved. One might say, for instance, that (c) is paradoxical, thus neither true nor false, thereby making (e) true and (d) false. However, nothing suggests that doing so is more appropriate than saying that (d) is faulty, in which case (c), rather than being paradoxical, is true and (e) false. The problem in this discourse can not intuitively be pinned down to any of its statements, so placing the blame becomes an exercise in arbitrariness. Importantly, it does not seem to tell us anything about the paradox manifested by the discourse (6).

The problem of this discourse does not emerge from any one of its statements. No, the problem arises at a different level altogether, it arises from the totality of the discourse, which is not a mere collection of statements, but involves also their mutual *interaction*. The problem lies in this referential structure, and only there: paradox emerges not so much from what statements say as from how they interact. It is more than a gap in truth-attribution, it is a *failure* of truth-attribution, the impossibility of realizing the truth-claims of particular statements. This is always due to their unfortunate interaction, and it is the situation which occurs in discourses where truth has malfunctioned, where the concept has failed to provide what we expect of it. In general, we can not pin down paradox beyond locating some boundary around the problematic structures of a discourse and isolating them from others which are not paradoxical or semantically dependent on the paradoxical ones. We will consider this issue in more detail in Section 2.6, but remark here that it is different from placing the blame on particular statements. Rather, it concerns the question of placing boundaries around discursive structures in such a way that a semantic link between them allows to pinpoint the smaller wholes wherein the paradox arises and, hopefully, also some other wholes were truth can be regained.

Returning to our examples, if we say that paradoxicality of (a) in (5) makes (b) true, then does not paradoxicality of (c) in 6 make (e) true in the same way? Both statements claim untruth of another, paradoxical statement. There seems to be no reason to declare (b) true and (e) paradoxical. Indeed, in a compositional, truth-functional semantics of statements,

there is may be place for such a difference. Once we accept paradox as the property of whole discourses, however, the situation becomes different. For in (5), there is an identifiable subdiscourse (a) – a smaller whole – which we intuitively recognize as responsible for all paradoxicality. Consequently, the paradox does not affect (b) in the same way as it affects all three statements of (6). For (b) is not needed for the substructure (a) to become paradoxical, (a) does it all on its own. The paradox of (6), on the other hand, consists of all three statements together. So (b) and (e) mean different things because, although literally identical, they are made different by their respective discourses.

We can thus accept the intuition that (b) is true, rephrasing its "not true" as meaning 'false or lacking truth-value'. It is made true when we draw new boundaries and consider the subdiscourse (a). Then, seeing (b) as being outside this paradoxical subdiscourse, we can conclude that it is correct *and* that it does not involve any referential failure, as happens with (a). In this way, (b) *becomes* true. (e), on the other hand, although literally indistinguishable from (b), is *necessarily* involved in such a failure, since it is part of a paradoxical discourse around which we cannot, intuitively speaking, draw any new boundaries without affecting the meaning also of statements inside the boundary. We cannot separate (e) from its paradoxical contexts and therefore its discourse necessarily invalidates its truth-claim.[7]

Summing up, admitting discourses into the ontology of truth leads to the designation of paradox as a *property of discourses*, not of individual statements. Precisely because of this, statements such as (b), claiming untruth of a paradox, can be seen as true, without forcing the truth of other, paradoxical statements, such as (a) or (e), which claim literally the same. This gives a strikingly simple and revenge-immune account of the paradoxes whereby we can say confidently – and truthfully! – that statements in paradoxical discourses have no truth-values – not because they have some other values, but because a paradoxical discourse amounts exactly to invalidating truth-claims of all involved statements.

### 2.5 Snow is white, but might be more – discourse as truthmaker for the resolution of contingent paradoxes

At this point, the reader might have started to worry about the apparent lack of statements which are true independently from their discourses. To see that they are still around, let us consider "Snow is white", a typical well-behaved statement, thought to be grounded in facts outside its discursive context and true independently of it. Agreeing to this, we only want to view it as a discourse, containing this single statement, which we call (s). The standard

[7]The question of how to draw boundaries around discursive structures is subtle. Some of its aspects are addressed in Section 2.6 and for a further illustration of the arising difficulties, we point to the work on so-called token-relativism in [Gai88, Gai92]. We think, in particular, that the technical aspects of this work can be better understood if one sees it as an attempt to give an account of boundaries and, consequently, that the metaphysical foundation upon which this project should be based is an ontology of discourses. But their role seems to have been overlooked also by those adopting Gaifman's token-relativism. Our statements might remind of tokens but the only feature they share with them is context dependency. The token-type distinction might be, perhaps, related to our distinction between actual and literal meaning, but these are only attributes of statements, while any significant counterpart of types is missing from our ontology.

On a more technical side, [Gai88, Gai92] introduce complex operational rules to avoid both putting blame on statements that appear harmless, like (b) in discourse (5), and assigning semantic values to statements that appear complicit in the paradoxical interactions, like those of discourse (6). Variants of this proposal do not seem to have reached any stable state and appear inconclusive, so we do not address it further here. In the appendix, however, we offer a formalization in a similar vein which relies on classical logic alone and, therefore, should be far less controversial.

notion, whereby (s) claims that snow is white and nothing more, is consistent with our view. The only difference concerns the added qualification "nothing more", which is no longer just a true *description* of the statement. Under our account, it is *part of it*, it is an essential feature of its nature.

Consider for comparison the following discourse.

(cl) This statement is false and so is the next one.

(s1) Snow is white. (7)

Here, there *is* something more and, consequently, the sentence "snow is white", as it appears in this discourse, does not express the statement (s), but some other one, (s1). The difference can be justified by the fact that while (s1) does resolve the liar statement (cl), no such claim is true of statement (s). In particular, there is nothing whatsoever about snow being white that inherently resolves contingent liars. Surely, this cannot be an essential aspect of the whiteness of snow? Indeed, if I say "this statement is false and the moon is made of cheese", I utter another contingent liar, and while this one is also resolved, it seems obvious that the whiteness of snow has nothing to do with it. So we conclude that it is the fact that snow is white *and* the fact that the contingent liar (cl) points to a statement which is made true by this fact, which *together* ensure that (cl) is resolved. Furthermore, the pointing – involving, as always, both pointer and pointee – is not only something we must attribute to (cl), it is also a part of what makes the statement (s1) what it is. Essentially so, and essentially different from the statement (s).[8]

To put it more concretely: a statement expressing the fact that snow is white in isolation, like (s), does not resolve any liars. The resolution of a contingent liar is precisely the *part of* (s1) making it special and distinct from (s). This is so by virtue of the fact that we have the statement (cl), referring to (s1). This, in particular, concerns also the nature and meaning of (s1). While (s) and (s1) are linguistically identical and say the same, their appearance in different discourses differentiates also their status in the real world of language-independent objects. So, while snow is still white, and we can still express that much, expressing this in isolation is something different from expressing it in a discursive context where it also serves to ground the truth of other statements. In this case, snow is still white, but this might be of minor significance compared to other aspects that are inherent to the statement expressing it, by virtue of the discourse of which it forms part.

While "Snow is white" can sometimes mean more than it does in an isolated statement (s), it seems unreasonable to think that it can become false. Rather, its truth seems to persist across all the discourses where statements saying the same might appear. In this sense, it seems different from the ungrounded statements, whose truth may be influenced by other

[8]One might try to view (s) and (s1) as the same statement which, only due to occurrences in different contexts, has different meanings. This might be just another way of saying the same, provided that one pays attention to the fact that the difference in meaning influences the truth-value. This is more apparent when we consider the first statement of (3) and the second statement of (4), or else the two statements of (5). Seeing them *only* as the same statement, i.e., reducing them to what they say, rather than what they mean, one is still forced to grant them different truth status. Since we view statement as a minimal truth-bearing unit, we find it appropriate to distinguish them *as statements*.

Certainly, there is a sense in which both (s) and (s1) say the same. But we are not inquiring here into this identity of literal meaning, and only observe that it does not always suffice for determining the truth-value and, consequently, does not exhaust the semantics.

statements to which they do not refer, as the second (and third) statement in the list (4). Now, let us change (7) to the following discourse:

$$\begin{array}{ll} \text{(dl)} & \text{This statement is false and the next statement is true.} \\ \text{(s2)} & \text{Snow is white.} \end{array} \tag{8}$$

The statement (s2) appears to be true but, unfortunately, this results now in a paradoxical discourse, which would be avoided if we could make (s2) false. The mere existence of this structure however, does not imply any change to the accepted whiteness of snow. We see no reason, in particular, why the holism of truth should extend that far, and we are not committed to any such extreme view. While discourses are one kind of objects responsible for making statements true, they are not the only kind. We can easily accept the possibly uncontroversial truth or falsehood of some statements, which can be seen as independent from the discourses where they appear. This might be the case of statements expressing empirical, non-discursive truths or, more generally, of propositions carrying a specific truth-value, as a consequence of all its actual utterances having always the same truth-value in all non-paradoxical discourses (like propositional tautologies or contradictions). We are not denying their existence, we only deny that they exhaust or explain all possibilities. In fact, they are rather special cases, and once their consequences are taken into due account, one is left with the vast, seemingly ungrounded, residuum of the discursive universe, which consists of discourses whose statements depend only mutually on each other. The holism of truth concerns primarily this residuum, and an ontology with discourses provides for their grounding in corresponding objects, allowing us to make sense of them in keeping with our intuitions about the workings of truth.[9]

However, looking at the matter from the perspective of the historical discourse of humankind, uncontroversial truths can happen to change truth-values. Even if the sentence expressing (s2) will be regarded as true in any discursive context, similarly uncontroversial truths might, one day, turn false, in order to make larger wholes non-paradoxical or consistent. If, and possibly how, the holism of truth might influence also the truth-values of statements that seem grounded in non-discursive facts, is a question asking for a separate treatment, if not a separate treatise. Here, we only note that while a non-discursive account of truth can hardly admit any such dependencies, at least not without ending in full blown relativism, an ontology of discourses provides for their grounding in corresponding properties of the world. The world is ripe with fact and while some facts may not be discursive, most of them are.

### 2.6 Subdiscourses as truthmakers when discourses fail

When considering discourse (8), it seems clear that we should conclude that even though discourses exist and inform our judgment about truth, snow is still white, and the discourse is still paradoxical. The paradox would disappear if we considered (s2) false, but, as we have emphasized, we are not proposing such an extreme coherence view of truth. Rather, we think the fact that (s2) does not, in any way, refer to (dl) – which we intuitively see as the cause of the paradox – provides a *discursive fact* that witnesses, by way of correspondence, to the inadmissibility of such overly holistic reasoning. We try to avoid paradoxes, but not at any cost, and discourses themselves seem the obvious place to look for the cues as to what is,

[9]This is reminiscent of Quine's epistemological holism, which allows to adjust various underdetermined parts of a theory, in order to save it in the face of apparently contradicting facts.

and what is not, an affordable price to pay. The situation can be quite involved though, as illustrated by the following example.

(n) The next statement is false.

(p) The previous statement is false and so is the next one. (9)

(l) This statement is false.

The paradoxical (l) is referred to from (p) and it is not intuitively clear how we should think about such references in the presence of paradox.[10] Still, unless one turns rather special, one will not say that the liar is false. We see little reason for being so special, and then, as the claim of falsity of (l) by (p) cannot be confirmed, (p) can hardly be considered true. However, we can safely view (n) as true, and this itself suffices for making (p) false, *irrespectively* of the status of (l).

Thus, within a discourse affected by paradox, we can sometimes find coherent subdiscourses, ones where truth-values can be distributed in a way respecting both what the statements say and their referential dependencies. With respect to discourses, coherence can be equated with non-paradoxicality, but for a coherent subdiscourse to be of interest, something more is needed, namely, a witness to the fact that the actual boundary is reasonable, that the coherence of the subdiscourse is at all *relevant* to the discourse. A plausible witness of such relevance can be provided logically, by establishing that a coherent distribution of truth-values to the statements of the subdiscourse is *logically immune against possible changes in truth-attributions outside the subdiscourse*. This is what the above example illustrates and what makes the assertions concerning the subdiscourse (n)-(p) relevant also for the analysis of the whole paradoxcial discourse (9). No matter what one might decide later about (l) – that it is, in fact, false or else that it is true, or neither or both – the possible truth of (n) and falsity of (p) are already sufficiently justified and remain unaffected.

Note that while this opens the door to non-explosive reasoning in the presence of paradox, the reasoning involved is classical. This is made possible by narrowing the attention to a coherent subdiscourse, which remains relatively independent from the truth-values of other, possibly problematic statements. A formal definition of this concept is given in appendix.

In all examples, we suggestively mark the boundaries of the discourses. In practice, the boundaries of a discourse can be unclear, but this fuzziness does not imply non-existence. Boundaries, much like the 'possible worlds' of Kripke's metaphysics of modality, are not in themselves metaphysically significant, they are merely stipulated. Even if one disagrees with this, our general assertions concerning the need for an ontology of discourses are not affected by such potential disagreement concerning the nature of their boundaries. The important objects are the discourses themselves, and they do exist, unless one wants to deny the existence of this paper or of today's news report.

However, the fact that we view discourses as independent entities does not mean that they can not be related in various ways, and the relationships that exist between a discourse and its smaller wholes, after we draw new boundaries, appear to be particularly significant. Coming to know anything about such relations is a non-trivial question and the concept of a coherent subdiscourse gives only one example. To illustrate the subtleties involved, let us say, tentatively, that a faithful subdiscourse, capable of informing us also of the larger whole, is

[10] Strictly speaking, paradoxical is not statement (l) but the subdiscourse consisting of exactly this statement, but this point should be clear enough by now, to admit such a more colloquial way of expression.

obtained simply by taking an arbitrary subsequence of a discourse's statements.[11] Recalling discourse (1) from the introduction:

(x) The last statement on this list is true.

(y) Snow is white and the first statement on this list is false.

the subsequence (y) would be one of its subdiscourses, which is itself paradoxical. This may still be acceptable but considering now the subdiscourse (x) shows that such a trivial notion of subdiscourse is far from adequate. For (x), when looked at as a discourse in its own right, is completely different from (x), looked at as a statement appearing in (1). In fact, it changes the truth status and is no longer (indirectly) self-negating but becomes self-affirming, a truth-teller.

One might brush this off, denouncing it as linguistic trickery, illustrating only the ambiguity of natural language or specificity of indexical references. However, if discourses play a fundamental role, as we have argued, such a dismissive attitude is unwarranted, and the phenomenon we point to is not of a purely linguistic, performative, nature. On our account, the question of how discourses are related becomes an important one, and its status as such does not derive from the particular linguistic constructions used to represent it, but from the very idea of an ontology of bounded discourses. For the metaphysical argument we are making in this paper, the mere recognition of the existence of boundaries between discourses suffices. The concept of a coherent subdiscourse, allowing to draw such boundaries in larger discourses, can serve as a first step towards a more complete account of the epistemology of discourses, which remains to be worked out.

### 2.7 Protected by being framed – discourse as truthmaker for absolute truths

The holism we observe here concerns only discourses, relative and limited totalities of statements considered together. It means that truth arises only from such totalities, and that it is threatened only by their possible malfunction, giving rise to paradox. However, even if the truth-concept does not succeed in every discourse, we always try to make place for it, we always attempt to form or interpret the discourse so that it provides grounds for truth-claims of all involved statements. Moreover, just like paradox is the limit of truth and arises holistically, truth of particular statements appears thanks to a similar effect, the overall correspondence between what truth tells us about the semantic status of the statements, and what the discursive structure itself has to tell us about their content. Truth and falsehood emerge in discourses, and they do so precisely when the truth-concept is applicable.

Does it mean that all truths are relative to context? Not at all, they are all definite. For a discourse not only surrounds its statements, the statements are *necessarily* contained in their discourses. If you change the discursive context, you are no longer in the presence of exactly the same statement. Still, asking for absolute truths, one might mean something more, one might like to have truth of statements that appear in the ultimate discourse, *The Discourse*, the union of all possible discourses, of everything that anyone has ever claimed, will ever

[11] Subsequence is more general than sublist, allowing removal of arbitrary elements. A (proper) subsequence is a (proper) subset of list's elements listed in the same order. A (proper) sublist, containing two elements $a, b$ from the list, must also contain all those lying between $a$ and $b$ on the list. E.g., proper sublists of the list $[x_1, x_2, x_3]$ are only 1-element lists, $[x_1, x_2]$ and $[x_2, x_3]$. Proper subsequences contain, in addition, also $[x_1, x_3]$.

claim, and, indeed, could ever possibly claim. Unfortunately, this behemoth constellation – assuming that it exists at all – is a heavy mess, riddled with paradox and contradiction, witnessed by the pernicious hubris of considering all things in the context of everything else. Presumably, no one would disagree that *The Discourse* is hard to make sense of, and many would agree that it is not worth trying. While our view does not explain what *The Discourse* might mean, we can point to its inherent paradoxicality, the impossibility of determining truth-values of all its statements, providing thus the ontological grounding for the prevailing intuition of its incomprehensibility.[12]

On our account, this gargantuan discourse – assuming it exists – is a separate entity, different from all its proper subdiscourses. So even if all statements say the same as they do in smaller discourses, their meaning is different. The fact that some statement in *The Discourse* lacks a truth-value does not preclude the possibility of some other statement, with the same literal meaning, carrying such a value in some smaller and coherent subdiscourse. Example (9) served to illustrate this phenomenon, where partial truth-attribution in a coherent subdiscourse can not be falsified by any of its extensions to larger parts of the discourse. More generally, our approach suggests a general, non-linguistic reason why analysis is so useful to the search for truth. By breaking things apart, one *changes the subject* – not completely, but only to some, hopefully appropriate degree – by directing attention at aspects of reality with less discursive content and, consequently, less prone to paradoxicality.[13]

Considering things separately, asking only how they are related in the non-discursive part of the world, is not the same, even in a metaphysical sense, as asking how they are related when they appear in *The Discourse*, whereby full metaphysical attention is demanded by all things at once. Thus, the irrelevance of *The Discourse* does not imply irrelevance of all its subdiscourses. They can be perfectly meaningful and interesting without the universal whole being so. In fact, it is not so that truth resides *also* in such limited discourses, truth resides *only* there, because it requires a corresponding, well-behaved, referential context. When a discourse expands too much and turns inconsistent or paradoxical, it becomes incapable of providing the grounding for truth. One then has to step back and look for reasonable limitations, smaller subdiscourses, where truth can either arise from the non-discursive grounds or succeed in bootstrapping itself onto the referential content of the discursive structure itself.

Moreover, the absolute truths are not in any need of *The Discourse*. Their absolute status arises from being grounded in some specific and limited discourse, concerned with absolute matters and carrying the same meaning in all contexts, protected by the discursive boundary that calls it into being. They have as little to do with the whiteness of snow as with the truth of the next statement on this list.

[12] One might argue that if *The Discourse* exists then it is not a discourse, by analogy to Russell's set of all sets not being a set. Let *Disc* denote collection of all discourses (assuming it is legitimate) and $X \prec Y$ denote that $X$ is a proper subdiscourse (subsequence) of $Y$. Obviously, no discourse is its own proper subdiscourse, i.e., for all $X \in \mathit{Disc} : X \not\prec X$. *The Discourse* contains all such discourses, is their union, $\mathsf{D} = \bigcup\{X \in \mathit{Disc} \mid X \not\prec X\}$. On the other hand, every proper subdiscourse $X$ of $\mathsf{D}$ is a possible discourse and, in particular, is not its own proper subdiscourse, i.e., $X \not\prec X$. In short, $X \prec \mathsf{D}$ iff $X \not\prec X$. And so, if *The Discourse* is a discourse then, substituting $\mathsf{D}$ for $X$, we obtain $\mathsf{D} \prec \mathsf{D}$ iff $\mathsf{D} \not\prec \mathsf{D}$.

[13] In Section 4.3 of the appendix, it is shown that paradox arises only from a purely discursive context.

## 3 Conclusions

Truth is fragile. Not only its attribution may be mistaken, but the concept itself is not granted and may fail to apply. It is a norm which nature tries to establish in every discourse. But nature sometimes fails – paradox can occur in any language having some minimal expressive power. Paradox is the failure of truth and, as such, it can not be an additional semantic value which, when assigned to some statements, can interact with others, allowing the concept of truth to sneak back in, through the back door as it were. No, when paradox occurs, it occurs as an inescapable *fact*, the statements fail to make sense by inherent necessity. However, when a discourse turns paradoxical, it is not because it loses its literal meaning (which it does not), but because the norm of truth becomes impossible to satisfy. Paradox is the end of truth, but not the end of meaning. Rather, it gives rise to new meanings, to the truth of new statements that reflect on it from a different vantage point. A paradoxical discourse suggests that we should look to other discourses for answers, that new boundaries need to be considered. In this way, a discourse sometimes *causes*, by way of paradox, a further circumscription of its own elements. Thus, a paradox need not be seen as a problem but can be seen as an opportunity to step beyond the current boundaries, surrounding the purported meaning of the discourse, to look for new meanings, grounded in discursive facts and, sometinmes, in the paradox itself.

This is not a solution to the paradoxes, at least not in the ordinary sense. In fact, as long as "solution" is taken to mean some logic capable of handling paradox along with the boolean values, we feel it is misguided to provide one, not because it is impossible, but because it is too easy. There are sufficiently many logics internalizing paradox, yet none of them seem to internalize a satisfactory notion of truth. While various ways of internalizing various truths in formal theories may be viable, it is perhaps time to recall Tarski's doubts whether it can be done in a philosophically satisfying manner.

It seems to us, however, that the problem is not to be found in any shortcomings of formal logic, but stems from certain untenable metaphysical assumptions underlying the search for formal truth-theories. Here, we have addressed what we think is the most glaring one, namely that a notion of bounded discourse is completely missing from the ontology of objects usually considered relevant for it. We have argued that they should be included because, besides explaining paradox as a failure of the truth-concept, discourses are real objects. Like this paper you are now reading, they are concretely all around us. We ignore them at our peril.

The proposed approach is immune against new paradoxes, for the simple reason that it does not attempt to fill any gaps in truth-attribution. Paradoxes are not meant to disappear, but are to be explained and circumscribed. For they are *real*. The task, then, is to offer an understanding of truth where paradox can happily reside alongside that which is coherent and substantial. We believe to have shown how discourses allow us to do this. They lead, in fact, to the following conclusion: *there are no paradoxical statements*. Only discourses are paradoxical, and paradox does not endow statements with new semantic values but renders their truth-claims invalid. It does so by reducing their meaning to the literal one, giving an impression of a possible truth-value, while being unable to provide any actual one.

A functioning truth-concept, on the other hand, is simply the absence of paradox, enabling a coherent distribution of truth-values among the discourse's statements. Truth arises from the totality of a discourse which succeeds in avoiding dissolution by paradox. In most cases, it is not any definite and final truth-value of all involved statements. The truth-concept only opens the possibility of truth-values, it does not always determine uniformly what is and what is not true. Definite truth-values arise only in some important special cases and, sometimes,

entirely due to discursive facts. But this is by no means constitutive for the concept of truth, which functions *absolutely* whenever it applies, also in cases when the absolute truth is that the truth-values are left undetermined by the discourse and other facts.

A functioning truth-concept concerns the totality of the discourse which conditions also the actual meaning of its statements. A statement's meaning is not exhausted by what it says; saying the same does not imply having the same actual meaning. If, for instance, you speak of snow being white in a context where a liar claims you are speaking a lie, this does not have the same meaning as when you speak of snow being white while talking to someone about skiing. This is not an observation about the situation in which you utter your statement, but concerns the *discursive context* in which it appears. This structure contributes to the very identity of the statements it contains. We can choose to think of statements and discourses in concrete terms, as linguistic constructions, or in terms of the abstract, as ideal objects, only vaguely reflected by our actual conversations. More plausibly, they are something in between, but for our argument nothing hinges on this. All we are saying is that referential structures are real, that they matter, and that they matter for truth.

The holism of truth and paradox is the ultimate consequence of this dependence on the discursive structure. The whole is found within the parts – the statement contains its discourse and does so as much and as necessarily, as the discourse contains its statements. This view is not imposed by our theory, but arises from the mere observation that the meaning of individual statements depends on the discourse in which they appear. Importantly, we do not take holism as a primitive notion, so we avoid any ultimate unity of everything, *The Discourse*, or the like. Although borders around discursive objects can be gradually dissolved in more and more elaborate relations and associations, we are not forbidden from drawing them. In some sense they are there already, the world around us is *disposed* to be segregated, perceived and analyzed. In fact, imposing a boundary around that which is considered relevant seems unavoidable, or else truth can not function. For truth is, above all else, *local*. It concerns specific things, which do not arise from what they are in themselves, but from the boundaries separating them from everything they are *not*. Such is also the nature of the primitive statement, bound to its discursive context where, in addition to what it says and what is being said about it, the meaning it conveys also depends – eventually – on all that which fails to concern it in any way.

# 4 Appendix

The following formalization demonstrates the sufficiency of classical logic for the analysis of paradox and truth. Let us only emphasize that its aim is not a definition of truth but its analysis. We are not interested in defining a truth-predicate, let alone its axiomatic formulation, but in a non-definitional explanation of the concept of truth. First we make some observations suggesting that a propositional formalization is both justified and adequate for representing the aletheic references in discourses – the aspect of their internal structure relevant for the concept of truth and critical for the emergence of paradox.

## 4.1 "This" is a statement: References are truth-carriers, not by proxy, but by nature

Formal representation of claims about truth relies on some mechanism allowing one to refer to statements and a stipulation about the semantic nature of referring. Concerning the latter,

Tarski's T-convention offers what is needed: referring to a statement by saying that it is true should be equivalent to claiming the statement itself. In predicate logic, this takes the form $T(\ulcorner\phi\urcorner) \leftrightarrow \phi$, where $\ulcorner\phi\urcorner$ is a name for $\phi$. But why do we use a special predicate to express this intuition about truth? Logic, after all, always models truth, does it not? Certainly, if one's aim is to define truth as the extension of a truth-predicate, such a predicate – modeling, so to speak, the truth of truth – is indispensable. But when the aim is to give a *description* of (a certain aspect of) truth, it is no longer so. For there appears to be no good reason to use a predicate to express the *descriptive* content of the T-schema, except for a purely syntactic accident: predicate logic does not allow names (constants) to stand for formulae. For this reason, such formal systems force the use of predicates in order to talk about statements, be it talk of their truth or some other properties. Propositional logic, on the other hand, provides the means for direct reference, capturing the admissibility of disquotation, expressed by the T-schema, simply as $\ulcorner\phi\urcorner \leftrightarrow \phi$, where $\ulcorner\phi\urcorner$ is now a new propositional constant.

The meaning of the two schemata appears very different. The former determines the extension of a truth predicate, the latter only ensures that reference $\ulcorner\phi\urcorner$ agrees with the referent $\phi$ with respect to truth-value. Yet, the latter has much more to do with the nature of truth than it might at first appear. In some sense, it also underlies the former, which gives the extension of the $T$-predicate precisely by means of the stipulation that referring, by means of the name $\ulcorner\phi\urcorner$, gives access to the truth-value of the referent $\phi$. That this is enough to give a definition of truth, reflecting its nature, rests on an extremely deflationary view which, moreover, is a failed one, as the paradoxes show.[14] The very fact that the equivalence fails, even in arithmetic and certainly in natural language, shows that tokens used to refer to statements must themselves be viewed as statements. The crux of the paradoxical references and self-references is that they fail to agree with their referents and, so to speak, start functioning "on their own". Without assuming that they have the capacity – or, at least, pretensions – to carry truth-values, the reasoning towards the liar-paradox undermines itself since then "this statement", as it were, is not a statement.

Thus, seeing references as statements relatively independent of the statements they refer to, the concept of truth can be identified as the mechanism by which the semantic link between the two strives to come into being, the *norm* that they should be interchangeable. A functioning truth-concept amounts exactly to the satisfaction of the expectation that all references and referent agree with respect to truth-value.

The T-schema is thus a postulate, the satisfaction of which ensures functioning of truth. Its bottom line amounts to the claim of the equivalence of a reference and its referent which, as illustrated by examples of semantic paradoxes, may fail. Taking it as definitional (or axiomatic) may be therefore warranted at most in restricted, formal languages. Doing so for truth proper amounts to letting the language supervene on reality, channeling the paradoxes into inconsistency of the theoretical framework and rendering them unreal and genuinely problematic.

### 4.2 A propositional representation of discourses

With every formula $\phi$, we associate some propositional constant $n_\phi$, which we take to be a statement of its reference. Whatever the subject matter or context, the statement of reference $n_\phi$ is meant to be equivalent with whatever it is a reference to, with its literal meaning $\phi$. The

[14] In this regard, it might be appropriate to recall Putnam's doubts that Tarski's treatment provides any significant insights concerning truth in nature (as opposed to in a formal construction) [Put85].

general principle of reference, underlying also the T-schema, becomes the simple equivalence

$$n_\phi \leftrightarrow \phi. \tag{10}$$

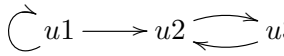
For instance, to represent the liar, let us name it $\lambda$. The liar says $\neg\lambda$, so we obtain an instance of (10), $\lambda \leftrightarrow \neg\lambda$. The unsatisfiability of this equivalence means that the truth-concept (10) fails – by way of logic, as it should – and paradox is recognized.

Generally, we form a discourse's truth-theory as a series of equivalences (10). E.g., the representation of discourse (4) becomes:

$$\begin{array}{lll} (u1) & \text{This statement is false and so is the next one.} & u1 \leftrightarrow \neg u1 \wedge \neg u2 \\ (u2) & \text{The next statement is false.} & u2 \leftrightarrow \neg u3 \\ (u3) & \text{The previous statement is false.} & u3 \leftrightarrow \neg u2 \end{array} \tag{11}$$

Discourse is paradoxical iff its truth-theory is inconsistent. The general applicability of this claim has been shown earlier, e.g., [Wal09, Dyr12, DW13], so here we only analyse simple examples, like the above discourse. It has the only model which, making $u2$ true and both $u1$ and $u3$ false, witnesses to its non-paradoxicality.

Truth-theories, represented as collections of statements of the form (10), can in turn be equivalently represented as directed graphs. This observation originates from [Coo04], and its technical ramifications have been developed further in [BGW12, DW13]. One of the results, established in [BGW12], is that directed graphs provide a normal form for propositional theories: every propositional theory has an equisatisfiable theory in *graph normal form* (which is essentially a truth-theory, consisting of a series of equivalences (10); in addition, right-hand-side of every equivalence is a conjunction of negated references, as exemplified by (11)). The graph is obtained by taking all references as nodes, and drawing an edge from a node, occurring on the left of an equivalence, to every node occurring under negation in the conjunction on the right-hand-side of the equivalenence. For the discourse (11), the corresponding graph becomes:



. (12)

Converesely, every digraph $\mathsf{G} = \langle G, E\rangle$, with $E \subseteq G \times G$, determines a logical theory $\mathsf{T}(\mathsf{G})$ making every vertex equivalent to the conjunction of the negations of all its out-neighbours:

$$\mathsf{T}(\mathsf{G}) = \{x \leftrightarrow \bigwedge_{y \in E(x)} \neg y \mid x \in G\},$$

where $E(x) = \{y \mid (x, y) \in E\}$. For sinks, i.e., nodes with $E(x) = \varnothing$, the empty conjunction amounts to the value **1** (true). These transformations give plausibility to the formal representation, according to which a discourse can be defined simply as a directed graph.

The true power and benefit of this representation arises once we consider discourses as combinatorial objects and analyze them in terms of their digraph structure. The graph normal form allows to import results from graph theory, which we can now interpret as results concerning the nature of paradox and truth. They all arise from the fact that a boolean assignment is a model of a theory in the graph normal form iff the set of propositional constants assigned **1** (true) forms a kernel of the corresponding digraph. A *kernel* of a digraph $\mathsf{G} = \langle G, E\rangle$ is a subset of nodes $K \subseteq G$ such that

$$E^\smile(K) = \mathsf{G} \setminus K,$$

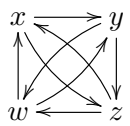
where $E^\smile$ is the converse of $E$, i.e., $E^\smile(y) = \{x \mid (x, y) \in E\}$. Equivalently, $K \subseteq G$ is a kernel if it is independent (no nodes in $K$ point to each other: $E^\smile(K) \subseteq \mathsf{G} \setminus K$) and absorbing (every node outside $K$ points into $K : \mathsf{G} \setminus K \subseteq E^\smile(K)$).[15]

Not every digraph has a kernel and this corresponds to the fact that the concept of truth fails for certain discourses. The liar, for instance, has a succint representation:

$$\mathsf{L} : \lambda \circlearrowright \qquad\qquad \mathsf{T}(\mathsf{L}) : \lambda \leftrightarrow \neg\lambda$$

Its paradoxicality can be now recognized, equivalently, by noting either that $\mathsf{L}$ has no kernel, or that $\mathsf{T}(\mathsf{L})$ is inconsistent. In the case of the simple liar, a logician might think the latter is easier to see, but consider the following example:

$\mathsf{G}$ :




| $\mathsf{T}(\mathsf{G})$ : | |
|---|---|
| $x \leftrightarrow \neg y \wedge \neg z$ | (x) The next two statements are false. |
| $y \leftrightarrow \neg z \wedge \neg w$ | (y) The next two statements are false. |
| $z \leftrightarrow \neg w \wedge \neg x$ | (z) The next and the first statement are false. |
| $w \leftrightarrow \neg z \wedge \neg y$ | (y) The first two statements are false. |

At first, it is unclear if the discourse is paradoxical, but this can be easily established as a special case of a simple fact about kernels in digraphs. In $\mathsf{G}$, every two vertices are related by an arc, as one says, the graph is *weakly complete*. For such digraphs, any kernel can consist of at most one vertex, or else it would fail to be independent. Furthermore, a kernel can be such a singleton vertex only when it is absorbing, i.e., pointed to by every other vertex of the graph. Now, noticing that $\mathsf{G}$ has no such vertex (each having indegree 2), one concludes that the discourse is paradoxical.

This is just one very simple example of reasoning facilitated by the graphical representation of discourses. There are several other techniques and results that can be imported from kernel theory, which can be found in [Dyr12, DW13]. The fundamental result, due to Richardson, [Ric53], is that a (finitary) digraph with no odd cycle always has a kernel. Thus, not only self-negation, in form of an odd cycle of negations, is able to create finitary paradoxes (as has been known formally since Tarski, and informally since Epimenides), but also such a self-negation is necessary for finitary paradoxes.[16]

### 4.3 Discursive holism and non-discursive facts

Our truth-concept admits 'external facts' or, as it may be more appropriate to put it here, statements whose truth-value appears independent from the rest of their discursive context and on which all references agree. Snow is white. This means that a claim that snow is not white is false, and one can continue this process of inducing truth-values to statements which are determined by the non-discursive facts, but are also more remote from the border where discourse meets them.

The graphical representation gives a precise picture of this coexistence of two, apparently, opposite views of truth: the grounded (correspondence) and the ungrounded (coherence)

[15] The notion of a kernel was introduced by Von Neumann and Moregenstern in [vNM47] as a solution concept in cooperative game theory. It has been recently rediscovered, [Dun95], as the so called "stable semantics" in the growing field of argumentation theory, which uses digraphs to represent argumentation scenarios.

[16] A finitary digraph may have infinitely many vertices, but every vertex can have at most finitely many neighbours. This excludes, for instance, Yablo's paradox, represented by the digraph $\langle \mathbb{N}, \{(x, y) \mid x < y\}\rangle$. It has no kernel for the just mentioned reason: being weakly complete, it has no absorbing vertex.

view. Sinks of the graph, representing such 'external facts', belong to every kernel and induce values to some internal nodes, following the simple schema, [Dyr12, DW13]:

$$\begin{array}{rcl} L_0 & = & \varnothing \\ L_{i+1} & = & sinks(\mathsf{G} \setminus E^{\smile}(L_i)) \end{array} \tag{13}$$

Fixed point, $\overline{\varnothing} = L_{i+1} = L_i$, is reached no later than at $i$ being the number of nodes in the graph.[17] Every kernel of any graph $\mathsf{G}$ consists of such a uniquely induced (grounded) local kernel $\overline{\varnothing}$ composed with a kernel of the ungrounded, sinkless residuum $\mathsf{G}^{\circ} = G \setminus (\overline{\varnothing} \cup E^{\smile}(\overline{\varnothing}))$:

$$Kr(\mathsf{G}) = \{K \cup \overline{\varnothing} \mid K \in Kr(\mathsf{G}^{\circ})\}. \tag{14}$$

That is, although contingent, external facts may infuence the discourse and, so to speak, release its internal paradoxicality, paradox never occurs exclusively because of such facts but only due to purely discursive dependencies in $\mathsf{G}^{\circ}$. External facts do not cause paradox, they can, at most, prevent it. A paradox exists only if there exists also a (part of a) discourse whose truth status is independent from the non-discursive reality. Even if this might have been accepted intuitively for a long time, (14) turns it into a mathematical theorem.

(14) does not merely show that the grounded (correspondence) view of truth may coexist with the ungrounded (coherence) view. It shows that, as long as we work with classical logic, the two actually coexist. In some cases, of fully grounded discourses, i.e., acyclic graphs with no infinite paths, $\mathsf{G}^{\circ}$ is empty and the induced $\overline{\varnothing}$ is its only kernel – all truths arise from the external facts. Sometimes, $\mathsf{G}^{\circ}$ may be empty even when the graph contains cycles, e.g., (15).(a) below. But these are rather special cases and, in general, external facts do not determine the whole discourse. E.g., example (b) below is paradoxical, with an unresolved liar left after inducing, while (c) is neither paradoxical nor fully determined, having a pair of opposite statements as the ungrounded residuum, whose legitimate truth-claims can be resolved as those of discourse (3) in Section 2.2.

(a) This statement is false and snow isn't white. $\quad \circlearrowright \mathbf{0} \longrightarrow \mathbf{1}$

(b) This statement is false and snow is white. $\quad \circlearrowright b \longrightarrow \mathbf{0} \longrightarrow \mathbf{1}$ (15)

(c) This statement is not false and snow is white. $\quad \overline{c} \rightleftarrows c \longrightarrow \mathbf{0} \longrightarrow \mathbf{1}$

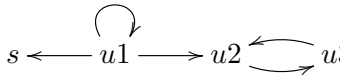
### 4.4 Coherent subdiscourses: local kernels

As we have remarked earlier, both the question of how to draw boundaries around discourses, and the question of how discourses are related, are difficult and deserve further attention, both philosophically and technically. Treating subdiscourses simply as subsequences of statements is, in many cases, not appropriate, at least not if one wishes a notion whereby subdiscourses have a well-behahaved semantic relationship to the discourses they form part of. In terms of our representation as digraphs, such a subdiscourse $\mathsf{G}'$ of a discourse $\mathsf{G}$ amounts to an *induced subgraph* of $\mathsf{G}$, namely, one with $G' \subseteq G$ and $E' = E \cap (G' \times G')$. Collecting in this way only some, arbitrary statements from a discourse, disregarding the rest, introduces the risk of changing the meaning of these statements, since they might refer to statements that are no longer present in the induced subgraph. Consider, for instance, the contingent liar $u1$ from (12). Its graph admits the liar as the subgraph induced by $u1$, and the paradoxicality

[17] It corresponds to Kripke's minimal fixed point.

of $u1$ emerges in this substructure as a necessary truth, differentiating its meaning from the meaning of the $u1$ appearing in the whole graph (12), even though they are literally the same.

Sometimes, however, subgraphs do preserve the semantic structure of the the original discourse, in the sense of it being *epistemically safe* to regard a semantic judgment made with respect to the smaller discourse as correct also for the bigger one. This, for instance, is the case in the following discourse $\mathsf{G}$:




(16)

Here, the contingent liar $u1$ is resolved by $s$, the statement of an empirical fact, like "snow is white", and the subgraph induced by $\{u1, s\}$ justifies this same observation. The observation is true in *both* discourses. It is tempting, then, to say that these discourses are related by this being so. While their statements are not the same, because the discourses are different, they are related in a semantically significant way. In particular, drawing the boundary around $\{u1, s\}$ in $\mathsf{G}$ is semantically faithful: the semantic judgment we make for the statements within this limited context, holds true also of the original statements in the whole from which this smaller context was abstracted.

In order to account for when this is possible, we consider the interaction that takes place across the boundary of the smaller structure. This interaction, since the boundary effectively removes it, can cause problems when we wish to let the smaller part inform us of the greater whole. But if the semantic judgment we make about the part does not depend on any such interaction and can be justified entirely by what is completely contained inside the smaller substructure, then we are entitled to export this semantic judgment to the larger discourse.

A formal representation of this intuition can be found in the notion of a local kernel, introduced in [NL71]. A *local kernel* is a subset of nodes $L \subseteq G$ such that

$$E(L) \subseteq E^{\smile}(L) \subseteq G \setminus L.$$

In particular, $L$ is a local kernel if it is independent (no two vertices in $L$ point to each other) and every vertex pointed to from $L$, points back into $L$. In discourse (12), $\{u2\}$ is one local kerenel and $\{u3\}$ another one. Each provides also example of a local kernel for (16), where also $\{s\}$ is a local kernel. Of course, every kernel is also a local kernel.

Now, it is not hard to see that a local kernel $L$ is a kernel in the subgraph induced by $L^+ = L \cup E(L)$ and, moreover, that it renders the possible interaction across the border between $L^+$ and $G \setminus L^+$ insignificant for $L^+$. One can consider all statements in $L$ true and and all in $E(L)$ false, and this judgment is immune against possible truth-assignments to the nodes in $G \setminus L^+$. For even if some statement $x \in L^+$ negates some $y \in G \setminus L^+$, so either $x \in L$, in which case $y$ negates something in $L$, or else $x \in E(L)$, in which case $x$ negates something in $L$. In either case, the status of $x$ is completely justified by the semantic status of statements that are present in $L^+$ and do not refer to anything outside of this local context. This appears a reasonable argument for viewing the conclusions about $L^+$ as a legitimate semantic insight also about the whole discourse.

One important consequence of this point of view is that it allows us to recognize well-behaved subdiscourses also inside discourses that are, as a whole, paradoxical. It provides thus means to prevent the spread of paradox which, otherwise, might afflict the entire discourse. For instance, in line with the informal analysis in Section 2.6, the paradoxical subdiscourse (l) of (9) need not prevent us from viewing the local kernel $\{n\}$ as true, and the subdiscourse $\{n\}^+ = \{n, p\}$ as locally meaningful. While it may be unclear how to handle apparently

meaningful statements in a paradoxical context, such a local judgment, made inside stricter boundaries and not dependent on the interaction across these boundaries, should be possible and imply a sound judgment also about the statements of the whole discourse. Truth might not live there but these statements give rise to a possible local view under which they become statements to which the notion of truth applies. Not the same statements, but related in a clear and meaningful way.

We should mention some complications, suggesting possible amendments, to the view of coherent subdiscourses provided by local kernels. Interaction across the border between $L^+$ and $G \setminus L^+$ is insignificant only for $L^+$, but it may have consequences for the rest of the discourse. Consider again the representation (12) of discourse (4), where $\{u3\}$ is a local kernel, i.e., $u3$ can be locally considered as true. Now, while the local view provided by placing a boundary around $\{u2, u3\}$ allows us to reach this semantic conclusion that seems, at first sight, unaffected by $u1$, it turns out that this is not so. For $u3$ can not be true in the discourse (4), as this would make the discourse paradoxical. Holism has its revenge, making the boundary inadmissible even if this can not be recognized by looking at the local interactions across it. The holism seems also right, for saying that the truth-value of $u3$ in the subdiscourse $\{u2, u3\}$ is completely unrelated to the larger discourse from which it arises, is not entirely correct. Instead, one should rather say that the truth of $u3$ in this smaller context, while arising as a possibility from restricting discourse (4), is nevertheless necessarily counterfactual, due to the presence of the self-negating statement $u1$: $u3$ could be true, if $u1$ were not there. Certainly, narrowing one's view, one may miss important aspects of the whole, but how to formulate this relationship at the level of discourses remains a challenge for further inquiry.

### 4.5 Analysing failures of truth

In a sense, the story of a paradoxical discourse ends in the moment one notices its paradoxicality. This is a fact about the world, made true by the discourse itself and invalidating truth-claims of its statements. Consequently, the analyses in this section do not intend to inform us about the norm of truth, which is expressed already in principle (10). They do not address the truth-concept but truth-crimes, where actual does not agree with the literal, where reference and referent part ways and can not meet again. But since discourses can be paradoxical in different ways, due to different reasons and, most importantly, due to different subdiscourses being paradoxical, a more fine-grained analysis may be of interest.

Such an analysis requires a relaxation of principle (10), since paradoxical discourses are exactly the ones where the principle is impossible to satisfy. It is now useful to introduce a reminiscence of the truth-predicate. This is perfectly possible using only propositonal means and, given the graph normal form, our adjustment on this point is merely a matter of finding a more suitable technical presentation of our claims, it does not involve any new logical principles. To this aim, we keep $n_\phi$ as the name referring to $\phi$, and introduce a new propositional constant, $T_{n_\phi}$, of which we think as the statement that $n_\phi$ is true. What should hold of this statement? Firstly, it should be equivalent to $n_\phi$ and, secondly, it should also be equivalent to whatever $n_\phi$ itself claims, i.e., to $\phi$ – reference should behave normally under the scope of truth. These two requirements are captured by the following schemata which, replacing the principle (10), result in an equiconsistent theory for any discourse:

$$\begin{array}{lccc} (\mathsf{R}) & T_{n_\phi} & \leftrightarrow & n_\phi \\ (\mathsf{L}) & T_{n_\phi} & \leftrightarrow & \phi. \end{array} \tag{17}$$

As an example, consider discourse (9), represented by the graph $n \leftrightarrows p \rightarrow l \circlearrowleft$. Its truth-theory consists now of two parts, $\mathsf{L}(9)$ and $\mathsf{R}(9)$:

| | discourse (9): | $\mathsf{L}(9)$: | $\mathsf{R}(9)$: |
|---|---|---|---|
| (n) | The next statement is false. | $T_n \leftrightarrow \neg p$ | $T_n \leftrightarrow n$ |
| (p) | The previous and the next statement are false. | $T_p \leftrightarrow \neg n \wedge \neg l$ | $T_p \leftrightarrow p$ |
| (l) | This statement is false. | $T_l \leftrightarrow \neg l$ | $T_l \leftrightarrow l$ |

(18)

The collection $\mathsf{L}(9)$ of the literal claims is trivially consistent – simply assign to each $T_x$ whatever results from evaluating the right-hand side of its equivalence. Even so, we can still recognize paradoxes and, importantly, various ways in which things can go wrong. For even if $\mathsf{L}(\mathsf{G})$ is consistent, when $\mathsf{G}$ is paradoxical it is no longer possible to satisfy also the referential dependencies, $\mathsf{R}(\mathsf{G})$: the intended equivalence $T_x \leftrightarrow x$ will fail for some $x$, and this can be seen as the failure of principle (10). This inconsistency of the whole truth-theory, i.e., paradoxicality of the discourse, can be observed as the negation of (some of) the referential equivalences ($\mathsf{R}$) following logically from the consistent, literal claims ($\mathsf{L}$), in the example, $\mathsf{L}(9) \models \neg(T_l \leftrightarrow l)$. Incidentally, this shows also that (l) is the problematic statement of this discourse, since the negation of the ($\mathsf{R}$)-schema of any other statement does not follow from $\mathsf{L}(9)$. (As $\{n\}$ is a local kernel, so $\{n, p\}$ forms a coherent subdiscourse.)

In this example, paradox arises from its last statement alone. The entailment $T_l \leftrightarrow \neg l \models \neg(T_l \leftrightarrow l)$ witnesses to the paradoxicality of the simple liar.

Claiming one's own paradoxicality is false. To represent such a claim, we have to express that a statement fails to have truth-value. In classical logic this requires a hack, and the hack we provide involves splitting the equivalence (10) into ($\mathsf{L}$) and ($\mathsf{R}$). The lack of truth-value amounts to the failure of (10), which is reflected as an inconsistency of the ($\mathsf{R}$)-schema with the literal theory ($\mathsf{L}$). The discourse (pl) "I have no truth-value", gives the truth-theory containing the referential schema, $\mathsf{R}(pl) : T_{pl} \leftrightarrow pl$ and the literal claim of its failure, $\mathsf{L}(pl) : T_{pl} \leftrightarrow \neg(T_{pl} \leftrightarrow pl)$. It has a unique model, where both $pl$ and $T_{pl}$ are false, witnessing thus to the non-paradoxicality of (pl).

As a more intricate example, consider the strong liar, (sl), "This statement is not true". Its "not true", meaning false or lacking truth-value, distinguishes it from the simple liar and gives its literal meaning: $\neg sl \vee \neg(T_{sl} \leftrightarrow sl)$. The resulting truth-theory is then:

$$\mathsf{R}(sl) : T_{sl} \leftrightarrow sl \qquad \mathsf{L}(sl) : T_{sl} \leftrightarrow \neg sl \vee \neg(T_{sl} \leftrightarrow sl).$$

This does not change the conclusion of paradoxicality, i.e., $\mathsf{L}(sl) \models \neg\mathsf{R}(sl)$, but it allows us to observe a few subtleties. The only model which respects the literal meaning $\mathsf{L}(sl)$ of (sl), makes $T_{sl}$ true, since $\neg(T_{sl} \leftrightarrow sl)$ must be true, with $sl = \mathbf{0}$ and $T_{sl} = \mathbf{1}$. Just as it should be, for it is a *compositional truth*, arising from the literal meaning of (sl), that it fails to behave normally with respect to truth. Of course, this immediatley implies paradoxicality, $\mathsf{R}(sl)$ becomes false, and this is as it should be – the literal $T_{sl}$ appears true at the cost of its dissociation from $sl$, just as in the case of other paradoxes. But here the important observation concers the impossibility of falsifying its literal meaning. This impossibility is not due to the paradoxicality of the discourse, but due to the truth of $T_{sl}$ arising directly and compositionally, in disregard of referential dependencies, from the literal claim (sl) makes concerning its own lack of truth-value, its failure to satisfy ($\mathsf{R}$). The simple liar lacks this strength, allowing to view its literal meaning, $\mathsf{L}(l) : T_l \leftrightarrow \neg l$, as either true or false, only never consistently with its $\mathsf{R}(l) : T_l \leftrightarrow l$.

Let us recall here discourse (5), as (sl) represents its statement (a). Statement (b), having the same literal meaning, obtains then the representation:

$$\mathsf{R}(b) : T_b \leftrightarrow b \qquad \mathsf{L}(b) : T_b \leftrightarrow \neg sl \vee \neg(T_{sl} \leftrightarrow sl).$$

In spite of the paradox arising from (sl), (b) is true, $\mathsf{L}(sl, b) \models T_b$, as in our informal analysis. Moreover, $T_b \leftrightarrow b$ is consistent with this literal theory, showing that this truth-value respects the truth-concept and is not, like the truth of $T_{sl}$, only a literal reminiscence of truth.[18]

This structured account, in terms of the relaxation ($\mathsf{R}$) and ($\mathsf{L}$) of (10) and the dissociation of the literal from the referential, leads to other examples of phenomena related to the strong liar. There is, for instance, a following dual of strong liar, (cl): "This statement has a truth-value and is false", with the following representation:

$$\mathsf{R}(cl) : T_{cl} \leftrightarrow cl \qquad \mathsf{L}(cl) : T_{cl} \leftrightarrow (T_{cl} \leftrightarrow cl) \wedge \neg cl$$

As the strong liar, this statement is paradoxical, but the only way to satisfy its literal meaning $\mathsf{L}(cl)$ is to make $T_{cl}$ false and $cl$ true. Agreeing to step beyond the paradox, this has the intuitive plausibility, for while there is a problem with the statement claiming its own falsity, this fact itself seems to make the statement false, in so far as the statement is also optimisitc about the non-pardoxality of its discursive context. Intuitively and literally, it is tempting to view the statement as simply false, it has the allure of falsehood in much the same way as the strong liar has the allure of truth.

A further variation on this theme is a strong truth-teller (tt): "This statement has a truth-value or it is true". Like in (sl) and (cl), the claim is not limited to the statement's truth, but refers also to the applicability of the truth-concept. In this case, however, truth functions properly, as can be seen from the consistency of its theory:

$$\mathsf{R}(tt) : T_{tt} \leftrightarrow tt \qquad \mathsf{L}(tt) : T_{tt} \leftrightarrow (T_{tt} \leftrightarrow tt) \vee tt$$

Interestingly, the literal claim $\mathsf{L}(tt)$ has only one model, which makes both $tt$ and $T_{tt}$ true. Then, $\mathsf{R}(tt)$ follows as a logical consequence of $\mathsf{L}(tt)$ and becomes superfluous. It is a *literal* fact that (tt) is actually true and that it is well-behaved with respect to truth. As its dual, we can consider the following (ct): "This statement does not have a truth-value and it is true", with the following theory:

$$\mathsf{R}(ct) : T_{ct} \leftrightarrow ct \qquad \mathsf{L}(ct) : T_{ct} \leftrightarrow \neg(T_{ct} \leftrightarrow ct) \wedge ct$$

The only model of the literal meaning makes both $T_{ct}$ and $ct$ false. As with (tt), the referential coherence $\mathsf{R}(ct)$ follows here from the literal meaning $\mathsf{L}(ct)$, only that now the statement is necessarily false.

## References

[BGW12] Marc Bezem, Clemens Grabmayer, and Michał Walicki. Expressive power of digraph solvability. *Annals of Pure and Applied Logic*, 162(3), 2012.

[18] This is what seems left from the contextualist "solution" of the strong liar. There is, indeed, a crucial difference between (sl) and (b) but it does not concern the evaluation context. It is the difference between referential structures of two distinct statements. Claiming one's own untruth is one statement, different from claiming untruth of another statement, identity of the literal meaning notwithstanding. The truth of (b) does not affect (sl) which, although literally true, remains paradoxical.

[Coo04] Roy Cook. Patterns of paradox. *The Journal of Symbolic Logic*, 69(3):767–774, 2004.

[Coo11] Roy Cook. The no-no paradox is a paradox. *Australasian Journal of Philosophy*, 89(3):467–482, 2011.

[Dun95] Phan Minh Dung. On the acceptability of arguments and its fundamental role in nonmonotonic reasoning, logic programming and $n$-person games. *Artificial Intelligence*, 77:321–357, 1995.

[DW13] Sjur Dyrkolbotn and Michał Walicki. Propositional discourse logic. *Synthese*, 2013. [to appear].

[Dyr12] Sjur Dyrkolbotn. *Argumentation, paradox and kernels in directed graphs*. PhD thesis, Institute of Informatics, University of Bergen, 2012.

[Gai88] Haim Gaifman. Operational pointer semantics: solution to self-referential puzzles. In Moshe Vardi, editor, *Theoretical Aspects of Reasoning about Knowledge*, pages 43–59. Morgan Kauffman, 1988.

[Gai92] Haim Gaifman. Pointers to truth. *The Journal of Philosophy*, 89(5):223–261, 1992.

[Kri81] Saul Kripke. *Naming and Necessity*. Blackwell Publishing, 1981.

[Mea67] George H. Mead. *Mind, Self and Society*. University of Chicago Press, Chicago, 3 edition, 1967.

[NL71] Victor Neumann-Lara. Seminúcleos de una digráfica. Technical report, Anales del Instituto de Matemáticas II, Universidad Nacional Autónoma México, 1971.

[Put85] Hilary Putnam. A comparison of something with something else. *New Literary History*, 17(1):61–79, 1985.

[Ric53] Moses Richardson. Solutions of irreflexive relations. *The Annals of Mathematics, Second Series*, 58(3):573–590, 1953.

[Sor03] Roy Sorensen. A definite no-no. *Liars and Heaps*, pages 225–229, 2003.

[vNM47] John von Neumann and Oscar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944 (1947).

[Wal09] Michał Walicki. Reference, paradoxes and truth. *Synthese*, 171:195–226, 2009.