Intermediate Microeconomics Lecture 12

Game Theory

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Strategic Interactions

- In a competitive market, consumers and firms are assumed to be price takers.
- In a monopoly market, there is only one firm in the market.
- But many markets/social activities involve only a small number of participants and economic agents can interact strategically in a variety of ways
  - sports games
  - international relations
  - several firms compete in a certain market
    - Oligopolies
    - Cartels

Strategic Interactions (cont.)

- The outcome to a participant depends not only on his own choice but also on what others do.
- When participants make decisions, they strategically take into account their opponents' behavior.
- Game theory provides a rigorous mathematical tool to study these situations.

Elements of a Game

- Players（博弈参与人）
  - decision making entities
  - firms, governments, individuals
- Strategies(策略) (or choices or actions) for each player
  - output, prices, advertising budget
- Payoffs（得益）to players as a function of strategy profiles
  - depend not only on own strategies, but on other's strategies as well

Examples of Simple Games

- Matching pennies（猜硬币博弈）
  - players: Player A, B
  - actions: Heads（正面）or Tails（反面）
  - payoffs:
    - player B gets \$1 and player A loses \$1 if pennies match
    - player B loses a dollar and player A gains \$1 if pennies don't match

Examples of Simple Games (cont.)

- Prisoner's dilemma (囚徒困境)
  - players: Prisoner A, B
  - actions: deny or confess
  - payoffs:
    - 0 if you confess while other denies
    - -1 if you both deny
    - -3 if you both confess
    - -6 if you deny and the other guy confess

Important Terminology

- Decision nodes
  - player make decisions at various points in a game
- Actions
  - the set of choices available each decision node in a game
- Pure Strategy（纯策略）
  - a rule that tells the player what actions to take at each of his information set in the game
- Mixed strategy（混合策略）
  - consists of a probability distribution on the set of pure strategies

Important Terminology (cont.)

- Common knowledge(公共知识)
  - if every player knows it, every player knows that every other player knows it, every player knows that every other player knows that every other player knows it, and so on
  - we usually assume that the complete description of the game is common knowledge
- Simultaneous move (同时行动)
  - if players in a game have to make their decision at the same times
- Sequential move (序贯行动)
  - if players make their decisions in a particular sequence, one after another
- Perfect information
  - every player at every decision node knows the decision taken previously by every other player

Classification of Games

- These are several ways of classifying games
  - by the number of players
    - 2-person v.s. n-person
  - by the number of strategies available to each of the players
    - finite games v.s. infinite games
  - by the nature of payoffs
    - zero-sum v.s. non-zero sum
  - the nature of preplay negotiation
    - cooperative games v.s. non-cooperative games
  - Interactions over time
    - dynamic game v.s. static games

Classification of Games (cont.)

- These are several ways of classifying games
  - by the nature of the states
    - stochastic v.s. deterministic game
  - perfect information vs. imperfect information game
    - Perfect information（完美信息博弈): at each move in the game, the player with the move knows the full history of the play of the game thus far
  - complete information v.s. incomplete information game
    - Incomplete information (非完全信息) : at least one player is uncertain about another player's payoff function (Asymmetric or private information)

Classification of Games (cont.)

	Complete	Incomplete
Static	Nash Equilibrium	Bayesian NE
Dynamic	Subgame Perfect NE	Perfect Bayesian NE

Ways to Describe a Game

- There are two ways to describe a game
  - the “Normal”or matrix form (标准型或距阵型)
    - there are only 2 (sometimes 3) players
    - there are a finite number of strategies
    - actions approximately simultaneous
  - the “Extensive”form (扩展型)
    - if actions are sequential

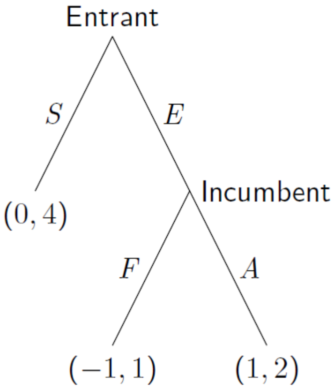
Ways to Describe a Game (cont.)

- Matching Pennies Game

		Player B	
		Heads	Tails
Player A	Heads	(-1,1)	(1,-1)
	Tails	(1,-1)	(-1,1)

Ways to Describe a Game (cont.)

- Chain store game



Dominant Strategies

- We can describe the prisoners' dilemma in its normal form

		Prisoner B	
		Confess	Deny
Prisoner A	Confess	(-3,-3)	(0,-6)
	Deny	(-6,0)	(-1,-1)

- If you were A, what would you choose?
  - If B chooses to confess, then you should confess because $-3 > -6$
  - If B chooses to deny, then you should confess because $0 > -1$
  - In sum, regardless of B's choice, A should (will) choose to confess

Dominant Strategies (cont.)

- The same logic applies to B's decision: regardless of A's choice, confession is better for B
- Thus in this game, we know both players will choose to confess, yielding a 3-year sentence in prison for each player
- In games, if a strategy is always strictly optimal for a player regardless of other players' choices, we call this strategy a strictly dominant strategy (严格占优策略) for this player
  - strategy s_i (strictly) dominates strategy s_i' if, for all possible strategy combinations of opponents, s_i yields a (strictly) higher payoff than s_i' to player i
- Confession is the strictly dominant strategy for both players in the prisoners' dilemma

Dominant Strategies (cont.)

- If a player has a strictly dominant strategy, this player will always play this strategy.
- If all players have a strictly dominant strategy, then the game is dominance-solvable, because we know every player will simply play their strictly dominant strategy.
- Strictly dominant strategy is natural and intuitive.
- But unfortunately, in many games, players do not have a strictly dominant strategy.

Nash Equilibrium

- Battle of the sexes:

		Wife: Soccer	Wife: Opera
Husband	Soccer	(2,1)	(0,0)
	Opera	(0,0)	(1,2)

- A couple wishes to go out for entertainment.
- The husband (row player) prefers soccer to opera while the wife (column player) prefers opera to soccer.
- They both prefer going together to going alone.

Nash Equilibrium (cont.)

- A Nash equilibrium of a N player normal form game is a strategy profile $(s_1, \cdots, s_n)$ such that for each player i, s_i is optimal given other players' strategy $(s_1, \cdots, s_{i-1}, s_{i+1}, \cdots, s_n)$
  - no player can improve her payoff by unilaterally deviating
- Technically, a Nash equilibrium is a strategy profile $(s_1^*, \cdots, s_n^*)$ such that for every player i :

  $$u_i(s_1^*, \cdots, s_{i-1}^*, s_i^*, s_{i+1}^*, \cdots, s_n^*) \geq u_i(s_1^*, \cdots, s_{i-1}^*, s_i, s_{i+1}^*, \cdots, s_n^*) \; \forall s_i \in S_i$$

  where u_i is the payoff function of player i
- For two player game, this boils down to a pair of strategies (s_1, s_2) such that
  - given s_2, s_1 is optimal for player 1
  - given s_1, s_2 is optimal for player 2.

Nash Equilibrium (cont.)

- In the battle of the sexes, there are four possible strategy profiles:
  - (S, S), Nash equilibrium
    - given that the husband goes to soccer, it is optimal for the wife to go to soccer
    - given that the wife goes to soccer, it is optimal for the husband to go to soccer
  - (O, O), Nash equilibrium
  - (S, O), not Nash equilibrium.
    - given that the husband goes to soccer, it is not optimal for the wife to go to opera
  - (O, S), not Nash equilibrium.
- The definition of Nash equilibrium does not say anything about which one should/will be played.

Nash Equilibrium (cont.)

- Consider again the prisoners' dilemma

		Prisoner B	
		Confess	Deny
Prisoner A	Confess	(-3,-3)	(0,-6)
	Deny	(-6,0)	(-1,-1)

- (Confess, Confess) is the only Nash equilibrium of this game.

Nash Equilibrium (cont.)

- The macthing pennies game:

		Player B	
		Heads	Tails
Player A	Heads	(-1,1)	(1,-1)
	Tails	(1,-1)	(-1,1)

- There is no Nash equilibrium in this game

Nash Equilibrium (cont.)

- Underlying assumptions of Nash equilibrium:
  - Players have predictions about other players' play and they optimally respond to their predictions.
  - Players' predictions are correct.
- Dominant Strategies
  - "I'm doing the best I can no matter what you do."
  - "You're doing the best you can no matter what I do."
- Nash Equilibrium
  - "I'm doing the best I can given what you are doing"
  - "You're doing the best you can given what I am doing."

Nash Equilibrium (cont.)

- Iterated elimination of strictly dominated strategies:
  - eliminate all strategies which are dominated, relative to opponent's strategies which have not yet been eliminated
- If iterated elimination of strictly dominated strategies yields a unique strategy profile, then this strategy profile is the unique Nash equilibrium

Nash Equilibrium (cont.)

- Example of Iterated Elimination:

		Player *II*	
		Left	Right
Player *I*	Top	1,2	4,1
	Middile	3,2	2,1
	Bottom	2,1	1,3

- Bottom is dominated by Middle (for Player I)
- Right is dominated by Left (for Player II)
- Top is dominated by Middle (for Player I)

Mixed Strategies

- To deal with non-existence of Nash equilibrium, we extend players' strategy spaces.
- We assume players in the matching pennies games can randomize between H and T.
- We use $p \circ H + (1-p) \circ T$ to denote the strategy which plays H with probability p and T with probability $1-p$, where$p \in [0,1]$
- This kind of strategies is called mixed strategies (混合策略)
  - If player i have K pure strategies available. Then a mixed strategy for player i is a probability distribution over those K strategies

Mixed Strategies (cont.)

- The strategies we studied previously, e.g. H and T in the matching pennies game, are called pure strategies (纯策略).
  - just special cases of mixed strategies, e.g. H is equivalent to $1 \circ H + 0 \circ T$
- There are many interpretations/justifications for mixed strategies.
  - Literally, randomization in players' brain.
  - Large population: fraction p of people play H and fraction $1 - p$ play T

Mixed Strategies (cont.)

- Use expected payoffs to evaluate payoffs from mixed strategies
- Player 1 adopts a mixed strategy σ_1
  - plays s_1 with probability p and s_1' with probability $1 - p$
- Player 2 adopts a mixed strategy σ_2
  - plays s_2 with probability q and s_2' with probability $1 - q$
- Then:
  - (s_1, s_2) is played with probability $p \times q$
  - (s_1, s_2') is played with probability $p \times (1 - q)$
  - (s_1', s_2) is played with probability $(1 - p) \times q$
  - (s_1', s_2') is played with probability $(1 - p) \times (1 - q)$

Mixed Strategies (cont.)

- Hence the expected utilities are given by

$$\begin{aligned} u_1(\sigma_1,\sigma_2) &= pqu_1(s_1,s_2)+p(1-q)u_1(s_1,s_2') \\ &\quad +(1-p)qu_1(s_1',s_2)+(1-p)(1-q)u_1(s_1',s_2') \end{aligned}$$

$$\begin{aligned} u_2(\sigma_1,\sigma_2) &= pqu_2(s_1,s_2)+p(1-q)u_2(s_1,s_2') \\ &\quad +(1-p)qu_2(s_1',s_2)+(1-p)(1-q)u_2(s_1',s_2') \end{aligned}$$

- For example, in matching pennies game, assume $\sigma_1 = \frac{1}{2}\circ H + \frac{1}{2}\circ T$ and $\sigma_2 = H$. Then

$$\begin{aligned} u_1(\sigma_1,\sigma_2) &= \frac{1}{2}u_1(H,H)+\frac{1}{2}u_1(T,H) \\ &= \frac{1}{2}\times(-1)+\frac{1}{2}\times 1 = 0 \end{aligned}$$

Mixed Strategies (cont.)

- A Nash equilibrium in mixed strategies is similarly defined.
- (σ_1, σ_2) is a Nash equilibrium if
  - given σ_2, σ_1 is optimal for player 1
  - given σ_1, σ_2 is optimal for player 2.
- In general, it is more difficult to find equilibria in mixed strategies than in pure strategies.
- For 2×2 games (2 players, 2 pure strategies for each) the best response correspondence (最优反应曲线) is a powerful tool to help us analyze

Mixed Strategies (cont.)

- Let's consider the matching pennies game again.
- Assume player 1 plays $p \circ H + (1-p) \circ T$ and player 2 plays $q \circ H + (1-q) \circ T$
- Then the payoff to player 2 is:

$$
\begin{aligned}
u_2(p, q) &= pq + p(1-q)(-1) + (1-p)q(-1) + (1-p)(1-q) \\
&= 2(2p-1)q + (1-2p)
\end{aligned}
$$

  - if $p < 1/2$, player 2's optimal choice is $q^* = 0(i.e.\ T)$
  - if $p > 1/2$, player 1's optimal choice is $q^* = 1(i.e.\ H)$
  - if $p = 1/2$, any choice of $q^* \in [0, 1]$ is optimal

Mixed Strategies (cont.)

- Player 2's optimal choice can be expressed as

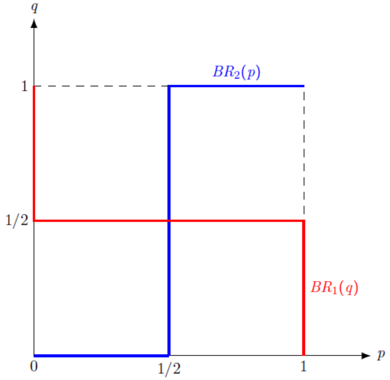
$$BR_2(p) = \begin{cases} \{0\} & \text{if } p < 1/2 \\ [0,1] & \text{if } p = 1/2 \\ \{1\} & \text{if } p > 1/2 \end{cases}$$

- Similarly, we can calculate player 1's optimal choice

$$BR_1(q) = \begin{cases} \{1\} & \text{if } q < 1/2 \\ [0,1] & \text{if } q = 1/2 \\ \{0\} & \text{if } q > 1/2 \end{cases}$$

Mixed Strategies (cont.)

- We can plot it in the $p - q$ plane



- The intersections of the two best response correspondences are Nash equilibria

Mixed Strategies (cont.)

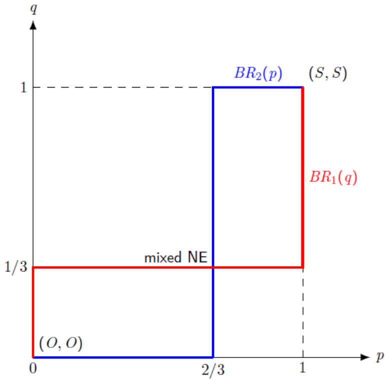
- Battle of the sexes:

		Wife	
		Soccer	Opera
Husband	Soccer	(2,1)	(0,0)
	Opera	(0,0)	(1,2)

- Any mixed Nash equilibrium?

Mixed Strategies (cont.)

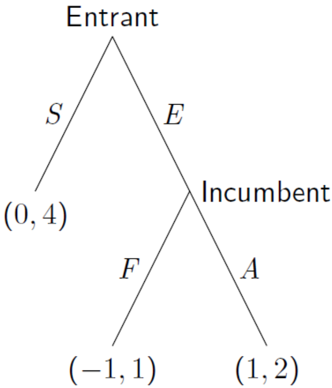
- One mixed strategy Nash equilibrium: player 1 plays $\frac{2}{3} \circ S + \frac{1}{3} \circ O$ and player 2 plays $\frac{1}{3} \circ S + \frac{2}{3} \circ O$



Games with Sequential Moves

- Consider the following chain store game.
- There is an entrant who chooses to either enter a market (E) or stay out (S)
- There is an incumbent who chooses to either fight (F) or accommodate (A) if the entrant enters
- Coca-Cola is an entrant while Pepsi is the incumbent in the Russian market around 1990.

Games with Sequential Moves (cont.)



Games with Sequential Moves (cont.)

- There are four possible strategy profiles. Which profile(s) do you think plausible?
- (E, F) is not
  - if the entrant enters, then it is optimal for the incumbent not to fight
- (S, A) is not
  - if the incumbent chooses to accommodate after entry, the entrant should enter instead of staying out

Games with Sequential Moves (cont.)

- (S, F) seems plausible:
  - given that the incumbent will fight after entry, it is optimal for the entrant to stay out
  - given that the entrant stays out, the choice of the incumbent is irrelevant (thus trivially optimal)
- (E, A) seems plausible too:
  - given that the incumbent will accommodate, it is optimal for the entrant to enter
  - given that the entrant enters, it is optimal for the incumbent to accommodate
- Therefore, these two profiles are Nash equilibria

Games with Sequential Moves (cont.)

- But in (S, F), the incumbent supposedly plans to fight in case of the entrant entering.
- It is like a threat: if you enter, I will fight with you!
- But is this threat credible?
- Probably not.
- In contrast, (E, A) does not suffer from this problem.
  - it is indeed the incumbent's best response to accommodate after entry.
- But to rule out (S, F), we need a stronger solution concept that incorporates the fact that actions are taken in sequence

Subgame Perfect Equilibrium

- A Nash equilibrium is said to be a subgame perfect Nash equilibrium (子博弈精炼纳什均衡) if and only if it is a Nash equilibrium in every subgame of the game
- A subgame(子博弈) in an extensive-form game has the following properties:
  - it begins at a node of the tree corresponding to an information set reduced to a singleton (the set contains only one set)(单结)
  - it encompasses all parts of the tree following the starting node
  - it never divides an information set

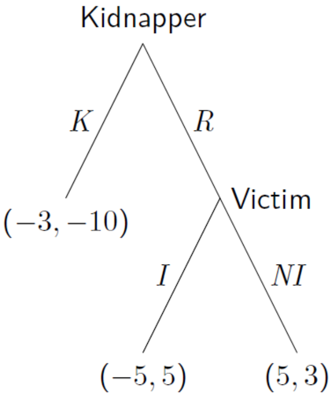
Backward Induction

- To get an SPNE for finite horizon games, we could work back from terminal nodes.
  - go to final "decision node" and assign action to the player that maximizes his payoff
  - reduce game by trimming tree at this node and making terminal payoffs at this node, the payoffs when the player whose turn it was takes best action
  - keep working backwards
- This procedure is called backward induction solution (倒推归纳法)
- (E, A) is the only backward induction solution to the entrant & incumbent game.

Another Example: Kidnapping

- The kidnapper has two choices after receiving the ransom:
  - release (R)
  - or kill the victim (K).
- After release, the victim has two choices
  - identify the kidnapper (I)
  - or refrain (NI)

Another Example: Kidnapping (cont.)



Another Example: Kidnapping (cont.)

- If the victim had been released, he would choose to identify the kidnapper
- Thus, the kidnapper would get -5 if he releases the victim
- It is then optimal for the kidnapper to kill the victim in the first place
- This is the only backward induction solution of this game
- The outcome from (R, NI) is socially optimal, but it is not consistent with backward induction
  - the victim can not commit to playing NI