# Intermediate Microeconomics Lecture 8

## Profit Maximization & Cost Minimization Problem

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# Profit Maximization

- There are two ways to investigate firm's profit maximization problem.
- One is a direct approach.
- The other is an indirect approach.
  - we divide a firm's profit maximization problem into two stages.
  - one is cost minimization and the other is profit maximization.

# Competitive Markets and Price Takers

- For now, we assume that firms are price takers (价格接受者) in both market for the factors of production it uses and market for the output goods it produces.
- Thus, from firms' point of view, prices for both factors and outputs are fixed, regardless of their demands in the factor markets and supplies in the output markets.
- Markets in which firms take prices as given are called competitive markets (竞争性市场).
  - these participating firms are called competitive firms

# Competitive Markets and Price Takers (cont.)

- The justification for price taking behavior is large markets.
  - because there are many many participants in the same market, each individual participant only have negligible effect on the market price
- In this course, we will always assume competitive factor markets. But later, we will see what happens if the market for the output is not competitive.

# Profits

- Suppose a firm sells $y$ units of output at price $p$
- To produce these outputs, the firm uses inputs $(x_1, \ldots, x_n)$ at price $(w_1, \ldots, w_n)$.
- The profits (利润) the firm receives can be expressed as

$$\pi = py - \sum_{i=1}^{n} w_i x_i$$

  - $py$ is the firm's revenue
  - $\sum_{i=1}^{n} w_i x_i$ is total cost

# Profits (cont.)

- In the expression for cost, we should be sure to include all of the factors of production used by the firm, valued at their market price.
  - if an individual works in his own firm, then his labor is an input and it should be counted as part of the costs
  - his wage rate is simply the market price of his labor – what he would be getting if he sold his labor on the open market
- Economic costs (经济成本) like these are often referred to as opportunity costs (机会成本).
  - the name comes from the idea that if you are using your labor, for example, you forgo the opportunity of employing it elsewhere
  - those lost wages are part of the cost of production

# Short-Run Profit Maximization

- Let's consider the short-run profit-maximization problem when input 2 is fixed at some level $x_2$.
- Let $f(x_1, x_2)$ be the production function for the firm.
- Then the profit maximization problem facing the firm can be written as

$$\max_{x_1} pf(x_1, \bar{x}_2) - wx_1 - w_2\bar{x}_2$$

- The first order condition is

$$pMP_1(x_1^*, \bar{x}_2) = w_1$$

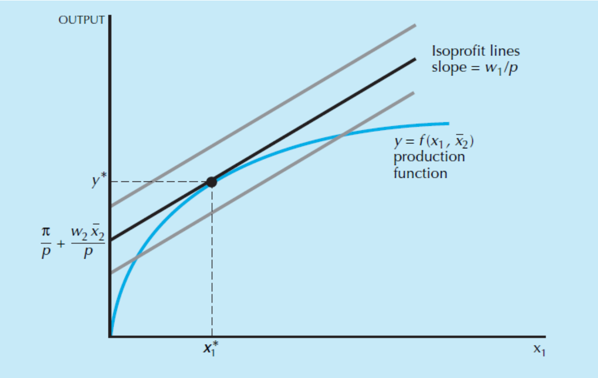
  - the value of the *MP* of a factor should equal its price

# Short-Run Profit Maximization (cont.)

- Recall $MP_1$ is interpreted as the additional output that can be produced if one more unit of input 1 is used.
- So $pMP_1(x_1, x_2)$ is the benefit of using one more unit of input 1, while $w_1$ is the cost.
  - if $pMP_1(x_1, x_2) > w_1$, the firm should use more input 1 and produce more
  - if $pMP_1(x_1, x_2) < w_1$, the firm should use less input 1 and produce less

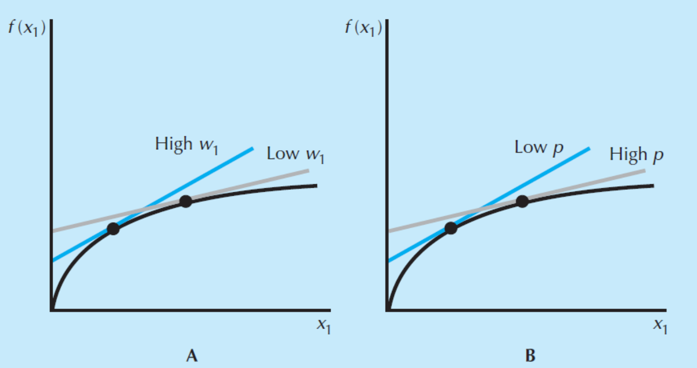
# Short-Run Profit Maximization (cont.)

- Isoprofit lines (等利润线): $y = \frac{\pi}{p} + \frac{w_2}{p}\bar{x}_2 + \frac{w_1}{p}x_1$



# Comparative Statics

- How does the optimal choice of factor 1 vary as we vary its factor price $w_1$? Waht if the output price decreases?



# Long-Run Profit Maximization

- In the long run, the firm is free to choose the level of all inputs.
- Thus, the long-run profit-maximization problem can be posed as

$$\max_{x_1, x_2} pf(x_1, x_2) - wx_1 - w_2 x_2$$

- This is basically the same as the short-run problem, but now both factors are free to vary.
- The first order condition is

$$pMP_1(x_1^*, x_2^*) = w_1$$

$$pMP_2(x_1^*, x_2^*) = w_2$$

# Cost Minimization v.s.Profit Maximization

- An important implication of the firm choosing a profit-maximization production plan is that there is no way to produce the same amounts of outputs at a lower total input costs
  - cost minimization is a necessary condition for profit maximization
- Reasons to study firm's cost minimization problem
  - leads us to a number of results that are technically useful
  - when the firm is not a price taker, we can no longer use the profit funtion for analysis
  - if the production set exhibits nondecreasing returns to scales, the profit maximization problem dose not work well

# Cost Minimization Problem

- Recall the (long-run) profit maximization is:

$$\max_{x_1,x_2} pf(x_1,x_2) - wx_1 - w_2x_2$$

- We can equivalently write this unconstrained maximization problem into a constrained maximization problem

$$\max_{y,x_1,x_2} py - wx_1 - w_2x_2$$
$$s.t.\ y = f(x_1,x_2)$$

- Firm's profit maximization problem can be divided into two steps:
  - first, for various levels of output, the firm choose the optimal factors that minimize production costs
  - second, the firm decides how much to produce to maximize its profit

# Cost Minimization Problem (cont.)

- We now consider the cost minimization problem (成本最小化问题)

$$\min_{x_1, x_2} wx_1 + w_2 x_2$$
$$s.t.\ \bar{y} = f(x_1, x_2)$$

- Assume $f$ is differentiable.
- Suppose $x_1^* > 0$ and $x_2^* > 0$ solves this problem. Using Lagrange method, we know

$$\frac{w_1}{w_2} = \frac{MP_1(x_1^*, x_2^*)}{MP_2(x_1^*, x_2^*)} = TRS_{12}(x_1^*, x_2^*)$$

# Cost Minimization Problem (cont.)

- At optimal, $TRS_{12}$ must equal factor price ratio
- This is very intuitive.
  - if $\frac{w_1}{w_2} > TRS_{12}(\tilde{x_1}, \tilde{x_2})$
  - imagine we lower $x_1$ by one unit.
  - we can increase $x_2$ by $TRS_{12}(x_1, x_2)$ units and get the same output level
  - the difference in costs is $-w_1 + TRS_{12}(x_1, x_2)w_2 < 0$
  - so $(\tilde{x_1}, \tilde{x_2})$ is not cost minimizing

# Cost Minimization Problem (cont.)

- Suppose that we want to plot all the combinations of inputs that have some given level of cost, $C$. We can write this as

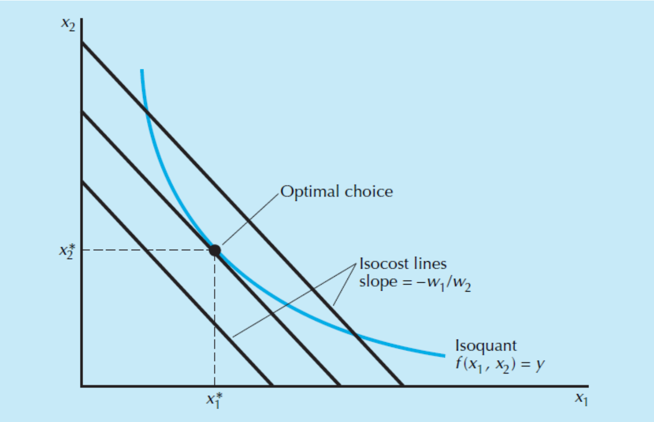
$$w_1 x_1 + w_2 x_2 = C$$

which can be rearranged to give

$$x_2 = \frac{C}{w_2} - \frac{w_1}{w_2} x_1$$

- As we let the number $C$ vary we get a whole family of isocost lines.
- The choice of factors that minimize production costs can be determined by finding the point on the isoquant that has the lowest associated isocost curve.

# Cost Minimization Problem (cont.)



# Cost Minimization Problem (cont.)

- The solution can be written as $x_1(w_1, w_2, y)$ and $x_2(w_1, w_2, y)$
  - e.g. if $f(x_1, x_2) = x_1^a x_2^b$, $x_1 = [(\frac{b}{a}\frac{w_1}{w_2})^{-b} y]^{\frac{1}{a+b}}$
  - called the conditional factor demand functions (有条件的要素需求函数), or derived factor demands (派生的要素需求)
- The correspongding cost function is $c(w_1, w_2, y)$

# Examples: Cobb-Douglas Technology

- Consider the Cobb-Douglas technology $f(x_1, x_2) = x_1^a x_2^b$ for some $a, b > 0$
- We have

$$MP_1(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_1} = a x_1^{a-1} x_2^b$$

$$MP_2(x_1, x_2) = \frac{\partial f(x_1, x_2)}{\partial x_2} = b x_1^a x_2^{b-1}$$

- So at the optimal:

$$\frac{w_1}{w_2} = \frac{a x_2}{b x_1}$$

# Examples: Cobb-Douglas Technology (cont.)

- Combined with

$$x_1^a x_2^b = y$$

we get

$$x_1 = [(\frac{b}{a}\frac{w_1}{w_2})^{-b} y]^{\frac{1}{a+b}}$$

- So the minimun cost is

$$\begin{aligned} c(y) &= w_1 x_1 + w_2 x_2 \\ &= \frac{a+b}{a} w_1 x_1 \\ &= \frac{a+b}{a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}} w_1^{\frac{a}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}} \end{aligned}$$

- If $a + b = 1$, we have $c(y) = \frac{1}{a^a b^b} w_1^a w_2^b y$

# Examples: Perfect Complements

- Consider $f(x_1, x_2) = \min\{x_1, x_2\}$
  - factors are perfect complements
- To produce $y$ units of output, we need at least $y$ units of $x_1$ and $y$ units of $x_2$
- So the minimal cost is

$$c(y) = w_1 y + w_2 y = (w_1 + w_2)y$$

# Examples: Perfect Substitutes

- Consider $f(x_1, x_2) = x_1 + x_2$
  - perfect substitutes
- To produce $y$ units of output, we can use $y$ units of $x_1$ and 0 unit of $x_2$, or 0 unit of $x_1$ and $y$ units of $x_2$
- So the minimal cost is

$$c(y) = \min\{w_1 y + w_2 \times 0, w_1 \times 0 + w_2 y\} = \min\{w_1, w_2\} y$$

- How about $f(x_1, x_2) = x_1 + 2x_2$?

# Short-run Costs

- In producers' theory, the difference between short-run and long-run is whether all factors in production are free to vary
  - in the short-run, some factors are fixed
- Suppose $x_2$ is fixed in the short-run. Then the firm's cost minimization problem becomes

$$\min_{x_1} wx_1 + w_2\bar{x}_2$$
$$s.t.\ y = f(x_1, \bar{x}_2)$$

  - the firm now can only choose $x_1$
- We write the short-run cost function as $c_s(w_1, w_2, y; \bar{x}_2)$
  - the minimal costs needed to produce $y$ units of the output holding the input of the second factor fixed at $\bar{x}_2$

# Short-run Costs (cont.)

- Consider Cobb-Douglas technology $f(x_1, x_2) = x_1^a x_2^b$ again
- Suppose $x_2 > 0$ is fixed in the short-run.
- Then, to produce $y$ units of the output, the firm needs at least

$$x_1 = (\frac{y}{\bar{x}_2^b})^{\frac{1}{a}}$$

  units of the first factor.
- The firm's short-run cost function is

$$c_s(w_1, w_2, y; x_2) = w_1(\frac{y}{\bar{x}_2^b})^{\frac{1}{a}} + w_2\bar{x}_2$$

# Short-run Costs (cont.)

- In the short-run, even if the firm does not produce, it bears strictly positive costs.
  - $c_s(w_1, w_2, 0; x_2) = w_2\bar{x}_2 > 0$
  - comes from the fixed inputs which can not be changed in the short-run
  - called fixed costs (固定成本；不变成本)

# Short-run Costs v.s. Long-run Costs

- Now let's consider the following minimization problem

$$\min_{x_2} c_s(w_1, w_2, y; x_2)$$

- Plugging in $c_s(w_1, w_2, y; x_2)$, we can write

$$\min_{x_2} w_1\left(\frac{y}{x_2^b}\right)^{\frac{1}{a}} + w_2 x_2$$

- We can show that the minimized cost is

$$\frac{a+b}{a^{\frac{a}{a+b}} b^{\frac{b}{a+b}}} w_1^{\frac{a}{a+b}} w_2^{\frac{b}{a+b}} y^{\frac{1}{a+b}}$$

which is exactly the $c(w_1, w_2, y)$ we have calculated before

# Short-run Costs v.s. Long-run Costs (cont.)

- In other words, for each $y > 0$, we have

$$c(w_1, w_{2,}y) = \min_{x_2} c_s(w_1, w_{2,}y; x_2)$$

  - the cost-minimizing amount of the variable factor in the long run is that amount that the firm would choose in the short run-if it happened to have the long-run cost-minimizing amount of the fixed factor
  - is true not only for this example, but for all production functions