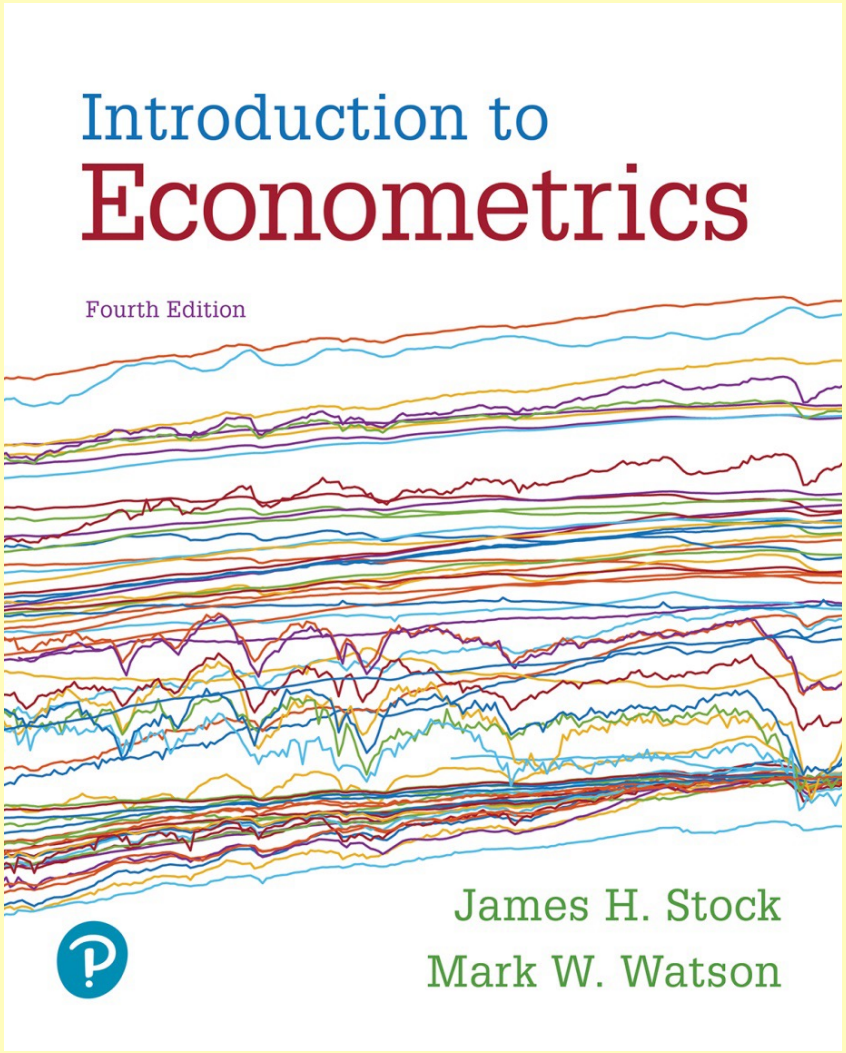
Introduction to Econometrics

Fourth Edition



Chapter 11

Regression with a Binary Dependent Variable

Outline

1. The Linear Probability Model
2. Probit and Logit Regression
3. Estimation and Inference in Probit and Logit
4. Application to Racial Discrimination in Mortgage Lending

Binary Dependent Variables: What's Different?

So far the dependent variable (Y) has been continuous:

- district-wide average test score
- traffic fatality rate

What if Y is binary?

- Y = get into college, or not; X = high school grades, SAT scores, demographic variables
- Y = person smokes, or not; X = cigarette tax rate, income, demographic variables
- Y = mortgage application is accepted, or not; X = race, income, house characteristics, marital status

Example: Mortgage Denial and Race The Boston Fed HMDA Dataset

- Individual applications for single-family mortgages made in 1990 in the greater Boston area
- 2380 observations, collected under Home Mortgage Disclosure Act (HMDA)

Variables

- Dependent variable:
  - Is the mortgage denied or accepted?
- Independent variables:
  - income, wealth, employment status
  - other loan, property characteristics
  - race of applicant

Binary Dependent Variables and the Linear Probability Model (SW Section 11.1) (1 of 3)

A natural starting point is the linear regression model with a single regressor:

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

But:

- What does β_1 mean when Y is binary? Is $\beta_1 = \frac{\Delta Y}{\Delta X}$?
- What does the line $\beta_0 + \beta_1 X$ mean when Y is binary?
- What does the predicted value $\hat{Y}$ mean when Y is binary? For example, what does $\hat{Y} = 0.26$ mean?

Binary Dependent Variables and the Linear Probability Model (SW Section 11.1) (2 of 3)

In the linear probability model, the predicted value of Y is interpreted as the predicted probability that $Y = 1$, and β_1 is the change in that predicted probability for a unit change in X. Here's the math:

Linear probability model: $Y_i = \beta_0 + \beta_1 X_i + u_i$

When Y is binary,

$$E(Y|X) = 1 \times \Pr(Y=1|X) + 0 \times \Pr(Y=0|X) = \Pr(Y=1|X)$$

Under LS assumption #1, $E(u_i|X_i) = 0$, so

$$E(Y_i|X_i) = E(\beta_0 + \beta_1 X_i + u_i|X_i) = \beta_0 + \beta_1 X_i,$$

so

$$\Pr(Y = 1|X) = \beta_0 + \beta_1 X_i$$

Binary Dependent Variables and the Linear Probability Model (SW Section 11.1) (3 of 3)

When Y *is binary, the linear regression model*

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

is called the ***linear probability model*** because

$$\Pr(Y = 1|X) = \beta_0 + \beta_1 X$$

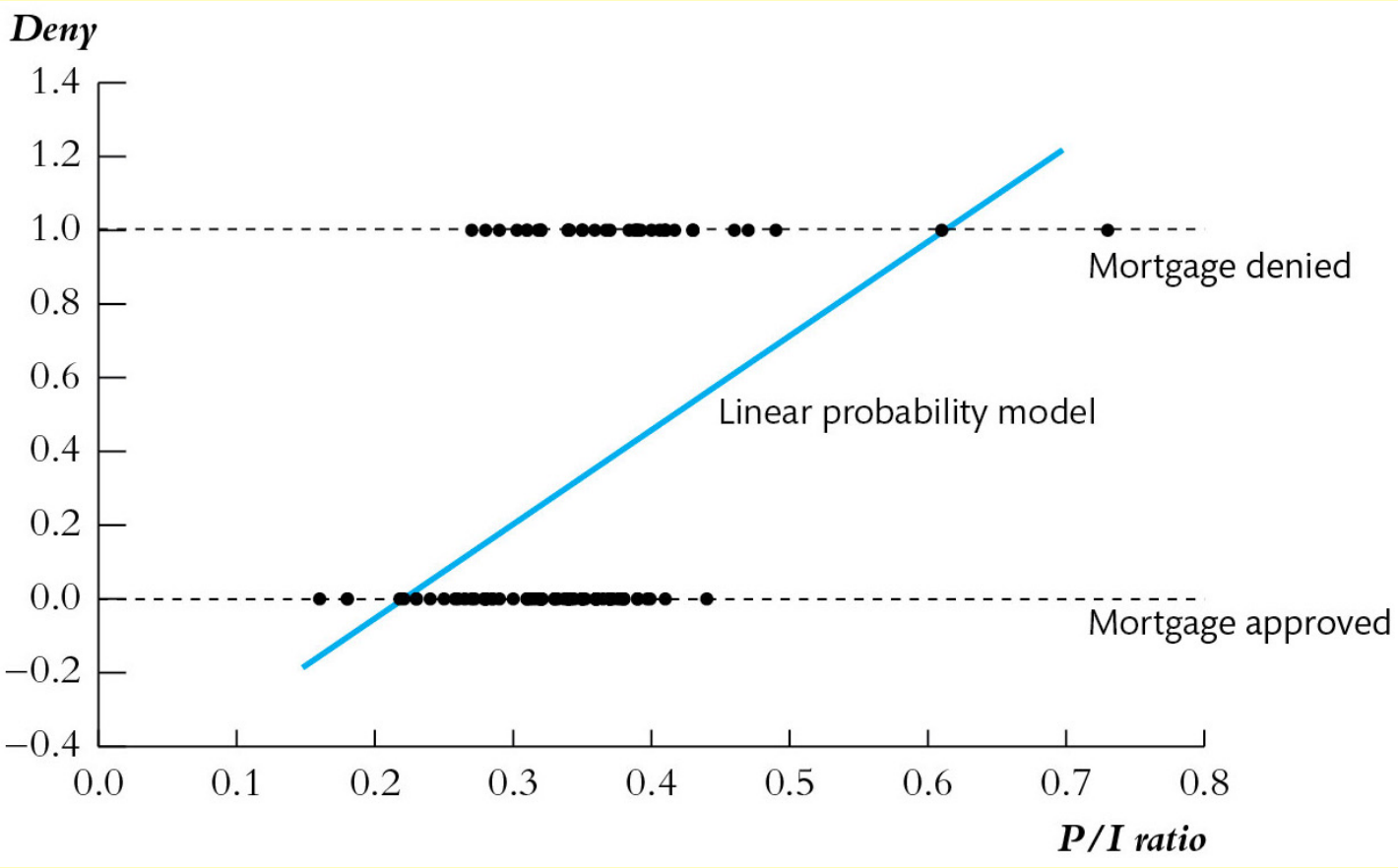
- The predicted value is a ***probability***:
  - $E(Y|X = x) = \Pr(Y = 1|X = x) =$ probability that $Y = 1$ given x
  - $\hat{Y} =$ the ***predicted probability*** that $Y_i = 1$, given X

- β_1 = difference in probability that $Y = 1$ associated with a unit difference in x:

$$\beta_1 = \frac{\Pr(Y = 1|X = x + \Delta x) - \Pr(Y = 1|X = x)}{\Delta x}$$

Example: linear probability model, HMDA data

Mortgage denial v. ratio of debt payments to income (P/I ratio) in a subset of the HMDA data set (*n* = 127)



Linear probability model: full HMDA data set (1 of 2)

$$\widehat{deny} = -.080 + .604 P/I\ ratio \qquad (n = 2380)$$
$$(.032)\ (.098)$$

- What is the predicted value for $P/I\ ratio = .3$?

$$\widehat{Pr}(deny = 1 | P/I ratio = .3) = -.080 + .604 \times .3 = .151$$

- Calculating "effects:" increase $P/I\ ratio$ from .3 to .4:

$$\widehat{Pr}(deny = 1 | P/I ratio = .3) = -.080 + .604 \times .4 = .212$$

The effect on the probability of denial of an increase in $P/I\ ratio$ from .3 to .4 is to increase the probability by .061, that is, by 6.1 percentage points. (*What is a "percentage point"?*)

Linear probability model: full HMDA data set (2 of 2)

Next include *black* as a regressor:

$$\widehat{deny} = -.091 + .559 P/I\ ratio + .177 black$$
$$(.032)\ (.098) \qquad\qquad (.025)$$

Predicted probability of denial:

- for black applicant with *P/I ratio* = .3:

$$\widehat{Pr}(deny = 1) = -.091 + .559 \times .3 + .177 \times 1 = .254$$

- for white applicant with *P/I ratio* = .3:

$$\widehat{Pr}(deny = 1) = -.091 + .559 \times .3 + .177 \times 0 = .077$$

- difference = .177 = 17.7 percentage points
- Coefficient on *black* is significant at the 5% level
- *Still plenty of room for omitted variable bias…*

The linear probability model: Summary

- The linear probability model models $\Pr(Y = 1|X)$ as a linear function of X
- Advantages:
  - simple to estimate and to interpret
  - inference is the same as for multiple regression (need heteroskedasticity-robust standard errors)
- Disadvantages:
  - A LPM says that the change in the predicted probability for a given change in X is the same for all values of X, but that doesn't make sense. Think about the HMDA example…
  - Also, LPM predicted probabilities can be <0 or >1!
- These disadvantages can be solved by using a *nonlinear* probability model: probit and logit regression

Probit and Logit Regression (SW Section 11.2) (1 of 5)

The problem with the linear probability model is that it models the probability of $Y = 1$ as being linear in X:

$$\Pr(Y = 1|X) = \beta_0 + \beta_1 X$$

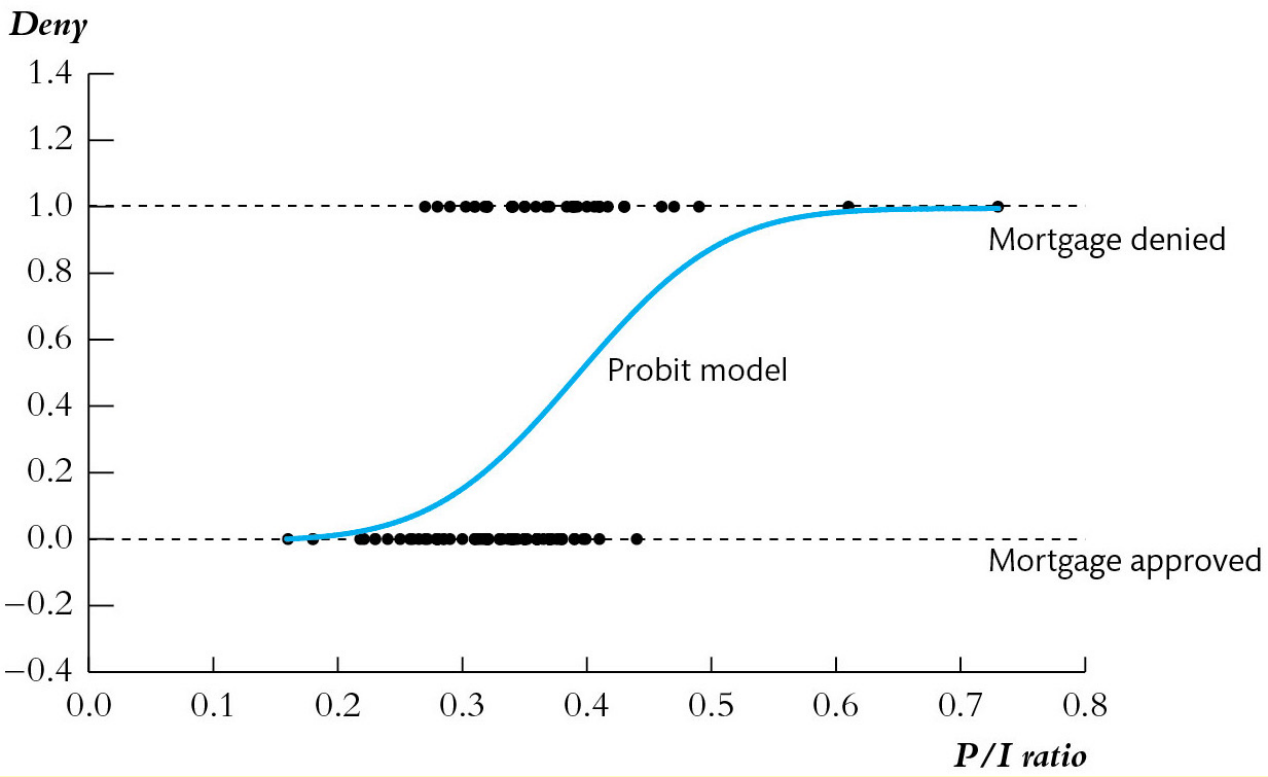
Instead, we want:

I. $\Pr(Y = 1|X)$ to be increasing in X for $\beta_1 > 0$, and

II. $0 \leq \Pr(Y = 1|X) \leq 1$ for all X

This requires using a nonlinear functional form for the probability. How about an "S-curve"…

Probit and Logit Regression (SW Section 11.2) (2 of 5)



- The probit model satisfies these conditions:
  - I. $\Pr(Y=1|X)$ to be increasing in X for $\beta_1 > 0$, and
  - II. $0 \leq \Pr(Y=1|X) \leq 1$ for all X

Probit and Logit Regression (SW Section 11.2) (3 of 5)

Probit regression models the probability that $Y = 1$ using the cumulative standard normal distribution function, $\Phi(z)$, evaluated at $z = \beta_0 + \beta_1 X$. The probit regression model is,

$$\Pr(Y = 1|X) = \Phi(\beta_0 + \beta_1 X)$$

where Φ is the cumulative normal distribution function

- $z = \beta_0 + \beta_1 X$ is the "z-value" or "z-index" of the probit model.

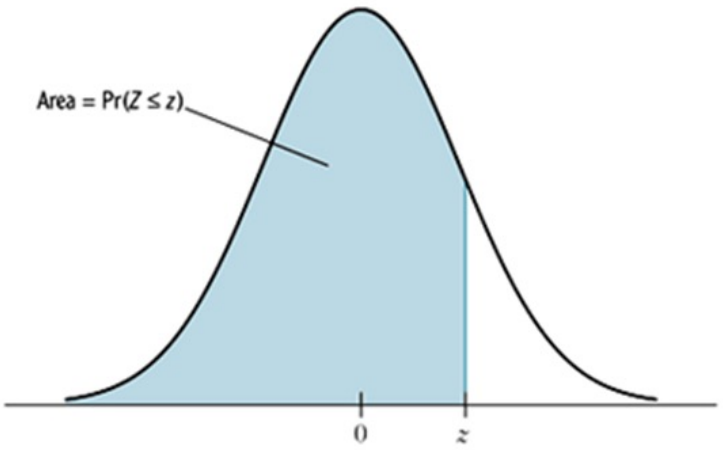
Example: Suppose $\beta_0 = -2$, $\beta_1 = 3$, $X = .4$, so

$$\Pr(Y = 1|X = .4) = \Phi(-2 + 3 \times .4) = \Phi(-0.8)$$

$\Pr(Y = 1|X = .4) =$ area under the standard normal density to left of $z = -.8$, which is…

Probit and Logit Regression (SW Section 11.2) (4 of 5)

TABLE 1 The Cumulative Standard Normal Distribution Function, $\Phi(z) = \Pr(Z \leq z)$



	Second Decimal Value of z									
z	0	1	2	3	4	5	6	7	8	9
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
−0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
−0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
−0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
−0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121

$$\Pr(z \leq -0.8) = .2119$$

Probit and Logit Regression (SW Section 11.2) (5 of 5)

Why use the cumulative normal probability distribution?

- The "S-shape" gives us what we want:
  - $\Pr(Y=1|X)$ is increasing in X for $\beta_1 > 0$
  - $0 \le \Pr(Y=1|X) \le 1$ for all X
- Easy to use – the probabilities are tabulated in the cumulative normal tables (and also are easily computed using regression software)
- Relatively straightforward interpretation:
  - $\beta_0 + \beta_1 X = z$-value
  - $\hat{\beta}_0 + \hat{\beta}_1 X$ is the predicted z-value, given X
  - β_1 is the change in the z-value for a unit change in X

STATA Example: HMDA data (1 of 5)

```
. probit deny p_irat, r
Iteration 0:   log likelihood =  -872.0853          We'll discuss this later
Iteration 1:   log likelihood =  -835.6633
Iteration 2:   log likelihood = -831.80534
Iteration 3:   log likelihood = -831.79234
Probit estimates                                  Number of obs   =       2380
                                                  Wald chi2(1)    =      40.68
                                                  Prob > chi2     =     0.0000
Log likelihood = -831.79234                       Pseudo R2       =     0.0462
------------------------------------------------------------------------------
             |               Robust
        deny |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      p_irat |    2.967908   .4653114     6.38   0.000     2.055914    3.879901
       _cons |   -2.194159   .1649721   -13.30   0.000    -2.517499    -1.87082
------------------------------------------------------------------------------
```

$$\Pr(deny = 1|P/Iratio) = \Phi(-2.19 + 2.97 \times P/I\ ratio)$$
$$(.16)\quad(.47)$$

STATA Example: HMDA data (2 of 5)

$$\widehat{\Pr(deny=1|P/Iratio)} = \Phi(-2.19 + 2.97 \times P/I\ ratio)$$
$$(.16) \qquad (.47)$$

- Positive coefficient: *Does this make sense*?
- Standard errors have the usual interpretation
- Predicted probabilities:

$$\widehat{\Pr(deny=1|P/Iratio = .3)} = \Phi(-2.19 + 2.97 \times .3)$$
$$= \Phi(-1.30) = .097$$

- Effect of change in *P/I ratio* from .3 to .4:

$$\widehat{\Pr(deny=1|P/Iratio = .4)} = \Phi(-2.19 + 2.97 \times .4)$$
$$= \Phi(-1.00) = .159$$

- Predicted probability of denial rises from .097 to .159

Probit regression with multiple regressors

$$\Pr(Y = 1|X_1, X_2) = \Phi(\beta_0 + \beta_1 X_1 + \beta_2 X_2)$$

- Φ is the cumulative normal distribution function.
- $z = \beta_0 + \beta_1 X_1 + \beta_2 X_2$ is the "z-value" or "z-index" of the probit model.
- β_1 is the effect on the z-score of a unit change in X_1, holding constant X_2 (when a causal interpretation is justified)

STATA Example: HMDA data (3 of 5)

```
. probit deny p_irat black, r
Iteration 0:   log likelihood =  -872.0853
Iteration 1:   log likelihood = -800.88504
Iteration 2:   log likelihood =  -797.1478
Iteration 3:   log likelihood = -797.13604
Probit estimates                                  Number of obs   =       2380
                                                  Wald chi2(2)    =     118.18
                                                  Prob > chi2     =     0.0000
Log likelihood = -797.13604                       Pseudo R2       =     0.0859
------------------------------------------------------------------------------
             |               Robust
        deny |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      p_irat |    2.741637   .4441633     6.17   0.000     1.871092    3.612181
       black |    .7081579   .0831877     8.51   0.000      .545113    .8712028
       _cons |   -2.258738   .1588168   -14.22   0.000    -2.570013   -1.947463
------------------------------------------------------------------------------
```

We'll go through the estimation details later…

STATA Example: HMDA data (4 of 5)

```
. probit deny p_irat black, r
Probit estimates                                  Number of obs   =       2380
                                                  Wald chi2(2)    =     118.18
                                                  Prob > chi2     =     0.0000
Log likelihood = -797.13604                       Pseudo R2       =     0.0859
------------------------------------------------------------------------------
             |               Robust
        deny |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      p_irat |   2.741637   .4441633     6.17   0.000     1.871092    3.612181
       black |   .7081579   .0831877     8.51   0.000      .545113    .8712028
       _cons |  -2.258738   .1588168   -14.22   0.000    -2.570013   -1.947463
------------------------------------------------------------------------------
.  sca z1 = _b[_cons]+_b[p_irat]*.3+_b[black]*0
.  display "Pred prob, p_irat=.3, white: " normprob(z1)
Pred prob, p_irat=.3, white: .07546603
```

NOTE

_b[_cons] is the estimated intercept (-2.258738)

_b[p_irat] is the coefficient on p_irat (2.741637)

sca creates a new scalar which is the result of a calculation

display prints the indicated information to the screen

STATA Example: HMDA data (5 of 5)

$$\widehat{\Pr(deny=1|P/I, black)}$$
$$= \Phi(-2.26 + 2.74 \times P/I\ ratio + .71 \times black)$$
$$(.16)\quad (.44)\qquad\qquad (.08)$$

- Is the coefficient on *black* statistically significant?
- Estimated effect of race for P/I *ratio* = .3:

$$\widehat{\Pr(deny=1|.3,1)} = \Phi(-2.26 + 2.74 \times .3 + .71 \times 1) = .233$$

$$\widehat{\Pr(deny=1|.3,0)} = \Phi(-2.26 + 2.74 \times .3 + .71 \times 0) = .075$$

- Difference in rejection probabilities = .158 (15.8 percentage points)
- *Still plenty of room for omitted variable bias!*

Logit Regression (1 of 2)

Logit regression models the probability of $Y = 1$, given X, as the cumulative standard *logistic* distribution function, evaluated at $z = \beta_0 + \beta_1 X$:

$$\Pr(Y = 1|X) = F(\beta_0 + \beta_1 X)$$

where F is the cumulative logistic distribution function:

$$F(\beta_0 + \beta_1 X) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X)}}$$

Because logit and probit use different probability functions, the coefficients (β's) are different in logit and probit.

Logit Regression (2 of 2)

$$\Pr(Y = 1|X) = F(\beta_0 + \beta_1 X)$$

where $$F(\beta_0 + \beta_1 X) = \frac{1}{1 + e^{-(\beta_0 + \beta_1 X)}}.$$

Example: $\beta_0 = -3, \beta_1 = 2, X = 0.4,$

so $\beta_0 + \beta_1 X = -3 + 2 \times 0.4 = -2.2$ so

$$\Pr(Y = 1|X = 0.4) = 1/(1 + e^{-(-2.2)}) = .0998$$

Why bother with logit if we have probit?

- The main reason is historical: logit is computationally faster & easier, but that doesn't matter nowadays
- In practice, logit and probit are very similar – since empirical results typically don't hinge on the logit/probit choice, both tend to be used in practice

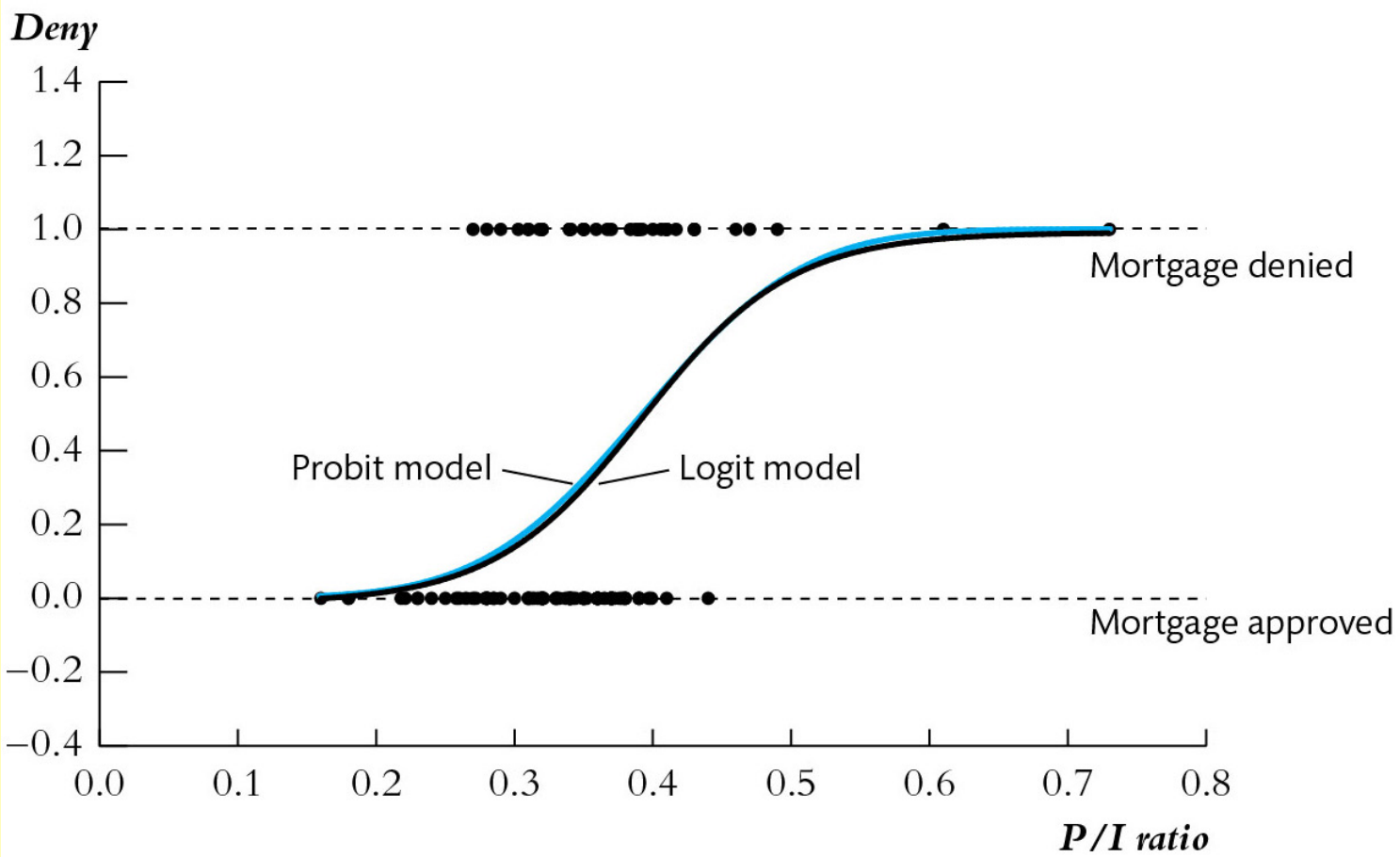
STATA Example: HMDA data

```
. logit deny p_irat black, r
Iteration 0:   log likelihood =  -872.0853          Later…
Iteration 1:   log likelihood =  -806.3571
Iteration 2:   log likelihood = -795.74477
Iteration 3:   log likelihood = -795.69521
Iteration 4:   log likelihood = -795.69521
Logit estimates                                   Number of obs   =       2380
                                                  Wald chi2(2)    =     117.75
                                                  Prob > chi2     =     0.0000
Log likelihood = -795.69521                       Pseudo R2       =     0.0876
------------------------------------------------------------------------------
             |               Robust
        deny |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
      p_irat |   5.370362   .9633435     5.57   0.000     3.482244    7.258481
       black |   1.272782   .1460986     8.71   0.000     .9864339     1.55913
       _cons |  -4.125558    .345825   -11.93   0.000    -4.803362   -3.447753
------------------------------------------------------------------------------
.  dis "Pred prob, p_irat=.3, white: "  ///
        1/(1+exp(-(_b[_cons]+_b[p_irat]*.3+_b[black]*0)))

Pred prob, p_irat=.3, white: .07485143
```

NOTE: the probit predicted probability is .07546603

The predicted probabilities from the probit and logit models are very close in these HMDA regressions



Example for class discussion (1 of 7)

Who are Hezbollah Militants?

Source: Alan Krueger and Jitka Maleckova, "Education, Poverty and Terrorism: Is There a Causal Connection?" *Journal of Economic Perspectives*, Fall 2003, 119–144.

Data set: See the article for details

Logit regression: 1 = died in Hezbollah military event

Table of logit results:

Example for class discussion (2 of 7)

TABLE 4 Characteristics of Hezbollah Militants and Lebanese Population of Similar Age

Characteristic	*Deceased Hezbollah Militants*	*Lebanese Population Age 15–38*
< Poverty	28%	33%
Education	Blank	Blank
Illiterate	0%	6%
Read and write	22%	7%
Primary	17%	23%
Preparatory	14%	26%
Secondary	33%	23%
University	13%	14%
High Studies	1%	1%
Age	Blank	Blank
Mean	22.17	25.57
[std. dev.]	(3.99)	(6.78)
15–17	2%	15%
18–20	41%	14%

Example for class discussion (3 of 7)

TABLE 4 (*Continued*)

Characteristic	*Deceased Hezbollah Militants*	*Lebanese Population Age 15–38*
21–25	42%	23%
26–30	10%	20%
31–38	5%	28%
Hezbollah	21%	NA
Education	Blank	Blank
System	Blank	Blank
Region of Residence	Blank	Blank
Beirut	42%	13%
Mount	0%	36%
Lebanon	Blank	Blank
Bekaa	26%	13%
Nabatieh	2%	6%
South	30%	10%
North	0%	22%

Example for class discussion (4 of 7)

TABLE 4 (*Continued*)

Characteristic	*Deceased Hezbollah Militants*	*Lebanese Population Age 15–38*
Marital Status	Blank	Blank
Divorced	1%	NA
Engaged	5%	NA
Married	39%	NA
Single	55%	NA

Notes: Sample size for Lebanese population sample is 120,796. Sample size for Hezbollah is 50 for poverty status, 78 for education, 81 for age (measured at death), 129 for education in Hezbollah system, 116 for region of residence and 75 for marital status.

Example for class discussion (5 of 7)

TABLE 5 Logistic Estimates of Participation in Hezbollah
(*dependent variable is 1 if individual is a deceased Hezbollah militant, and 0 otherwise; standard errors shown in parentheses*)

	All of Lebanon:				*Heavily Shiite Regions:*	
	Unweighted Estimates		*Weighted Estimates*		*Weighted Estimates*	
Blank	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*
Intercept	−4.886	−5.910	−5.965	−6.991	−4.658	−5.009
	(0.365)	(0.391)	(0.230)	(0.255)	(0.232)	(0.261)
Attended Secondary School or Higher (1 = yes)	0.281	0.171	0.281	0.170	0.220	0.279
	(0.191)	(0.193)	(0.159)	(0.164)	(0.159)	(0.167)
Poverty (1 = yes)	−0.335	−0.167	−0.335	−0.167	−0.467	−0.500
	(0.221)	(0.223)	(0.158)	(0.162)	(0.159)	(0.166)
Age	−0.083	−0.083	−0.083	−0.083	−0.083	−0.082
	(0.015)	(0.015)	(0.008)	(0.008)	(0.008)	(0.008)

Example for class discussion (6 of 7)

TABLE 5 (*Continued*)

	All of Lebanon:				*Heavily Shiite Regions:*	
	Unweighted Estimates		*Weighted Estimates*		*Weighted Estimates*	
Blank	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*
Beirut (1 = yes)	—	2.199 (0.219)	—	2.200 (0.209)	—	0.168 (0.222)
South Lebanon (1 = yes)	—	2.187 (0.232)	—	2.187 (0.221)	—	1.091 (0.221)
Pseudo R-Square	0.020	0.091	0.018	0.080	0.021	0.033
Sample Size	120,925	120,925	120,925	120,925	34,826	34,826

Note: Sample pools together observations on 129 deceased Hezbollah fighters and the general Lebanese population from 1996 PHS. Weights used in columns 3 and 4 are the relative share of Hezbollah militants in the population to their share in the sample and relative share of PHS respondents in the sample to their share in the population. Weight is 0.273 for Hezbollah sample and .093 for PHS sample.

Example for class discussion (7 of 7)

Compute the effect of schooling by comparing predicted probabilities using the logit regression in column (3):

$$\Pr(Y = 1|\text{secondary} = 1,\ \text{poverty} = 0,\ \text{age} = 20)$$
$$-\Pr(Y = 0|\text{secondary} = 0,\ \text{poverty} = 0,\ \text{age} = 20):$$

$$\Pr(Y = 1|\text{secondary} = 1,\ \text{poverty} = 0,\ \text{age} = 20)$$
$$= 1/[1 + e^{-(-5.965 + .281 \times 1 - .335 \times 0 - .083 \times 20)}]$$
$$= 1/[1 + e^{7.344}] = .000646 \textit{ Does this make sense?}$$

$$\Pr(Y = 1|\text{secondary} = 0,\ \text{poverty} = 0, \text{age} = 20)$$
$$= 1/[1 + e^{-(-5.965 + .281 \times 1 - .335 \times 0 - .083 \times 20)}]$$
$$= 1/[1 + e^{7.625}] = .000488 \textit{ Does this make sense?}$$

Predicted change in probabilities

$$\Pr(Y = 1|\text{secondary} = 1,\ \text{poverty} = 0, \text{age} = 20)$$
$$- \Pr(Y = 1|\text{secondary} = 1, \text{poverty} = 0, \text{age} = 20)$$
$$= .000646 - .000488 = .000158$$

Both these statements are true:

- The probability of being a Hezbollah militant increases by 0.0158 percentage points, if secondary school is attended.
- The probability of being a Hezbollah militant increases by 32%, if secondary school is attended (.000158/.000488 = .32).
- *These sound so different! What is going on?*

Estimation and Inference in the Logit and Probit Models (SW Section 11.3)

We'll focus on the probit model:

$$\Pr(Y = 1|X) = \Phi(\beta_0 + \beta_1 X)$$

- Estimation and inference
  - How can we estimate β_0 and β_1?
  - What is the sampling distribution of the estimators?
  - Why can we use the usual methods of inference?
- First motivate via nonlinear least squares
- Then discuss *maximum likelihood* estimation (what is actually done in practice)

Probit estimation by nonlinear least squares

Recall OLS: $\min_{b_0,b_1} \sum_{i=1}^{n} [Y_i - (b_0 + b_1 X_i)]^2$

- The result is the OLS estimators $\hat{\beta}_0$ and $\hat{\beta}_1$

  Nonlinear least squares extends the idea of OLS to models in which the parameters enter nonlinearly:

$$\min_{b_0,b_1} \sum_{i=1}^{n} [Y_i - \Phi(b_0 + b_1 X_i)]^2$$

How to solve this minimization problem?

- Calculus doesn't give an explicit formula for the NLLS estimators.
- Instead, the minimization problem is solved *numerically* using the computer (specialized minimization algorithms)
- Nonlinear least squares isn't actually used in practice. A more efficient estimator (smaller variance) is…

The Maximum Likelihood Estimator of the Coefficients in the Probit Model

The ***likelihood function*** is the **conditional density** of $Y_1,\ldots,Y_n$ given $X_1,\ldots, X_n$, treated as a function of the unknown parameters β_0 and β_1.

- The maximum likelihood estimator (MLE) is the value of (β_0, β_1) that maximize the likelihood function.
- The MLE is the value of (β_0, β_1) that best describe the full distribution of the data.
- In large samples, the MLE is:
  - Consistent
  - Normally distributed
  - Efficient (has the smallest variance of all consistent estimators)

Special case: The probit MLE with no X (1 of 6)

$$Y = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases} \quad \text{(Bernoulli distribution)}$$

Data: $Y_1, \ldots, Y_n$, i.i.d.

Derivation of the likelihood starts with the density of Y_1:

$$\Pr(Y_1 = 1) = p \text{ and } \Pr(Y_1 = 0) = 1-p$$

so

$$\Pr(Y_1 = y_1) = p^{y_1}(1-p)^{1-y_1} \quad (\textit{verify this for } y_1 = 0, 1!)$$

Special case: The probit MLE with no X (2 of 6)

Joint density of (Y_1, Y_2): because Y_1 and Y_2 are independent,

$$\begin{aligned}\Pr(Y_1 = y_1, Y_2 = y_2) &= \Pr(Y_1 = y_1) \times \Pr(Y_2 = y_2) \\ &= [p^{y_1}(1-p)^{1-y_1}] \times [p^{y_2}(1-p)^{1-y_2}] \\ &= p^{(y_1+y_2)}(1-p)^{[2-(y_1+y_2)]}\end{aligned}$$

Joint density of $(Y_1, \ldots, Y_n)$:

$$\begin{aligned}&\Pr(Y_1 = y_1, Y_2 = y_2, \ldots, Y_n = y_n) \\ &= [p^{y_1}(1-p)^{1-y_1}] \times [p^{y_2}(1-p)^{1-y_2}] \times \ldots \times [p^{y_n}(1-p)^{1-y_n}] \\ &= p^{\sum_{i=1}^n y_i}(1-p)^{\left(n-\sum_{i=1}^n y_i\right)}\end{aligned}$$

Special case: The probit MLE with no X (3 of 6)

The likelihood is the joint density, treated as a function of the unknown parameters, which here is p:

$$f(p;Y_1,\ldots,Y_n) = p^{\sum_{i=1}^{n} Y_i}(1-p)^{\left(n-\sum_{i=1}^{n} Y_i\right)}$$

The MLE maximizes the likelihood. Its easier to work with the logarithm of the likelihood, ln[f ($p;Y_1,\ldots,Y_n$)]:

$$\ln[f(p;Y_1,\ldots,Y_n)] = \left(\sum_{i=1}^{n} Y_i\right)\ln(p) + \left(n-\sum_{i=1}^{n} Y_i\right)\ln(1-p)$$

Maximize the log likelihood by setting the derivative = 0:

$$\frac{d\ln[f(p;Y_1,\ldots,Y_n)]}{dp} = \left(\sum_{i=1}^{n} Y_i\right)\frac{1}{p} + \left(n-\sum_{i=1}^{n} Y_i\right)\left(\frac{-1}{1-p}\right) = 0$$

Solving for p yields the MLE; that is, $\hat{p}^{MLE}$ satisfies,

Special case: The probit MLE with no *X* (4 of 6)

$$\left(\sum_{i=1}^{n} Y_i\right)\frac{1}{\hat{p}^{MLE}} + \left(n - \sum_{i=1}^{n} Y_i\right)\left(\frac{-1}{1-\hat{p}^{MLE}}\right) = 0$$

or

$$\left(\sum_{i=1}^{n} Y_i\right)\frac{1}{\hat{p}^{MLE}} = \left(n - \sum_{i=1}^{n} Y_i\right)\frac{1}{1-\hat{p}^{MLE}}$$

or

$$\frac{\bar{Y}}{1-\bar{Y}} = \frac{\hat{p}^{MLE}}{1-\hat{p}^{MLE}}$$

or

$$\hat{p}^{MLE} = \bar{Y} = \text{fraction of 1's}$$

Whew… a lot of work to get back to the first thing you would think of using…but the nice thing is that this whole approach generalizes to more complicated models…

Special case: The probit MLE with no X (5 of 6)

$$\hat{p}^{MLE} = \bar{Y} = \text{fraction of 1's}$$

- For Y_i i.i.d. Bernoulli, the MLE is the "natural" estimator of p, the fraction of 1's, which is $\bar{Y}$
- We already know the essentials of inference:
  - In large n, the sampling distribution of $\hat{p}^{MLE} = \bar{Y}$ is normally distributed
  - Thus inference is "as usual": hypothesis testing via t-statistic, confidence interval as $\pm 1.96SE$

Special case: The probit MLE with no X (6 of 6)

- The theory of maximum likelihood estimation says that $\hat{p}^{MLE}$ is the ***most*** efficient estimator of p – of *all* consistent estimators! – at least for large n. (Much stronger than the Gauss-Markov theorem). For this reason the MLE is primary estimator used for models that in which the parameters (coefficients) enter nonlinearly.
- STATA note: to emphasize requirement of large-n, the printout calls the t-statistic the z-statistic; instead of the F-statistic, the *chi-squared* statistic ($= q \times F$).

We are now ready to turn to the MLE of probit coefficients, in which the probability is conditional on X.

The Probit Likelihood with one *X* (1 of 3)

The derivation starts with the density of Y_1, given X_1:

$$\Pr(Y_1 = 1|X_1) = \Phi(\beta_0 + \beta_1 X_1)$$

$$\Pr(Y_1 = 0|X_1) = 1 - \Phi(\beta_0 + \beta_1 X_1)$$

so

$$\Pr(Y_1 = y_1|X_1) = \Phi(\beta_0 + \beta_1 X_1)^{y_1}[1 - \Phi(\beta_0 + \beta_1 X_1)]^{1-y_1}$$

The probit likelihood function is the joint density of $Y_1,\ldots,Y_n$ given $X_1,\ldots,X_n$, treated as a function of β_0, β_1:

$$f(\beta_0, \beta_1; Y_1,\ldots,Y_n|X_1,\ldots,X_n)$$
$$= \{\Phi(\beta_0 + \beta_1 X_1)^{Y_1}[1 - \Phi(\beta_0 + \beta_1 X_1)]^{1-Y_1}\} \times$$
$$\ldots \times \{\Phi(\beta_0 + \beta_1 X_n)^{Y_n}[1 - \Phi(\beta_0 + \beta_1 X_n)]^{1-Y_n}\}$$

The Probit Likelihood with one *X* (2 of 3)

$$f(\beta_0, \beta_1; Y_1, \ldots, Y_n | X_1, \ldots, X_n)$$
$$= \{\Phi(\beta_0 + \beta_1 X_1)^{Y_1}[1 - \Phi(\beta_0 + \beta_1 X_1)]^{1-Y_1}\} \times$$
$$\ldots \times \{\Phi(\beta_0 + \beta_1 X_n)^{Y_n}[1 - \Phi(\beta_0 + \beta_1 X_n)]^{1-Y_n}\}$$

- $\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}$ maximize this likelihood function.
- But we can't solve for the maximum explicitly! So the MLE must be maximized using numerical methods
- As in the case of no X, in large samples:
  - $\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}$ are consistent
  - $\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}$ are normally distributed
  - $\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}$ are asymptotically efficient – among all estimators (assuming the probit model is the correct model)

The Probit Likelihood with one *X* (3 of 3)

- Standard errors of $\hat{\beta}_0^{MLE}$, $\hat{\beta}_1^{MLE}$ are computed automatically...
- Testing, confidence intervals proceeds as usual
- This all extends to multiple X's, for details see SW App. 11.2

The Logit Likelihood with one *X*

- The only difference between probit and logit is the functional form used for the probability: Φ is replaced by the cumulative logistic function.
- Otherwise, the likelihood is similar; for details see SW App. 11.2
- As with probit,
  - $\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}$ are consistent
  - $\hat{\beta}_0^{MLE}, \hat{\beta}_1^{MLE}$ are normally distributed
  - Their standard errors can be computed
  - Testing confidence intervals proceeds as usual

Measures of Fit for Logit and Probit

The R^2 and $\overline{R}^2$ don't make sense here (*why*?). So, two other specialized measures are used:

1. The ***fraction correctly predicted*** = fraction of Y's for which the predicted probability is >50% when $Y_i = 1$, or is <50% when $Y_i = 0$.
2. The ***pseudo-R^2*** measures the improvement in the value of the log likelihood, relative to having no X's (see SW App. 11.2). The pseudo-R^2 simplifies to the R^2 in the linear model with normally distributed errors.

Application to the Boston HMDA Data (SW Section 11.4)

- Mortgages (home loans) are an essential part of buying a home.
- Is there differential access to home loans by race?
- If two otherwise identical individuals, one white and one black, applied for a home loan, is there a difference in the probability of denial?

The HMDA Data Set

- Data on individual characteristics, property characteristics, and loan denial/acceptance
- The mortgage application process circa 1990-1991:
  - Go to a bank or mortgage company
  - Fill out an application (personal+financial info)
  - Meet with the loan officer
- Then the loan officer decides – by law, in a race-blind way. Presumably, the bank wants to make profitable loans, and (if the incentives inside the bank or loan origination office are right – a big if during the mid-2000s housing bubble!) the loan officer doesn't want to originate defaults.

The Loan Officer's Decision

- Loan officer uses key financial variables:
  - *P/I ratio*
  - housing expense-to-income ratio
  - loan-to-value ratio
  - personal credit history
- The decision rule is nonlinear:
  - loan-to-value ratio $> 80\%$
  - loan-to-value ratio $> 95\%$ (what happens in default?)
  - credit score

Regression Specifications (1 of 6)

$$\Pr(deny=1|black, \text{ other } X's) = \ldots$$

- linear probability model
- probit

Main problem with the regressions so far: potential omitted variable bias. The following variables (i) enter the loan officer decision *and* (ii) are or could be correlated with race:

- wealth, type of employment
- credit history
- family status

Fortunately, the HMDA data set is very rich…

Regression Specifications (2 of 6)

TABLE 11.1 Variables Included in Regression Models of Mortgage Decisions

Variable	Definition	Sample Average
Financial Variables		
P/I ratio	Ratio of total monthly debt payments to total monthly income	0.331
housing expense-to-income ratio	Ratio of monthly housing expenses to total monthly income	0.255
loan-to-value ratio	Ratio of size of loan to assessed value of property	0.738
consumer credit score	1 if no "slow" payments or delinquencies 2 if one or two slow payments or delinquencies 3 if more than two slow payments 4 if insufficient credit history for determination 5 if delinquent credit history with payments 60 days overdue 6 if delinquent credit history with payments 90 days overdue	2.1
mortgage credit score	1 if no late mortgage payments 2 if no mortgage payment history 3 if one or two late mortgage payments 4 if more than two late mortgage payments	1.7
public bad credit record	1 if any public record of credit problems (bankruptcy, charge-offs, collection actions) 0 otherwise	0.074

Regression Specifications (3 of 6)

TABLE 11.1 (*Continued*)

Additional Applicant Characteristics		
denied mortgage insurance	1 if applicant applied for mortgage insurance and was denied, 0 otherwise	0.020
self-employed	1 if self-employed, 0 otherwise	0.116
single	1 if applicant reported being single, 0 otherwise	0.393
high school diploma	1 if applicant graduated from high school, 0 otherwise	0.984
unemployment rate	1989 Massachusetts unemployment rate in the applicant's industry	3.8
condominium	1 if unit is a condominium, 0 otherwise	0.288
black	1 if applicant is black, 0 if white	0.142
deny	1 if mortgage application denied, 0 otherwise	0.120

Regression Specifications (4 of 6)

TABLE 11.2 Mortgage Denial Regressions Using the Boston HMDA Data

Dependent variable: *deny* = 1 if mortgage application is denied, = 0 if accepted; 2380 observations.

Regression Model	LPM	Logit	Probit	Probit	Probit	Probit
Regressor	(1)	(2)	(3)	(4)	(5)	(6)
black	0.084** (0.023)	0.688** (0.182)	0.389** (0.098)	0.371** (0.099)	0.363** (0.100)	0.246 (0.448)
P/I ratio	0.449** (0.114)	4.76** (1.33)	2.44** (0.61)	2.46** (0.60)	2.62** (0.61)	2.57** (0.66)
housing expense-to-income ratio	−0.048 (0.110)	−0.11 (1.29)	−0.18 (0.68)	−0.30 (0.68)	−0.50 (0.70)	−0.54 (0.74)
medium loan-to-value ratio (0.80 ≤ *loan-value ratio* ≤ 0.95)	0.031* (0.013)	0.46** (0.16)	0.21** (0.08)	0.22** (0.08)	0.22** (0.08)	0.22** (0.08)
high loan-to-value ratio (*loan-value ratio* > 0.95)	0.189** (0.050)	1.49** (0.32)	0.79** (0.18)	0.79** (0.18)	0.84** (0.18)	0.79** (0.18)
consumer credit score	0.031** (0.005)	0.29** (0.04)	0.15** (0.02)	0.16** (0.02)	0.34** (0.11)	0.16** (0.02)
mortgage credit score	0.021 (0.011)	0.28* (0.14)	0.15* (0.07)	0.11 (0.08)	0.16 (0.10)	0.11 (0.08)
public bad credit record	0.197** (0.035)	1.23** (0.20)	0.70** (0.12)	0.70** (0.12)	0.72** (0.12)	0.70** (0.12)
denied mortgage insurance	0.702** (0.045)	4.55** (0.57)	2.56** (0.30)	2.59** (0.29)	2.59** (0.30)	2.59** (0.29)

Regression Specifications (5 of 6)

TABLE 11.2 (*Continued*)

Dependent variable: *deny* = 1 if mortgage application is denied, = 0 if accepted; 2380 observations.

Regression Model	LPM	Logit	Probit	Probit	Probit	Probit
Regressor	(1)	(2)	(3)	(4)	(5)	(6)
self-employed	0.060** (0.021)	0.67** (0.21)	0.36** (0.11)	0.35** (0.11)	0.34** (0.11)	0.35** (0.11)
single				0.23** (0.08)	0.23** (0.08)	0.23** (0.08)
high school diploma				−0.61** (0.23)	−0.60* (0.24)	−0.62** (0.23)
unemployment rate				0.03 (0.02)	0.03 (0.02)	0.03 (0.02)
condominium					−0.05 (0.09)	
black × P/I ratio						−0.58 (1.47)
black × housing expense-to-income ratio						1.23 (1.69)
additional credit rating indicator variables	no	no	no	no	yes	no
constant	−0.183** (0.028)	−5.71** (0.48)	−3.04** (0.23)	−2.57** (0.34)	−2.90** (0.39)	−2.54** (0.35)

Regression Specifications (6 of 6)

TABLE 11.2 (*Continued*)

***F*-Statistics and *p*-Values Testing Exclusion of Groups of Variables**

	(1)	(2)	(3)	(4)	(5)	(6)
applicant single; high school diploma; industry unemployment rate				5.85 (< 0.001)	5.22 (0.001)	5.79 (< 0.001)
additional credit rating indicator variables					1.22 (0.291)	
race interactions and black						4.96 (0.002)
race interactions only						0.27 (0.766)
difference in predicted probability of denial, white vs. black (percentage points)	8.4%	6.0%	7.1%	6.6%	6.3%	6.5%

These regressions were estimated using the n = 2380 observations in the Boston HMDA data set described in Appendix 11.1. The linear probability model was estimated by OLS, and probit and logit regressions were estimated by maximum likelihood. Standard errors are given in parentheses under the coefficients, and p-values are given in parentheses under the F-statistics. The change in predicted probability in the final row was computed for a hypothetical applicant whose values of the regressors, other than race, equal the sample mean. Individual coefficients are statistically significant at the *5% or **1% level.

Summary of Empirical Results

- Coefficients on the financial variables make sense.
- *Black* is statistically significant in all specifications
- Race-financial variable interactions aren't significant.
- Including the covariates sharply reduces the effect of race on denial probability.
- LPM, probit, logit: similar estimates of effect of race on the probability of denial.
- Estimated effects are large in a "real world" sense.

Remaining Threats to Internal, External Validity

Internal validity

1. Omitted variable bias?
2. Wrong functional form?
3. Errors-in-variables bias?
4. Sample selection bias?
5. Simultaneous causality bias?

What do you think?

External validity

These data are from Boston in 1990-91. Do you think the results also apply today, where *you* live?

Conclusion (SW Section 11.5)

- If Y_i is binary, then $E(Y|X) = \Pr(Y = 1|X)$
- Three models:
  - **linear probability model** (linear multiple regression)
  - **probit** (cumulative standard normal distribution)
  - **logit** (cumulative standard logistic distribution)
- LPM, probit, logit all produce predicted probabilities
- Effect of ΔX is change in conditional probability that $Y = 1$. For logit and probit, this depends on the initial X
- Probit and logit are estimated via maximum likelihood
  - Coefficients are normally distributed for large n
  - Large-n hypothesis testing, conf. intervals is as usual