Intermediate Microeconomics
Spring 2025

Week 13b: Uncertainty and Risk Aversion (II)

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The State-Preference Approach

- The approach taken in this chapter up to this point has not used the basic model of utility-maximization subject to a budget constraint.

- We will develop new techniques to incorporate the standard choice-theoretic framework

States of the World

- Outcomes of any random event can be categorized into a number of states of the world
  - "good times" or "bad times"

- Contingent commodities are goods delivered only if a particular state of the world occurs
  - "\$1 in good times" or "\$1 in bad times"

- It is conceivable that an individual could purchase a contingent commodity
  - buy a promise that someone will pay you \$1 if tomorrow turns out to be good times
  - this good will probably cost less than \$1

Utility Analysis

- Assume that there are two contingent goods
  - wealth in good times (W_g) and wealth in bad times (W_b)
  - individual believes the probability that good times will occur is π

Utility Analysis

- The expected utility associated with these two contingent goods is

$$V(W_g, W_b) = \pi U(W_g) + (1 - \pi) U(W_b)$$

- This is the value that the individual wants to maximize given his initial wealth (W)

Prices of Contingent Commodities

- Assume that the person can buy \$1 of wealth in good times for p_g and \$1 of wealth in bad times for p_b

- His budget constraint is

$$W = p_g W_g + p_b W_b$$

- The price ratio p_g / p_b shows how this person can trade dollars of wealth in good times for dollars in bad times

Fair Markets for Contingent Goods

- If markets for contingent wealth claims are well-developed and there is general agreement about π, prices for these goods will be actuarially fair
  - that is, they will equal the underlying probabilities:

$$p_g = \pi \text{ and } p_b = (1 - \pi)$$

- The price ratio will reflect the odds in favor of good times

$$\frac{p_g}{p_b} = \frac{\pi}{1-\pi}$$

Risk Aversion

- If contingent claims markets are fair, a utility-maximizing individual will opt for a situation in which $W_g = W_b$
- he will arrange matters so that the wealth obtained is the same no matter what state occurs
- can also show the math…

Risk Aversion

- Maximization of utility subject to a budget constraint requires that

$$MRS = \frac{\partial V / \partial W_g}{\partial V / \partial W_b} = \frac{\pi U'(W_g)}{(1-\pi)U'(W_b)} = \frac{p_g}{p_b}$$

- If markets for contingent claims are fair, $\frac{p_g}{p_b} = \frac{\pi}{1-\pi}$

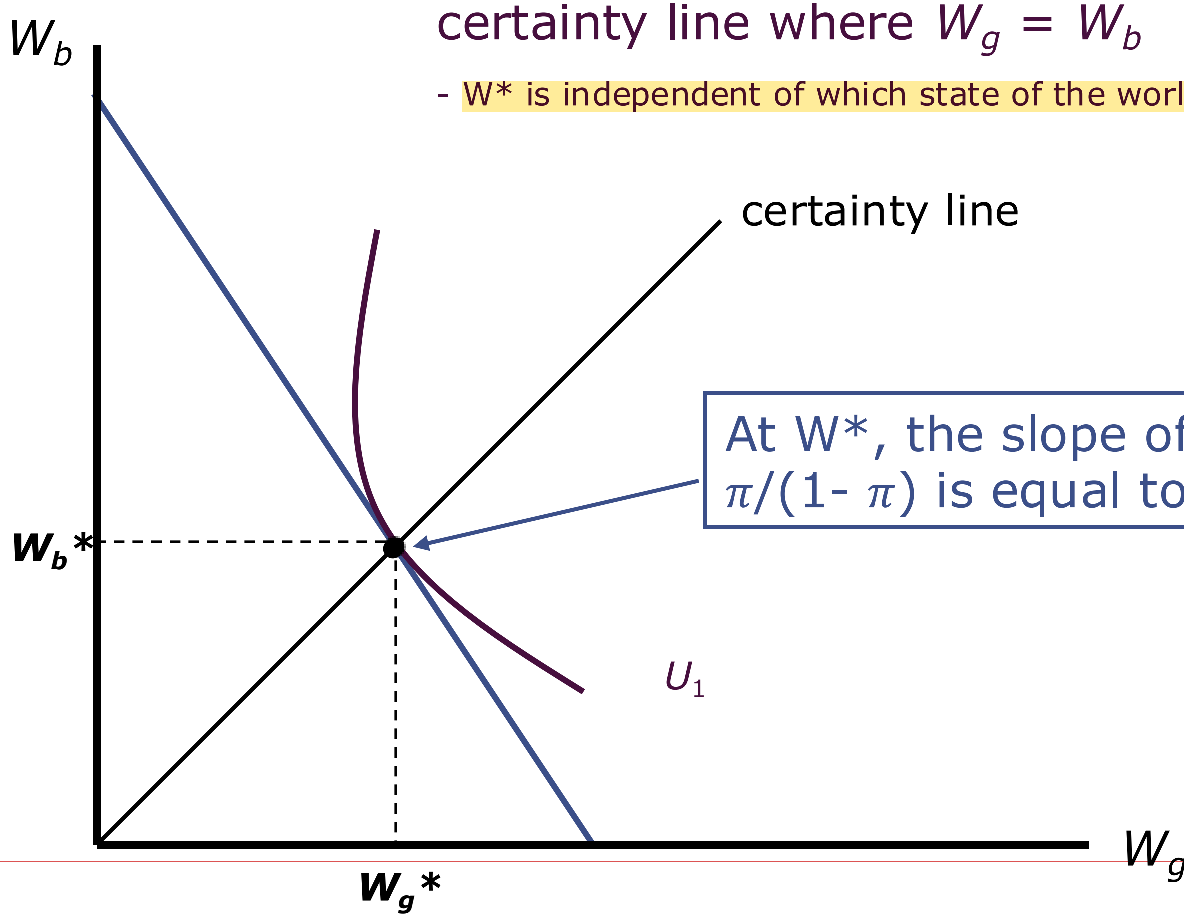
$$\frac{U'(W_g)}{U'(W_b)} = 1$$

$$W_g = W_b$$

Risk Aversion

The individual maximizes utility on the certainty line where $W_g = W_b$

- W* is independent of which state of the world occurs

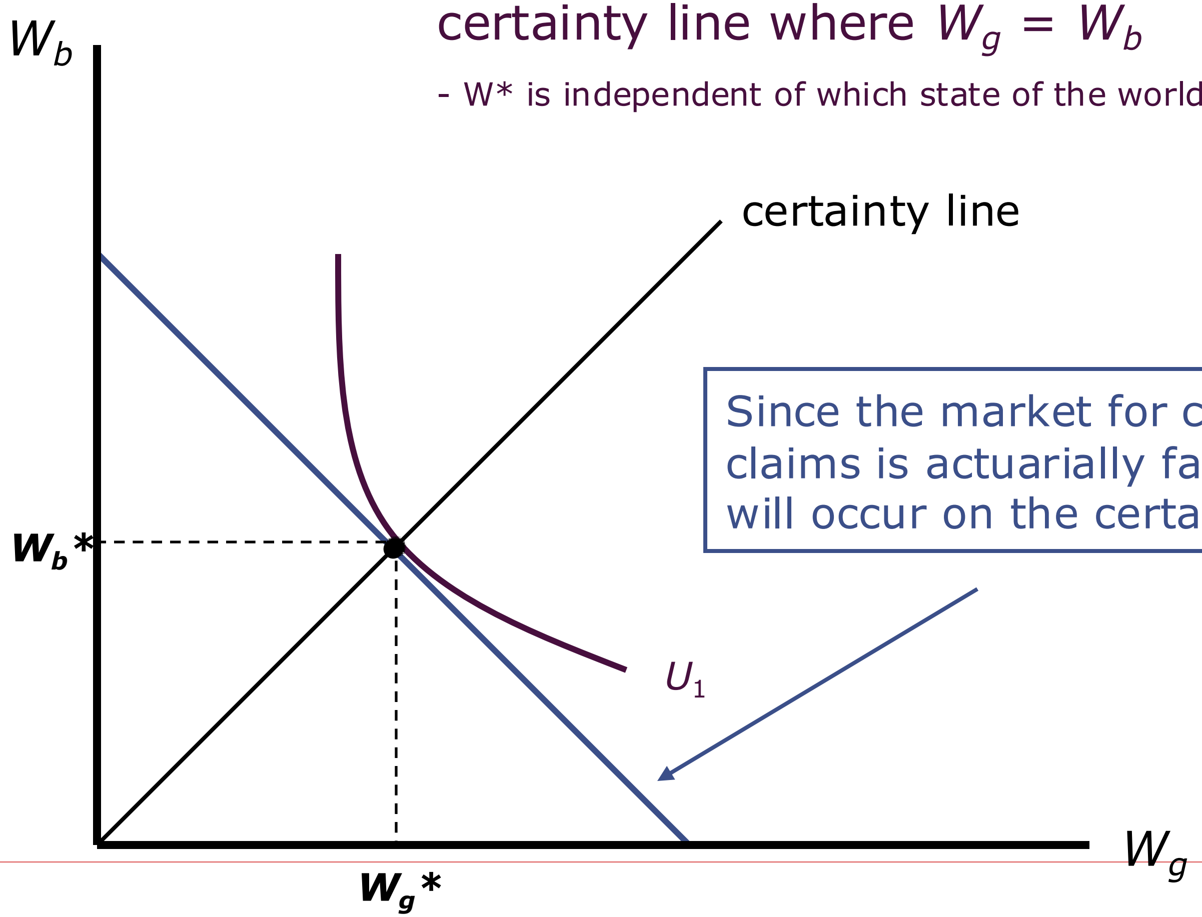


At W*, the slope of the indifference curve $\pi/(1-\pi)$ is equal to the price ratio p_g/p_b

Risk Aversion

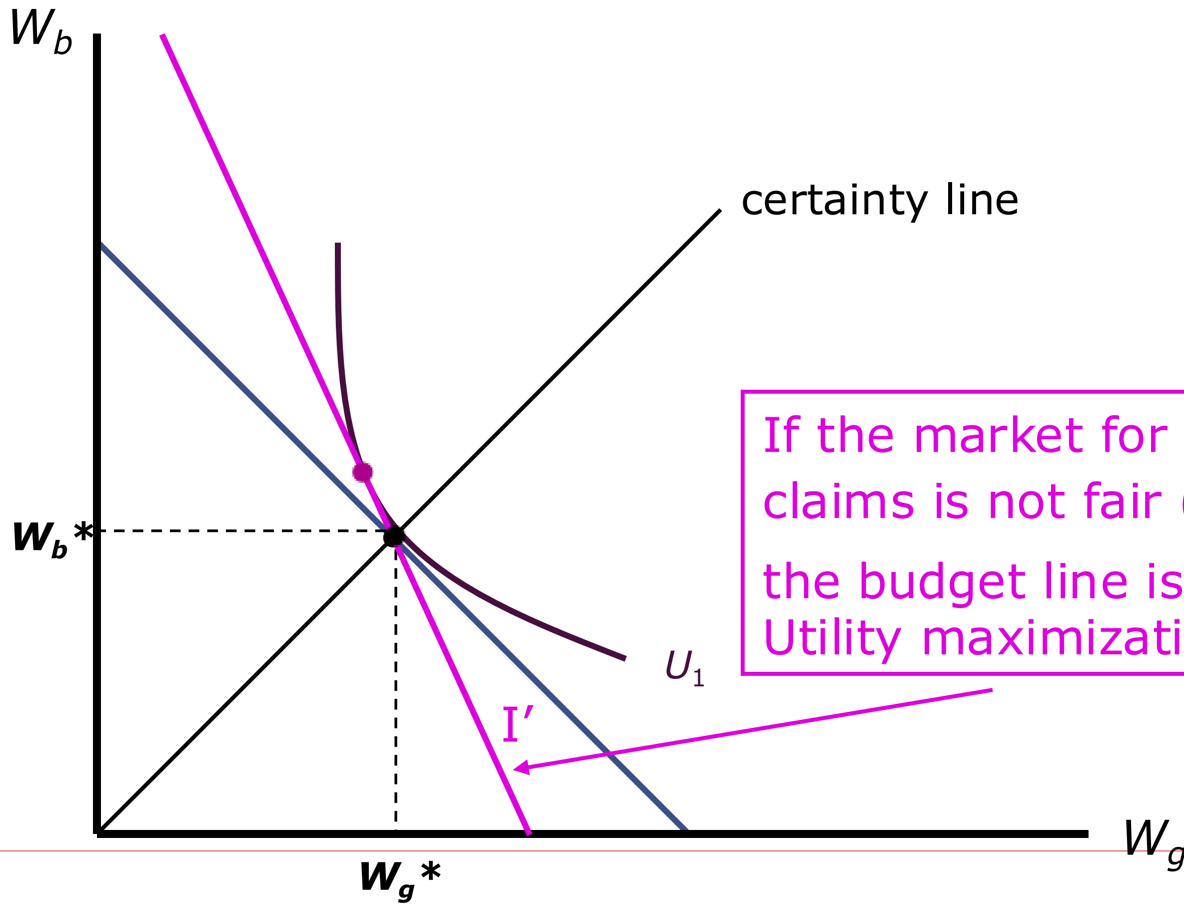
The individual maximizes utility on the certainty line where $W_g = W_b$

- W* is independent of which state of the world occurs



Since the market for contingent claims is actuarially fair, utility maximization will occur on the certainty line $W^*=W_b=W_g$

Risk Aversion



If the market for contingent claims is not fair ($\frac{p_g}{p_b} \neq \frac{\pi}{1-\pi}$), the budget line is as I′, Utility maximization occur at $W_b > W_g$.

Risk Aversion and Risk Premiums

- Consider two people, each of whom starts with an initial wealth of W^*

- Each seeks to maximize an expected utility function of the form

$$V(W_g, W_b) = \pi \frac{W_g^R}{R} + (1-\pi)\frac{W_b^R}{R}$$

- This utility function exhibits constant relative risk aversion

Risk Aversion and Risk Premiums

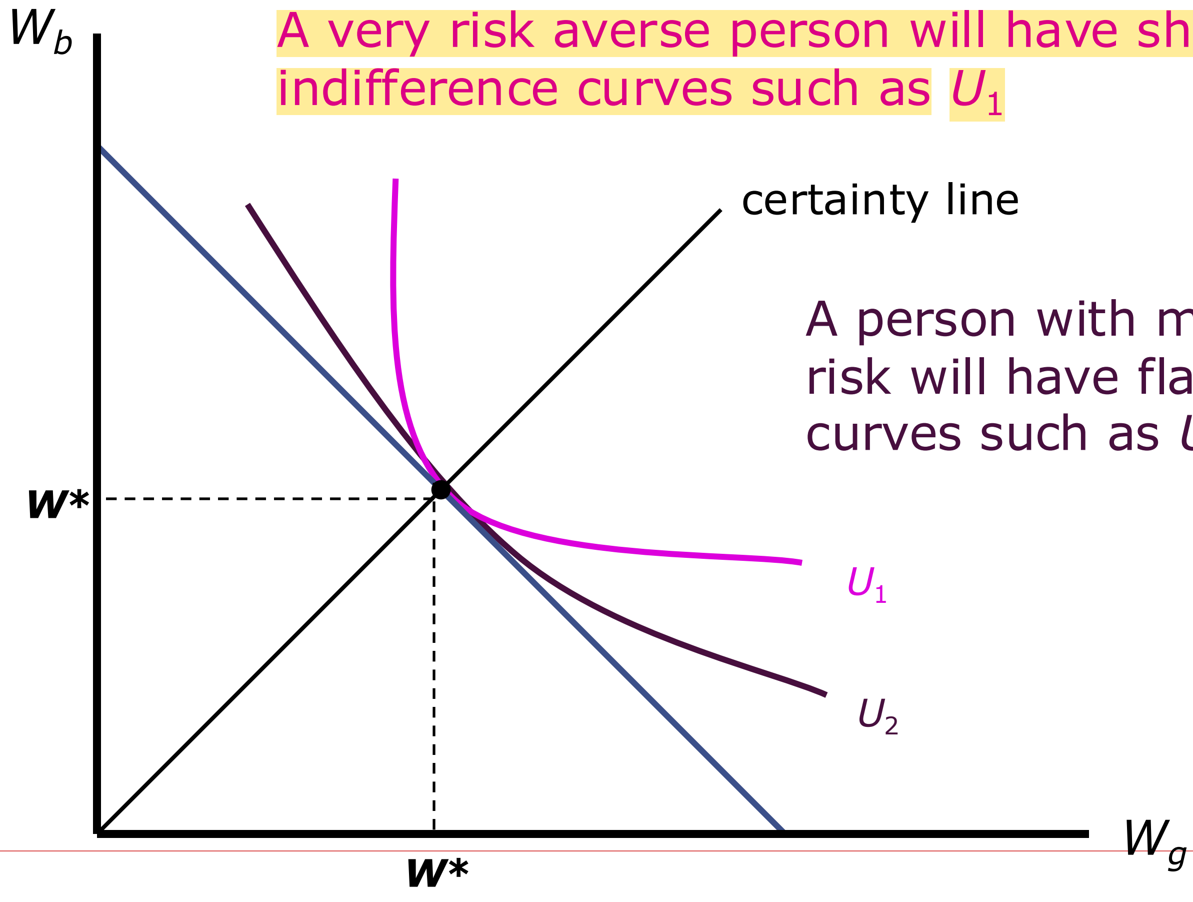
$$V(W_g, W_b) = \pi \frac{W_g^R}{R} + (1 - \pi) \frac{W_b^R}{R}$$

- The parameter *R* determines both the degree of risk aversion and the degree of curvature of indifference curves implied by the function
  - a very risk averse individual will have a large negative value for *R*

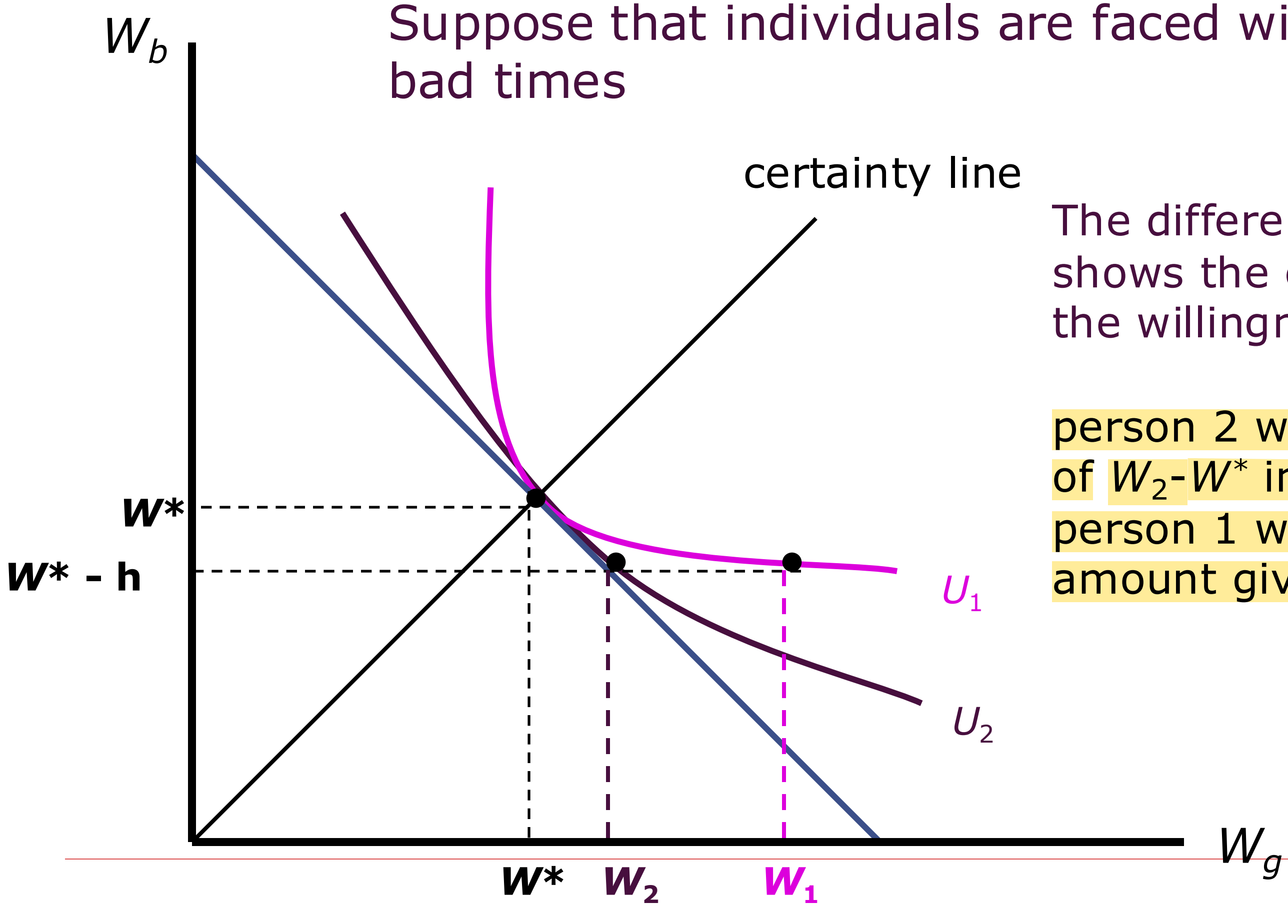
Risk Aversion and Risk Premiums

A very risk averse person will have sharply curved indifference curves such as U_1

A person with more tolerance for risk will have flatter indifference curves such as U_2



Risk Aversion and Risk Premiums



Suppose that individuals are faced with losing *h* dollars in bad times

The difference between W_1 and W_2 shows the effect of risk aversion on the willingness to accept risk

person 2 will require compensation of W_2-W^* in good times, whereas person 1 will require a larger amount given by W_1-W^*.

Insurance in the State-Preference Model

- Consider a person with wealth of \$100,000 who faces a 25% chance of losing his automobile worth \$20,000
  - wealth with no theft (W_g) = \$100,000 and probability of no theft = 0.75
  - wealth with a theft (W_b) = \$80,000 and probability of a theft = 0.25

Insurance in the State-Preference Model

- If we assume logarithmic utility, then

$$E(U) = 0.75U(W_g) + 0.25U(W_b)$$

$$E(U) = 0.75\ ln\ W_g + 0.25\ ln\ W_b$$

$$E(U) = 0.75\ ln\ (100{,}000) + 0.25\ ln\ (80{,}000)$$

$$E(U) = 11.45714$$

Insurance in the State-Preference Model

- The budget constraint is written in terms of the prices of the contingent commodities

$$p_g W_g^* + p_b W_b^* = p_g W_g + p_b W_b$$

- Assuming that these prices equal the probabilities of these two states

$$0.75(100,000) + 0.25(80,000) = 95,000$$

- The expected value of wealth = $95,000

Insurance in the State-Preference Model

- The individual will move to the certainty line and receive an expected utility of

$$E(U) = \ln 95{,}000 = 11.46163$$

- to be able to do so, the individual must be able to transfer \$5,000 = -(95,000-100,000) in extra wealth in good times into \$15,000 = 95,000-80,000 of extra wealth in bad times
  - a fair insurance contract will allow this
  - the wealth changes promised by insurance (dW_b/dW_g) = 15,000/-5,000 = -3

A Policy with a Deductible

- Suppose that the insurance policy costs $4,900, but requires the person to incur the first $1,000 of the loss

$$W_g = 100{,}000 - 4{,}900 = 95{,}100$$

$$W_b = 80{,}000 - 4{,}900 + 19{,}000 = 94{,}100$$

$$E(U) = 0.75\ ln\ 95{,}100 + 0.25\ ln\ 94{,}100$$

$$E(U) = 11.46004$$

- The policy still provides higher utility than doing nothing

The Demand for Insurance

- π = probability of accident,
- W = income,
- L = loss if the accident happened,
- M = amount insured,
- θ = premium percentage, then insurance premium = θM.

- There are two parties in the insurance market: insured (i.e., consumer) and insurer (i.e., insurance company)

The Demand for Insurance

- Insured
  - wealth at good times =
  - wealth at bad times =
  - The insured's objective is to choose M to maximize expected utility:
  - To the insured, $\pi, W, \theta,$ and L are exogenous (i.e., given).
  - To maximize utility, the first-order condition is

The Demand for Insurance

- Insured
  - wealth at good times =
  - wealth at bad times =

  - The insured's objective is to choose M to maximize expected utility:

$$\max_{M} \pi U(W - \theta M - L + M) + (1 - \pi) U(W - \theta M)$$

  - To the insured, $\pi, W, \theta,$ and L are exogenous (i.e., given).
  - The first-order condition is

$$\pi(1 - \theta) U'(W - \theta M - L + M) - (1 - \pi)\theta U'(W - \theta M) = 0$$

The Demand for Insurance

- Example: If $U(W) = \log W$, then the optimal M^* is given by

The Demand for Insurance

- Example: If $U(W)$ = log$W$, then the optimal $M*$ is given by

$$M^* = \frac{(1-\pi)\theta L + (\pi - \theta)W}{(1-\theta)\theta}$$

The Demand for Insurance

- To the insurer,
  - revenue = θM,
  - Cost = $\pi M + (1 - \pi)0 = \pi M$,
  - Hence, expected profit = $\theta M - \pi M$.
  - If insurance is actuarially fair (e.g., the insurance market is perfectly competitive), then the expected profit = 0, which implies $\theta = \pi$.
  - As a result, the premium percentage = probability of payout to the insured.

The Demand for Insurance

- Back to the insured,
- If the insurance is actuarially fair, then substituting $\theta = \pi$ into the first-order condition,
- One obtains

$$\pi(1-\pi)U'(W-\pi M-L+M)-(1-\pi)\pi U'(W-\pi M)=0$$
$$U'(W-\pi M-L+M)=U'(W-\pi M)$$

- This equation means that the insured will achieve an optimum by choosing M to equalize the marginal utility of income across the two states (the accident state and the no-accident state).

The Demand for Insurance

- From mathematics, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ if f(x) is a monotonic function.
- Assuming U''< 0, U' is thus monotonic, hence $W - \pi M - L + M = W - \pi M$, which implies that $M = L$. The insured is fully insured because $M = L$.

The Demand for Insurance

- From mathematics, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$ if f(x) is a monotonic function.
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- As $\theta = \pi$ and $M = L$; the insured's utility = $\pi U(W - \theta M - L + M) + (1 - \pi)U(W - \theta M) = U(W - \pi L)$.
- Regardless of whether the accident occurs, the insured receives $U(W - \pi L)$ with certainty. Insurance essentially *removes* uncertainty!

- Example: If $U(W)$ = log$W$, and $\theta = \pi$, then M^* = L.

Methods for Reducing Uncertainty and Risk

- Four different methods that individuals can take to mitigate the problem of risk and uncertainty:
  - Insurance
  - Diversification
  - Flexibility
  - Information

Diversification

- Two firms, A and B.
- Shares cost \$10.
- With prob. 1/2 A's profit is \$100 and B's profit is \$20.
- With prob. 1/2 A's profit is \$20 and B's profit is \$100.
- You have \$100 to invest.
- How?

Diversification

- (1) Buy only firm A's stock?
- \$100/10 = 10 shares.
- You earn \$1000 with prob. 1/2 and \$200 with prob. 1/2.
- Expected earning: \$500 + \$100 = \$600

Diversification

- (2) Buy only firm B's stock?
- $100/10 = 10 shares.
- You earn $1000 with prob. 1/2 and $200 with prob. 1/2.
- Expected earning: $500 + $100 = $600

Diversification

- (3) Buy 5 shares in each firm?
- You earn $600 for sure.
- Diversification has maintained expected earning and lowered risk.

- Typically, diversification lowers expected earnings in exchange for lowered risk.

Flexibility

- Flexibility
  - Allows the person to adjust the initial decision depending on how the future unfolds
  - The more uncertain the future, the more valuable this flexibility
  - Keeps the decision-maker from being tied to one course of action
    - And instead provides a number of options

Flexibility

- Financial option contract
  - Offers the right, but not the obligation, to buy or sell an asset
    - During some future period
    - At a certain price
- Real option
  - An option arising in a setting outside of financial markets

Flexibility

- All options share three fundamental attributes
  - Specify the underlying transaction
  - Specify a period over which the option may be exercised
  - Specifies a price

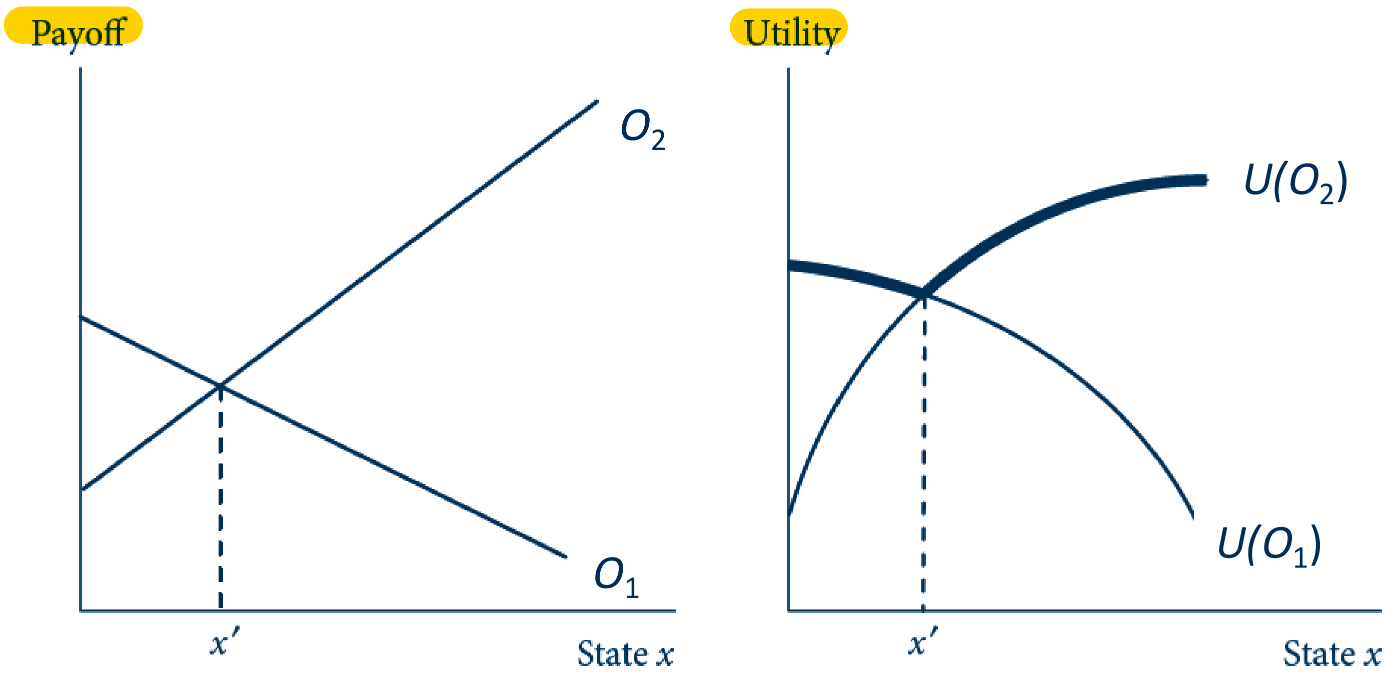
Flexibility

- Model of real options
  - Let x embody all the uncertainty in the economic environment
  - The individual has some number, $i = 1,\ldots,n$, of choices currently available
  - $O_i(x)$ = payoffs provided by choice i
    - (x) allows each choice to provide a different pattern of returns depending on how the future turns out

Flexibility

- Model of real options
  - No flexibility
    - Choose the single alternative that is best on average
    - Expected utility from this choice: $\max\{E[U(O_1)], \ldots, E[U(O_n)]\}$
  - Flexibility
    - Choose the best alternative
    - Expected utility: $E\{\max[U(O_1), \ldots, U(O_n)]\}$

The Nature of a Real Option



(a) Payoffs from alternatives

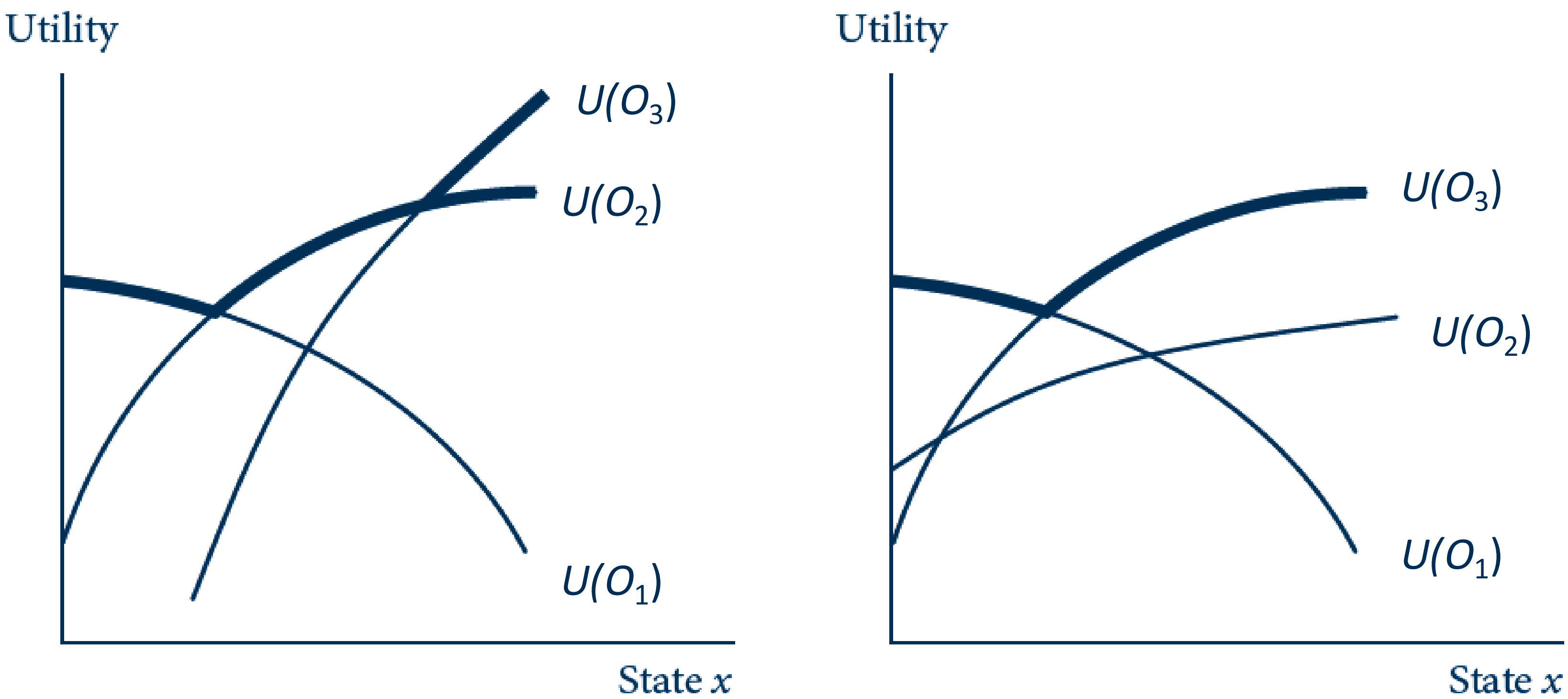
(b) Utilities from alternatives

- Panel (a) shows the payoffs and panel (b) shows the utilities provided by two alternatives across states of the world (x).
- If the decision has to be made upfront, the individual chooses the single curve having the highest expected utility.
- If the real option to make either decision can be preserved until later, the individual can obtain the expected utility of the upper envelope of the curves, shown in bold.

Flexibility

- More options are better (generally)
  - Options give the holder the right—but not the obligation—to choose them

More Options Cannot Make the Individual Decision-Maker Worse Off



(a) Additional valuable option

(b) Additional worthless option

- The addition of a third alternative to the two drawn in the previous figure is valuable in (a) because it shifts the upper envelope (shown in bold) of utilities up.
- The new alternative is worthless in (b) because it does not shift the upper envelope, but the individual is not worse off for having it.

Flexibility

- Computing the value of a real option
  - Let F be the fee that has to be paid
    - For the ability to choose the best alternative *after* x has been realized instead of *before*

  - The individual would be willing to pay the fee as long as:

$E\{max[U(O_1(x)-F),\ldots,U(O_n(x)-F)]\} \geq max\{E[U(O_1(x)]\ ,\ldots,\ E[U(O_n(x)]\}$

Practice example: Value of a Flexible-Fuel Car

- $O_1(x) = 1-x$
  - The payoff from a fossil-fuel–only car
- $O_2(x) = x$
  - The payoff from a electric-only car

- State of the world, x
  - Reflects the relative importance of electricity compared with fossil fuels over the car's lifespan
  - Random variable, uniformly distributed between 0 and 1

Value of a Flexible-Fuel Car

- Probability density function (PDF) is f(x) = 1
  - When the uniform random variable ranges between 0 and 1

- Suppose first that the car buyer is risk neutral
  - Utility level = payoff level
  - Forced to choose a electric vehicle
  - Expected utility:

$$E[O_2] = \int_0^1 O_2(x) f(x) dx = \frac{1}{2}$$

$$\text{with } f(x) = 1$$

Value of a Flexible-Fuel Car

- Risk neutrality
  - Flexible-fuel car is available
  - Buyer: either $O_1(x)$ or $O_2(x)$, whichever is higher under the latter circumstances
  - Expected utility:

$$E[\max(O_1, O_2)] = \int_0^1 \max(1-x, x) f(x) dx =$$

$$= \int_0^{1/2} (1-x) dx + \int_{1/2}^1 x dx = 3/4$$

  - This individual is willing to pay 0.25 for the flexible-fuel car.

Value of a Flexible-Fuel Car

- Risk aversion, $U(x)=\sqrt{x}$
  - Expected utility from an electric vehicle:

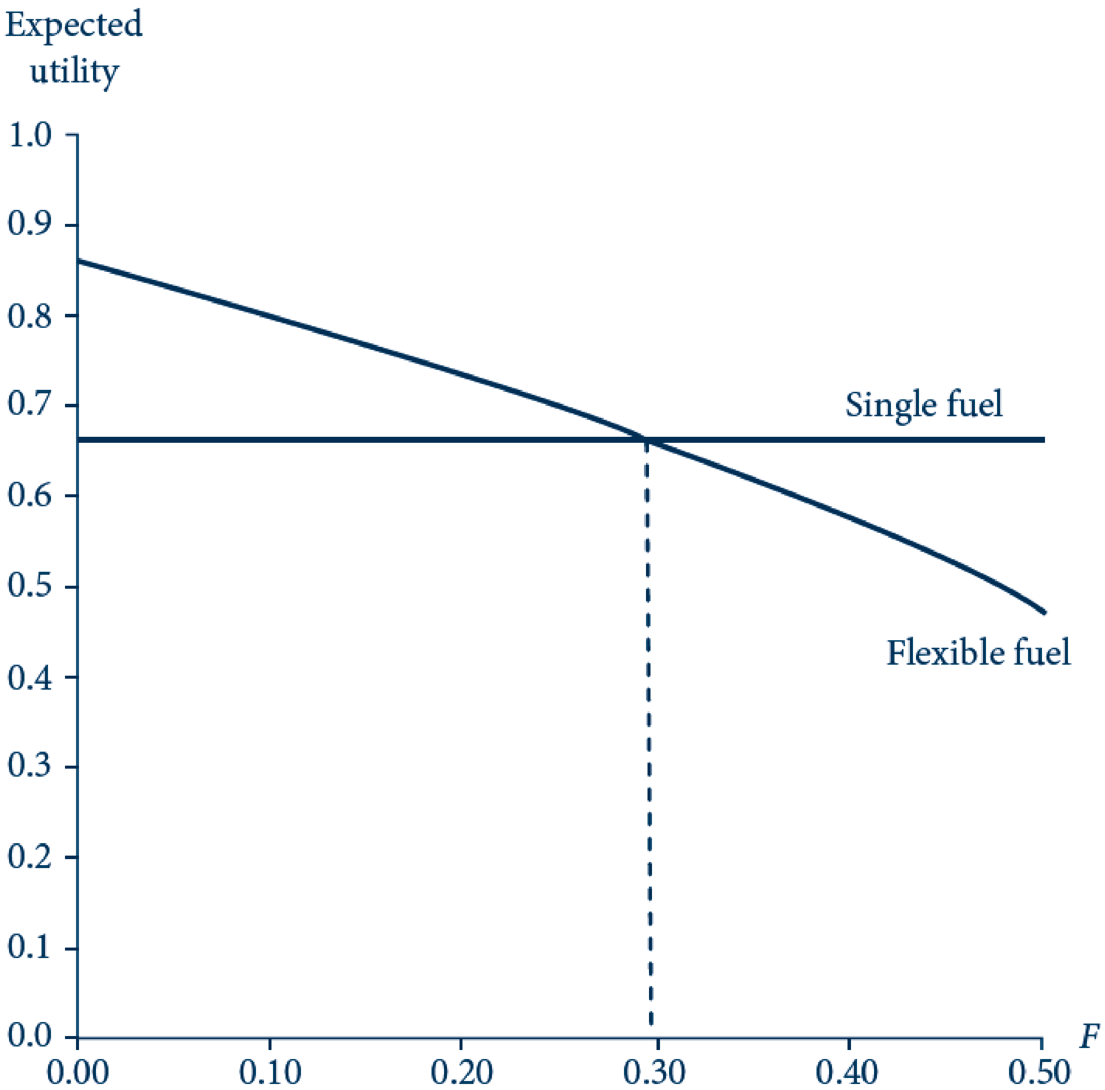
$$E[U(O_2)]=\int_0^1 \sqrt{O_2(x)}\, f(x)dx=\int_0^1 x^{1/2}dx=2/3$$

  - Expected utility from a fossil-fuel car; $E[U(O_1)]=2/3$
  - Expected utility from a flexible-fuel car that costs F more than a single-fuel car:

$$E\{\max[U(O_1(x)-F),U(O_2(x)-F)]\}=$$

$$=\int_0^1 \max(\sqrt{1-x-F},\sqrt{x-F})\, f(x)dx=\int_0^{1/2}\sqrt{1-x-F}\,dx+\int_{1/2}^1\sqrt{x-F}\,dx$$

Graphical Method for Computing the Premium for a Flexible-Fuel Car



To find the maximum premium F that the risk-averse buyer would be willing to pay for the flexible-fuel car, we plot the expected utility from a single-fuel car and from the flexible-fuel car, and see the value of F where the curves cross.

Value of a Flexible-Fuel Car

- we see that this value of F is slightly less than 0.3 (0.294 to be more precise).

- Therefore, the risk-averse buyer is willing to pay a premium of 0.294 for the flexible-fuel car, which is also the option value of this type of car.

- Scaling up by $10,000 for more realistic monetary values, the price premium would be $2,940.
- This is $440 more than the risk-neutral buyer was willing to pay. Thus, the option value is greater in this case for the risk-averse buyer.