Intermediate Macroeconomics: Problem Set 2 Solution

1 Can I Borrow from You? (Midterm 2023)

Consider the following infinite-period model with two types of agents. Both types have the same utility function over their lifetime consumption:

$$U^A = \sum_{t=0}^{\infty} \beta^t u(c_t^A), \quad U^B = \sum_{t=0}^{\infty} \beta^t u(c_t^B)$$

where U^i, c_t^i represents the type i agents' utility and consumption, respectively. The difference between the two types is that:

- Type A agents earn real income y in odd periods (i.e. $t = 1, 3, 5, \ldots$), but no income in even periods (i.e. $t = 0, 2, 4, \ldots$).
- Type B agents are just the opposite, receiving income y in even periods and no income in odd periods.

Agents are allowed to save/borrow with real interest rate r, which they take as given. **Assume there are equal numbers of type A and type B agents**, and their utility function $u(\cdot)$ satisfies $u' > 0, u'' < 0$.

Note that the budget constraint for type A agents in period $t = 0$ is:

$$c_0^A + b_0^A = 0,$$

where b_0^A represents type A agents' savings in period 0. Similarly, the budget constraint for type B agents in period $t = 0$ is:

$$c_0^B + b_0^B = y.$$

Questions:

a. (4 points) Based on the information above, write down type A and type B agents' budget constraints in $t = 1, 2$.

b. (4 points) Write down the intertemporal budget constraints for type A and type B agents, which should take the following form:

$$\sum_{t=0}^{\infty} \frac{c_t^i}{(1+r)^t} = PI_i,$$

where PI_i represents type i agents' permanent income, and should be expressed as a function of y and r. Show that when $r > 0$, type B agents have higher permanent income than type A agents.

(Hint: you may find the following formula helpful.)

$$1 + \alpha + \alpha^2 + \alpha^3 + \cdots = \frac{1}{1-\alpha}, \quad 0 < \alpha < 1$$

c. (4 points) Define type A agent's optimization problem, and write down the corresponding Lagrangian function and first order conditions.

d. (3 points) Using the first order condition with respect to c_t^A, derive the Euler Equation:

$$\frac{u'(c_t^A)}{\beta u'(c_{t+1}^A)} = ?$$

From part e onwards, assume the utility function takes the logarithmic form: $u(c_t^i) = \ln c_t^i$.

e. (3 points) Using your answers in b) and d), solve c_0^A as a function of y, r and β.

f. (3 points) Write down the market clearing conditions in this economy.

g. (4 points, **Difficult**) Calculate the interest rate level r^* at the competitive equilibrium, when both types of agents optimize their utility functions subject to budget constraints, and all markets clear.

h. (5 points) Show that in a steady state with $c_{t+1}^A = c_t^A = c^A$ and $c_{t+1}^B = c_t^B = c^B$, type A agents would occasionally borrow from type B agents, but type B agents never borrow from type A agents. Calculate the amount that type A agents borrow in both odd and even periods.

Solution:

a. Type A:

$$c_1^A + b_1^A = b_0^A(1+r) + y$$
$$c_2^A + b_2^A = b_1^A(1+r)$$

Type B:

$$c_1^B + b_1^B = b_0^B(1+r)$$
$$c_2^B + b_2^B = b_1^B(1+r) + y$$

b. Type A:

$$\sum_{t=0}^{\infty} \frac{c_t^A}{(1+r)^t} = PI_A = y\frac{(1+r)}{(1+r)^2 - 1}$$

Type B:

$$\sum_{t=0}^{\infty} \frac{c_t^B}{(1+r)^t} = PI_B = y\frac{(1+r)^2}{(1+r)^2-1}$$

When $r > 0$, $PI_B > PI_A$.

c. Type A agent's optimization problem:

$$\max_{\{c_t^A\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t^A)$$
$$s.t. \sum_{t=0}^{\infty} \frac{c_t^A}{(1+r)^t} = y\frac{(1+r)}{(1+r)^2-1}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t^A) + \lambda \left(y\frac{(1+r)}{(1+r)^2-1} - \sum_{t=0}^{\infty} \frac{c_t^A}{(1+r)^t} \right)$$

First order conditions:

$$[c_t^A] \;\; \beta^t u'(c_t^A) - \frac{\lambda}{(1+r)^t} = 0$$

d. From the FOC Condition:

$$\frac{u'(c_t^A)}{u'(c_{t+1}^A)} = \beta(1+r)$$

e. Given the logarithmic form of the utility function,

$$\frac{c_{t+1}^A}{c_t^A} = \beta(1+r)$$

i.e.,

$$c_t^A = c_0^A \beta^t (1+r)^t$$

By the B.C. of Type A, we get:

$$\sum_{t=0}^{\infty} \beta^t c_0^A = y\frac{(1+r)}{(1+r)^2-1}$$

$$c_0^A = y\frac{(1+r)(1-\beta)}{(1+r)^2-1}$$

f. Goods market clearing:

$$c_t^A + c_t^B = y \quad \forall t$$

Bond market clearing:

$$b_t^A + b_t^B = 0$$

g. We can calculate that

$$c_0^B = y\frac{(1+r)^2(1-\beta)}{(1+r)^2-1}$$

From the clearinig condition of the goods market:

$$y\frac{(1+r)(1-\beta)}{(1+r)^2-1} + y\frac{(1+r)^2(1-\beta)}{(1+r)^2-1} = y$$

By solving the equation:

$$\begin{aligned}
(1+r)(1-\beta)+(1+r)^2(1-\beta) &= (1+r)^2-1 \\
(1+r)(1-\beta)(2+r) &= r(r+2) \\
(1+r)(1-\beta) &= r \\
\beta &= \frac{1}{(1+r)} \\
r^* &= \frac{1}{\beta}-1
\end{aligned}$$

h. This part does not need a correct answer from last part.

In a steady state, $\beta(1+r)=1$, plug in the expression for c_0^A

$$c_0^A = y\frac{(1+r)(1-\beta)}{(1+r)^2-1} = \frac{\beta}{1+\beta}y$$

Therefore

$$b_0^A = 0 - c_0^A = -\frac{\beta}{1+\beta}y$$

Similarly we can find

$$c_0^B = \frac{1}{1+\beta}y, \;\; b_0^B = \frac{\beta}{1+\beta}y$$

From the $t=1$ budget constraint, plug in b_0^A, we have

$$\begin{aligned}
b_1^A &= b_0^A(1+r^*)+y-c_1^A \\
&= -\frac{\beta}{1+\beta}y\frac{1}{\beta}+y-\frac{\beta}{1+\beta}y \\
&= 0
\end{aligned}$$

From the bond market clearing condition: $b_1^B = 0$

All subsequent period are similar:

$$b_t^A = \begin{cases} -\frac{\beta}{1+\beta}y & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}$$

In other words, type A agents borrow $\frac{\beta}{1+\beta}y$ from type B agents in even periods and repay $\frac{1}{1+\beta}y$ in odd periods; type B agents lend $\frac{\beta}{1+\beta}y$ to type A agents in even periods and collect the loan (plus interest) $\frac{1}{1+\beta}y$ in odd periods.

2 Cash-In-Advance Model with Two Goods (Midterm 2022)

Consider the following cash-in-advance model. There are two consumption goods, C_t and F_t, in the economy. The former good C_t can be paid by any means, but the latter good F_t can only be paid in cash. Both goods have identical prices P_t in each period. The good F_t is subject to the following CIA constraint:

$$P_t F_t \leq M_{t-1}.$$

There are no other saving technology except money in this economy. The representative agent is endowed with real income Y_t each period, and maximizes her utility function by choosing C_t, F_t and the real money balance m_t:

$$\max_{\{C_t, F_t, m_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t [u(C_t) + u(F_t)]$$
$$\text{s.t.} \quad C_t + F_t + m_t = Y_t + m_{t-1}\frac{P_{t-1}}{P_t} + \frac{\tau_t}{P_t}$$
$$[\text{+CIA Constraint}]$$

The monetary authority sets the following policy:

$$M_t = M_{t-1} + \tau_t = (1+\mu)M_{t-1},$$

Where μ is the money growth rate, and $\tau_t = \mu M_{t-1}$ is the additional cash printed each period. Finally, define the inflation rate as

$$\pi_t = \frac{P_{t+1}}{P_t} - 1.$$

Questions:

a. Write the CIA constraint in real terms (by dividing P_t on both sides), and form the representative agent's Lagrangian function. (For Lagrange multipliers, please use $\lambda_{1,t}$ for the budget constraint and $\lambda_{2,t}$ for the CIA constraint.)

b. Find the first order conditions with respect to $[C_t]$, $[F_t]$, and $[m_t]$.

c. Combine the three first-order conditions into one equation, which does not contain the Lagrange multipliers.

d. When the CIA constraint is binding, compare the marginal utilities $u'(F_t)$ and $u'(C_t)$. Explain why they are (or are not) equal.

e. Suppose the real income is $Y_t = Y$ in each period, and the economy is at a steady state:

$$C_{t+1} = C_t = C^*, \quad F_{t+1} = F_t = F^*$$

Using the answers from parts c and d, what do you think should be the optimal money growth rate μ? Explain.

Solution:

a. $F_t \le \frac{M_{t-1}}{P_t}$ or $F_t \le m_{t-1}\frac{P_{t-1}}{P_t}$ 2 分

b.

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t [u(C_t) + u(F_t)] + \sum_{t=0}^{\infty} \lambda_{1,t}[Y_t + m_{t-1}\frac{P_{t-1}}{P_t} + \frac{\tau_t}{P_t} - C_t - F_t - m_t] + \sum_{t=0}^{\infty} \lambda_{2,t}[m_{t-1}\frac{P_{t-1}}{P_t} - F_t]$$

$$\begin{aligned}
[C_t] \quad & \beta^t u'(C_t) - \lambda_{1,t} = 0 \\
[F_t] \quad & \beta^t u'(F_t) - \lambda_{1,t} - \lambda_{2,t} = 0 \\
[m_t] \quad & -\lambda_{1,t} + (\lambda_{1,t+1} + \lambda_{2,t+1})\frac{P_t}{P_{t+1}} = 0
\end{aligned}$$

c.

$$u'(C_t) = \beta u'(F_{t+1})\frac{P_t}{P_{t+1}}$$

d. From the FOC condition, we find that:

$$\begin{aligned}
\beta^t u'(C_t) &= \lambda_{1,t} \\
\beta^t u'(F_t) &= \lambda_{1,t} + \lambda_{2,t}
\end{aligned}$$

Since now the the CIA constraint is binding, we know that $\lambda_{2,t} > 0$, therefore it is natural to get:

$$u'(F_t) > u'(C_t)$$

e. From the FOC condition and at the steady state, we have:

$$\begin{aligned}
u'(c^*) &= \beta u'(F^*)\frac{P_t}{P_{t+1}} \\
u'(c^*) &= \beta u'(F^*)\frac{1}{1+\mu}
\end{aligned}$$

In the optimal state, we do not want the CIA constraint to distort the agent's choices, therefore, the goverment needs to make sure that the CIA constraint is not binding. In this case,

$$u'(F_t) = u'(C_t)$$

Combining the equations, $\mu = \beta - 1$