Intermediate Macro: Lecture 22

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Review

Last time we studied **dynamic programming** in finite horizon. We discussed the following topics:

- Define the value function;
- Rewrite sequence problem into Bellman Equation;
- Take first order condition, and use backwards induction to solve value function.

This time we will study dynamic programming in **infinite horizon**.

A Traveler's Guide to Dynamic Programming

In the following slides, I will present a "cookbook" style approach to dynamic programming problems.

- **Step 1: Define the sequence problem.**
  May be difficult in some cases (like the McCall model). But whenever possible, do this step first.
- **Step 2: Formulate the Bellman Equation.**
- **Step 3: Check mathematical conditions.**

Cookbook-2

- **Step 4: Solve for the value and policy functions**
- **Step 5: Characterize the optimality conditions, policy function and Euler Equation**
- **Step 6: Characterize the Steady State**

Step 1: Sequence Problem

The generic sequence problem looks like this:

$$\max_{\{y_t, x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, y_t)$$

$$\text{s.t.} \quad y_t \in \Gamma(x_t)$$

$$x_{t+1} = h(x_t, y_t)$$

$$x_0 \text{ given}$$

Where x_t is the **state variable(s)** and y_t is the **control variable(s)**. $\beta^t F(x_t, y_t)$ is the **objective function**, while $F(x_t, y_t)$ is the **instantaneous return** function. $\Gamma(x_t)$ is the **feasibility set**, and $h(x_t, y_t)$ is the **law of motion** for the state variable.

Example: Neoclassical Growth Model

$$
\begin{aligned}
\max_{\{c_t, i_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\
\text{s.t.} \quad & c_t + i_t \leq f(k_t) \\
& k_{t+1} = (1-\delta)k_t + i_t \\
& k_0 \text{ given}
\end{aligned}
$$

Questions:

- What are the state/choice variables?
- What is the feasibility set?

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Questions:

- What are the state/choice variables?
- What is the feasibility set?

Answer:

- State: k_t; Control: k_{t+1}, c_t, i_t
- Feasibility set:

$$\Gamma(k_t) = \{(c_t, i_t) : c_t + i_t \leq f(k_t)\}$$

Step 1: Sequence Problem

- It is often useful to write the sequence problem such that the only control is the next period's state variable. The general sequence problem will instead look like

$$\begin{aligned} \max_{\{x_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \\ \text{s.t.} \quad & x_{t+1} \in \Gamma(x_t) \\ & x_0 \text{ given} \end{aligned}$$

- Example: NCG model:

$$\boxed{\begin{aligned} \max_{\{k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u\left(f(k_t) - k_{t+1} + (1-\delta)k_t\right) \\ \text{s.t.} \quad & k_{t+1} \in [(1-\delta)k_t, (1-\delta)k_t + f(k_t)] \\ & k_0 \text{ given} \end{aligned}}$$

Step 2: Bellman Equation

- Define value function the same way (dropping the constraint for simplicity):

$$V(x_t) = \max_{\{x_{\tau+1}\}_{\tau=t}^{\infty}} \sum_{\tau=t}^{\infty} \beta^{\tau-t} F(x_\tau, x_{\tau+1})$$

  Value function at t: the total discounted value of instantaneous return function $F(\cdot,\cdot)$ from t to ∞.

- Using Principle of Optimality, we can rewrite this value function into:

$$V(x_t) = \max_{x_{t+1}} \left\{ F(x_t, x_{t+1}) + \beta \left(\max_{\{x_{t+1}\}_{\tau=t+1}^{\infty}} \sum_{\tau=t+1}^{\infty} \beta^{\tau-(t+1)} F(x_\tau, x_{\tau+1}) \right) \right\}$$

Step 2: Bellman Equation

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- By the same definition of value function, we get the Bellman Equation:

$$\underbrace{V(x_t)}_{\text{Value in } t} = \max_{x_{t+1} \in \Gamma(x_t)} \underbrace{F(x_t, x_{t+1})}_{\text{Inst.return from } t \text{ to } t+1} + \underbrace{\beta V(x_{t+1})}_{\text{discounted value in } t+1}$$

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- Example: NCG model:

$$V(k_t) = \max_{k_{t+1} \in \Gamma(k_t)} u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta V(k_{t+1})$$

Where $\Gamma(k_t) = [(1-\delta)k_t, (1-\delta)k_t + f(k_t)]$

Step 3: Math Conditions

To guarantee existence and uniqueness of a solution to the Bellman Equation, we should then check the following conditions:

1. $\beta \in (0, 1)$
   This condition is usually satisfied by assumption.
2. $\Gamma(x_t)$ **is non-empty for all** x_t**.**
   Meaning: there exist at least one feasible choice, no matter what the state is.
3. $F(\cdot)$ **is bounded on the state space.**
   Equivalent condition: if the state space is **compact** (closed and bounded), then $F(\cdot)$ is **bounded** on the state space.

Step 3: Math Conditions

Again, take the Neoclassical Growth model as an example:

- $\beta \in (0, 1)$. This is satisfied by assumption.
- $\Gamma(k_t)$ is non-empty. This is true because $k_{t+1]} = (1 - \delta)k_t$ is always feasible, when investment is zero.
- $F(\cdot)$ is bounded on the state space. $u(\cdot)$ and $f(\cdot)$ are not bounded themselves. But there is an $\bar{k}$ such that

  $$f(\bar{k}) = \delta \bar{k}$$

  And capital k_t is always in the range $[0, \bar{k}]$, which is a compact set. (Why?)

Step 3: Math Conditions

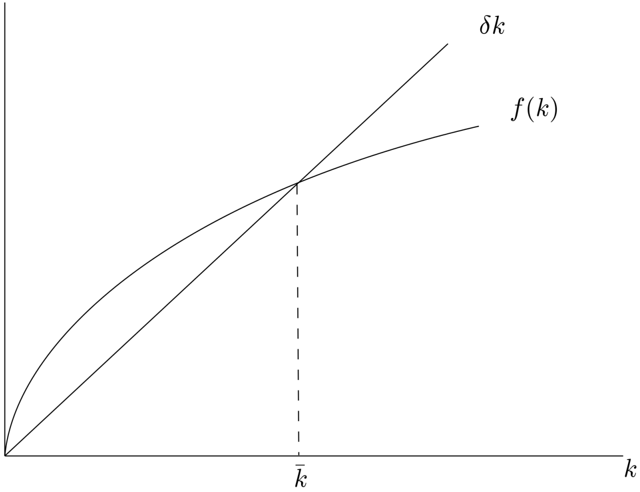

Figure: Bounding the state space in the Neoclassical Growth Model

Step 4: Solve for the value and policy functions.

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  1. Guess and verify
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Example: NCG model with log utility $u(c) = \log c$, Cobb-Douglas production $f(k) = k^\alpha$ and full depreciation $\delta = 1$. The Bellman equation for this model is

$$V(k) = \max_{k'} \ \log(k^\alpha - k') + \beta V(k')$$

Note that I **dropped time subscripts**, and used k and k' to denote kapital in current and next period.

Step 4: Solve for the value and policy functions.

$$V(k) = \max_{k'} \ \log(k^\alpha - k') + \beta V(k')$$

- **Method 1: Guess and Verify**
  Guess that

$$V(k) = a + b \log k$$

- Substitute our guess for the value function into the Bellman equation

$$a + b \log k = \max_{k' \in [0, k^\alpha]} \log(k^\alpha - k') + \beta(a + b \log k')$$

Method 1: Guess and Verify

$$a + b \log k = \max_{k' \in [0, k^\alpha]} \log(k^\alpha - k') + \beta(a + b \log k')$$

- Take first order condition with respect to k':

$$[k'] : \quad \frac{-1}{k^\alpha - k'} + \beta \frac{b}{k'} = 0$$

- Solving for k' gives the policy function

$$k' = g(k) = \frac{\beta b}{1 + \beta b} k^\alpha$$

Plug the policy function back to the value function to see if our guess is correct.

Method 1: Guess and Verify

$$k' = g(k) = \frac{\beta b}{1+\beta b} k^{\alpha}$$

- The Bellman Equation becomes:

$$a + b \log k = \log\left(k^{\alpha} - \frac{\beta b}{1+\beta b}\right) + \beta\left(a + b\log\left(\frac{\beta b}{1+\beta b}k\right)\right)$$

- Rearrange terms:

$$a + b\log k = \underbrace{\beta a + \log\left(\frac{1}{1+\beta b}\right) + \beta b \log\left(\frac{\beta b}{1+\beta b}\right)}_{=a} + \underbrace{(\alpha + \beta b \alpha)}_{=b} \log k$$

Method 1: Guess and Verify

$$a + b\log k = \underbrace{\beta a + \log\left(\frac{1}{1+\beta b}\right) + \beta b \log\left(\frac{\beta b}{1+\beta b}\right)}_{=a} + \underbrace{(\alpha + \beta b \alpha)}_{=b} \log k$$

- The right hand side has the form we are looking for.

$$b = (\alpha + \beta b \alpha)$$
$$\Rightarrow b = \frac{\alpha}{1-\beta\alpha}$$

and

$$a = \beta a + \log\left(\frac{1}{1+\beta b}\right) + \beta b \log\left(\frac{\beta b}{1+\beta b}\right)$$
$$\Rightarrow a = \frac{1}{1-\beta}\left[\log(1-\beta\alpha) + \frac{\beta\alpha}{1-\beta\alpha}\log(\beta\alpha)\right]$$

- So we've verified and also solved the value function.

Method 1: Guess and Verify

- Finally, the policy function can is equal to

$$g(k) = \frac{\beta b}{1+\beta b} k^{\alpha} = (\beta\alpha) k^{\alpha}$$

By plugging in

$$b = \frac{\alpha}{1-\beta\alpha}$$

Questions about Guess and Verify

- How can we reliably guess the correct form of value function?
- Usually in exercises or problem sets, all you need to know is how to verify a guess.
- When doing research, however, researchers generally use another method called **Value Function Iteration**, because the value function gets very complicated and become difficult to guess.

Method 2: Value Function Iteration

- The intuition is as follows. Starting with any bounded function $V^0(x)$ (for example, $V^0(x) = 0$ for all x). Then iteratively apply the operator

$$V^{n+1}(x) = TV^n(x) = \max_{y \in \Gamma(x)} F(x, y) + \beta V^n(y)$$

- Stop the iteration when the two functions V^{n+1} and V^n are sufficiently close.
- The true value function V satisfies $V = TV$, where T is the operator described above.
- This is an extremely powerful tool that numerically approximate the solution, especially in cases when analytical solutions are difficult.

Step 5: Optimality Condition, Policy Function and Euler Equation

- For the general form of Bellman Equation:

$$V(x_t) = \max_{x_{t+1}\in\Gamma(x_t)} F(x_t, x_{t+1}) + \beta V(x_{t+1})$$

- The first order condition is

$$[x_{t+1}]: \quad F_2(x_t, x_{t+1}) + \beta V'(x_{t+1}) = 0$$

- And the envelope condition is

$$V'(x_t) = F_1(x_t, x_{t+1})$$

- Putting the two together gives us the Euler equation

$$F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$$

Example: The NCG model

- Consider the NCG model

$$V(k_t) = \max_{k_{t+1} \in \Gamma(k_t)} u(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta V(k_{t+1})$$

- The first order condition is

$$[k_{t+1}]: \quad -u'(f(k_t) + (1-\delta)k_t - k_{t+1}) + \beta V'(k_{t+1}) = 0$$

- and the envelope condition is

$$V'(k_t) = u'(f(k_t) + (1-\delta)k_t - k_{t+1})(f'(k_t) + 1 - \delta)$$

- Putting them together and substitute in $c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$ to get

$$u'(c_t) = \beta u'(c_{t+1})(f'(k_{t+1}) + 1 - \delta)$$

- Which is the usual Euler Equation.

Step 6: Characterize the Steady State

Recall the Euler Equation with general form:

$$F_2(x_t, x_{t+1}) + \beta F_1(x_{t+1}, x_{t+2}) = 0$$

- Solve for the steady state by setting $x_t = x_{t+1} = x_{t+2}$:

$$F_2(x^*, x^*) + \beta F_1(x^*, x^*) = 0$$

- Example: For NCG model, at the steady state, the Euler equation yields

$$\beta(f'(k^*) + 1 - \delta) = 1$$

  From which we can solve the steady state k^* as usual.

Steps 7 and 8?

- After solving the steady state, we can study the transition dynamics of this system by log-linearizing the difference equations.
- The idea is that we look at the how does a **percentage deviation from steady state** for one variable affect another.
- I will omit the discussion here, but instead include these parts as extra credits for the empirical project. (Will be posted on Thursday and due the last day of class.)

Next time

Next time I'll discuss the following topics:

- Stochastic Dynamic Programming: **having a random variable as state.**
- We'll return to the McCall model of labor search that I introduced at the beginning!