Intermediate Microeconomics
Spring 2025

Week 15b: Introduction to game theory

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Basic Concepts

- Players: each decision maker
- Strategy: each course of action open to a player
  - It may be a simple action or a complex plan of action
  - S_i is the set of strategies open to player i
  - s_i is the strategy chosen by player i, $s_i \in S_i$

Basic Concepts

- Payoffs
  - Are measured in levels of utility obtained by the players
  - Players are assumed to prefer higher payoffs to lower ones
    - $U_1(s_1,s_2)$ denotes player 1's payoff assuming she follows s_1 and player 2 follows s_2
    - $U_2(s_1, s_2)$ would be player 2's payoff under the same circumstances

FIGURE 8.1 Normal Form for the Prisoners' Dilemma

		Suspect 2	
		Fink	Silent
Suspect 1	Fink	$U_1 = 1, U_2 = 1$	$U_1 = 3, U_2 = 0$
	Silent	$U_1 = 0, U_2 = 3$	$U_1 = 2, U_2 = 2$

Two suspects simultaneously choose to fink or be silent. Rows refer to player 1's actions and columns to player 2's. Each box is an outcome; the first entry is player 1's payoff and the second player 2's in that outcome.

Nash Equilibrium

- Nash equilibrium
  - Involves strategic choices that, once made, provide no incentives for players to alter their behavior
  - Best choice for each player given the other players' equilibrium strategies

Nash Equilibrium

- Best response
  - s_i is the best response for player *i* to rivals' strategies s_{-i}, denoted $s_i \in BR_i(s_{-i})$ if

$$u_i(s_i,s_{-i}) \geq u_i(s'_i,s_{-i}) \text{ for all } s'_i \in S_i$$

- A Nash equilibrium
  - Is a strategy profile $(s^*_1,s^*_2,\ldots s^*_n)$
  - Such that s^*_i is a best response to other players' equilibrium strategies, s^*_{-i} or

$$s^*_i \in BR_i(s^*_{-i})$$

Nash Equilibrium in Prisoners' Dilemma

- Finking is player 1's best response to player 2's finking
- If player 2 finks
  - If player 1 also finks, his payoff is 1
  - If player 1 is silent, his payoff is 0
    - Best response: fink
- Players are symmetric
  - The same logic applies for player 2
  - Best response for player 2: fink

FIGURE 8.2 Underlining Procedure in the Prisoners' Dilemma

		Suspect 2	
		Fink	Silent
Suspect 1	Fink	$\underline{U_1 = 1}$, $\underline{U_2 = 1}$	$\underline{U_1 = 3}$, $U_2 = 0$
	Silent	$U_1 = 0$, $\underline{U_2 = 3}$	$U_1 = 2$, $U_2 = 2$

The first step is to underline player 1's best responses. Player 1 prefers to fink if 2 finks, so we underline $U_1 = 1$ in the upper left box. Player 1 prefers to fink if 2 is silent, so we underline $U_1 = 3$ in the upper right box. The next step is to underline player 2's best responses. Player 2 prefers to fink if 1 finks, so we underline $U_2 = 1$ in the upper left box. Player 2 prefers to fink if 1 is silent, so we underline $U_2 = 3$ in the lower left box. The final step is to circle any box with both payoffs underlined, here showing the Nash equilibrium involves both finking.

Dominant Strategies

- Dominant strategy
  - A strategy that is a best response to any strategy the other players might choose
  - Finking is a dominant strategy for both players

$$s^*_i \in BR_i(s_{-i}) \text{ for all } s_{-i}$$

- When a dominant strategy exists, it is the unique Nash equilibrium

Battle of the Sexes

- A wife and husband may either go to the ballet or to a boxing match
  - Both prefer spending time together
  - The wife prefers ballet and the husband prefers boxing
- There are two Nash equilibria
  - Both going to the ballet
  - Both going to boxing
- There is no dominant strategy

FIGURE 8.4 Underlining Procedure in the Battle of the Sexes

		Player 2 (Husband)	
		Ballet	Boxing
Player 1 (Wife)	Ballet	2, 1	0, 0
	Boxing	0, 0	1, 2

The underlining procedure yields two Nash equilibria in pure strategies: both go to ballet and both go to boxing.

Mixed Strategies

- Pure strategy
  - When a player chooses one action of another with certainty
- Mixed strategies
  - Players may randomly select from several possible actions

Mixed Strategies

- Reasons for studying mixed strategies
  - Some games have no Nash equilibria in pure strategies but will have one in mixed strategies
  - Strategies involving randomization are familiar and natural in certain settings
  - It is possible to “purify” mixed strategies

Mixed Strategies

- Suppose that player i has a set of M possible actions, $A_i = \{a^1_i,\ldots a^m_i,\ldots,a^M_i\}$
  - A mixed strategy is a probability distribution over the M actions,

    $s_i = (\sigma^1_i,\ldots,\sigma^m_i,\ldots,\sigma^M_i)$
  - σ^m_i indicates the probability of player i playing action a^m_i
    - $0 \leq \sigma^m_i \leq 1$
    - $\sigma^1_i+\ldots+\sigma^m_i+\ldots+\sigma^M_i = 1$

Mixed Strategies

- Battle of the Sexes
  - $A_1 = A_2 = \{\text{ballet, boxing}\}$
  - A mixed strategy (σ, 1-σ)
    - σ = probability that the player chooses ballet
    - (1/3, 2/3): Player plays ballet with probability 1/3 and boxing with probability 2/3
    - (1/2, 1/2): Player is equally likely to play ballet or boxing
    - (1, 0): Player chooses ballet with certainty
    - (0, 1): Player chooses boxing with certainty

Mixed Strategies

- A pure strategy
  - Is a special case of a mixed strategy
  - Only one action is played with positive probability
- Strictly mixed strategies
  - Mixed strategies that involve two or more actions being played with positive probability

EXAMPLE 8.2 Expected Payoffs in the Battle of the Sexes

- Suppose the wife chooses mixed strategy (1/9,8/9) and the husband chooses (4/5,1/5)
- The wife's expected payoff is

$$U_1\left(\left(\frac{1}{9},\frac{8}{9}\right),\left(\frac{4}{5},\frac{1}{5}\right)\right)=\left(\frac{1}{9}\right)\left(\frac{4}{5}\right)U_1(\text{ballet,ballet})+\left(\frac{1}{9}\right)\left(\frac{1}{5}\right)U_1(\text{ballet,boxing})$$

$$+\left(\frac{8}{9}\right)\left(\frac{4}{5}\right)U_1(\text{boxing,ballet})+\left(\frac{8}{9}\right)\left(\frac{1}{5}\right)U_1(\text{boxing,boxing})$$

$$=\left(\frac{1}{9}\right)\left(\frac{4}{5}\right)(2)+\left(\frac{1}{9}\right)\left(\frac{1}{5}\right)(0)+\left(\frac{8}{9}\right)\left(\frac{4}{5}\right)(0)+\left(\frac{8}{9}\right)\left(\frac{1}{5}\right)(1)=\frac{16}{45}$$

EXAMPLE 8.2 Expected Payoffs in the Battle of the Sexes

- Suppose the wife chooses mixed strategy (w,1-w) and the husband chooses (h,1-h)
  - The wife plays ballet with probability w and the husband with probability h
- Her expected payoff becomes $U((w, 1-w), (h, 1-h))=$

$$(w)(h)U_1(\text{ballet,ballet}) + (w)(1-h)U_1(\text{ballet,boxing}) +$$

$$+ (1-w)(h)U_1(\text{boxing,ballet}) +$$

$$+ (1-w)(1-h)U_1(\text{boxing,boxing}) =$$

$$= (w)(h)(2) + (w)(1-h)(0) + (1-w)(h)(0) + (1-w)(1-h)(1)$$

$$= 1 - h - w + 3hw$$

EXAMPLE 8.3 Mixed-Strategy Nash Equilibrium in the Battle of the Sexes

- General mixed strategy: the wife chooses (w,1-w) and the husband chooses (h,1-h)
- Wife's expected payoff:

  $U_1((w, 1-w), (h, 1-h)) = 1 - h - w + 3hw = (3h-1)w + 1 - h$

- The wife's best response depends on *h*
  - If $h < 1/3$, she should set $w = 0$
  - If $h > 1/3$, she should set $w = 1$
  - If $h = 1/3$, her expected payoff is the same no matter what value of w she chooses

EXAMPLE 8.3 Mixed-Strategy Nash Equilibrium in the Battle of the Sexes

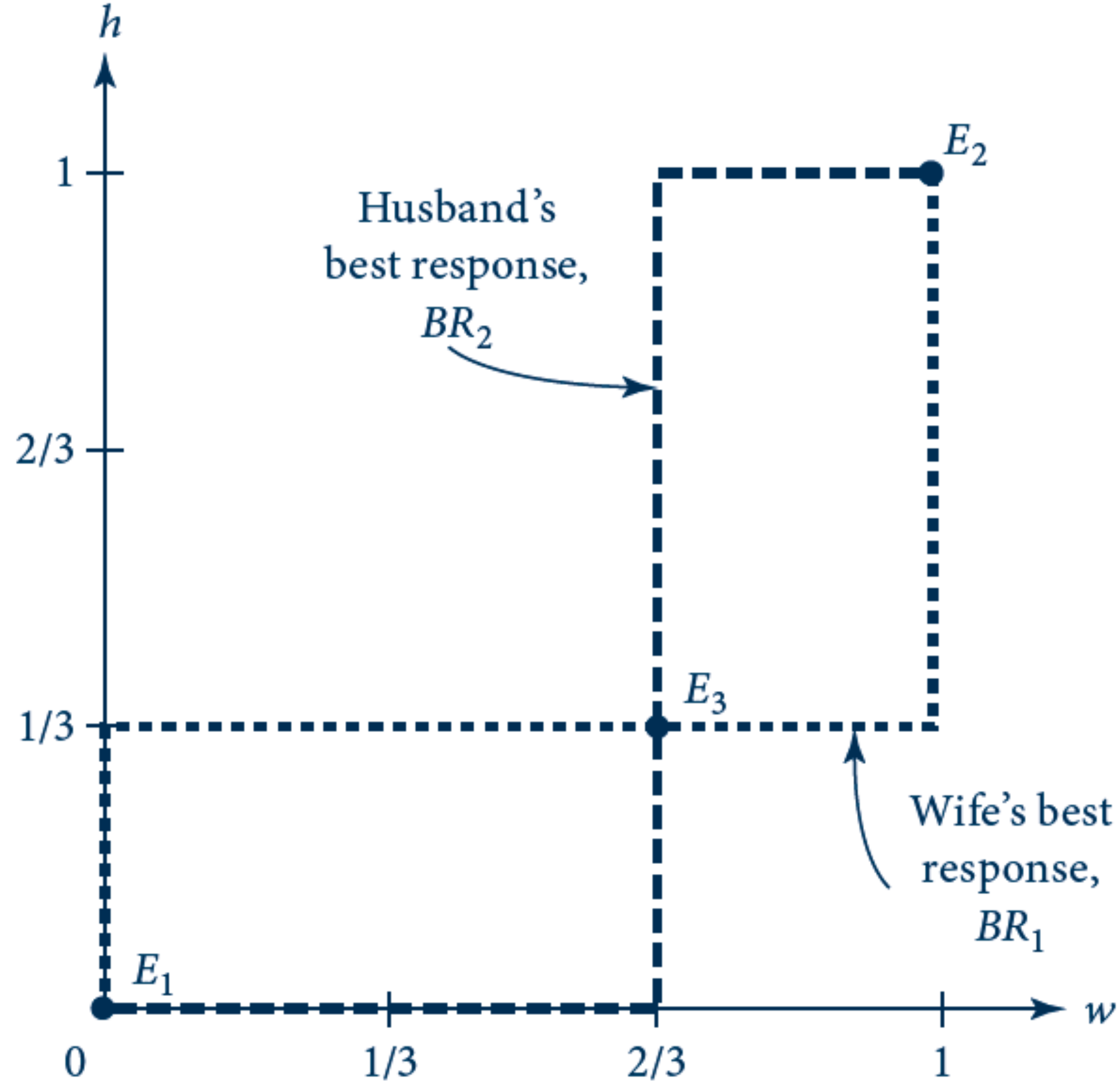
- Husband's expected payoff:

  $U_2((h, 1\text{-}h), (w, 1\text{-}w)) = 2 – 2h – 2w + 3hw = (3w\text{-}2)h + 2\text{-}2w$

- The husband's expected payoff is maximized
  – when $w < 2/3$, he should set $h = 0$
  – when $w > 2/3$, he should set $h = 1$
  – when $w = 2/3$, his expected payoff is the same no matter what value of h he chooses

FIGURE 8.6 Nash Equilibria in Mixed Strategies in the Battle of the Sexes

Ballet is chosen by the wife with probability w and by the husband with probability h. Players' best responses are graphed on the same set of axes. The three intersection points E_1, E_2, and E_3 are Nash equilibria. The Nash equilibrium in strictly mixed strategies, E_3, is $w^*=2/3$ and $h^*=1/3$.



Existence Of Equilibrium

- Nash proved the existence of a Nash equilibrium in all finite games
  - The existence theorem does not guarantee the existence of a pure-strategy Nash equilibrium
  - It does guarantee that, if a pure-strategy Nash equilibrium does not exist, a mixed-strategy Nash equilibrium does

Continuum of Actions

- Some settings
  - Are more realistically modeled via a continuous range of actions
- Using calculus to solve for Nash equilibria
  - Makes it possible to analyze how the equilibrium actions vary with changes in underlying parameters

Tragedy of the Commons

- Solve for the Nash equilibrium when the game involves a continuum of actions
  - Write down the payoff for each player as a function of all players' actions
  - Compute the first-order condition associated with each player's payoff maximum
    - Equation - can be rearranged into the best response of each player as a function of all other players' actions
    - Solve the system of N equations for the N unknown equilibrium actions

EXAMPLE 8.4 Tragedy of the Commons

- The "Tragedy of the Commons"
  - Describes the environmental problem of overuse that arises when scarce resources are treated as common property
  - Two herders decide how many sheep to graze on the village commons
  - The commons is quite small and can rapidly succumb to overgrazing
  - q_i = the number of sheep chosen by herder i
  - Per-sheep value of grazing on the commons is

$$v(q_1, q_2) = 120 - (q_1 + q_2)$$

EXAMPLE 8.4 Tragedy of the Commons

- The "Tragedy of the Commons"
  - The normal form is a listing of the herders' payoff functions

$$U_1(q_1,q_2) = q_1 v(q_1,q_2) = q_1(120 - q_1 - q_2)$$

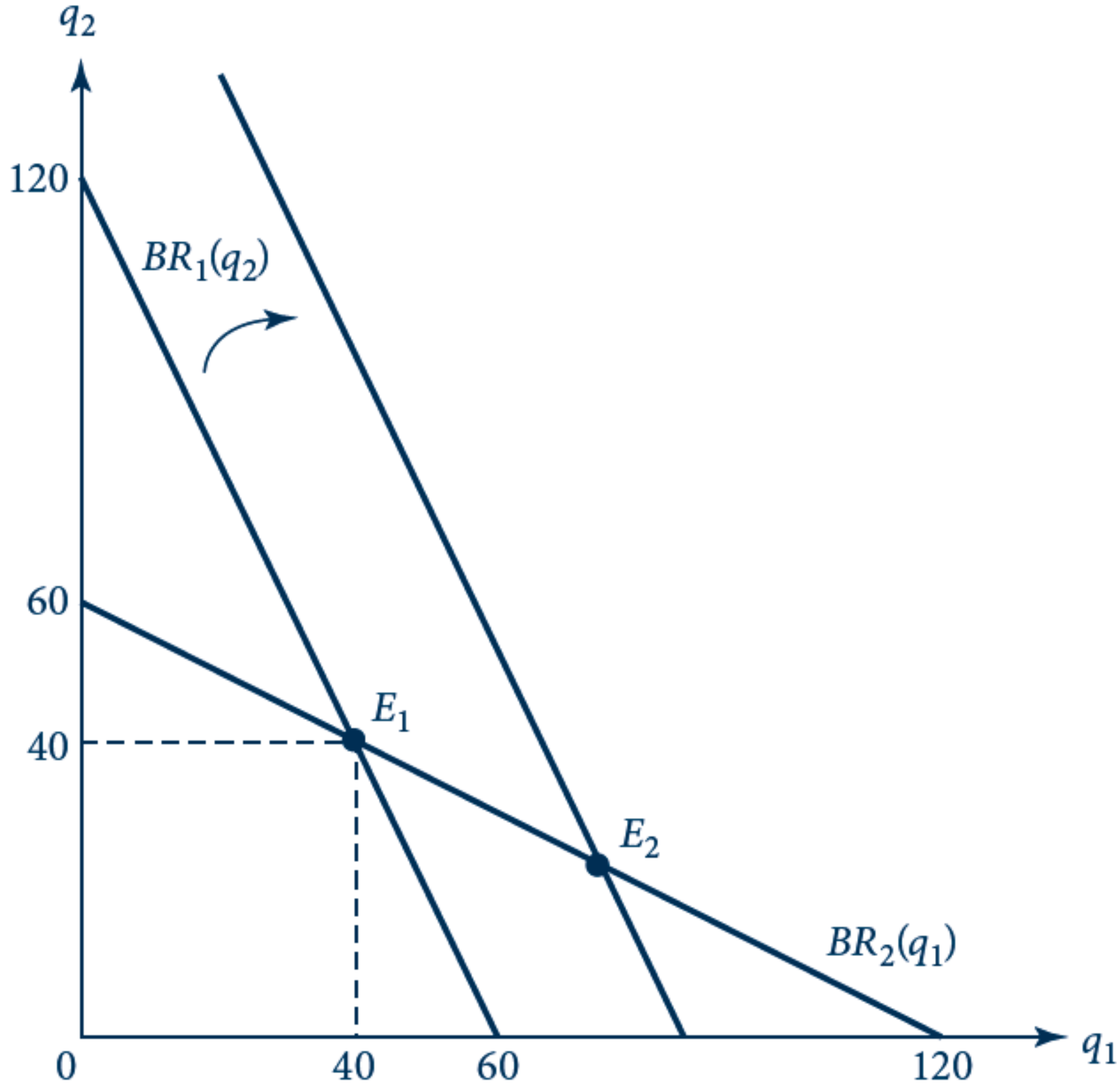
$$U_2(q_1,q_2) = q_2 v(q_1,q_2) = q_2(120 - q_1 - q_2)$$

- Solve for the Nash equilibrium
  - Solve herder 1's maximization problem and get his best-response function: $q_1 = 60 - q_2/2 = BR_1(q_2)$
  - Solve herder 2's maximization problem and get his best-response function: $q_2 = 60 - q_1/2 = BR_2(q_1)$
  - The Nash equilibrium: $q^*_1 = q^*_2 = 40$, payoff=1,600

FIGURE 8.7 Best-Response Diagram for the Tragedy of the Commons

The intersection, E_1, between the two herders' best responses is the Nash equilibrium.

An increase in the per-sheep value of grazing for herder 1 (no change for herder 2) shifts out herder 1's best response, resulting in a Nash equilibrium E_2 in which herder 1 grazes more sheep (and herder 2, fewer sheep) than in the original Nash equilibrium.



EXAMPLE 8.4 Tragedy of the Commons

- The Nash equilibrium is not the best use of the commons
  - If both herders grazed 30 sheep each, their payoffs would rise
- Solving a joint-maximization problem will lead to the higher payoffs

$$\max_{q_1,q_2}\{(q_1+q_2)v(q_1,q_2)\}$$

  - $q_1 = q_2 = 30$, or any $q_1 + q_2 = 60$

Sequential Games

- In some games, the order of moves matters
  - A player that can move later in the game can see how others have played up to that point

Sequential Battle of the Sexes

- Suppose the wife chooses first
  - And the husband observes her choice before making his
  - Her possible strategies haven't changed
  - His possible strategies have expanded
    - For each of his wife's actions, he can choose one of two actions

TABLE 8.1 Husband's contingent strategies

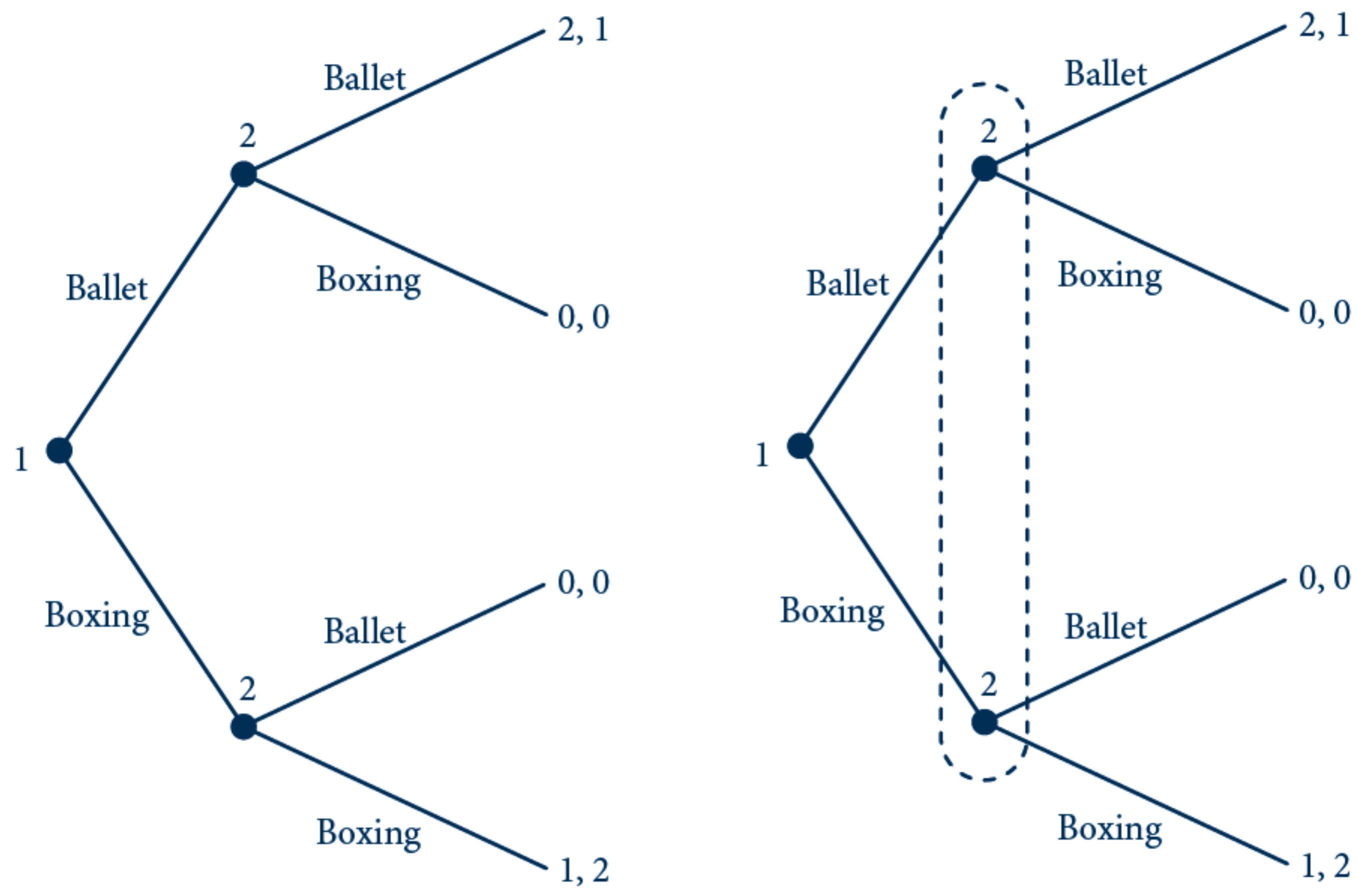
Contingent Strategy	Written in Conditional Format
Always go to the ballet	(ballet | ballet, ballet | boxing)
Follow his wife	(ballet | ballet, boxing | boxing)
Do the opposite	(boxing | ballet, ballet | boxing)
Always go to boxing	(boxing | ballet, boxing | boxing)

FIGURE 8.8 Normal Form for the Sequential Battle of the Sexes

		Husband			
		(Ballet | Ballet Ballet | Boxing)	(Ballet | Ballet Boxing | Boxing)	(Boxing | Ballet Ballet | Boxing)	(Boxing | Ballet Boxing | Boxing)
Wife	Ballet	2, 1	2, 1	0, 0	0, 0
	Boxing	0, 0	1, 2	0, 0	1, 2

The column player (husband) has more complicated, contingent strategies in the sequential Battle of the Sexes. The normal form expands to reflect his expanded strategy space.

FIGURE 8.9 Extensive Form for the Battle of the Sexes



(a) Sequential version

(b) Simultaneous version

In the sequential version (a), the husband moves second, after observing his wife's move. In the simultaneous version (b), he does not know her choice when he moves, so his decision nodes must be connected in one information set.

Sequential Battle of the Sexes

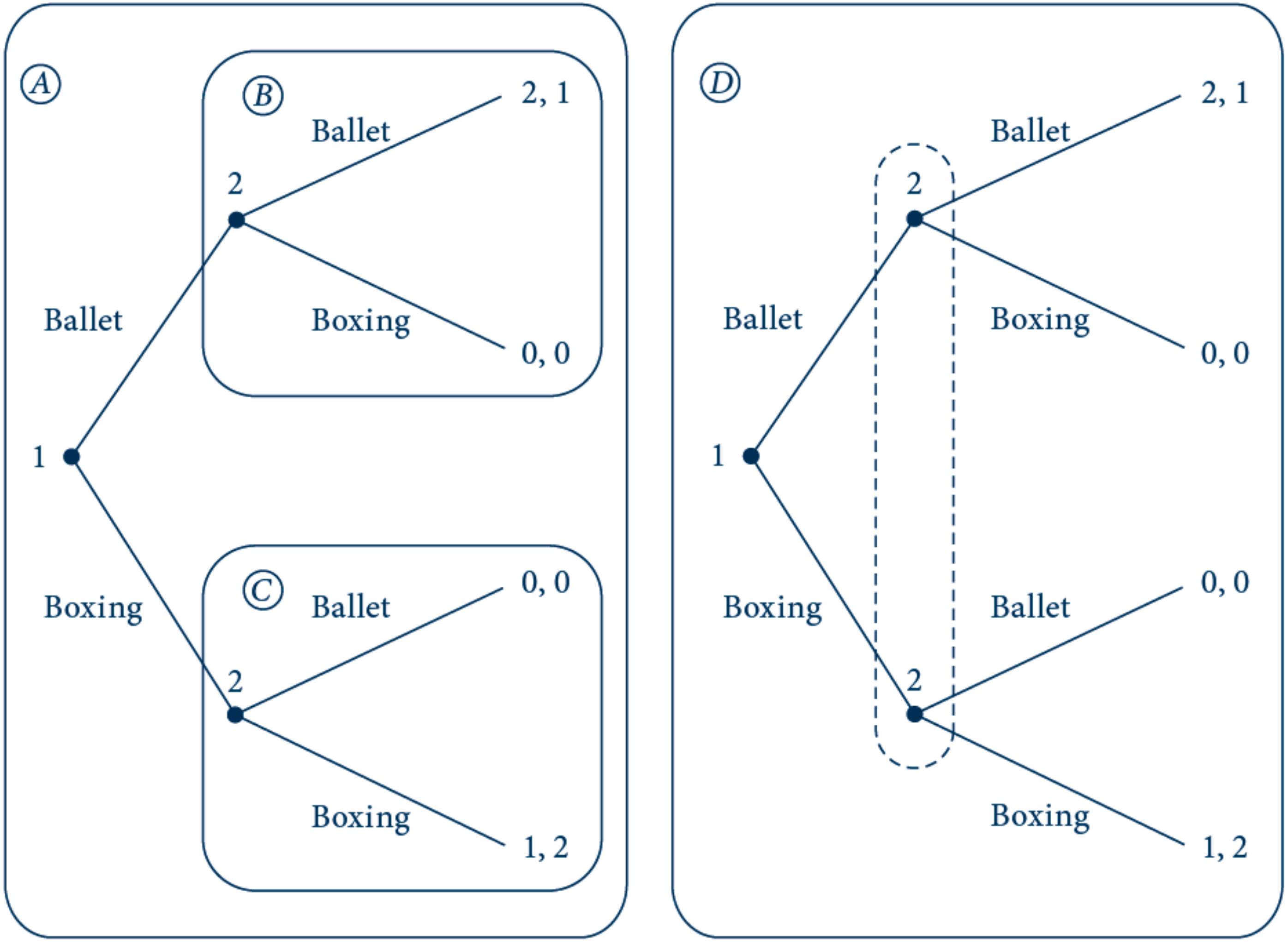
- There are three pure-strategy Nash equilibria
  1. Wife plays ballet, husband plays (ballet | ballet, ballet | boxing)
  2. Wife plays ballet, husband plays (ballet | ballet, boxing | boxing)
  3. Wife plays boxing, husband plays (boxing | ballet, boxing | boxing)

Subgame-Perfect Equilibrium

- A subgame
  - A part of the extensive form beginning with a decision node and including everything to the right of it
- A proper subgame
  - Starts at a decision node not connected to another in an information set

FIGURE 8.11 Proper Subgames in the Battle of the Sexes

The sequential version in (a) has three proper subgames, labeled A, B, and C. The simultaneous version in (b) has only one proper subgame: the whole game itself, labeled D.



(a) Sequential

(b) Simultaneous

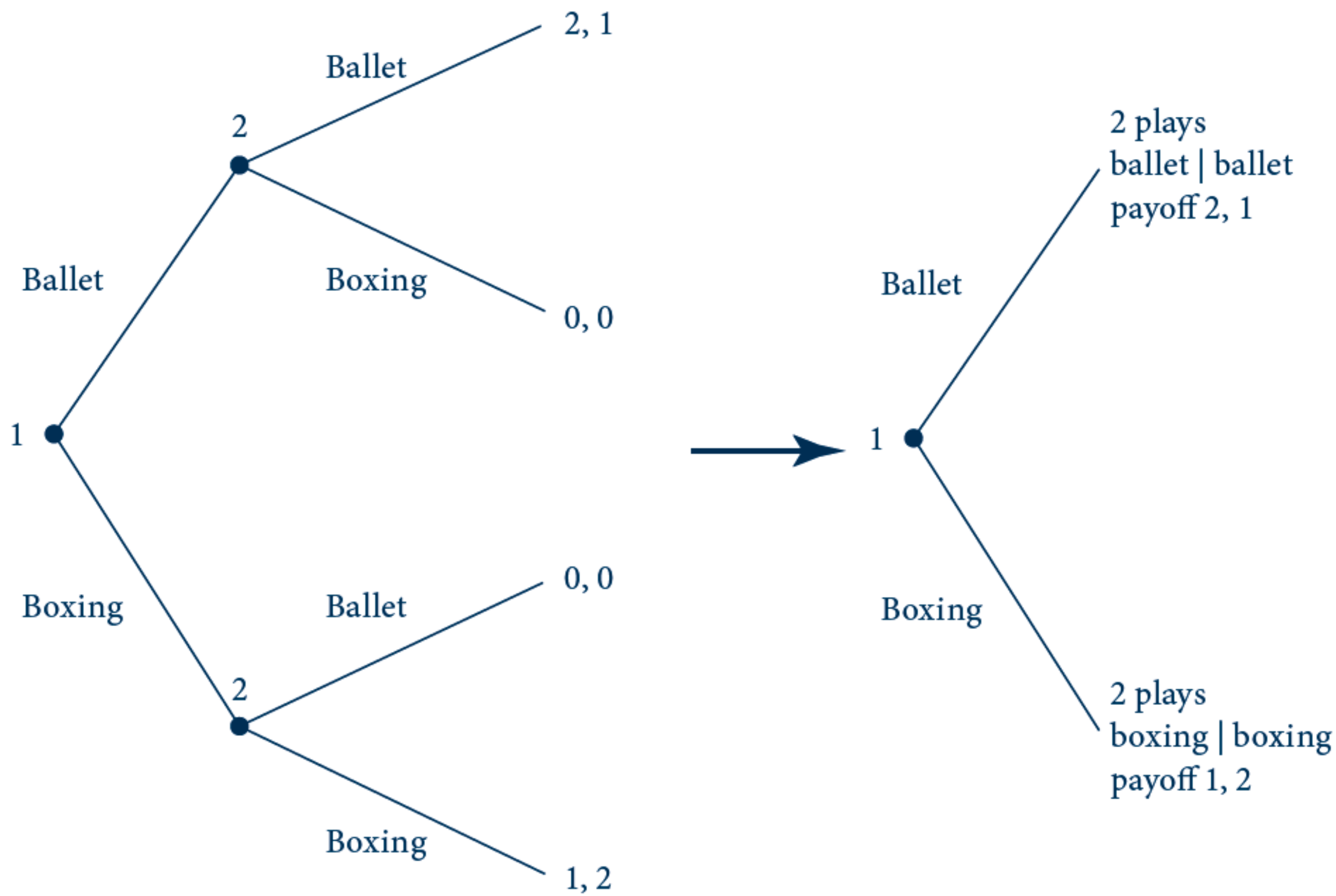
Subgame-Perfect Equilibrium

- A subgame-perfect equilibrium
  - A strategy profile $(s^*_1, s^*_2, \ldots, s^*_n)$ that constitutes a Nash equilibrium for every proper subgame
  - Is always a Nash equilibrium
  - Rules out any empty threat in a sequential game

Backward Induction

- Backward induction
  - A shortcut for finding the perfect-subgame equilibrium directly
  - Working backwards from the end of the game to the beginning
    - Compute the Nash equilibria for the bottommost subgames at the husband’s decision nodes
    - Substitute his equilibrium strategies for the subgames themselves
    - The resulting game is a simple decision problem for the wife

FIGURE 8.12 Applying Backward Induction



The last subgames (where player 2 moves) are replaced by the Nash equilibria on these subgames. The simple game that results at right can be solved for player 1's equilibrium action.

Repeated Games

- Stage game
  - Simple constituent game that is played repeatedly
- Repeated play
  - Opens up the possibility of cooperation in equilibrium
  - Players can adopt trigger strategies
    - Cooperate as long as everyone else does

Finitely Repeated Games

- Repeat a stage game for a known, finite number of times
  - May not increase the possibility for cooperation
- Selten's theorem
  - For any stage game with a unique Nash equilibrium
  - The unique subgame-perfect equilibrium of the finitely repeated game
  - Involves playing Nash equilibrium every period

Finitely Repeated Games

- If the stage game has multiple Nash equilibria
  - It may be possible to achieve cooperation in a finitely repeated game
  - Players can use trigger strategies to maintain cooperation
    - Threaten to play the Nash equilibrium that yields a worse outcome for the player who deviates from cooperation

Infinitely Repeated Games

- Players can sustain cooperation in infinitely repeated games
  - By using trigger strategies
  - The trigger strategy must be severe enough to deter deviation
- Suppose both players use the following specific trigger strategy in the Prisoners' Dilemma:
  - Continue being silent if no one has deviated; fink forever afterward if anyone has deviated to fink in the past.

Infinitely Repeated Games

- Both players follow a trigger strategy in the Prisoners' Dilemma
  - If both players are silent every period, the payoff over time would be

$$V^{eq} = 2 + 2\delta + 2\delta^2 + \ldots = 2/(1-\delta)$$

  - If a player deviates and then the other finks every period, that player's payoff is

$$V^{dev} = 3 + 1\delta + 1\delta^2 + \ldots = 3 + \delta/(1-\delta)$$

  - Trigger strategies form a perfect-subgame equilibrium: $V^{eq} \geq V^{dev}$, so $\delta \geq ½$

Infinitely Repeated Games

- Both players follow a trigger strategy in the Prisoners' Dilemma
  - Trigger strategies form a perfect-subgame equilibrium: $V^{eq} \geq V^{dev}$, so $\delta \geq ½$
  - Players will find continued cooperative play desirable provided they do not discount future gains from such cooperation too highly.
  - If $\delta < 1/2$, then no cooperation is possible in the infinitely repeated Prisoners' Dilemma; the only subgame-perfect equilibrium involves finking every period.

Infinitely Repeated Games

- Grim strategy
  - The trigger strategy in which players revert to the harshest punishment possible
  - Revert to stage-game Nash equilibrium forever
  - Elicits cooperation for the lowest value of δ for any strategy
- A tit-for-tat strategy
  - Involves only one round of punishment for cheating

Infinitely Repeated Games

- As δ approaches 1
  - Grim-strategy punishments become infinitely harsh
    - Because they involve an unending stream of undiscounted losses

Practice example

- The following game is a version of the Prisoners' Dilemma, but the payoffs are slightly different.

		Suspect 2	
		Fink	Silent
Suspect 1	Fink	0, 0	3, –1
	Silent	–1, 3	1, 1

- What is the Nash equilibrium?

Practice example

- Suppose the stage game is repeated infinitely many times.
- Compute the discount factor required for their suspects to be able to cooperate on silent each period.
- Outline the trigger strategies you are considering for them.

Incomplete Information

- Some players have information about the game that others do not

- Players that lack full information
  - Will try to use what they do know to make inferences about what they do not

Simultaneous Bayesian Games

- A two-player
  - Simultaneous-move game
  - In which player 1 has private information
  - But player 2 does not
- We can model private information
  - By introducing player types

Player Types and Beliefs

- Player 1
  - Can be of a number of possible types, *t*
  - Knows his own type
- Player 2
  - Is uncertain about *t*
  - Must decide strategy based on her beliefs about *t*

Player Types and Beliefs

- The game begins at a chance node
  - At which a value of t_k is drawn for player 1 from a set of possible types,

  $T = \{t_1, \ldots, t_k, \ldots, t_n\}$
  - $Pr(t_k)$ = probability of drawing type t_k
  - Player 1 sees which type is drawn
  - Player 2 does not see which type is drawn
    - Only knows the probabilities

Player Types and Beliefs

- Since player 1 observes t before moving
  - His strategy can be conditioned on t
    - Let $s_1(t)$ be 1's strategy contingent on his type
    - Player 2's strategy is an unconditional one, s_2
- Player 1's type may affect player 2's payoff in two ways
  - Directly
  - Indirectly through player 1's strategy

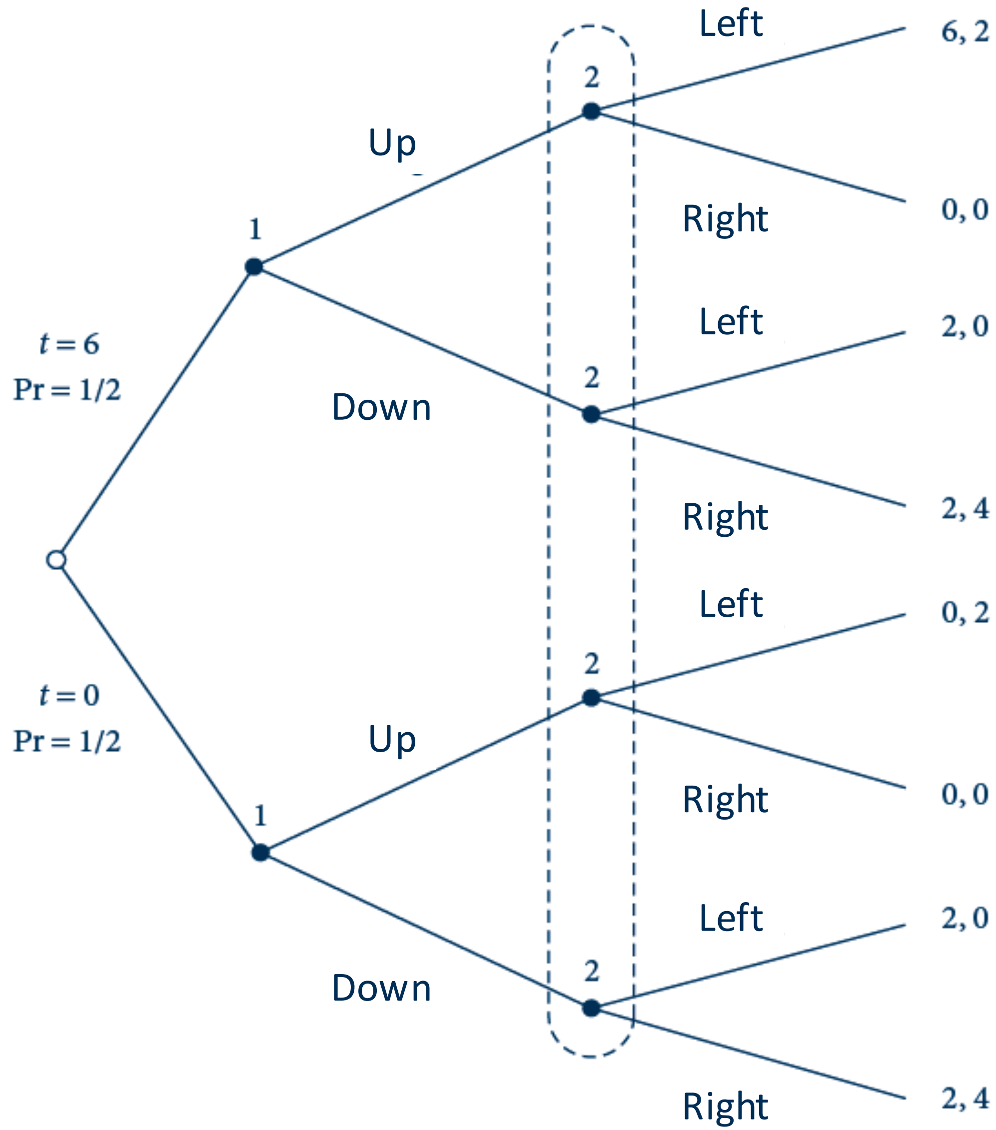
FIGURE 8.13 Simple Game of Incomplete Information

		Player 2	
		Left	Right
Player 1	Up	t, 2	0, 0
	Down	2, 0	2, 4

In this game, all payoffs are known to both players except for t in the upper left.
Player 2 only knows the distribution: an equal chance that $t = 0$ or $t = 6$.
Player 1 knows the realized value of t, equivalent to knowing his or her type.

FIGURE 8.14 Extensive Form for Simple Game of Incomplete Information

This figure translates Figure 8.13 into an extensive-form game. The initial chance node is indicated by an open circle. Player 2's decision nodes are in the same information set because she does not observe player 1's type or action before moving.



Bayesian-Nash Equilibrium

- Equilibrium requires that
  - 1's strategy be the best response for each and every one of his types
  - 2's strategy maximize an expected payoff
    - The expectation is taken with respect to her beliefs about 1's type

Bayesian-Nash Equilibrium

- Bayesian-Nash equilibrium
  - In a two-player simultaneous move game in which player 1 has private information,
  - Is a strategy profile $(s^*_1(t), s^*_2)$ such that

player 1 $U_1(s^*_1(t), s^*_2, t) \geq U_1(s'_1, s^*_2, t)$ for all $s'_1 \in S_1$

and **player 2**

$$\sum_{t_k \in T} \Pr(t_k) U_2\left(s_2^*, s_1^*(t_k), t_k\right) \geq \sum_{t_k \in T} \Pr(t_k) U_2\left(s_2^{'}, s_1^*(t_k), t_k\right)$$

$$\text{for all } s_2^{'} \in S_2$$

EXAMPLE 8.5 Bayesian–Nash Equilibrium of Game in Figure 8.14

- Two possible candidates for an equilibrium in pure strategies
  - 1 plays (*Up*|*t*=6, *Down*|*t*=0) and 2 plays *Left*
    - Not an equilibrium
  - 1 plays (*Down*|*t*=6, *Down*|*t*=0) and 2 plays *Right*
    - A Bayesian-Nash equilibrium

EXAMPLE 8.6 Tragedy of the Commons as a Bayesian Game

- Herder 1
  - Has private information regarding his value of grazing per sheep, $v_1(q_1,q_2,t)=t-(q_1+q_2)$
  - His type is
    - t=130 (the "high" type) with probability 2/3
    - t=100 (the "low" type) with probability 1/3
  - Value-maximization problem:

$$\max_{q_1}\{q_1 v_1(q_1,q_2,t)\} = \max_{q_1}\{q_1(t-q_1-q_2)\}$$

- First-order condition: $t-2q_1-q_2=0$
- So, $q_{1H}=65-q_2/2$ and $q_{1L}=50-q_2/2$

EXAMPLE 8.6 Tragedy of the Commons as a Bayesian Game

- Herder 2
  - Expected payoff:

$$\frac{2}{3}[q_2(120 - q_{1H} - q_2)] + \frac{1}{3}[q_2(120 - q_{1L} - q_2)] = q_2(120 - \bar{q}_1 - q_2)$$

$$\text{where} \quad \bar{q}_1 = \frac{2}{3}q_{1H} + \frac{1}{3}q_{1L}$$

$$q_2 = 60 - \frac{\bar{q}_1}{2} = 30 + \frac{q_2}{4}$$

$$q_2^* = 40, q_{1H}^* = 45, q_{1L}^* = 30$$

FIGURE 8.15 Equilibrium of the Bayesian Tragedy of the Commons

Best responses for herder 2 and both types of herder 1 are drawn as thick solid lines; the expected best response as perceived by 2 is drawn as the thick dashed line. The Bayesian–Nash equilibrium of the incomplete-information game is given by points A and C; Nash equilibria of the corresponding full-information games are given by points A’ and C’ .