Intermediate Microeconomic
Spring 2025

Part three: Production and supply

Week 5(b): Profit maximization

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Profit Maximization

- A profit-maximizing firm
  - Chooses both its inputs and its outputs
    - With the sole goal of achieving maximum economic profits
  - Seeks to maximize the difference between total revenue and total economic costs

Profit Maximization

- A profit-maximizing firm: makes decisions in a "marginal" way
  - Examine the marginal profit obtainable from producing one more unit of hiring one additional labor
    - If the marginal profit >0, the extra output will be produced or the extra labor will be hired
    - If the marginal profit = 0, it would not be profitable to go further

Output Choice

- Total revenue for a firm, $R(q) = p(q) \cdot q$
- Economic costs incurred, $C(q)$
  - In the production of q
- Economic profits, π
  - The difference between total revenue and total costs

$$\pi(q) = R(q) - C(q) = p(q) \cdot q - C(q)$$

Output Choice

- Maximize profits, choose q:
  - Necessary condition to choosing *q*
  - Set the derivative of the π function with respect to *q* equal to zero

$$\frac{d\pi}{dq} = \pi'(q) = \frac{dR}{dq} - \frac{dC}{dq} = 0$$

$$\frac{dR}{dq} = \frac{dC}{dq}$$

Output Choice

- Marginal revenue, MR
  - The change in total revenue *R* resulting from a change in output *q*

Marginal revenue = *MR* = *dR/dq*

- Profit maximization
  - Choose output q^* at which $MR(q^*)=MC(q^*)$

$$MR = \frac{dR}{dq} = \frac{dC}{dq} = MC$$

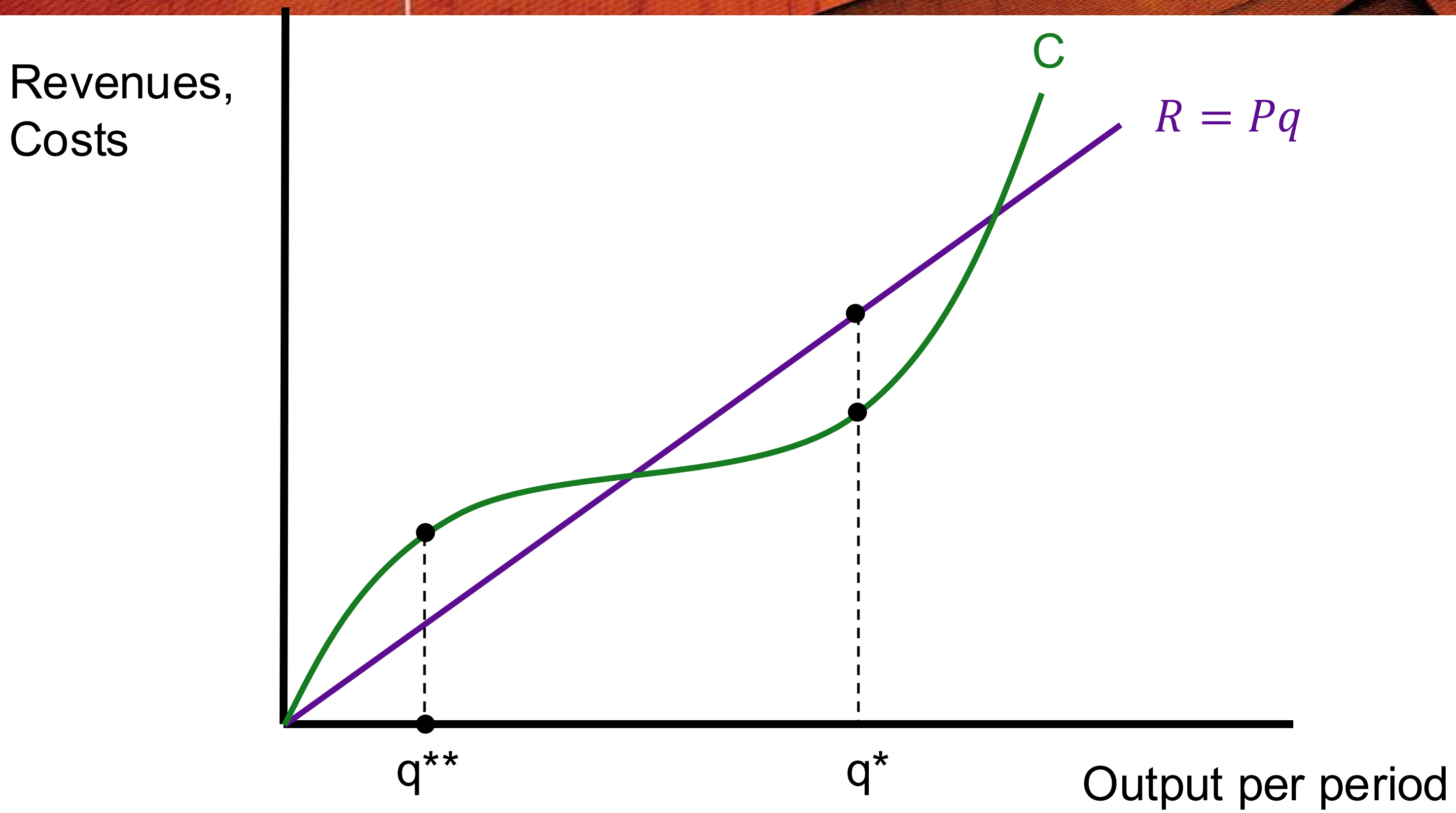
Second-Order Conditions

- *MR = MC*
  - Is only a necessary condition for profit maximization
- For sufficiency, it is also required:

$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q^*} = \left.\frac{d\pi'(q)}{dq}\right|_{q=q^*} < 0$$

  - “marginal” profit must decrease at the optimal level of output, q*
    - For q<q*, $\pi'(q) > 0$
    - For q>q*, $\pi'(q) < 0$

Marginal Revenue Must Equal Marginal Cost for Profit Maximization



- Profits, defined as **revenues (R) minus costs (C)**, reach a maximum when the slope of the revenue function (marginal revenue) is equal to the slope of the cost function (marginal cost).

- This equality is **only** a necessary condition for a maximum, as may be seen by **comparing points q*** (a true maximum) **and q**** (a local minimum), points at which marginal revenue equals marginal cost.

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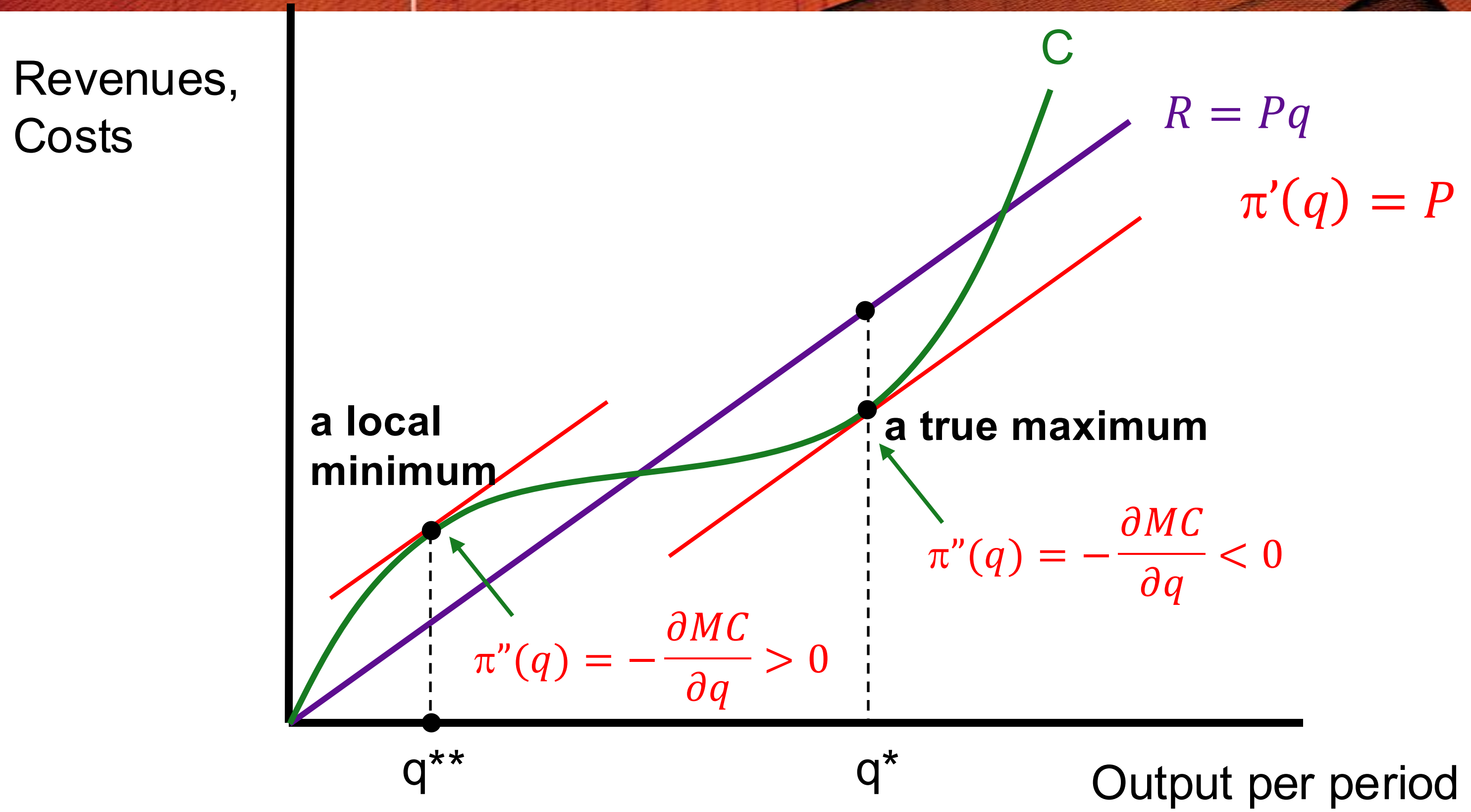

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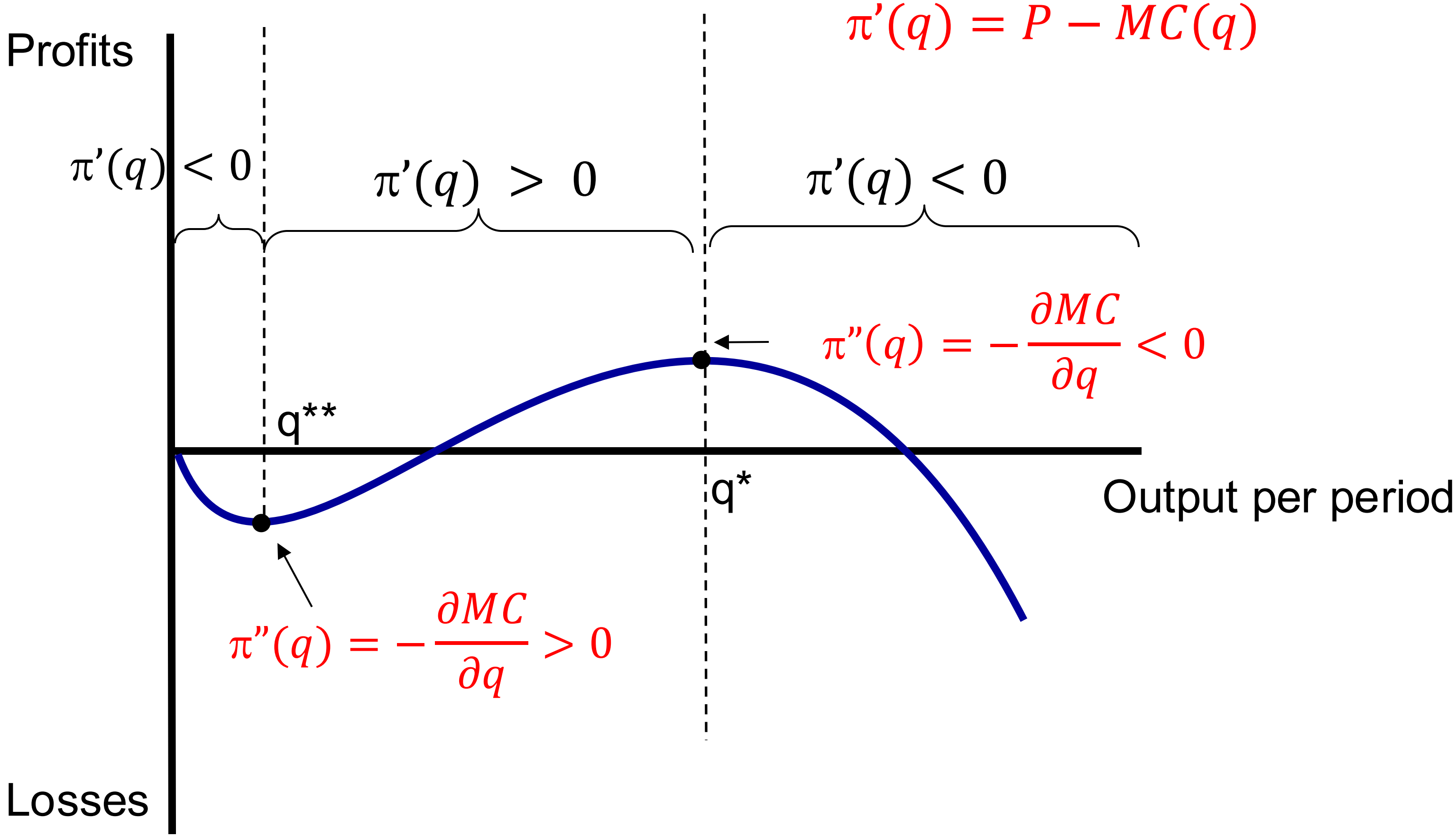
$$R = Pq, \qquad MR = P$$
$$\pi'(q) = P - MC(q)$$



Marginal Revenue

- Marginal revenue
  - If a firm faces a downward-sloping demand curve: more output can only be sold if the firm reduces the good's price (not assuming price-taking behaviors)

$$MR(q) = \frac{dR}{dq} = \frac{d[p(q) \cdot q]}{dq} = p + q \cdot \frac{dp}{dq}$$

Marginal Revenue

- Marginal revenue is a function of output
  - If price does not change as quantity increases
    - $dp/dq = 0$, MR = p
    - The firm is a price taker
  - If price decreases as quantity increases
    - $dp/dq < 0$, MR < p

EXAMPLE 11.1 Marginal Revenue from a Linear Demand Function

- Demand curve for a sandwich is

$$q = 100 - 10p$$

- Solving for price: $p = -q/10 + 10$
- Total revenue: $R = pq = -q^2/10 + 10q$
- Marginal revenue: $MR = dR/dq = -q/5 + 10$
  - MR < p for all values of q
- If the average and marginal costs are constant ($4)
  - Profit maximizing quantity: MR = MC, so q*=30
  - Price = $7, and profits = $90

Marginal Revenue and Elasticity

- The concept of marginal revenue is directly related to the elasticity of the demand curve facing the firm
- The price elasticity of demand is equal to the percentage change in quantity that results from a one percent change in price

$$e_{q,p} = \frac{dq/q}{dp/p} = \frac{dq}{dp} \cdot \frac{p}{q}$$

Marginal Revenue and Elasticity

- This means that

$$MR = p + \frac{q \cdot dp}{dq} = p\left(1 + \frac{q}{p} \cdot \frac{dp}{dq}\right) = p\left(1 + \frac{1}{e_{q,p}}\right)$$

  - if the demand curve slopes downward, $e_{q,p} < 0$ and $MR < p$
  - if the demand is elastic, $e_{q,p} < -1$ and marginal revenue will be positive
    - if the demand is infinitely elastic (flat), $e_{q,p} = -\infty$ and marginal revenue will equal price

Marginal Revenue and Elasticity

$e_{q,p} < -1$	$MR > 0$
$e_{q,p} = -1$	$MR = 0$
$e_{q,p} > -1$	$MR < 0$

The Inverse Elasticity Rule

- Because $MR = MC$ when the firm maximizes profit, we can see that

$$MC = p\left(1 + \frac{1}{e_{q,p}}\right)$$

the mark up

$$\frac{p - MC}{p} = -\frac{1}{e_{q,p}}$$

- The gap between price and marginal cost will fall as the demand curve facing the firm becomes more elastic ($|e_{q,p}|$ increases).

The Inverse Elasticity Rule

$$\frac{p - MC}{p} = -\frac{1}{e_{q,p}}$$

- If $e_{q,p}$ > -1, MC < 0, which cannot happen in real world.
- This means that firms will choose to operate only at points on the demand curve where demand is elastic ($e_{q,p}$ <-1)

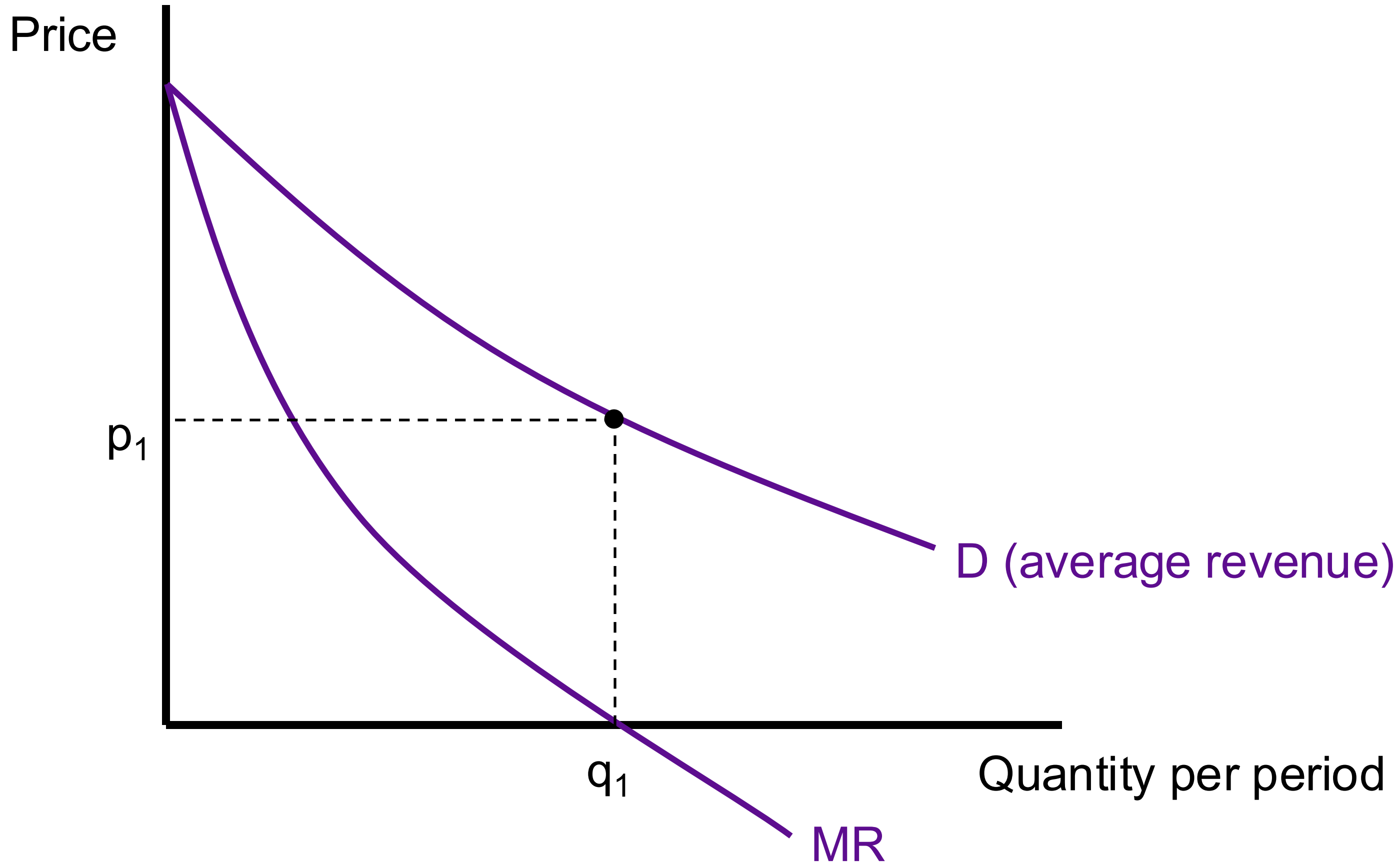
Average Revenue Curve

- Assume
  - That the firm must sell all its output at one price
  - So, we can think of the demand curve facing the firm as its average revenue curve
    - Shows the revenue per unit yielded by alternative output choices

Marginal Revenue Curve

- Marginal revenue curve
  - Shows the extra revenue provided by the last unit sold
  - Below the demand curve
    - In the case of a downward-sloping demand curve

FIGURE 11.2 Market Demand Curve and Associated Marginal Revenue Curve



Because the demand curve is negatively sloped, the marginal revenue curve will fall below the demand (''average revenue'') curve.

$$MR = p + \frac{q \cdot dp}{dq} = p\left(1 + \frac{q}{p} \cdot \frac{dp}{dq}\right) = p\left(1 + \frac{1}{e_{q,p}}\right)$$

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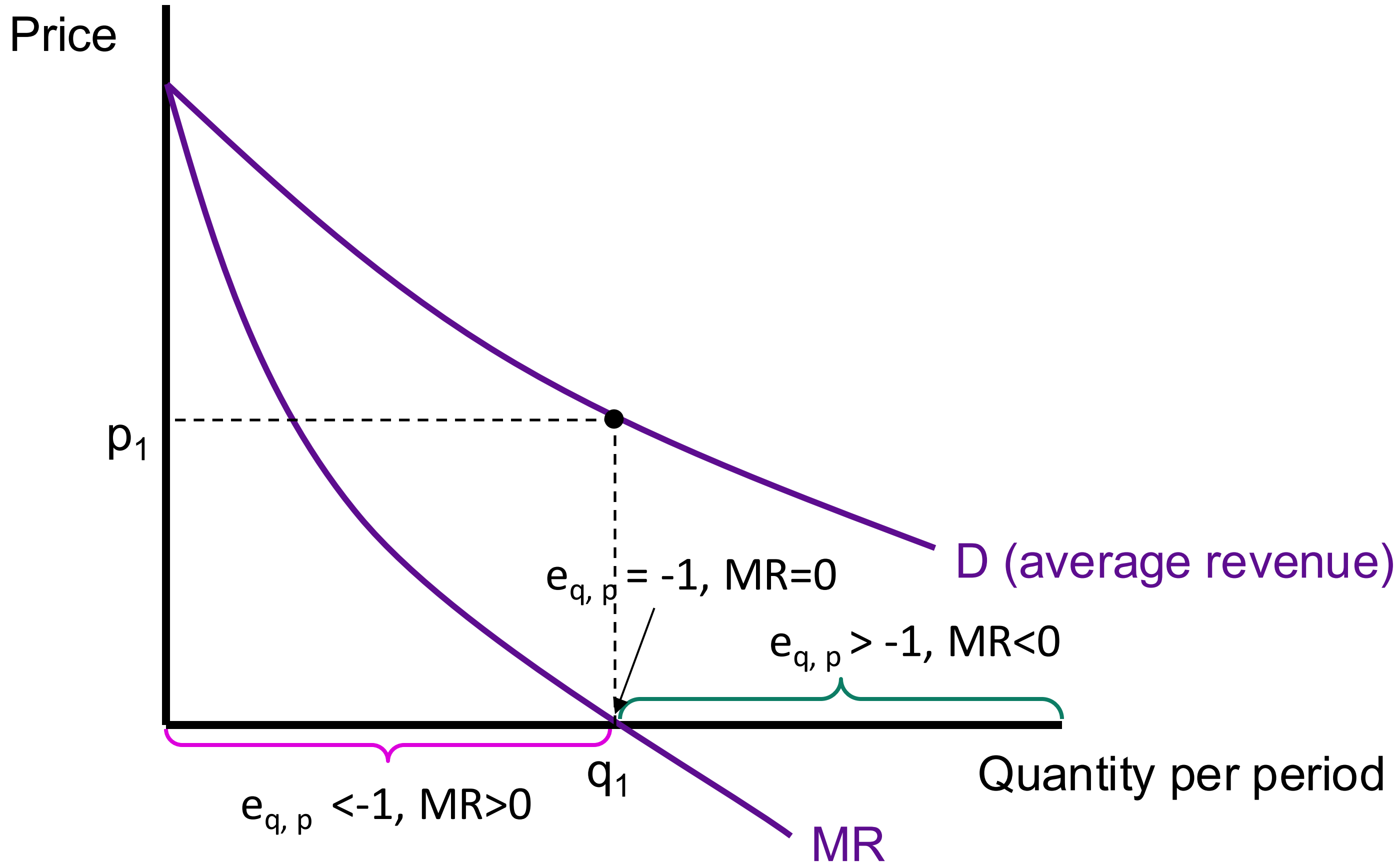


- Because the demand curve is negatively sloped, the marginal revenue curve will fall below the demand curve.
- For output levels beyond q_1, MR is negative.
- At q_1, total revenues ($p_1 \cdot q_1$) are a maximum;
- beyond this point, additional increases in q cause total revenues to decrease because of the decreases in price.

Marginal Revenue Curve

- When the demand curve shifts
  - The marginal revenue curve associated with it shifts as well
- A marginal revenue curve
  - Cannot be calculated without referring to a specific demand curve

EXAMPLE 11.2 The Constant Elasticity Case

- Demand function of the form: $q = ap^b$
  - Has a constant price elasticity of demand = $-b$

$$e_{q,p} = \frac{dq/q}{dp/p} = \frac{dq}{dp} \cdot \frac{p}{q}$$

  - What is the MR?

Profit Functions

- A firm's economic profit can be expressed as a function of inputs

$$\pi = pq - C(q) = pf(k,l) - vk - wl$$

- Only the variables *k* and *l* are under the firm's control
  - the firm chooses levels of these inputs in order to maximize profits
    - treats *p*, *v*, and *w* as fixed parameters in its decisions

Profit Functions

- A firm's <u>profit function</u> shows its maximal profits as a function of the prices that the firm faces

$$\Pi(p,v,w) = \underset{k,l}{Max}\,\pi(k,l) = \underset{k,l}{Max}[pf(k,l) - vk - wl]$$

Properties of the Profit Function

- Homogeneity
  - the profit function is homogeneous of degree one in all prices
    - with pure inflation, a firm will not change its production plans and its level of profits will keep up with that inflation

Properties of the Profit Function

- Nondecreasing in output price
  - a firm could always respond to a rise in the price of its output by not changing its input or output plans
    - profits must rise

Properties of the Profit Function

- Nonincreasing in input prices
  - if the firm responded to an increase in an input price by not changing the level of that input, its costs would rise
    - profits would fall

Properties of the Profit Function

- Convex in output prices
  - the profits obtainable by averaging those from two different output prices will be at least as large as those obtainable from the average of the two prices

$$\frac{\Pi(p_1,v,w)+\Pi(p_2,v,w)}{2} \geq \Pi\left[\frac{p_1+p_2}{2},v,w\right]$$

Envelope Results

- Apply the envelope theorem
  - To see how profits respond to changes in output and input prices

$$\Pi(P,v,w) = \max_{k,l} \pi(k,l) = \max_{k,l}[Pf(k,l) - vk - wl]$$

$$\frac{\partial \Pi(P,v,w)}{\partial P} = q(P,v,w)$$

$$\frac{\partial \Pi(P,v,w)}{\partial v} = -k(P,v,w)$$

$$\frac{\partial \Pi(P,v,w)}{\partial w} = -l(P,v,w)$$

Producer Surplus in the Short Run

- Profit function is nondecreasing in output prices
  - If $P_2 > P_1$, $\Pi(P_2,\ldots) \geq \Pi(P_1,\ldots)$
  - The welfare gain to the firm of from the price change:

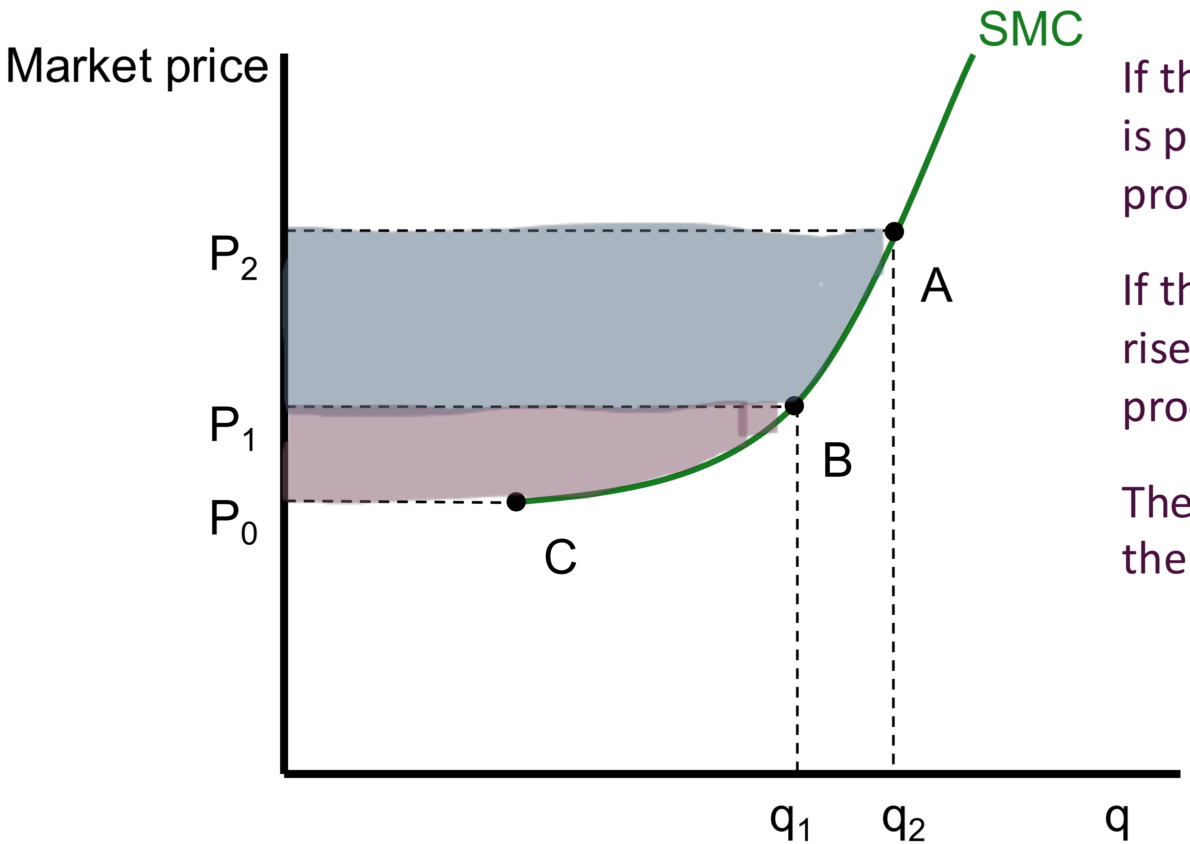
$$\text{welfare gain} = \Pi(P_2,\ldots) - \Pi(P_1,\ldots)$$

Producer Surplus in the Short Run

- Producer surplus
  - The extra return that producers make by making transactions at the market price
    - Over and above what they would earn if nothing were produced
  - The area below the market price and above the supply curve

FIGURE 11.4 Changes in Short-Run Producer Surplus

Measure Firm Profits



If the market price is p_1, the firm will produce q_1.

If the market price rises to p_2, the firm will produce q_2

The firm's profits rise by the blue shaded area

- If price increases from P_1 to P_2, then the increase in the firm's profits is given by area P_2ABP_1.
- At a price of P_1, the firm earns short-run producer surplus given by area P_0CBP_1.
- This measures the increase in short-run profits for the firm when it produces q_1 rather than shutting down when price is P_0 or below.

Producer Surplus in the Short Run

- Mathematically, we can use the envelope theorem results

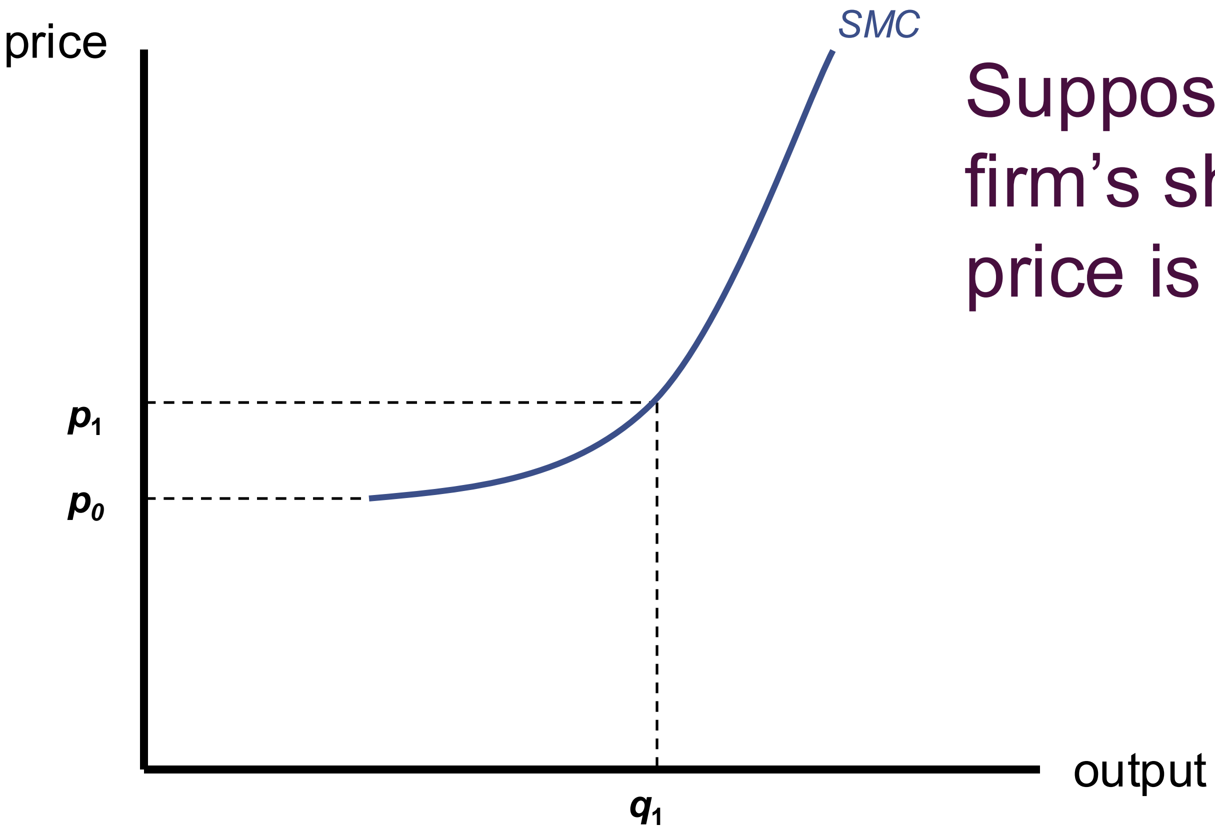
$$\text{welfare gain } = \Pi(P_2,\ldots) - \Pi(P_1,\ldots) =$$

$$= \int_{p_1}^{p_2} \frac{\partial \Pi}{\partial P} dP = \int_{P_1}^{P_2} q(P) dP$$

Producer Surplus in the Short Run

- We can measure how much the firm values the right to produce at the prevailing price relative to a situation where it would produce no output

Producer Surplus in the Short Run



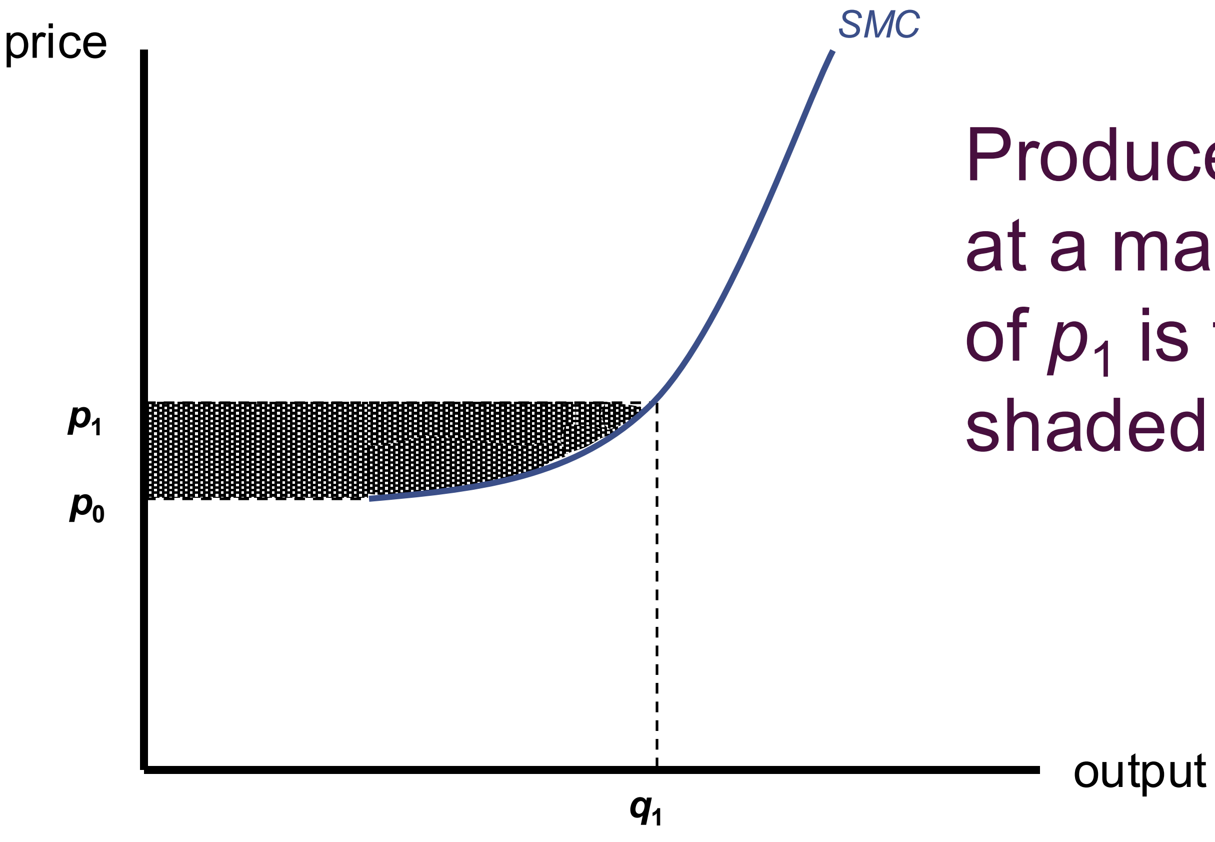
Suppose that the firm's shutdown price is p_0

Producer Surplus in the Short Run

- The extra profits available from facing a price of p_1 are defined to be producer surplus

$$\text{producer surplus} = \Pi(p_1, \ldots) - \Pi(p_0, \ldots) = \int_{p_0}^{p_1} q(p)\,dp$$

Producer Surplus in the Short Run



Producer surplus at a market price of p_1 is the shaded area

Producer Surplus in the Short Run

- Producer surplus is the extra return that producers make by making transactions at the market price over and above what they would earn if nothing was produced
  - the area below the market price and above the supply curve

Producer Surplus in the Short Run

- Because the firm produces no output at the shutdown price, $\Pi(p_0,\ldots) = -vk_1$
  - – profits at the shutdown price are equal to the firm's fixed costs
- This implies that

$$\text{producer surplus} = \Pi(p_1,\ldots) - \Pi(p_0,\ldots)$$
$$= \Pi(p_1,\ldots) - (-vk_1) = \Pi(p_1,\ldots) + vk_1$$

  - – producer surplus is equal to current profits plus short-run fixed costs

EXAMPLE 11.4 A Short-Run Profit Function

- Cobb–Douglas production function, $q = k^{\alpha} l^{\beta}$
  - With $k = k_1$ in the short-run
  - Profits are $\pi = P k_1^{\alpha} l^{\beta} - v k_1 - w l$

- Find the profit function $\Pi(P, v, w)$
- Find the short-run supply function $q(P, v, w, k_1)$
- For $\alpha = \beta = 0.5$, $v = 3$, $w = 12$, $k_1 = 80$, find the producer surplus in the short run at $P = 12$

Profit Maximization and Input Demand

- A firm's output
  - Is determined by the amount of inputs it chooses to employ
- Relationship between inputs and outputs
  - Summarized by the production function
- A firm's economic profit
  - Can be expressed as a function of inputs

$$\pi(k,l) = Pq - C(q) = Pf(k,l) - (vk + wl)$$

Profit Maximization and Input Demand

- The first-order conditions for a maximum:

$$\partial\pi/\partial k = P[\partial f/\partial k] - v = 0$$

$$\partial\pi/\partial l = P[\partial f/\partial l] - w = 0$$

  - Also imply cost minimization: *MRTS* = *w*/*v*
- A profit-maximizing firm
  - Should hire any input up to the point at which
    - Its marginal contribution to revenues is equal to the marginal cost of hiring the input

Profit Maximization and Input Demand

- Marginal revenue product
  - The extra revenue a firm receives when it uses one more unit of an input
  - In the price-taking case,
  - $MRP_l = Pf_l$
  - $MRP_k = Pf_k$

Profit Maximization and Input Demand

- Second-order conditions:

$$\pi_{kk} = f_{kk} < 0$$

$$\pi_{ll} = f_{ll} < 0$$

$$\pi_{kk}\pi_{ll} - \pi_{kl}^2 = f_{kk}f_{ll} - f_{kl}^2 > 0$$

  – Capital and labor must exhibit sufficiently diminishing marginal productivities so that marginal costs rise as output expands

Input Demand Functions

- In principle, the first-order conditions can be solved to yield input demand functions

$$\text{Capital Demand} = k(p,v,w)$$

$$\text{Labor Demand} = l(p,v,w)$$

- These demand functions are unconditional
  - they implicitly allow the firm to adjust its output to changing prices

Single-Input Case

- We expect $\partial l/\partial w \leq 0$
  - diminishing marginal productivity of labor
- The first order condition for profit maximization was

$$\partial \pi/\partial l = p[\partial f/\partial l] - w = 0$$

- Taking the total differential, we get

$$dw = p \cdot \frac{\partial f_l}{\partial l} \cdot \frac{\partial l}{\partial w} \cdot dw$$

Single-Input Case

- This reduces to

$$1 = p \cdot f_{ll} \cdot \frac{\partial l}{\partial w}$$

- Solving further, we get

$$\frac{\partial l}{\partial w} = \frac{1}{p \cdot f_{ll}}$$

- Since $f_{ll} \leq 0$, $\partial l/\partial w \leq 0$

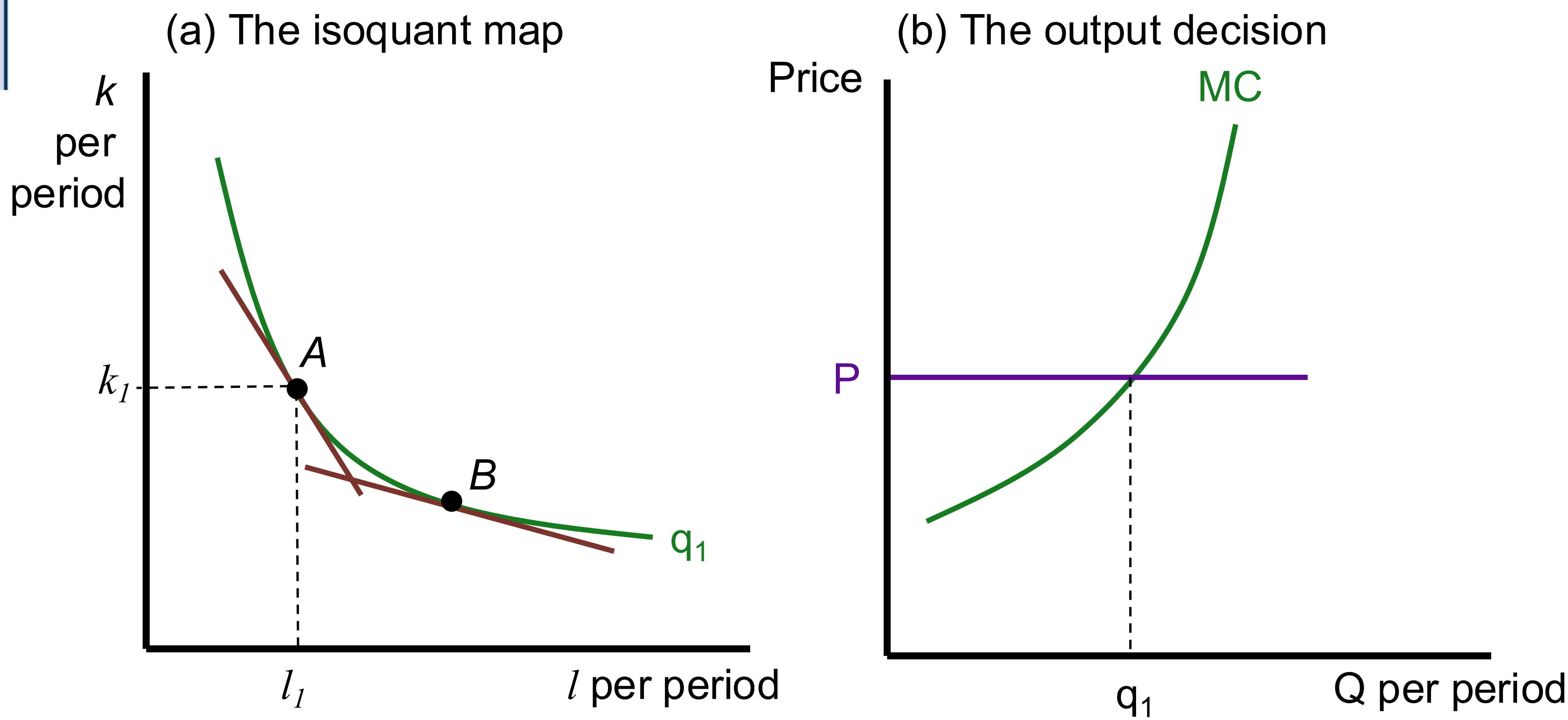
Two-Input Case

- For the case of two (or more inputs), the story is more complex
  - If there is a decrease in *w*, there will not only be a change in l but also a change in *k* as a new cost-minimizing combination of inputs is chosen
    - When *k* changes, the entire f_l function changes
- But, even in this case, $\partial l(P,v,w)/\partial w \leq 0$

Two-Input Case

- When *w* falls
  - Substitution effect
    - If output is held constant, there will be a tendency for the firm to want to substitute l for *k* in the production process
  - Output effect
    - A change in *w* will shift the firm's expansion path
    - The firm's cost curves will shift and a different output level will be chosen

FIGURE 11.5 The Substitution and Output Effects of a Decrease in the Price of a Factor



When the price of labor falls,

- the substitution effect would cause more labor to be purchased if output were held constant. This is shown as a movement from point A to point B in (a). At point B, the cost-minimizing condition ($MRTS = w/v$) is satisfied for the new, lower w.

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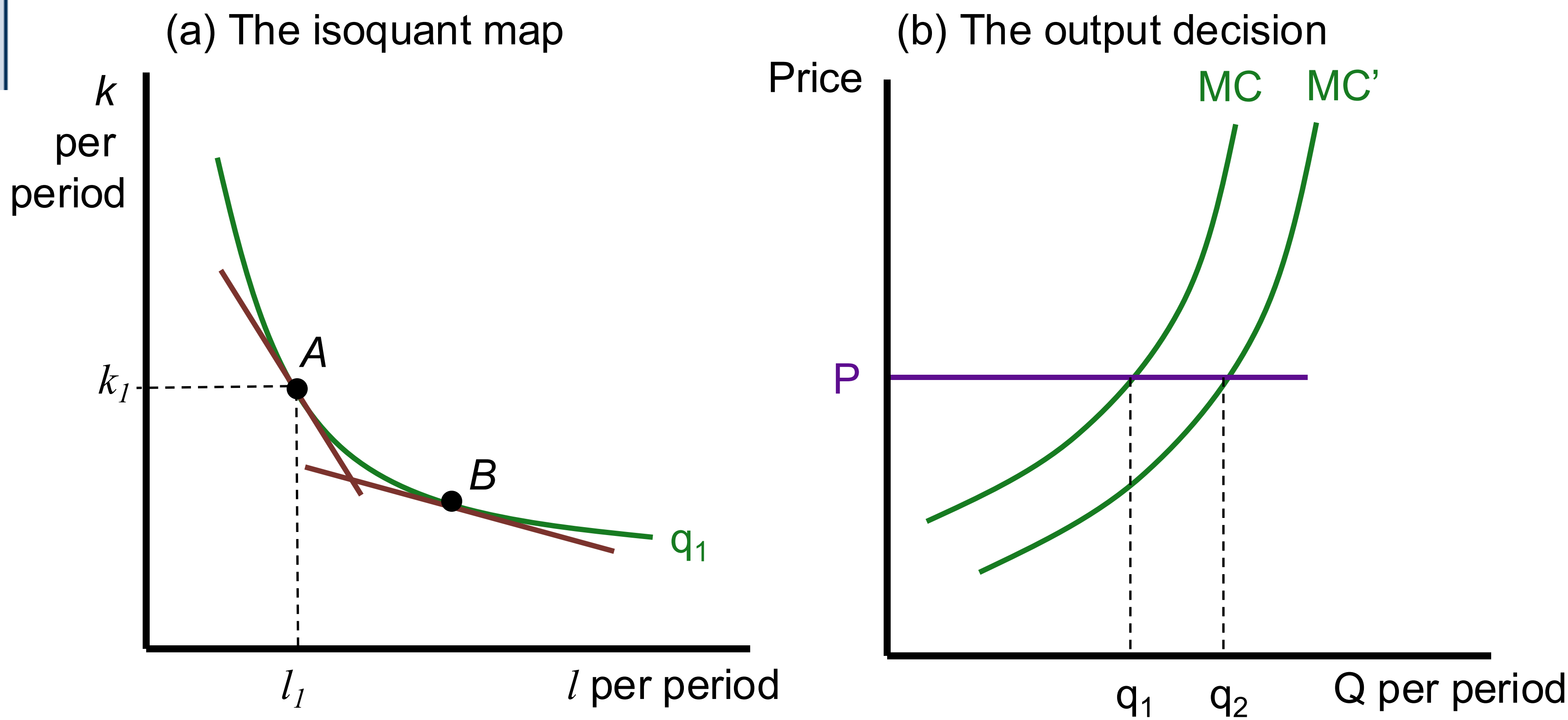

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- This change in w/v will also shift the firm's expansion path and its marginal cost curve. A normal situation might be for the MC curve to shift downward in response to a decrease in w as shown in (b). With this new curve (*MC'*) a higher level of output (q_2) will be chosen.

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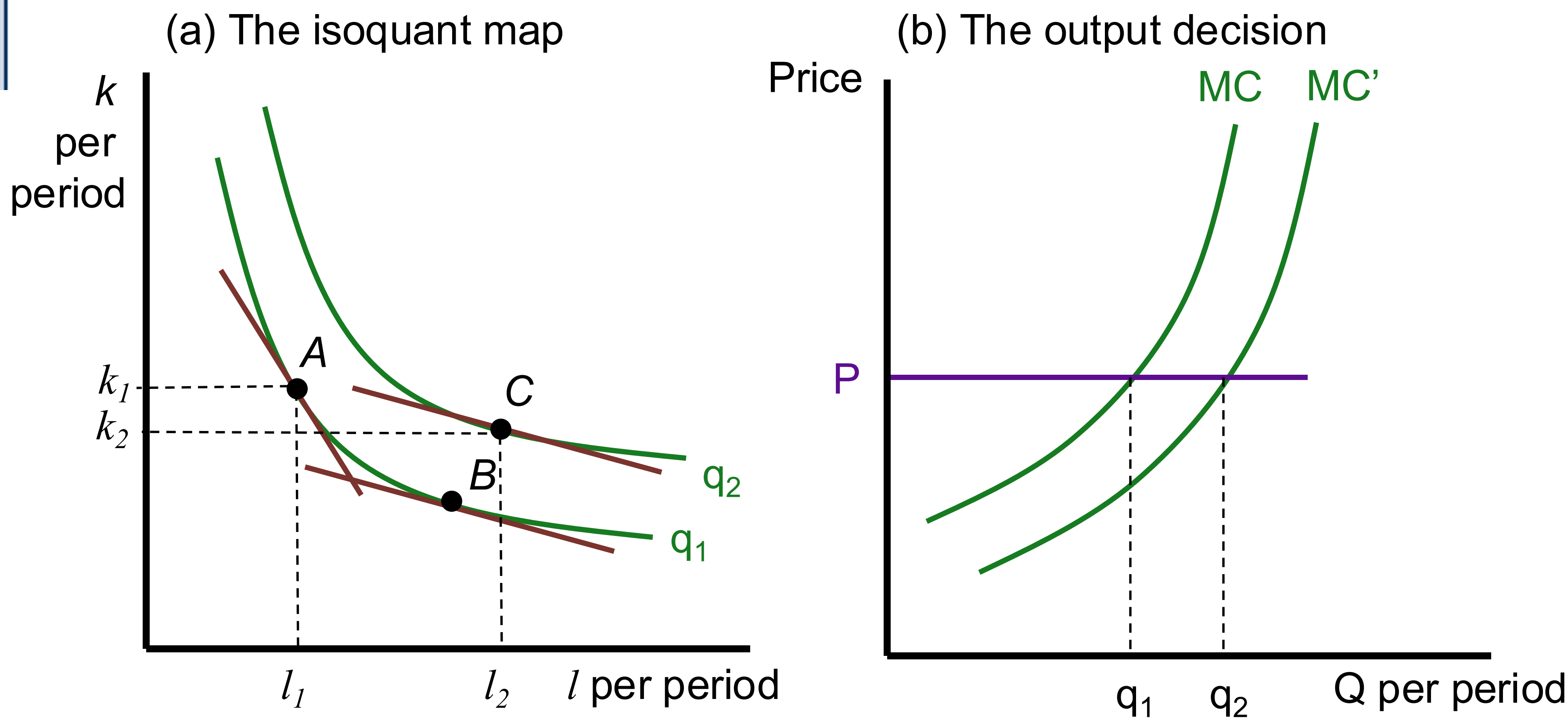

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- This change in w/v will also shift the firm's expansion path and its marginal cost curve. A normal situation might be for the MC curve to shift downward in response to a decrease in w as shown in (b). With this new curve (*MC'*) a higher level of output (q_2) will be chosen.
- Consequently, the hiring of labor will increase (to l_2), also from this output effect.

Substitution and Output Effects

- When the price of an input falls
  - Two effects cause the quantity demanded of that input to rise:

1. The substitution effect causes any given output level to be produced using more of the input
2. The fall in costs causes more of the good to be sold, thereby creating an additional output effect that increases demand for the input

Cross-Price Effects

- How capital usage responds to a wage change
  - No definite statement can be made
  - A fall in the wage will lead the firm to substitute away from capital
  - The output effect will cause more capital to be demanded as the firm expands production

Substitution and Output Effects

- Two concepts of demand for any input
  - Conditional input demand for labor, $l^c(v,w,q)$
  - Unconditional input demand for labor, $l(P,v,w)$
  - At the profit-maximizing level of output

$$l(P,v,w) = l^c(v,w,q) = l^c(v,w,\ q(P,v,w))$$

Substitution and Output Effects

- Differentiation with respect to *w* yields

$$\frac{\partial l(P,v,w)}{\partial w} = \underbrace{\underbrace{\frac{\partial l^c(v,w,q)}{\partial w}}_{\text{substitution effect}} + \underbrace{\frac{\partial l^c(v,w,q)}{\partial q} \cdot \frac{\partial q(P,v,w)}{\partial w}}_{\text{output effect}}}_{\text{total effect}}$$