Intermediate Macroeconomics: Problem Set 1

1. Nominal and Real GDP (15 points)

There are only two goods in the economy - apples and iPads. In Year 1, 110 iPads are sold at \$420 each and 9,500apples are sold at \$1 each. In Year 2, 160 iPads are sold at \$520 each and 8,000 apples are sold at \$1.25 each.

(a) Compute the nominal GDP for both years.

(b) Compute the real GDP for Year 2 using Year 1 as the base year. What is the growth rate of real GDP?

(c) Suppose in Year 2 iPads are upgraded, being twice as productive (faster, more features, etc.) as the Year 1 version. How would this change your answers to (a), (b), and why?

Solution:

(a)

$$\text{Nominal GDP Year 1 } (\text{GDP}_n^1) = 110 \times 420 + 9500 \times 1 = 55700$$

$$\text{Nominal GDP Year 2 } (\text{GDP}_n^2) = 160 \times 520 + 8000 \times 1.25 = 93200$$

(b)

$$\text{Real GDP Year 2 } (\text{GDP}_r^2) = 160 \times 420 + 8000 \times 1 = 75200$$

$$g_r = \frac{\text{GDP}_r^2 - \text{GDP}_n^1}{\text{GDP}_n^1} = \frac{75200 - 55700}{55700} = 0.35$$

(c) No change to the answer, since quality improvements are only captured (possibly incompletely) through prices.

2. National Accounting (30 points)

Consider an economy with a coal producer, a steel producer, and some consumers who want to buy steel (as final goods). In a given year, the coal producer produces 18 tons of coal and sells it for \$6 per ton. The coal producer pays \$60 in wages. The steel producer uses 28 tons of coal as input for steel production, purchased at the market price. Of this, 18 tons of coal comes from the domestic producer and 10 are imported. The steel producer produces 12 tons of steel and sells it for $22 per ton. Domestic consumers buy 9 tons of steel, and 3 tons are exported. The steel producer pays $50 in wages. All profits are distributed to domestic consumers.

(a) Calculate GDP using all three approaches discussed in class, and show that the results are the same.

(b) Gross National Product (GNP) is defined as the total value of all final products and services produced in a given period by the means of production owned by a country's entities. Calculate the GNP of this economy.

(c) Suppose that the coal producer is instead owned by a foreign nation, and profits of the coal producers are distributed abroad. Compute GNP and GDP in this case. Briefly explain the changes compared to part (b).

Solution:

(a) **1. Value-added approach:**

$$(108 - 0) + (264 - 168) = 204$$

2. Expenditure approach:

$$\underbrace{22 \times 9}_{\text{(Consumption)}} + \underbrace{0}_{\text{(Investment)}} + \underbrace{0}_{\text{Government Spending}} + \underbrace{22 \times 3}_{\text{(Export)}} - \underbrace{10 \times 6}_{\text{(Import)}} = 204$$

3. Income approach:

$$\underbrace{(60 + 50)}_{\text{(Total wage income)}} + \underbrace{(18 \times 6 - 60)}_{\text{(Profit of coal firm)}} + \underbrace{(22 \times 12 - 28 \times 6 - 50)}_{\text{(Profit of steel firm)}} = 204$$

(b)

$$\text{GNP} = 12 \times 22 - 10 \times 6 = 204$$

(c)

GDP is still equal to 204.

$$\text{GNP} = \underbrace{12 \times 22}_{\text{(Total output)}} + \underbrace{60}_{\text{(Wage income from abroad)}} - \underbrace{28 \times 6}_{\text{(Factor cost paid to abroad)}} = 156$$

Alternatively, since the foreigners are receiving $48 in coal industry profits as income:

$$\text{GNP} = \text{GDP} - 48 = 156$$

3. Calculating Growth Rates (20 points)

For this exercise, visit the National Bureau of Statistics and FRED Economics Data's websites to answer the following questions.

(1) Calculate the average annual growth rate of (nominal) GDP per capita between 1980 and 2020 for China and U.S.

(2) Assume that U.S. GDP per capita grows at 2% over the next 50 years. What annual growth rate will bring China's GDP per capita equal to the United States by 2075?

4. A Two-Period Fertility Model with Transfers (30 Points)

(Inspired by Ward and Butz, 1980) Assume a household lives for two periods, $t = 1, 2$. In each period t, it consumes $c_t \geq 0$ out of a fixed income y_t. It also decides how many children $n_t \geq 0$ to have, paying a per-child cost $p_t > 0$. In addition, parents receive a transfer $w \geq 0$ in period 2 for each child born in period 1. The utility from consumption and having children is represented by $u(\cdot)$ and $v(\cdot)$. We let $\beta \in (0, 1)$ denote the household's discount factor.

The household's objective is to maximize:

$$U = u(c_1) + \beta u(c_2) + v(n_1 + n_2),$$

subject to the budget constraints:

$$c_1 + p_1 n_1 = y_1, \quad \text{and} \quad c_2 + p_2 n_2 = y_2 + w n_1.$$

Questions:

(a) Write down the Lagrangian for the household's problem.

(b) Write down the first-order conditions for the Lagrangian and show that, in equilibrium, we have:

$$u'(c_1)p_1 = \beta u'(c_2)(w + p_2).$$

Explain the intuition behind this equation.

(c) From part (c) onward, assume $u(c) = \ln c$ and $v(n) = n$. Solve for the optimal consumption and number of children $\{c_1^*, c_2^*, n_1^*, n_2^*\}$ as functions of the model parameters $(y_1, y_2, p_1, p_2, w, \beta)$.

(d) Analyze how total fertility $n_1 + n_2$ changes if current income y_1, future income y_2, or the per-child transfer w increases.

Solution:

(a) Lagrangian Function:

The Lagrangian for the household's problem is:

$$\mathcal{L} = u(c_1) + \beta u(c_2) + v(n_1 + n_2) + \lambda_1(y_1 - c_1 - p_1 n_1) + \lambda_2(y_2 + w n_1 - c_2 - p_2 n_2),$$

where λ_1 and λ_2 are the Lagrange multipliers.

(b) First-Order Conditions:

The first-order conditions with respect to c_1, c_2, n_1, n_2 are:

$$\frac{\partial \mathcal{L}}{\partial c_1} = u'(c_1) - \lambda_1 = 0 \quad \Rightarrow \quad \lambda_1 = u'(c_1),$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = \beta u'(c_2) - \lambda_2 = 0 \quad \Rightarrow \quad \lambda_2 = \beta u'(c_2),$$

$$\frac{\partial \mathcal{L}}{\partial n_1} = v'(n_1 + n_2) - \lambda_1 p_1 + \lambda_2 w = 0,$$

$$\frac{\partial \mathcal{L}}{\partial n_2} = v'(n_1 + n_2) - \lambda_2 p_2 = 0.$$

From the first-order conditions, we can derive:

$$u'(c_1) p_1 = \beta u'(c_2)(w + p_2).$$

Intuition: This equation balances the marginal cost of having a child in period 1 (p_1) with the discounted marginal benefit in period 2 ($w + p_2$). The household considers both the transfer w and the cost p_2 of raising a child in period 2.

(c) Solving with Log Utility:

Assume $u(c) = \ln c$ and $v(n) = n$. Substituting these into the Lagrangian:

1. From the budget constraints:

$$c_1 = y_1 - p_1 n_1, \quad c_2 = y_2 + w n_1 - p_2 n_2.$$

2. Substituting $u'(c_1) = \frac{1}{c_1}$ and $u'(c_2) = \frac{1}{c_2}$ into the condition:

$$\frac{p_1}{c_1} = \beta \frac{w + p_2}{c_2}.$$

3. Solving for c_1^* and c_2^*: From $v'(n_1 + n_2) = \lambda_1 p_1 - \lambda_2 w = \lambda_2 p_2 = 1$, and using $\lambda_1 = \frac{1}{c_1}$, $\lambda_2 = \frac{\beta}{c_2}$, we solve the system for c_1^* and c_2^*:

$$c_1^* = \frac{p_1 p_2}{w + p_2}$$

$$c_2^* = \beta p_2$$

4. Optimal fertility:

$$n_1^* = \frac{y_1 - c_1^*}{p_1}, \quad n_2^* = \frac{y_2 + wn_1^* - c_2^*}{p_2}$$

$$n_1^* = \frac{y_1}{p_1} - \frac{p_2}{w + p_2}$$

$$n_2^* = \frac{y_2}{p_2} + \frac{wy_1}{p_1 p_2} - \frac{w}{w + p_2} - \beta$$

(d) Comparative Statics:

- If y_1, y_2, w increase: Both increase $n_1 + n_2$

Extra Credit: Growth Rate Approximations (5 points)

A function $F(x)$ can be approximated by the Taylor series expansion around any value a to the n-th order. That is:

$$F(x) = \sum_{n=0}^{\infty} \frac{\partial^n F(a)}{\partial x^n} \frac{(x-a)^n}{n!}$$

When $a = 0$, the first-order Taylor approximation of $F(x)$ is:

$$F(x) \approx F(0) + xF'(0)$$

(a) Using first-order Taylor approximations, show that the growth rate of a variable X, defined as:

$$g = \frac{X_{t+1} - X_t}{X_t}$$

can be approximated by $g \approx \log X_{t+1} - \log X_t$.

Solution:

$$1 + g = \frac{X_{t+1}}{X_t}$$

$$\log(1 + g) = \log(X_{t+1}) - \log(X_t)$$

The first-order Taylor approximation for $\log(1 + g)$ around 0 is:

$$\log(1 + g) \approx \log(1) + g = g$$

Therefore:

$$g = \frac{X_{t+1} - X_t}{X_t} \approx \log(X_{t+1}) - \log(X_t)$$