宏观经济学

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上次课上的思考题

- **（Boy born on a Tuesday）** A family has two children. One is a boy born on a Tuesday. What is the probability that the family has two boys?
- **（Monty Hall Problem）** On a game show, you're given the choice of three doors. Behind one door is a car, and the others, goats. You pick a door, say #1. The host, who knows what's behind the doors, opens another door, say #3, which has a goat. He then says to you, "Do you want to pick door #2?" **Is it to your advantage to switch your choice?**

Boy born on a Tuesday

Our goal is to compute the conditional probability:

- $P(B = 2|BT \geq 1) = \frac{P(B=2 \cap BT \geq 1)}{P(BT \geq 1)} = \frac{P(BT \geq 1|B=2)P(B=2)}{P(BT \geq 1)}$

First, let's compute the probability that at least one boy is born on a Tuesday. It is equal to the probability that the first kid is a boy born on Tuesday (1/14) plus the probability that the first child is **not** a boy born on Tuesday but the second child is (13/14∗1/14):

$$P(BT \geq 1) = \frac{1}{14} + \left(1 - \frac{1}{14}\right)\frac{13}{14} = \frac{27}{196}$$

Boy born on a Tuesday

The probability that the family has two boys **and** at least one of them is born on a Tuesday is equal to:

$$
\begin{aligned}
P(B = 2 \cap BT \geq 1) \quad &= P(BT \geq 1 | B = 2) P(B = 2) \\
&= \left[1 - \left(\frac{6}{7}\right)^2\right] \frac{1}{4} \\
&= \frac{13}{196}
\end{aligned}
$$

Boy born on a Tuesday

Finally, the conditional probability we want is

$$P(B = 2|BT \geq 1) = \frac{P(B = 2 \cap BT \geq 1)}{P(BT \geq 1)} = \frac{13}{27}$$

Monty Hall Problem

We want to compute the conditional probability that the car is behind the second door, after the host opens the third door:

$$P(C = 2|O = 3) = \frac{P(C = 2 \cap O = 3)}{P(O = 3)} = \frac{P(O = 3|C = 2)P(C = 2)}{P(O = 3)}$$

First compute the probability that the game host opens door 3:

$$\begin{aligned} P(O = 3) \quad &= P(O = 3|C = 1)P(C = 1) + P(O = 3|C = 2)P(C = 2) + P(O = 3|C = 3)P(C = 3) \\ &= \frac{1}{2} \times \frac{1}{3} + 1 \times \frac{1}{3} + 0 \\ &= \frac{1}{2} \end{aligned}$$

Monty Hall Problem

Why does $P(O=3|C=2)=1$? Because if you choose door $\#1$ and the car is behind door $\#2$, the host has no choice but to open door $\#3$. He can't open the door you picked or open the door with the car; otherwise he is ruining the game.

The conditional probability we want is:

$$P(C=2|O=3) \quad = \frac{P(O=3|C=2)P(C=2)}{P(O=3)}$$

$$= \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}$$

一个带有风险的两期模型

- 第0期，无任何风险。代理人可以购买债券 b_0（利率为 r）
- 第1期，经济体可能处在好的状态（概率π_g），或坏的状态（概率π_b）；$\pi_b + \pi_g = 1$
- 效用函数：

$$\begin{aligned} U(c_0, c_g, c_b) &= u(c_0) + \beta \mathbb{E}[u(c_1)] \\ &= u(c_0) + \beta[\pi_g u(c_g) + \pi_b u(c_b)] \end{aligned}$$

- 第1期的消费 c_1 表示为 c_g 和 c_b, 取决于下期状态

模型求解

- 预算约束:

$$
\begin{aligned}
c_0 + b_0 &= y_0 \\
c_g &= y_g + (1+r)b_0 \\
c_b &= y_b + (1+r)b_0
\end{aligned}
$$

- 代入$b_0 = y_0 - c_0$

$$
\begin{aligned}
c_g &= y_g + (1+r)(y_0 - c_0) \\
c_b &= y_b + (1+r)(y_0 - c_0)
\end{aligned}
$$

拉格朗日函数（简单）

$$\begin{aligned}\mathcal{L} = \quad & u(c_0) + \beta[\pi_g u(c_g) + \pi_b u(c_b)] + \\ & \lambda_g[y_g + (1+r)(y_0 - c_0) - c_g] + \\ & \lambda_b[y_b + (1+r)(y_0 - c_0) - c_b]\end{aligned}$$

- 一阶条件

$$u'(c_0) = [\lambda_g + \lambda_b](1+r)$$

$$[c_g] \quad \pi_g \beta u'(c_g) = \lambda_g$$

$$[c_b] \quad \pi_b \beta u'(c_b) = \lambda_b$$

欧拉方程与利息

- 欧拉方程：

$$u'(c_0) = \beta(1+r)[\pi_g u'(c_g) + \pi_b u'(c_b)]$$

- 从市场出清条件得知 $c_0 = y_0, c_b = y_b, c_g = y_g$，那么利率满足

$$1 + r = \frac{u'(y_0)}{\beta[\pi_g u'(y_g) + \pi_b u'(y_b)]} = \frac{u'(y_o)}{\beta\mathbb{E}[u'(y_1)]}$$

利息的决定

$$1 + r = \frac{u'(y_0)}{\beta[\pi_g u'(y_g) + \pi_b u'(y_b)]} = \frac{u'(y_o)}{\beta \mathbb{E}[u'(y_1)]}$$

$$\frac{\partial r}{\partial y_g} > 0, \quad \frac{\partial r}{\partial y_b} > 0$$

$$\frac{\partial r}{\partial y_0} < 0, \quad \frac{\partial r}{\partial \pi_g} > 0$$

如果明天的预期收入相对今天增加，家庭的储蓄欲望会降低，也就是说需要更高的利率才能使得市场出清!

风险资产

- 刚才模型中的债券b_0算是一种“安全资产”，不管世界是好与坏的状态下，都会给出同样的回报。
- 除了这种稳定的资产以外，许多生活中常见的资产是具有风险的（股票、基金、人力资本）。
- 问题：如何给这些风险资本定价?

State contingent claims（或有索取权）

- 假设有 S 种可能不同的状态，以$\{1,2,\ldots,S\}$ 表示。
- 定义： A **state-contingent claim (bond)** is an asset that delivers one unit of a good in a future state s and 0 otherwise.

Contingent (adj.) 依情况而定的

- Denote the price of a state s bond to be $q(s)$

资产定价

- 假设我们知道所有的state-contingent claims的价格 $q(s)$

- 那么我们可以对下面的资产进行尝试定价：
  - Pays 1 unit of good in state 1? $\boldsymbol{q(1)}$
  - Pays 2 units of good in state 1? $\boldsymbol{2q(1)}$
  - Pays 1 unit of good in state 1, and 1 unit of good in state 2? $\boldsymbol{q(1)+q(2)}$
  - Pays $x(s)$ units in state s, for each state $s=1,2,\ldots,S$? $\boldsymbol{\sum_{s=1}^{S} q\,(s)x(s)}$

- 理论上，我们可以用state-contingent claims 对任何的风险资产进行定价。

资产定价-2

- 问题：如何给state contingent claims定价？
- 框架：回到两期模型的例子：
- 第0期收入y_0固定，第1期两种可能的状态 $\{g, b\}$, 对应的收入分别为$\{y_g, y_b\}$.
- 假设有如下三种资产可以购买：
  - A safe bond with price 1, which pays $1 + r$ units of good regardless of state
  - A good state claim, that only pays 1 in good state
  - A bad state claim, that only pays 1 in bad state

资产定价-3

Asset	p_0	payout in g	payout in b	Quantity
Bond	1	$1+r$	$1+r$	b_1
Claim(good)	q_g	1	0	x_g
Claim(bad)	q_b	0	1	x_b

- 大家认为哪种资产价格更高？猜一下：

Good state claim or bad state claim?

资源约束

$$
\begin{aligned}
BC_0: &\quad c_0 + b_1 + q_g x_g + q_b x_b = y_0 \\
BC_g: &\quad c_g = y_g + (1+r)b_1 + x_g \\
BC_b: &\quad c_b = y_b + (1+r)b_1 + x_b
\end{aligned}
$$

$$
\begin{aligned}
c_g &= y_g + x_g + (1+r)[y_0 - c_0 - q_g x_g - q_b x_b] \\
c_b &= y_b + x_b + (1+r)[y_0 - c_0 - q_g x_g - q_b x_b]
\end{aligned}
$$

解出模型

$$\mathcal{L} = u(c_0) + \beta[\pi_g u(c_g) + \pi_b u(c_b)] +$$
$$\lambda_g[y_g - c_g + x_g + (1+r)(y_0 - c_0 - q_g x_g - q_b x_b)] +$$
$$\lambda_b[y_b - c_b + x_b + (1+r)(y_0 - c_0 - q_g x_g - q_b x_b)]$$

一阶条件：

$[c_0]$: $$u'(c_0) = (\lambda_g + \lambda_b)(1+r)$$

$[c_g]$: $$\beta\pi_g u'(c_g) = \lambda_g$$

$[c_b]$: $$\beta\pi_b u'(c_b) = \lambda_b$$

$[x_g]$: $$\lambda_g[1 - (1+r)q_g] = \lambda_b(1+r)q_g$$

$[x_b]$: $$\lambda_b[1 - (1+r)q_b] = \lambda_g(1+r)q_b$$

解出模型-2

- $[c_0], [c_g], [c_b]$ →欧拉方程

$$u'(c_0) = (1+r)\beta[\pi_g u'(c_g) + \pi_b u'(c_b)]$$

- 调整$[x_g]$ 和 $[x_b]$, 得到

$$q_b = \frac{\lambda_b}{(1+r)(\lambda_g + \lambda_b)}$$

解出模型-3

- 把$[c_g]$ $[c_b]$ 代入上页最后两个公式，得到

$$q_g = \frac{\beta\pi_g u'(c_g)}{u'(c_0)}$$

$$q_b = \frac{\beta\pi_b u'(c_b)}{u'(c_0)}$$

- 调整公式，得到：

$$\pi_i \frac{1}{q_i} = \frac{u'(c_0)}{\beta u'(c_i)}$$

- 左边：state i claim 的预期回报率；
- 右边：当下期状态为i时，两期消费的MRS

理解资产价格

- 以good state claim举例：

$$q_g = \frac{\beta \pi_g u'(c_g)}{u'(c_0)}$$

- 如果$\pi_g = 0$，good state claim的价格为0, 因为明天永远不会更好（！）

- 如果$\pi_g = 1$,

$$q_g = \frac{\beta u'(c_g)}{u'(c_0)} = \frac{1}{1+r}$$

第二个等式来自于欧拉方程，此时不存在不确定性， good state claim和债券等价

市场出清

- 在竞争均衡条件下，市场出清条件依然成立：

$$b_1 = 0, \quad x_g = x_b = 0, \quad c_g = y_g, c_b = y_b, c_0 = y_0$$

- 资产价格表示为：

$$q_g = \frac{\beta \pi_g u'(y_g)}{u'(y_0)}$$

$$q_b = \frac{\beta \pi_b u'(y_b)}{u'(y_0)}$$

比较两种风险资产的价格

- 如果 $u(c) = \log(c)$, 那么

$$q_g = \frac{\beta \pi_g y_0}{y_g}, \; q_b = \frac{\beta \pi_b y_0}{y_b}$$

- 假设 $\pi_g = \pi_b = 0.5$，两种状态发生的可能性相等。
- 因为 $y_g > y_b$，我们可以得到 $q_g < q_b$
- 思考: Why does the state-contingent claim for good state is **cheaper** than the one for bad state?

比较风险资产的价格-2

- 两种风险资产的预期收益都是$\frac{1}{2}$, 因为 $\pi_g = \pi_b = 0.5$。
- 但是，bad-state claim的价格更高，因为它可以被看做一种保险，在形势不好的时候依然支付1单位消费品。
- 风险厌恶偏好，使得代理人相比“锦上添花”更喜欢“雪中送炭”。

回到模型

- 我们已经算出了两种风险资产的价格q_b 和 q_g。那稳定债券b_0的价格如何计算?

- 答案：可以从两种风险资产的价格推算。

$$\begin{aligned} q &= (1+r)q_b + (1+r)q_g \\ &= (1+r)\left[\frac{\beta\pi_g u\prime(y_g)+\beta\pi_b u\prime(y_b)}{u\prime(y_0)}\right] \\ &= (1+r)\left[\frac{1}{1+r}\right](from\ Euler\ Equation) \\ &= 1 \end{aligned}$$

- 结果符合我们初始的假设

例子

- 刚才的例子里，稳定债券可以由两种具有风险的资产组合得来。
- 当然，我们也可以依此计算其他任何风险资产的价格
- 例子：$\pi_g = 0.5, u(c) = log(c), \beta = 0.95, y_0 = 1, y_g = 1.2, y_b = 0.8$
- 问题：一支股票在经济形势好的时候会支付1.5单位消费品，经济形势坏的时候会支付1单位消费品。请问，这支股票的价格是多少？

例子

- 在这个特定的例子中，因为效用给出了具体形式，我们知道 $q_g = \frac{\beta\pi_g y_0}{y_g}, q_b = \frac{\beta\pi_b y_0}{y_b}$

- 可以计算

$$p_s = 1.5q_g + q_b$$
$$= 1.5\left[\beta\pi_g \frac{y_0}{y_g}\right] + \left[\beta\pi_b \frac{y_0}{y_b}\right]$$
$$= 1.1875$$

以及

$$q_g = 0.396, q_b = 0.594$$

预期回报率

- 股票的预期回报率为：

$$\frac{E[\text{payoff}]}{p_s} = \frac{0.5 * 1.5 + 0.5 * 1}{1.1875} = 1.0526$$

也就是说这支股票的预期回报率为5.2%

- 稳定债券的回报率是多少？

$$1 + r = \frac{1}{q_g + q_b} = \frac{1}{0.396 + 0.594} = 1.011$$

也就是1.1%

风险与收益

- 我们从刚才的例子观察到，这支股票的预期收益要高于稳定债券
- 风险越大，收益就越大？
- 例子：另一支股票在经济形势好的时候会支付1单位消费品，经济形势坏的时候会支付1.5单位消费品。
- 这支股票的价格是$p_{s2} = q_g + 1.5q_b = 1.2865 > p_s$，而且收益率为

$$\frac{0.5 * 1.5 + 0.5 * 1}{1.2865} = 0.9716 < 1$$

风险资产的定价

- In this example, price of an asset that pays x_g and x_b in good and bad states are given by

$$P = x_g q_g + x_b q_b$$
$$= x_g \pi_g \frac{\beta u'(y_g)}{u'(y_0)} + x_b \pi_b \frac{\beta u'(y_b)}{u'(y_0)}$$

- 资产价格取决于今天和明天消费的边际替代率$\frac{\beta u'(y_i)}{u'(y_0)}$。这个系数在有时候也会被称为定价核（pricing kernel）或者随机贴现因子（stochastic discount factor）

收益的周期性

- 我们看到，相比稳定债券来说，如果资产的收益多少和经济形势的好坏正相关，这样的资产价格往往会更低，预期回报会更高
- 这样的资产无法帮助投资者“对冲风险”，反而会加剧风险的影响程度。因此，投资者需要更高的回报率来补偿承担的额外风险
- 需要多高的回报率，由投资者的风险厌恶程度决定。风险厌恶程度越强的投资者，会要求风险资产有更高的回报率。

完全市场（complete market）

- If there exists a contingent claim for **every state of nature**, we say the markets are **complete**.
- 当市场不完全时，无法完美地对每一种风险资产定价，可能会产生套利机会（arbitrage opportunity）
- 现实生活中，风险资产的种类是有限的，因此市场可能是不完全的。

例子：Equity Premium Puzzle

- “股权溢价之谜”，由 Mehra 和 Prescott 1985年的论文提出
- 两位作者基于一般均衡模型，计算了1889-1978年美国股票的历史平均收益与无风险国债的平均收益之差，发现这一差值在6%左右
- 如此大的收益率差距，无法通过合理的风险厌恶偏好来解释。
- 许多国家都发现了类似的观察结论，如英国这一差值在4.6%左右，日本在3.3% （Mehra and Prescott (2003)）

例子：Equity Premium Puzzle

- 有许多论文针对这一问题进行了后续的研究，到今天也是在学界被不断讨论的问题之一

- 可能的原因：
  - 不完全市场
  - 流动性不同
  - 非CRRA或异质性偏好
  - 损失厌恶

Mean Preserving Spread

- 回到一个两期模型， $\pi_g = \pi_b = 0.5$，假设第1期的收入为 $y_g = \overline{y} + a$ (good state) or $y_b = \overline{y} - a$ (bad state)

- 第1期的预期收入是 $E[y_1] = \overline{y}$, 与a 无关

- However, if a increase, there's more **variance** in y_1. This is called a "**mean-preserving spread**" for y_1.

社会风险对资产价格的影响

- 假设效用函数是 $u_c = \log(c)$。两种state-contingent claims 的价格分别是

$$q_g = \beta\pi_g \frac{y_0}{\bar{y} + a}$$

$$q_b = \beta\pi_b \frac{y_0}{\bar{y} - a}$$

- 当 a 上升时, q_b 会变得更高，q_g 会变得更低，规避风险的需求催生了价格的变化

社会风险对利率的影响

- 无风险的债券利率为：

$$1+r=\frac{1}{q_b+q_g}=\frac{1}{0.5\beta y_0\left(\frac{1}{\overline{y}+a}+\frac{1}{\overline{y}-a}\right)}=\frac{\overline{y}^2-a^2}{\beta y_0\overline{y}}$$

- 当社会风险程度越高时，利率越低（为什么？）

预防性储蓄（Precautionary Savings）

- 上节课，我们讨论过预期收入对利率的影响；预期收入越高，家庭的跨期储蓄欲望越低，需要更高的利率使市场出清

- 这里，社会风险增加不影响预期收入水平，却可以影响家庭的储蓄欲望和利率水平

- 从经济学直觉上说，这是由于家庭有“预防性储蓄”动机，即通过储蓄为明天可能出现的坏结果做出准备。如果明天可能的坏结果更差，家庭则有动机增加储蓄，未雨绸缪；这种增加的储蓄动机降低了利率水平。

风险对预期回报的影响

- 假设一种风险资产的回报是:

$$\text{Pays } x + b \text{ in good times}$$
$$\text{pays } x - b \text{ in bad times}$$

- $E(\text{payoff}) = 0.5(x+b) + 0.5(x-b) = x$
- 可以计算这种风险资产的期望回报:

$$E[1+\tilde{r}] = \frac{x}{q_g(x+b) + q_b(x-b)}$$
$$= \frac{x}{(q_g + q_b)x_b - b(q_b - q_g)}$$

- 如果$b > 0$，那么风险越大，资产的预期回报率越高!

Exercise: Pricing a "Lucas Tree"

Professor Robert Lucas Jr. in his 1978 *Econometrica* paper introduced a model to price "trees", which are assets that provide dividend each period. Let's practice with a simplified version of the famous "Lucas Tree" model.

Start with an infinite-period economy with a fixed supply of 1 tree, which produces non-storable consumption good (fruit) d_t each period, where d_t follows a Markov process with $Pr(d_{t+1}|d_t)$. The tree can be traded in perfectly competitive market at P_t in each period t. The representative household's problem is

$$\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \quad \mathbb{E}_0\left[\sum_{t=0}^{\infty} \beta^t \, u(C_t)\right]$$

$$\text{s.t.} \quad C_t + P_t a_{t+1} = (P_t + d_t) a_t$$

Exercise: Pricing a “Lucas Tree"

1. Write down the market clearing condition for trees and consumption good (fruit). **(Hint: fruits are not storable.)**

2. From （2） to (4), suppose that d_t is known for all t (i.e. there is no uncertainty about the dividend). Write down the Lagrangian of the problem.

3. Solve the first order condition and find the Euler Equation between consumption c_t and c_{t+1}, and find an expression of P_t as a function of C_t, C_{t+1}, P_{t+1} and d_{t+1}.

4. Recursively substitute P_{t+1} to the P_t equation you acquired in last part, and show that

$$P_t = \sum_{j=1}^{\infty} \beta^j \frac{U'(C_{t+j})}{U'(C_t)} d_{t+j}$$

5. Compute the price of a Lucas tree with **fixed** dividend $d_t = d$ each period.