Intermediate Microeconomics

Spring 2025

Week 1(b): Math Review

Yuanning Liang

Use of mathematics in economics

- Mathematization of economics
  - Took off in 1950s (Paul Samuelson, John Hicks, Kenneth Arrow, ...)
  - Ph.D. program admission requirements include linear algebra, multivariable calculus, real analysis, etc.
  - Some economists are actually mathematicians by training.
  - Anti-Mathematization or highly skeptical of mathematization (e.g., Ronald Coase)
    - Mechanical
    - Lack of economics
    - Tend not to care about the real world
  - The mathematization trend cannot be stopped or reversed

Intuition versus Mathematics

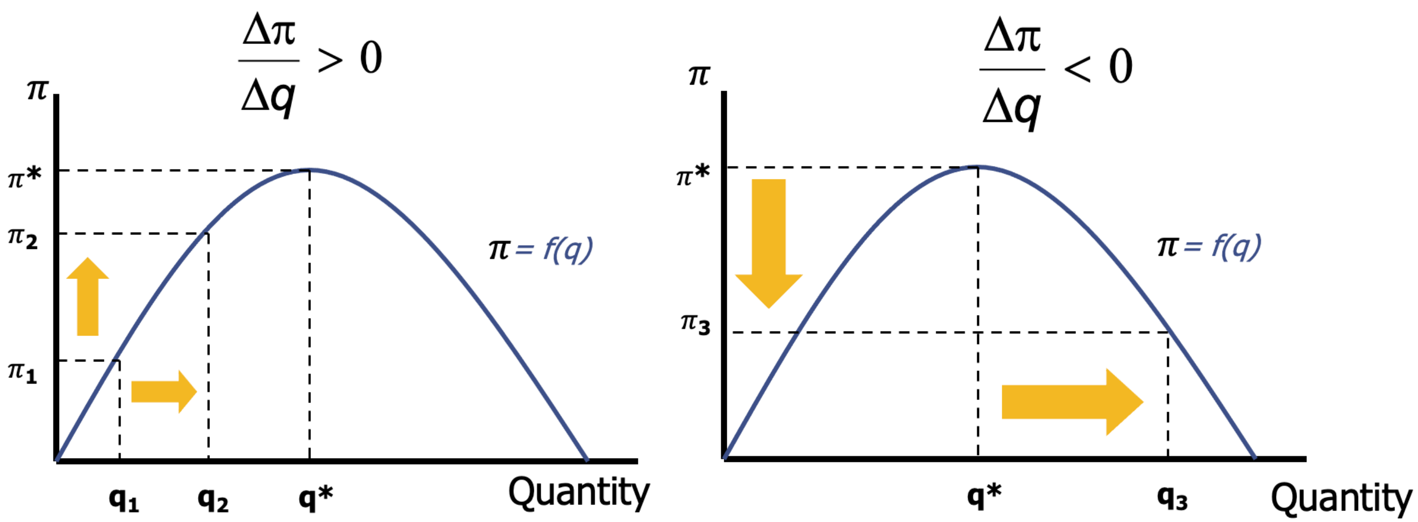
- Excellent math grades are neither necessary nor sufficient to be a great economist.

- Intuition
  - The act of knowing or sensing without the use of rational processes; immediate cognition.
  - Know something directly without analytic reasoning, bridging the gap between the conscious and nonconscious parts of our mind, and also between instinct and reason.
- Intuition is extremely valuable.

- Unfortunately, intuition is difficult (or impossible) to teach. It may be innate.
- The value of mathematics in economics
  - Intuition may not work all the time
  - Intuition can be wrong (e.g., Diminishing Marginal Utility $\nRightarrow$ Diminishing Marginal Rate of Substitution)
  - Mathematics is precise

Maximization of a function of one variable

- Suppose that a manager of a firm desires to maximize the profits received from selling a particular good.
- Suppose that the profits (π) received depend only on the quantity (q) of the good sold. Mathematically,
  - $\pi = f(q)$.
  - an increase from q_1 to q_2 leads to a rise in π, so keep moving to the right
  - an increase from q^* to q_3 leads to a drop in π, so q^* must be optimum



Derivatives

- **Math check**
- **Question: Consider function $\boldsymbol{\pi(q) = 2q + \sqrt{q}}$**
  - Find $\frac{\partial \pi(q)}{\partial q}$

- Value of derivatives at a point

$$\frac{d\pi}{dq} = \frac{df}{dq} = \lim_{h \to 0} \frac{f(q_1 + h) - f(q_1)}{h}$$

$$\boxed{\left.\frac{d\pi}{dq}\right|_{q=q_1} > 0} \qquad \boxed{\left.\frac{d\pi}{dq}\right|_{q=q_3} < 0} \qquad \boxed{\left.\frac{d\pi}{dq}\right|_{q=q*} = 0}$$

Rules for Finding Derivatives

1. If b is a constant, then $\frac{db}{dx} = 0$

2. If b is a constant, then $\frac{d[bf(x)]}{dx} = bf'(x)$

3. If b is constant, then $\frac{dx^b}{dx} = bx^{b-1}$

4. $\frac{d \ln x}{dx} = \frac{1}{x}$

5. $\frac{da^x}{dx} = a^x \ln a$ for any constant a

   A particular case of this rule is $\frac{de^x}{dx} = e^x$

Rules for Finding Derivatives

- Now suppose that $f(x)$ and $g(x)$ are two functions of x and that $f'(x)$ and $g'(x)$ exist. Then:

$$6.\ \frac{d[f(x)+g(x)]}{dx} = f'(x) + g'(x)$$

$$7.\ \frac{d[f(x)\cdot g(x)]}{dx} = f(x)g'(x) + f'(x)g(x)$$

$$8.\ \frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

provided that $g(x) \neq 0$

Rules for Finding Derivatives

- Finally, if $y = f(x)$ and $x = g(z)$ and if both $f'(x)$ and $g'(z)$ exist, then

$$9.\quad \frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{df}{dx} \cdot \frac{dg}{dz}$$

- This is the *chain rule.* Some examples are

$$10.\quad \frac{de^{ax}}{dx} = \frac{de^{ax}}{d(ax)} \cdot \frac{d(ax)}{dx} = e^{ax} \cdot a = ae^{ax}$$

$$11.\quad \frac{d[\ln(ax)]}{dx} = \frac{d[\ln(ax)]}{d(ax)} \cdot \frac{d(ax)}{dx} = \frac{1}{ax} \cdot a = \frac{1}{x}$$

$$12.\quad \frac{d[\ln(x^2)]}{dx} = \frac{d[\ln(x^2)]}{d(x^2)} \cdot \frac{d(x^2)}{dx} = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$$

First-order condition for a maximum

- If a manager could estimate the function *f(q)* from some sort of real-world data, it would theoretically be possible to find the point where *df/dq = 0.* At this optimal point (say, *q*),*

$$\left.\frac{df}{dq}\right|_{q=q^*} = 0$$

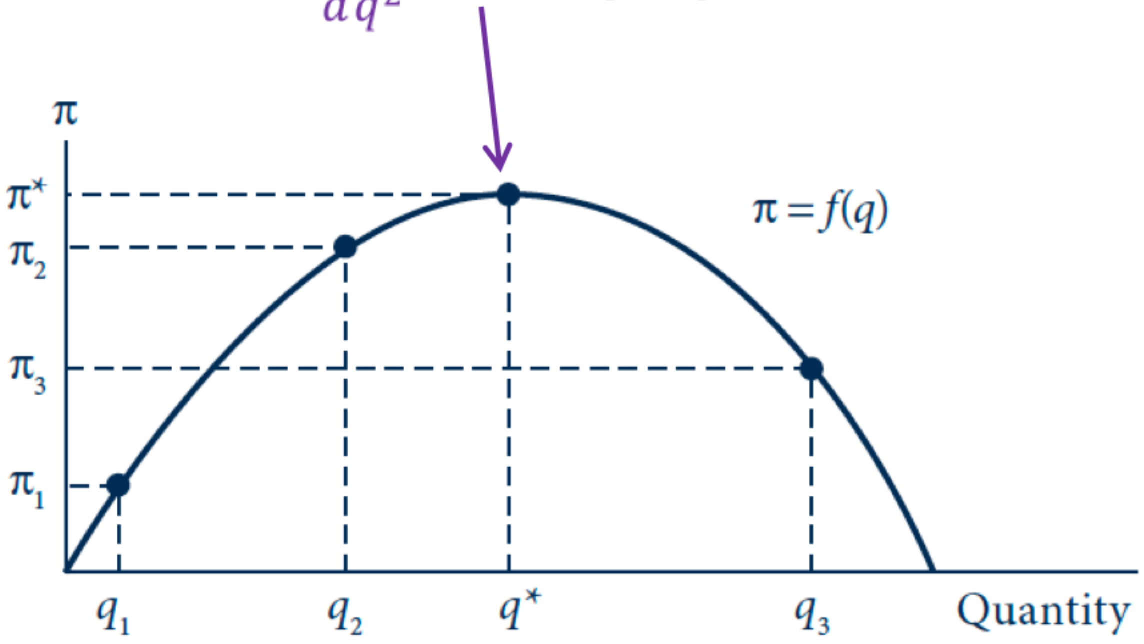
Second-order conditions

- Second derivatives:

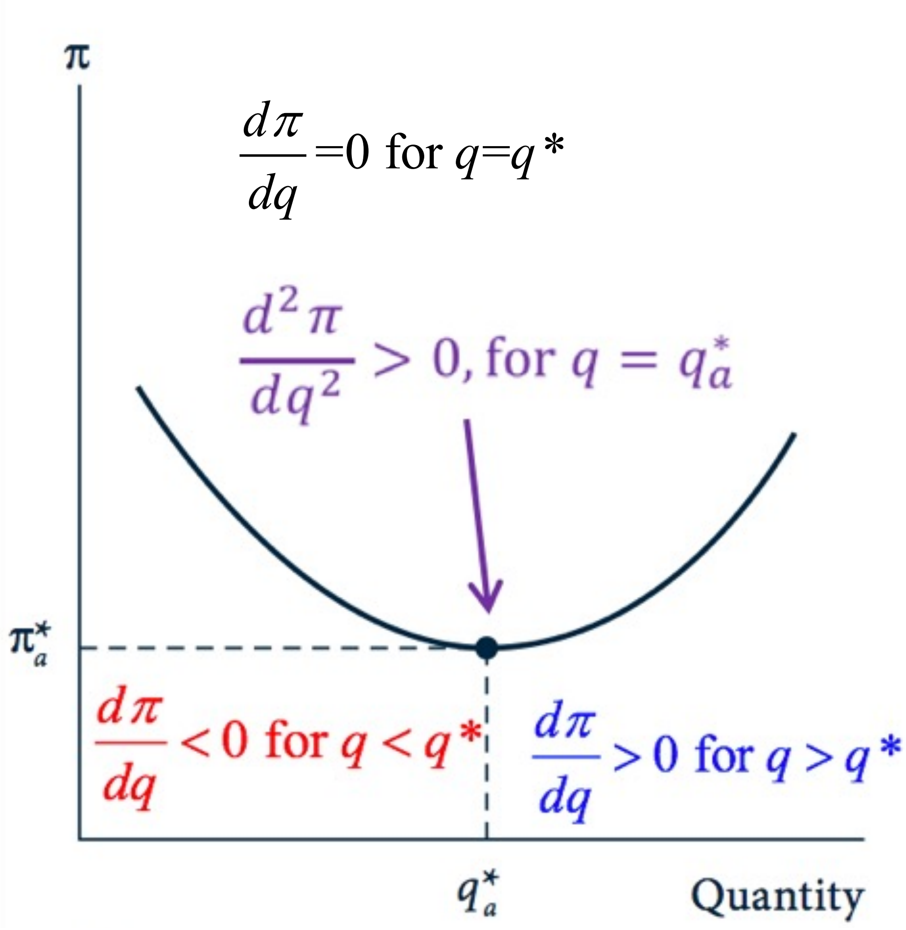
$$\frac{d^2\pi}{dq^2} \text{ or } \frac{d^2 f}{dq^2} \text{ or } f''(q).$$

Second-order conditions

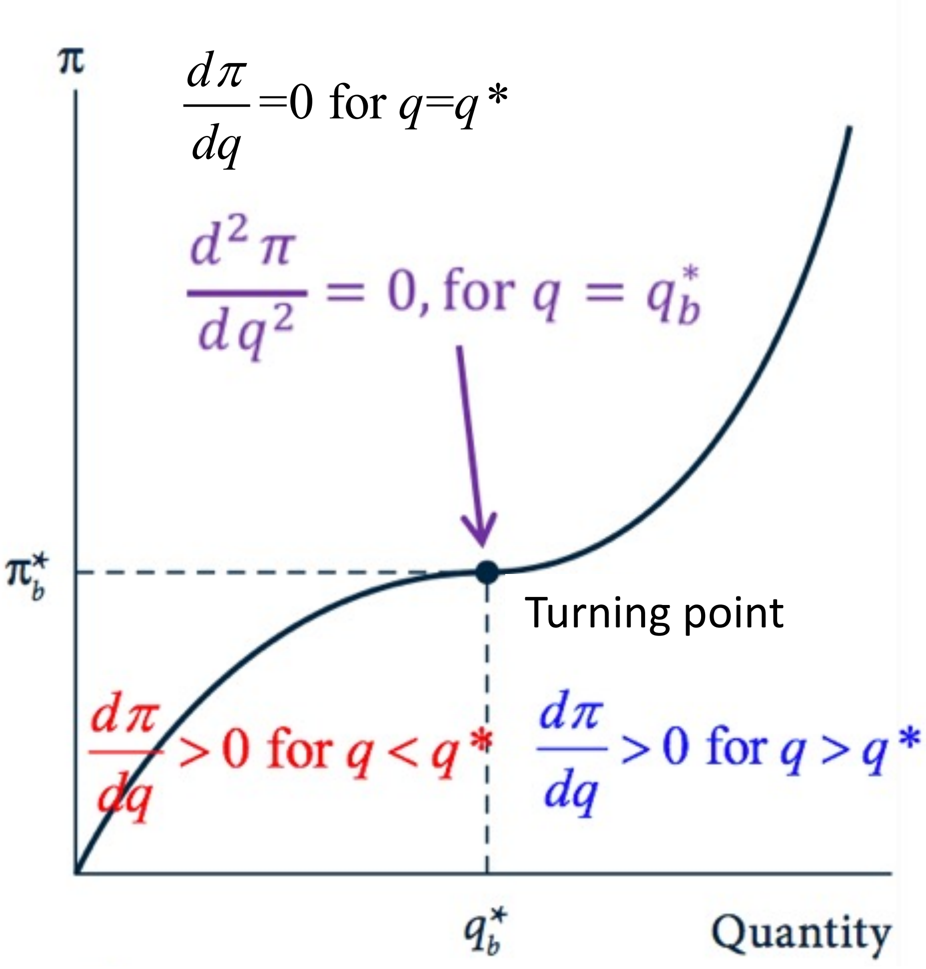
$\frac{d\pi}{dq} = 0$ for $q = q^*$ $\frac{d^2\pi}{dq^2} < 0$, for $q = q^*$



Second-order conditions



Second-order conditions



Functions of Several Variables

- Economic problems seldom involve functions of only a single variables.
- Most goals of interest to economic agents depend on several variables, and trade-offs must be made among these variables.

- Consumer *utilities* depend on the amount of each good consumed.
- A firm's *production function* depends on the quantity of labor, capital, and land devoted to production.

- This dependence of one variable (*y)* *on a series of other variables* $(x_1, x_2, \ldots, x_n)$ is denoted by

$$y = f(x_1, x_2, \ldots, x_n).$$

Partial derivatives

- Usually, the only directional slopes of interest are those that are obtained by increasing one of the x's while holding all the other variables constant.

- The partial derivatives can be denoted by

$$\frac{\partial y}{\partial x_1} \quad \text{or} \quad \frac{\partial f}{\partial x_1} \quad \text{or} \quad f_{x_1} \quad \text{or} \quad f_1$$

- A somewhat more formal definition of the partial derivative is

$$\left.\frac{\partial f}{\partial x_1}\right|_{\bar{x}_2,\ldots,\bar{x}_n} = \lim_{h \to 0} \frac{f(x_1 + h, \bar{x}_2, \ldots, \bar{x}_n) - f(x_1, \bar{x}_2, \ldots, \bar{x}_n)}{h}$$

Calculating partial derivatives

1. If $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2, \text{then}$

$$\frac{\partial f}{\partial x_1} = f_1 = 2ax_1 + bx_2$$

$$\frac{\partial f}{\partial x_2} = f_2 = bx_1 + 2cx_2.$$

Calculating partial derivatives

2. If $y = f(x_1, x_2) = e^{ax_1 + bx_2}$, then

$$\frac{\partial f}{\partial x_1} = f_1 =$$

$$\frac{\partial f}{\partial x_2} = f_2 =$$

Calculating partial derivatives

3. If $y = f(x_1, x_2) = a \ln x_1 + b \ln x_2$, then

$$\frac{\partial f}{\partial x_1} = f_1 =$$

$$\frac{\partial f}{\partial x_2} = f_2 =$$

Second-order partial derivatives

- Written as $\frac{\partial(\frac{\partial f}{\partial x_j})}{\partial x_i}$, or simply as $\frac{\partial^2 f}{\partial x_j \partial x_i} = f_{ij}$.

- For example, $y = f(x_1, x_2) = a \ln x_1 + b \ln x_2$
    - $f_{11} =$
    - $f_{12} =$
    - $f_{21} =$
    - $f_{22} =$

- Young's theorem: under general conditions, the order in which partial differentiation is conducted to evaluate second-order partial derivatives does not matter. That is

$$f_{ij} = f_{ji}$$

First-order conditions for a maximum

- A necessary condition for a point to be a local maximum is that at this point:

$$f_1 = f_2 = \ldots = f_n = 0$$

- A point at which the equation above holds is called a *critically point* of the function.
- It is not necessarily a maximum point unless certain second-order conditions (to be discussed later) hold.

Finding a maximum

- y represents an individual's health (measured on a scale of 0 to 10)
- x_1 and x_2 are daily dosages of two health-enhancing drugs
- Find values of x_1 and x_2 that make y as large as possible.

- Suppose that y is function of x_1 and x_2 given by

$$y = -(x_1 - 1)^2 - (x_2 - 2)^2 + 10$$

Taking partial derivatives of y with respect to x_1 and x_2 and applying necessary conditions of finding a maximum yields

$$\frac{\partial y}{\partial x_1} = -2x_1 + 2 = 0$$
$$\frac{\partial y}{\partial x_2} = -2x_2 + 4 = 0$$

Or

$$x_1^* = 1,$$
$$x_2^* = 2.$$

Elasticity – A general definition

- Economists use elasticities A LOT.

- Elasticities focus on the proportional effect of a change in one variable on another.
- Unit-free – the units "cancel out"

- Suppose that y is a function of x: y(x)
- Then the elasticity of y with respect to x, denoted by $e_{y,x}$, is defined as $e_{y,x} = \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}} = \frac{\Delta y}{\Delta x} \cdot \frac{x}{y} = \frac{\partial y}{\partial x} \cdot \frac{x}{y}$

Elasticity

1. If y = a + bx + other terms,

   In this case,

$$e_{y,x} = \frac{\partial y}{\partial x} \cdot \frac{x}{y} = b \cdot \frac{x}{y} = b \cdot \frac{x}{a + bx + \cdots}$$

Elasticity

2. If $y = ax^b$ (exponential form),

$$e_{y,x} = \frac{dy}{dx} \cdot \frac{x}{y} = abx^{b-1} \cdot \frac{x}{ax^b} = b$$

The elasticity is a constant, independent of where it is measured.

Elasticity

A logarithmic transformation of this equation ($y = ax^b$) provides a convenient alternative definition of elasticity.

$$\ln y = \ln a + b \ln x$$

$$e_{y,x} = \frac{d \ln y}{d \ln x} = b$$

“logarithmic transformation” is frequently used as it is the easiest way of proceed in calculating elasticities.

The chain rule

- If $y = f[x_1(a), x_2(a), x_3(a)]$

$$\frac{dy}{da} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{da} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{da} + \frac{\partial f}{\partial x_3} \cdot \frac{dx_3}{da}$$

- Example:
  - Pizza 1 costs *p* per pie; pizza 2 costs *2p* per pie; pizza 3 costs *3p* per pie.
  - The consumer wants to spend \$30 to each type of pizza.
  - Question: how will the underlying price *p* affect the total number of pizza purchased?

$x_1 = 30/p$, $x_2 = 30/2p$, $x_3 = 30/3p$.

Total pizza purchased (y) = $f[x_1(p), x_2(p), x_3(p)] = x_1(p) + x_2(p) + x_3(p)$

Applying the chain rule, we get

$$\frac{dy}{dp} = \mathrm{f}_1 \cdot \frac{dx_1}{dp} + \mathrm{f}_2 \cdot \frac{dx_2}{dp} + \mathrm{f}_3 \cdot \frac{dx_3}{dp} = -30\mathrm{p}^{-2} - 15p^{-2} - 10p^{-2} = -55p^{-2}$$

Question:

Suppose that initially *p = 5,* what happens to total pizza purchases if *p* increases by 0.05?

$\Delta y = -55 \times 0.05^{-2} = -0.11$

Implicit function theorem

隐函数定理

- Suppose that $z = f(x, y)$=c, which is a constant.
- If we hold the value of z constant, we have created an implicit relationship between x and y (y = g(x)) showing how changes in them must be related to keep the value of the function constant.
  - z = f(x, g(x)) = c
- Using the chain rule to differentiate the relationship with respect to x yields:
  - $0 = f_x + f_y \cdot \frac{dg(x)}{dx}$
  - $\frac{dg(x)}{dx} = \boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}$

Implicit function theorem

- A production possibility frontier for two goods of the form:

$$2x^2 + y^2 = 225$$

- Can be rewritten as:

$$f(x, y) = 2x^2 + y^2 - 225 = 0$$

- the trade-off between *x* and *y* is

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = \frac{-4x}{2y} = \frac{-2x}{y}$$

The Envelope Theorem

- It is a major application of the idea of implicit functions.

- It is about how the optimal value for a particular function changes when a parameter of the function changes.
  - The effects that changing the market price of a commodity will have on an individual's purchases

- Illustrate through an example…

A specific example of the Envelope Theorem

- Suppose that *y* is a function of *x*

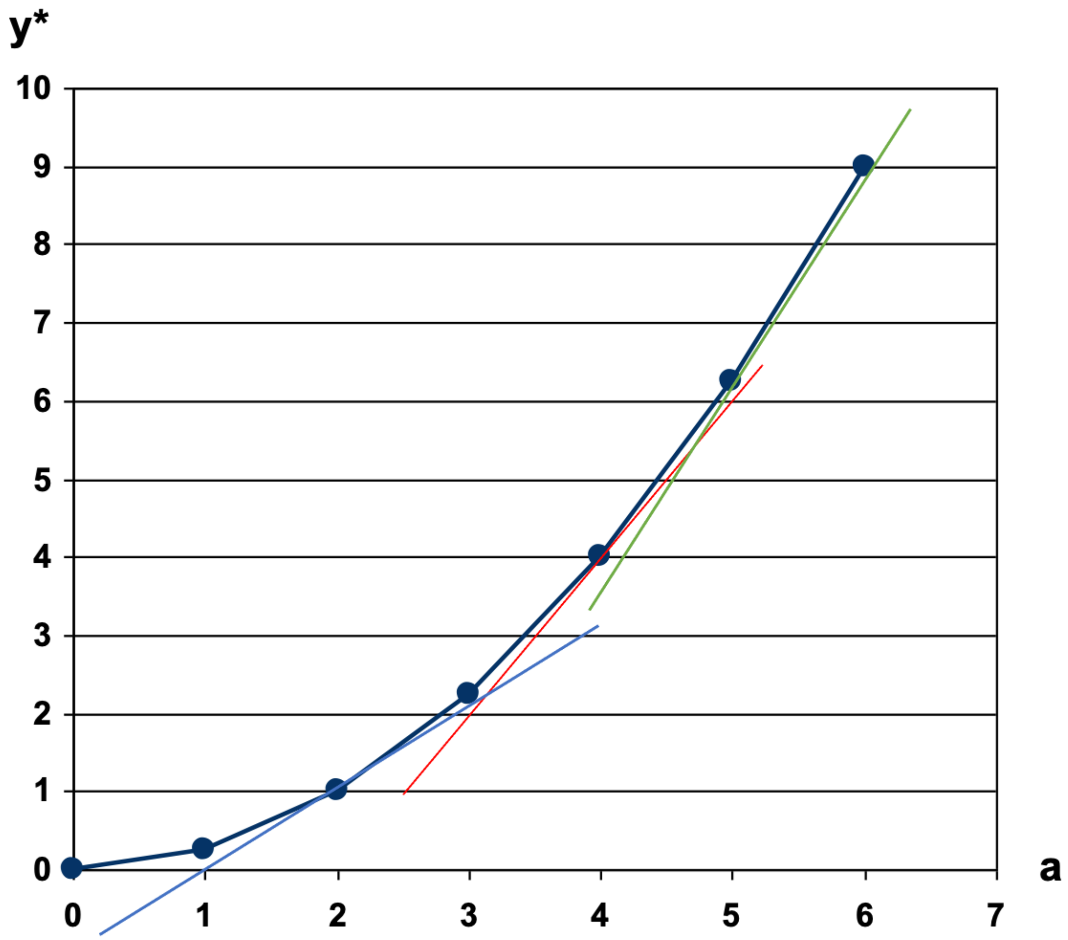
$$y = -x^2 + ax$$

- For different values of *a*, this function represents a family of inverted parabolas.
- If *a* is assigned a specific value, then *y* becomes a function of *x* only and the value of *x* that maximizes *y* can be calculated:

Optimal values of y and x for alternative values of a in $y = -x^2 + ax$

Value of *a*	Value of *x**	Value of *y**
0	0	0
1	1/2	1/4
2	1	1
3	3/2	9/4
4	2	4
5	5/2	25/4
6	3	9

Illustration of the Envelope theorem



- As *a* increases, the maximal value for *y* (*y**) increases.
- The relationship between *a* and *y** is quadratic.
- This figure also shows that the slope of the curve is positive and increases in a.

The direct, but maybe time-consuming approach

- We are interested in how y* changes when *a* changes.

- First, we can calculate the slope of the function $y = -x^2 + ax$ directly.
  - Find x* as a function of *a*

- Then, substituting this value of x* in $y = -x^2 + ax$ gives y* as a function of *a*

- Lastly, calculate $\frac{dy*}{da}$.

The envelope shortcut

- For small changes in *a*, *dy*/da* can be computed by holding *x at its optimal value* and simply calculating $\frac{\partial y}{\partial x}$ from the objective function directly.

$$\frac{dy*}{da} = \frac{\partial y}{\partial a}\Big|_{x=x^*(a)} = \frac{\partial(-x^2+ax)}{\partial a}\Big|_{x=x^*(a)} = x^*(a)$$

- Substitute in x* = a/2,

$$\frac{dy*}{da} = x^*(a) = a/2$$

The Envelope Theorem

- The Envelope Theorem states that the change in the optimal value of a function with respect to a parameter of that function

  -- can be found by **partially differentiating** the objective function while holding *x* (or several *x*'s) at its optimal value

$$\frac{dy^*}{da} = \frac{\partial y}{\partial a}\{x = x^*(a)\}$$

- The notation reminds that $\frac{\partial y}{\partial a}$ must be computed at the optimal value of x.

Constrained maximization

- So far we focus on finding the maximum value of y without restricting the choices of the x's available.
- However,
  - Managers choose output *x* to maximize profit *y* → need all x's positive
  - Choices are constrained by purchasing power (budget constraint)
- y may not be as large as it could be

Lagrange multiplier method

- Introduce the Lagrange multiplier as an additional variable to be solved corresponding to the constraint on the x's.

- The formal problem:
- Find the values of $x_1, x_2, \dots, x_n$ that maximize

$$y = f(x_1, x_2, \dots, x_n)$$

Subject to a constraint on x's

$$g(x_1, x_2, \dots, x_n) = 0$$

Set up the Lagrangian expression

$$\mathcal{L} = f(x_1, x_2, \dots, x_n) + \boldsymbol{\lambda} g(x_1, x_2, \dots, x_n)$$

- λ is the Lagrange multiplier.
- When the constraint holds (or, the constraint is binding), $\mathcal{L} = f$, because $g(x_1, x_2, \dots, x_n) = 0$.

- If we restrict to only the values of the x's that satisfy the constraint,
- Finding the constrained maximum value of f is equivalent to finding a critical value of $\mathcal{L}$

First-order conditions

$$
\left.\begin{aligned}
\boldsymbol{d}\mathcal{L}/dx_1 &= f_1 + \boldsymbol{\lambda} g_1 = 0\\
\boldsymbol{d}\mathcal{L}/dx_2 &= f_2 + \boldsymbol{\lambda} g_2 = 0\\
&.\\
&.\\
&.\\
\boldsymbol{d}\mathcal{L}/dx_n &= f_n + \boldsymbol{\lambda} g_n = 0\\
\boldsymbol{d}\mathcal{L}/d\boldsymbol{\lambda} &= g(x_1, x_2, \dots, x_n) = 0
\end{aligned}\right\}
$$

n+1 equations for n+1 unknown variables

The solution will have **two properties**:

1. the *x's* will **obey** the constraint
2. these *x's* will make the value of ***L*** (and therefore *f*) **as large as possible**

Interpretation of the Lagrange multiplier

- The Lagrange multiplier is more than just a math "trick".

- Important economic interpretation:
- Rewrite the first-order conditions as

$$\frac{f_1}{-g_1} = \frac{f_2}{-g_2} = \cdots = \frac{f_n}{-g_n} = \lambda$$

  - the numerators = the *marginal benefit* that one more unit of x_i will have for the function f
  - the denominators = the added burden (or *marginal cost*) on the constraint of using more x_i

Interpretation of the Lagrange multiplier

- At the optimal choices for the *x*'s, the ratio of the marginal benefit of increasing x_i to the marginal cost of increasing x_i should be the same for every *x*

- λ is the **common cost-benefit ratio** for all of the x's

$$\lambda = \frac{\text{marginal benefit of } x_i}{\text{marginal cost of } x_i}$$

Lagrange multiplier = shadow price

- The Lagrangian multiplier, provides a measure of how the ***relaxation in the constraint*** will affect the value of *y*

- $\boldsymbol{\lambda}$ is called a "***shadow price***" to the constraint:
  - A high value of $\boldsymbol{\lambda}$ → y could be increased substantially by relaxing the constraint
  - A low value of $\boldsymbol{\lambda}$ → there is not much to be gained by relaxing the constraint
  - $\boldsymbol{\lambda}$=0 implies that the constraint is *not binding* → same as solving the unconstrained maximization problem

An example: optimal fences and constrained maximization

- Suppose a farmer had a certain length of fence (*P*) and wished to enclose the largest possible rectangular area
  - Let *x* be the length of one side
  - Let *y* be the length of the other side

- What shape area should the farmer choose?
- The problem is to choose x and y to maximize A = x*y, subject to the constraint that the perimeter is fixed at *P=2x+2y.*

$$\begin{cases} y - 2\lambda = 0 \\ x - 2\lambda = 0 \\ 2x + 2y - P = 0 \end{cases}$$

$$xy - \lambda(2x + 2y - P)$$

Duality

- Any **<u>constrained maximization</u>** problem has associated with it a <u>dual</u> problem in **<u>constrained minimization</u>** that focuses attention on the constraints in the original problem

Original Problem	Dual Problem
• Individuals maximize utility subject to a budget constraint	• Individuals minimize the expenditure needed to achieve a given level of utility
• Firms minimize the cost of inputs to produce a given level of output	• Firms maximize output for a given cost of inputs purchased

Still that example

- What is the dual problem in words?
  - Ans: For a given area of the rectangular field, the farmer wishes to minimize the fence required to surround it.
- How to set up the math problem?
  - Minimizes what?
  - Subject to what?
- How to set up the Lagrange expression?
  - First-order conditions?
  - Solve for x and y and lambda
  - How would you interpret this lambda?

Second-order conditions and curvature

- We now discuss the sufficient conditions for an optimum and their relationship to second-order conditions.
- We will also discuss the economic explanations for these curvature conditions.

- Functions of two variables x_1 and x_2.

$$y = f(x_1, x_2)$$

→ Ensure that the *dy* is decreasing for all movements through the critical point

Unconstrained maximization

- A formal analysis:

Total differential of the function is

$$dy = f_1 dx_1 + f_2 dx_2$$

The total differential of dy function is

$$d^2y = (f_{11}dx_1 + f_{12}dx_2)dx_1 + (f_{21}dx_1 + f_{22}dx_2)dx_2$$

Or $$d^2y = f_{11}dx_1^2 + f_{12}dx_2dx_1 + f_{21}dx_1\,dx_2 + f_{22}dx_2^2$$

By Young's theorem, $f_{12} = f_{21,}$

Rearrange terms,

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

$$d^2y = f_{11}dx_1^2 + 2f_{12}dx_1dx_2 + f_{22}dx_2^2$$

- For this equation to be unambiguously negative for any change in the x's, f_{11} and f_{22} *must be negative.*

  - If $dx_2 = 0$, then $d^2y = f_{11}{dx_1}^2$
    - for $d^2y < 0$, it must be $f_{11} < 0$
  - If $dx_1 = 0$, then $d^2y = f_{22}{dx_2}^2$
    - for $d^2y < 0$, it must be $f_{22} < 0$
  - If neither dx_1 nor dx_2 is 0, then d^2y is unambiguously negative if and only if

$$f_{11}f_{22} - f_{12}^2 > 0$$

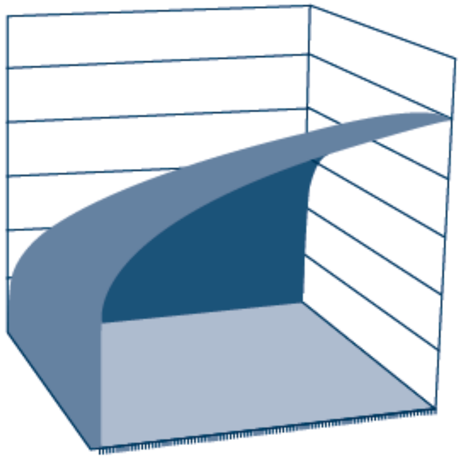
the second partial derivatives (f_{11} and f_{22}) must be sufficiently negative so that they outweigh any possible perverse effects from the cross-partial derivatives ($f_{12} = f_{21}$)

Concave functions

- Functions that obey such a condition are called (strictly) *concave functions*.
  - Functions with one variable: $f''(x) < 0$
  - Functions with two variables: $f_{11} < 0$ (and $f_{22} < 0$) , $f_{11} f_{22} - f_{12}^2 > 0$
  - Functions with multiple variables: Hessian matrix is negative definite (not required in this course)

- In three dimensions, concave functions resemble

inverted teacups.



(a) $k = 0.2$

Constrained maximization

- Choose x_1 and x_2 to maximize

$$y = f(x_1, x_2)$$

- Subject to the linear constraint

$$c - b_1 x_1 - b_2 x_2 = 0$$

- We can set up the Lagrangian

$$L = f(x_1, x_2) + \lambda(c - b_1 x_1 - b_2 x_2)$$

Constrained maximization

- First-order conditions:

$$f_1 - \lambda b_1 = 0$$
$$f_2 - \lambda b_2 = 0$$
$$c - b_1 x_1 - b_2 x_2 = 0$$

- To ensure that the point is a local maximum, we need to use the "second" total differential:

$$d^2y = f_{11} dx_1^2 + 2 f_{12} dx_1 dx_2 + f_{22} dx_2^2 < 0$$

Constrained maximization

- Because of the constraint, not all possible small changes in the x's are permissible.
- Only those that satisfy the constraint can be considered valid alternatives to the critical point.

- We must calculate the total differential of the constraint:

$$-b_1 dx_1 - b_2 dx_2 = 0$$
$$dx_2 = -(b_1/b_2) dx_1$$

- These are allowable relative changes in x_1 and x_2.

Constrained maximization

We know $f_1/f_2 = b_1/b_2$ from FOC, combining with the equation above yields

$$dx_2 = -(f_1/f_2)\, dx_1$$

We can take dx_2 into d^2y expression,

$$d^2y = f_{11}dx_1^2 - 2f_{12}(f_1/f_2)dx_1^2 + \; f_{22}({f_1}^2/{f_2}^2)dx_1^2$$

Combining terms gives

$$d^2y = (f_{11}\,{f_2}^2 - 2f_{12}f_1f_2 + f_{22}{f_1}^2)[dx_1^2/\,{f_2}^2]$$

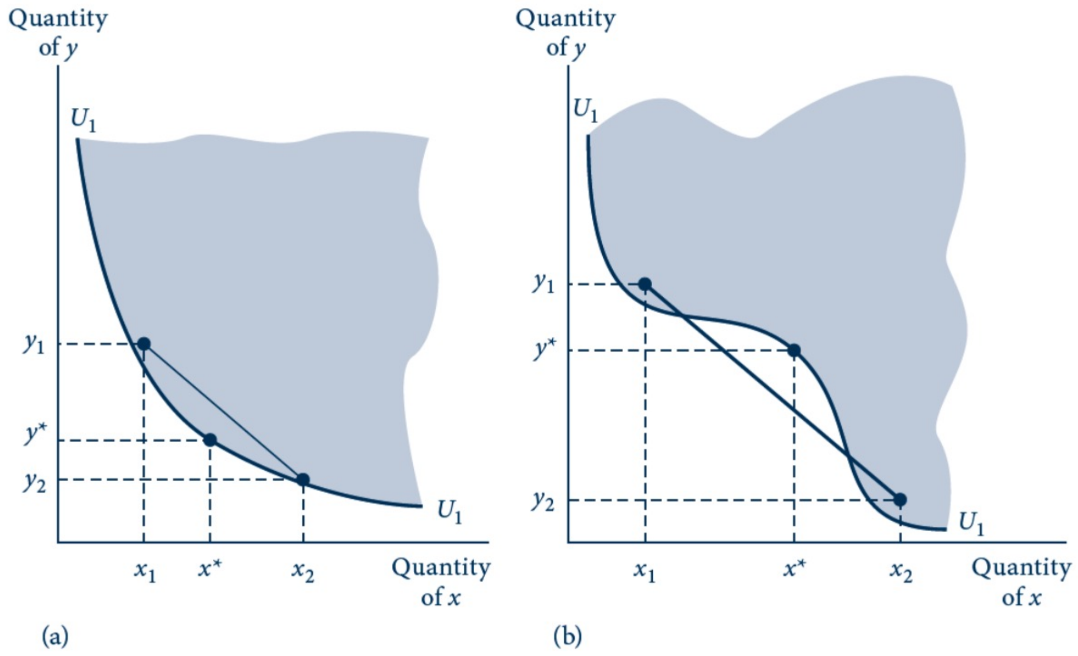
Consequently, for $d^2y < 0$, it must be the case that

$$f_{11}\,{f_2}^2 - 2f_{12}f_1f_2 + f_{22}{f_1}^2 < \; 0$$

<u>Functions satisfy this condition are called *quasi-concave functions.*</u>

Quasi-concave functions

- The set of all points for which such a function takes on a value greater than any specific constant is a convex set.
- i.e., any two points in the set can be joined by a line contained completely within the set.



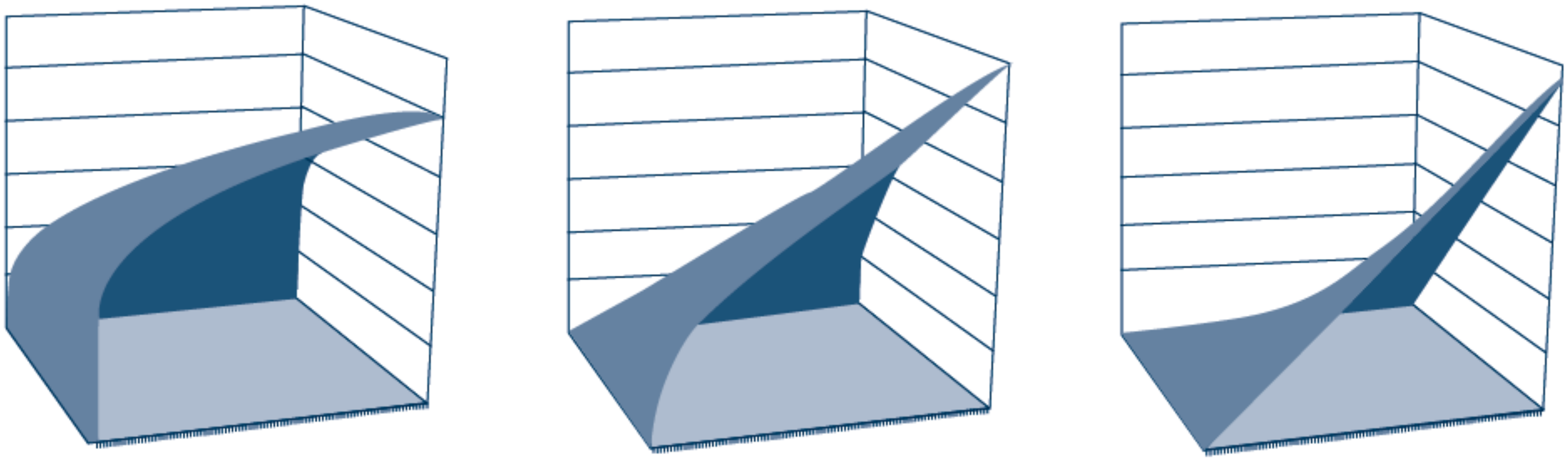
Concave and Quasi-Concave Functions

- $y = f(x_1, x_2) = (x_1 \cdot x_2)^k$

  - Is this function quasi-concave?

  - Is this function concave?

Concave and Quasi-Concave Functions



(a) $k = 0.2$ (b) $k = 0.5$ (c) $k = 1.0$

In all three cases these functions are quasi-concave.

But only for k = 0.2 is the function strictly concave.
The case k = 1.0 clearly shows nonconcavity because the function is not below its tangent plane.