Intermediate Microeconomics

Spring 2025

Week 13: Uncertainty and Risk Aversion

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How to make decisions under uncertainty?

- How to model uncertainty?

- Random variable
  - X = random variable, x = a realization of X
  - Example: X = outcome of throwing a dice, a realization can be any number between 1 and 6

- If a lottery offers n distinct prizes and the probabilities of winning the prizes are π_i (i=1,…,n) then

$$\sum_{i=1}^{n} \pi_i = 1$$

Expected Value

- For a lottery (X) with prizes $x_1, x_2, \ldots, x_n$ and the probabilities of winning $\pi_1, \pi_2, \ldots \pi_n$, the expected value of the lottery is

$$E(X) = \pi_1 x_1 + \pi_2 x_2 + \ldots + \pi_n x_n$$

$$E(X) = \sum_{i=1}^{n} \pi_i x_i$$

- The expected value is a weighted sum of the outcomes
  - the weights are the respective probabilities

Expected value

- If X is a continuous random variable,

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

- Where $f(x)$ = probability density function of X

Expected Value

- Games which have an expected value of zero (or cost their expected values) are called <u>actuarially fair games</u>

Fair Games

- People are generally unwilling to play fair games
- There may be a few exceptions
  - when very small amounts of money are at stake
  - when there is utility derived from the actual play of the game
    - we will assume that this is not the case

St. Petersburg Paradox

- A coin is flipped until a head appears
- If a head appears on the n^{th} flip, the player is paid $\$2^n$

$$x_1 = \$2,\ x_2 = \$4,\ x_3 = \$8, \ldots,\ x_n = \$2^n$$

- The probability of getting a head on the i^{th} trial is $(½)^i$

$$\pi_1 = ½,\ \pi_2 = ¼, \ldots,\ \pi_n = 1/2^n$$

St. Petersburg Paradox

- The expected value of the St. Petersburg paradox game is infinite

$$E(X)=\sum_{i=1}^{\infty}\pi_i x_i=\sum_{i=1}^{\infty}2^i\left(\frac{1}{2}\right)^i$$

$$E(X)=1+1+1+\ldots+1=\infty$$

- Because no player would pay a lot to play this game, it is not worth its infinite expected value

Expected Utility

- Individuals do not care directly about the dollar values of the prizes
  - they care about the utility that the dollars provide

- If we assume diminishing marginal utility of wealth, the St. Petersburg game may converge to a finite expected utility value
  - this would measure how much the game is worth to the individual

Daniel Bernoulli’s Solution to St.Petersburg’s Paradox

- Assume declining marginal utility of income, then the expected utility value (“moral value”) of the game may be finite.

- For example, consider $U(x) = \log(x)$, then

$$\text{expected utility} = \sum_{i=1}^{\infty} \frac{1}{2^i} \log\left(2^i\right) = \sum_{i=1}^{\infty} \frac{i}{2^i} \log 2 = (\log 2)\ \left(\sum_{i=1}^{\infty} \frac{i}{2^i}\right)$$

$$= 2\log 2 = 1.39$$

- Therefore, this individual will be willing to offer at most \$M to play the game, i.e., $U(M) = \log M = 1.39$, i.e., $M = e^{1.39} = 4.015$

Expected Utility

- Expected utility can be calculated in the same manner as expected value

$$E(X) = \sum_{i=1}^{n} \pi_i U(x_i)$$

- Because utility may rise less rapidly than the dollar value of the prizes, it is possible that expected utility will be less than the monetary expected value

The von Neumann-Morgenstern Theorem

- Suppose that there are n possible prizes that an individual might win ($x_1,\ldots,x_n$) arranged in ascending order of desirability
  - x_1 = least preferred prize $\Rightarrow U(x_1) = 0$ (arbitrary)
  - x_n = most preferred prize $\Rightarrow U(x_n) = 1$

- The point of the von Neumann-Morgenstern theorem is to show that there is a reasonable way to assign specific utility numbers to the other prizes available

The von Neumann-Morgenstern Theorem

- Consider the following experiment.
- Ask the individual to state the probability π_i,
- at which he or she would be indifferent between x_i with certainty,
- and a gamble offering prizes of x_n with probability π_i and x_1 with probability $1 - \pi_i$.

- The von Neumann-Morgenstern method is to define the utility of x_i as the expected utility of the gamble that the individual considers equally desirable to x_i

$$U(x_i) = \pi_i \cdot U(x_n) + (1 - \pi_i) \cdot U(x_1)$$

The von Neumann-Morgenstern Theorem

- Since $U(x_n) = 1$ and $U(x_1) = 0$

$$U(x_i) = \pi_i \cdot 1 + (1 - \pi_i) \cdot 0 = \pi_i$$

- The utility number attached to any other prize is simply the probability of winning it

- Note that this choice of utility numbers is arbitrary

Expected Utility Maximization

- A rational individual will choose among gambles based on their expected utilities (the expected values of the von Neumann-Morgenstern utility index)

- Expected utility function = von Neumann-Morgenstern utility function

Martingale/Double-up Gambling Strategy

- A gambler bets B and will win B if a coin comes up head and loses it if it comes up tail.

- The martingale or double-up strategy is that the gambler doubles his bet after every loss, therefore the first win will recover all previous losses plus a profit that is equal to the initial bet.

- Let B = initial bet, x_i = profit when the first head appears on the i^{th} flip and π_i= probability that the first head appears on the i^{th} flip.

Martingale/Double-up Gambling Strategy

$$
\begin{aligned}
x_1 &= B \\
x_2 &= 2B - B = B \\
x_3 &= 2(2B) - (B + 2B) = B \\
x_4 &= 2(2(2B)) - (B + 2B + 2(2B)) = B \\
&\vdots \\
x_n &= 2^{n-1}B - (1 + 2 + 2^2 + 2^3 + ... + 2^{n-3} + 2^{n-2})B = 2^{n-1}B - (2^{n-1} - 1)B = B
\end{aligned}
$$

and $\pi_1 = \frac{1}{2}$, $\pi_2 = \frac{1}{4}, \pi_3 = \frac{1}{8}, \pi_4 = \frac{1}{16}, ..., \pi_n = \frac{1}{2^n}$. Thus,

$$E(X) = \sum_{i=1}^{\infty} \pi_i x_i = \sum_{i=1}^{\infty} \frac{1}{2^i} B = B \sum_{i=1}^{\infty} \frac{1}{2^i} = B.$$

This strategy generates a finite expected value, which equals the initial bet.

$$EU(X) = \sum_{i=1}^{\infty} \pi_i U(x_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} U(B) = U(B) \sum_{i=1}^{\infty} \frac{1}{2^i} = U(B).$$

- Question: Is the martingale strategy a winning strategy?

Expected Utility Maximization

- Consider two gambles:
  - first gamble offers x_2 with probability q and x_3 with probability (1-q)

$$\text{expected utility (1)} = q \cdot U(x_2) + (1-q) \cdot U(x_3)$$

  - second gamble offers x_5 with probability t and x_6 with probability (1-t)

$$\text{expected utility (2)} = t \cdot U(x_5) + (1-t) \cdot U(x_6)$$

Expected Utility Maximization

- Substituting the utility index numbers gives

$$\text{expected utility (1)} = q \cdot \pi_2 + (1\text{-}q) \cdot \pi_3$$

$$\text{expected utility (2)} = t \cdot \pi_5 + (1\text{-}t) \cdot \pi_6$$

- The individual will prefer gamble 1 to gamble 2 if and only if

$$q \cdot \pi_2 + (1\text{-}q) \cdot \pi_3 > t \cdot \pi_5 + (1\text{-}t) \cdot \pi_6$$

Expected Utility Maximization

- If individuals obey the von Neumann-Morgenstern axioms of behavior in uncertain situations, they will act as if they choose the option that maximizes the expected value of their von Neumann-Morgenstern utility index

Risk Aversion

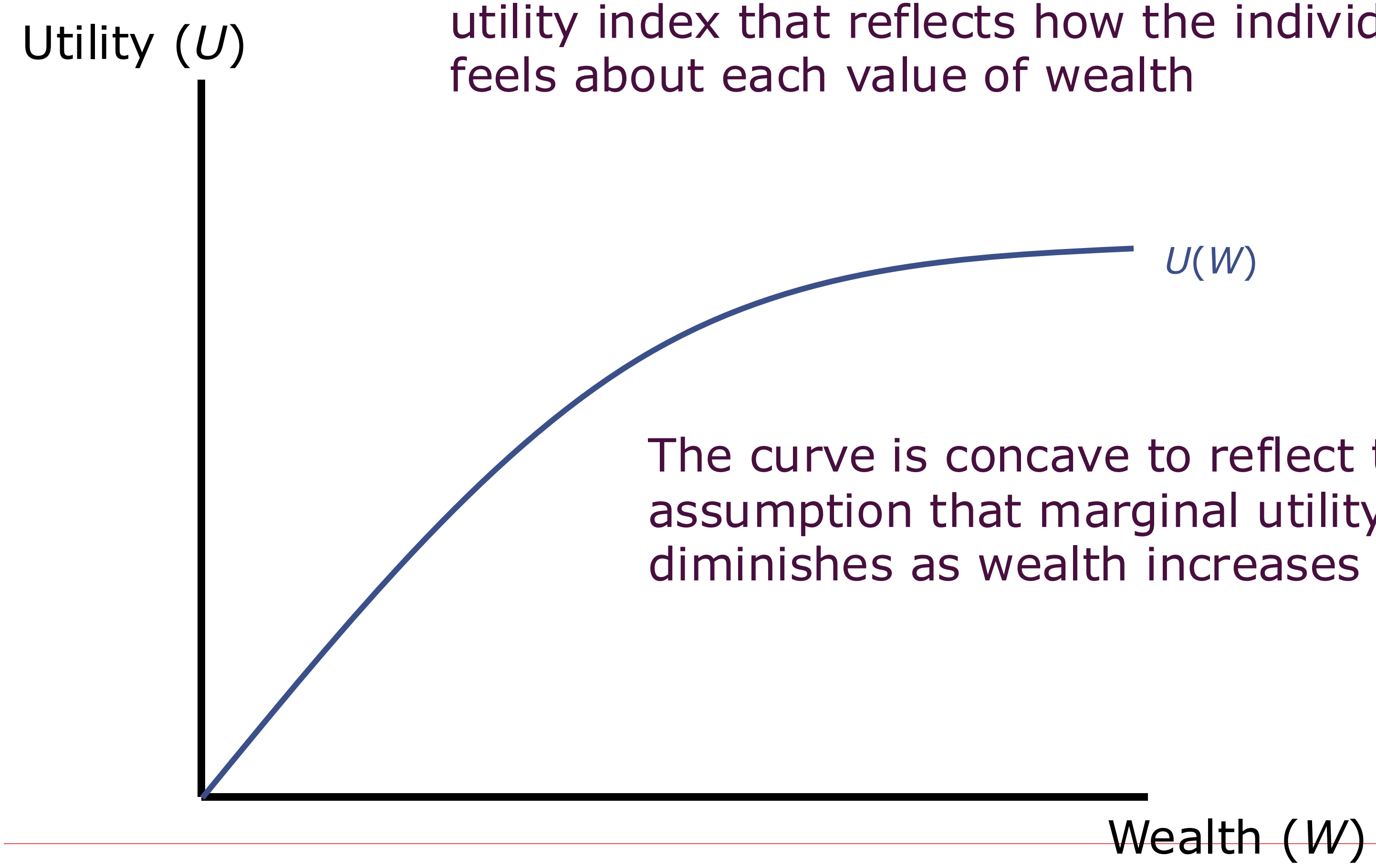
- Two lotteries may have the same expected value but differ in their riskiness
  - flip a coin for \$1 versus \$1,000
- <u>Risk</u> refers to the variability of the outcomes of some uncertain activity
- When faced with two gambles with the same expected value, individuals will usually choose the one with lower risk

Risk Aversion

- In general, we assume that the marginal utility of wealth falls as wealth gets larger
  - for example, few people would choose to take a $10,000 bet on the outcome of a coin flip, even though the average payoff is 0. The reason is that the gamble's money prizes do not completely reflect the utility provided by the prizes.
  - a flip of a coin for $1,000 promises a small gain in utility if you win, but a large loss in utility if you lose
  - a flip of a coin for $1 is inconsequential as the gain in utility from a win is not much different as the drop in utility from a loss

Risk Aversion

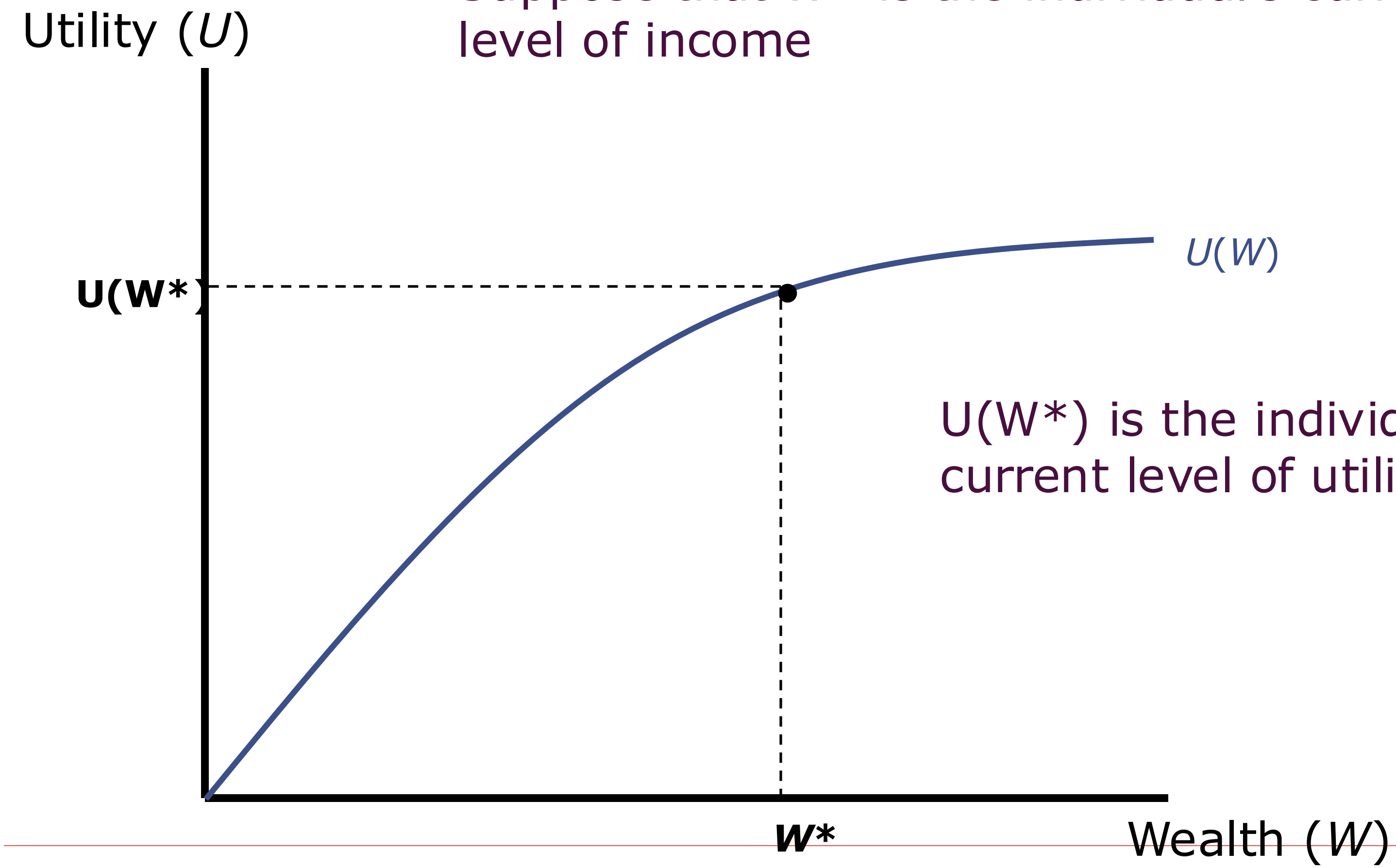
$U(W)$ is a von Neumann-Morgenstern utility index that reflects how the individual feels about each value of wealth



The curve is concave to reflect the assumption that marginal utility diminishes as wealth increases

Risk Aversion

Suppose that W^* is the individual's current level of income



U(W*) is the individual's current level of utility

Risk Aversion

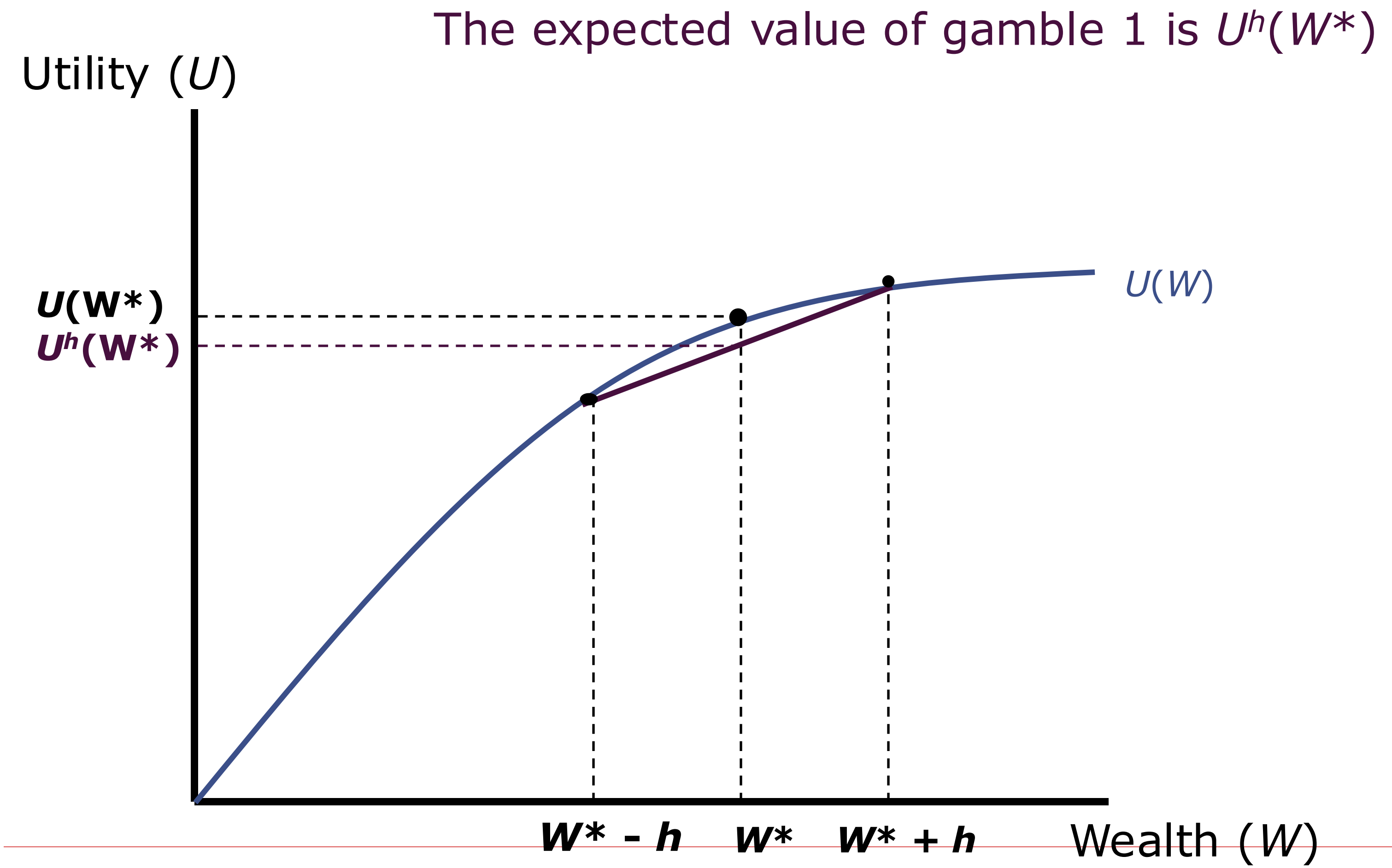
- Suppose that the person is offered two fair gambles:
  - a 50-50 chance of winning or losing $*h*

$$U^h(W^*) = \tfrac{1}{2}\, U(W^* + h) + \tfrac{1}{2}\, U(W^* - h)$$

  - a 50-50 chance of winning or losing $2*h*

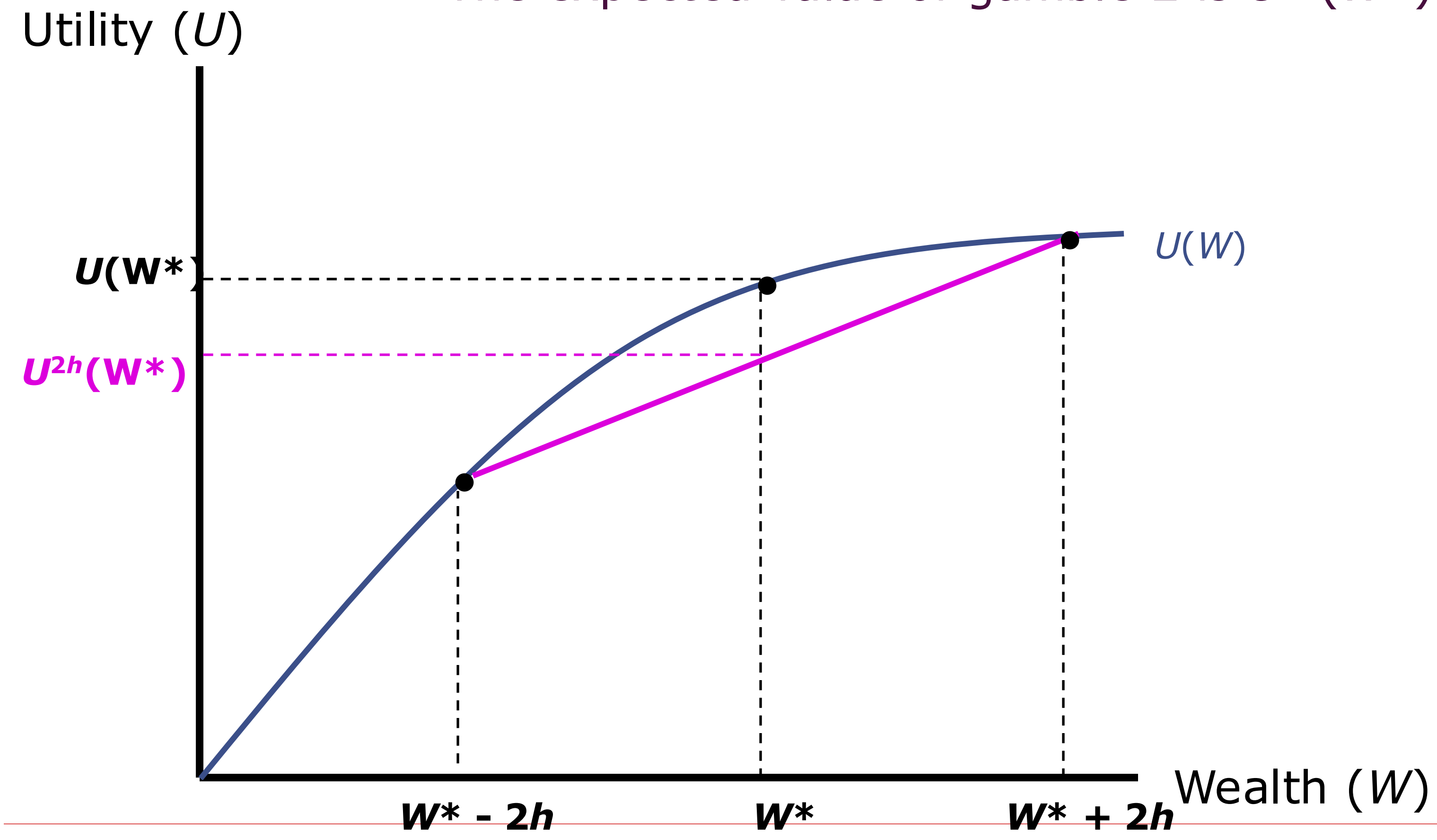
$$U^{2h}(W^*) = \tfrac{1}{2}\, U(W^* + 2h) + \tfrac{1}{2}\, U(W^* - 2h)$$

Risk Aversion



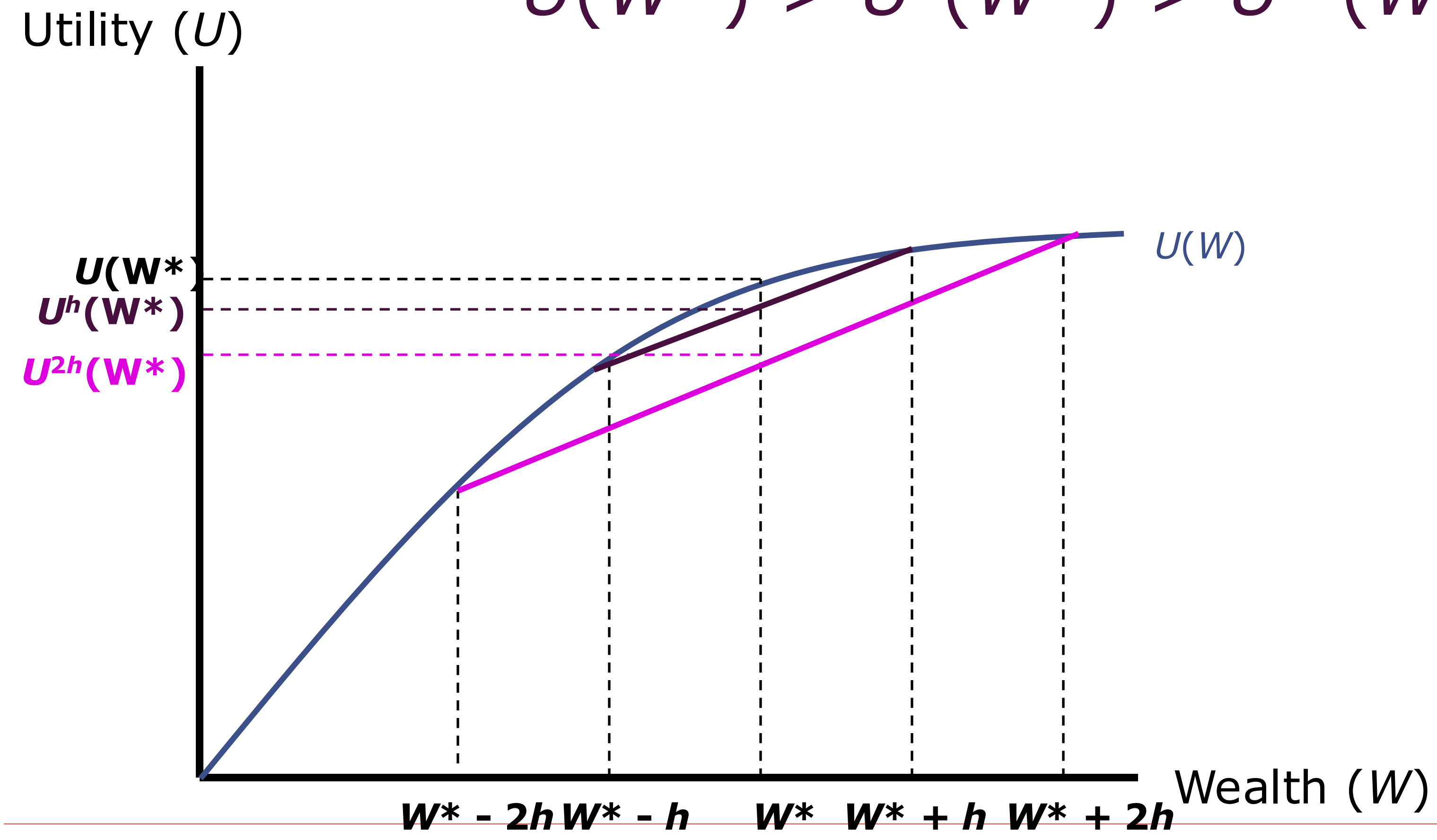
Risk Aversion

The expected value of gamble 2 is $U^{2h}(W^*)$



Risk Aversion

$$U(W^*) > U^h(W^*) > U^{2h}(W^*)$$



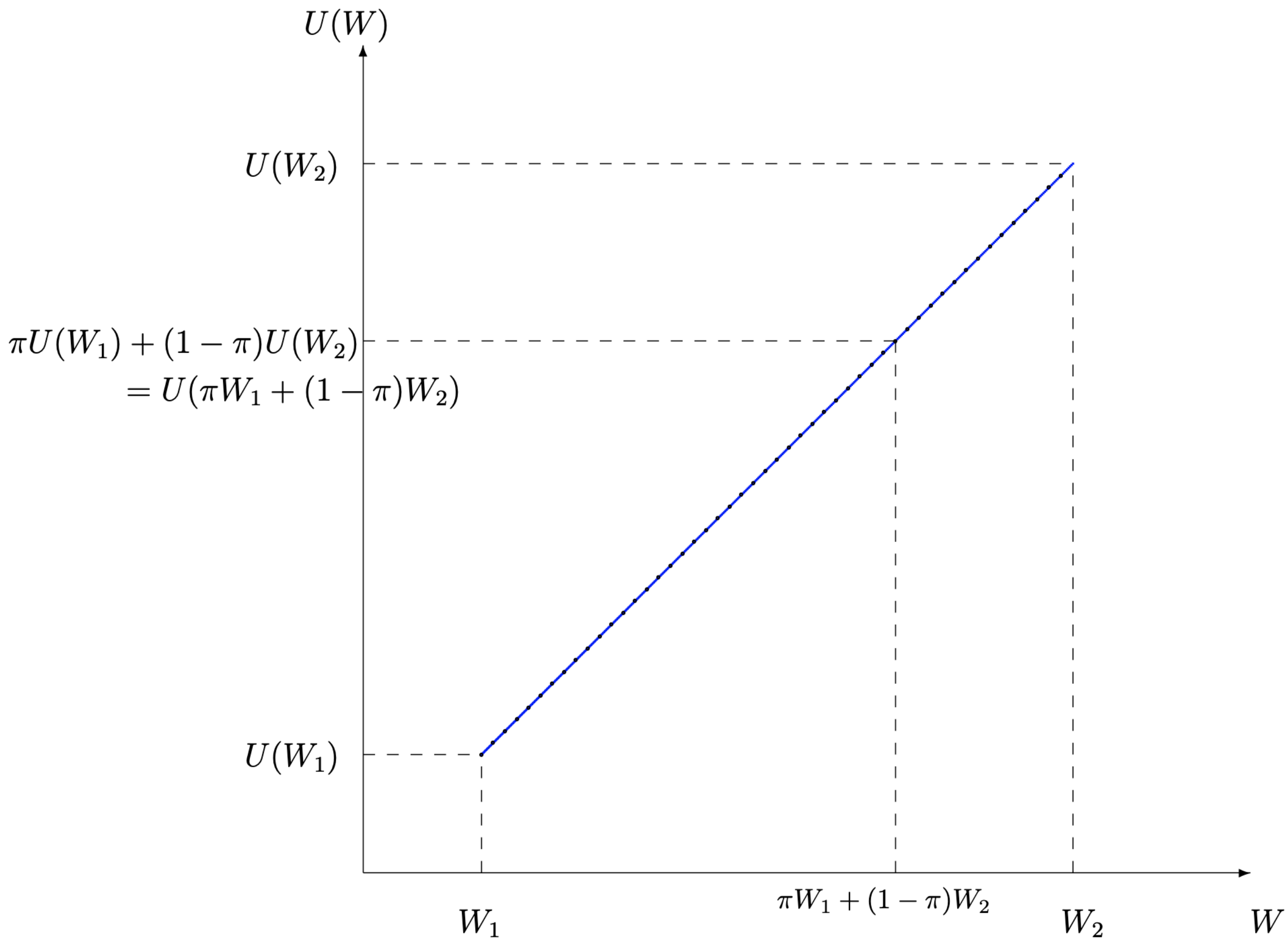
Risk Aversion

- The person will prefer current wealth to that wealth combined with a fair gamble
- The person will also prefer a small gamble over a large one

Risk Averse:	$\pi U(W_1) + (1-\pi)U(W_2) < U(\pi W_1 + (1-\pi)W_2)$ reject a fair gamble	
Risk Loving:	$\pi U(W_1) + (1-\pi)U(W_2) > U(\pi W_1 + (1-\pi)W_2)$ accept a fair gamble	
Risk Neutral:	$\pi U(W_1) + (1-\pi)U(W_2) = U(\pi W_1 + (1-\pi)W_2)$ indifferent to rejecting or accepting a fair gamble	

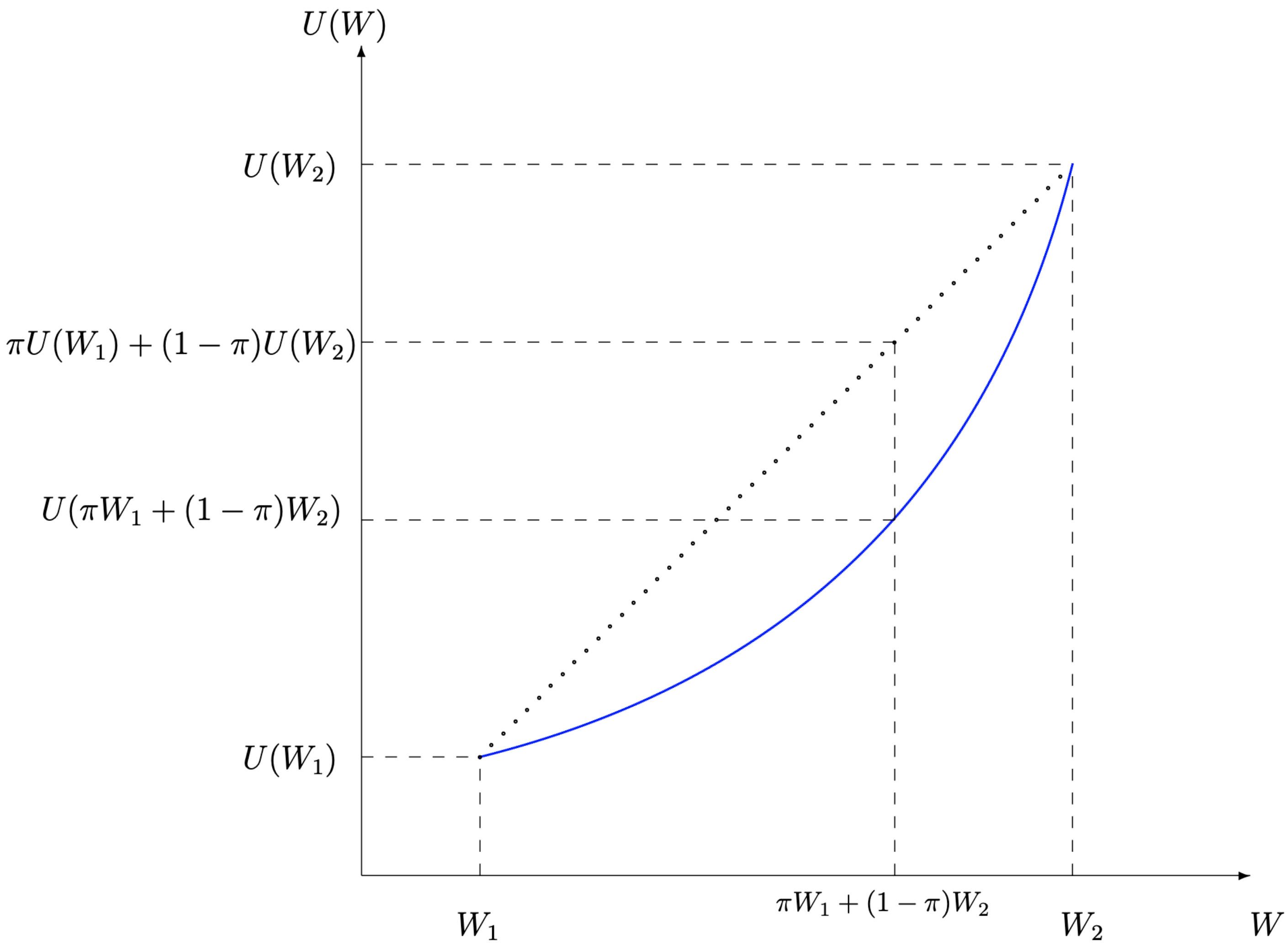
Risk Neutral

$\pi U(W_1) + (1-\pi)U(W_2) = U(\pi W_1 + (1-\pi)W_2)$



Risk Loving

$$\pi U(W_1) + (1-\pi)U(W_2) > U(\pi W_1 + (1-\pi)W_2)$$

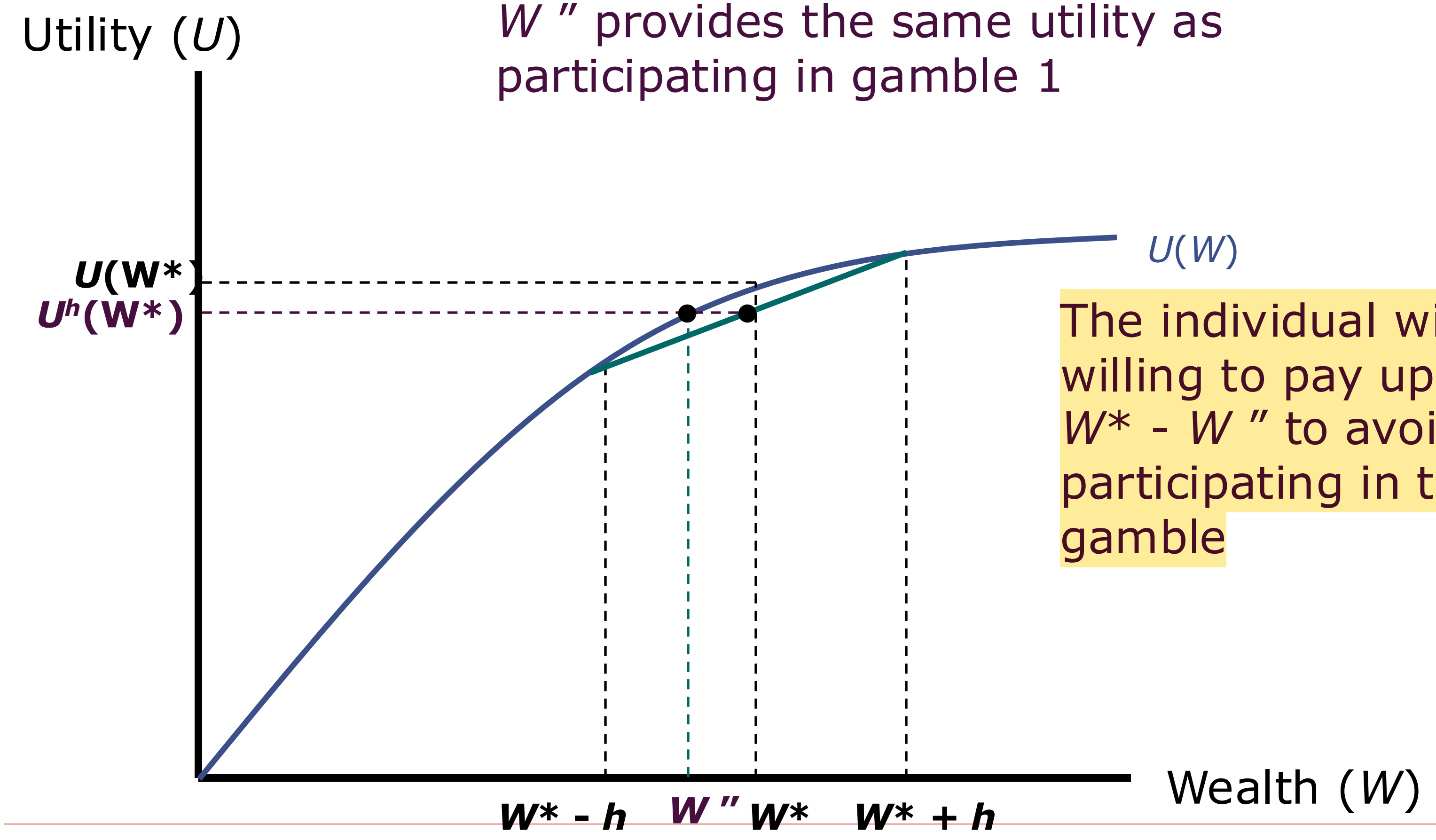


Risk Aversion and Insurance

- The person might be willing to pay some amount to avoid participating in a gamble
- This helps to explain why some individuals purchase insurance

Risk Aversion and insurance

W ″ provides the same utility as participating in gamble 1



The individual will be willing to pay up to *W** - *W ″* to avoid participating in the gamble

Willingness to Pay for Insurance

- Consider a person with a current wealth of \$100,000 who faces a 25% chance of losing his automobile worth \$20,000
- Suppose also that the person's von Neumann-Morgenstern utility index is

$$U(W) = ln\ (W)$$

Willingness to Pay for Insurance

- The person's expected utility will be

$$E(U) = 0.75U(100{,}000) + 0.25U(80{,}000)$$

$$E(U) = 0.75\, ln(100{,}000) + 0.25\, ln(80{,}000)$$

$$E(U) = 11.45714$$

- In this situation, a fair insurance premium would be $5,000 (25% of $20,000)

Willingness to Pay for Insurance

- The individual will likely be willing to pay more than \$5,000 to avoid the gamble. How much will he pay?

$$E(U) = U(100{,}000 - x) = ln(100{,}000 - x) = 11.45714$$

$$100{,}000 - x = e^{11.45714}$$

$$x = 5{,}426$$

- The maximum premium is \$5,426

Measuring Risk Aversion

- The most commonly used risk aversion measure was developed by J.W.Pratt in the 1960s.

$$r(W) = -\frac{U''(W)}{U'(W)}$$

- For risk averse individuals, $U''(W) < 0$
  - $r(W)$ will be positive for risk averse individuals
  - $r(W)$ is invariant with respect to linear transformations of the utility function.

Measuring Risk Aversion

- The Pratt measure of risk aversion is proportional to the amount an individual will pay to avoid a fair gamble

Measuring Risk Aversion

- Suppose the winnings from such a fair bet are denoted by the random variable h (which takes on both positive and negative values).

$$E(h) = 0$$

- Let p be the size of the insurance premium that would make the individual exactly indifferent between taking the fair bet h and paying p with certainty to avoid the gamble

$$E[U(W + h)] = U(W - p)$$

Measuring Risk Aversion

- We now expand both sides of the equation using Taylor's series

- Because p is a fixed amount, we can use a simple linear approximation to the right-hand side

$$U(W - p) = U(W) - pU'(W) + \text{higher order terms}$$

Measuring Risk Aversion

- For the left-hand side, we need to use a quadratic approximation to allow for the variability of the gamble (h)

$$E[U(W + h)] = E[U(W) + hU'(W) + h^2/2\ U''(W) + \text{higher order terms}]$$

$$= U(W) + E(h)U'(W) + E(h^2)/2\ U''(W) + \text{higher order terms}$$

Measuring Risk Aversion

- Remembering that $E(h)=0$, dropping the higher order terms, and substituting k for $E(h^2)/2$, we get

$$U(W) - pU'(W) \cong U(W) + kU''(W)$$

$$p \cong -\frac{kU''(W)}{U'(W)} = kr(W)$$

- the amount that a risk-averse individual is willing to pay to avoid a fair bet is approximately proportional to Pratt's risk aversion measure.

Measuring Risk Aversion

- Because insurance premiums paid are observable in the real world, these are often used to estimate individuals' risk aversion coefficients or to compare such coefficients among groups of individuals.

- Therefore, it is possible to use market information to learn a bit about attitudes toward risky situations.

Risk Aversion and Wealth

- An important question is whether risk aversion increases or decreases with wealth.

- It is not necessarily true that risk aversion declines as wealth increases
  - diminishing marginal utility would make potential losses less serious for high-wealth individuals
  - however, diminishing marginal utility also makes the gains from winning gambles less attractive
  - the net result depends on the shape of the utility function

Risk Aversion and Wealth: examples

- If utility is quadratic in wealth

$$U(W) = a + bW + cW^2$$

where $b > 0$ and $c < 0$

- What is Pratt's risk aversion measure?

- Does risk aversion increase or decrease as wealth increases?

increase

Risk Aversion and Wealth: examples

- If utility is quadratic in wealth

$$U(W) = a + bW + cW^2$$

where $b > 0$ and $c < 0$

- Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{-2c}{b + 2cW}$$

- Risk aversion increases as wealth increases

Risk Aversion and Wealth

- If utility is logarithmic in wealth

$$U(W) = ln\ (W)$$

where $W > 0$

- What is Pratt's risk aversion measure?

- Does risk aversion increase or decrease as wealth increases?

Risk Aversion and Wealth

- If utility is logarithmic in wealth

$$U(W) = ln\ (W\ )$$

where $W > 0$

- Pratt's risk aversion measure is

$$r(W) = -\frac{U"(W)}{U(W)} = \frac{1}{W}$$

- Risk aversion decreases as wealth increases

decrease

Risk Aversion and Wealth

- If utility is exponential

$$U(W) = -e^{-AW}$$

where A is a positive constant

- What is Pratt's risk aversion measure?

- Does risk aversion increase or decrease as wealth increases?

Risk Aversion and Wealth

- If utility is exponential

$$U(W) = -e^{-AW}$$

where A is a positive constant

- Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U(W)} = \frac{A^2 e^{-AW}}{Ae^{-AW}} = A$$

- Risk aversion is constant as wealth increases

constant
所以，risk averse measure与wealth的关系取决于utility function的形式

Relative Risk Aversion

- The relative risk aversion formula is

$$rr(W) = Wr(W) = -W\frac{U''(W)}{U'(W)}$$

Relative Risk Aversion

- The power utility function

$$U(W,R) = \begin{cases} W^R/R & \text{for } R < 1,\ R \neq 0 \\ \ln W & \text{for } R = 0 \end{cases}$$

exhibits diminishing absolute risk aversion

$$r(W) = -\frac{U''(W)}{U'(W)} = -\frac{(R-1)W^{R-2}}{W^{R-1}} = -\frac{(R-1)}{W}$$

but constant relative risk aversion

$$rr(W) = Wr(W) = -(R-1) = 1-R$$