Intermediate Microeconomic

Spring 2025

Part three: Production and supply

Week 4(a): Producer theory basics

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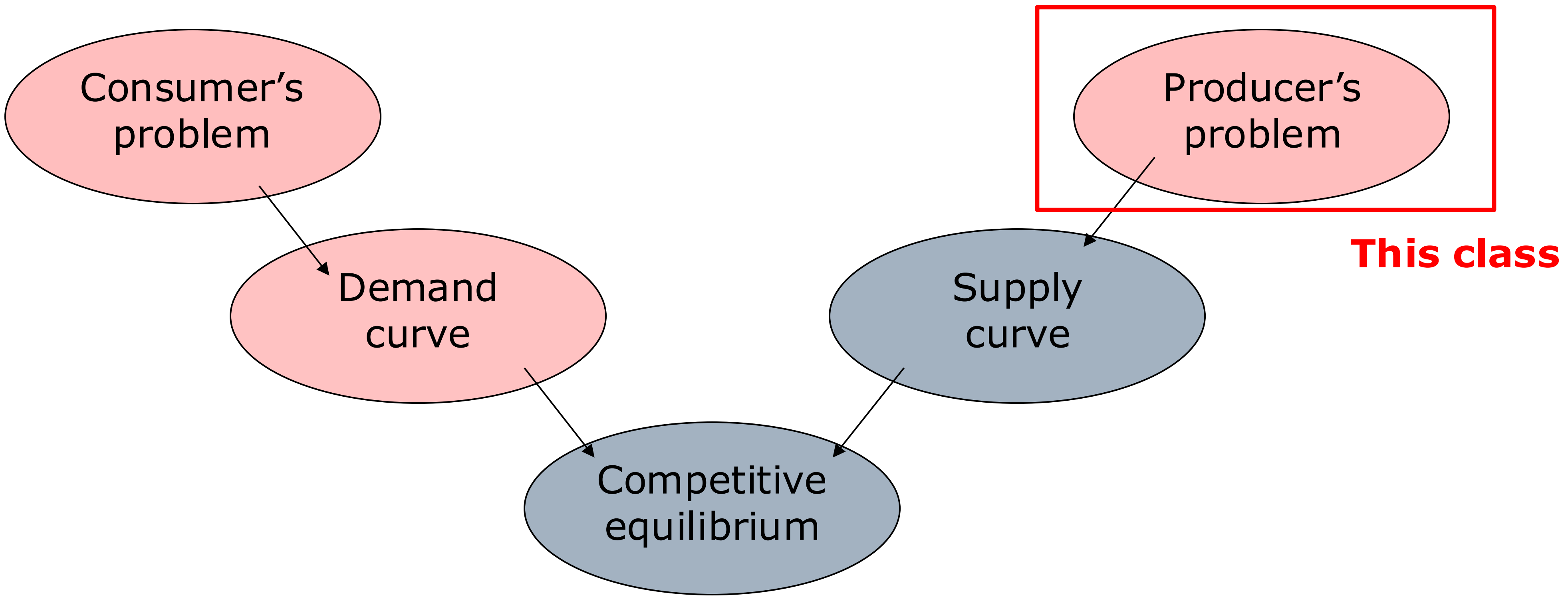
Big Picture


Consumer's problem
Demand curve
Producer's problem
This class
Supply curve
Competitive equilibrium

Producer's Problem

- A producer wakes up in the morning. The world is such that she can use two inputs, capital (K) and labor (L), to produce an output (Q). Her production technology is summarized by Q = F(K, L). She looks outside the window and observe:

  v = price of capital

  w = price of labor

  p = price of output

- Producer's **Profit Maximization Problem**: find the optimal choice of (Q*, K*, L*) that maximizes firm's profit: pQ – vK – wL

Producer’s Problem: Clarification

- You might wonder: don’t firms set price for their products?
  - You should consider the producer’s problem as a “managerial” problem: given the price of output and inputs, what is the best production decision to maximize profits
  - More formally, known as the **price taking assumption**

- There are more complicated models with pricing strategy
  - When firms have market power (e.g., monopoly), then they simultaneously choose price and quantity.
  - But even in that case, they still maximize profits
  - We begin our study with price-taking firms; we return to pricing problems later

Producer’s Problem

- Producer’s **Profit Maximization Problem (PMP)**: Given the production technology Q=F(K,L), find the optimal choice of output (Q*), capital (K*), labor(L*) that maximizes firm’s profit: pQ – vK – wL

- If you think about it, PMP has two parts
  - How many output Q should I produce?
  - Given I want to produce Q, what’s the optimal combination of K and L that I should choose

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We will first study how to solve this part of the problem.

Producer's Problem

- Producer's **Profit Maximization Problem (PMP)**: Given the production technology $Q=F(K,L)$, find the optimal choice of output (Q^*), capital (K^*), labor(L^*) that maximizes firm's profit: $pQ - vK - wL$

- If you think about it, PMP has two parts
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Known as the "Cost Minimization Problem" (CMP)

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Known as the "Cost Minimization Problem" (CMP)

- A profit-maximizing producer is *necessarily* cost-minimizing

Technology

- The center of producer's problem is its **technology**, or **production function** Q=F(K,L)

- Three important aspects of the production function
  - Technology sets
  - Isoquant curves
  - Returns to scale

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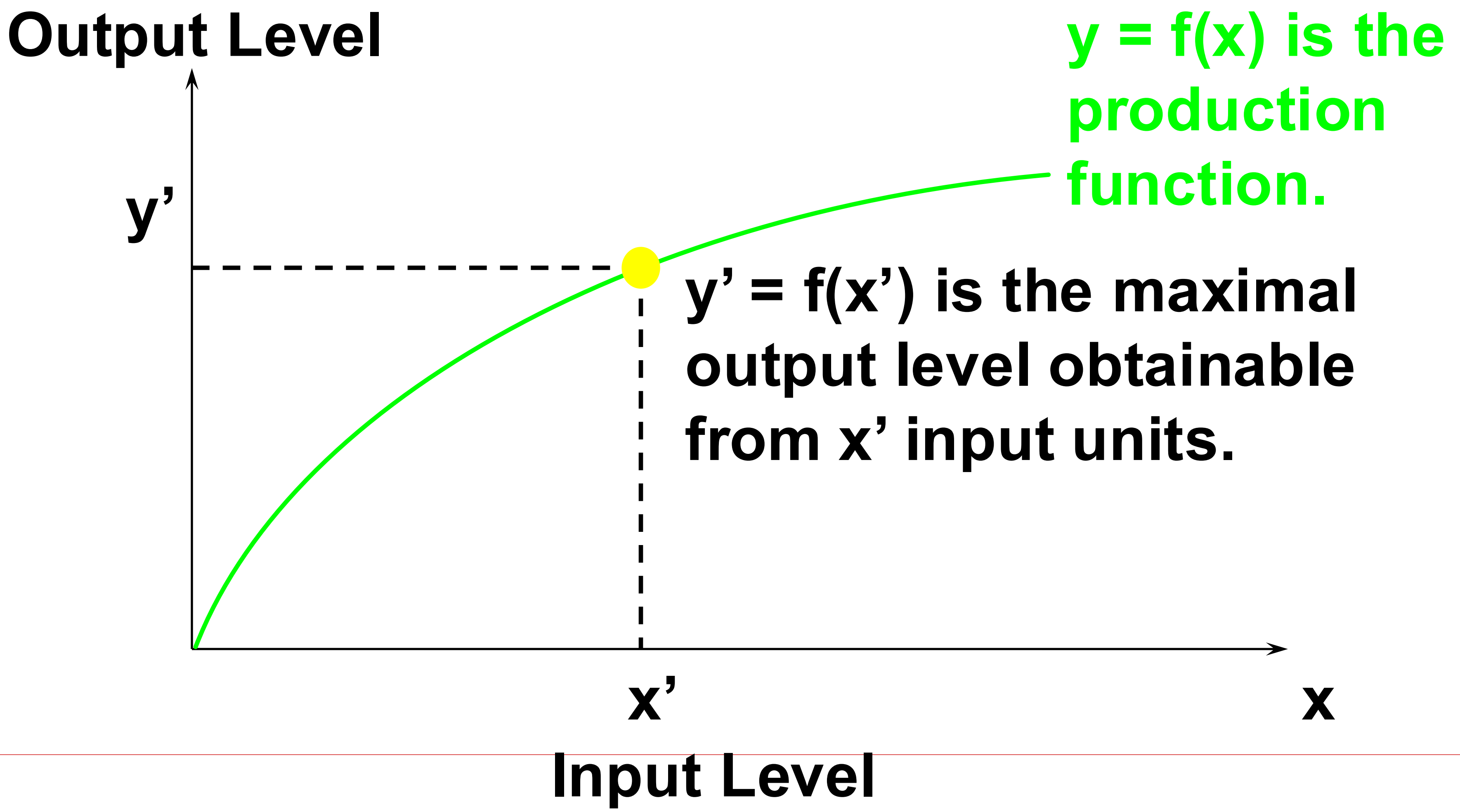
Production Functions



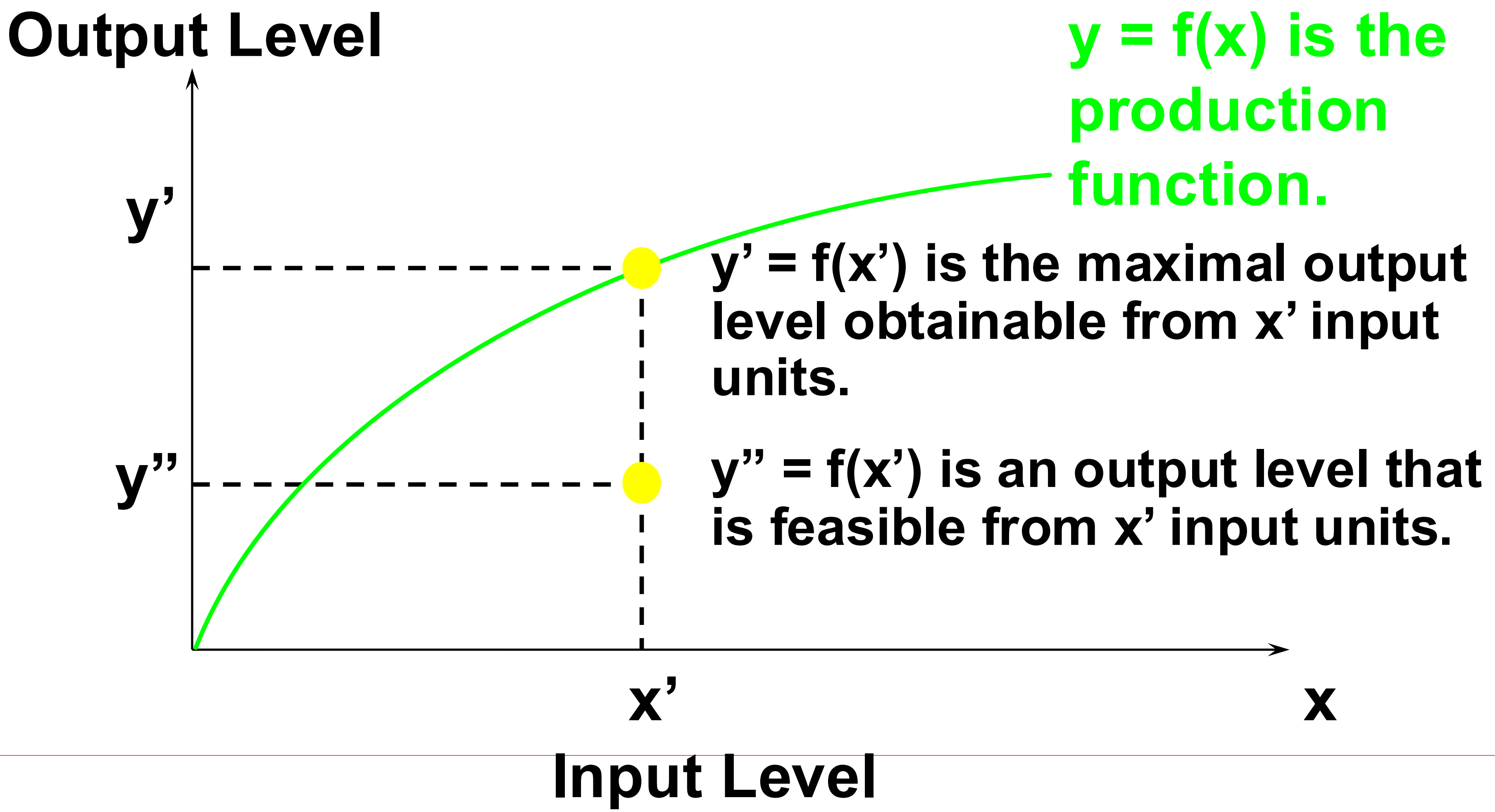
Technology Sets

- A production plan is an input bundle and an output level; $(x_1, \dots, x_n, y)$.
- A production plan is feasible if

$$\mathbf{y \le f(x_1, \cdots, x_n)}$$

- The collection of all feasible production plans is the technology set.

Technology Sets

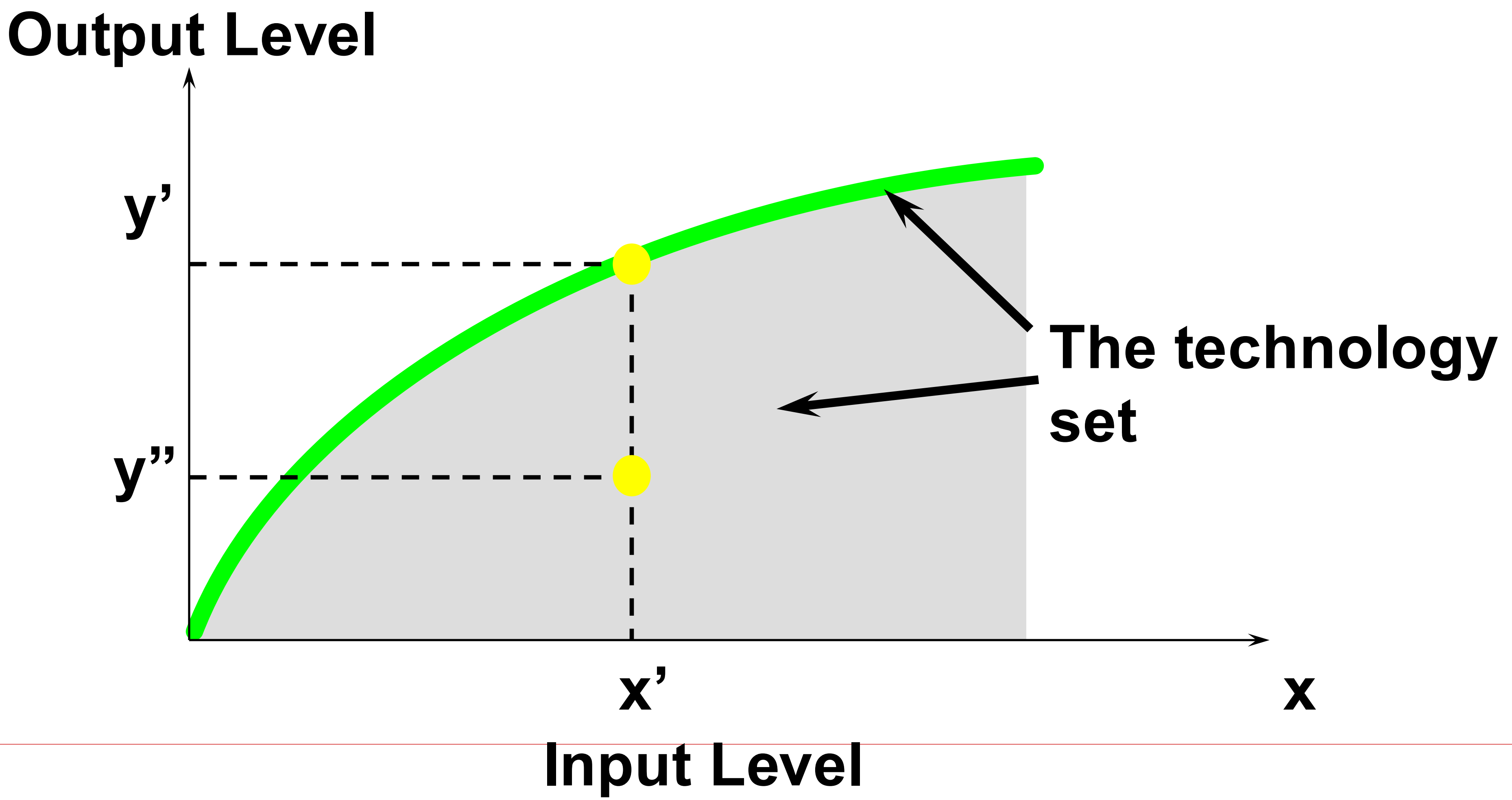


Technology Sets

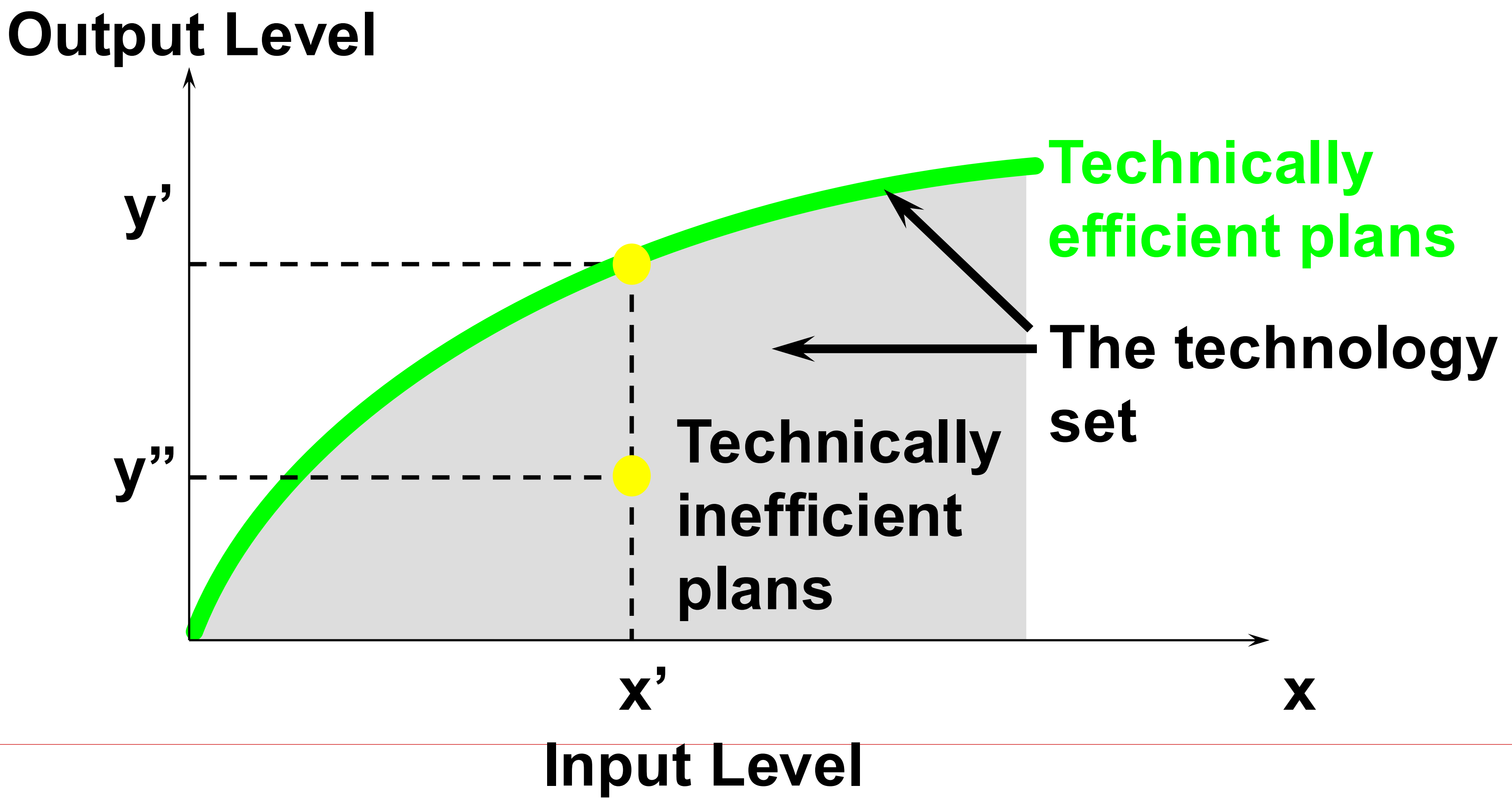
The technology set is

$$T = \{(x_1, \cdots, x_n, y) \mid y \le f(x_1, \cdots, x_n) \text{ and } x_1 \ge 0, \ldots, x_n \ge 0\}.$$

Technology Sets



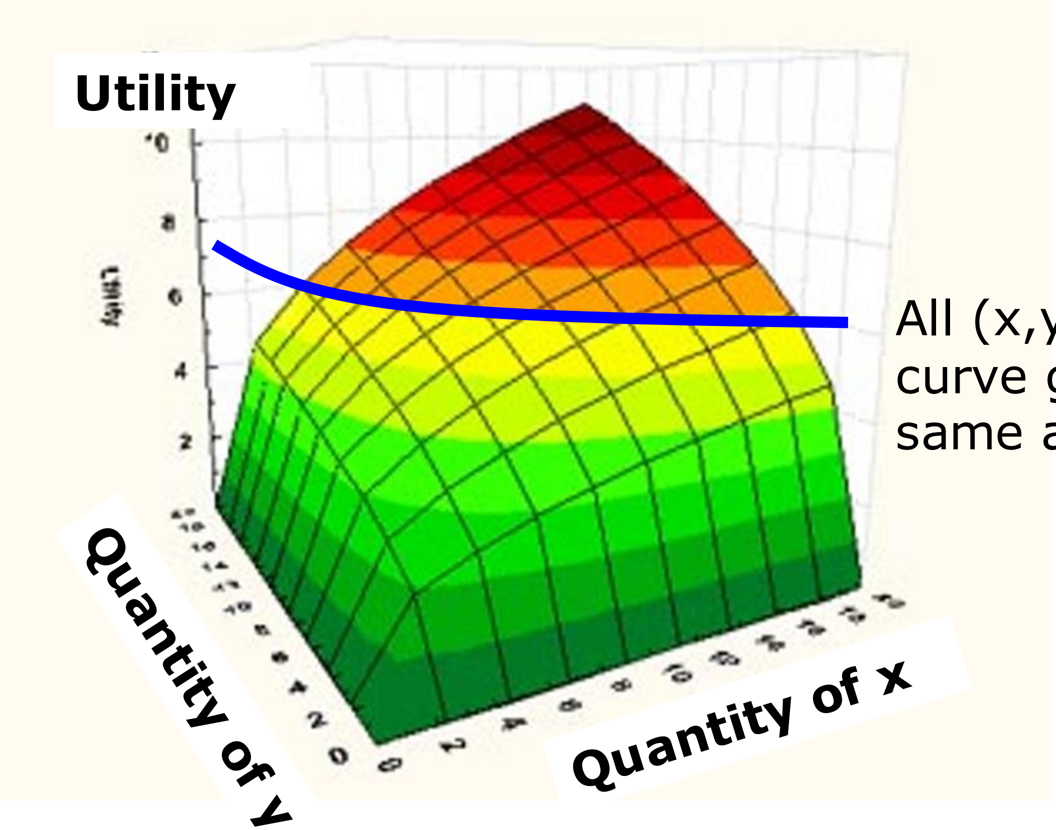
Technology Sets



Technology

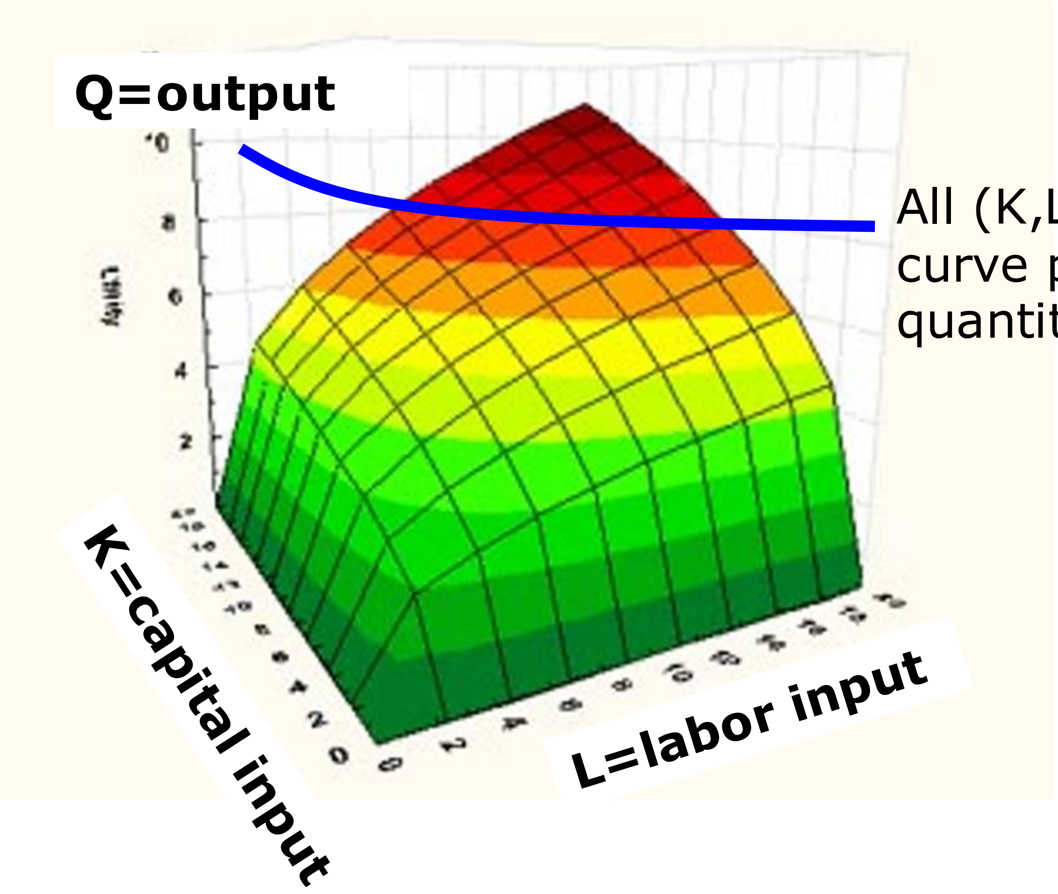
- The center of producer's problem is its **technology**, or **production function** Q=F(K,L)
- Three important aspects of the production function
  - Technology sets
  - Isoquant curves
  - Returns to scale

Recall from consumer theory



All (x,y) bundles along this curve gives the consumer the same amount of utility

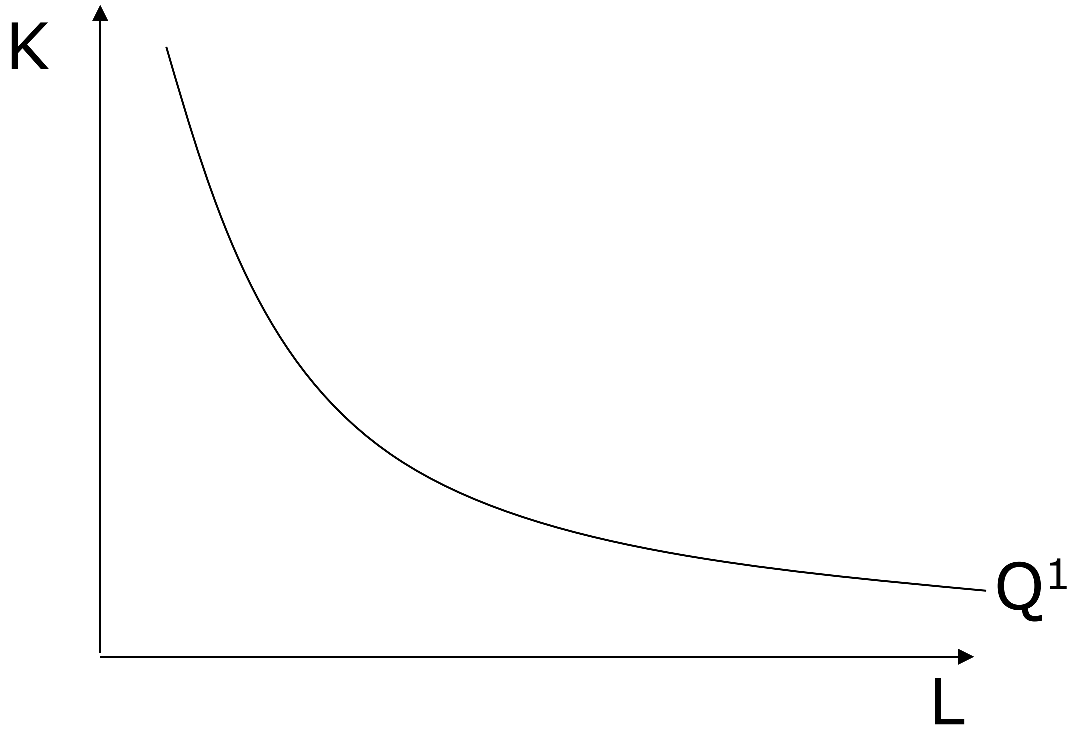
Production Function



All (K,L) pairs along this curve produces the same quantity

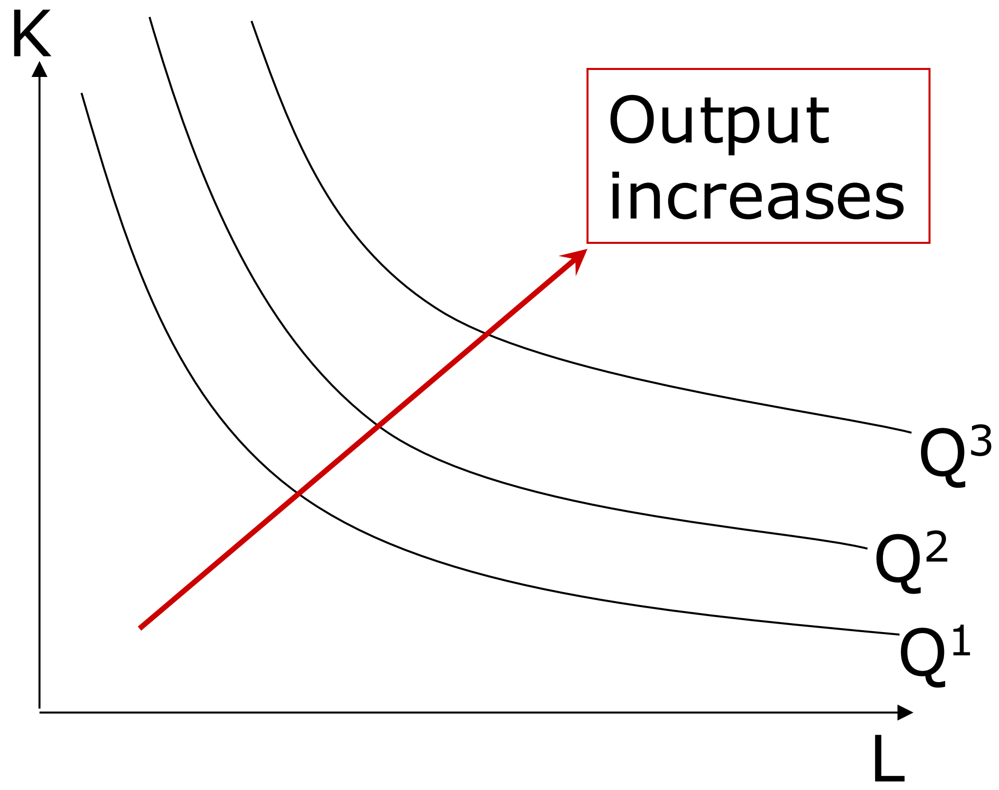
Production "isoquants"

- A key feature of the production function Q=F(K,L): "isoquants"
- An ***isoquant*** is a curve connecting combinations of K and L that produce the same output, Q.
- An isoquant is all (K, L) pairs such that $F(K, L) = Q^1$.



Isoquants and output

- Greater inputs produce more output.
- As isoquants move up and to the right, output increases.
- Production isoquant map looks like a utility indifference curve map, but they differ in two important ways.



Isoquants vs. indifference curves

- A production isoquant has nothing to do with preferences – it has to do with technical feasibility, i.e., technology.
- The "quantity" associated with an isoquant has intrinsic meaning (units of good produced) while utility is an ordinal measure.
- The distance between isoquants matters.
  - A utility difference of 3 doesn't tell us much, but a quantity difference of 3 does.
  - $F(K,L) = K^{\frac{2}{3}}L^{\frac{1}{3}}$ and $F(K,L) = \frac{2}{3}\ln(K) + \frac{1}{3}\ln(L)$ do NOT represent the same production process.

The Marginal Product of an Input

- If you increase K or L by a small amount, output $F(K,L)$ increases: $\frac{\partial F(K,L)}{\partial K} > 0$; $\frac{\partial F(K,L)}{\partial L} > 0$.
- We call the amount by which output increases the **marginal product** of that input.
- From now on, use notation: $MP_K = \frac{\partial F(K,L)}{\partial K}$; $MP_L = \frac{\partial F(K,L)}{\partial L}$
- Frequent assumption is **declining marginal product**:

$$\frac{\partial MP_K}{\partial K} = \frac{\partial^2 F(K,L)}{\partial K^2} < 0; \qquad \frac{\partial MP_L}{\partial L} = \frac{\partial^2 F(K,L)}{\partial L^2} < 0$$

Average Physical Product

- Labor productivity
  - Often means average productivity

- Average product of labor

$$AP_l = \frac{\text{output}}{\text{labor input}} = \frac{q}{l} = \frac{f(k,l)}{l}$$

AP_l also depends on the amount of capital employed

A Two-Input Production Function

- Suppose the production function for flyswatters can be represented by

$$q = f(k,l) = 600k^2l^2 - k^3l^3$$

- To construct MP_l and AP_l, we must assume a value for k
  - let $k = 10$

- What is the marginal productivity MP_l, and the average productivity AP_l?

A Two-Input Production Function

- The production function becomes

$$q = 60{,}000l^2 - 1000l^3$$

- The marginal productivity function is

$$MP_l = \partial q/\partial l = 120{,}000l - 3000l^2$$

which diminishes as l increases

- This implies that q has a maximum value:

$$120{,}000l - 3000l^2 = 0$$

$$l = 40$$

- Labor input beyond $l = 40$ reduces output

A Two-Input Production Function

- To find average productivity, we hold k=10 and solve

$$AP_l = q/l = 60{,}000l - 1000l^2$$

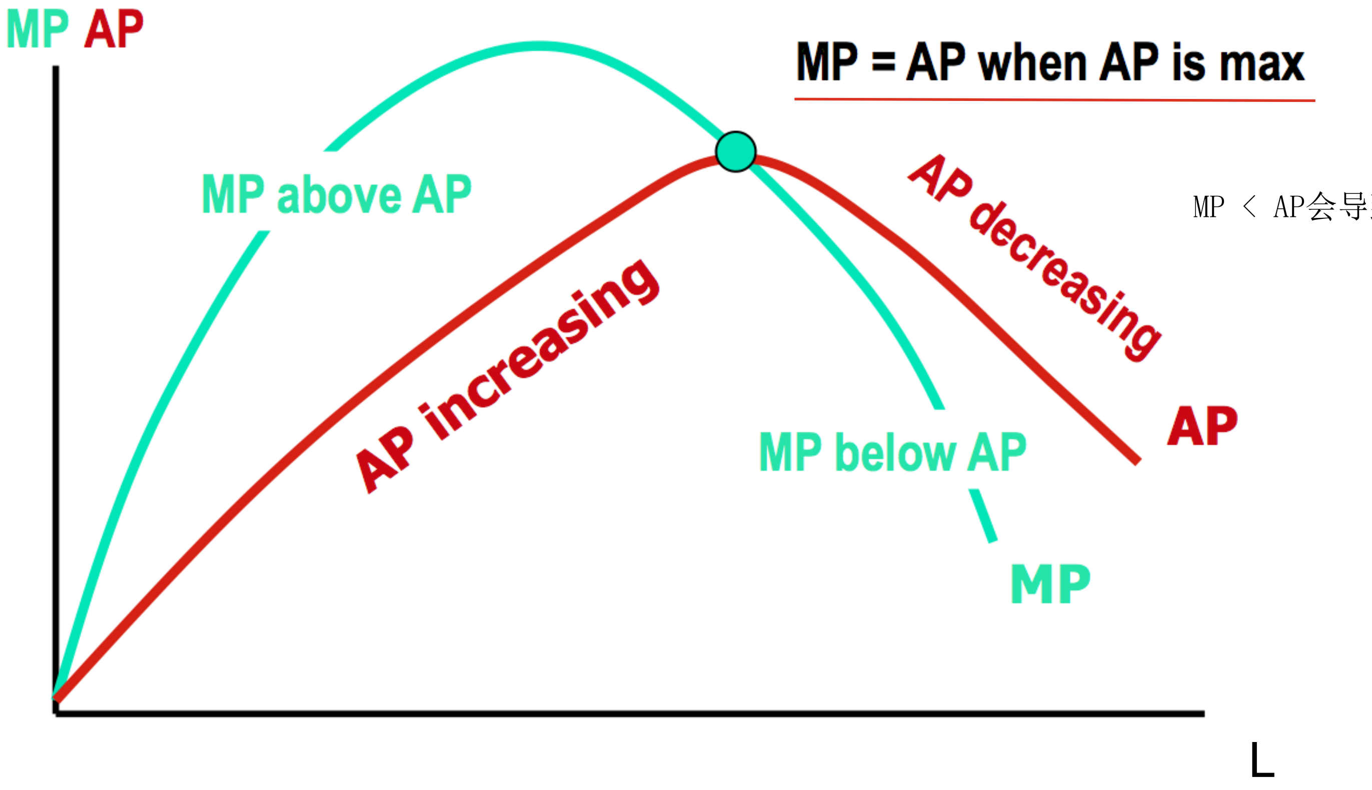
- AP_l reaches its maximum where

$$\partial AP_l/\partial l = 60{,}000 - 2000l = 0$$

$$l = 30$$

- when l = 30, both AP_l and MP_l are equal to 900,000
- Thus, when AP_l is at its maximum, AP_l and MP_l are equal

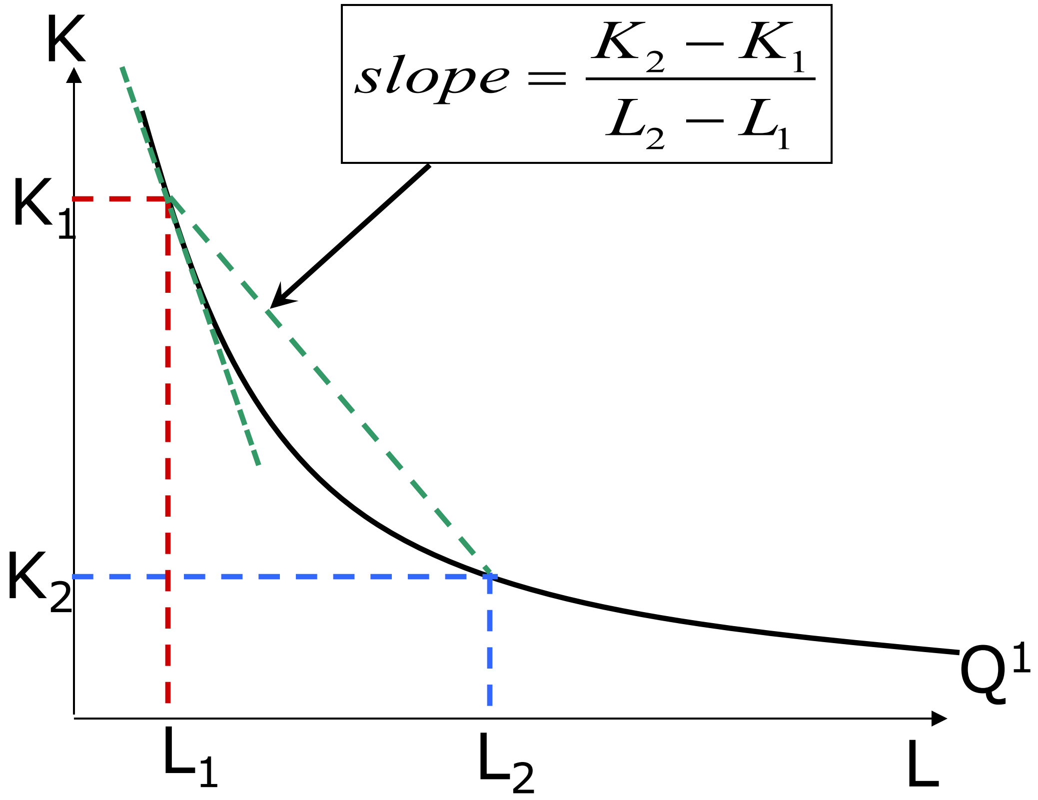
Marginal product and average product



MP < AP会导致AP减小

Marginal Rate of Technical Substitution

- Marginal Rate of Technical Substitution (MRTS) characterizes substitutability between inputs.
- If one factor increases, by how much can you reduce the other factor to keep Q constant?
- When L increases from L_1 to L_2, K reduces from K_1 to K_2.
- Ratio $\Delta K/\Delta L$.
- For a small change, the ratio $\Delta K/\Delta L$ is the slope of the isoquant.



Marginal Rate of Technical Substitution

- The MRTS is the |slope| of the production isoquant.

$$\boxed{MRTS = -\frac{dK}{dL} = \left| \frac{\partial F}{\partial L} \Big/ \frac{\partial F}{\partial K} \right| = MP_L / MP_K}$$

- "if we hire one more workers, how many fewer machines would we need to maintain output?"

- Intuitively, the answer should depend on how productive the workers are relative to machines at the current level of output.

Understanding Check

- **Derive MRTS of production function**

$$\mathbf{Q} = K^{0.5}L^{0.5}$$

MRTS and Marginal Productivities

- Take the total differential of the production function:

$$dq = \frac{\partial f}{\partial l} \cdot dl + \frac{\partial f}{\partial k} \cdot dk = MP_l \cdot dl + MP_k \cdot dk$$

- Along an isoquant dq = 0, so

$$MP_l \cdot dl = -MP_k \cdot dk$$

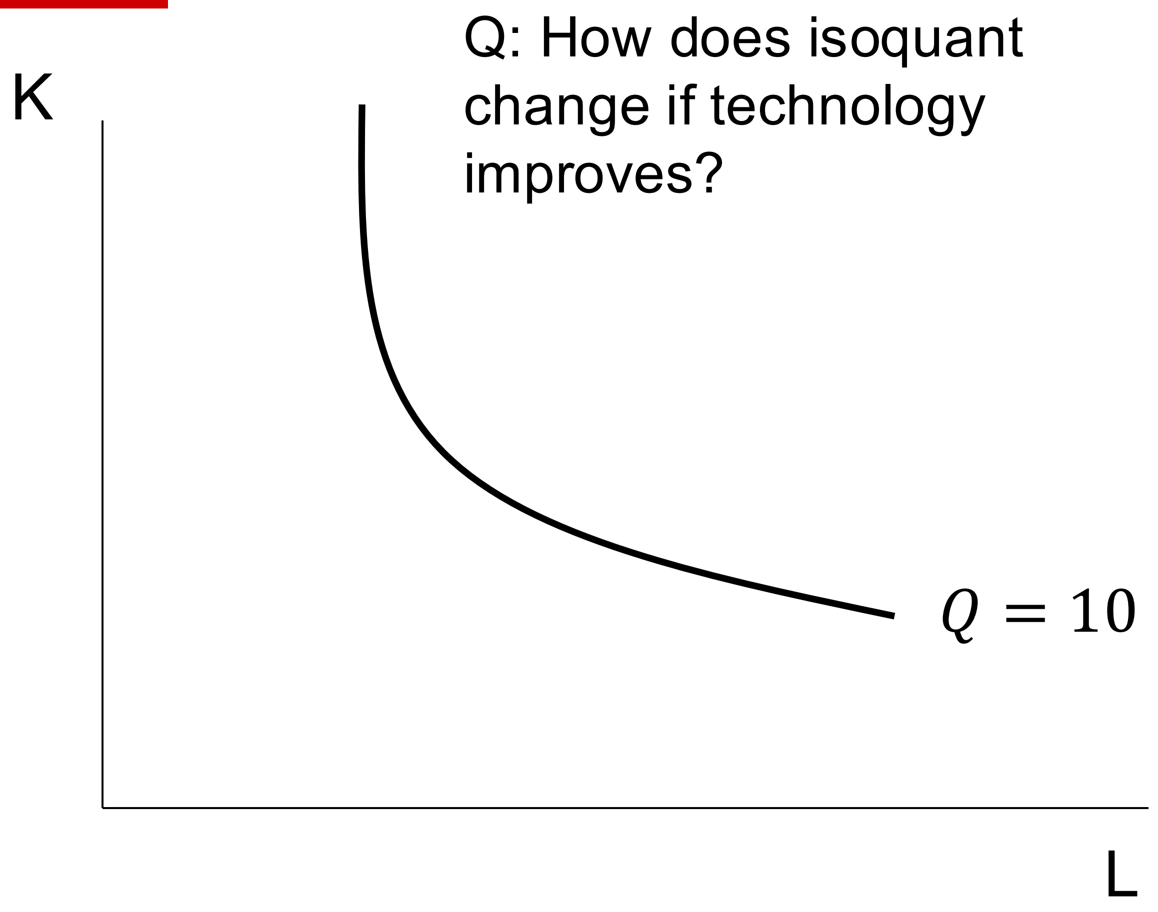
$$\text{MRTS}\,(l \text{ for } k) = \left.\frac{-dk}{dl}\right|_{q=q_0} = \frac{MP_l}{MP_k}$$

Summary

- If a firm is not minimizing cost, it cannot be maximizing profit.
- **Isoquants** describe which combinations of inputs produce the same level of output
- **Marginal product of an input** is the increase in output when an input increases by a small amount
- **Marginal rate of technical substitution (MRTS)** describes how inputs can be interchanged at a particular point in the production process.

Technological Progress changes F(K,L)

- If the firm develops new production techniques, machines, etc., then $F(K,L)$ will change.



Technological Progress changes F(K,L)

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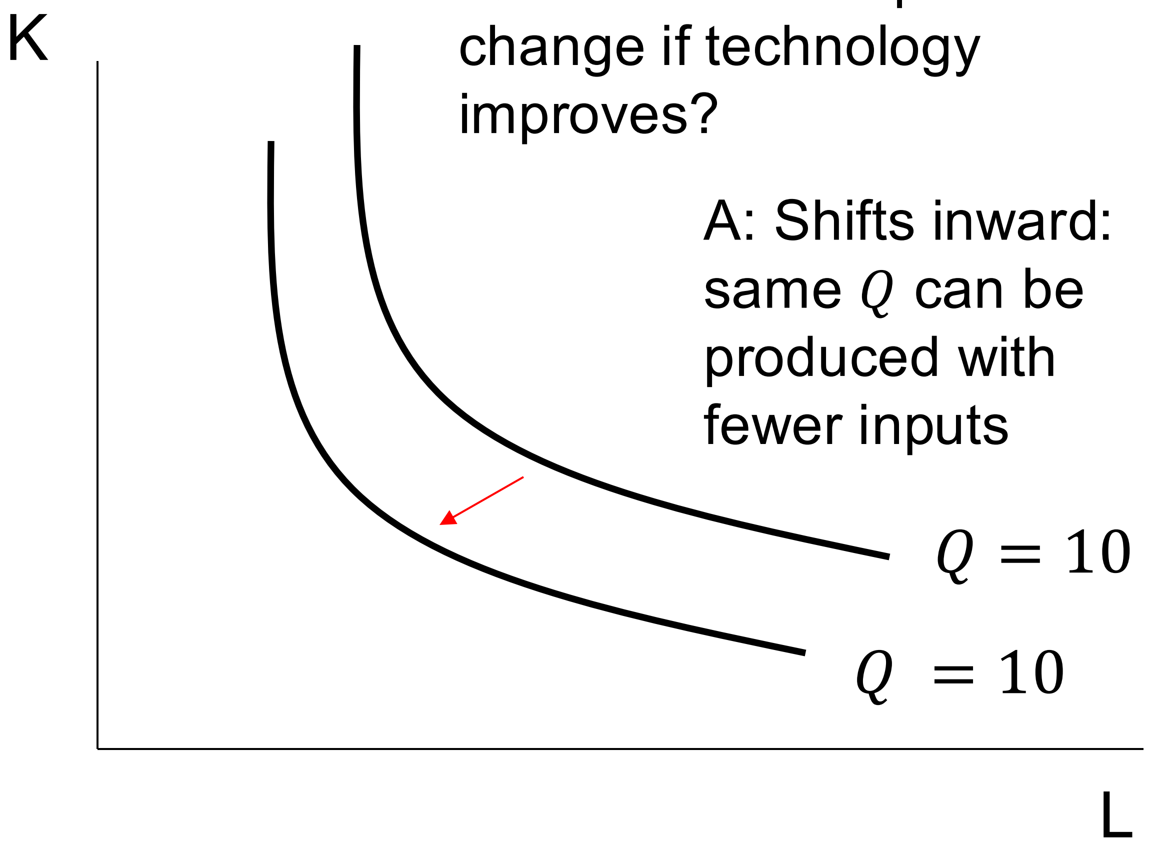
- **Questions:**
- **Can isoquants that represent the same technology cross?**



Q: How does isoquant change if technology improves?

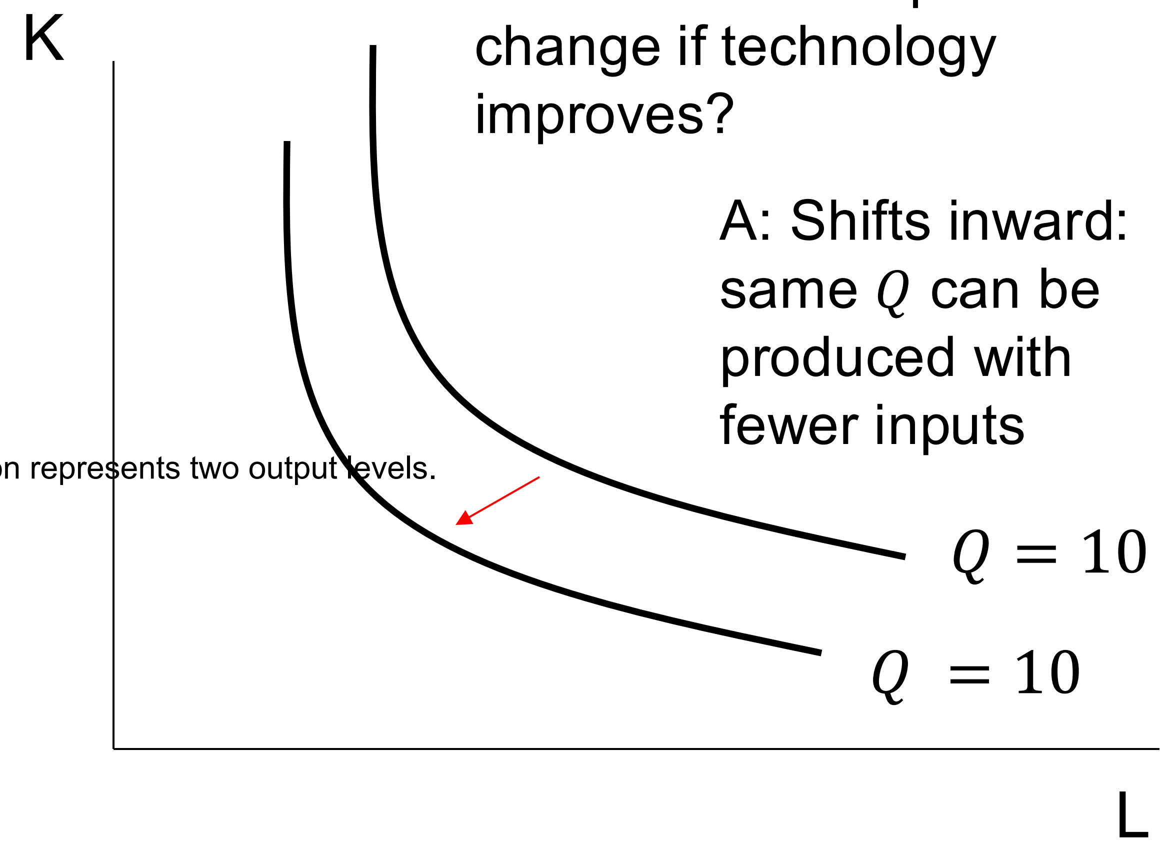
A: Shifts inward: same Q can be produced with fewer inputs

Technological Progress changes F(K,L)

- If the firm develops new production techniques, machines, etc., then $F(K,L)$ will change.
- **Questions:**
- **Can isoquants that represent the same technology cross?** No. The intersection represents two output levels.
- **What about two different technologies?**

Q: How does isoquant change if technology improves?

A: Shifts inward: same Q can be produced with fewer inputs



Production Function

- The center of producer's problem is its technology, or production function $Q=F(K,L)$

- Three important aspects of the production function
  - Technology sets
  - Isoquant curves
  - **Returns to scale**
    - suppose that all inputs are doubled, would output double?

Returns to scale

- Returns to scale have been of interest to economists since the days of Adam Smith

- Smith identified two forces that come into operation as inputs are doubled
  - greater division of labor and specialization of function
  - loss in efficiency because management may become more difficult given the larger scale of the firm

Returns to Scale

- A production function's ***Returns to Scale*** captures how efficiently production can be scaled up.

Effect on Output	Returns to Scale
$f(tk,tl) = tf(k,l)$	Constant
$f(tk,tl) < tf(k,l)$	Decreasing
$f(tk,tl) > tf(k,l)$	Increasing

Understanding check

- **Suppose that the function relating the quantity of automobiles to units of labor and capital is:**

$$Q(K, L) = 2K^{\frac{1}{2}}L^{\frac{1}{2}}$$

- **Does this exhibit IRS, CRS or DRS?**

Understanding check

- **Suppose that the function relating the quantity of automobiles to units of labor and capital is:**

$$Q(K, L) = 0.5K^{\frac{1}{2}} + L^{\frac{1}{2}}$$

- **Does this exhibit IRS, CRS or DRS?**

Returns to scale and Q

- The same firm may have different returns to scale at different output levels. A common pattern is IRS for low Q, DRS for high Q.
  - Low Q, can't make good use of capital.
  - At high Q, you are overtaxing capital.

- CRS is in many ways the benchmark.
  - If a single firm can operate many identical factories, then it should achieve (at least) CRS.
  - IRS if the firm or industry is too small to operate at the "right" scale.
  - DRS if there is some factor in limited supply that prohibits replication.

Returns to scale and Q

- It is possible for a production function to exhibit constant returns to scale for some levels of input usage and increasing or decreasing returns for other levels

- economists refer to the degree of returns to scale with the implicit notion that only a *fairly narrow range* of variation in input usage and the related level of output is being considered

Constant Returns to Scale

- Constant returns-to-scale production functions are homogeneous of degree one in inputs

$$f(tk,tl) = t^1 f(k,l) = tq$$

- This implies that the marginal productivity functions are homogeneous of degree zero
  - if a function is homogeneous of degree k, its derivatives are homogeneous of degree k-1
  - Proof?

Returns to Scale with *n* inputs

- Returns to scale can be generalized to a production function with *n* inputs

$$q = f(x_1, x_2, \ldots, x_n)$$

- If all inputs are multiplied by a positive constant t, we have

$$f(tx_1, tx_2, \ldots, tx_n) = t^k f(x_1, x_2, \ldots, x_n) = t^k q$$

  - If $k = 1$, we have constant returns to scale
  - If $k < 1$, we have decreasing returns to scale
  - If $k > 1$, we have increasing returns to scale

Elasticity of Substitution

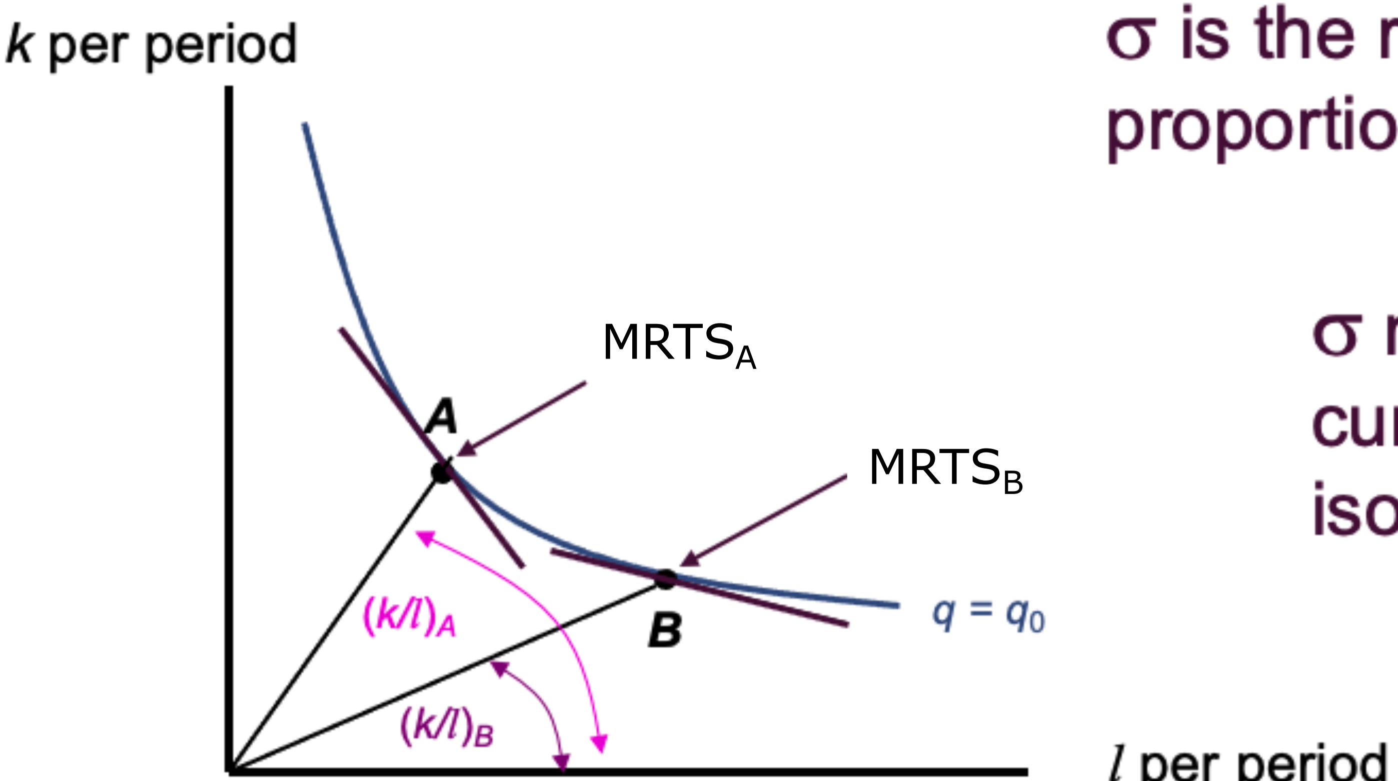
- The elasticity of substitution (σ) measures the proportionate change in k/l relative to the proportionate change in the *MRTS* along an isoquant

$$\sigma = \frac{\%\Delta(k/l)}{\%\Delta RTS} = \frac{d(k/l)}{dRTS} \cdot \frac{RTS}{k/l} = \frac{\partial \ln(k/l)}{\partial \ln RTS}$$

- The value of σ will always be positive because k/l and MRTS move in the same direction

Elasticity of Substitution

Both MRTS and k/l will change as we move from point A to point B



σ is the ratio of these proportional changes

σ measures the curvature of the isoquant how curve is this isoquant

Elasticity of Substitution

- If σ is high, the *MRTS* will not change much relative to k/l
  - the isoquant will be relatively flat
- If σ is low, the *MRTS* will change by a substantial amount as k/l changes
  - the isoquant will be sharply curved
- It is possible for σ to change along an isoquant or as the scale of production changes

Examples of commonly used production functions

- 1. Linear production function
- 2. Fixed proportions (Leontief production function)
- 3. Cobb-Douglas production function

The Linear Production Function

- Linear production function:

$$q = f(k,l) = \alpha k + \beta l$$

  - Returns to scale for t>1?
  - $MRTS$ = ?
  - σ = ? \infin

Fixed Proportions or Leontief production function

- Fixed proportions production function:

$$q = \min(\alpha k, \beta l) \quad \alpha, \beta > 0$$

  - Returns to scale for t>1?
  - *MRTS* ?
  - σ = ? 0

Cobb-Douglas Production Function

- Cobb-Douglas production function:

$$q = f(k,l) = Ak^{\alpha}l^{\beta} \quad A,\alpha,\beta > 0$$

  - Returns to scale?
  - *MRTS* ?
  - σ = ? [1]