Intermediate Macroeconomics: Problem Set 3

Due Tuesday, April 1

1. Solow Model with "Learning by Doing" (Midterm 2022)

Productivity growth does not always occur as a result of deliberate efforts, and sometimes comes as a side effect of conventional economic activity. In this version of Solow model, we will endogenize productivity by considering this particular type of knowledge accumulation, known as *learning-by-doing.*

Production technology is given by

$$Y_t = (K_t)^{\alpha}(A_t N_t)^{1-\alpha}$$

Where K_t, N_t, and A_t denote capital, labor and technology respectively, with $\alpha \in (0,1)$. In addition, productivity is a function of capital stock:

$$A_t = BK_t^{\phi}$$

Where $B > 0, \phi > 0$ are positive constants. The household has fixed savings rate, denoted by s. Depreciation rate for capital is $\delta \in (0,1)$.

a. (7 points) Write down the resource constraint for this economy. (Hint: it's simple.)

b. (7 points) Write down the law of motion for capital. (Hint: use the fact that the household invests a fixed proportion of output each period.)

c. (6 points) Does the production function have constant return to scale with respect to (K_t, N_t)? Why or why not?

From part d) - f), assume that population growth rate is zero (i.e. $N_t = 1$ for all t).

d. (6 points) Draw a phase diagram and solve the steady state for capital, K^*.

e. (4 points) At the steady state, compute the slope for the law of motion: $\frac{\partial K_{t+1}}{\partial K_t}|_{K_t=K^*}$. Compare this slope and the slope of the 45^o line, and show that the steady state is locally stable if $\phi < 1$.

f. (5 points) When $\phi < 1$, discuss how steady state capital K^* changes with exogenous increases in B, s and δ.

2. Industrial Policy in a Discrete-Time Ramsey Model[1]

We consider a closed economy in discrete time $t = 0, 1, 2, \ldots$ with population normalized to one. A representative household maximizes its discounted utility

$$\sum_{t=0}^{\infty} \beta^t U(c_t), \quad 0 < \beta < 1,$$

where $U(\cdot)$ follows a CRRA specification:

$$U(c) = \begin{cases} \dfrac{c^{1-\sigma}}{1-\sigma}, & \sigma \neq 1, \\ \ln(c), & \sigma = 1, \end{cases}$$

and $\sigma > 0$. Production is given by a Cobb-Douglas function $y_t = A\,k_t^{\alpha}$ with $0 < \alpha < 1$, where k_t is per capita capital. The law of motion for capital is $k_{t+1} = (1-\delta)k_t + i_t$, with $\delta \in (0, 1)$. We introduce an *industrial policy* by letting the government tax a fraction $\tau \in [0, 1]$ of output, i.e. $\tau\, y_t$, and invest it in strategic sectors. This does not raise current output but increases next period's productivity from A to $A + \phi(\tau)$, where for concreteness we assume

$$\phi(\tau) = \gamma\, \tau^{\eta}, \quad \gamma > 0,\ 0 < \eta < 1.$$

The government's budget is balanced each period, so all revenue $\tau\, y_t$ is spent immediately on these targeted activities.

Questions:

(a) Write down the modified per capita resource constraint that accounts for the government tax $\tau\, y_t$. Briefly discuss why this policy leads to a lower disposable output for the household today, but potentially higher productivity in future periods.

(b) Formulate the representative household's optimization problem and derive its Euler equation. First show the case with $\tau = 0$, then discuss how the introduction of a one-time only industrial policy with $\tau > 0$ modifies the Euler Equation.

(c) Suppose the industial policy is implemented indefinitely. Use the Euler equation and the capital accumulation equation to find the steady-state conditions for capital k^* and consumption c^*.

[1] Thanks to the generous help from ChatGPT o1 in helping me creating this problem.

Discuss how changes in τ (and thus $\phi(\tau)$) may affect the long-run level of k^* and c^*. Under what conditions could a higher τ reduce long-run welfare?

(d) **(Difficult)** Implement or adapt the following Python code, which:

- Searches a *grid* of τ values between 0 and 1,
- Solves the steady-state equation for $k^*(\tau)$,
- Computes $c^*(\tau)$ and $W(\tau)$,
- Identifies the τ^* that maximizes $W(\tau)$.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import fsolve

beta  = 0.96
sigma = 2.0
A     = 1.0
alpha = 0.33
delta = 0.08
gamma = 1.0
eta   = 0.5

def U(c):
if c <= 0: return -1e10
return c**(1.0 - sigma)/(1.0 - sigma)

def phi(tau):
return gamma * tau**eta

def solve_steady_state_k(tau, k_init=5.0):
def euler_resid(k):
if k <= 0: return 1e10
m_ret = (1 - delta) + (1 - tau)*alpha*(A + phi(tau))*k**(alpha - 1)
return 1.0 - beta*m_ret
k_sol = fsolve(euler_resid, k_init, xtol=1e-12, maxfev=500)[0]
if k_sol <= 0 or abs(euler_resid(k_sol)) > 1e-6:
k_sol = np.nan
```

```
return k_sol

def steady_state_consumption(k_star, tau):
if np.isnan(k_star): return np.nan
return (1 - tau)*(A + phi(tau))*k_star**alpha - delta*k_star

def steady_state_welfare(k_star, tau):
if np.isnan(k_star): return -1e10
c_star = steady_state_consumption(k_star, tau)
if c_star <= 0 or np.isnan(c_star): return -1e10
return U(c_star)/(1.0 - beta)

tau_grid = np.linspace(0.001, 0.999, 100)
w_vals, k_vals, c_vals = [], [], []
best_tau, best_w = None, -1e10

for tau in tau_grid:
k_star = solve_steady_state_k(tau)
w = steady_state_welfare(k_star, tau)
c_star = steady_state_consumption(k_star, tau)
w_vals.append(w)
k_vals.append(0 if np.isnan(k_star) else k_star)
c_vals.append(0 if np.isnan(c_star) else c_star)
if w > best_w:
best_w, best_tau = w, tau

print(f"Optimal tau* = {best_tau:.4f}")

plt.figure()
plt.plot(tau_grid, w_vals)
plt.axvline(best_tau, linestyle="--")
plt.xlabel("tau")
plt.ylabel("Welfare")
plt.title("Welfare vs. tau")
plt.show()

plt.figure()
plt.plot(tau_grid, k_vals, label="k*")
```

```
plt.plot(tau_grid, c_vals, label="c*")
plt.axvline(best_tau, linestyle="--", color="gray")
plt.xlabel("tau")
plt.ylabel("k*, c*")
plt.title("k* and c* vs. tau")
plt.legend()
plt.show()
```

Run your code to find τ^*. Report the resulting τ^*, $k^*(\tau^*)$, and $c^*(\tau^*)$. Discuss how parameters such as γ (gamma) η (eta), and α (alpha) change the optimal tax rate $\tau^\star$.

Hint: you may consult ChatGPT, DeepSeek or other AI tools for assistance with this question. Make sure to briefly describe which tool did you use, what did you ask and whether you find the AI tool useful.