

Intermediate Microeconomics

Spring 2025

Week 13: Uncertainty and Risk Aversion

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How to make decisions under uncertainty?

- How to model uncertainty?
- Random variable
 - X = random variable, x = a realization of X
 - Example: X = outcome of throwing a dice, a realization can be any number between 1 and 6
- If a lottery offers n distinct prizes and the probabilities of winning the prizes are π_i ($i=1,\dots,n$) then

$$\sum_{i=1}^n \pi_i = 1$$

Expected Value

- For a lottery (X) with prizes x_1, x_2, \dots, x_n and the probabilities of winning $\pi_1, \pi_2, \dots, \pi_n$, the expected value of the lottery is

$$E(X) = \pi_1 x_1 + \pi_2 x_2 + \dots + \pi_n x_n$$

$$E(X) = \sum_{i=1}^n \pi_i x_i$$

- The expected value is a weighted sum of the outcomes
 - the weights are the respective probabilities

Expected value

- If X is a continuous random variable,

$$E(x) = \int_{-\infty}^{\infty} xf(x)dx$$

- Where $f(x)$ = probability density function of X

Expected Value

- Games which have an expected value of zero (or cost their expected values) are called actuarially fair games

Fair Games

- People are generally unwilling to play fair games
- There may be a few exceptions
 - when very small amounts of money are at stake
 - when there is utility derived from the actual play of the game
 - we will assume that this is not the case

St. Petersburg Paradox

- A coin is flipped until a head appears
- If a head appears on the n^{th} flip, the player is paid $\$2^n$

$$x_1 = \$2, x_2 = \$4, x_3 = \$8, \dots, x_n = \$2^n$$

- The probability of getting a head on the i^{th} trial is $(1/2)^i$

$$\pi_1 = 1/2, \pi_2 = 1/4, \dots, \pi_n = 1/2^n$$

St. Petersburg Paradox

- The expected value of the St. Petersburg paradox game is infinite

$$E(X) = \sum_{i=1}^{\infty} \pi_i x_i = \sum_{i=1}^{\infty} 2^i \left(\frac{1}{2}\right)^i$$

$$E(X) = 1 + 1 + 1 + \dots + 1 = \infty$$

- Because no player would pay a lot to play this game, it is not worth its infinite expected value

Expected Utility

- Individuals do not care directly about the dollar values of the prizes
 - they care about the utility that the dollars provide

- If we assume diminishing marginal utility of wealth, the St. Petersburg game may converge to a finite expected utility value
 - this would measure how much the game is worth to the individual

Daniel Bernoulli's Solution to St. Petersburg's Paradox

- Assume declining marginal utility of income, then the expected utility value ("moral value") of the game may be finite.

- For example, consider $U(x) = \log(x)$, then

$$\begin{aligned}\text{expected utility} &= \sum_{i=1}^{\infty} \frac{1}{2^i} \log(2^i) = \sum_{i=1}^{\infty} \frac{i}{2^i} \log 2 = (\log 2) \left(\sum_{i=1}^{\infty} \frac{i}{2^i} \right) \\ &= 2 \log 2 = 1.39\end{aligned}$$

- Therefore, this individual will be willing to offer at most \$M to play the game, i.e., $U(M) = \log M = 1.39$, i.e., $M = e^{1.39} = 4.015$

Expected Utility

- Expected utility can be calculated in the same manner as expected value

$$E(X) = \sum_{i=1}^n \pi_i U(x_i)$$

- Because utility may rise less rapidly than the dollar value of the prizes, it is possible that expected utility will be less than the monetary expected value

The von Neumann-Morgenstern Theorem

- Suppose that there are n possible prizes that an individual might win (x_1, \dots, x_n) arranged in ascending order of desirability
 - x_1 = least preferred prize $\Rightarrow U(x_1) = 0$ (arbitrary)
 - x_n = most preferred prize $\Rightarrow U(x_n) = 1$

- The point of the von Neumann-Morgenstern theorem is to show that there is a reasonable way to assign specific utility numbers to the other prizes available

The von Neumann-Morgenstern Theorem

- Consider the following experiment.
- Ask the individual to state the probability π_j ,
- at which he or she would be indifferent between x_j with certainty,
- and a gamble offering prizes of x_n with probability π_j and x_1 with probability $1 - \pi_j$.
- The von Neumann-Morgenstern method is to define the utility of x_j as the expected utility of the gamble that the individual considers equally desirable to x_j

$$U(x_j) = \pi_j \cdot U(x_n) + (1 - \pi_j) \cdot U(x_1)$$

The von Neumann-Morgenstern Theorem

- Since $U(x_n) = 1$ and $U(x_1) = 0$

$$U(x_i) = \pi_i \cdot 1 + (1 - \pi_i) \cdot 0 = \pi_i$$

- The utility number attached to any other prize is simply the probability of winning it
- Note that this choice of utility numbers is arbitrary

Expected Utility Maximization

- A rational individual will choose among gambles based on their expected utilities (the expected values of the von Neumann-Morgenstern utility index)
- Expected utility function = von Neumann-Morgenstern utility function

Martingale/Double-up Gambling Strategy

- A gambler bets B and will win B if a coin comes up head and loses it if it comes up tail.
- The martingale or double-up strategy is that the gambler doubles his bet after every loss, therefore the first win will recover all previous losses plus a profit that is equal to the initial bet.
- Let B = initial bet, x_i = profit when the first head appears on the i^{th} flip and π_i = probability that the first head appears on the i^{th} flip.

Martingale/Double-up Gambling Strategy

$$x_1 = B$$

$$x_2 = 2B - B = B$$

$$x_3 = 2(2B) - (B + 2B) = B$$

$$x_4 = 2(2(2B)) - (B + 2B + 2(2B)) = B$$

\vdots

$$x_n = 2^{n-1}B - (1 + 2 + 2^2 + 2^3 + \dots + 2^{n-3} + 2^{n-2})B = 2^{n-1}B - (2^{n-1} - 1)B = B$$

and $\pi_1 = \frac{1}{2}, \pi_2 = \frac{1}{4}, \pi_3 = \frac{1}{8}, \pi_4 = \frac{1}{16}, \dots, \pi_n = \frac{1}{2^n}$. Thus,

$$E(X) = \sum_{i=1}^{\infty} \pi_i x_i = \sum_{i=1}^{\infty} \frac{1}{2^i} B = B \sum_{i=1}^{\infty} \frac{1}{2^i} = B.$$

This strategy generates a finite expected value, which equals the initial bet.

$$EU(X) = \sum_{i=1}^{\infty} \pi_i U(x_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} U(B) = U(B) \sum_{i=1}^{\infty} \frac{1}{2^i} = U(B).$$

- Question: Is the martingale strategy a winning strategy?

Expected Utility Maximization

□ Consider two gambles:

- first gamble offers x_2 with probability q and x_3 with probability $(1-q)$

$$\text{expected utility (1)} = q \cdot U(x_2) + (1-q) \cdot U(x_3)$$

- second gamble offers x_5 with probability t and x_6 with probability $(1-t)$

$$\text{expected utility (2)} = t \cdot U(x_5) + (1-t) \cdot U(x_6)$$

Expected Utility Maximization

- Substituting the utility index numbers gives

$$\text{expected utility (1)} = q \cdot \pi_2 + (1-q) \cdot \pi_3$$

$$\text{expected utility (2)} = t \cdot \pi_5 + (1-t) \cdot \pi_6$$

- The individual will prefer gamble 1 to gamble 2 if and only if

$$q \cdot \pi_2 + (1-q) \cdot \pi_3 > t \cdot \pi_5 + (1-t) \cdot \pi_6$$

Expected Utility Maximization

- If individuals obey the von Neumann-Morgenstern axioms of behavior in uncertain situations, they will act as if they choose the option that maximizes the expected value of their von Neumann-Morgenstern utility index

Risk Aversion

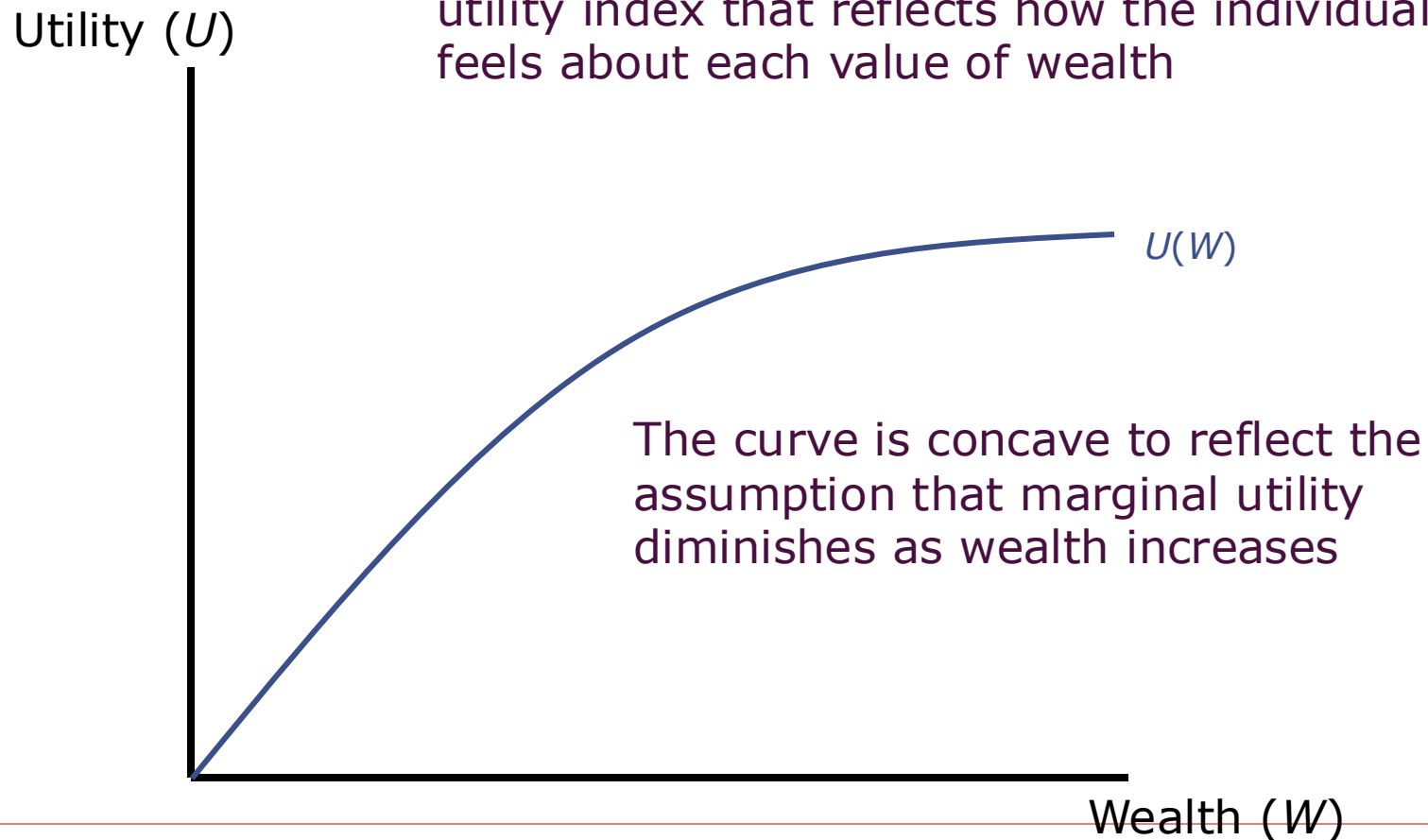
- Two lotteries may have the same expected value but differ in their riskiness
 - flip a coin for \$1 versus \$1,000
- Risk refers to the variability of the outcomes of some uncertain activity
- When faced with two gambles with the same expected value, individuals will usually choose the one with lower risk

Risk Aversion

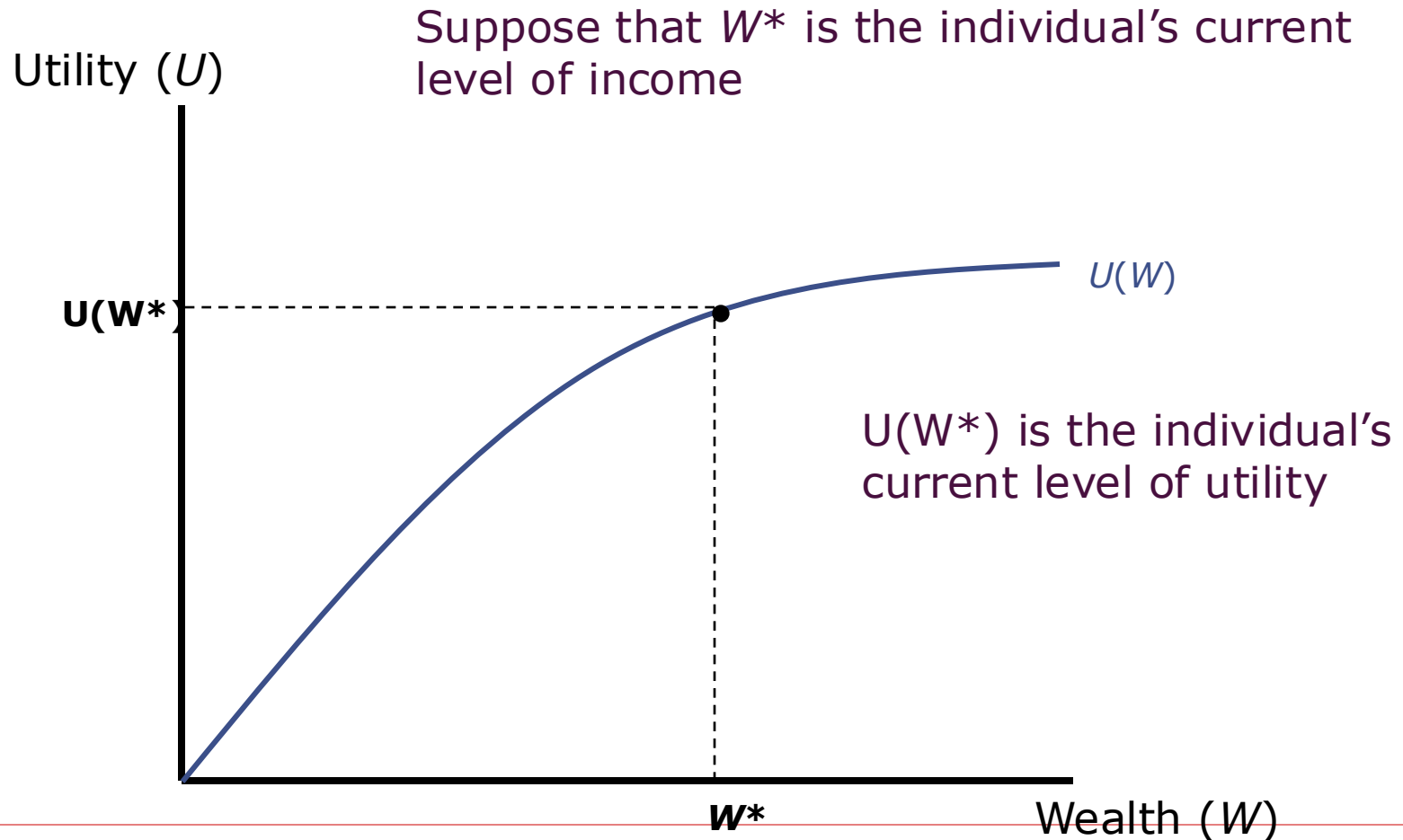
- In general, we assume that the marginal utility of wealth falls as wealth gets larger
 - for example, few people would choose to take a \$10,000 bet on the outcome of a coin flip, even though the average payoff is 0. The reason is that the gamble's money prizes do not completely reflect the utility provided by the prizes.
 - a flip of a coin for \$1,000 promises a small gain in utility if you win, but a large loss in utility if you lose
 - a flip of a coin for \$1 is inconsequential as the gain in utility from a win is not much different as the drop in utility from a loss

Risk Aversion

$U(W)$ is a von Neumann-Morgenstern utility index that reflects how the individual feels about each value of wealth



Risk Aversion



Risk Aversion

□ Suppose that the person is offered two fair gambles:

- a 50-50 chance of winning or losing \$ h

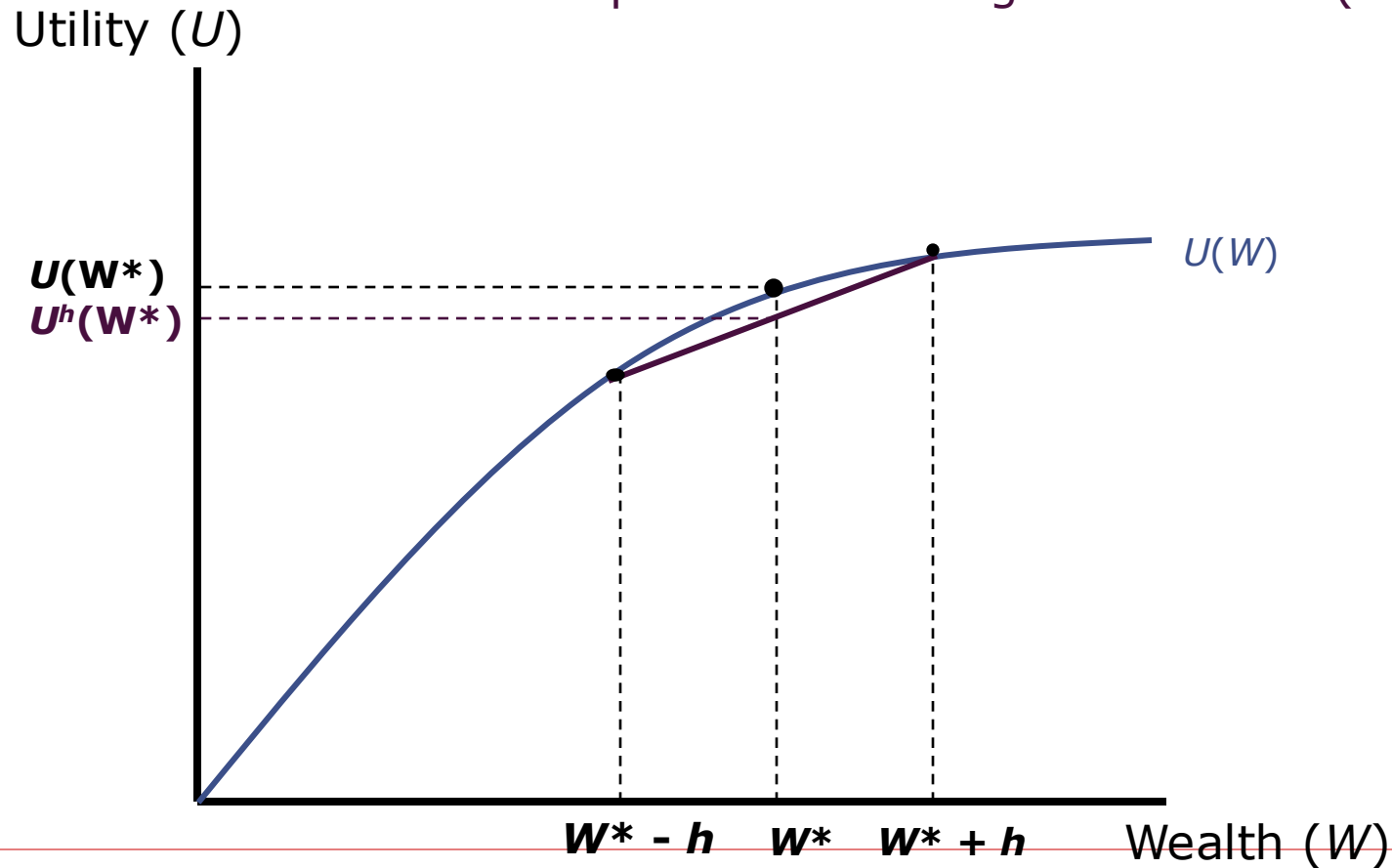
$$U^h(W^*) = \frac{1}{2} U(W^* + h) + \frac{1}{2} U(W^* - h)$$

- a 50-50 chance of winning or losing \$ $2h$

$$U^{2h}(W^*) = \frac{1}{2} U(W^* + 2h) + \frac{1}{2} U(W^* - 2h)$$

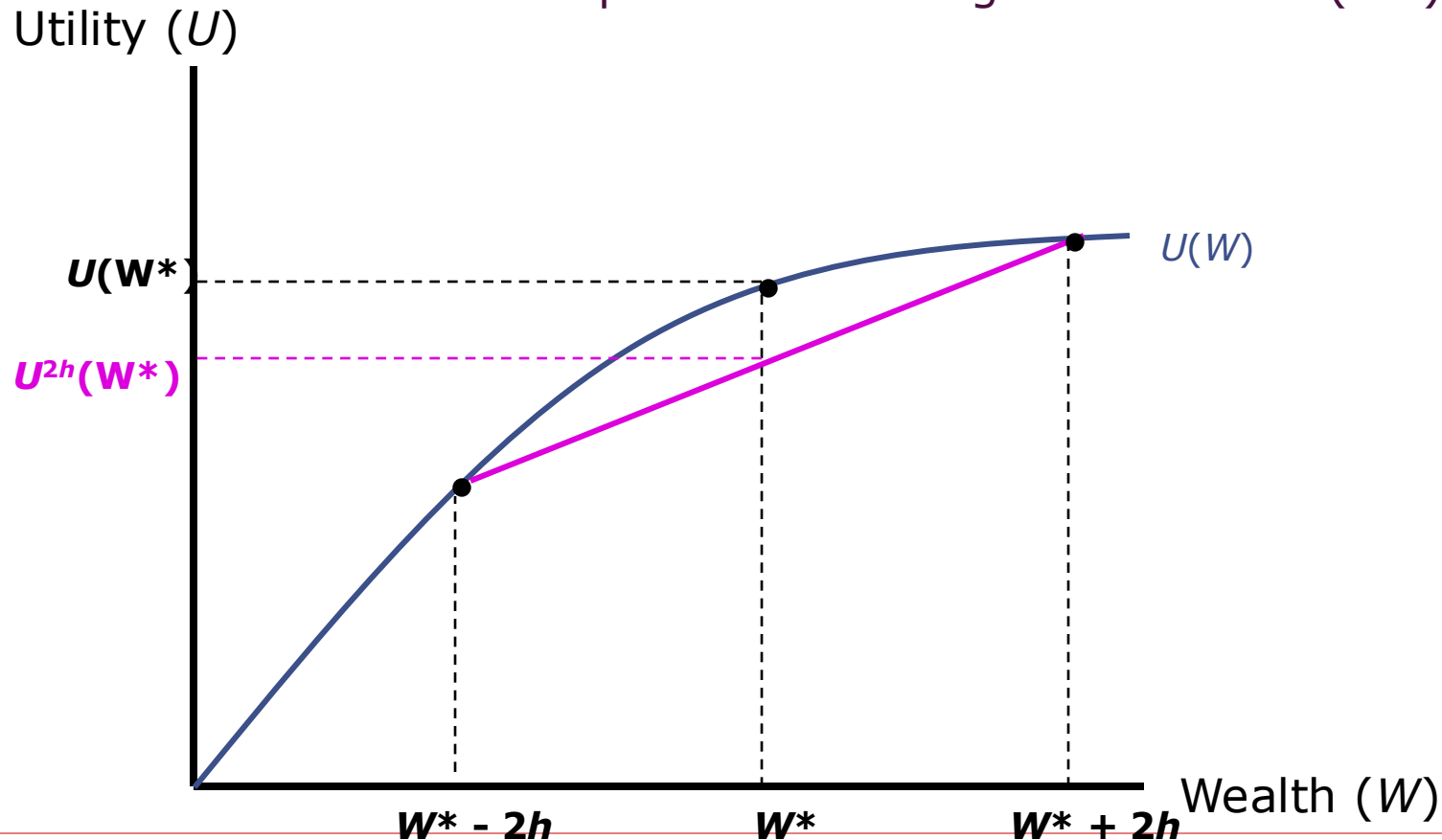
Risk Aversion

The expected value of gamble 1 is $U^h(W^*)$



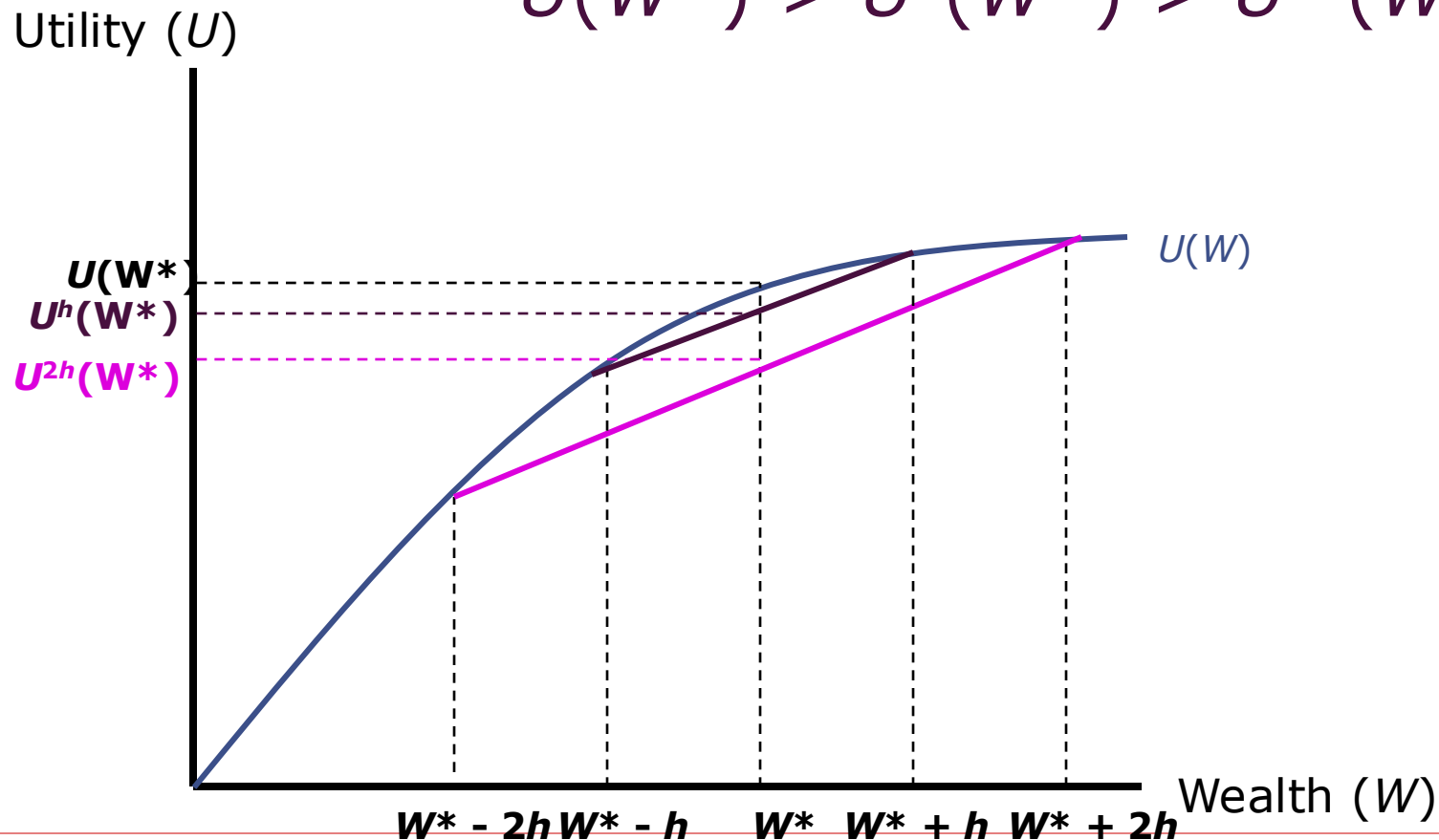
Risk Aversion

The expected value of gamble 2 is $U^{2h}(W^*)$



Risk Aversion

$$U(W^*) > U^h(W^*) > U^{2h}(W^*)$$



Risk Aversion

- The person will prefer current wealth to that wealth combined with a fair gamble
- The person will also prefer a small gamble over a large one

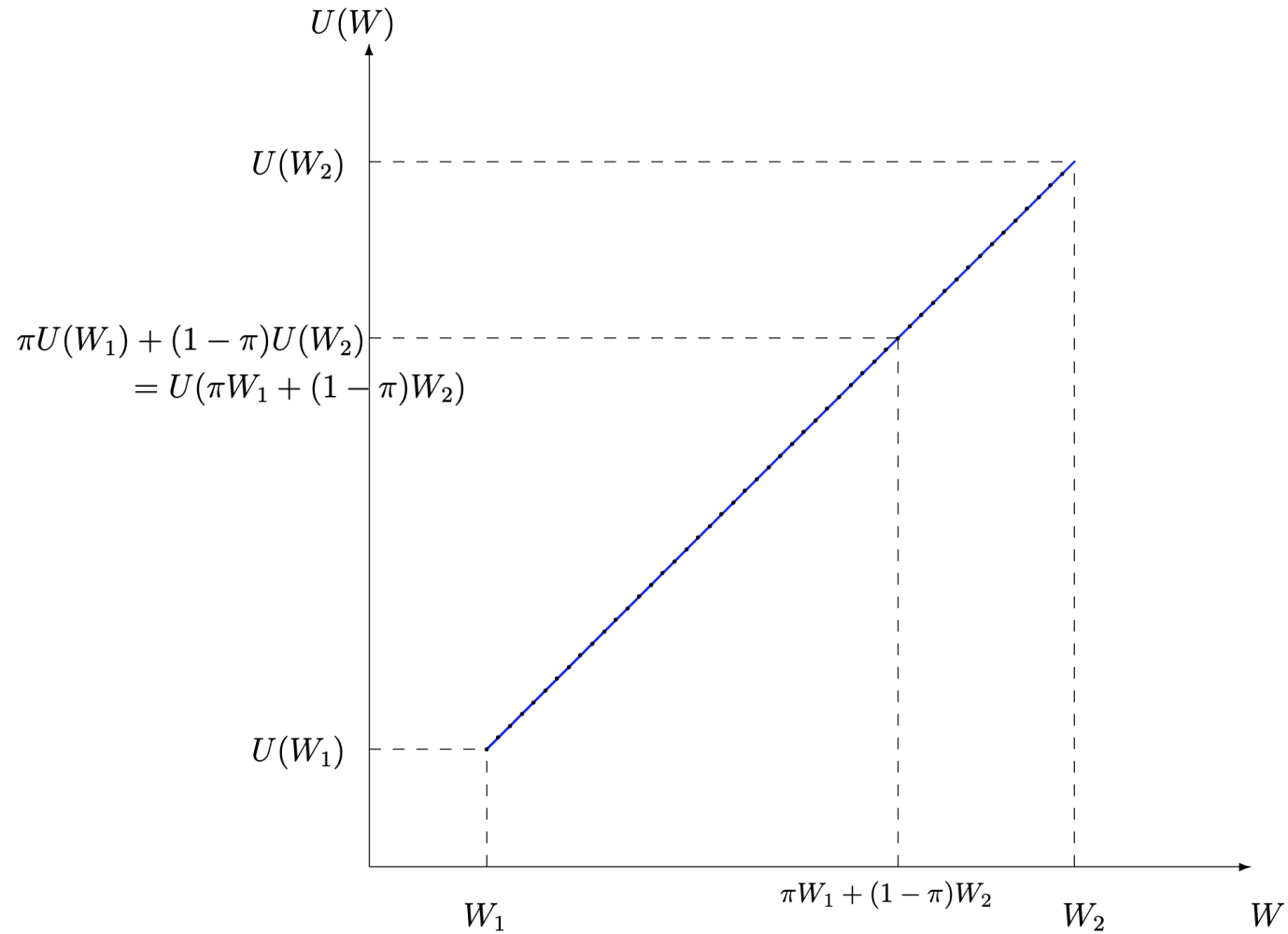
Risk Averse: $\pi U(W_1) + (1 - \pi)U(W_2) < U(\pi W_1 + (1 - \pi)W_2)$
reject a fair gamble

Risk Loving: $\pi U(W_1) + (1 - \pi)U(W_2) > U(\pi W_1 + (1 - \pi)W_2)$
accept a fair gamble

Risk Neutral: $\pi U(W_1) + (1 - \pi)U(W_2) = U(\pi W_1 + (1 - \pi)W_2)$
indifferent to rejecting or accepting a fair gamble

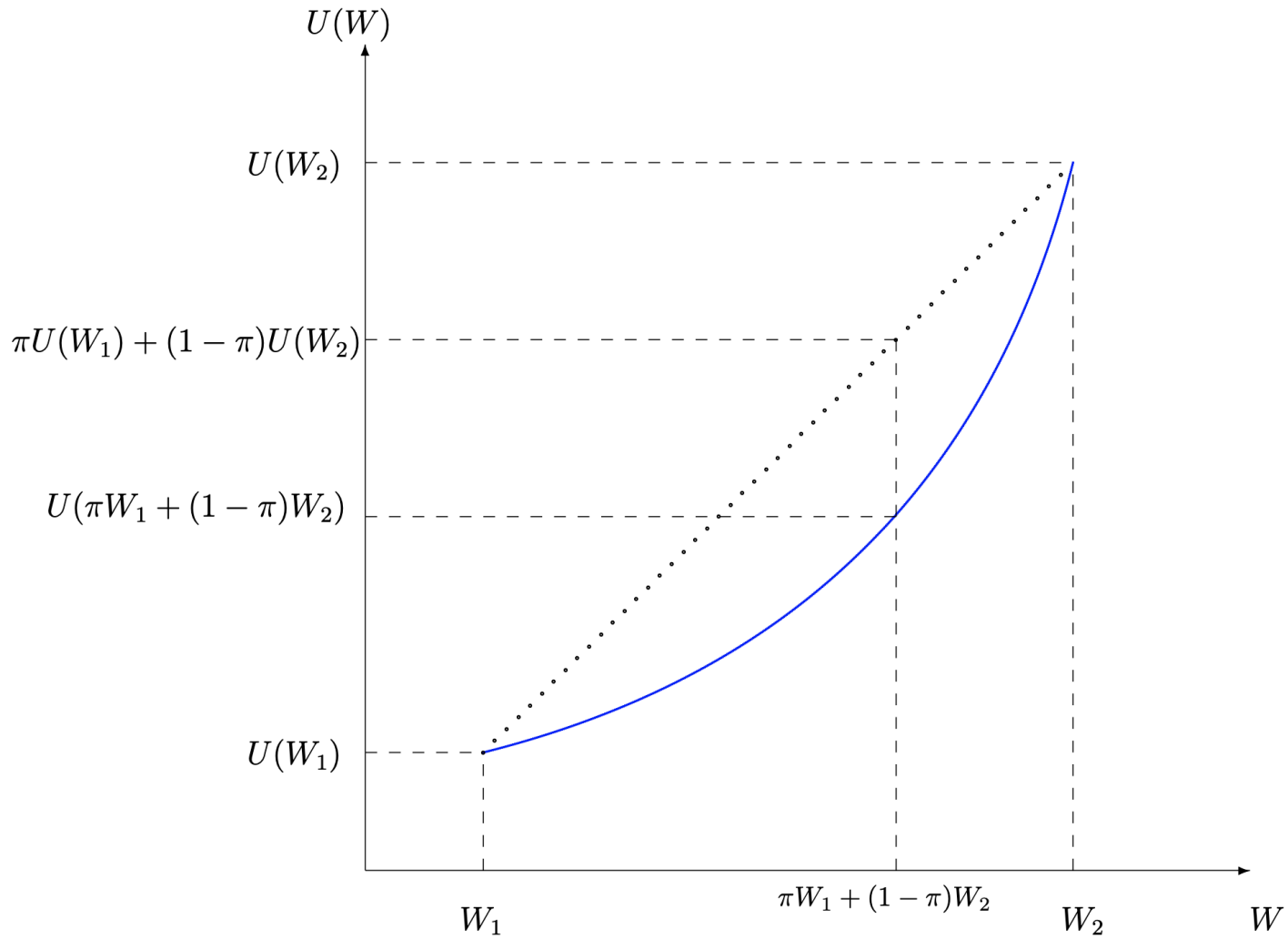
Risk Neutral

$$\pi U(W_1) + (1 - \pi)U(W_2) = U(\pi W_1 + (1 - \pi)W_2)$$



Risk Loving

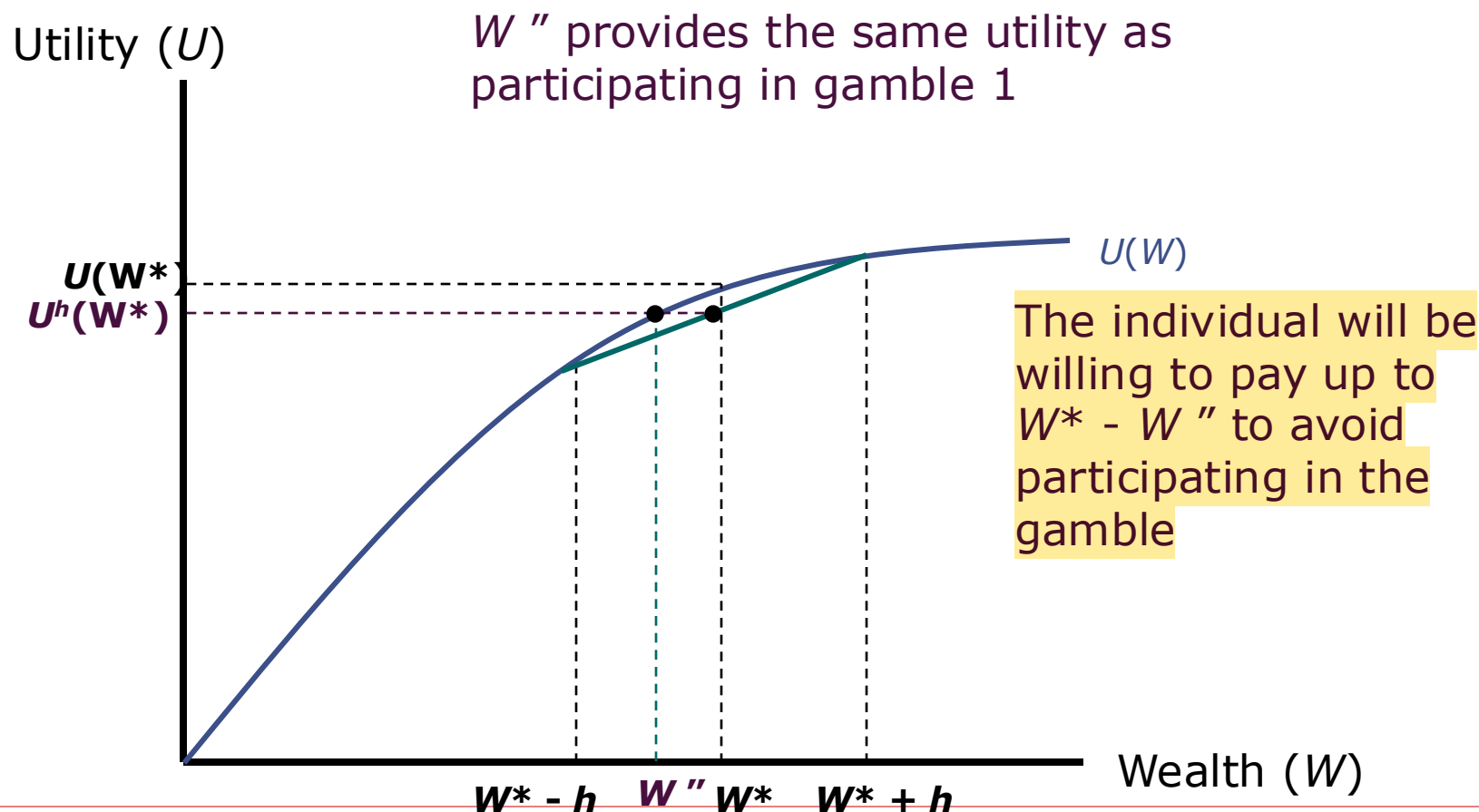
$$\pi U(W_1) + (1 - \pi)U(W_2) > U(\pi W_1 + (1 - \pi)W_2)$$



Risk Aversion and Insurance

- The person might be willing to pay some amount to avoid participating in a gamble
- This helps to explain why some individuals purchase insurance

Risk Aversion and insurance



Willingness to Pay for Insurance

- Consider a person with a current wealth of \$100,000 who faces a 25% chance of losing his automobile worth \$20,000
- Suppose also that the person's von Neumann-Morgenstern utility index is

$$U(W) = \ln (W)$$

Willingness to Pay for Insurance

- The person's expected utility will be

$$E(U) = 0.75U(100,000) + 0.25U(80,000)$$

$$E(U) = 0.75 \ln(100,000) + 0.25 \ln(80,000)$$

$$E(U) = 11.45714$$

- In this situation, a fair insurance premium would be \$5,000 (25% of \$20,000)

Willingness to Pay for Insurance

- The individual will likely be willing to pay more than \$5,000 to avoid the gamble. How much will he pay?

$$E(U) = U(100,000 - x) = \ln(100,000 - x) = 11.45714$$

$$100,000 - x = e^{11.45714}$$

$$x = 5,426$$

- The maximum premium is \$5,426

Measuring Risk Aversion

- The most commonly used risk aversion measure was developed by J.W.Pratt in the 1960s.

$$r(W) = -\frac{U''(W)}{U'(W)}$$

- For risk averse individuals, $U''(W) < 0$
 - $r(W)$ will be positive for risk averse individuals
 - $r(W)$ is invariant with respect to linear transformations of the utility function.

Measuring Risk Aversion

- The Pratt measure of risk aversion is proportional to the amount an individual will pay to avoid a fair gamble

Measuring Risk Aversion

- Suppose the winnings from such a fair bet are denoted by the random variable h (which takes on both positive and negative values).

$$E(h) = 0$$

- Let p be the size of the insurance premium that would make the individual exactly indifferent between taking the fair bet h and paying p with certainty to avoid the gamble

$$E[U(W + h)] = U(W - p)$$

Measuring Risk Aversion

- We now expand both sides of the equation using Taylor's series
- Because p is a fixed amount, we can use a simple linear approximation to the right-hand side

$$U(W - p) = U(W) - pU'(W) + \text{higher order terms}$$

Measuring Risk Aversion

- For the left-hand side, we need to use a quadratic approximation to allow for the variability of the gamble (h)

$$E[U(W + h)] = E[U(W) + hU'(W) + h^2/2 U''(W) + \text{higher order terms}]$$

$$= U(W) + E(h)U'(W) + E(h^2)/2 U''(W) + \text{higher order terms}$$

Measuring Risk Aversion

- Remembering that $E(h)=0$, dropping the higher order terms, and substituting k for $E(h^2)/2$, we get

$$U(W) - pU'(W) \cong U(W) + kU''(W)$$

$$p \cong -\frac{kU''(W)}{U'(W)} = kr(W)$$

- the amount that a risk-averse individual is willing to pay to avoid a fair bet is approximately proportional to Pratt's risk aversion measure.

Measuring Risk Aversion

- Because insurance premiums paid are observable in the real world, these are often used to estimate individuals' risk aversion coefficients or to compare such coefficients among groups of individuals.
- Therefore, it is possible to use market information to learn a bit about attitudes toward risky situations.

Risk Aversion and Wealth

- An important question is whether risk aversion increases or decreases with wealth.
- It is not necessarily true that risk aversion declines as wealth increases
 - diminishing marginal utility would make potential losses less serious for high-wealth individuals
 - however, diminishing marginal utility also makes the gains from winning gambles less attractive
 - the net result depends on the shape of the utility function

Risk Aversion and Wealth: examples

- If utility is quadratic in wealth

$$U(W) = a + bW + cW^2$$

where $b > 0$ and $c < 0$

- What is Pratt's risk aversion measure?
- Does risk aversion increase or decrease as wealth increases?

Risk Aversion and Wealth: examples

- If utility is quadratic in wealth

$$U(W) = a + bW + cW^2$$

where $b > 0$ and $c < 0$

- Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{-2c}{b + 2cW}$$

- Risk aversion increases as wealth increases

Risk Aversion and Wealth

- If utility is logarithmic in wealth

$$U(W) = \ln(W)$$

where $W > 0$

- What is Pratt's risk aversion measure?
- Does risk aversion increase or decrease as wealth increases?

Risk Aversion and Wealth

- If utility is logarithmic in wealth

$$U(W) = \ln(W)$$

where $W > 0$

- Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{1}{W}$$

- Risk aversion decreases as wealth increases

decrease

Risk Aversion and Wealth

- If utility is exponential

$$U(W) = -e^{-AW}$$

where A is a positive constant

- What is Pratt's risk aversion measure?
- Does risk aversion increase or decrease as wealth increases?

Risk Aversion and Wealth

- If utility is exponential

$$U(W) = -e^{-AW}$$

where A is a positive constant

- Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U'(W)} = \frac{A^2 e^{-AW}}{A e^{-AW}} = A$$

- Risk aversion is constant as wealth increases

constant 所以，risk averse measure与wealth的关系取决于utility function的形式

Relative Risk Aversion

□ The relative risk aversion formula is

$$rr(W) = Wr(W) = -W \frac{U''(W)}{U'(W)}$$

Relative Risk Aversion

- The power utility function

$$U(W, R) = \begin{cases} W^R/R & \text{for } R < 1, R \neq 0 \\ \ln W & \text{for } R = 0 \end{cases}$$

exhibits diminishing absolute risk aversion

$$r(W) = -\frac{U''(W)}{U'(W)} = -\frac{(R-1)W^{R-2}}{W^{R-1}} = -\frac{(R-1)}{W}$$

but constant relative risk aversion

$$rr(W) = Wr(W) = -(R-1) = 1-R$$