Intermediate Macroeconomics: Problem Set 4 Solution

Due Thursday, May 8

1. RBC Model with Consumption Habits (60 points)

In the standard RBC model studied in class, utility in period t depends only on consumption in that period and is not affected by past consumption. Here we consider an RBC model with consumption habits. Assume the representative agent's utility function is

$$u(C_t, C_{t-1}, L_t) = \gamma \log\big(C_t - \phi\, C_{t-1}\big) + (1-\gamma) \log\big(1 - L_t\big),$$

where C_t is consumption in period t, C_{t-1} is consumption in period $t-1$, L_t is labor supply in period t, and $\gamma \in (0,1)$ is the consumption–leisure weight. The agent's budget constraint is

$$C_t + I_t = W_t\, L_t + R_t^k\, K_t,$$

where I_t is total investment (savings), W_t is the real wage, R_t^k is the real rental rate of capital, and K_t is the capital stock. **In parts (a)–(g), assume no stochastic technology shocks exist in the economy.**

a. Assume capital depreciates at rate δ. Write down the law of motion for capital.

Solution:

资本的运动方程由资本存量、投资和折旧率描述：

$$K_{t+1} = (1-\delta)K_t + I_t$$

其中，δ 为资本的折旧率，I_t 为投资。

b. Substitute the investment I_t from the capital accumulation equation into the agent's budget constraint to obtain a new budget constraint.

Solution:

代理人预算约束为：

$$C_t + I_t = W_t L_t + R_t^K K_t$$

将 I_t 从资本的运动方程中代入，得到：

$$C_t + K_{t+1} - (1-\delta)K_t = W_t L_t + R_t^K K_t$$

整理后，可写为：

$$C_t + K_{t+1} = W_t L_t + R_t^K K_t + (1-\delta)K_t$$

c. State the agent's dynamic optimization problem and construct the Lagrangian.

Solution:

代理人最大化效用函数：

$$\max \ \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\gamma \ln(C_t - \phi C_{t-1}) + (1-\gamma)\ln(1-L_t)\right]$$

其约束条件为：

$$C_t + K_{t+1} = W_t L_t + R_t^K K_t + (1-\delta)K_t$$

构造拉格朗日函数：

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{\gamma \ln(C_t - \phi C_{t-1}) + (1-\gamma)\ln(1-L_t) + \lambda_t \left[W_t L_t + R_t^K K_t + (1-\delta)K_t - C_t - K_{t+1}\right]\right\}$$

d. Take derivatives of the Lagrangian and write down the first-order conditions with respect to C_t, K_{t+1}, and L_t.

Solution:

对 C_t、K_{t+1} 和 L_t 分别求导，得到一阶条件：1. 对 C_t：

$$\beta^t \left[\gamma \frac{1}{C_t - \phi C_{t-1}} - \lambda_t\right] - \beta^{t+1}\left[\gamma\phi \frac{1}{C_{t+1} - \phi C_t}\right] = 0$$

2. 对 K_{t+1}：

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\lambda_t \beta^t + \beta^{t+1}\lambda_{t+1}\left[R_{t+1}^K + 1 - \delta\right] = 0$$

3. 对 L_t：

$$\frac{\partial \mathcal{L}}{\partial L_t} = -\frac{1-\gamma}{1-L_t} + \lambda_t W_t = 0$$

e. Using the first-order conditions for C_t and K_{t+1}, derive the intertemporal equilibrium (Euler) equation,

$$\frac{?}{?} = \beta\big[R^k_{t+1} + 1 - \delta\big].$$

Then show the special case when $\phi = 0$. How should we interpret this result?

Solution:

结合 C_t 和 K_{t+1} 的一阶条件，得到：

$$\frac{\frac{1}{C_t-\phi C_{t-1}} - \beta\phi\frac{1}{C_{t+1}-\phi C_t}}{\frac{1}{C_{t+1}-\phi C_t} - \beta\phi\frac{1}{C_{t+2}-\phi C_{t+1}}} = \beta\left(R^K_{t+1} + 1 - \delta\right)$$

当 $\phi = 0$ 时，公式简化为：

$$\frac{C_{t+1}}{C_t} = \beta\left(R^K_{t+1} + 1 - \delta\right)$$

这表明当前消费的边际效用等于未来消费的贴现值调整后的边际效用。

f. Suppose firms have a Cobb–Douglas production function

$$Y_t = A_t\, K_t^{\alpha}\, L_t^{1-\alpha}.$$

Write down the firm's profit-maximization problem and derive expressions for the factor prices W_t and R^k_t from the first-order conditions.

Solution:

厂商最大化利润问题：

$$\max\ \Pi_t = A_t K_t^{\alpha} L_t^{1-\alpha} - W_t L_t - R^K_t K_t$$

一阶条件：1. 对 K_t：

$$R^K_t = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}$$

2. 对 L_t：

$$W_t = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}$$

g. List the seven equations that characterize the dynamics of the seven endogenous variables

$$\{Y_t,\ C_t,\ I_t,\ K_t,\ L_t,\ R_t^k,\ W_t\}.$$

(Hint: just list the equations; do not log-linearize or solve them.)

Solution:

根据以上推导，七个方程为：

1. 生产函数：

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

2. 资本运动方程：

$$K_{t+1} = (1-\delta)K_t + I_t$$

3. 资源约束：

$$C_t + I_t = Y_t$$

4. 劳动市场条件：

$$(1-\gamma)\frac{1}{1-L_t} = \gamma\left[\frac{1}{C_t-\phi C_{t-1}} - \beta\phi\frac{1}{C_{t+1}-\phi C_t}\right]W_t$$

5. 跨期均衡方程：

$$\frac{\frac{1}{C_t-\phi C_{t-1}} - \beta\phi\frac{1}{C_{t+1}-\phi C_t}}{\frac{1}{C_{t+1}-\phi C_t} - \beta\phi\frac{1}{C_{t+2}-\phi C_{t+1}}} = \beta\left(R_{t+1}^K + 1 - \delta\right)$$

6. 资本的回报：

$$R_t^K = \alpha A_t K_t^{\alpha-1} L_t^{1-\alpha}$$

7. 工资：

$$W_t = (1-\alpha)A_t K_t^{\alpha} L_t^{-\alpha}$$

h. Now let the stochastic technology shock $\tilde{A}_t$ follow an AR(1) process, with $g = 0$:

$$\begin{aligned}\log A_t &= \log \bar{A} + \tilde{A}_t,\\ \tilde{A}_t &= \rho_A\,\tilde{A}_{t-1} + \epsilon_t.\end{aligned}$$

Compare this habit-formation model to the baseline RBC model: for a given technology shock that occurs once, will the instantaneous response of consumption C_t be larger or smaller? How about investment I_t? (Hint: it's not necessary to solve the problem; using words or intuition is sufficient.)

Solution:

当 $\phi > 0$ 时，消费习惯会导致消费对冲击的反应更加平滑，因为上一期的消费会增加本期的效用，而这一特性使得消费的瞬时反应小于基准 RBC 模型。投资的瞬时反应大于基准 RBC 模型，因为消费的变化被抑制，更多资源用于投资。

2. Two-Period Model with Labor Income Tax and Government (Final 2024, 40 points)

Consider an economy that lasts for 2 periods, $t = 0, 1$. There is one representative household in this economy with 1 unit of time to allocate between labor n_t and leisure l_t. The utility function for the household is given by:

$$U(c_0, n_0, c_1, n_1) = \log(c_0) + \theta \log(1 - n_0) + \beta[\log(c_1) + \theta \log(1 - n_1)]$$

The household can choose to save/borrow s between time period 0 and 1 by purchasing a "treasury bond", whose interest rate r is taken as given by the household. The wage for household's labor services is (w_0, w_1), but in both periods the household faces labor income taxes with rate (τ_0, τ_1) .

The household's $t = 0$ budget constraint is:

$$c_0 + s = w_0 n_0 (1 - \tau_0)$$

There is a representative firm with constant return to scale production function that only uses labor as input, i.e.

$$y_t = A n_t$$

where A is fixed. There is a government with **fixed** expenditures (g_0, g_1) in each period. The goverment expenditures are financed by labor income taxes (τ_0, τ_1) and "treasury bonds" b. The government's budget constraints are given by:

$$g_0 = w_0 n_0 \tau_0 + b$$

$$g_1 + b(1 + r) = w_1 n_1 \tau_1$$

You can think of the "treasury bond" as government borrowing from households in $t = 0$ and repaying the debt in $t = 1$ with some interest rate r. The bond market clearing condition is:

$$b = s$$

a. Write the household's period $t = 1$ budget constraint, as well as its intertemporal budget constraint.

Solution:

$$c_1 = s(1 + r) + w_1 n_1 (1 - \tau_1)$$

Intertemporal constraint:

$$c_0 + \frac{c_1}{1 + r} = w_0 n_0 (1 - \tau_0) + \frac{w_1 n_1 (1 - \tau_1)}{1 + r}$$

b. Solve the household's problem and derive the first-order conditions for labor (n_0, n_1) and consumption (c_0, c_1) in both periods. Show that the Euler Equation for consumption does not depend on tax rates.

Solution:

$$\mathcal{L} = \log(c_0) + \theta \log(1 - n_0) + \beta[\log(c_1) + \theta \log(1 - n_1)] + \lambda \left(w_0 n_0 (1 - \tau_0) + \frac{w_1 n_1 (1 - \tau_1)}{1 + r} - c_0 - \frac{c_1}{1 + r} \right)$$

FOC:

$$[c_0] \quad \frac{1}{c_0} = \lambda \tag{1}$$

$$[c_1] \quad \frac{\beta}{c_1} = \frac{\lambda}{1 + r} \tag{2}$$

$$[n_0] \quad \frac{-\theta}{1 - n_0} + \lambda w_0 (1 - \tau_0) = 0 \tag{3}$$

$$[n_1] \quad \frac{-\beta\theta}{1 - n_1} + \lambda w_1 \frac{1 - \tau_1}{1 + r} = 0 \tag{4}$$

Euler equation:

$$\frac{c_1}{c_0} = \beta(1 + r)$$

which doesn't depend on tax rates

c. Show that the firm's problem and market clearing conditions imply that $w_0 = w_1 = A$. Combine this result and the first order conditions to show how equilibrium consumption-leisure ratio $\frac{c_0^*}{1-n_0^*}$ in period 0 depends on tax rate τ_0.

Solution:

$$\frac{c_0}{1-n_0} = \frac{w_0(1-\tau_0)}{\theta} = \frac{A(1-\tau_0)}{\theta}$$

which means the ratio goes down when τ_0 goes up.

d. Suppose the labor tax rates are $(\bar{\tau}_0, \bar{\tau}_1)$ before, with $\bar{\tau}_0 > 0$ and $\bar{\tau}_1 > 0$. A new administration wants to cut taxes in period 0 by setting $\tau_0 = 0$. Discuss the impact of this policy on equilibrium consumption (c_0^*, c_1^*) and labor supply (n_0^*, n_1^*).

Solution:

We'll need to solve the whole model. Another relationship between c_1 and $1 - n_1$ is:

$$\frac{c_1}{1-n_1} = \frac{A(1-\tau_1)}{\theta}$$

From the equations above, we can solve get

$$n_0 = 1 - \frac{\theta c_0}{A(1-\tau_0)}$$

and

$$n_1 = 1 - \frac{\theta c_1}{A(1-\tau_1)} = 1 - \frac{\theta\beta(1+r)c_0}{A(1-\tau_1)}$$

Plug in the intertemporal BC, we get

$$\begin{aligned}
c_0 + \beta c_0 &= A\left(1 - \frac{\theta c_0}{A(1-\tau_0)}\right)(1-\tau_0) + A\left(1 - \frac{\theta\beta(1+r)c_0}{A(1-\tau_1)}\right)\frac{1-\tau_1}{1+r} \\
&= A\left(1 - \tau_0 + \frac{1-\tau_1}{1+r}\right) - \theta c_0 - \beta\theta c_0 \\
\Rightarrow \quad c_0 &= \frac{A}{1+\beta+\theta+\beta\theta}\left(1 - \tau_0 + \frac{1-\tau_1}{1+r}\right)
\end{aligned}$$

Then $c_1 = \beta(1+r)c_0$, and we can calculate the rest using the equations above.

Add the government budget constraint with household's $t = 0$ constraint, along with $b = s$, gives us

$$c_0 + g_0 = w_0 n_0 = A - \frac{\theta c_0}{(1 - \tau_0)} \tag{5}$$

$$\Rightarrow c_0 = \frac{A - g_0}{1 + \frac{\theta}{1 - \tau_0}} \tag{6}$$

Similarly, we can get

$$c_1 + g_1 = w_1 n_1 = A - \frac{\theta c_1}{(1 - \tau_1)} \tag{7}$$

$$\Rightarrow c_1 = \frac{A - g_1}{1 + \frac{\theta}{1 - \tau_1}} \tag{8}$$

Plug in the expression for n_0 and n_1 to get:

$$n_0 = 1 - \frac{\theta(A - g_0)}{A(1 - \tau_0 + \theta)} \tag{9}$$

$$n_1 = 1 - \frac{\theta(A - g_1)}{A(1 - \tau_1 + \theta)} \tag{10}$$

If τ_0 decrease, τ_1 increase, this will cause c_0 to increase, c_1 to decrease; n_0 to increase, and n_1 to decrease.