Intermediate Microeconomic

Spring 2025

Part two: Choice and Demand

Week 2(b): Utility Maximization and Choice

Yuanning Liang

Consumer Theory

- A consumer wakes up in the morning. The world has two goods (commodities) for consumption: x and y. He looks outside the window and observe prices Px and Py. He reaches into his pocket and find M dollars there; he knows his preferences over goods x and y.

- Consumer's problem: find the <u>best</u> <u>affordable</u> combination of goods x and y for him to eat

- Two building blocks of the consumer theory:
  - **Preference**
  - **Budget constraint**

Optimization principle

- To maximize utility, given a fixed amount of income to spend, an individual will buy the goods and services:
  - that exhaust his or her total income
  - for which the rate of trade-off between any goods (the *MRS*) is equal to the rate at which goods can be traded for one another in the marketplace

A numerical illustration

- Assume that the individual's *MRS* = 1
  - What does that imply?

A numerical illustration

- Assume that the individual's *MRS* = 1
  - What does that imply?
  - willing to trade one unit of *x* for one unit of *y*

- Suppose the price of *x* = \$2 and the price of *y* = \$1

- The individual can be made better off by?

Budget Constraint

- Consumers' ability to consume is limited by what they can afford. Let p_x be the price of x and p_y be the price of y, and suppose the consumer has m units of wealth to spend on x and y.

- Then the consumer's choice must satisfy the ***budget constraint***:

$$\boldsymbol{p_x x + p_y y \leq m}$$

- If consumers maximize their utility, there is no satiation and no savings, then the budget constraint should hold with equality. So we usually write budget constraint as:

$$\boldsymbol{p_x x + p_y y = m}$$

Budget Constraint

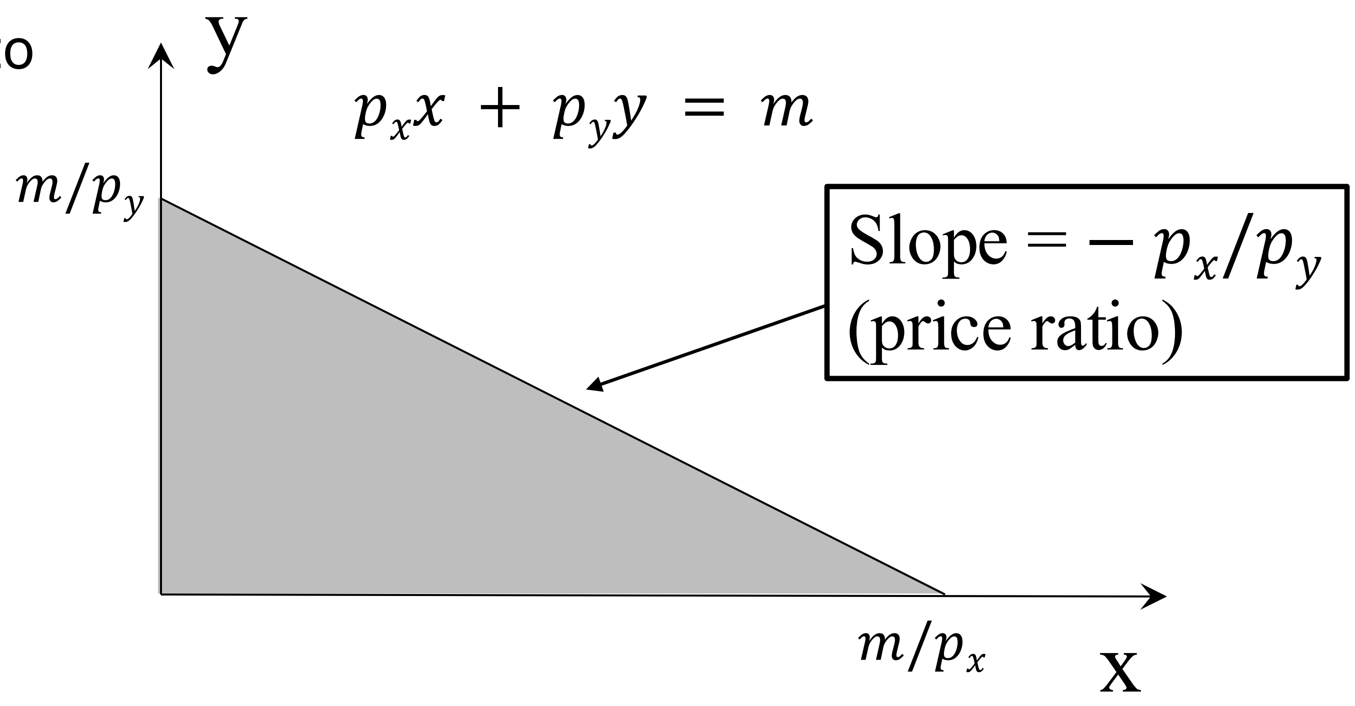
- **Question: Cookies (x) cost $1 each and milk (y) costs $2.5 per gallon. You have $10 to spend. Write down your budget constraint.**

Graphing budget constraint

- Rearrange the budget equation to get:

$$y = -p_x/p_y\ x + m/p_y$$

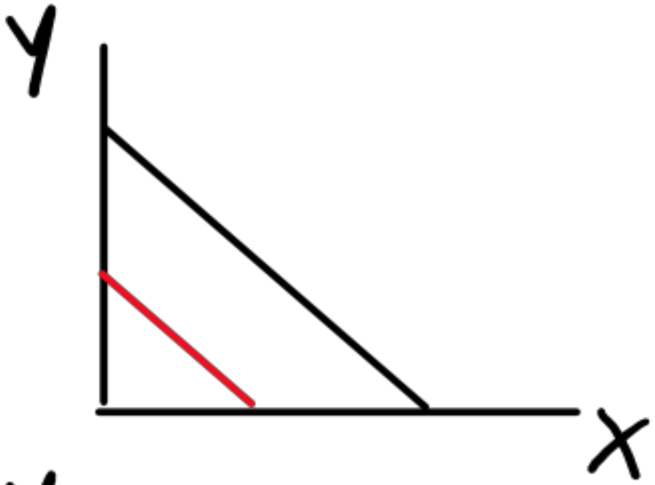
- Slope is $-p_x/p_y$.
- Spend all m on x, can buy m/p_x units.
- Spend all m on y, can buy m/p_y units.
- Set of **feasible choices** in gray.



What if m increases from m_1 to m_2?

What if p_x increases from p_{x1} to p_{x2}?

Question: Which of the following graphs shows income fell? (Black is original, Red is new)

A.


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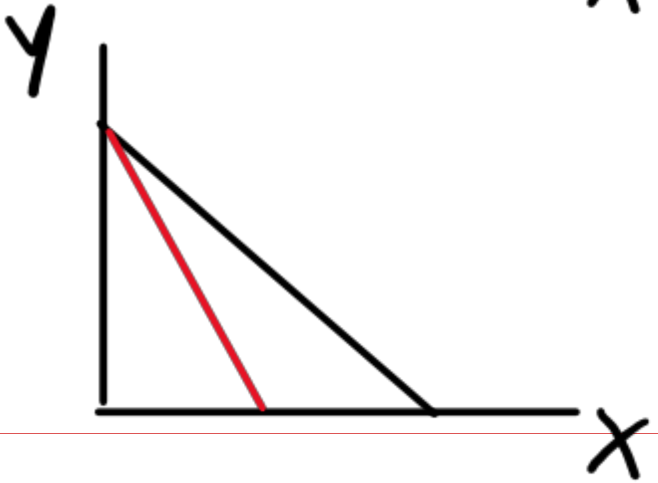
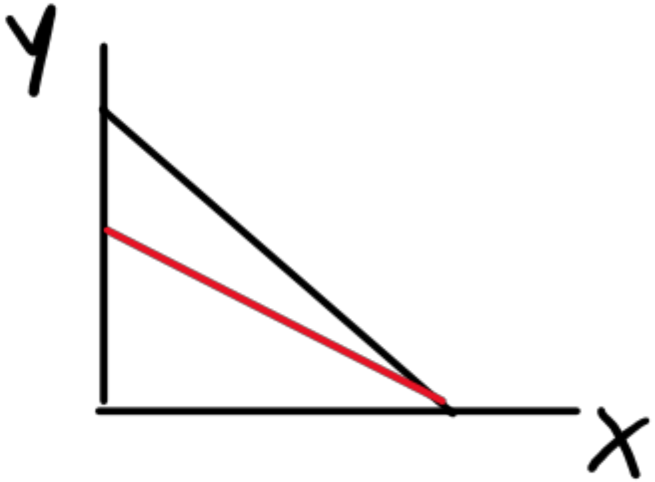


C.


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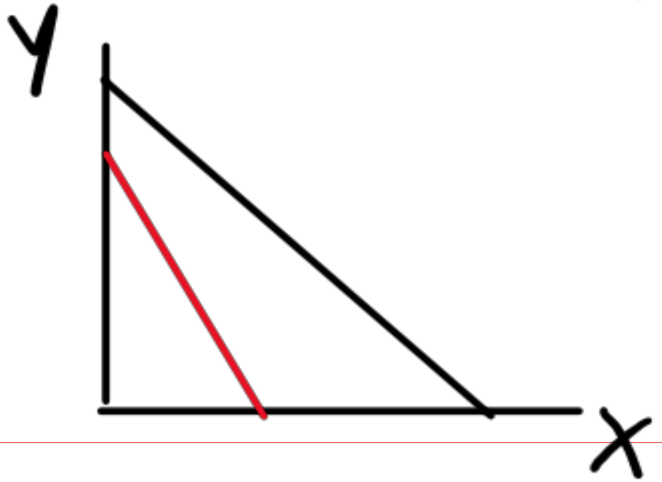
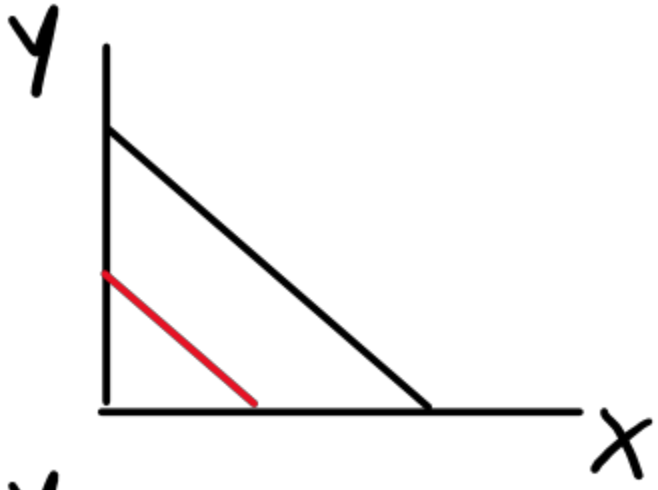


Question: Which of the following graphs shows the price of Y increased? (Black is original, Red is new)

A.



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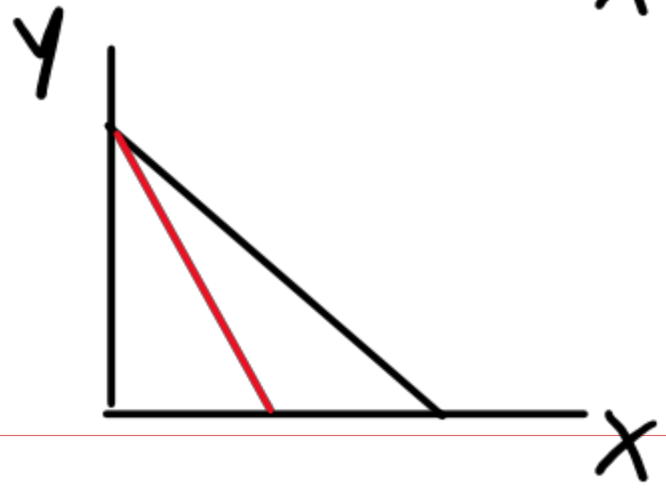

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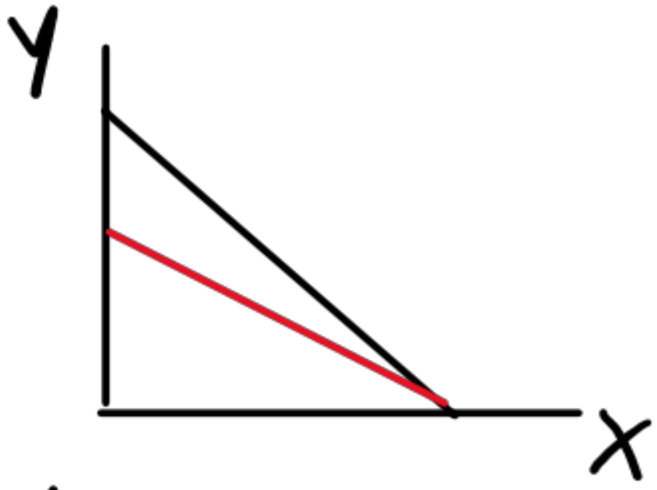

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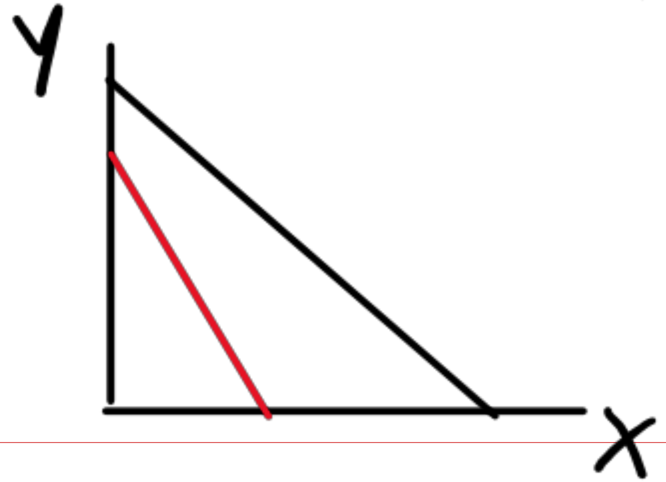
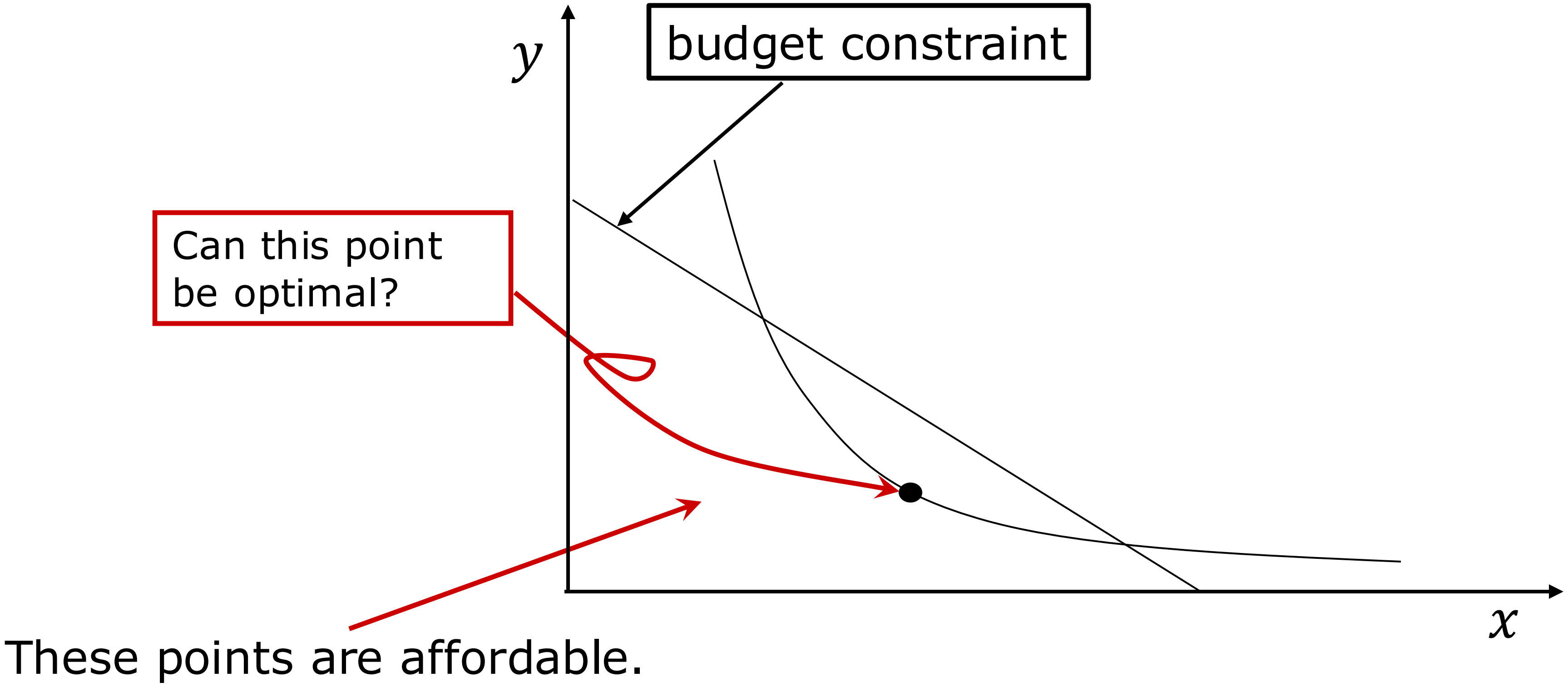


Utility Maximization Problem

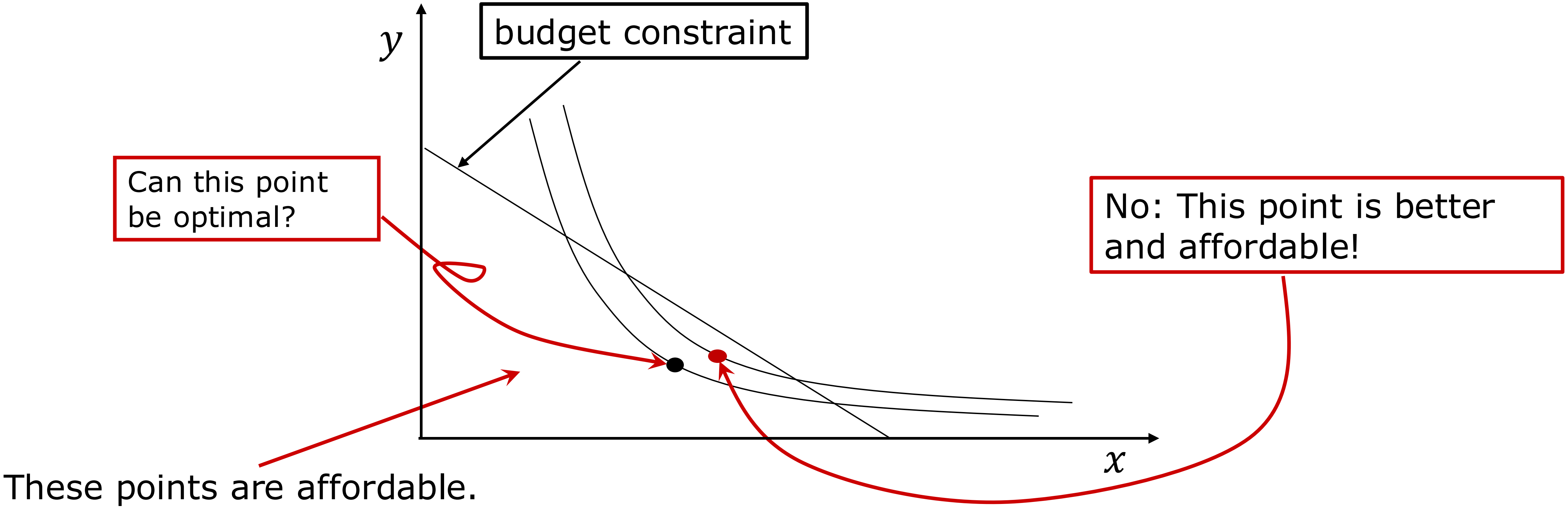
- The consumer has preferences represented by a utility function.
- The consumption bundles available are those that satisfy the budget constraint.
- Given this constraint, the consumer chooses the bundle that maximizes his utility function.
- We call this the consumer's **Utility Maximization Problem (UMP)**:

$$\boxed{\begin{gathered} \max_{x,y} u(x,y) \\ \text{subject to: } p_x x + p_y y \leq m \\ x, y \geq 0 \end{gathered}}$$

Solving the UMP Graphically



Solving the UMP Graphically



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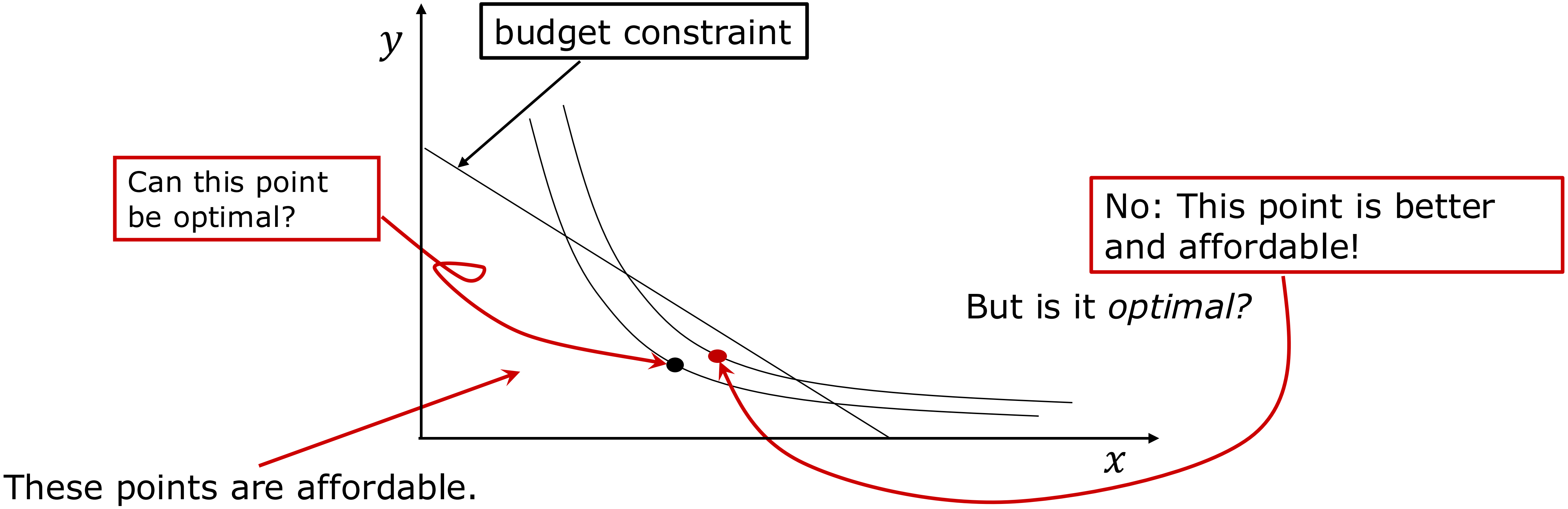

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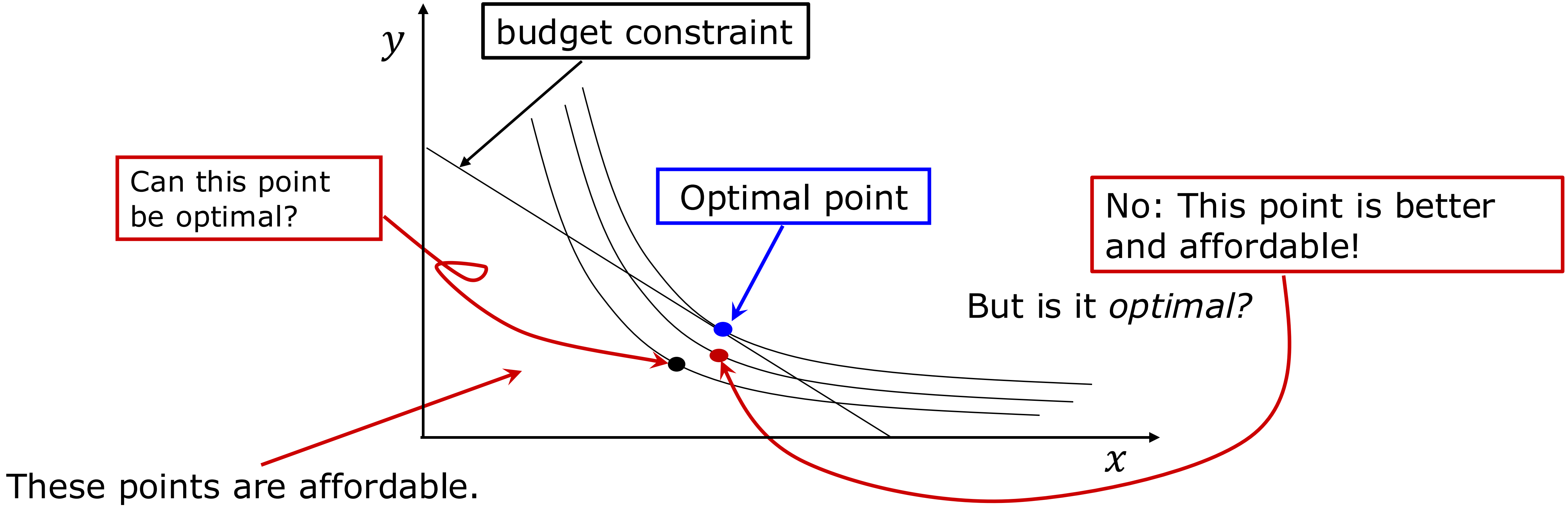


- The optimal point is one where there is no <u>affordable</u> point that is <u>better</u>. This is a point of **tangency** between the budget line and an indifference curve.

Solving the UMP algebraically

- This means that the following two conditions determine the optimal point (x*,y*):

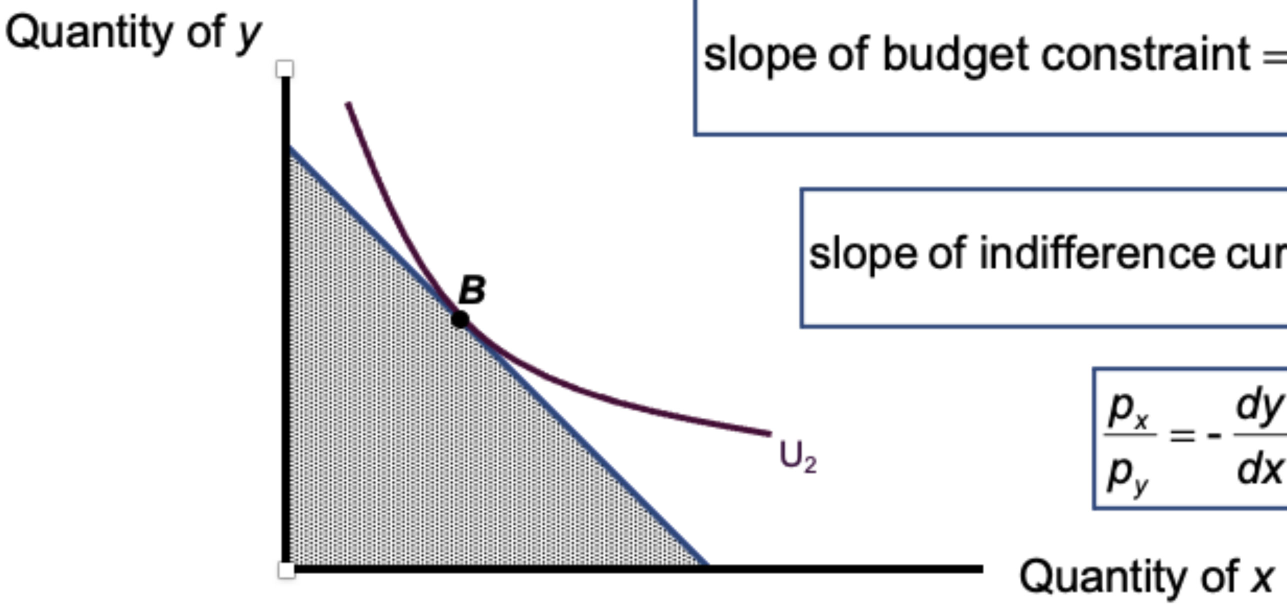
1. (x*,y*) is on the budget line: $p_x x^* + p_y y^* = m$.

2. The |slope of the budget line| = |the slope of the indifference curve| (which is |MRS|):

$$MRS = \frac{\frac{\partial U(x^*, y^*)}{\partial x}}{\frac{\partial U(x^*, y^*)}{\partial y}} = \frac{p_x}{p_y}$$

First-order conditions for a maximum

- Utility is maximized where the indifference curve is tangent to the budget constraint



$$\text{slope of budget constraint} = -\frac{p_x}{p_y}$$

$$\text{slope of indifference curve} = \left.\frac{dy}{dx}\right|_{U=\text{constant}}$$

$$\frac{p_x}{p_y} = -\left.\frac{dy}{dx}\right|_{U=\text{constant}} = MRS$$

Interpreting the tangency condition

- Let U_x denote $\partial U/\partial x$ and U_y denote $\partial U/\partial y$. (In general, $U_n = \partial U/\partial n$.)
- Rearrange the tangency condition to get: $\frac{U_x}{p_x} = \frac{U_y}{p_y}$.
- Note $\frac{U_x}{p_x} = U_x \cdot \frac{1}{p_x}$

How much x you can buy if you spend 1 more dollar on x

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How much x you can buy if you spend 1 more dollar on x

How much utility you get if you consumer a little more of x

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How much x you can buy if you spend 1 more dollar on x

How much utility you get if you consumer a little more of x

- So U_x/p_x is the additional utility from spending another dollar on x.

Interpreting the tangency condition

- $\frac{U_x}{p_x} = \frac{U_y}{p_y}$

- At the optimum, the additional utility from spending another dollar on x must equal the additional utility from spending another dollar on y. Why must this be?

- Question: What happens if $U_x/p_x > U_y/p_y$? What should the consumer do?

Solving the UMP algebraically: example

- Consider $U(x, y) = xy$

- Step 1. Invoke optimal condition:

$$\frac{U_x}{p_x} = \frac{U_y}{p_y}$$

  $U_x = \partial U/\partial x = y$, $U_y = \partial U/\partial y = x$, so

$$\frac{y^*}{p_x} = \frac{x^*}{p_y}$$

  Rearranging this gives:

$$y^* = \frac{p_x x^*}{p_y}$$

Solving the UMP algebraically: example

- Step 2: Substitute $y^* = \frac{p_x x^*}{p_y}$ into budget constraint:

$$p_x x^* + p_y y^* = m$$

$$p_x x^* + p_y \frac{p_x x^*}{p_y} = m$$

$$2p_x x^* = m$$

Demand for x

$$\boxed{x^* = \frac{m}{2p_x}}$$

Demand for y

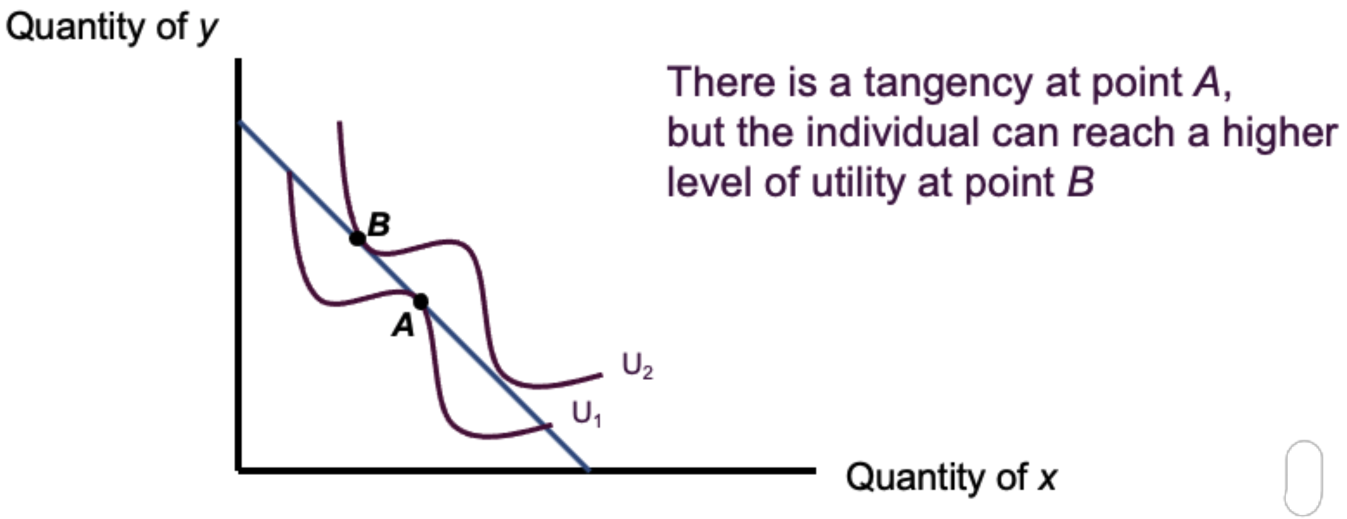
$$\boxed{y^* = \frac{p_x x^*}{p_y} = \frac{m}{2p_y}}$$

- Note that the solutions are ***functions*** of prices (p_x, p_y) and wealth (m).

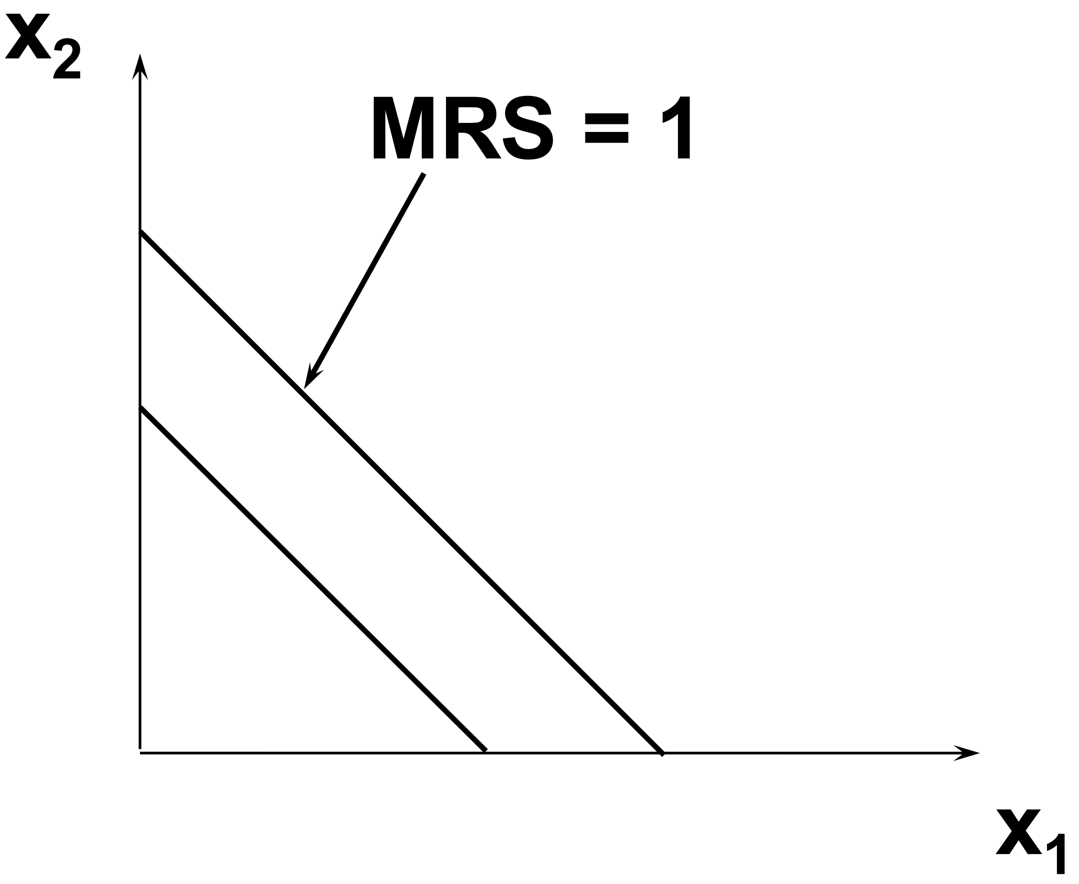
Second-order conditions for a maximum

- The tangency rule is only necessary but not sufficient
  - We need *MRS* to be diminishing
  - then indifference curves are strictly convex

- If *MRS* is not diminishing, then we must check second-order conditions to ensure that we are at a maximum



Examples of Corner Solutions -- the Perfect Substitutes Case



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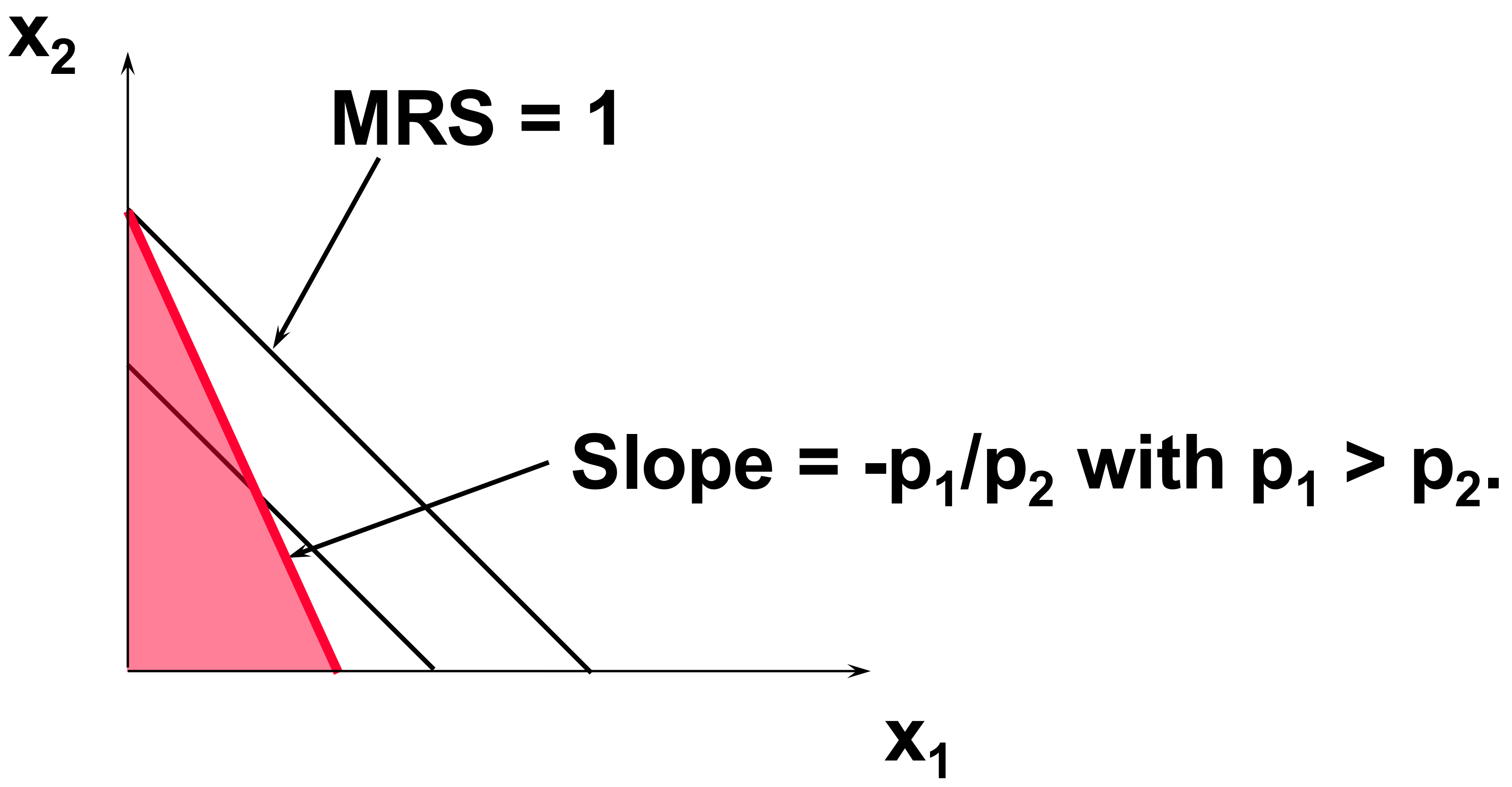

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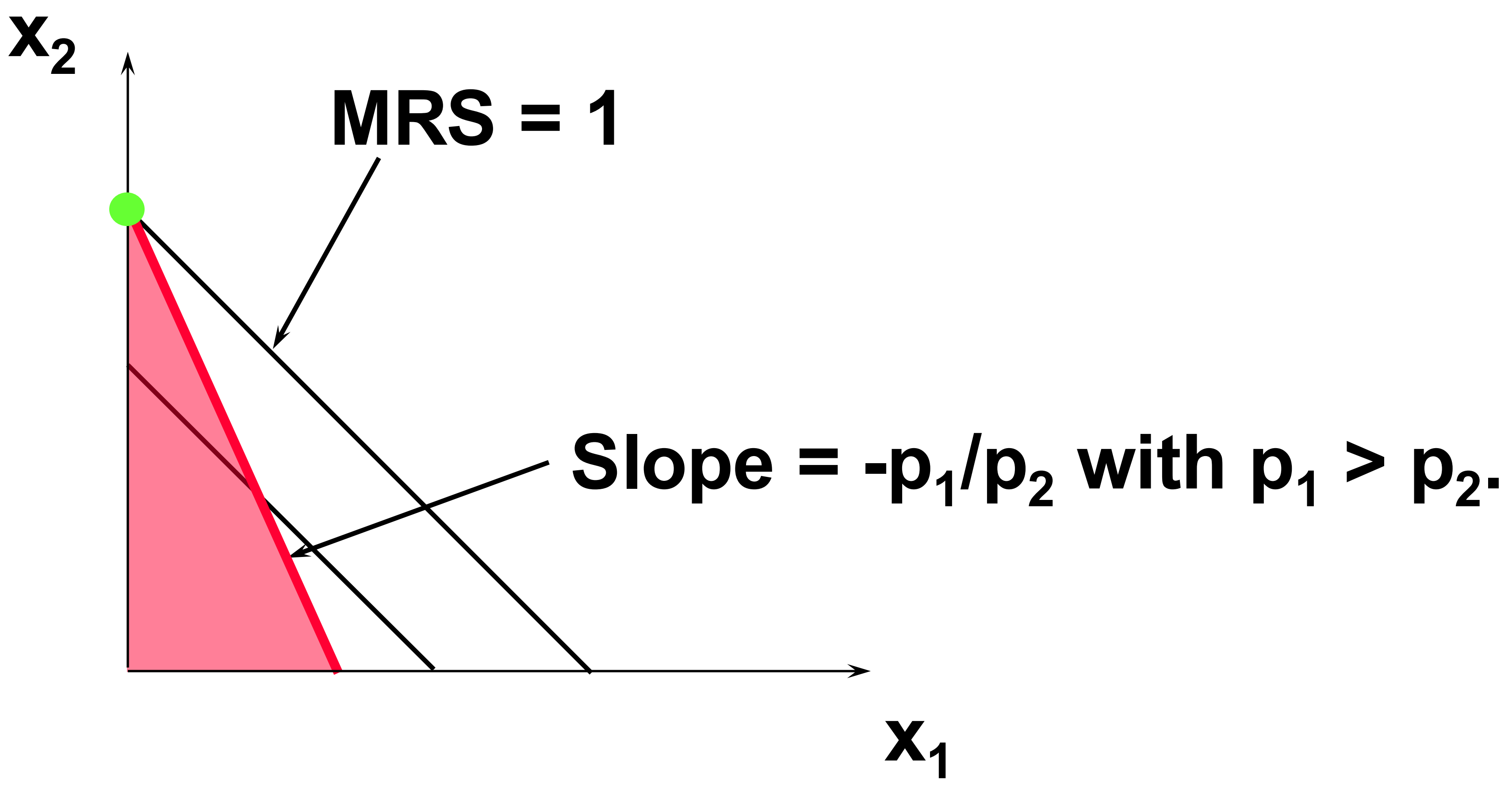

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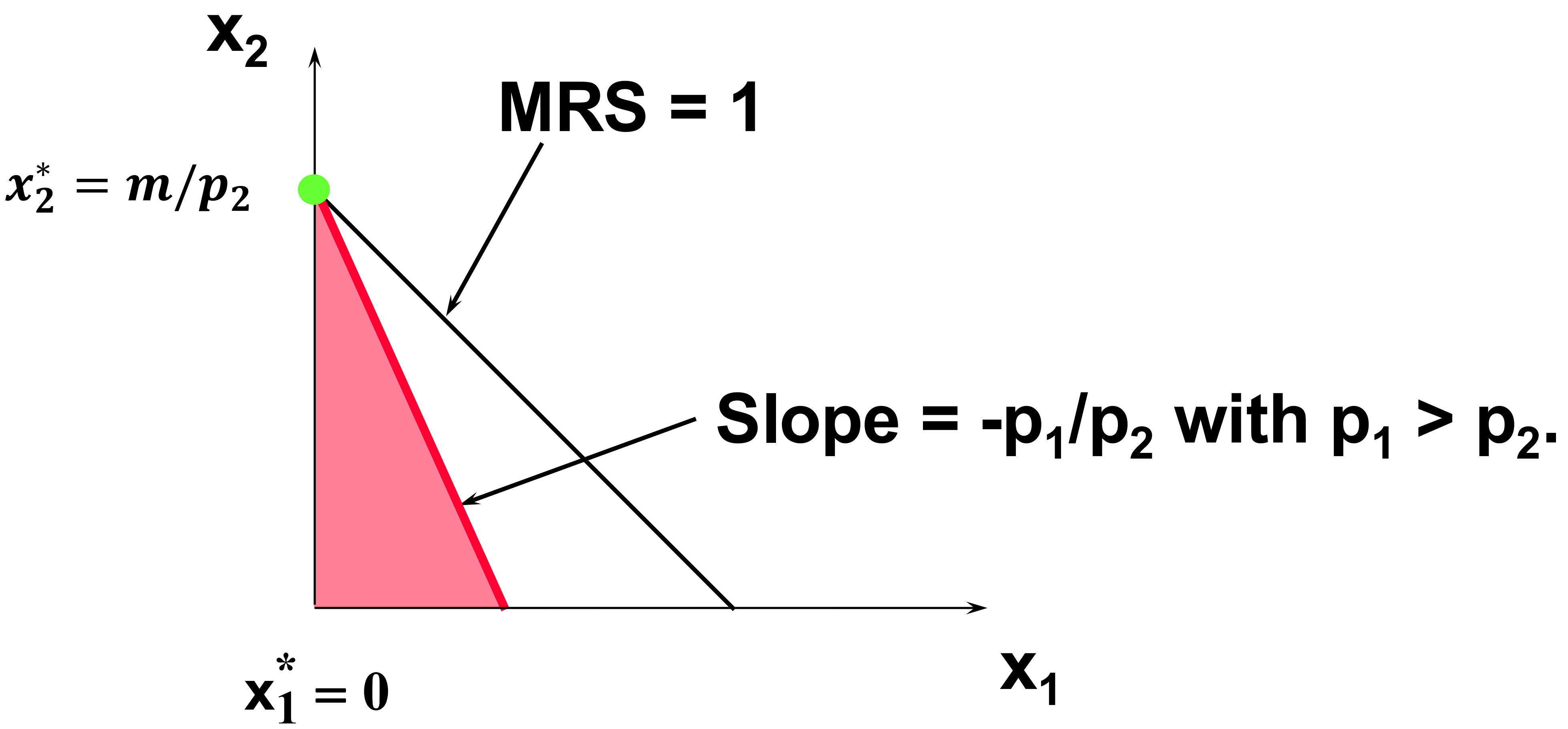

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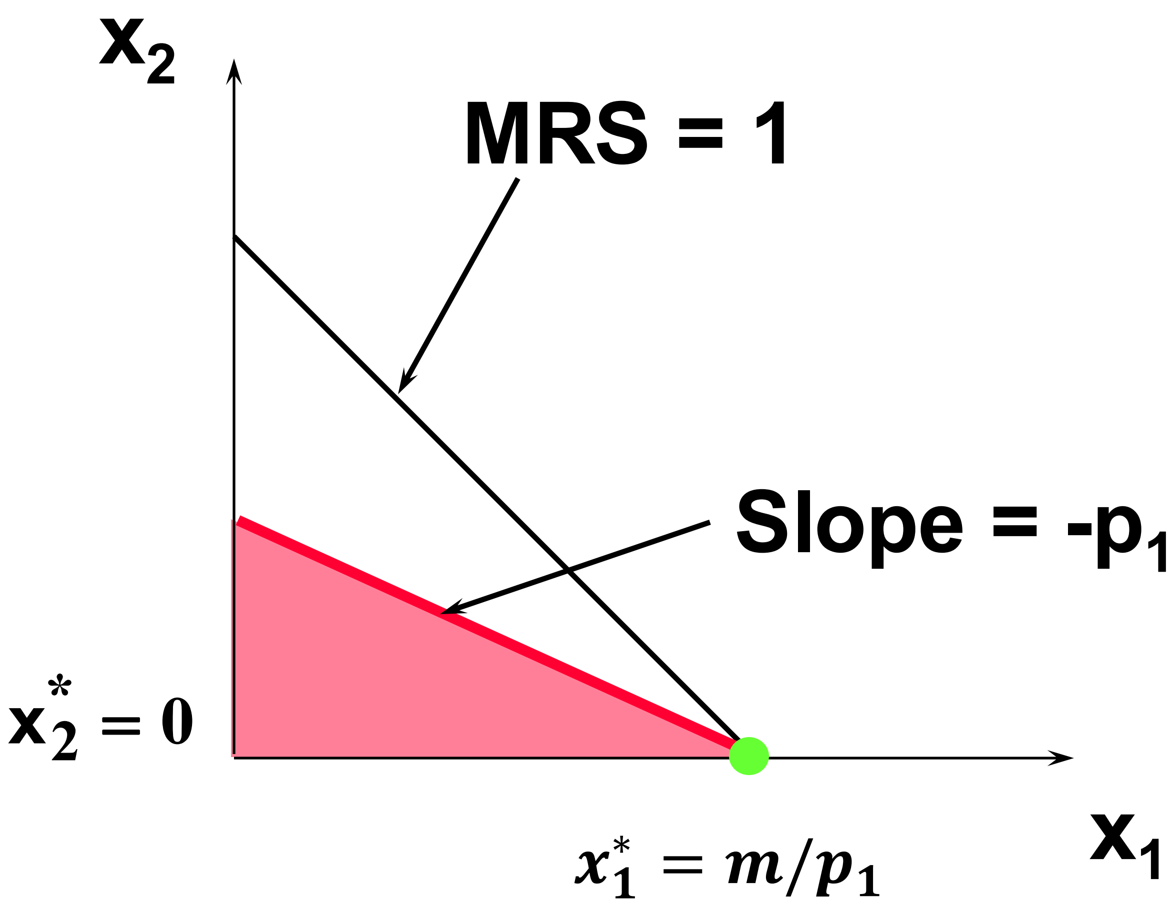

Examples of Corner Solutions -- the Perfect Substitutes Case

So when $U(x_1,x_2) = x_1 + x_2$, the most preferred affordable bundle is (x_1^*,x_2^*) where

$$(x_1^*, x_2^*) = \left(\frac{m}{p_1}, 0\right) \quad \textbf{if } p_1 < p_2$$

and

$$(x_1^*, x_2^*) = \left(0, \frac{m}{p_2}\right) \quad \textbf{if } p_1 > p_2.$$

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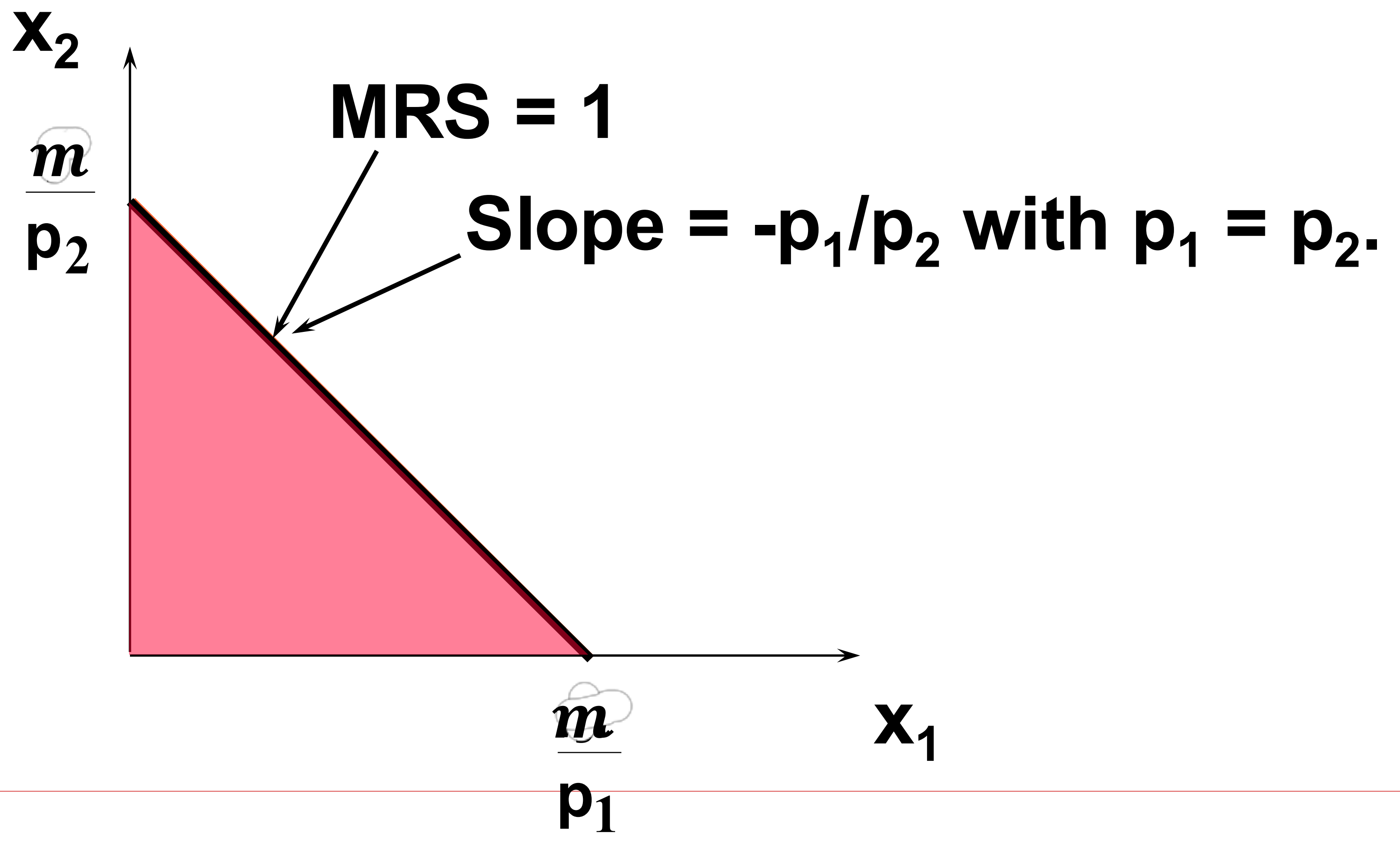

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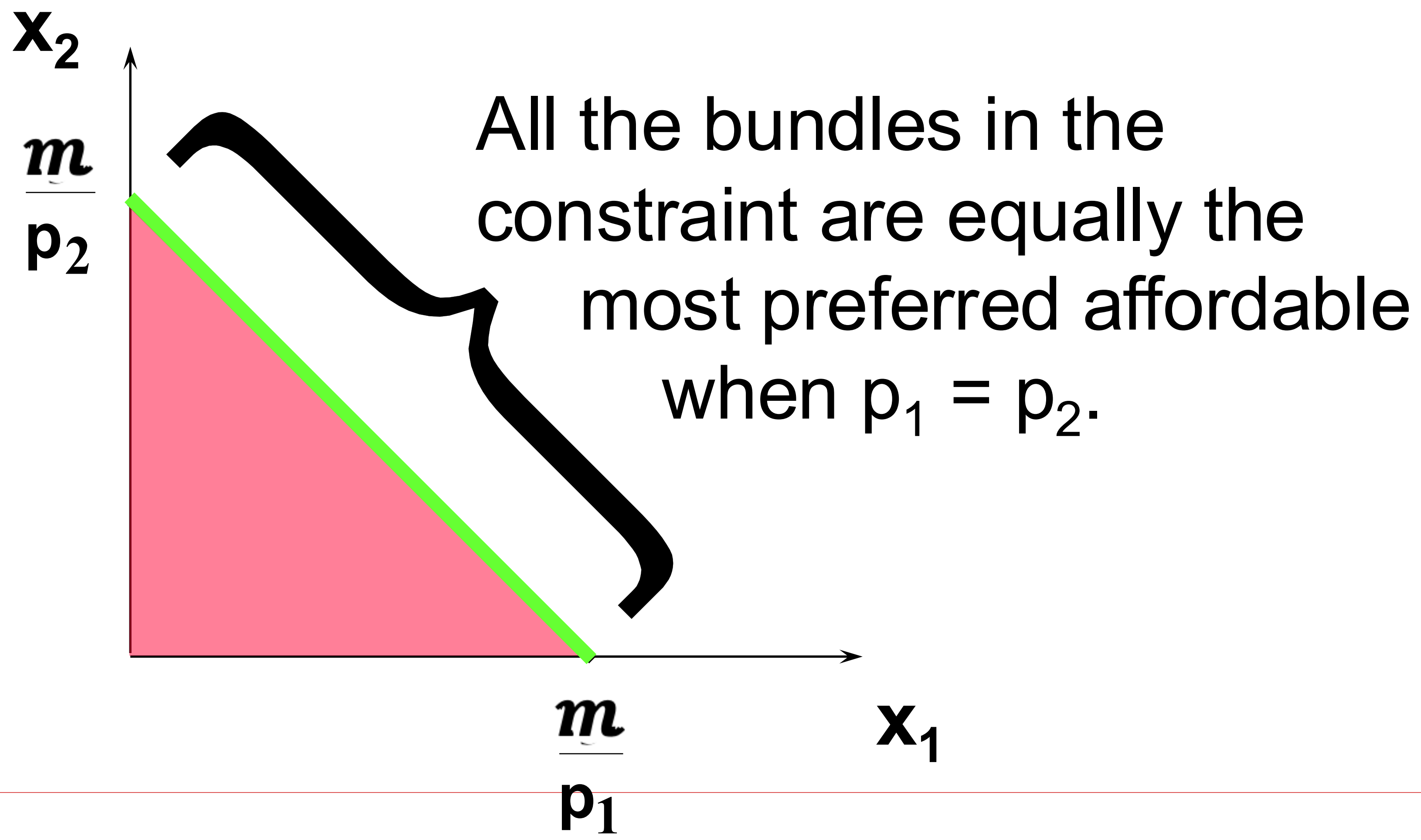
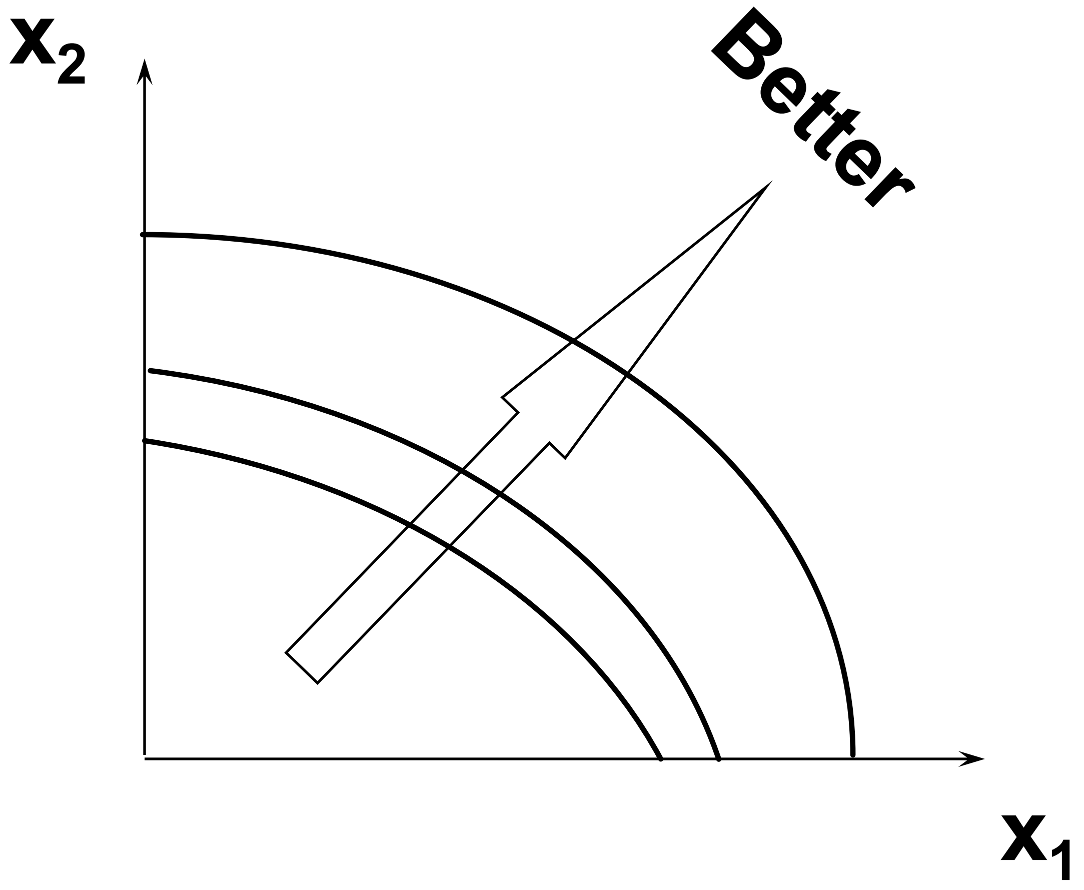


Examples of Corner Solutions
-- the Non-Convex Preferences Case



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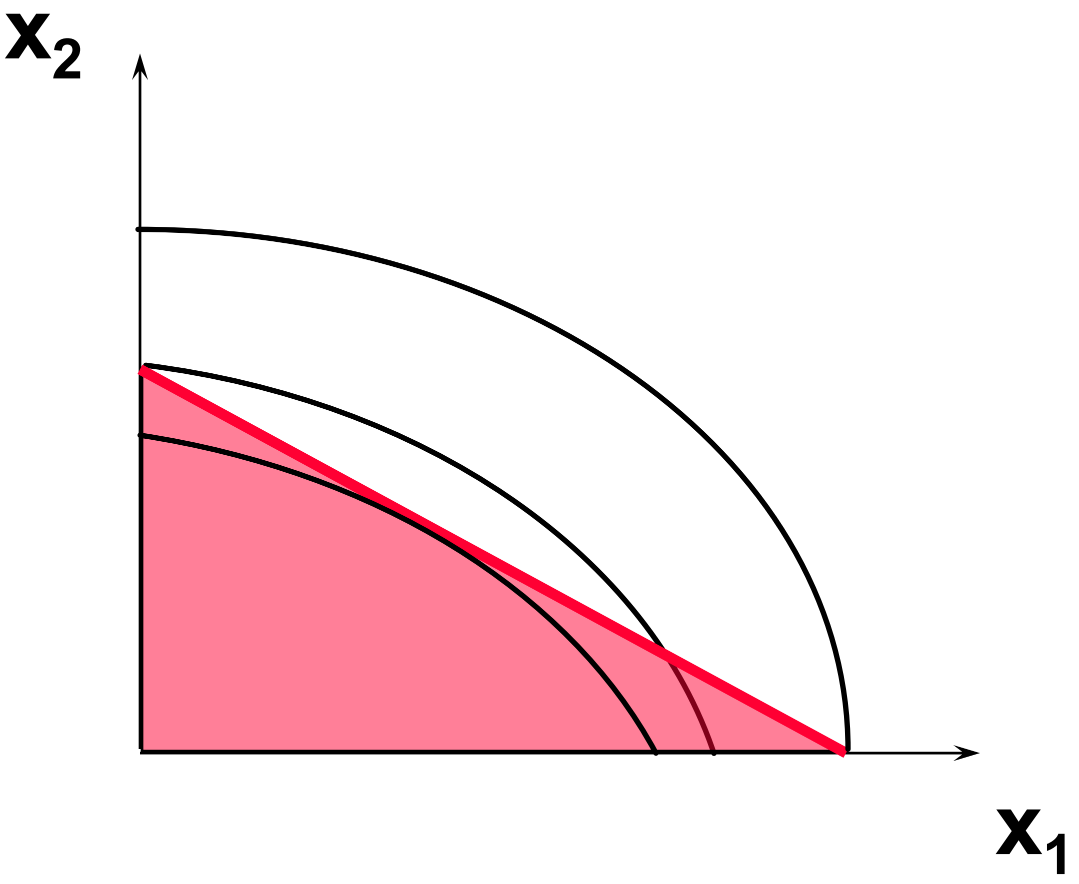

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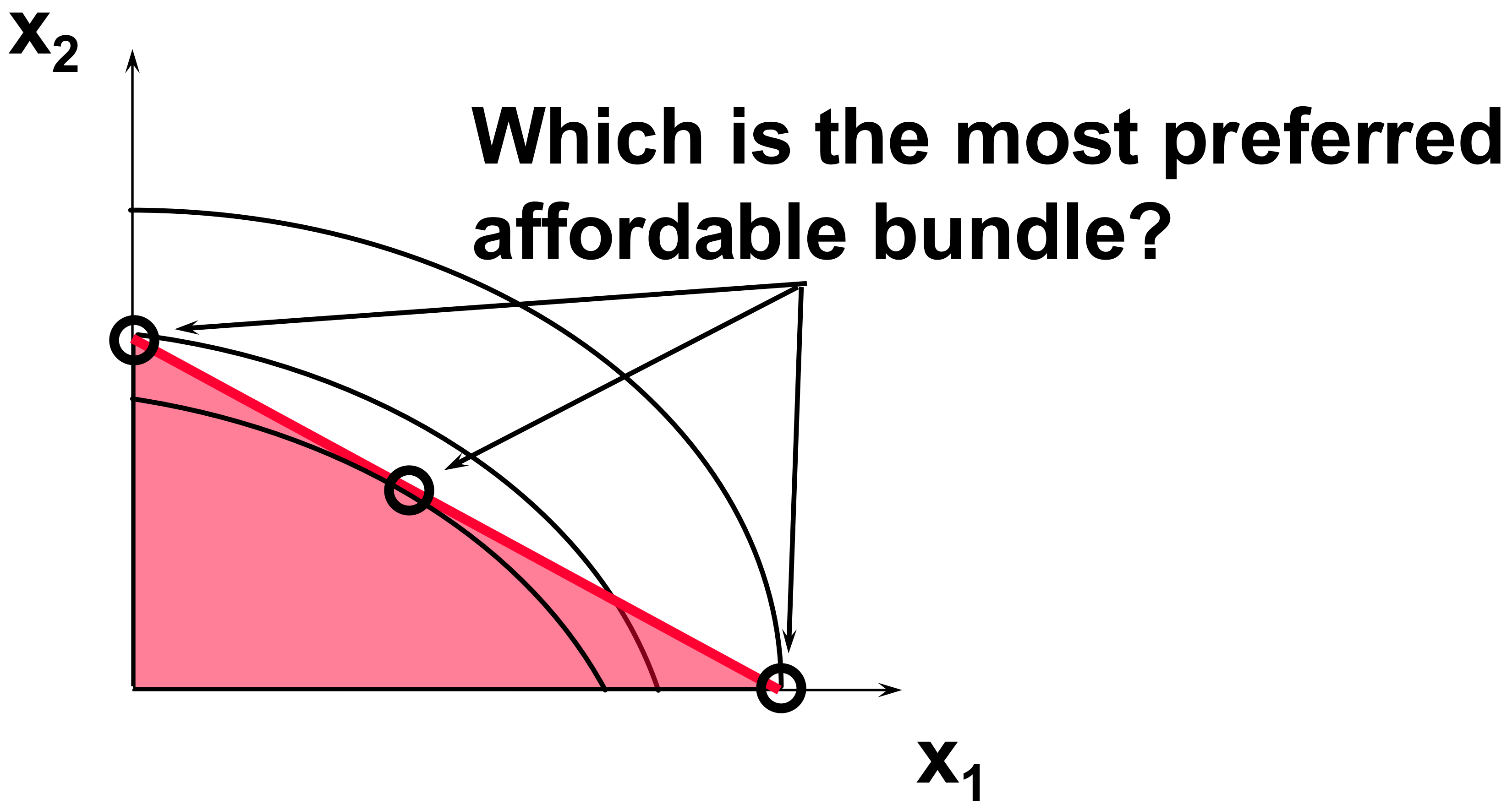

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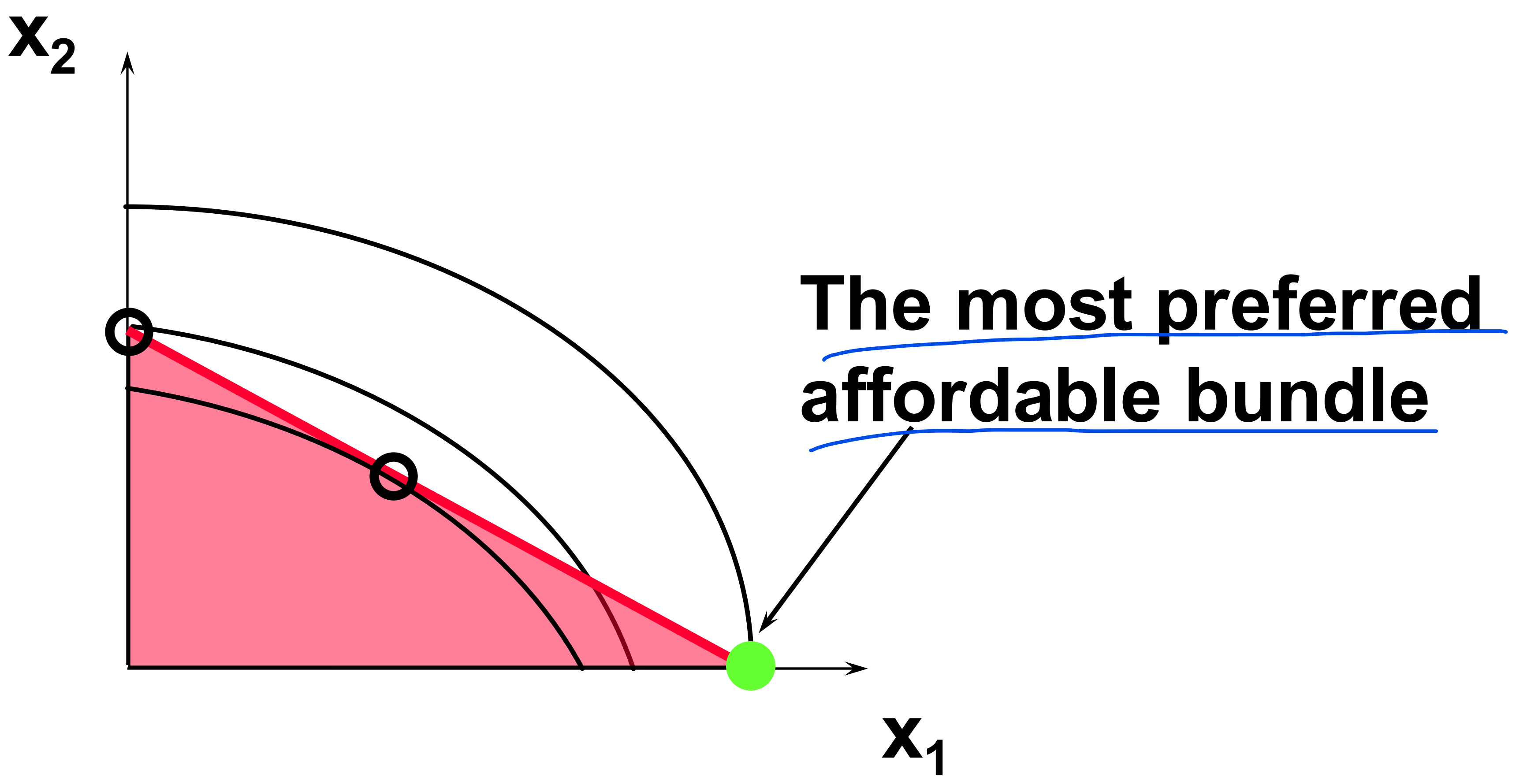

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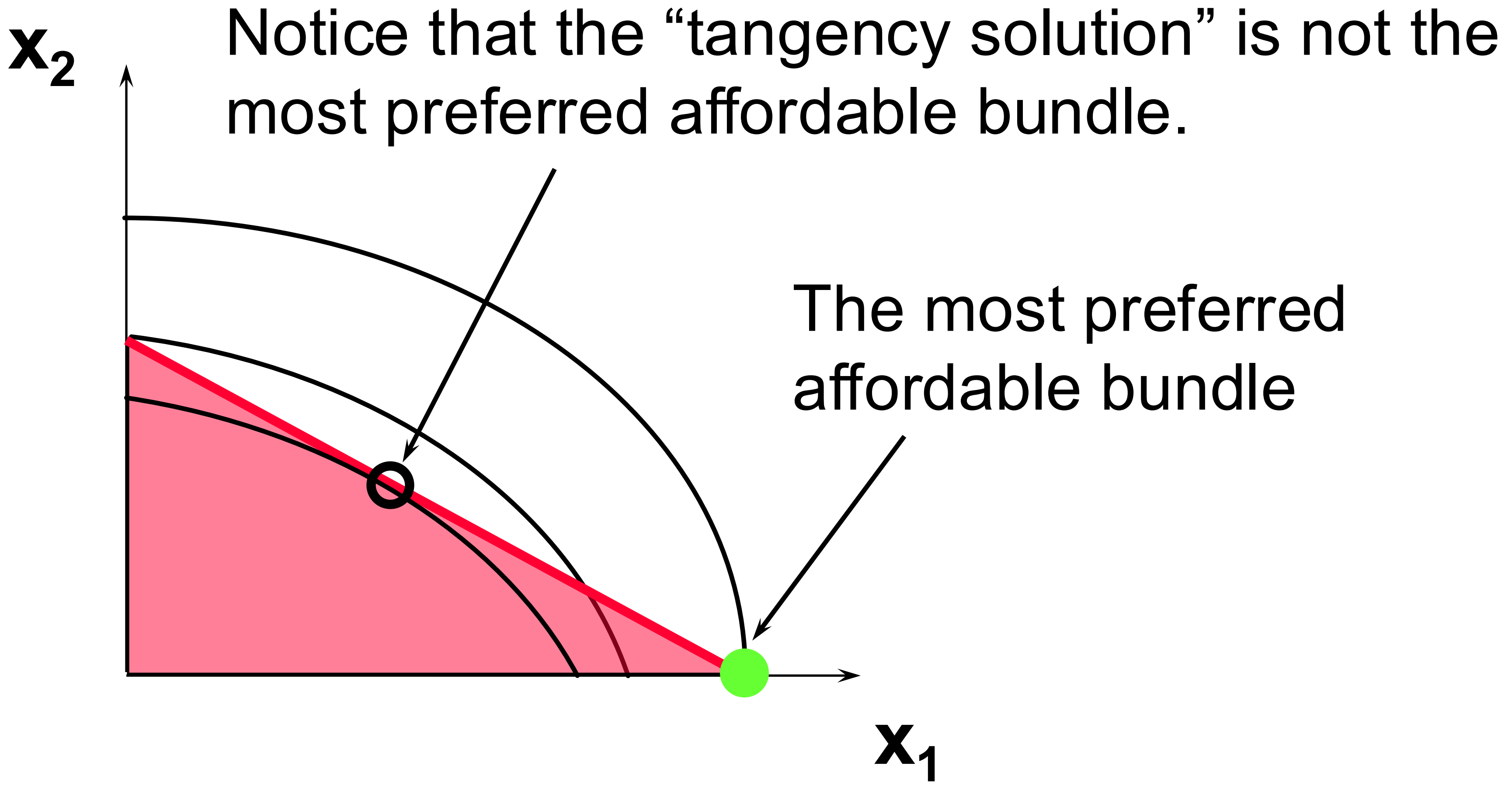
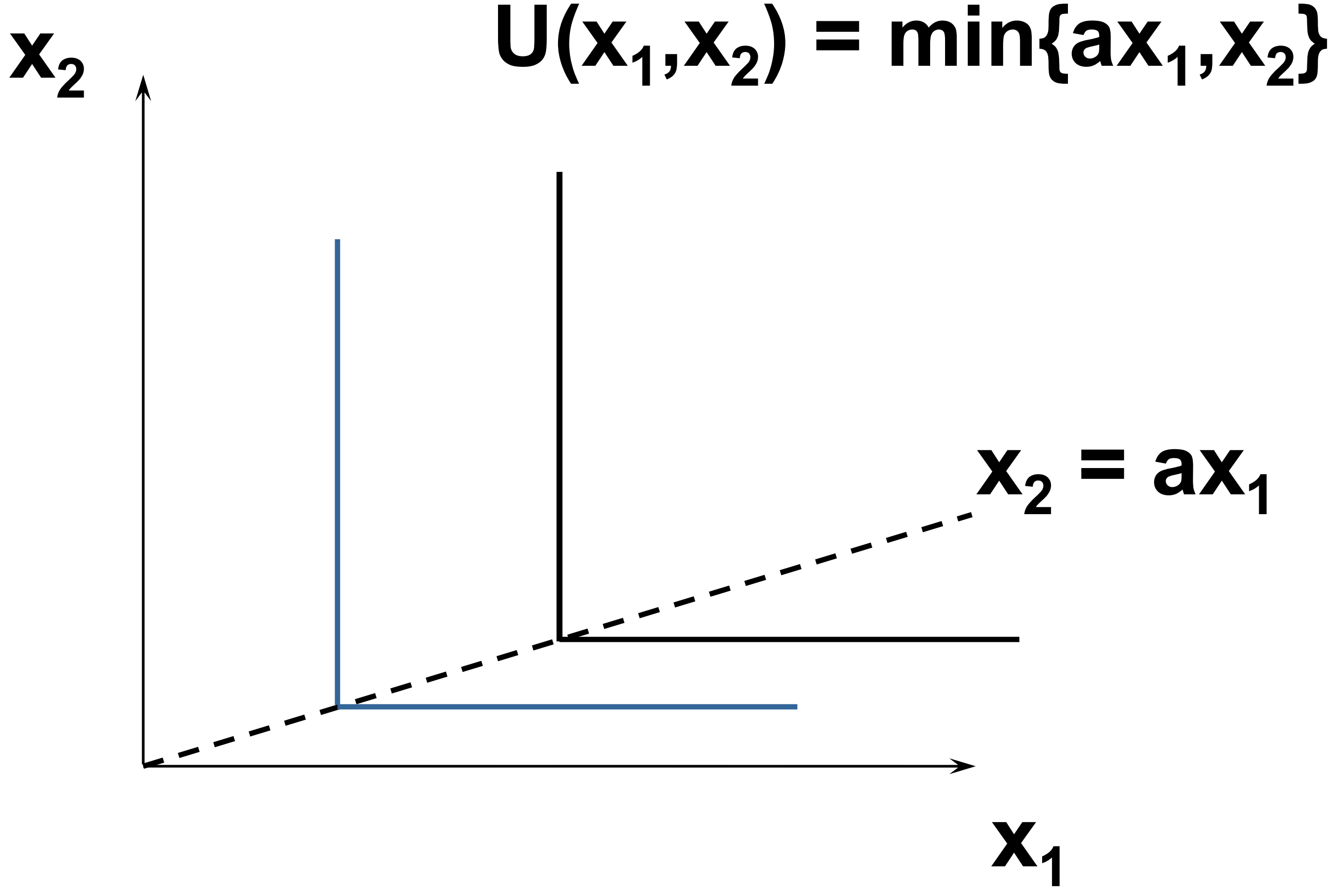


Examples of 'Kinky' Solutions -- the Perfect Complements Case



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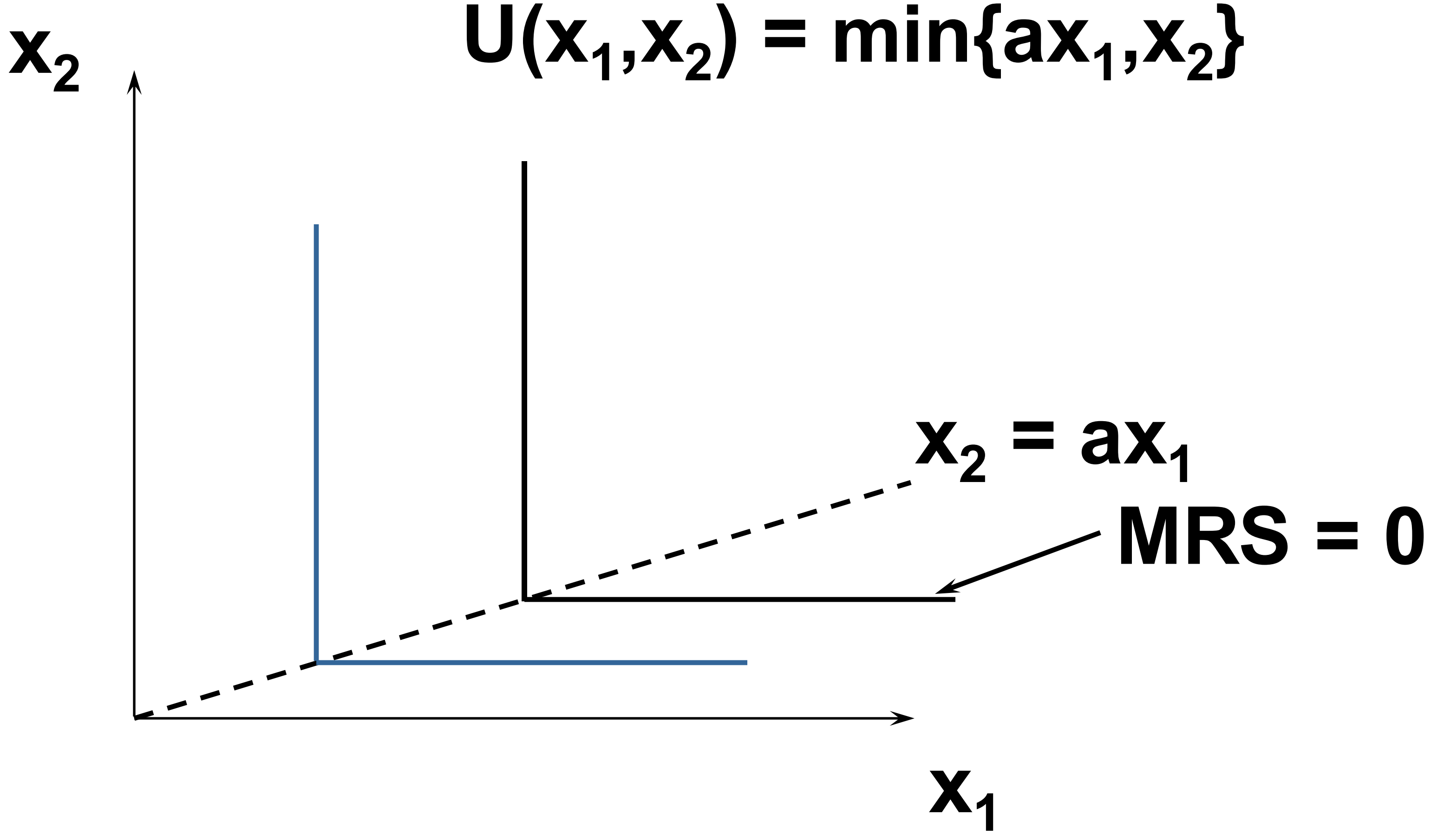

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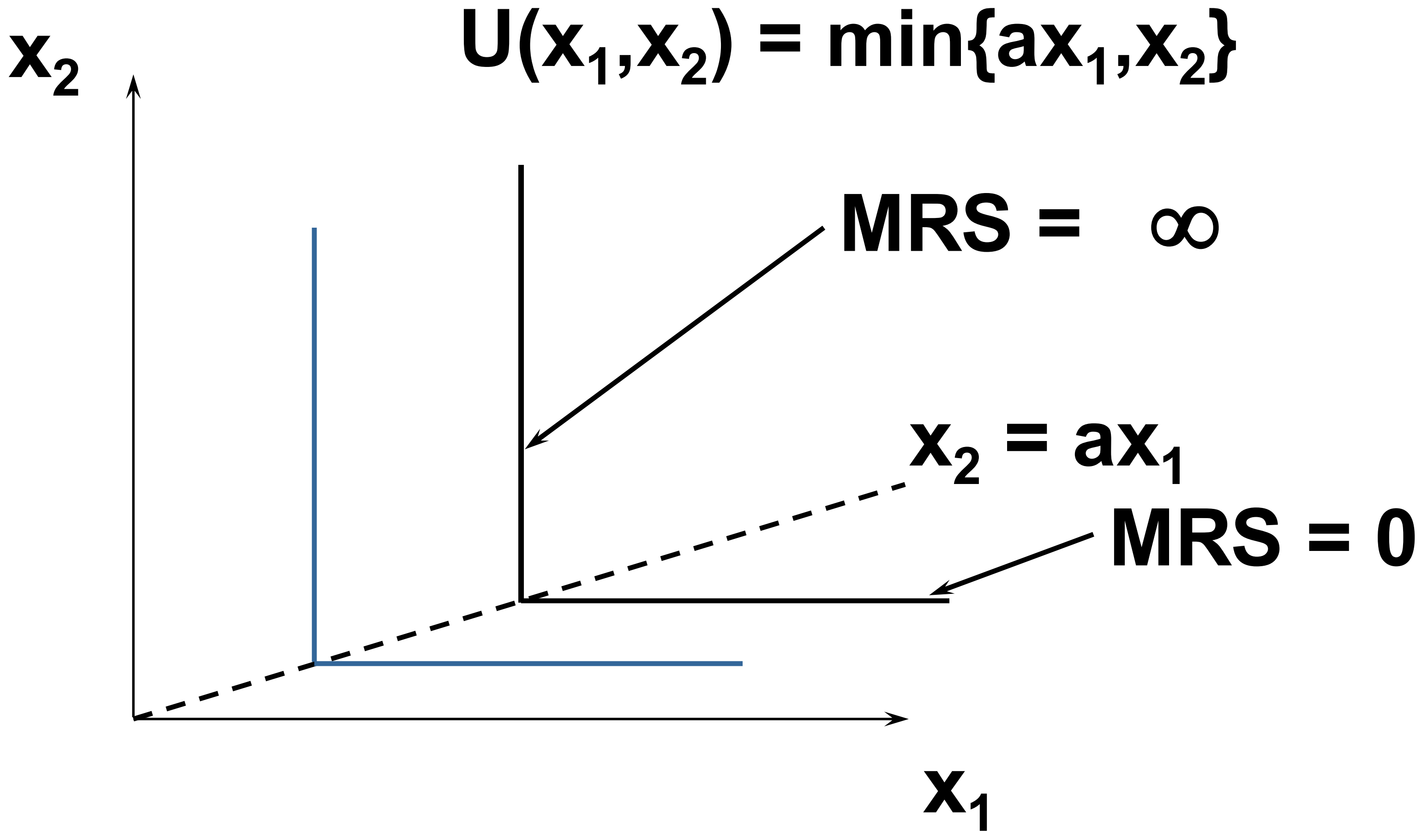

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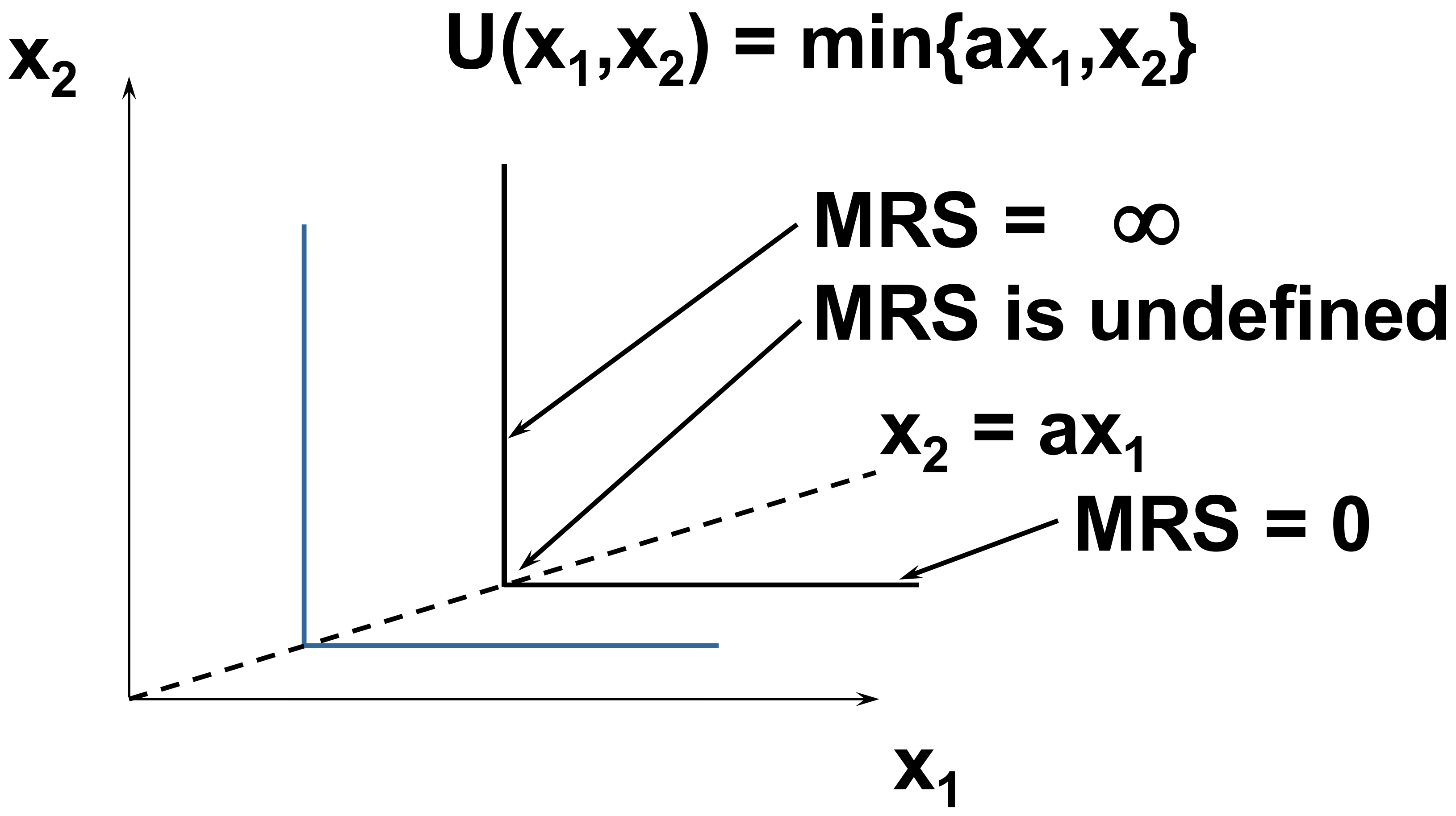

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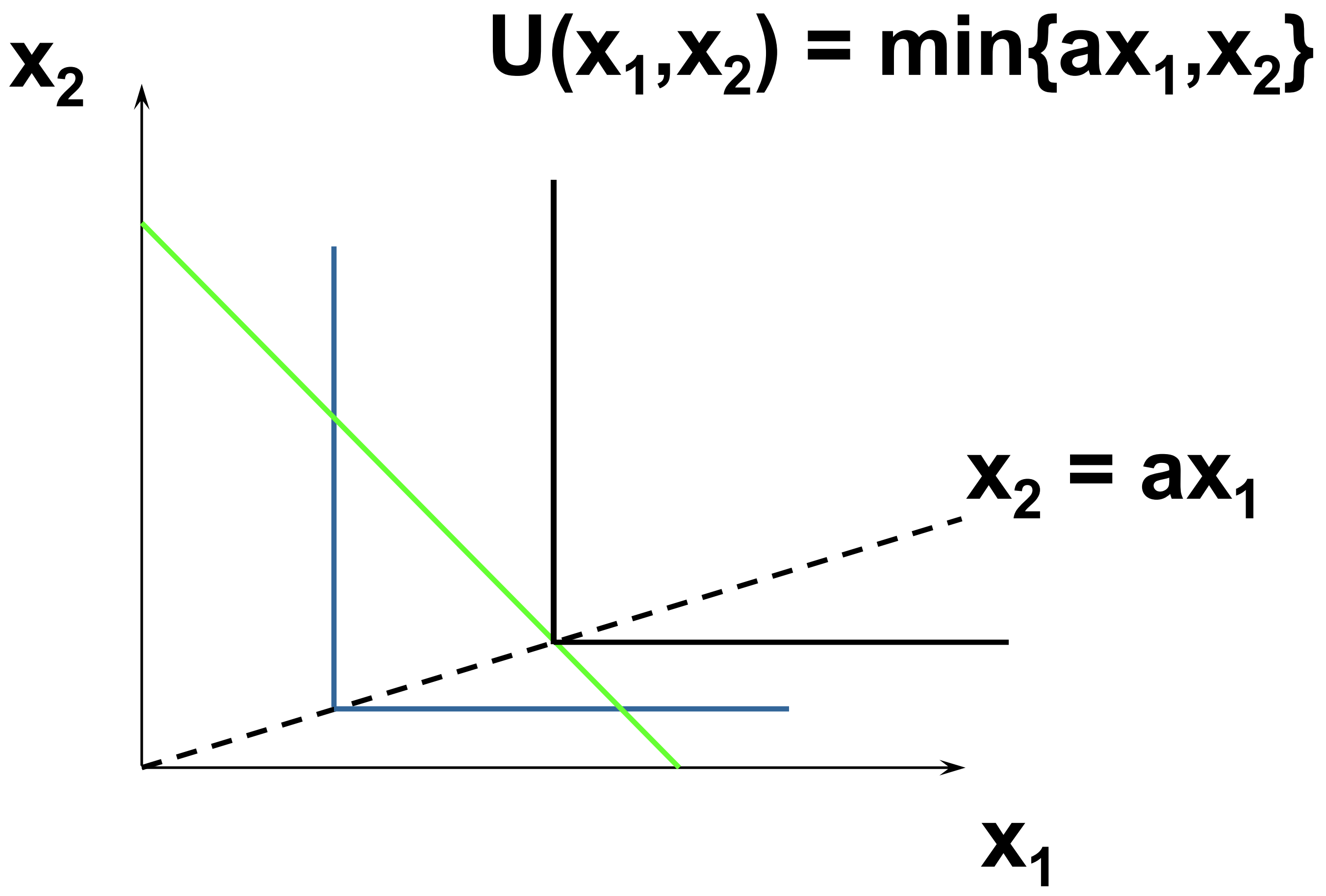
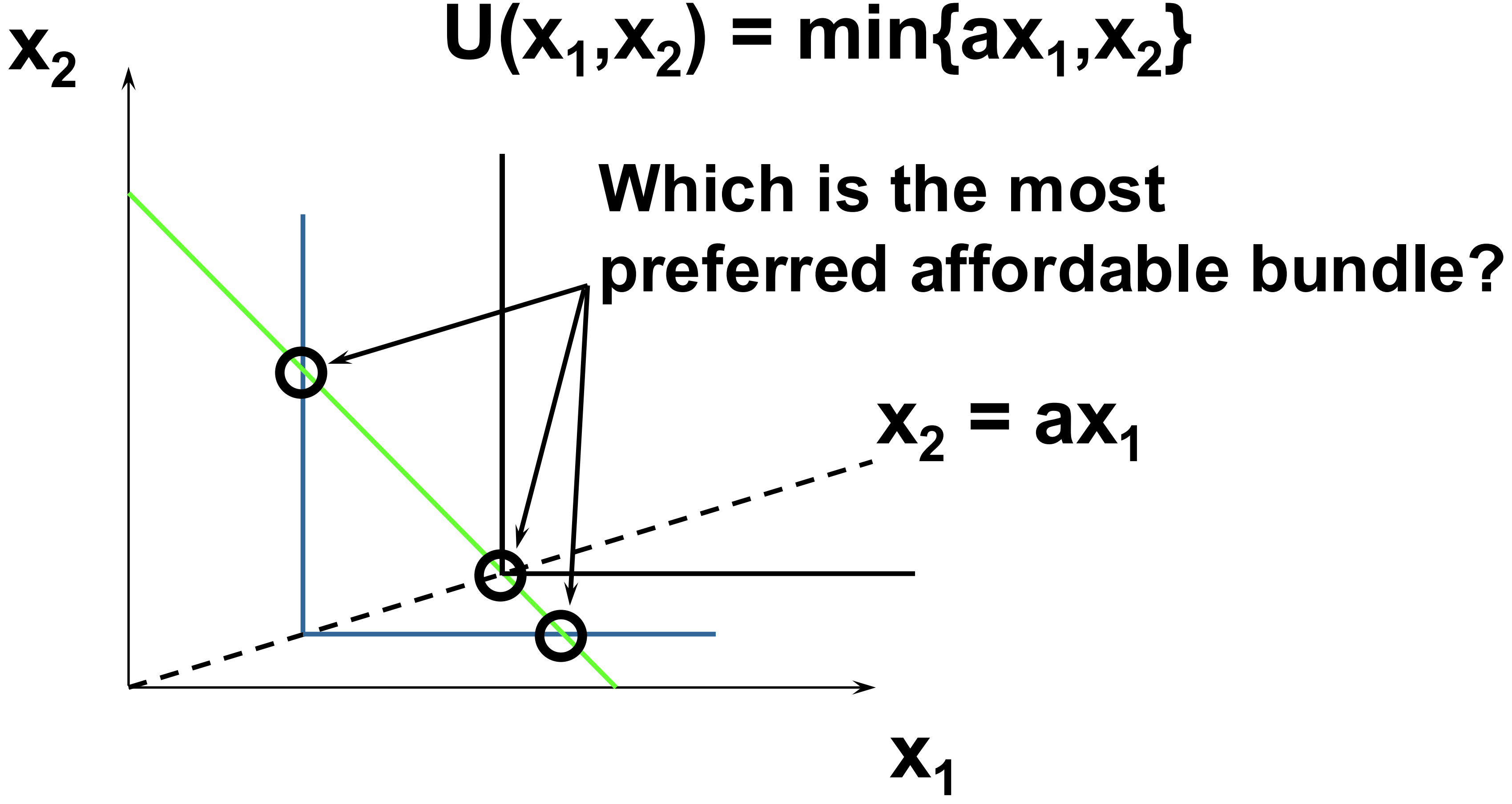


Examples of 'Kinky' Solutions -- the Perfect Complements Case

$$U(x_1,x_2) = \min\{ax_1,x_2\}$$



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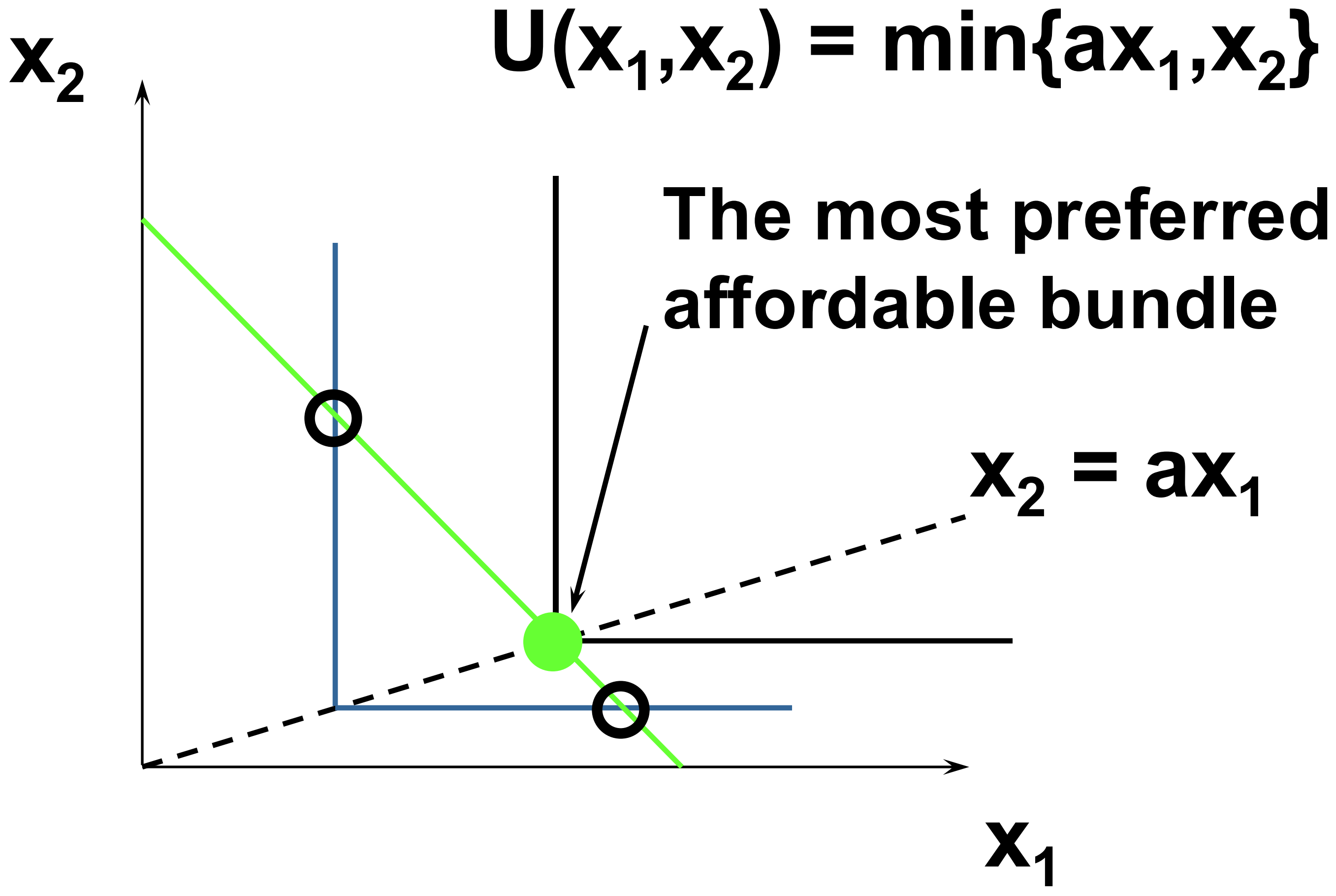

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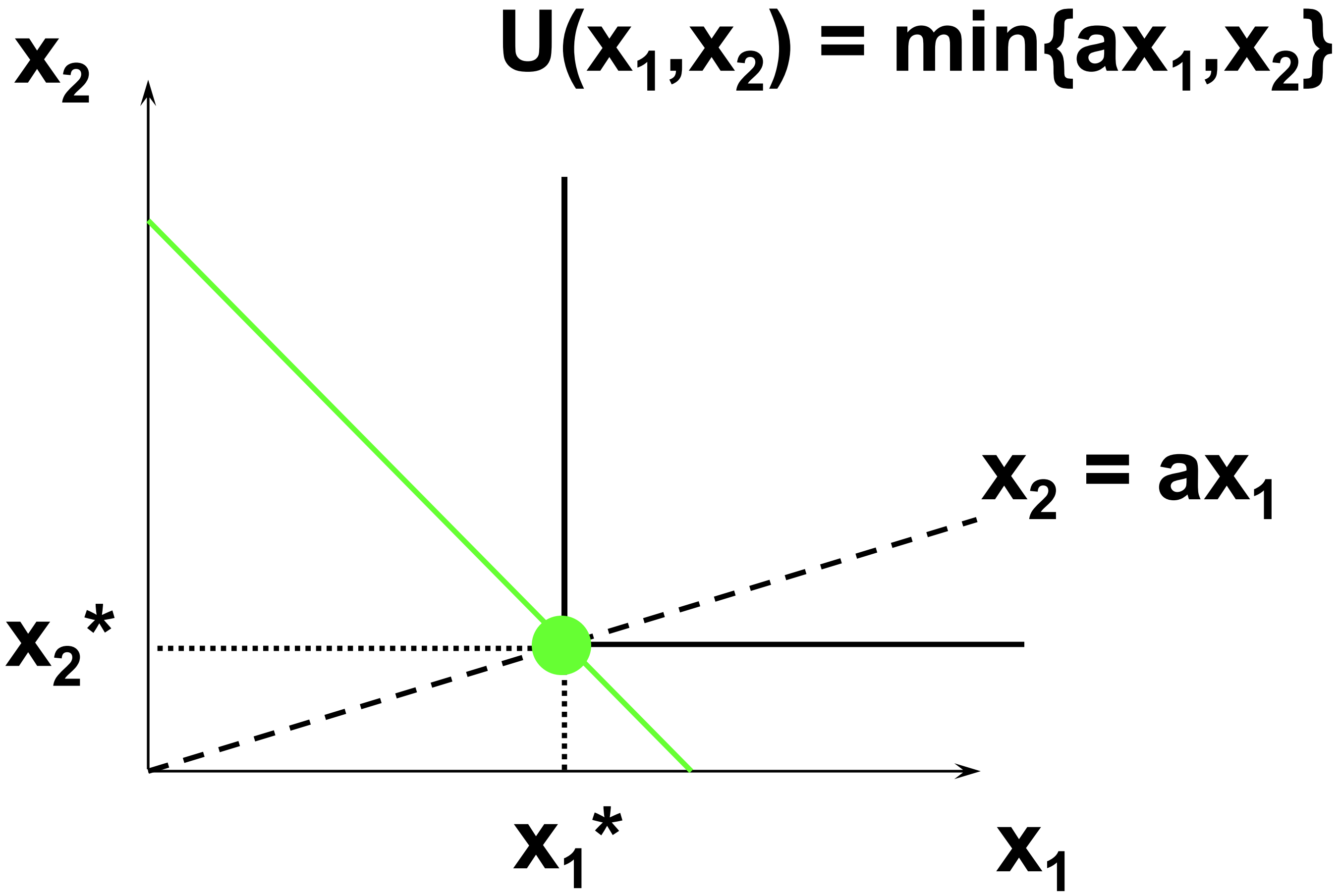
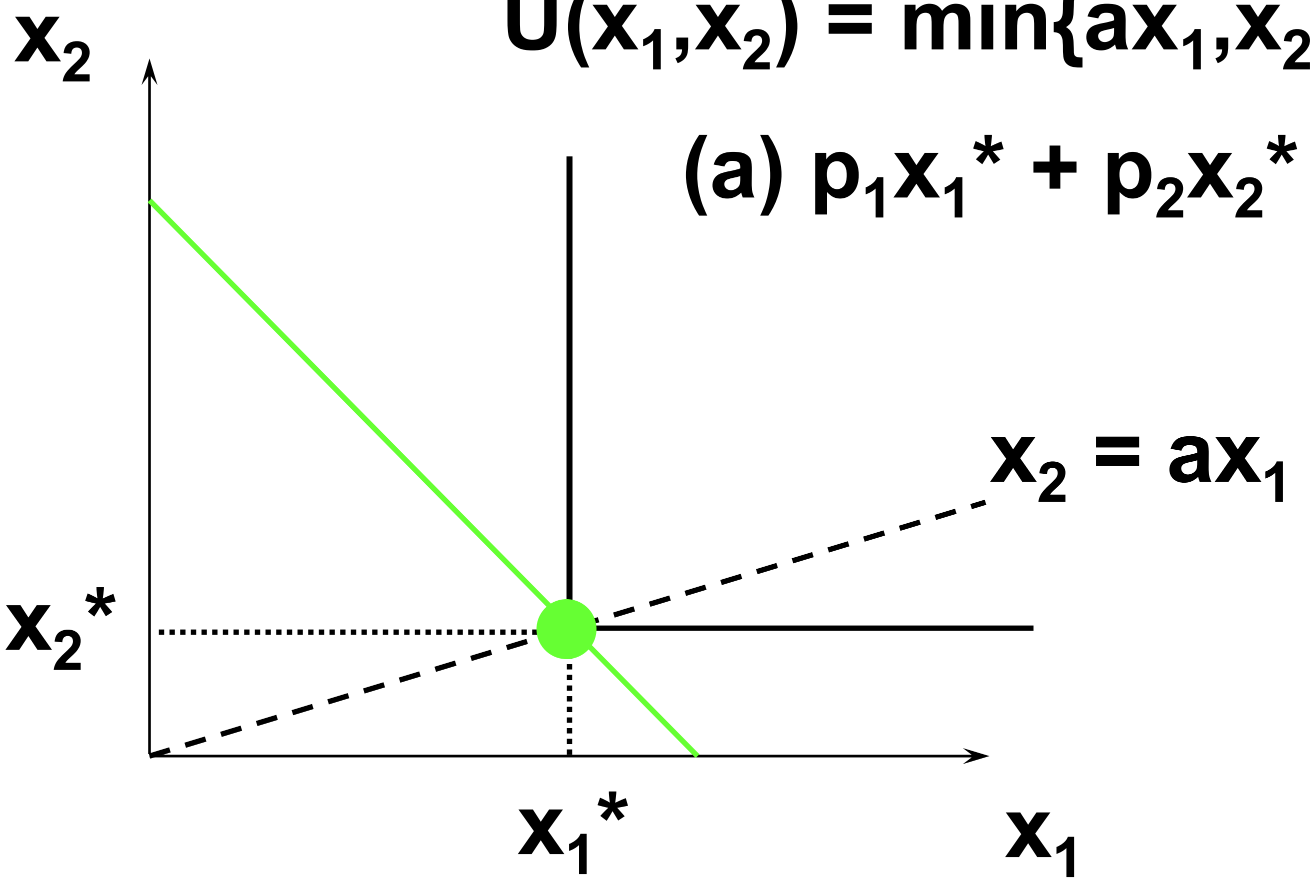


Examples of 'Kinky' Solutions -- the Perfect Complements Case

$U(x_1,x_2) = \min\{ax_1,x_2\}$

(a) $p_1x_1^* + p_2x_2^* = m$



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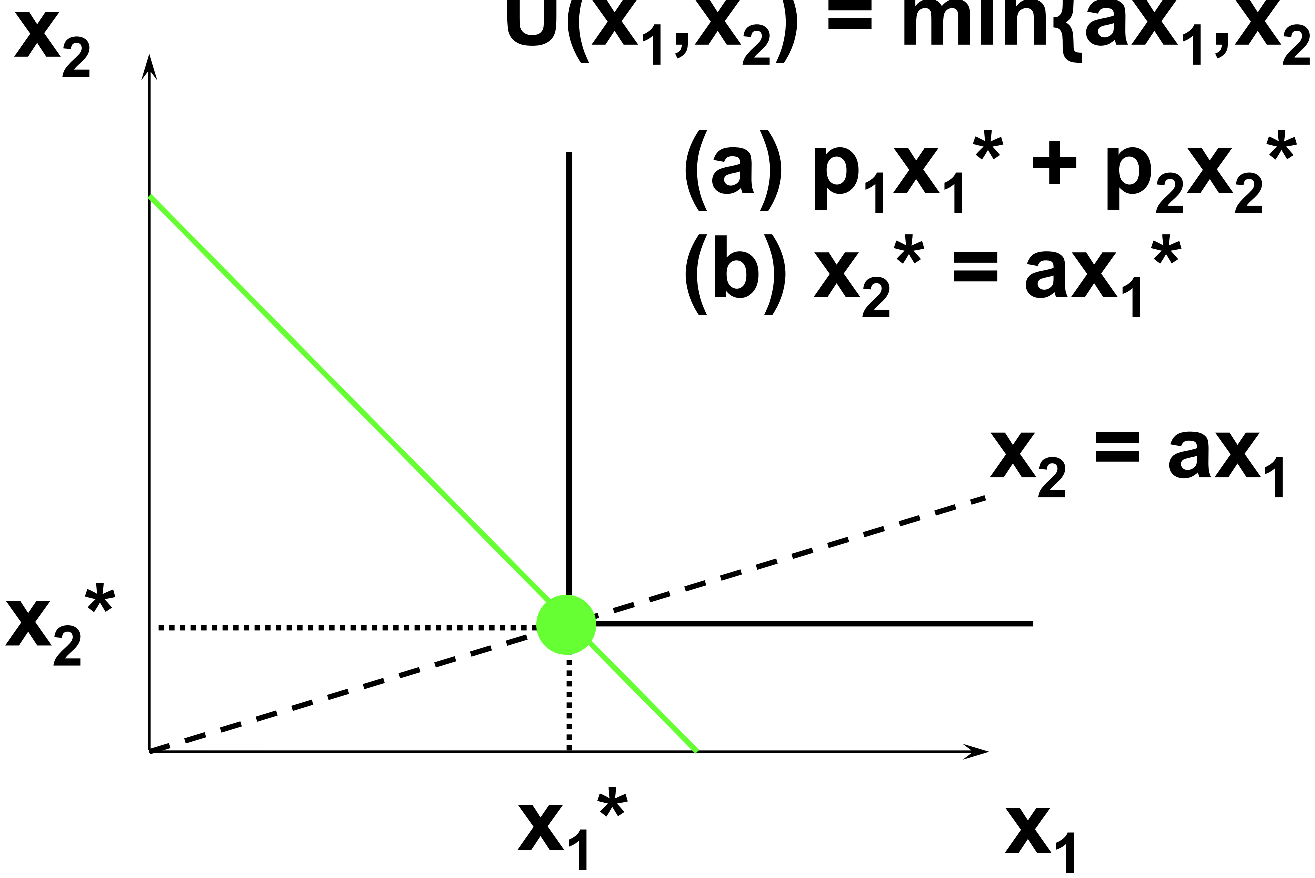
$$U(x_1,x_2) = \min\{ax_1,x_2\}$$

(a) $p_1x_1^* + p_2x_2^* = m$

(b) $x_2^* = ax_1^*$



Examples of ‘Kinky’ Solutions -- the Perfect Complements Case

(a) $p_1x_1^* + p_2x_2^* = m$; (b) $x_2^* = ax_1^*$.

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(a) $p_1x_1^* + p_2x_2^* = m$; (b) $x_2^* = ax_1^*$.

Substitution from (b) for x_2^* in (a) gives $p_1x_1^* + p_2ax_1^* = m$

Examples of 'Kinky' Solutions -- the Perfect Complements Case

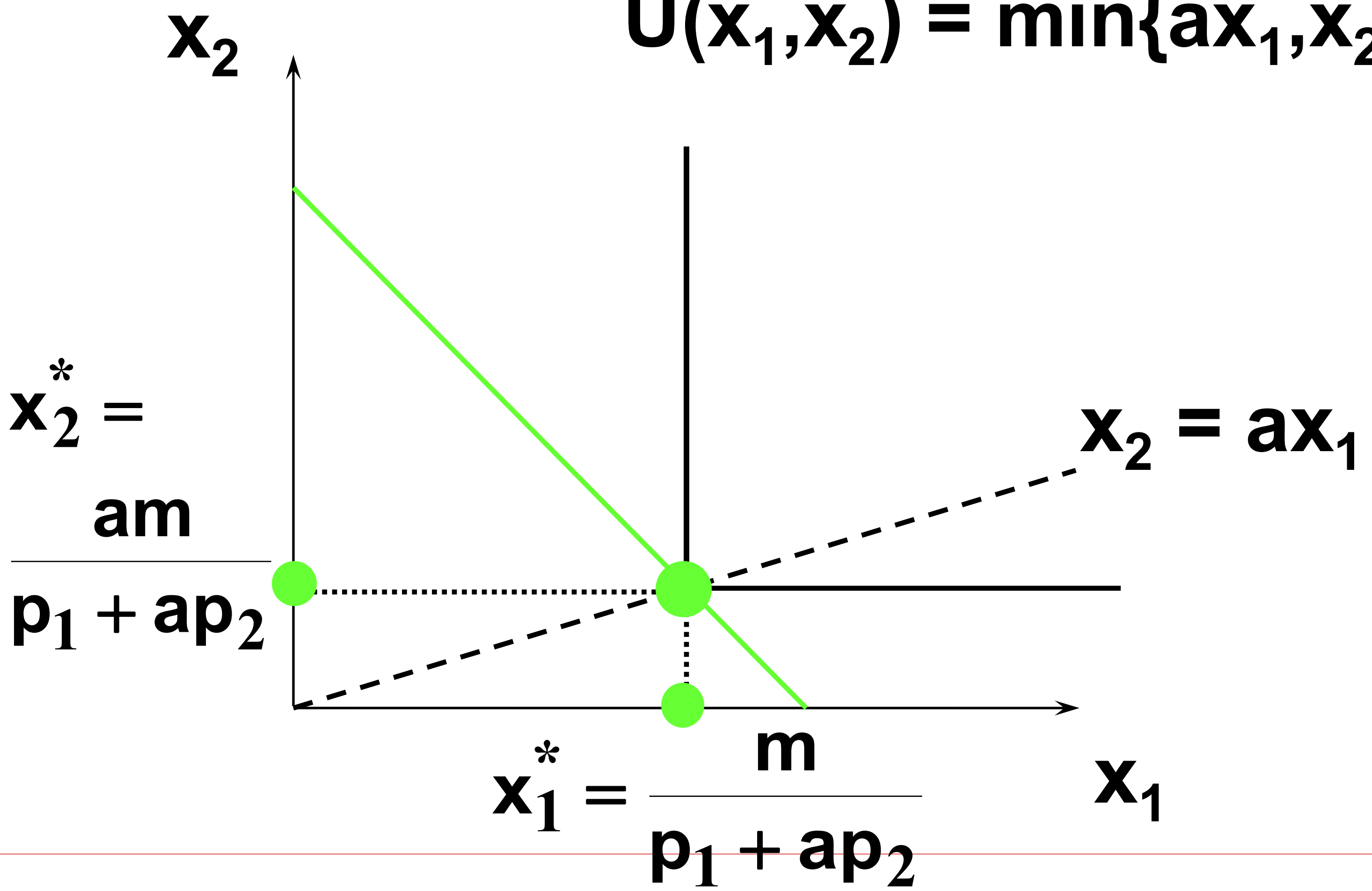
(a) $p_1x_1^* + p_2x_2^* = m$; (b) $x_2^* = ax_1^*$.

Substitution from (b) for x_2^* in (a) gives $p_1x_1^* + p_2ax_1^* = m$

which gives $$x_1^* = \frac{m}{p_1 + ap_2}; \; x_2^* = \frac{am}{p_1 + ap_2}.$$

Examples of ‘Kinky’ Solutions -- the Perfect Complements Case

$U(x_1,x_2) = \min\{ax_1,x_2\}$



The *n*-Good Case

- The individual's objective is to maximize

$$\text{utility} = U(x_1, x_2, \ldots, x_n)$$

subject to the budget constraint

$$I = p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

- Set up the Lagrangian:

$$\mathbf{L} = U(x_1, x_2, \ldots, x_n) + \lambda(I - p_1x_1 - p_2x_2 - \ldots - p_nx_n)$$

The n-Good Case

- First-order conditions for an interior maximum:

$$\partial \mathbf{L}/\partial x_1 = \partial U/\partial x_1 - \lambda p_1 = 0$$

$$\partial \mathbf{L}/\partial x_2 = \partial U/\partial x_2 - \lambda p_2 = 0$$

$$\vdots$$

$$\partial \mathbf{L}/\partial xn = \partial U/\partial xn - \lambda pn = 0$$

$$\partial \mathbf{L}/\partial \lambda = I - p1x1 - p2x2 - \ldots - pnxn = 0$$

Implications of First-Order Conditions

- For any two goods,

$$\frac{\partial U / \partial x_i}{\partial U / \partial x_j} = \frac{p_i}{p_j}$$

- This implies that at the optimal allocation of income

$$MRS\,(x_i \text{ for } x_j) = \frac{p_i}{p_j}$$

Interpreting the Lagrangian Multiplier

$$\lambda = \frac{\partial U / \partial x_1}{p_1} = \frac{\partial U / \partial x_2}{p_2} = \ldots = \frac{\partial U / \partial x_n}{p_n}$$

$$\lambda = \frac{MU_{x_1}}{p_1} = \frac{MU_{x_2}}{p_2} = \ldots = \frac{MU_{x_n}}{p_n}$$

- λ is the marginal utility of an extra dollar of consumption expenditure
  - the marginal utility of income

Cobb-Douglas Demand Functions

- Cobb-Douglas utility function:

$$U(x,y) = x^{\alpha}y^{\beta}, \quad \alpha + \beta = 1$$

- Setting up the Lagrangian:

$$\mathbf{L} = x^{\alpha}y^{\beta} + \lambda(I - p_x x - p_y y)$$

- First-order conditions:

$$\partial\mathbf{L}/\partial x = \alpha x^{\alpha-1}y^{\beta} - \lambda p_x = 0$$

$$\partial\mathbf{L}/\partial y = \beta x^{\alpha}y^{\beta-1} - \lambda p_y = 0$$

$$\partial\mathbf{L}/\partial\lambda = I - p_x x - p_y y = 0$$

Cobb-Douglas Demand Functions

- First-order conditions imply:

$$\alpha y/\beta x = p_x/p_y$$

- Since $\alpha + \beta = 1$:

$$p_y y = (\beta/\alpha)p_x x = [(1- \alpha)/\alpha]p_x x$$

- Substituting into the budget constraint:

$$I = p_x x + [(1- \alpha)/\alpha]p_x x = (1/\alpha)p_x x$$

Cobb-Douglas Demand Functions

- Solving for x yields

$$x^* = \frac{\alpha I}{p_x}$$

Expenditure on X = Px*X
Expenditure share of x =
Px*X/I = alpha

- Solving for y yields

$$y^* = \frac{\beta I}{p_y}$$

Expenditure on Y = Py*Y
Expenditure share of y =
Py*Y/I = beta

- The individual will allocate α percent of his income to good x and β percent of his income to good y

Cobb-Douglas Demand Functions

- The Cobb-Douglas utility function is limited in its ability to explain actual consumption behavior
  - the share of income devoted to particular goods often changes in response to changing economic conditions

- A more general functional form might be more useful in explaining consumption decisions

CES Demand

- Assume that $\delta = 0.5$

$$U(x,y) = x^{0.5} + y^{0.5}$$

- Setting up the Lagrangian:

$$\mathbf{L} = x^{0.5} + y^{0.5} + \lambda(I - p_x x - p_y y)$$

- First-order conditions:

$$\partial\mathbf{L}/\partial x = 0.5x^{-0.5} - \lambda p_x = 0$$

$$\partial\mathbf{L}/\partial y = 0.5y^{-0.5} - \lambda p_y = 0$$

$$\partial\mathbf{L}/\partial\lambda = I - p_x x - p_y y = 0$$

CES Demand

- This means that

$$(y/x)^{0.5} = p_x/p_y$$

- Substituting into the budget constraint, we can solve for the demand functions

$$x^* = \frac{I}{p_x[1+\frac{p_x}{p_y}]} \qquad y^* = \frac{I}{p_y[1+\frac{p_y}{p_x}]}$$

CES Demand

- In these demand functions, the share of income spent on either *x* or *y* is not a constant
  - depends on the ratio of the two prices

- The higher is the relative price of *x* (or *y*), the smaller will be the share of income spent on *x* (or *y*)

Perfect Complements

$$U(x,y) = Min(x,4y)$$

- The person will choose only combinations for which $x = 4y$
- This means that

$$I = p_x x + p_y y = p_x x + p_y(x/4)$$

$$I = (p_x + 0.25p_y)x$$

Perfect Complements

- Hence, the demand functions are

$$x^* = \frac{I}{p_x + 0.25p_y}$$

$$y^* = \frac{I}{4p_x + p_y}$$

Indirect Utility Function

- It is often possible to manipulate first-order conditions to solve for optimal values of $x_1, x_2, \ldots, x_n$
- These optimal values will depend on the prices of all goods and income

$$x^*_1 = x_1(p1, p2, \ldots, pn, I)$$
$$x^*_2 = x_2(p1, p2, \ldots, pn, I)$$
$$\vdots$$
$$x^*_n = x_n(p1, p2, \ldots, pn, I)$$

Indirect Utility Function

- We can use the optimal values of the x's to find the indirect utility function

$$\text{maximum utility} = U(x^*_1, x^*_2, \ldots, x^*_n)$$

- Substituting for each x^*_i, we get

$$\text{maximum utility} = V(p_1, p_2, \ldots, p_n, I)$$

- The optimal level of utility will depend <u>indirectly</u> on prices and income
  - if either prices or income were to change, the maximum possible utility will change

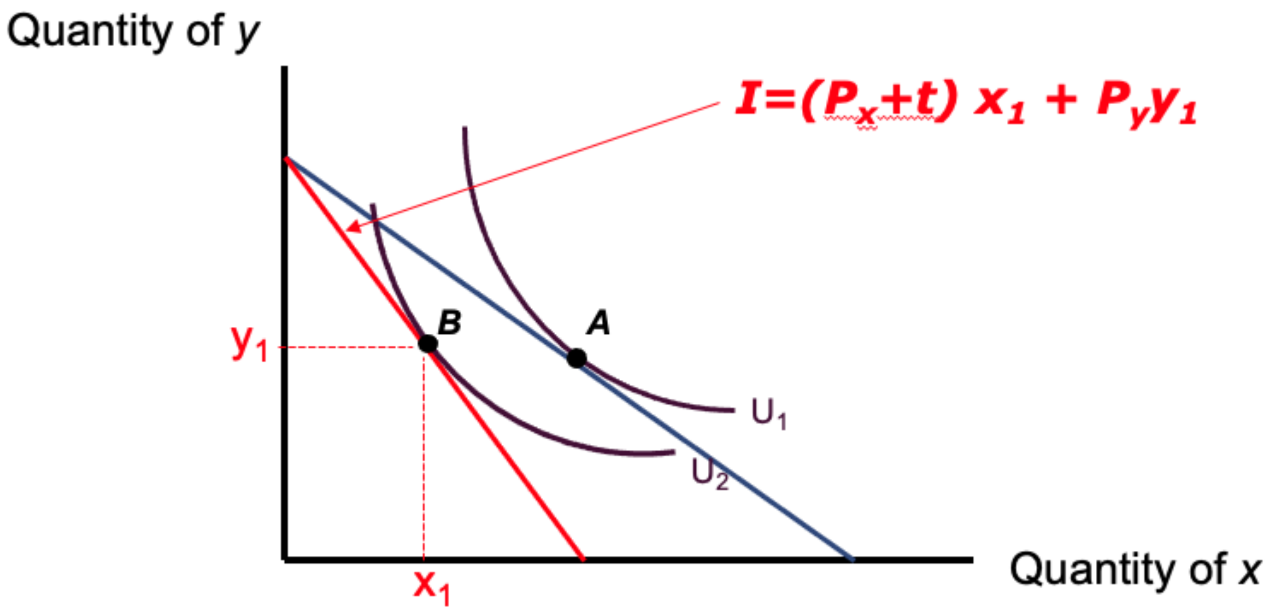
Application

The Lump Sum Principle

- Taxes on an individual's general purchasing power are superior to taxes on a specific good
- From intuition:
  - an income tax allows the individual to decide freely how to allocate remaining income
  - a tax on a specific good will reduce an individual's purchasing power and distort his choices

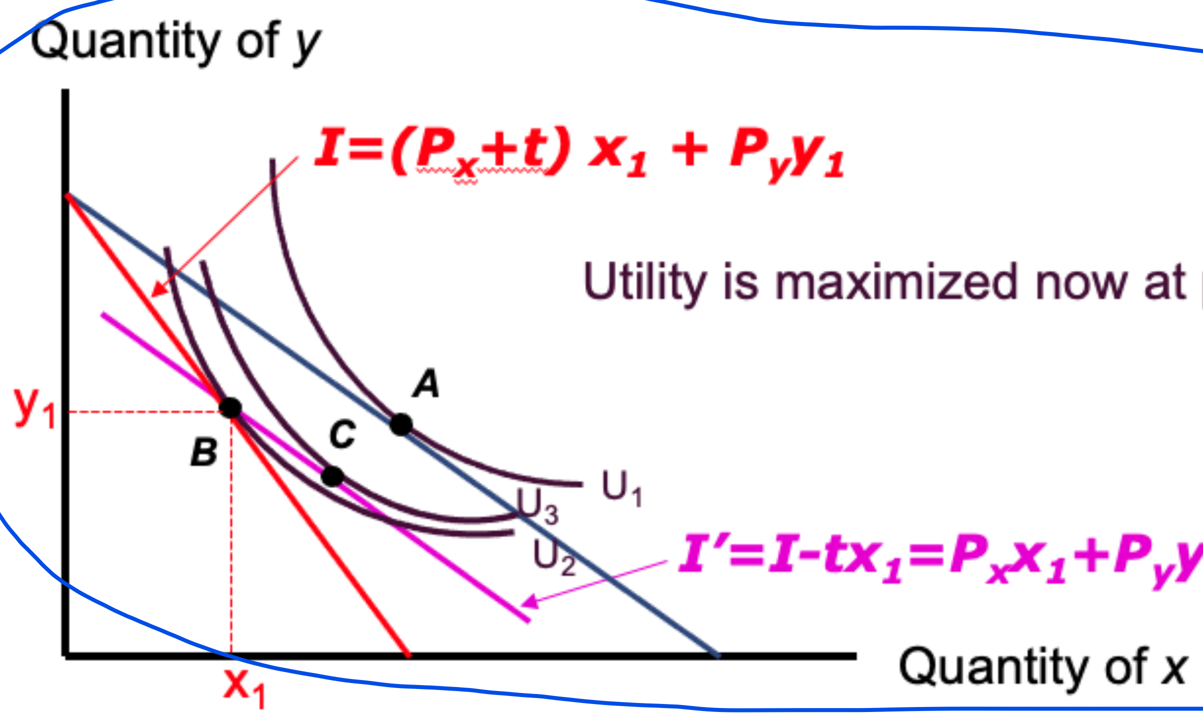
The Lump Sum Principle

- A tax on good *x* would shift the utility-maximizing choice from point *A* to point *B*



The Lump Sum Principle

- An income tax that collected the same amount would shift the budget constraint to *I'*



Indirect Utility and the Lump Sum Principle

- If the utility function is Cobb-Douglas with $\alpha = \beta = 0.5$, we know that

$$x^* = \frac{I}{2p_x} \qquad y^* = \frac{I}{2p_y}$$

So the indirect utility function is

$$V(p_x, p_y, I) = (x^*)^{0.5}(y^*)^{0.5} = \frac{I}{2p_x^{0.5}p_y^{0.5}}$$

$$\text{If } p_x = 1,\ \ p_y = 4,\ \text{I} = 8,\ \text{then } V = \frac{8}{2 \cdot 1 \cdot 2} = 2$$

Indirect Utility and the Lump Sum Principle

$$x^* = \frac{I}{2p_x} \qquad y^* = \frac{I}{2p_y} \qquad p_x = 1,\ \ p_y = 4,\ \ \mathrm{I} = 8$$

$$V(p_x, p_y, I) = (x^*)^{0.5}(y^*)^{0.5} = \frac{I}{2p_x^{0.5}p_y^{0.5}}$$

Then V = 2.

- If a tax of $1 was imposed on good *x*
  - the individual will purchase $x^*=2$
  - indirect utility will fall from 2 to 1.41
- An equal-revenue tax will reduce income to $6
  - indirect utility will fall from 2 to 1.5

Another example -- Perfect complements

- If the utility function is fixed proportions with $U = Min(x,4y)$, we know that

$$x^* = \frac{I}{p_x + 0.25p_y} \qquad y^* = \frac{I}{4p_x + p_y}$$

So the indirect utility function is

$$V(p_x, p_y, I) = Min(x^*, 4y^*) = x^* = \frac{I}{p_x + 0.25p_y}$$

$$= 4y^* = \frac{4I}{4p_x + p_y} = \frac{I}{p_x + 0.25p_y}$$

Indirect Utility and the Lump Sum Principle

$p_x = 1, \;\; p_y = 4, \;\; \mathrm{I} = 8$

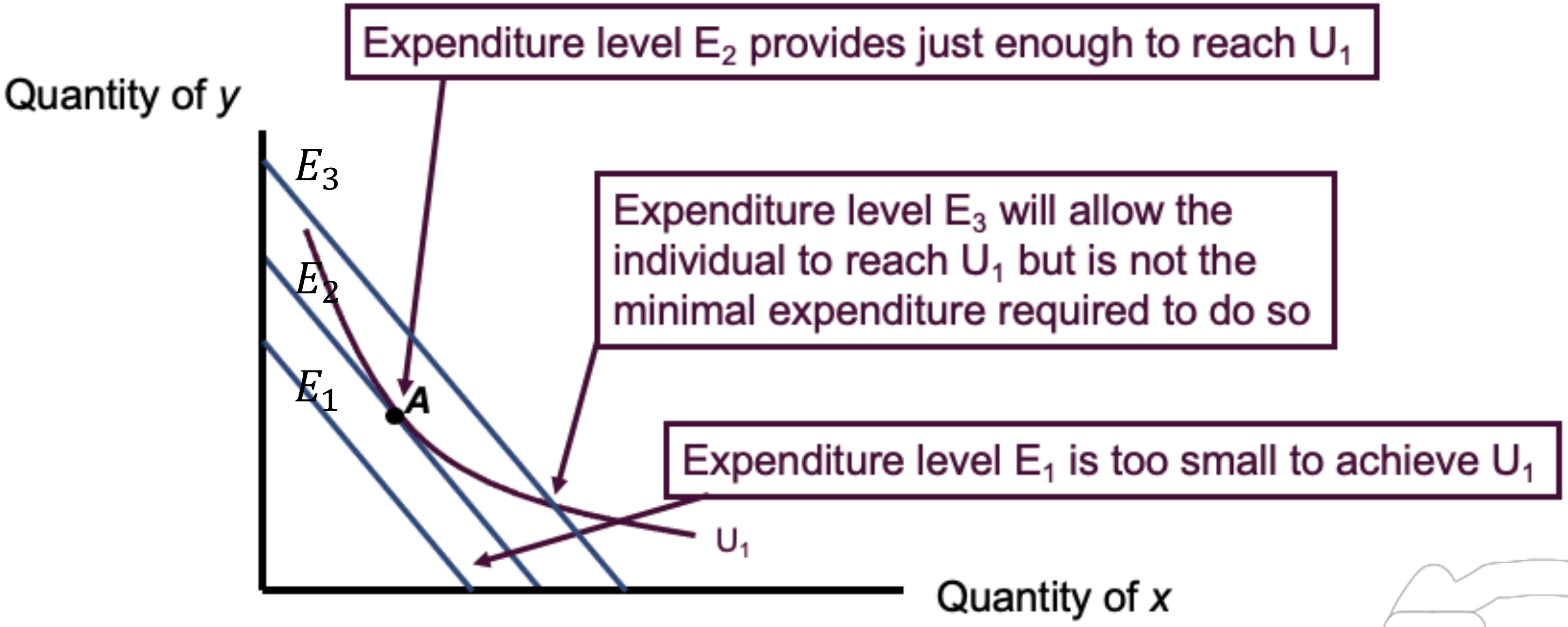
- If a tax of $1 was imposed on good *x*
  - indirect utility will fall from 4 to 8/3
- An equal-revenue tax will reduce income to $16/3
  - indirect utility will fall from 4 to 8/3
- Since preferences are rigid, the tax on *x* does not distort choices

Expenditure Minimization

- Dual minimization problem for utility maximization
  - allocating income in such a way as to achieve a given level of utility with the minimal expenditure
  - this means that the goal and the constraint have been reversed

Expenditure Minimization

Point *A* is the solution to the dual



Expenditure Minimization

- The individual's problem is to choose $x_1, x_2, \ldots, x_n$ to minimize

$$\text{total expenditures} = E = p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

subject to the constraint

$$\text{utility} = U_1 = U(x_1, x_2, \ldots, x_n)$$

- The optimal amounts of $x_1, x_2, \ldots, x_n$ will depend on the prices of the goods p_i's and the required utility level U_1.

Expenditure Function

- The expenditure function shows the minimal expenditures necessary to achieve a given utility level for a particular set of prices

$$\text{minimal expenditures} = E(p_1,p_2,\ldots,p_n,U_1)$$

Expenditure Minimization

- Construct the Lagrangian

$$L = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n + \lambda(U_0 - U(x_1, x_2, \cdots, x_n))$$

- FOCs:

$$\partial L/\partial x_1 = p_1 - \lambda\, \partial U/\partial x_1 = 0$$
$$\partial L/\partial x_2 = p_2 - \lambda\, \partial U/\partial x_2 = 0$$
$$\vdots$$
$$\partial L/\partial x_n = p_n - \lambda\, \partial U/\partial x_n = 0$$
$$\partial L/\partial \lambda = U_0 - U(x_1, x_2, \cdots, x_n) = 0$$

Expenditure Minimization

MC of 1 more unit of x_i

$$\lambda = \frac{p_1}{\partial U / \partial x_1} = \frac{p_2}{\partial U / \partial x_2} = \ldots = \frac{p_n}{\partial U / \partial x_n}$$

MU of 1 more unit of x_i

- In expenditure minimization problem:
  - λ is the marginal expenditure required to reach an extra unit of utility
- In utility maximization problem:
  - λ is the marginal utility of an extra dollar of consumption expenditure

How are indirect utility function and expenditure function related?

- The indirect utility function in the two-good, Cobb-Douglas case is

$$V(p_x, p_y, I) = \frac{I}{2p_x^{0.5}p_y^{0.5}}$$

If we interchange the role of utility and income (expenditure), we will have the expenditure function

$$E(p_x,p_y,U) = 2px^{0.5}py^{0.5}U$$

How are indirect utility function and expenditure function related?

- For the fixed-proportions case, the indirect utility function is

$$V(p_x, p_y, I) = \frac{I}{p_x + 0.25 p_y}$$

If we again switch the role of utility and expenditures, we will have the expenditure function

$$E(p_x,p_y,U) = (p_x + 0.25p_y)U$$

- The expenditure function and the indirect utility function are inversely related