Homework 4

Walker Bagley

November 13, 2023

B. Homework Exercises

(1) Heat Equation IBVP

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < l,\ t > 0 \\ u(0,t) = u(l,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 < x < l \end{cases}$$

(a) $l = \pi$ and $f(x) = 5\sin x - \sqrt{3}\sin 9x$

$$u(x,t) = 5e^{-t}\sin x - \sqrt{3}e^{-81t}\sin 9x$$

(b) $l = 1$ and $f(x) = 7\sin \pi x - \sqrt{3}\sin 9\pi x$

$$u(x,t) = 7e^{-\pi^2 t}\sin \pi x - \sqrt{3}e^{-81\pi^2 t}\sin 9\pi x$$

(2) Wave Equation IBVP

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < l,\ t > 0 \\ u(0,t) = u(l,t) = 0 & t > 0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & 0 < x < l \end{cases}$$

(a) $f(x) = \sin(\frac{n\pi x}{l})$ and $g(x) = 0$

$$\begin{aligned} B_n &= \frac{2}{l}\int_0^l \sin^2\left(\frac{n\pi x}{l}\right)\, dx = \frac{2}{l}\left(\frac{x}{2} - \frac{l}{4\pi n}\sin\left(\frac{2n\pi x}{l}\right)\right\rvert_0^l \\ &= \frac{2}{l}\left(\frac{l}{2}\right) = 1 \\ b_n &= 0 \end{aligned}$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi t}{l}\right)$$

(b) $l = \pi,\ f(x) = -4\sin 3x$ and $g(x) = 2\sin 5x$

$$u(x,t) = -4\sin 3x\cos 3t + \frac{2}{5}\sin 5x \sin 5t$$

(5) Full Fourier Series

$$f(x) = \begin{cases} x^2 & 0 < x < \pi \\ 0 & -\pi < x < 0 \end{cases}$$

1. $-\pi < x \leq 0$

$$
\begin{aligned}
A_0 &= \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot 1dx = 0 \\
A_n &= \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot \cos nx \ dx = 0 \\
B_n &= \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot \sin nx \ dx = 0
\end{aligned}
$$

$$
f(x) = 0 = \frac{1}{2} \cdot 0 + \sum_{n=1}^{\infty} 0 \cdot \sin nx + 0 \cdot \cos nx \ dx = 0 + \sum_{n=1}^{\infty} 0 = 0
$$

2. $0 \leq x < \pi$

$$
\begin{aligned}
A_0 &= \frac{1}{\pi}\int_{0}^{\pi} x^2 dx = \frac{\pi^2}{3} \\
A_n &= \frac{1}{\pi}\int_{0}^{\pi} x^2 \cos nx \ dx = \frac{x^2}{n\pi} \sin nx|_0^\pi - \frac{1}{\pi}\int_0^\pi \frac{\sin nx}{n} 2xdx \\
&= \frac{2x}{n^2\pi} \cos nx|_0^\pi - \frac{2}{n^2\pi}\int_0^\pi \cos nx = (-1)^n \frac{2}{n^2} - \frac{2}{n^2\pi} \sin nx|_0^\pi = (-1)^n \frac{2}{n^2} \\
B_n &= 0
\end{aligned}
$$

$$
f(x) = x^2 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} (-1)^n \frac{2}{n^2} \cos nx
$$

3. $x = -\pi, \pi, 2\pi, 3\pi \ldots$

$$
\begin{gathered}
f(x) = \frac{1}{2}[f(x^-) + f(x^+)] \\
\text{Case 1: } x = 2k\pi \quad k \in \mathbb{Z} \\
f(x) = f(2k\pi) = \frac{1}{2}[f(2k\pi^-) + f(2k\pi^+)] \\
= \frac{1}{2}[0 + 0] = 0 \\
\text{Case 2: } x = (2k+1)\pi \quad k \in \mathbb{Z} \\
f(x) = f((2k+1)\pi) = \frac{1}{2}[f((2k+1)\pi^-) + f((2k+1)\pi^+)] \\
= \frac{1}{2}[\pi^2 + 0] = \frac{\pi^2}{2}
\end{gathered}
$$

(6) Heat Equation IBVP

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = x^2 + 3\sin 8x & 0 < x < \pi \end{cases}$$

Fourier Sine Series of x^2

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_0^\pi x^2 \sin nx\ dx = \frac{2}{\pi}\left[\left(-\frac{x^2}{n}\cos nx\right|_0^\pi + \int_0^\pi \frac{2x}{n}\cos nx\ dx\right] \\ &= \frac{2}{\pi}\left[(-1)^{n+1}\frac{\pi^2}{n} + \left(\frac{2x}{n^2}\sin nx\right|_0^\pi - \int_0^\pi \frac{2}{n^2}\sin nx\ dx\right] = \frac{2}{\pi}\left[(-1)^{n+1}\frac{\pi^2}{n} - \left(\frac{2}{n^3}\cos nx\right|_0^\pi\right] \\ &= (-1)^{n+1}\frac{2\pi}{n} + (-1)^n\frac{4}{\pi n^3} - \frac{4}{\pi n^3} = (-1)^{n+1}\frac{2\pi}{n} + [(-1)^n - 1]\frac{4}{\pi n^3} \end{aligned}$$

$$f(x) = x^2 = \sum_{n=1}^\infty \left[(-1)^{n+1}\frac{2\pi}{n} + [(-1)^n - 1]\frac{4}{\pi n^3}\right]\sin nx$$

$$u(x,t) = \sum_{n=1}^\infty \left[(-1)^{n+1}\frac{2\pi}{n} + [(-1)^n - 1]\frac{4}{\pi n^3}\right] e^{-n^2 t}\sin nx + 3e^{-64t}\sin 8x$$

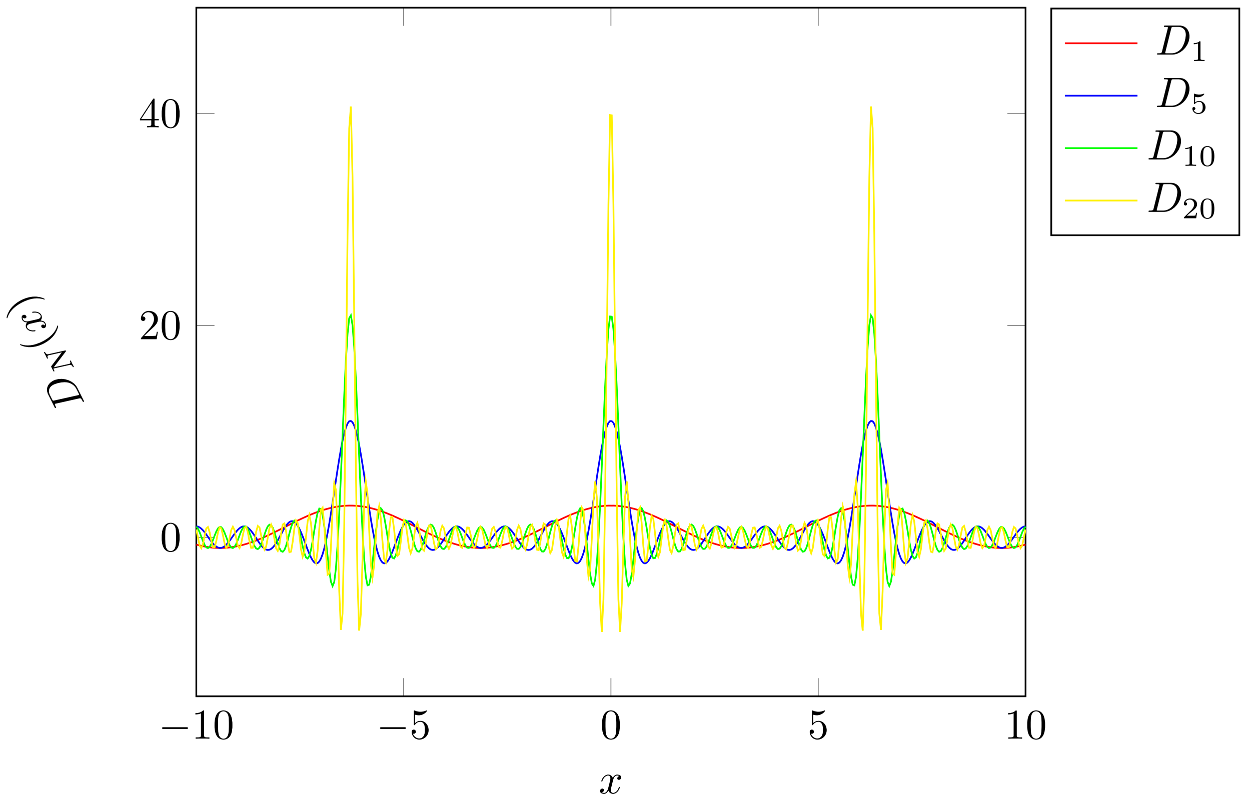
(7) Wave Equation IBVP

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u(0,t) = u(\pi,t) = 0 & t > 0 \\ u(x,0) = \sin 4x - 10\sin 7x,\ u_t(x,0) = x & 0 < x < \pi \end{cases}$$

$$\begin{aligned} B_n &= \frac{2}{\pi}\left[\int_0^\pi \sin 4x \sin nx\ dx - \int_0^\pi 10\sin 7x\sin nx\ dx\right] = 0 \\ b_n &= \frac{2}{\pi}\int_0^\pi x\sin nx\ dx = \frac{2}{\pi}\left[\left(-\frac{x}{n}\cos nx\right|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\ dx\right] \\ &= \frac{2}{\pi}\left[(-1)^{n+1}\frac{\pi}{n} + \frac{1}{n^2}\left(\sin nx\right|_0^\pi\right] = (-1)^{n+1}\frac{2}{n} \end{aligned}$$

$$u(x,t) = \sin 4x\cos 4t - 10\sin 7x\cos 7t + \sum_{n=1}^\infty (-1)^{n+1}\frac{2}{n^2}\sin nx\sin nt$$

(8) Dirichlet Kernel Plots



(9) Heat Equation IBVP

$$\begin{cases} u_t - u_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = x^2 + 3\cos 8x & 0 < x < \pi \end{cases}$$

Recall the FS of x^2 from class

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4(-1)^n \frac{\cos nx}{n^2} + 3\cos 8x$$

$$u(x,t) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4(-1)^n \frac{\cos nx}{n^2} e^{-n^2 t} + 3e^{-64t}\cos 8x$$

(10) Heat Equation IBVP

$$\begin{cases} u_t - u_{xx} = 0 & -\pi < x < \pi,\ t > 0 \\ u(-\pi, t) = u(\pi, t),\ u_x(-\pi, t) = u_x(\pi, t) & t > 0 \\ u(x, 0) = |x| + 3\cos 8x + 5\sin 9x & -\pi < x < \pi \end{cases}$$

Calculate the FS of $|x|$

$$A_0 = \frac{2}{\pi}\int_0^{\pi} x dx = \pi$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\ dx = \frac{2}{\pi}\left[\left(\frac{x}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx\ dx\right]\right.$$

$$= \frac{2}{\pi}\frac{1}{n^2}\left(\cos nx|_0^{\pi} = [(-1)^n - 1]\frac{2}{\pi n^2} = \right.$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\ dx = \frac{2}{\pi}\left[\left(-\frac{x}{n}\cos nx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos nx\ dx\right]\right.$$

$$= \frac{2}{\pi}\left[(-1)^{n+1}\frac{\pi}{n} + \frac{1}{n^2}(\sin nx|_0^{\pi}\right] = (-1)^{n+1}\frac{2}{\pi}$$

$$f(x) = |x| = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{4\cos((2n-1)x)}{\pi(2n-1)^2}$$

$$u(x,t) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{4\cos((2n-1)x)}{\pi(2n-1)^2}e^{-(2n-1)^2 t} + 3e^{-64t}\cos 8x + 5e^{-81t}\sin 9x$$