Final Exam

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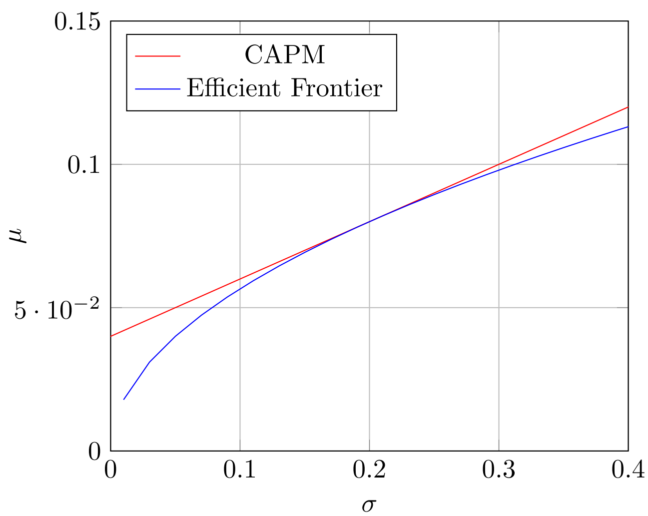
May 10, 2024

2. 10pts

$$
\begin{aligned}
\text{Sharpe Ratio } &= \frac{0.08 - 0.04}{0.2} = \frac{0.04}{0.2} = 0.2 \\
\text{CAPM line: } \mu &= 0.2\sigma + 0.04 \\
0.1 &= 0.2\sigma + 0.04 \\
\sigma &= 0.3 \\
0.3^2 &= k_s^2 \cdot 0.2^2 \\
k &= \begin{cases} k_s = 1.5 \\ k_b = -0.5 \end{cases}
\end{aligned}
$$



3. 10pts

Using the following average returns which were calculated from closing price 5 years ago to today:

1. Apple: 28.21%
2. GE: 27.11%
3. Coca Cola: 22.01%

And the following standard deviations for each company's returns:

1. Apple: 31.5%
2. GE: 40.9%
3. Coca Cola: 20.9%

We have the following correlation matrix:

$$\begin{bmatrix} 1 & 0.31 & 0.16 \\ 0.31 & 1 & 0.18 \\ 0.16 & 0.18 & 1 \end{bmatrix}$$

Which we can generate the covariance matrix from by multiplying by the standard deviations for each company:

$$\begin{bmatrix} 0.0922 & 0.0399 & 0.0105 \\ 0.0399 & 0.1673 & 0.0154 \\ 0.0105 & 0.0154 & 0.0437 \end{bmatrix}$$

Then the notebook gives us the following portfolio breakdown:

1. Apple: -158.23%
2. GE: -43.14%
3. Coca Cola: 301.36%

And a standard deviation of 75.71%.

4. 5pts

$$\begin{aligned} u(x,t) &= e^{ikx-k^2t} \\ u_t &= -k^2e^{ikx-k^2t} \\ u_x &= ike^{ikx-k^2t} \\ u_{xx} &= i^2k^2e^{ikx-k^2t} = -k^2e^{ikx-k^2t} \\ u_t - u_{xx} &= -k^2e^{ikx-k^2t} - (-k^2e^{ikx-k^2t}) = k^2e^{ikx-k^2t} - k^2e^{ikx-k^2t} = 0 \end{aligned}$$

$$\begin{aligned} u(x,t) &= e^{ikx+k^2t} \\ u_t &= k^2e^{ikx+k^2t} \\ u_x &= ike^{ikx+k^2t} \\ u_{xx} &= i^2k^2e^{ikx+k^2t} = -k^2e^{ikx+k^2t} \\ u_t + u_{xx} &= k^2e^{ikx-k^2t} - k^2e^{ikx-k^2t} = 0 \end{aligned}$$

5. 5pts

a.

$$\begin{aligned} V(t,s) &= As + Be^{rt} \\ \frac{\partial V}{\partial t} &= Bre^{rt} \\ \frac{\partial V}{\partial s} &= A \\ \frac{\partial^2 V}{\partial s^2} &= 0 \\ \frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} - rV &= Bre^{rt} + Ars - r(As + Be^{rt}) \\ &= Bre^{rt} + Ars - Ars - Bre^{rt} = 0 \end{aligned}$$

b.

$$\begin{aligned}
V(t,s) &= Be^{ct}s^2 \\
\frac{\partial V}{\partial t} &= Bce^{ct}s^2 \\
\frac{\partial V}{\partial s} &= 2Be^{ct}s \\
\frac{\partial^2 V}{\partial s^2} &= 2Be^{ct} \\
\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 V}{\partial s^2} - rV &= Bce^{ct}s^2 + rs(2Be^{ct}s) + \frac{1}{2}\sigma^2 s^2(2Be^{ct}) - r(Be^{ct}s^2) \\
&= Be^{ct}(cs^2 + 2rs^2 + \sigma^2 s^2 - rs^2) \\
cs^2 + rs^2 + \sigma^2 s^2 &= 0 \\
s^2(c + r + \sigma^2) &= 0 \\
c &= -(r + \sigma^2)
\end{aligned}$$

7. 10pts

$$\begin{aligned}
V(s,t) &= sN(d_+(s,T-t)) - Ke^{-r(T-t)}N(d_-(s,T-t)) \\
d_\pm(s,\tau) &= \frac{1}{\sigma\sqrt{\tau}}\left[\ln\left(\frac{s}{K}\right) + (r \pm \frac{1}{2}\sigma^2)\tau\right] \\
N(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}y^2}dy
\end{aligned}$$

$$\begin{aligned}
\frac{\partial d_\pm}{\partial s} &= \frac{1}{s\sigma\sqrt{T-t}} \\
\frac{\partial d_\pm}{\partial t} &= \frac{(r \pm \frac{1}{2}\sigma^2)(t-T) + \ln\left(\frac{s}{K}\right)}{2\sigma(T-t)\sqrt{T-t}}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V}{\partial t} &= sN'(d_+)d_{+t} - Kre^{-r(T-t)}N(d_-) - Ke^{-r(T-t)}N'(d_-)d_{-t} \\
\frac{\partial V}{\partial s} &= N(d_+) + sN'(d_+)d_{+s} - Ke^{-r(T-t)}N'(d_-)d_{-s} \\
&= N(d_+) + \frac{1}{\sigma\sqrt{T-t}}N'(d_+) - \frac{1}{s\sigma\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-) \\
rs\frac{\partial V}{\partial s} &= rsN(d_+) + \frac{rs}{\sigma\sqrt{T-t}}N'(d_+) - \frac{r}{\sigma\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-) \\
\frac{\partial^2 V}{\partial s^2} &= N'(d_+)d_{+s} + \frac{1}{\sigma\sqrt{T-t}}N''(d_+)d_{+s} + \frac{1}{s^2\sigma\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-) - \frac{1}{s\sigma\sqrt{T-t}}Ke^{-r(T-t)}N''(d_-)d_{-s} \\
&= \frac{1}{s\sigma\sqrt{T-t}}N'(d_+) + \frac{1}{s\sigma^2(T-t)}N''(d_+) + \frac{1}{s^2\sigma\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-) - \frac{1}{s^2\sigma^2(T-t)}Ke^{-r(T-t)}N''(d_-) \\
\frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial s^2} &= \frac{s\sigma}{2\sqrt{T-t}}N'(d_+) + \frac{s}{2(T-t)}N''(d_+) + \frac{\sigma}{2\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-) - \frac{1}{2(T-t)}Ke^{-r(T-t)}N''(d_-) \\
-rV &= -rsN(d_+) + Kre^{-r(T-t)}N(d_-)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{1}{2}\sigma^2 s^2\frac{\partial^2 V}{\partial s^2} - rV =& sN'(d_+)d_{+t} - Ke^{-r(T-t)}N'(d_-)d_{-t} + \frac{rs}{\sigma\sqrt{T-t}}N'(d_+) - \frac{r}{\sigma\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-) \\
&+ \frac{s\sigma}{2\sqrt{T-t}}N'(d_+) + \frac{s}{2(T-t)}N''(d_+)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma}{2\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-)-\frac{1}{2(T-t)}Ke^{-r(T-t)}N''(d_-)\\
=&sN'(d_+)d_{+t}-Ke^{-r(T-t)}N'(d_-)d_{-t}+\frac{2rs+\sigma^2 s}{2\sigma\sqrt{T-t}}N'(d_+)\\
&+\frac{s}{2(T-t)}N''(d_+)+\frac{\sigma^2-2r}{2\sigma\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-)-\frac{1}{2(T-t)}Ke^{-r(T-t)}N''(d_-)\\
=&sN'(d_+)d_{+t}-Ke^{-r(T-t)}N'(d_-)d_{-t}+\frac{2(2r+\sigma^2)(T-t)}{4\sigma(T-t)\sqrt{T-t}}sN'(d_+)\\
&+\frac{s}{2(T-t)}N''(d_+)+\frac{\sigma^2-2r}{2\sigma\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-)-\frac{1}{2(T-t)}Ke^{-r(T-t)}N''(d_-)\\
=&-Ke^{-r(T-t)}N'(d_-)d_{-t}+\frac{(2r+\sigma^2)(T-t)+2\ln\left(\frac{s}{K}\right)}{4\sigma(T-t)\sqrt{T-t}}sN'(d_+)\\
&+\frac{s}{2(T-t)}N''(d_+)+\frac{2(\sigma^2-2r)(T-t)}{4\sigma(T-t)\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-)-\frac{1}{2(T-t)}Ke^{-r(T-t)}N''(d_-)\\
=&\frac{(2r+\sigma^2)(T-t)+2\ln\left(\frac{s}{K}\right)}{4\sigma(T-t)\sqrt{T-t}}sN'(d_+)+\frac{s}{2(T-t)}N''(d_+)\\
&+\frac{(\sigma^2-2r)(T-t)+2\ln\left(\frac{s}{K}\right)}{4\sigma(T-t)\sqrt{T-t}}Ke^{-r(T-t)}N'(d_-)-\frac{1}{2(T-t)}Ke^{-r(T-t)}N''(d_-)\\
=&\frac{\sigma^2(T-t)+2\ln\left(\frac{s}{K}\right)}{4\sigma(T-t)\sqrt{T-t}}(sN'(d_+)+Ke^{-r(T-t)}N'(d_-))+\frac{s}{2(T-t)}N''(d_+)\\
&+\frac{r}{2\sigma\sqrt{T-t}}(sN'(d_+)-Ke^{-r(T-t)}N'(d_-))-\frac{1}{2(T-t)}Ke^{-r(T-t)}N''(d_-)\\
=&\left(\frac{\sigma}{2\sqrt{T-t}}+\frac{\ln\left(\frac{s}{K}\right)}{2\sigma(T-t)\sqrt{T-t}}\right)(sN'(d_+)+Ke^{-r(T-t)}N'(d_-))\\
&+\frac{r}{2\sigma\sqrt{T-t}}(sN'(d_+)-Ke^{-r(T-t)}N'(d_-))\\
&+\frac{1}{2(T-t)}(sN''(d_+)-Ke^{-r(T-t)}N''(d_-))\\
=&\left(\frac{\sigma}{2\sqrt{T-t}}\right)(sN'(d_+)+Ke^{-r(T-t)}N'(d_-))\\
&+\frac{1}{2(T-t)}(sN''(d_+)-Ke^{-r(T-t)}N''(d_-))\\
=&0
\end{aligned}
$$

8. 10pts

a.

$$
\begin{aligned}
d_1 &= \frac{\ln\left(\frac{100}{160}\right)+(0.07+\frac{1}{2}0.2^2)\cdot 5}{0.2\cdot\sqrt{5}}=\frac{-0.47+0.45}{0.4472}=-0.04473\\
d_2 &= -0.04473-0.2\cdot\sqrt{5}=-0.04473-0.4472=-0.4919\\
V_0^{ECO} &= 100N(-0.04473)-160e^{-0.07\cdot 5}N(-0.4919)\\
&= 100(0.48216147)-160e^{-0.35}(0.31137979)\\
&= 48.216147-112.75(0.31137979)=48.216147-35.1081\\
&= \$13.11
\end{aligned}
$$

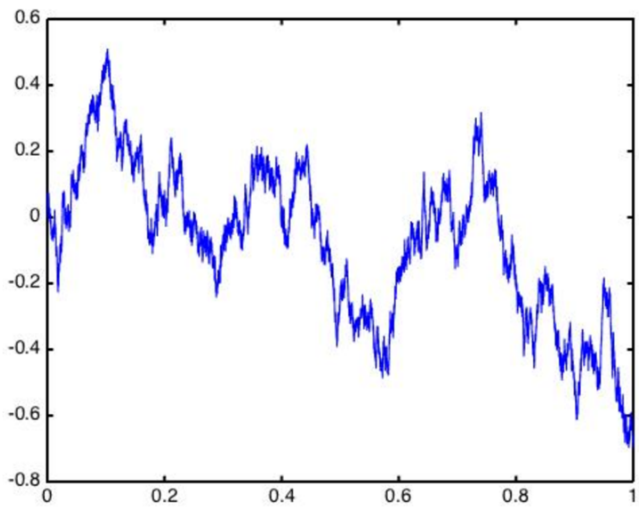
b.

$$
d_1=\frac{\ln\left(\frac{160e}{160}\right)+(0.07+\frac{1}{2}0.2^2)\cdot 4}{0.2\cdot\sqrt{4}}=\frac{1+0.36}{0.4}=3.4
$$

$$
\begin{aligned}
d_2 &= 3.4 - 0.2 \cdot \sqrt{4} = 3 \\
V_0^{ECO} &= 160eN(3.4) - 160e^{-0.07 \cdot 4}N(3) \\
&= 160e(0.99966307) - 160e^{-0.28}(0.9986501) \\
&= 434.77855 - 120.9254(0.9986501) = 434.77855 - 120.76216 \\
&= \$314.02
\end{aligned}
$$

9. 10pts

a.



1. $W(0) = 0$
2. Each path $\omega \in \Omega$ is continuous on the interval $[0, \infty)$ in $\mathbb{R}$
3. For $0 \le s < t$, the difference $W_t - W_s$ is normally distributed with mean 0 and variance $t - s$
4. It has independent increments, meaning $W_1 - W_0, W_2 - W_1 \ldots W_n - W_{n-1}$ are all independent random variables

b.

$$
\begin{aligned}
\mathbb{E}(W_s W_t) &= Cov(W_s, W_t) - \mathbb{E}(W_s)\mathbb{E}(W_t) = Cov(W_s - W_0, W_t - W_s + W_s - W_0) - 0 \\
&= Cov(W_s - W_0, W_s - W_0) + Cov(W_s - W_0, W_t - W_s) = s + 0 = s
\end{aligned}
$$

c.

$$
\begin{aligned}
f(t) &= 3W_2(\omega)\mathbf{1}_{[2,3)}(t) + 8W_3(\omega)\mathbf{1}_{[3,4)}(t) \\
\int_0^4 f(t)dW_t &= 3W_2(W_3 - W_2) + 8W_3(W_4 - W_3) \\
\mathbb{E}(\int_0^4 f(t)dW_t) &= 3\mathbb{E}(W_2)\mathbb{E}(W_3 - W_2) + 8\mathbb{E}(W_3)\mathbb{E}(W_4 - W_3) = 0
\end{aligned}
$$

12. 10pts

$$\begin{aligned}
g(t,x) &= \frac{1}{25}x^{25} \\
Y_t &= \frac{1}{25}W_t^{25} \\
\frac{\partial g}{\partial x} &= x^{24} \\
\frac{\partial^2 g}{\partial x^2} &= 24x^{23} \\
dY_t &= W_t^{24}dW_t + 12W_t^{23}dt \\
\frac{1}{25}W_t^{25} &= \frac{1}{25}W_0^{25} + \int_0^T W_t^{24}dW_t + 12\int_0^T W_s^{23}ds \\
\int_0^T W_t^{24}dW_t &= \frac{1}{25}W_t^{25} - 12\int_0^T W_s^{23}ds
\end{aligned}$$

14. 10pts

$$\begin{aligned}
\ln\left(\frac{S_t}{100}\right) &= \int_0^t 0.05 + 0.02\cos s - \frac{1}{2}0.3^2 ds + 0.3W_t \\
&= 0.02\sin t + (0.05 - \frac{1}{2}0.3^2)t + 0.3W_t \\
&= 0.02\sin t + 0.005t + 0.3W_t \\
S_t &= 100e^{0.02\sin t + 0.005t + 0.3W_t}
\end{aligned}$$

15. 10pts

$$\begin{aligned}
V_t &= 500000e^{(0.1-\frac{1}{2}0.2^2)t+0.2W_t} = 500000e^{0.08t+0.2W_t} \\
V_{10} &= 500000e^{0.8+0.2W_{10}} \\
P(0.8 + 0.2W_{10} > 1) &= P(W_{10} > 1) \\
&= 37.59\%
\end{aligned}$$

16. 5pts

a.

$$\begin{aligned}
\frac{dA}{dt} &= 0.08 \cdot A(t) + S(t) \\
A(0) &= A_0 = 0
\end{aligned}$$

b.

$$\begin{aligned}
\frac{dA}{dt} - 0.08 \cdot A(t) &= S \\
A(t) &= e^{0.08t}V(t) \\
\frac{dA}{dt} &= 0.08e^{08t}V(t) + e^{0.08t}V'(t) \\
0.08e^{08t}V(t) + e^{0.08t}V'(t) - 0.08e^{0.08t}V(t) &= S \\
e^{0.08t}V'(t) &= S \\
V'(t) &= e^{-0.08t}S
\end{aligned}$$

$$
\begin{aligned}
V(t) &= A_0 + \int_0^t Se^{-0.08\tau} d\tau \\
A(t) &= A_0 e^{0.08t} + \int_0^t Se^{0.08(t-\tau)} d\tau \\
A(t) &= \int_0^t Se^{0.08(t-\tau)} d\tau
\end{aligned}
$$

c.

$$
\begin{aligned}
3000000 &= \int_0^{35} Se^{0.08(35-\tau)} d\tau = \int_0^{35} Se^{2.8-0.08\tau} d\tau \\
&= \left(-\frac{S}{0.08} e^{2.8-0.08\tau} \right|_0^{35} \\
&= -\frac{S}{0.08}(e^{2.8-0.08\cdot 35} - e^{2.8}) = -\frac{S}{0.08}(1-e^{2.8}) \\
&= \frac{S}{0.08}(e^{2.8}-1) \\
S &= \frac{3000000 \cdot 0.08}{e^{2.8}-1} = \frac{240000}{15.4446} = \$15,539.36
\end{aligned}
$$

19. 5pts

$\Delta t = \frac{1}{4}$, $\Delta x = \frac{1}{2}$