# Homework 3

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## Section 2.2

### 2.2.5

Let $\{I_n\}$ be a nested family of finite closed intervals ($I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n$). Prove that there is a point $p$ contained in all the intervals.

*Proof.* Let the interval $I_1 = [a_1, b_1]$. Then we consider the interval $I_2 = [a_2, b_2]$. We know $a_2 \geq a_1$ and $b_2 \leq b_1$, so we can divide all points $p \in I_1$ into two subsets. If we consider the set $IN_1 = p | p \in I_1$, then we can formulate $OUT_2 = \{p | p \in IN_1 \wedge p \notin I_2\}$ and $IN_2 = \{p | p \in IN_1 \wedge p \in I_2\}$. We can do the same thing with $I_3 = [a_3, b_3]$ and so on and so forth, until we have that $IN_n = \{p | p \in IN_1 \wedge p \in IN_2 \wedge \ldots \wedge p \in IN_{n-1}\}$. Thus, we have some $p \in IN_n$ that is contained in all the intervals. $\square$

### 2.2.6

Let $a, b \in \mathbb{R}_+$ and define $a_1 = \frac{a+b}{2}$ and $b_1 = \sqrt{ab}$. Then define $a_{n+1} = \frac{a_n + b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$. Prove that $a_n$ and $b_n$ converge to the same limit.

*Proof.* WLOG we can assume $a > b$ since $a_1, b_1$ do not change depending on whether $a$ or $b$ is larger. Let's first show that $a_n \geq b_n$ by induction:

Base case: $n = 1$. $a_1 = \frac{a+b}{2}$ and $b_1 = \sqrt{ab}$. Then

$$\begin{aligned} a_1 \geq b_1 &\Leftrightarrow a_1 = \frac{a+b}{2} \geq \sqrt{ab} \\ &\Leftrightarrow a + b \geq 2\sqrt{ab} \\ &\Leftrightarrow a - 2\sqrt{ab} + b \geq 0 \\ &\Leftrightarrow (\sqrt{a} - \sqrt{b})^2 \geq 0 \\ &\Leftrightarrow \sqrt{a} - \sqrt{b} \geq 0 \end{aligned}$$

Since $a > b$, then we know that $\sqrt{a} - \sqrt{b} \geq 0$ and thus $a_1 \geq b_1$.
Inductive step: $n = k + 1$. Assume $a_k \geq b_k$. We have that $a_{k+1} = \frac{a_k + b_k}{2}$ and $b_{k+1} = \sqrt{a_k b_k}$. Following the same algebra as the base case and using our inductive hypothesis, we can easily see that $a_{k+1} \geq b_{k+1}$.

From here we can show $a_n$ is decreasing and $b_n$ is increasing:

Since $a_{n+1} = \frac{a_n + b_n}{2}$ and $a_n \geq b_n$, $a_{n+1} \leq \frac{a_n + a_n}{2} = a_n$. So $a_n$ is decreasing.
Since $b_{n+1} = \sqrt{a_n b_n}$ and $a_n \geq b_n$, $b_{n+1} \geq \sqrt{b_n b_n} = b_n$. So $b_n$ is increasing.

Using these facts, we know $b < b_1 \leq b_2 \leq \ldots b_k \leq a_k \ldots \leq a_2 \leq a_1 < a$. By MCT, $a_n$ and $b_n$ must both converge to the same limit. $\square$

### 2.2.7

Let $a, b \in \mathbb{Z}_+$ and define $a_1 = \frac{a+b}{2}$ and $b_1 = \frac{2ab}{a+b}$. Then define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \frac{2a_nb_n}{a_n+b_n}$. Prove that $a_n$ and $b_n$ converge to $\sqrt{ab}$.

*Proof.* Very similar to above. WLOG we can assume $a > b$ since $a_1, b_1$ do not change depending on whether $a$ or $b$ is larger. Let's first show that $a_n \geq b_n$ by induction:

Base case: $n = 1$. $a_1 = \frac{a+b}{2}$ and $b_1 = \frac{2ab}{a+b}$. Then

$$\begin{aligned} a_1 \geq b_1 &\Leftrightarrow \frac{a+b}{2} \geq \frac{2ab}{a+b} \\ &\Leftrightarrow (a+b)^2 \geq 4ab \\ &\Leftrightarrow a^2 + 2ab + b^2 - 4ab \geq 0 \\ &\Leftrightarrow a^2 - 2ab + b^2 \geq 0 \\ &\Leftrightarrow (a-b)^2 \geq 0 \\ &\Leftrightarrow a - b \geq 0 \end{aligned}$$

Since $a > b$, then we know that $a - b \geq 0$ and thus $a_1 \geq b_1$.
Inductive step: $n = k + 1$. Assume $a_k \geq b_k$. We have that $a_{k+1} = \frac{a_k+b_k}{2}$ and $b_{k+1} = \frac{2a_kb_k}{a_k+b_k}$. Following the same algebra as the base case and using our inductive hypothesis, we can easily see that $a_{k+1} \geq b_{k+1}$.

From here we can show $a_n$ is decreasing and $b_n$ is increasing:

Since $a_{n+1} = \frac{a_n+b_n}{2}$ and $a_n \geq b_n$, $a_{n+1} \leq \frac{a_n+a_n}{2} = a_n$. So $a_n$ is decreasing.
Since $b_{n+1} = \frac{2a_nb_n}{a_n+b_n}$ and $a_n \geq b_n$, $b_{n+1} \geq \frac{2a_nb_n}{a_n+a_n} = b_n$. So $b_n$ is increasing.

Using these facts, we know $b < b_1 \leq b_2 \leq \ldots b_k \leq a_k \ldots \leq a_2 \leq a_1 < a$. By MCT, $a_n$ and $b_n$ must both converge to the same limit. So, let's take $\epsilon > 0$ and consider

$$\begin{aligned} |a_n - \sqrt{ab}| < \epsilon &\Leftrightarrow \frac{a_{n-1} + b_{n-1}}{2} - \sqrt{ab} < \epsilon \\ b_n \leq a_n &\Leftrightarrow \frac{a_{n-1} + b_{n-1}}{2} - \sqrt{ab} \leq \frac{a_{n-1} + a_{n-1}}{2} - \sqrt{ab} \\ &\Leftrightarrow \frac{a_{n-1} + a_{n-1}}{2} - \sqrt{ab} = a_{n-1} - \sqrt{ab} \\ a_{n-1} - \sqrt{ab} < \epsilon &\Leftrightarrow a_{n-1} < \epsilon + \sqrt{ab} \end{aligned}$$

So if we choose $N$ s.t. $a_{N-1} < \epsilon + \sqrt{ab}$ then we have that $|a_n - \sqrt{ab}| < \epsilon$ for $n > N$. □

## Section 2.4

### 2.4.1

Let $S$ be a set of real numbers that is bounded above and $\{a_n\}$ be a sequence of real numbers such that each $a_n$ is an upper bound of $S$. Assume $a_n \to a$. Prove that $a$ is an upper bound of $S$.

*Proof.* Since $a_n \to a$, take $\epsilon > 0$ s.t. $a_n - \epsilon < a < a_n + \epsilon$ for $n > N$. Also, $a_n$ is an upper bound, so $\forall s \in S$, $a_n \geq s$. Then we have that $a - \epsilon \leq a_n - \epsilon < a$, so as $\epsilon \to 0$, we have $\forall s \in S$ that $s < a$, so $a$ is an upper bound. □

### 2.4.2

Find the supremum and infimum of each of the following sets and discuss whether they are elements of the sets.

**(a)**

$S = \{1/n : n \text{ is a positive integer}\}$

$\sup S = \frac{1}{1} = 1$
$\inf S = \frac{1}{\infty} = 0$

The supremum is in $S$ since $1 = \frac{1}{1} = \frac{1}{n}$ when $n = 1$.
The infimum is not in $S$ since $\frac{1}{n} \neq 0$ for any $n$.

**(b)**

$S = \{2^n : n \text{ is an integer}\}$
$\sup S = 2^{\infty} = \infty$
$\inf S = 2^{-\infty} = 0$

The supremum is not in $S$ since $\infty \neq 2^n$ for any $n$.
The infimum is not in $S$ since $0 \neq 2^n$ for any $n$.

**(c)**

$S = \{x^2 : -1 < x < 1\}$
$\sup S = 1^2 = 1$
$\inf S = 0^2 = 0$

The supremum is not in $S$ since $1 = (\pm 1)^2$ but $1 \notin S$.
The infimum is in $S$ since $0 = x^2$ when $x = 0$.

## 2.4.3

Let $S$ be a set that is bounded above and $a$ be an upper bound of $S$. Prove that the following statements are equivalent.

(a) Any upper bound $a'$ of $S$ satisfies $a' \geq a$.

(b) For every $\epsilon > 0$, there exists an $x \in S$ such that $x > a - \epsilon$.

(c) There exists a sequence $\{x_n\} \subset S$ such that $x_n \to a$.

*Proof.* First by proving (a) $\Leftrightarrow$ (b):
$\Rightarrow$: Suppose (a) holds. Then take some $\epsilon > 0$ and suppose (b) is false. Then there is no $x \in S$ so that $x > a - \epsilon$, so $\forall x \in S$, $x \leq a - \epsilon$, making it an upper bound. Then $a' = a - \epsilon < a$, contradicting (a).
$\Leftarrow$: Suppose (b) holds. Then suppose (a) is false so that $\exists$ upper bound $a'$ s.t. $a' < a$. Letting $a' = a - \epsilon$ with $\epsilon > 0$ and since $a'$ is an upper bound, $\forall x \in S$, $x \leq a' = a - \epsilon$. So there is no $x \in S$ where $x > a - \epsilon$, contradicting (b).

Second by proving (b) $\Leftrightarrow$ (c):
$\Rightarrow$: Suppose (b) holds. Then suppose (c) is false so that there is no sequence of $S$ which converges to $a$. Then there must be some $\epsilon > 0$ s.t. there is no $x \in S$ where $x > a - \epsilon$, contradicting (b).
$\Leftarrow$: Suppose (c) holds. Then take some $\epsilon > 0$ and suppose (b) is false. Then there is no $x \in S$ so that $x > a - \epsilon$, so $\forall x \in S$, $x \leq a - \epsilon$, making it an upper bound. Then it is impossible for a sequence of $S$ to converge to $a$, contradicting (c).

By transitivity, (a) $\Leftrightarrow$ (c) and (a), (b), (c) are equivalent. $\square$

## 2.4.7

Let $S$ be a set of real numbers and $\alpha$ be a real number. Define $\alpha S = \{\alpha x : x \in S\}$.

### (a)

If $\alpha$ is positive, prove $\sup(\alpha S) = \alpha \sup S$ and $\inf(\alpha S) = \alpha \inf S$.

*Proof.* Suppose $\alpha > 0$, then we know that $\forall x \in S$, $\sup S \geq x$. Then it follows that $\alpha \sup S \geq \alpha x$ for all $x \in S$. So this matches the definition of supremum and thus $\sup(\alpha S) = \alpha \sup S$. We also know that $\forall x \in S$ $\inf S \leq x$. Then it follows that $\alpha \inf S \leq \alpha x$ for all $x \in S$. So this matches the definition of infimum and thus $\inf(\alpha S) = \alpha \inf S$. □

### (b)

If $\alpha$ is negative, prove $\sup(\alpha S) = \alpha \inf S$ and $\inf(\alpha S) = \alpha \sup S$.

*Proof.* Suppose $\alpha < 0$, then we know that $\forall x \in S$, $\inf S \leq x$. Then it follows that $\alpha \inf S \geq \alpha x$ for all $x \in S$. So this matches the definition of supremum and thus $\sup(\alpha S) = \alpha \inf S$. We also know that $\forall x \in S$ $\sup S \geq x$. Then it follows that $\alpha \sup S \leq \alpha x$ for all $x \in S$. So this matches the definition of infimum and thus $\inf(\alpha S) = \alpha \sup S$. □

## 2.4.9

Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences and define sets $A$, $B$, and $C$ by $A = \{a_n\}$, $B = \{b_n\}$, and $C = \{a_n + b_n\}$. Prove $\inf A + \inf B \leq \inf C$, $\sup C \leq \sup A + \sup B$. Construct an example to show that strict inequalities may hold.

*Proof.* For infimum:

$$\begin{aligned} \inf A &\leq a_n \\ \inf A + b_n &\leq a_n + b_n \\ \inf(\inf A + b_n) &\leq \inf(a_n + b_n) \\ \inf A + \inf B &\leq \inf C \end{aligned}$$

For supremum:

$$\begin{aligned} a_n &\leq \sup A \\ a_n + b_n &\leq \sup A + b_n \\ \sup(a_n + b_n) &\leq \sup(\sup A + b_n) \\ \sup C &\leq \sup A + \sup B \end{aligned}$$

□

Example: $a_n = n$ and $b_n = \frac{1}{n}$. We can clearly see that $\inf A = 1$, $\inf B = 0$ and $\inf C = 2$ so $\inf A + \inf B < \inf C$. A similar example can be shown for supremum by multiplying $a_n, b_n$ by $-1$.