Homework 6

Walker Bagley

March 27, 2024

B. Homework Exercises

(1) Power Utility

(a) $\gamma = 1/4$

$$\begin{aligned}
u &= \frac{140}{100} = 1.4 \\
d &= \frac{90}{100} = 0.9 \\
P &= \frac{q((1+r)-d)}{p(u-(1+r))} = \frac{0.3(1.05-0.9)}{0.7(1.4-1.05)} = \frac{0.045}{0.245} = 0.1837 \\
P^{-\frac{1}{\gamma}} &= P^{-4} = 0.1837^{-4} = 878.6467 \\
k_s &= \frac{(P^{-\frac{1}{\gamma}}-1)(1+r)}{(u-(1+r)) + P^{-\frac{1}{\gamma}}((1+r)-d)} \\
&= \frac{1.05(878.6467-1)}{(1.4-1.05)+878.6467(1.05-0.9)} \\
&= \frac{921.529035}{0.35+878.6467\cdot 0.15} = \frac{921.529035}{132.147} = 6.9735 \\
k_b &= 1 - k_s = -5.9735
\end{aligned}$$

(b) $\gamma = 3/4$

$$\begin{aligned}
P^{-\frac{1}{\gamma}} &= P^{-\frac{4}{3}} = 0.1837^{-\frac{4}{3}} = 9.5779 \\
k_s &= \frac{1.05(9.5779-1)}{(1.4-1.05)+9.5779(1.05-0.9)} \\
&= \frac{9.0068}{0.35+9.5779\cdot 0.15} = \frac{9.0068}{1.7867} = 5.041 \\
k_b &= 1 - k_s = -4.041
\end{aligned}$$

(2) Logarithmic Utility

(a) Proof

$$\begin{aligned}
U(C) &= A\ln C \\
U'(C) &= \frac{A}{C}
\end{aligned}$$

$$\begin{aligned}
\frac{pAu}{C_1(H)} + \frac{qAd}{C_1(T)} &= \frac{pA(1+r)}{C_1(H)} + \frac{qA(1+r)}{C_1(T)} \\
\frac{pAu}{C_1(H)} - \frac{pA(1+r)}{C_1(H)} &= \frac{qA(1+r)}{C_1(T)} - \frac{qAd}{C_1(T)}
\end{aligned}$$

$$
\begin{aligned}
p\frac{A}{C_1(H)}[u-(1+r)] &= q\frac{A}{C_1(T)}[(1+r)-d] \\
\frac{C_1(T)}{C_1(H)} &= \frac{q((1+r)-d)}{p(u-(1+r))} = P \\
P &= \frac{k_s(d-(1+r))+(1+r)}{k_s(u-(1+r))+(1+r)} \\
P[k_s(u-(1+r))+(1+r)] &= k_s(d-(1+r))+(1+r) \\
k_s[P(u-(1+r))-(d-(1+r))] &= (1-P)(1+r) \\
k_s &= \frac{(1-P)(1+r)}{((1+r)-d)+P(u-(1+r))} \\
k_b = 1-k_s &= 1-\frac{(1-P)(1+r)}{((1+r)-d)+P(u-(1+r))} \\
&= \frac{((1+r)-d)+P(u-(1+r))-(1-P)(1+r)}{((1+r)-d)+P(u-(1+r))} \\
&= \frac{((1+r)-d)+Pu-P(1+r)+(P-1)(1+r)}{((1+r)-d)+P(u-(1+r))} \\
&= \frac{Pu-d}{((1+r)-d)+P(u-(1+r))}
\end{aligned}
$$

(b) Application

$$
\begin{aligned}
u &= \frac{140}{100} = 1.4 \\
d &= \frac{90}{100} = 0.9 \\
P &= \frac{0.3(1.05-0.9)}{0.7(1.4-1.05)} = \frac{0.045}{0.245} = 0.1837 \\
k_s &= \frac{1.05(1-0.1837)}{(1.05-0.9)+0.1837(1.4-1.05)} = \frac{0.8571}{0.15+0.1837\cdot 0.35} \\
&= \frac{0.8571}{0.2143} = 4 \\
k_b &= 1-k_s = -3
\end{aligned}
$$