Exam 1 Cheatsheet

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Linearity

$$\begin{aligned} Lu &= f \\ L(cu) &= cL(u) \\ L(u+v) &= L(u) + L(v) \end{aligned}$$

Linear 1-O (Coefficients)

$$\begin{aligned} au_x + bu_y = 0 \Rightarrow \frac{dy}{dx} = \frac{b}{a} \Rightarrow y = \frac{b}{a}x + c \\ c = ay - bx \Rightarrow u(x,y) = f(bx - ay) \end{aligned}$$

Transport Equation

$$\begin{aligned} u_t + cu_x = 0 &\Rightarrow u(x,t) = f(x - ct) \\ \text{Initial condition: } u(x,0) = \phi(x) &\Rightarrow u(x,t) = \phi(x - ct) \end{aligned}$$

Classification of 2-O

$$\begin{aligned} 0 &= a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + LO \\ D &= a_{12}^2 - a_{11}a_{22} \text{ and change vars } (x,y) \to (\xi, \eta) \\ \text{Elliptic: } D &< 0 \Rightarrow \partial_\xi^2 u + \partial_\eta^2 u + LO = 0 \\ \text{Parabolic: } D &= 0 \Rightarrow \partial_\xi^2 u + LO = 0 \\ \text{Hyperbolic: } D &> 0 \Rightarrow \partial_\xi^2 u - \partial_\eta^2 u + LO = 0 \end{aligned}$$

Useful Formulae

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-\lambda z^2} dz &= \sqrt{\frac{\pi}{\lambda}} \quad \lambda > 0 \\ \mathscr{E}rf(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz \\ \text{Fourier Transform: } \widehat{\phi}(\xi) &= \int_{-\infty}^{\infty} e^{-i\xi x} \phi(x) dx \\ \text{Inverse Fourier Transform: } f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \widehat{f}(\xi) d\xi \\ \int_{-l}^{l} f(x) dx &= \begin{cases} 0 & f \text{ is odd} \\ 2\int_0^l f(x)dx & f \text{ is even} \end{cases} \\ \sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \end{aligned}$$

Wave Equation

General Solution

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \quad -\infty < x < \infty \\ u(x,t) &= f(x+ct) + g(x-ct) \end{aligned}$$

IVP Solution

$$u_{tt} - c^2 u_{xx} = 0 \quad u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x)$$

$$\text{d'Alembert's Formula: } u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

IVP & Forcing Solution

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= f(x,t) \quad u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) \\ u(x,t) &= v(x,t) + w(x,t) \quad v \text{ is IVP sol, } w \text{ is forcing sol} \\ v(x,t) &= \text{ d'Alembert's formula} \\ w(x,t) &= \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \\ u(x,t) &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \end{aligned}$$

Half Line

$$\begin{gathered} u_{tt} - c^2 u_{xx} = 0 \quad u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x) \\ u(0,t) = 0 \quad 0 < x < \infty \\ \text{Extend } \phi, \psi \text{ oddly} \Rightarrow \phi_{odd}, \psi_{odd} \\ \text{We then have two cases: } u(x,t) = \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds & 0 < ct < x \\ \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)ds & 0 < x < ct \end{cases} \end{gathered}$$

Interval

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 0 \quad 0 < x < l \quad t > 0 \\ u(x,0) &= \phi(x) \quad u_t(x,0) = \psi(t) \quad u(0,t) = u(l,t) = 0 \quad t > 0 \\ u(x,t) &= X(x)T(t) \Rightarrow XT'' = c^2 X'' T \Rightarrow \frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda \\ &= \begin{cases} X'' + \lambda X = 0 \quad X(0) = X(l) = 0 \\ T'' + \lambda c^2 T = 0 \end{cases} \\ \lambda > 0 &\Rightarrow \lambda = \mu^2 \quad \mu > 0 \\ X(x) &= c_1 \cos(\mu x) + c_2 \sin(\mu x) \quad c_1 = 0 \\ X(x) &= c_2 \sin(\mu x) \Rightarrow X(l) = 0 = c_2 \sin(\mu l) \quad \mu l = n\pi \Rightarrow \mu = \frac{n\pi}{l} \\ X_n(x) &= \sin\left(\frac{n\pi x}{l}\right) \\ T_n(t) &= A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right) \\ u_n(x,t) &= \left[A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right] \sin\left(\frac{n\pi x}{l}\right) \end{aligned}$$

Heat Diffusion Equation

IVP Solution

$$
\begin{aligned}
u_t - ku_{xx} &= 0 \quad -\infty < x < \infty \quad t > 0 \\
u(x,0) &= \phi(x) \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2} \phi(y) dy \quad \text{(Physical Solution)} \\
u(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} e^{-\xi^2 t} \widehat{\phi}(\xi)\, d\xi \quad \text{(Spectral Solution)}
\end{aligned}
$$

IVP & Forcing Solution

$$
\begin{aligned}
u_t - ku_{xx} &= f(x,t) \quad -\infty < x < \infty \quad t > 0 \\
u(x,0) &= \phi(x) \qquad u(x,t) = v(x,t) + w(x,t) \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2} \phi(y) dy + \int_0^t \frac{1}{\sqrt{4\pi k(t-\tau)}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2} f(y,\tau) dy d\tau
\end{aligned}
$$

Extra u Term

$$
\begin{aligned}
u_t - ku_{xx} + c(t)u &= f(x,t) \quad -\infty < x < \infty \quad t > 0 \\
u(x,0) &= \phi(x) \qquad u(x,t) = v(t)w(x,t)
\end{aligned}
$$

$$
\text{Then 2 equations: } \begin{cases} w_t - kw_{xx} = f(x,t) \\ v' + c(t)v = 0 \end{cases} \Rightarrow \quad \begin{matrix} w \text{ from other solution} \\ v(t) = e^{-\int c(t)dt} \end{matrix}
$$

Half Line

$$
\begin{aligned}
u_t - ku_{xx} &= 0 \quad 0 < x < \infty \quad t > 0 \\
u(x,0) &= \phi(x) \quad 0 < x < \infty \quad u(0,t) = 0 \quad t > 0 \\
\text{Extend } \phi \text{ oddly: } \phi_{odd}(x) &= \begin{cases} \phi(x) & x > 0 \\ 0 & x = 0 \\ -\phi(-x) & x < 0 \end{cases} \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \left[e^{-\frac{1}{4kt}(x-y)^2} - e^{-\frac{1}{4kt}(x+y)^2} \right] \phi(y) dy
\end{aligned}
$$

Interval

$$
\begin{aligned}
u_t - ku_{xx} &= 0 \quad 0 < x < l \quad t > 0 \\
u(x,0) &= \phi(x) \quad 0 < x < l \quad u(0,t) = u(l,t) = 0 \quad t > 0 \\
u(x,t) &= X(x)T(t) \Rightarrow XT' = kX''T \Rightarrow \frac{T'}{kT} = \frac{X''}{X} = -\lambda \\
&= \begin{cases} X'' + \lambda X = 0 \quad X(0) = X(l) = 0 \\ T' + \lambda kT = 0 \end{cases} \\
\lambda > 0 &\Rightarrow \lambda = \mu^2 \quad \mu > 0 \\
X(x) &= c_1 \cos(\mu x) + c_2 \sin(\mu x) \quad c_1 = 0 \\
X(x) &= c_2 \sin(\mu x) \Rightarrow X(l) = 0 = c_2 \sin(\mu l) \quad \mu l = n\pi \Rightarrow \mu = \frac{n\pi}{l} \\
X_n(x) &= \sin\left(\frac{n\pi x}{l}\right) \qquad T_n(t) = e^{-\lambda kt} = e^{-\left(\frac{n\pi}{l}\right)^2 kt} \\
u_n(x,t) &= B_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\left(\frac{n\pi x}{l}\right)
\end{aligned}
$$

Fourier Series

General Formula

$$\text{Fourier Series: } f(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left[A_n \cos\left(\frac{n\pi x}{l}\right) + B_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$\text{Fourier Coefficients: } = \begin{cases} A_n & = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \quad n = 0, 1, 2, \ldots \\ B_n & = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \quad n = 1, 2, \ldots \end{cases}$$

Even/Odd f

$$f \text{ is even: } \begin{cases} A_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ B_n = 0 \end{cases}$$

$$f \text{ is odd: } \begin{cases} A_n = 0 \\ B_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{cases}$$