Homework 4

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February 11, 2022

1. (a) True - $a \in \{a, b, c\}$
   (b) False - $\{a\} \notin \{a, b, c\}$ because $a \notin b \wedge a \notin c$ and $a \notin a$
   (c) False - $a, b, c \notin a$ so it cannot be a subset of the set containing a, b, and c by the definition of a subset
   (d) True - both a and b are elements of both sets, and there are no other elements in either set, so each of them is a subset of the other
   (e) True - $a = a$
   (f) True - by applying the Axiom of Union several times, you end up with a as a subset of a, which is always true
   (g) True - the set on the right just evaluates to $S = \{a, b\}$, so therefore $a \in S$
   (h) False - the set on the right evaluates to $S = \{a, b, c\}$, and $\{a, b, c\}$ cannot be an element
   (i) False - $a \notin a$, so $a \notin \{\{a\}\}$
   (j) False - none of the elements in either set are contained by the other, which violates the definition of a subset

2. *Proof.* $\forall x(x \subseteq x)$

$$\begin{aligned}
&\text{Let } x, y \text{ be sets according to Axiom 0}\\
&\text{Consider } y = \{a | a \in x\}\\
&\text{By Axiom 1, } x = y\text{, but also, } x = x\\
&\text{So, by the theorem of subsets, } x \subseteq y \wedge y \subseteq x\\
&\text{Substituting } x \text{ in for } y\text{, we are left with } x \subseteq x \wedge x \subseteq x\\
&\text{Therefore, with simplification } \forall x(x \subseteq x)
\end{aligned}$$

□

3. *Proof.* $\forall x \forall y \forall z((x \subseteq y \wedge y \subseteq z) \rightarrow x \subseteq z)$

$$\begin{aligned}
&\forall x \forall y \forall z(x \subseteq y \wedge y \subseteq z) \text{ from the statement}\\
&\text{Let } U, V, W \text{ be sets such that } U \subseteq V \wedge V \subseteq W\\
&\text{This means that } \forall s(s \in U \rightarrow s \in V) \wedge \forall t(t \in V \rightarrow t \in W)\\
&\text{Consider the set } a \text{ such that } a \in U \rightarrow a \in V \wedge a \in V \rightarrow a \in W\\
&\text{By the first order logic, } a \in U \rightarrow a \in W\\
&\text{Thus, } \forall b(b \in U \rightarrow b \in W)\\
&\text{By the definition of a subset, } U \subseteq W\\
&\text{Therefore, } \forall x \forall z(x \subseteq z)
\end{aligned}$$

□

4. (a) X does not necessarily contain any elements, although it can. X can be an empty set that still satisfies the condition of the Axiom of Existence that X = X
   (b) Yes, because the Axiom of Extensionality guarantees that for any set X, there is another set that is equal and contains the same elements.

5. *Proof.* $\forall x \forall z(x \subseteq z \rightarrow \exists y(x \subseteq y \wedge y \subseteq z))$

$$\begin{aligned}
&\forall x \forall z(x \subseteq z) \text{ is given}\\
&\text{Consider sets } U, V \text{ such that } U \subseteq V\\
&\text{By the definition of a subset, } \forall a(a \in U \rightarrow a \in V)\\
&\text{Let } Q \text{ be a set that exists such that } Q = U\\
&\text{Thus as we have proved, } U \subseteq Q\\
&\text{So, } (a \in U \rightarrow a \in Q) \wedge (a \in Q \rightarrow a \in V)\\
&\text{By the definition of a subset, } U \subseteq Q \wedge Q \subseteq V\\
&\text{Therefore, } \exists y(x \subseteq y \wedge y \subseteq z)
\end{aligned}$$

□