Homework 1

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Section 1.2

1.2.1

Prove $||x| - |y|| \leq |x - y|$ for $x, y \in \mathbb{R}$

Proof.

$$\begin{aligned} |x - y + y| \leq |x - y| + |y| \Rightarrow |x| - |y| \leq |x - y| \\ |y - x + x| \leq |y - x| + |x| \Rightarrow |y| - |x| \leq |y - x| \end{aligned}$$

We know that $|x - y| = |y - x|$ and $|y| - |x| = -(|x| - |y|)$ so

$$\begin{aligned} |x| - |y| &\leq |x - y| \\ -(|x| - |y|) &\leq |x - y| \end{aligned}$$

Thus, we know by definition of absolute value that $||x| - |y|| \leq |x - y|$. □

1.2.2

Prove $2ab \leq a^2 + b^2$ for $a, b \in \mathbb{N}$

Proof.

$$\begin{aligned} 2ab \leq a^2 + b^2 &\Rightarrow 0 \leq a^2 - 2ab + b^2 \\ &\Rightarrow 0 \leq (a - b)^2 \end{aligned}$$

Due to the square, we will always have $0 \leq n$ for some n positive number. □

1.2.3

Prove $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ for $a, b, c, d \in \mathbb{N}$ where $ad < bc$

Proof.

$$\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d} \Rightarrow a + \frac{ad}{b} < a + c < \frac{bc}{d} + c$$

Then we want to show the following two inequalities:

$$\begin{aligned} a + \frac{ad}{b} < a + c &\Rightarrow \frac{ad}{b} < c \\ &\Rightarrow ad < bc \\ a + c < \frac{bc}{d} + c &\Rightarrow a < \frac{bc}{d} \\ &\Rightarrow ad < bc \end{aligned}$$

It was given that $ad < bc$ so both inequalities hold. □

1.2.4

Prove $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$

Proof. By induction. Base case: $n = 1$

$$1^2 = 1 = \frac{1}{6} \cdot 2 \cdot 3 = \frac{1}{6}(1)(1+1)(2 \cdot 1 + 1)$$

Inductive Hypothesis: assume that for $k \in \mathbb{N}$, $1^2 + 2^2 + \cdots + k^2 = \frac{1}{6}k(k+1)(2k+1)$.

$$\begin{aligned} 1^2 + 2^2 + \cdots + (k+1)^2 &= 1^2 + 2^2 + \cdots + k^2 + (k+1)^2 \\ &= \frac{1}{6}k(k+1)(2k+1) + (k+1)^2 \quad \text{by IH} \\ &= \frac{1}{6}(k+1)[k(2k+1) + 6(k+1)] \\ &= \frac{1}{6}(k+1)(2k^2 + 7k + 6) \\ &= \frac{1}{6}(k+1)(k+2)(2k+3) \end{aligned}$$

This is what we would expect for $k + 1$. □

Section 2.1

2.1.1

(a)

Prove $\lim_{n\to\infty} \frac{\sin n}{\sqrt{n}} = 0$

Proof.

$$\begin{aligned} \left| \frac{\sin n}{\sqrt{n}} - 0 \right| < \epsilon &\Rightarrow \left| \frac{\sin n}{\sqrt{n}} \right| < \epsilon \\ &\Rightarrow \frac{|\sin n|}{|\sqrt{n}|} < \epsilon \\ &\Rightarrow |\sin n| < \epsilon \cdot \sqrt{n} \\ &\Rightarrow |\sin n| \leq 1 < \epsilon \cdot \sqrt{n} \\ &\Rightarrow \frac{1}{\epsilon} < \sqrt{n} \\ &\Rightarrow \frac{1}{\epsilon^2} < n \end{aligned}$$

Take $N > \frac{1}{\epsilon^2}$. □

(b)

Prove $\lim_{n\to\infty} \frac{2^n}{n!} = 0$

Proof.

$$\begin{aligned} \left| \frac{2^n}{n!} - 0 \right| < \epsilon &\Rightarrow \left| \frac{2^n}{n!} \right| < \epsilon \\ &\Rightarrow \frac{2^n}{n!} < \epsilon \end{aligned}$$

If we expand $a_n = \frac{2^n}{n!}$, we can see

$$\begin{aligned}\frac{2^n}{n!} &= \frac{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}{1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n} \\ &= \frac{2}{1} \cdot \frac{2}{2} \cdot \frac{2}{3} \cdots \frac{2}{n-2} \cdot \frac{2}{n-1} \cdot \frac{2}{n}\end{aligned}$$

Observe that $a_n = a_{n-1} \cdot \frac{2}{n}$ and $a_{n-1} < 1$ for $n > 5$. So we can say that when $n > 5$, $a_n < \frac{2}{n}$. Working with this...

$$\begin{aligned}\frac{2^n}{n!} < \epsilon &\Rightarrow \frac{2^n}{n!} < \frac{2}{n} < \epsilon \\ &\Rightarrow \frac{2}{\epsilon} < n\end{aligned}$$

Take $N > \frac{2}{\epsilon}$. □

(c)

Prove $\lim_{n\to\infty}(\sqrt{n+4} - \sqrt{n}) = 0$

Proof. Because $\sqrt{n+4} > \sqrt{n}$

$$\begin{aligned}|\sqrt{n+4} - \sqrt{n} - 0| < \epsilon &\Rightarrow |\sqrt{n+4} - \sqrt{n}| < \epsilon \\ &\Rightarrow \sqrt{n+4} - \sqrt{n} < \epsilon \\ &\Rightarrow \frac{4}{\sqrt{n+4} + \sqrt{n}} < \epsilon \\ &\Rightarrow \frac{4}{\sqrt{n+4} + \sqrt{n}} < \frac{2}{\sqrt{n}} < \epsilon \\ &\Rightarrow \frac{2}{\epsilon} < \sqrt{n} \\ &\Rightarrow \frac{4}{\epsilon^2} < n\end{aligned}$$

Take $N > \frac{4}{\epsilon^2}$. □

2.1.2

Prove $|a_n| \to |a|$ given $a_n \to a$

Proof. Given $a_n \to a$, we have that for any ϵ, $|a_n - a| < \epsilon$ for $n > N$. $|a_n| \to |a|$ implies that for any ϵ', $||a_n| - |a|| < \epsilon'$ for $n > N'$. By the reverse triangle inequality proved in (1.2.1), we know that $||a_n| - |a|| \le |a_n - a| < \epsilon$. So we take $\epsilon' = \epsilon$ and $N' = N$. □

2.1.3

Prove $a_n b_n \to 0$ given a_n is bounded and $b_n \to 0$

Proof. Since a_n is bounded, we know that $\exists B$ s.t. $|a_n| \le B$ for all n. Additionally, since $b_n \to 0$, we have that for any ϵ, $|b_n| < \epsilon$ for $n > N$. Then

$$\begin{aligned}|a_n b_n - 0| < \epsilon' &\Rightarrow |a_n b_n| < \epsilon' \\ &\Rightarrow |a_n||b_n| < \epsilon' \\ &\Rightarrow |a_n||b_n| < B\epsilon\end{aligned}$$

Since ϵ is very small, we can take $\epsilon' = B\epsilon$ for $n > N$. □

2.1.4

Prove $\sqrt{a_n} \to \sqrt{a}$ given a_n is positive and $a_n \to a$

Proof. Since $a_n \to a$, we know that for any ϵ, $|a_n - a| < \epsilon$ when $n > N$. So

$$\begin{aligned} |\sqrt{a_n} - \sqrt{a}| < \epsilon' &\Rightarrow \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \epsilon' \\ &\Rightarrow \left| \frac{a_n - a}{\sqrt{a_n} + \sqrt{a}} \right| < \left| \frac{a_n - a}{\sqrt{a}} \right| < \epsilon' \\ &|a_n - a| < \epsilon' \sqrt{a} \end{aligned}$$

Since $|a_n - a| < \epsilon$, we take $\epsilon' = \frac{\epsilon}{\sqrt{a}}$ for $n > N' = N$. □