Homework 6

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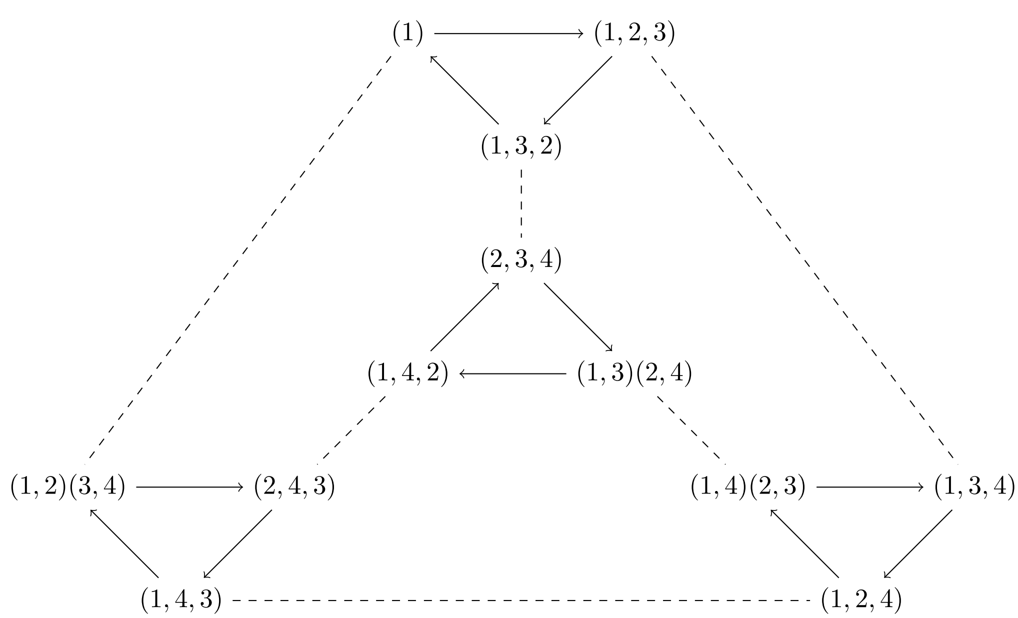
Section 9

1. $\{1,5,2\},\{3\},\{4,6\}$
6. $\{3n \mid n \in \mathbb{Z}\},\{3n+1 \mid n \in \mathbb{Z}\},\{3n-1 \mid n \in \mathbb{Z}\}$
7. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 3 & 5 & 8 & 6 & 2 & 7 \end{pmatrix}$
10. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} = (1,8)(3,6,4)(5,7) = (1,8)(3,4)(3,6)(5,7)$
13. (a) 4
    (b) A cycle's order equals its length
    (c) σ has order 6, τ has order 4
    (d) Exercise 10: $(1,8)(3,6,4)(5,7)$ has order 6
    Exercise 11: $(1,3,4)(2,6)(5,8,7)$ has order 6
    Exercise 12: $(1,3,4,7,8,6,5,2)$ has order 8
    (e) The order of a given permutation is equal to the least common multiple of the lengths of its disjoint cycles
14. Maximum order for an element of S_5 is 6: two disjoint cycles of lengths 2 and 3
15. Maximum order for an element of S_6 is 6: two disjoint cycles of lengths 2 and 3 or cycle of length 6



19.

20. For a permutation σ of a set A, an orbit of σ represents the equivalence class for $a, b \in A$ where $a \sim b$ if $b = \sigma^n(a)$ for some $n \in \mathbb{Z}$

21. A cycle is a permutation with at most one orbit containing at least one element

22. The alternating group A_n is the subgroup of S_n consisting of all even permutations

23. (a) False
    (b) True
    (c) False
    (d) False
    (e) False
    (f) False
    (g) True
    (h) True
    (i) True
    (j) False

29. *Proof.*

$$\begin{aligned}
\text{Suppose } H \text{ has no odd permutations, then all are even} \\
\text{Now suppose } o \in H \text{ is an odd permutation} \\
\text{Consider } \sigma : H \to H \text{ with } \sigma(h) = oh \text{ for all } h \in H \\
\text{This is 1-1 since } \sigma(h_1) = \sigma(h_2) \Rightarrow oh_1 = oh_2 \Rightarrow h_1 = h_2 \\
\text{This is also onto since for all } h \in H \text{ we can find } o^{-1}h \in H \\
\text{When multiplying permutations, } (odd)(odd) = (even) \text{ and } (even)(odd) = (odd) \\
\sigma \text{ then pairs each odd permutation with an even one, so half are even}
\end{aligned}$$

□

34. *Proof.*

$$\begin{aligned}
\text{If } \sigma \text{ is odd, then it can be written as an odd number of transpositions} \\
\text{When generating } \sigma^2\text{, every second element of } \sigma \text{ is skipped in } \sigma^2 \\
\text{This is because two transpositions of } \sigma \text{ are applied instead of 1} \\
\text{But since the length of } \sigma \text{ is odd, the cycle continues, going through the elements of } \sigma \\
\text{This time, the cycle goes through the second elements of } \sigma\text{, spaced apart by 2 transpositions} \\
\text{Then } \sigma^2 \text{ is a cycle of the same length as } \sigma
\end{aligned}$$

□