Homework 3

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1. (a) Prove that any regular expression with quantifiers has an equivalent expression without quantifiers

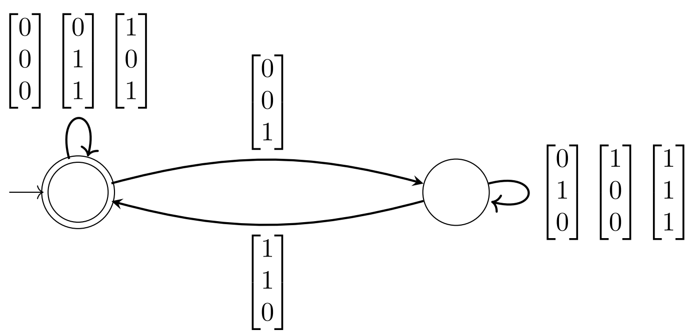
   Proof. Consider some regular expression with quantifiers $q = rst$ where r, t are any other regular expressions and $s = \alpha^{\{m,n\}}$ is the quantified section of q. By definition, s matches all strings $w^{(1)} \cdots w^{(l)}$ where $m \leq l \leq n$ and $w^{(i)} = \alpha$ for all $1 \leq i \leq l$. So, we can then define $\bar{s}$ as $\bar{s} = \bigcup_{i=m}^{n} w^{(1)} \cdots w^{(i)} = [w^{(1)} \cdots w^{(n)}] \cup [w^{(1)} \cdots w^{(n+1)}] \cup \cdots [w^{(1)} \cdots w^{(m)}]$. This is in itself a regular expression representing s without quantifiers, so then we reconstruct $q = r\bar{s}t$. □

   (b) Prove that a language with backreferences is not regular

   Proof. Consider the language given by a Unix regular expression $R = (a+)b\backslash 1$. Suppose, towards a contradiction that R is regular. By the pumping lemma, there is some $p \geq 1$ such that any $s \in R$ where $|s| \geq p$ can be written as $s = xyz$ with $|y| > 0$, $|xy| \leq p$ and for all i, $xy^iz \in R$. Let $s = a^pba^p$. Because $|xy| \leq p$, y consists of at least one a. Let $i = 2$, so according to the lemma $xyyz \in R$, but xyy contains more a's than z, so $xyyz \notin R$ and we have a contradiction. □

2. (a) Show that B is regular

   Proof. Consider the DFA below that accepts $B = \{w \in \Sigma_3{}^* |$ the bottom row of w is the sum of the top two rows$\}$. As a result, we know that B is regular.



□

   (b) Prove that ADD is not regular

   Proof. Consider the language given by $ADD = \{x = y + z | x, y, z \in \{0,1\}^*$ and $x = y + z$ is true$\}$. Suppose, towards a contradiction that ADD is regular. By the pumping lemma, there is some $p \geq 1$ such that any $s \in ADD$ where $|s| \geq p$ can be written as $s = tuv$ with $|u| > 0$, $|tu| \leq p$ and for all i, $tu^iv \in ADD$. Let s be $x^p = y^p + z^p$. Because $|tu| \leq p$, u contains at least one character in x^p. Let $i = 0$, so according to the lemma $tv \in ADD$, but tv does not make $x = y + z$ true, so $tv \notin ADD$ and we have a contradiction. □

3. (a) Prove that B is a regular language

Proof. Let B be the language defined by $B = \{1^k w | w \in \{0,1\}^*$ and w contains at least k 1s for $k \geq 1\}$. Then any string in B must begin with a 1, and have at least one other 1 after it. So, we can write this condition as a regular expression $B = 1(0|1)^*1(0|1)^*$. The existence of this regular expression means that B is regular. □

(b) Prove that C is not a regular language

Proof. Consider the language given by $C = \{1^k w | w \in \{0,1\}^*$ and w contains at most k 1s for $k \geq 1\}$. Suppose, towards a contradiction that C is regular. By the pumping lemma, there is some $p \geq 1$ such that any $s \in C$ where $|s| \geq p$ can be written as $s = xyz$ with $|y| > 0$, $|xy| \leq p$ and for all i, $xy^i z \in C$. Let $s = 1^p 0 1^p$. Because $|xy| \leq p$, y consists of at least one 1 before the 0. Letting $i = 0$, according to the lemma, $xz \in C$, but we know that there are now fewer 1s before the 0 than after, so $xz \notin C$ and we have a contradiction. □