Homework 5

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1. (a) $\{\varnothing\}$
   (b) $\{\varnothing, \{0\}, \{1\}, \{3\}, \{0,1\}, \{0,3\}, \{1,3\}, \{0,1,3\}\}$
   (c) $\{a \in \mathbb{N} | a \text{ is even}\}$
   (d) 4
   (e) 3
2. (a) *Proof.* $\forall x(x \cup \varnothing = x)$

$$\text{Let } x \text{ be a set}$$
$$\text{By the definition of union, } x \cup \varnothing = \{a | a \in x \vee a \in \varnothing\}$$
$$\text{We know that } \forall a(a \notin \varnothing)$$
$$\text{Because } a \notin \varnothing \text{, we can simplify } a \in x \vee a \in \varnothing \text{ to } a \in x$$
$$\text{Therefore, } x \cup \varnothing = \{a | a \in x\}$$
$$\text{By definition, } \{a | a \in x\} = x$$
$$\text{Finally, } x \cup \varnothing = x$$
$$\forall x(x \cup \varnothing = x)$$

□

   (b) *Proof.* $\forall x(x \cap \varnothing = \varnothing)$

$$\text{Let } x \text{ be a set}$$
$$\text{By the definition of intersection, } x \cap \varnothing = \{a | a \in x \wedge a \in \varnothing\}$$
$$\text{We know that } \forall a(a \notin \varnothing)$$
$$\text{Because } a \notin \varnothing\text{, } a \in x \wedge a \in \varnothing \text{ is always false}$$
$$\text{Therefore, } x \cap \varnothing = \varnothing$$
$$\text{So, } \forall x(x \cap \varnothing = \varnothing)$$

□

   (c) *Proof.* $\forall x \forall y(x \cap y \subseteq x)$

$$\text{Let } x, y \text{ be sets}$$
$$\text{By the definition of intersection, } x \cap y = \{a | a \in x \wedge a \in y\}$$
$$\text{If } a \in x \cap y\text{, then } a \in x$$
$$\text{Therefore, by the definition of a subset, } x \cap y \subseteq x$$
$$\text{Finally, } \forall x \forall y(x \cap y \subseteq x)$$

□

   (d) *Proof.* $\forall x \forall y(x \subseteq x \cup y)$

$$\text{Let } x, y \text{ be sets}$$
$$\text{By the definition of union, } x \cup y = \{a | a \in x \vee a \in y\}$$
$$\text{So, } \forall a(a \in x \to a \in x \cup y)$$
$$\text{By this statement, } x \subseteq x \cup y$$
$$\text{Therefore, } \forall x \forall y(x \subseteq x \cup y)$$

□

   (e) *Proof.* $\forall x \forall y(x \cap y \subseteq x \cup y)$

$$\text{Let } x, y \text{ be sets}$$
$$\text{By proof (c), } x \cap y \subseteq x$$
$$\text{By proof (d), } x \subseteq x \cup y$$
$$\text{By the definition of a subset, } a \in x \cap y \to a \in x$$
$$\text{By the definition of a subset, } a \in x \to a \in x \cup y$$
$$\text{By applying the hypothetical, } a \in x \cap y \to a \in x \cup y$$
$$\text{According to the def. of subset, } x \cap y \subseteq x \cup y$$
$$\text{Therefore, } \forall x \forall y(x \cap y \subseteq x \cup y)$$

□

3. *Proof.* $\forall x \forall y((x \cup y) \backslash (x \cap y) = (x \backslash y) \cup (y \backslash x))$

$$
\begin{aligned}
&(x \cup y) \backslash (x \cap y) = && \\
&= \{a | a \in x \vee a \in y\} \backslash \{a | a \in x \wedge a \in y\} && \text{Def. of union} \\
&= \{a | (a \in x \vee a \in y) \wedge \neg(a \in x \wedge a \in y)\} && \text{Def. of set subtraction} \\
&= \{a | (a \in x \vee a \in y) \wedge (a \notin x \vee a \notin y)\} && \text{DeMorgan's} \\
&= \{a | ((a \in x \vee a \in y) \wedge a \notin x) \vee ((a \in x \vee a \in y) \wedge a \notin y)\} && \text{Distribution} \\
&= \{a | ((a \in x \wedge a \notin x) \vee (a \in y \wedge a \notin x)) \vee ((a \in x \wedge a \notin y) \vee (a \in y \wedge a \notin y))\} && \text{Distribution} \\
&= \{a | (a \in y \wedge a \notin x) \vee (a \in x \wedge a \notin y)\} && \text{Negation and Domination} \\
&= \{a | a \in (y \backslash x) \vee a \in (x \backslash y)\} && \text{Def. of set subtraction} \\
&= (x \backslash y) \cup (y \backslash x) && \text{Def. of union}
\end{aligned}
$$

□

4. *Proof.* $\forall x(P(x) \not\subseteq x)$

$$
\begin{aligned}
\text{Let } x \text{ be a set} \\
\text{Assume } P(x) \subseteq x \\
\text{By the definition of the power set, } x \in P(x) \\
\text{By def. of subset, if } x \in P(x) \wedge P(x) \subseteq x, \text{ then } x \in x \\
\text{We have proved that } \forall x(x \notin x) \\
\text{Therefore, } \forall x(P(x) \not\subseteq x)
\end{aligned}
$$

□

5. (a) *Proof.* $\forall x \forall y(P(x) \cup P(y) \subseteq P(x \cup y))$

$$
\begin{aligned}
\text{Let } x, y \text{ be sets} \\
\text{By definition of power set, } P(x) \cup P(y) = \{a | a \subseteq x\} \cup \{b | b \subseteq y\} \\
\text{By definition of union, } P(x) \cup P(y) = \{a | a \subseteq x \vee a \subseteq y\} \\
\text{By def. of power set, } P(x \cup y) = \{a | a \subseteq x \cup y\} \\
\text{By proof (2d), we know that } \forall x \forall y(x \subseteq x \cup y) \\
\text{So, if } a \subseteq x \wedge x \subseteq x \cup y, \text{ then by subset transitivity, } a \subseteq x \cup y \\
\text{Similarly, if } a \subseteq y \wedge y \subseteq x \cup y, \text{ then by subset transitivity, } a \subseteq x \cup y \\
\text{Thus, } a \in P(x) \cup P(y) \rightarrow a \in P(x \cup y) \\
\text{By def. of subset, } P(x) \cup P(y) \subseteq P(x \cup y) \\
\text{Therefore, } \forall x \forall y(P(x) \cup P(y) \subseteq P(x \cup y))
\end{aligned}
$$

□

(b) *Proof.* $\exists x \exists y(P(x) \cup P(y) \neq P(x \cup y))$

$$
\begin{aligned}
\text{Let } x, y \text{ be sets s.t. } x = \{0\} \text{ and } y = \{1\} \\
P(x) = \{\varnothing, \{0\}\} \\
P(y) = \{\varnothing, \{1\}\} \\
P(x) \cup P(y) = \{\varnothing, \{0\}, \{1\}\} \\
P(x \cup y) = \{\varnothing, \{0\}, \{1\}, \{0, 1\}\} \\
\text{In this case, } P(x) \cup P(y) \neq P(x \cup y)
\end{aligned}
$$

□