Homework 7

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1. (a) *Proof.* $A \cup B$ is countable

$$\begin{aligned}
&\text{Let } A \text{ be a finite set and } B \text{ be a countable set}\\
&\text{Because } A \text{ is finite, } (\exists f : A \to n) \text{ for some } n \in \mathbb{N} \text{ s.t. } f \text{ is a bijection}\\
&\text{Consider the function } g : n \to \mathbb{N}, \text{ where } g(n) = n \in \mathbb{N} \text{ which is injective}\\
&\text{So by def. of composition, } g \circ f : A \to \mathbb{N}\\
&\text{Then, by def. of countability, } A \text{ must be countable}\\
\\
&\text{Because } A, B \text{ are both countable, } |A| \leq |\mathbb{N}| \wedge |B| \leq |\mathbb{N}|\\
&\text{Let } f_a, f_b \text{ be injections s.t. } f_a : A \to \mathbb{N} \text{ and } f_b : B \to \mathbb{N}\\
&\text{Recall that } |\mathbb{N}_{even}| = |\mathbb{N}_{odd}| = |\mathbb{N}|\\
&\text{So, we have injections } f_e, f_o \text{ s.t. } f_e : \mathbb{N} \to \mathbb{N}_{even} \text{ and } f_o : \mathbb{N} \to \mathbb{N}_{odd}\\
&\text{Let } \varphi : A \cup B \to \mathbb{N} \text{ given by } (\varphi(a) = f_e(f_a(a)) \text{ if } a \in A) \wedge (\varphi(b) = f_o(f_b(b)) \text{ if } b \in A \cup B \backslash A)\\
&\text{Let } a, b \in A \cup B \text{ and assume } \varphi(a) = \varphi(b)\\
&\text{Since these are equal, we know } \varphi(a) \text{ and } \varphi(b) \text{ are both either even or odd}\\
&\text{Case 1: If } \varphi(a) \text{ and } \varphi(b) \text{ are both even, we know } a \in A \wedge b \in A\\
&\text{This means } \varphi(a) = f_e(f_a(a)) \wedge \varphi(b) = f_e(f_a(b))\\
&\text{So, because } f_e, f_a \text{ are injections, } a = b\\
&\text{Case 2: If } \varphi(a) \text{ and } \varphi(b) \text{ are both odd, we know } (a \in A \cup B \backslash A) \wedge (b \in A \cup B \backslash A)\\
&\text{This means } \varphi(a) = f_o(f_b(a)) \wedge \varphi(b) = f_o(f_b(b))\\
&\text{So, because } f_o, f_b \text{ are injections, } a = b\\
&\text{Thus, by case 1 and case 2, } \varphi \text{ is an injection}\\
&\text{Therefore, by def. of countability, } A \cup B \text{ is countable}
\end{aligned}$$

$\square$

(b) *Proof.* $A \cap B$ is finite

$$\begin{aligned}
&\text{Let } A \text{ be a finite set and } B \text{ be a countable set}\\
&\text{Because } A \text{ is finite, } |A| = |n| \text{ for some } n \in \mathbb{N}\\
&\text{Because } n \in \mathbb{N}, |n| < |\mathbb{N}|\\
&\text{So, } |A| < |\mathbb{N}|\\
&\text{By def. of intersection, } (\forall x \in A \cap B)(x \in A \wedge x \in B)\\
&\text{So, } (\forall a \in A \cap B)(a \in A)\\
&\text{Because of this, } |A \cap B| \leq |A|\\
&\text{Since } |A| < |\mathbb{N}|, |A \cap B| < |\mathbb{N}|\\
&\text{Thus, } A \cap B \text{ is finite}
\end{aligned}$$

$\square$

2. *Proof.* $A := \{f | f : \mathbb{N} \to \mathscr{H}\}$ is uncountable

$$\begin{aligned}
&\text{Towards a contradiction, assume } |A| \leq |\mathbb{N}|\\
&\text{So we have } \varphi : \mathbb{N} \to A \text{ s.t. } \varphi \text{ is a surjection}\\
&\text{Consider } (\forall i \in \mathbb{N})(\varphi(i) = a_i = a_{i0}\ a_{i1}\ a_{i2}\ ...\ a_{ij})\\
&\text{Where } a_i \in A \text{ and } a_{ij} = a_i(j) \text{ and } a_{ij} \in \mathscr{H}\\
&\text{Consider } \tilde{a} = \tilde{a}_0\ \tilde{a}_1\ \tilde{a}_2\ ...\ \tilde{a}_j \text{ defined by } \tilde{a}_i := \{p \text{ if } a_{ii} = w \wedge w \text{ if } a_{ii} \neq w\}\\
&\text{Observe that, for any } i \in \mathbb{N}, \tilde{a} \neq a_i \text{ because } \tilde{a}(i) = \tilde{a}_i \neq a_{ii} = a_i(i)\\
&\text{This means } (\forall i \in \mathbb{N})(\varphi(i) \neq \tilde{a})\\
&\text{However, } \tilde{a} \in X, \text{ so } \varphi \text{ is not a surjection}\\
&\text{Therefore, } |A| > |\mathbb{N}|\\
&\text{So, } A \text{ is uncountable}
\end{aligned}$$

$\square$

3. *Proof.* $|\{f|(\exists n \in \mathbb{N})(f : n \to \{0,1\})\}| = |\mathbb{N}| = \aleph_0$

$$\text{Let } X := \{f|(\exists n \in \mathbb{N})(f : n \to \{0,1\})\}$$
$$\text{Consider the set } S \text{ of all finite binary sequences}$$
$$\text{Ordered s.t. } S := \{0, 1, 00, 01, 10, 11, 000, 001, 010, 011...\}$$
$$\text{Let } g : \{0,1\} \to n \text{ be a function s.t. } (\forall s \in S)(g(f(s_i)) = i)$$
$$\text{So, } g(f(n)) = n \text{ and therefore } g \circ f = id_A$$
$$\text{By definition, } f \text{ is a monomorphism}$$
$$\text{Let } h : \{0,1\} \to n \text{ be an function s.t. } (\forall s \in S)(f(h(s_i)) = s_i)$$
$$\text{So, } f \circ g = id_B$$
$$\text{Then by definition, } f \text{ is an epimorphism}$$
$$\text{As we have shown, } f \text{ is an isomorphism, so } f : \mathbb{N} \to \{0,1\} \text{ is a bijection}$$
$$\text{This means that } |X| = |\mathbb{N}| = \aleph_0$$

□

4. *Proof.* $P(\mathbb{N})$ is not countable

$$\text{By def, } A \text{ is countable if } |A| \leq |\mathbb{N}|$$
$$\text{By Cantor's Theorem, } |A| < |P(A)|$$
$$\text{So, } |\mathbb{N}| < |P(\mathbb{N})|$$
$$\text{Thus, } |P(\mathbb{N})| \not\leq |\mathbb{N}|, \text{ so } P(\mathbb{N}) \text{ is uncountable}$$

□

5. *Proof.* $\forall X \exists Y (|X| < |Y|)$

$$\text{Let } X \text{ be a set}$$
$$\text{By Cantor's Theorem, } |(P(X)| > |X|$$
$$\text{Therefore, there is no set with maximal cardinality}$$

□