Homework 1

Walker Bagley

September 4, 2023

A. Textbook Problems

Section 1.1

3. (a) order = 2, linear inhomogeneous
   (b) order = 2, linear homogeneous
   (c) order = 3, nonlinear
   (d) order = 2, linear inhomogeneous
   (e) order = 2, linear homogeneous
   (f) order = 1, nonlinear
   (g) order = 1, linear homogeneous
   (h) order = 4, linear inhomogeneous

4. *Proof.* If we have solutions u_1, u_2 such that $Lu = g$, then we have $L(u_1 - u_2) = L(u_1) - L(u_2) = g - g = 0$ $\square$

11.

$$\begin{aligned}
u(x,y) &= f(x)g(y) \\
u_x &= f'(x)g(y) \\
u_y &= f(x)g'(y) \\
u_{xy} &= f'(x)g'(y) \\
uu_{xy} &= u_x u_y \\
f(x)g(y)f'(x)g'(y) &= f'(x)g(y)f(x)g'(y) \\
f(x)g(y)f'(x)g'(y) &= f(x)g(y)f'(x)g'(y)
\end{aligned}$$

12.

$$\begin{aligned}
u_n(x,y) &= \sin(nx)\sinh(ny) \\
u_x &= n\cos(nx)\sinh(ny) \\
u_{xx} &= -n^2\sin(nx)\sinh(ny) \\
u_y &= n\sin(nx)\cosh(ny) \\
u_{yy} &= n^2\sin(nx)\sinh(ny) \\
u_{xx} + u_{yy} &= -n^2\sin(nx)\sinh(ny) + n^2\sin(nx)\sinh(ny) \\
&= 0
\end{aligned}$$

Section 1.2

1.

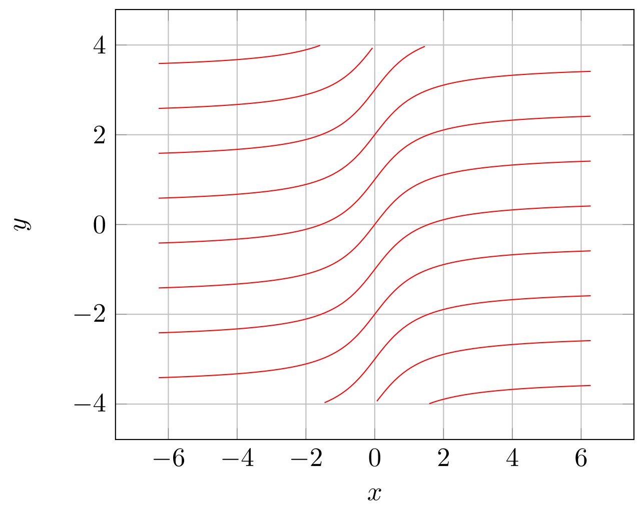
$$2u_t + 3u_x = 0 \qquad\qquad u(x,0) = \sin(x)$$
$$u_t + \frac{3}{2}u_x = 0$$
$$\frac{dx}{dt} = \frac{3}{2} \to x = \frac{3}{2}t + c$$
$$u(x,t) = u(\frac{3}{2}t + c, t)$$
$$u(x,0) = u(c,0) = sin(x)$$
$$u(x,t) = \sin\left(x - \frac{3}{2}t\right)$$

2.

$$3u_y + u_{xy} = 0$$
$$v = u_y \qquad\qquad u_x = v\frac{\partial y}{\partial x}$$
$$3v + \frac{\partial}{\partial y}\left[v\frac{\partial y}{\partial x}\right] = 0$$
$$3v + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + v\frac{\partial}{\partial y}\left[\frac{\partial y}{\partial x}\right] = 0$$
$$v_x + \left(3 + \frac{\partial}{\partial y}\left[\frac{\partial y}{\partial x}\right]\right)v = 0$$
$$\frac{\partial v}{\partial x} = -\left(3 + \frac{\partial}{\partial y}\left[\frac{\partial y}{\partial x}\right]\right)v$$
$$ln(v) = -3x + c$$
$$v = u_y = Ce^{-3x}$$
$$u(x,y) = f(Cye^{-3x})$$

3.

$$(1+x^2)u_x + u_y = 0$$
$$\frac{dy}{dx} = \frac{1}{1+x^2} \to y = \tan^{-1}(x) + c$$
$$c = y - \tan^{-1}(x)$$
$$u(x,y) = f(y - \tan^{-1}(x))$$



7. (a)

$$\begin{aligned}
yu_x + xu_y &= 0 \qquad\qquad u(0,y) = e^{-y^2} \\
\frac{dy}{dx} &= \frac{x}{y} \\
\frac{1}{2}y^2 &= \frac{1}{2}x^2 + c \to y^2 = x^2 + C \\
y &= \pm\sqrt{x^2 + C} \to C = y^2 - x^2 \\
u(x,y) &= f(y^2 - x^2) \to u(0,y) = f(y^2) = e^{-y^2} \\
u(x,y) &= e^{x^2 - y^2}
\end{aligned}$$

(b) The solution is uniquely determined when $y > 0$ or when $y < 0$

8.

$$\begin{aligned}
au_x + bu_y + cu &= 0 \\
s &= ax + by \qquad\qquad t = bx - ay \\
u_x &= au_s + bu_t \\
u_y &= bu_s - au_t \\
a(au_s + bu_t) + b(bu_s - au_t) + cu &= a^2u_s + abu_t + b^2u_s - abu_t + cu \\
(a^2 + b^2)u_s &= -cu \\
u &= f(C)e^{-\frac{cs}{a^2+b^2}} = f(C)e^{-\frac{c(ax+by)}{a^2+b^2}} \\
C &= t = bx - ay \\
u(x,y) &= e^{-\frac{c(ax+by)}{a^2+b^2}} f(bx - ay)
\end{aligned}$$

B. Homework Exercises

1.) (Superposition Principle)

Proof.

$$\begin{aligned}
&u_1, u_2, \ldots, u_n \text{ solve } Lu = 0 \\
&\text{So we know that for any solutions } v, w \text{ that } L(v + w) = Lv + Lw \\
&\text{We also know that for any solution } v \text{ and constant } c \text{ that } L(cv) = cLv \\
&\text{Combining these, any two solutions } v, w \text{ and constants } c, d \text{ yield } L(cv + dw) = cLv + dLw \\
&\text{Then we can say for a series of constants } c_1, c_2, \ldots, c_n \text{ that } L(\Sigma_1^n c_i u_i) = c_1Lu_1 + c_2Lu_2 + \ldots + c_nLu_n \\
&= c_1 * 0 + c_2 * 0 + \ldots + c_n * 0 = 0 + 0 + \ldots + 0 = 0 \\
&\text{So } \Sigma_1^n c_i u_i \text{ is a solution to } Lu = 0
\end{aligned}$$

□

4.) (KdV Solitons)

$$
\begin{aligned}
u(x,t) &= f(x-ct) \qquad\qquad f(x) = \frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}x\right] \\
\text{Let } v &= \frac{\sqrt{c}}{2}(x-ct) && (1)\\
u(x,t) &= \frac{c}{2}\operatorname{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct)\right] = \frac{c}{2}\operatorname{sech}^2(v) && (2)\\
u_t &= 2*\frac{c}{2}\operatorname{sech}(v)\left(-\frac{c\sqrt{c}}{2}\operatorname{sech}(v)\tanh(v)\right)\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v) && (3)\\
u_x &= 2*\frac{c}{2}\operatorname{sech}(v)\left(-\frac{\sqrt{c}}{2}\operatorname{sech}(v)\tanh(v)\right)\\
&= -\frac{c\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v)\\
u_{xx} &= -2*\frac{c\sqrt{c}}{2}\operatorname{sech}(v)\tanh(v)\left[-\frac{\sqrt{c}}{2}\operatorname{sech}(v)\tanh(v)\right] - \frac{c\sqrt{c}}{2}\operatorname{sech}^2(v)\left[\frac{\sqrt{c}}{2}\operatorname{sech}^2(v)\right]\\
&= \frac{c^2}{2}\operatorname{sech}^2(v)\tanh^2(v) - \frac{c^2}{4}\operatorname{sech}^4(v) && (4)\\
u_{xxx} &= 2*\frac{c^2}{2}\operatorname{sech}(v)\tanh^2(v)\left[-\frac{\sqrt{c}}{2}\operatorname{sech}(v)tanh(v)\right]\\
&+ 2*\frac{c^2}{2}\operatorname{sech}^2(v)\tanh(v)\left[\frac{\sqrt{c}}{2}\operatorname{sech}^2(v)\right]\\
&- 4*\frac{c^2}{2}\operatorname{sech}^3(v)\left[-\frac{\sqrt{c}}{2}\operatorname{sech}(v)tanh(v)\right]\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh^3(v) + \frac{c^2\sqrt{c}}{2}\operatorname{sech}^4(v)\tanh(v) + \frac{c^2\sqrt{c}}{2}\operatorname{sech}^4(v)\tanh(v)\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh^3(v) + c^2\sqrt{c}\operatorname{sech}^4(v)\tanh(v) && (5)\\
u_t + u_{xxx} + 6uu_x &= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v) - \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh^3(v) + c^2\sqrt{c}\operatorname{sech}^4(v)\tanh(v)\\
&+ 6\left[\frac{c}{2}\operatorname{sech}^2(v)\right]\left[-\frac{c\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v)\right]\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v) - \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh^3(v) + c^2\sqrt{c}\operatorname{sech}^4(v)\tanh(v)\\
&- \frac{3c^2\sqrt{c}}{2}\operatorname{sech}^4(v)\tanh(v)\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v) - \frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh^3(v) - \frac{c^2\sqrt{c}}{2}\operatorname{sech}^4(v)\tanh(v)\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v)[1-\tanh^2(v)-\operatorname{sech}^2(v)]\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v)[1-(\tanh^2(v)+\operatorname{sech}^2(v))]\\
&= -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v)[1-1] = -\frac{c^2\sqrt{c}}{2}\operatorname{sech}^2(v)\tanh(v)*0\\
&= 0
\end{aligned}
$$

5.) (Exponential Solutions to Linear KdV)

$$\begin{aligned}
u(x,t) &= e^{ikx+ik^3t} && (6)\\
u_t &= ik^3e^{ikx+ik^3t} && (7)\\
u_x &= ike^{ikx+ik^3t}\\
u_{xx} &= i^2k^2e^{ikx+ik^3t} = -k^2e^{ikx+ik^3t}\\
u_{xxx} &= -ik^3e^{ikx+ik^3t} && (8)\\
u_t + u_{xxx} &= ik^3e^{ikx+ik^3t} - ik^3e^{ikx+ik^3t} = 0
\end{aligned}$$

6.) (Exponential Solutions to LS)

$$\begin{aligned}
u(x,t) &= e^{ikx-ik^2t} && (9)\\
u_t &= -ik^2e^{ikx-ik^2t} && (10)\\
u_x &= ike^{ikx-ik^2t}\\
u_{xx} &= i^2k^2e^{ikx-ik^2t} = -k^2e^{ikx-ik^2t} && (11)\\
iu_t + u_{xx} &= i(-ik^2e^{ikx-ik^2t}) + (-k^2e^{ikx-ik^2t})\\
&= -i^2k^2e^{ikx-ik^2t} - k^2e^{ikx-ik^2t}\\
&= k^2e^{ikx-ik^2t} - k^2e^{ikx-ik^2t} = 0
\end{aligned}$$

7.) (Exponential Solutions to Heat Equation)

$$\begin{aligned}
u(x,t) &= e^{ikx-k^2t} && (12)\\
u_t &= -k^2e^{ikx-k^2t} && (13)\\
u_x &= ike^{ikx-k^2t}\\
u_{xx} &= i^2k^2e^{ikx-k^2t} = -k^2e^{ikx-k^2t} && (14)\\
u_t - u_{xx} &= -k^2e^{ikx-k^2t} - (-k^2e^{ikx-k^2t})\\
&= -k^2e^{ikx-k^2t} + k^2e^{ikx-k^2t} = 0
\end{aligned}$$