Homework 7

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Section 10

1.
$$\begin{aligned} 4\mathbb{Z} &= \{\dots, -8, -4, 0, 4, 8, \dots\} \\ 1 + 4\mathbb{Z} &= \{\dots, -7, -3, 1, 5, 9, \dots\} \\ 2 + 4\mathbb{Z} &= \{\dots, -6, -2, 2, 6, 10, \dots\} \\ 3 + 4\mathbb{Z} &= \{\dots, -5, -1, 3, 7, 11 \dots\} \end{aligned}$$

2.
$$\begin{aligned} 4\mathbb{Z} &= \{\dots, -8, -4, 0, 4, 8, \dots\} \\ 2 + 4\mathbb{Z} &= \{\dots, -6, -2, 2, 6, 10, \dots\} \end{aligned}$$

3.
$$\begin{aligned} \langle 2 \rangle &= \{0, 2, 4, 6, 8, 10\} \\ 1 + \langle 2 \rangle &= \{1, 3, 5, 7, 9, 11\} \end{aligned}$$

4.
$$\begin{aligned} \langle 4 \rangle &= \{0, 4, 8\} \\ 1 + \langle 4 \rangle &= \{1, 5, 9\} \\ 2 + \langle 4 \rangle &= \{2, 6, 10\} \\ 3 + \langle 4 \rangle &= \{3, 7, 11\} \end{aligned}$$

6. $\{\rho_0, \mu_2\}, \{\rho_1, \delta_2\}, \{\rho_2, \mu_1\}, \{\rho_3, \delta_1\}$

7. $\{\rho_0, \mu_2\}, \{\rho_1, \delta_1\}, \{\rho_2, \mu_1\}, \{\rho_3, \delta_2\}$

12. $[\mathbb{Z}_{24} : \langle 3 \rangle] = 3$
$$\begin{aligned} \langle 3 \rangle &= \{0, 3, 6, 9, 12, 15, 18, 21\} \\ 1 + \langle 3 \rangle &= \{1, 4, 7, 10, 13, 16, 19, 22\} \\ 2 + \langle 3 \rangle &= \{2, 5, 8, 11, 14, 17, 20, 23\} \end{aligned}$$

13. $[S_3 : \langle \mu_1 \rangle] = 3$
$$\begin{aligned} \langle \mu_1 \rangle &= \{\rho_0, \mu_1\} \\ \rho_1 \langle \mu_1 \rangle &= \{\rho_1 \rho_0, \rho_1 \mu_1\} = \{\rho_1, \mu_3\} \\ \rho_2 \langle \mu_1 \rangle &= \{\rho_2 \rho_0, \rho_2 \mu_1\} = \{\rho_2, \mu_2\} \end{aligned}$$

15. $\sigma = (1, 2, 5, 4)(2, 3) = (1, 2, 3, 5, 4)(6)$
$[S_5 : \langle \sigma \rangle] = \frac{|G|}{|\sigma|} = \frac{120}{5} = 24$

19. (a) True
    (b) True
    (c) True
    (d) False
    (e) True
    (f) False
    (g) True
    (h) True
    (i) False
    (j) True

20. Impossible by definition of abelian

21. Need to have only one coset, i.e. $G = H$, so $G = \mathbb{Z}_2$ and $H = \mathbb{Z}_2$

22. $\langle 0 \rangle$ of $\mathbb{Z}_6$

23. Impossible to divide a group into more partitions than it has elements.

24. Impossible since 4 does not divide 6

27. *Proof.*

$$\begin{aligned}
&\text{Let } f : H \to Hg \text{ be a function defined by } f(h) = hg \text{ for all } h \in H\\
&\text{Then if } f(a) = f(b),\ ag = bg \Rightarrow agg^{-1} = bgg^{-1} \Rightarrow a = b \text{ by right cancellation}\\
&\text{So } f \text{ is one to one}\\
&\text{Now take some arbitrary element } a \in Hg\\
&\text{Then } a = hg \text{ for some } h \in H \text{ by definition of a coset}\\
&\text{Then } f(h) = hg = a, \text{ so } f \text{ is onto}
\end{aligned}$$

□

30. Counterexample: $G = D_3$ and $H = \{e, \mu_1\}$ so $\rho_1 H = \mu_3 H$ but $H\rho_1 \neq H\mu_3$

31. *Proof.*

$$\begin{aligned}
&\text{Assume } Ha = Hb\\
&\text{Then } Haa^{-1} = Hba^{-1} \Rightarrow H = Hba^{-1}\\
&\text{So } ba^{-1} = e \text{ which means that } a = b\\
&\text{Since } e \in H \text{ for any } H,\ a \in Ha\\
&\text{Then } b \in Ha
\end{aligned}$$

□

32. *Proof.*

$$\begin{aligned}
\text{For some } h \in H,\ aH = bH &\Rightarrow b = ah\\
&\Rightarrow a^{-1}b = a^{-1}ah\\
&\Rightarrow a^{-1}b = h\\
&\Rightarrow a^{-1}bb^{-1} = hb^{-1}\\
&\Rightarrow a^{-1} = hb^{-1}\\
\text{Then } Ha^{-1} = Hb^{-1}&
\end{aligned}$$

□

33. *Proof.*

$$\begin{aligned}\text{Assume } aH &= bH \\ \text{Then } a^{-1}aH = a^{-1}bH \Rightarrow H &= a^{-1}bH \\ \text{So } a^{-1}b = e \text{ which means that } a &= b \\ \text{Then we have } aaH = bbH \Rightarrow a^2H &= b^2H\end{aligned}$$

□

34. *Proof.*

Let H be a proper subgroup of G

Since $|G| = pq$, any proper subgroup H will have $|H| = p$ or $|H| = q$ by Lagrange's Theorem

Since all groups of prime order are cyclic, any proper subgroup of G is cyclic

□

37. *Proof.*

Consider a group G which has an order $n \geq 2$ and no nontrivial subgroups

By the Lagrange theorem, has subgroups of orders that divide its order

Then we have two subgroups of G with orders 1 and n, but these are trivial

Since there are no nontrivial subgroups of G, then no integer divides n other than 1 and itself

By definition then, n is a prime number and G must be finite

□

41. *Proof.*

For any $a \in [0,1)$, $a + \mathbb{Z} = \{\ldots, a-2, a-1, a, a+1, a+2, \ldots\}$

So if $a = x$ then there exists an x s.t. $0 \leq x < 1$

For any $z \in \mathbb{R}$ we can represent $z = a + y$ where $y \in \mathbb{Z}$ and $a \in [0,1)$

For any left coset of $\mathbb{Z}$ with $z \in \mathbb{R}$ then $\exists x$ s.t. $0 \leq x < 1$

□

43. (a) *Proof.*

Reflexive: $a \sim a \Rightarrow a = hak$ when $h, k = e$

Symmetric: $a \sim b \Rightarrow a = hbk$ for some $h \in H$ and $k \in K$

We know $h^{-1} \in H$ and $k^{-1} \in K$ by definition

So $h^{-1}ak^{-1} = h^{-1}hbkk^{-1} = b \Rightarrow b \sim a$

Transitive: $a \sim b \Rightarrow a = h_1bk_1$ and $b \sim c \Rightarrow b = h_2ck_2$ for $h_1, h_2 \in H$ and $k_1, k_2 \in K$

Then $a = h_1h_2bk_2k_1$ where $h_1h_2 \in H$ and $k_1k_2 \in K$ by definition

So $a \sim c$

□

(b) $\overline{a} = \{b \in G \mid b = hak \text{ for some } h \in H, k \in K\} = HaK$