Homework 10

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April 28, 2023

Section 18

3. $(11)(-4) = -44 = 1$

5. $(2,3)(3,5) = (6,15) = (1,6)$

7. $n\mathbb{Z}$ is closed under addition and multiplication: $a = nk$ and $b = nl$, $a + b = n(k + l)$ and $ab = kln^2$ so both $a + b$ and ab are multiples of n. It is a commutative ring, following from $\mathbb{Z}$. $1 \notin n\mathbb{Z}$ so it does not have unity. There are no multiplicative inverses, so it is not a field.

9. $\mathbb{Z} \times \mathbb{Z}$ is closed under addition and multiplication by components: $(a,b) + (c,d) = (a + c, b + d)$ and $(a,b)(c,d) = (ac, bd)$ and obviously $a + c, b + d, ac, bd \in \mathbb{Z}$. It is a commutative ring, following from $\mathbb{Z}$. $(1,1) \in \mathbb{Z} \times \mathbb{Z}$ so it has unity, but there are not multiplicative inverses, so it is not a field.

11. $\{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$ is closed under addition and multiplication: $n = a + b\sqrt{2}$ and $m = c + d\sqrt{2}$, then $n + m = (a + c) + (b + d)\sqrt{2}$ and $nm = (a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$. It is commutative following from $\mathbb{R}$. It has unity since $1 + 0\sqrt{2} = 1$ is contained in the ring. Multiplicative inverses are not in the ring, so it cannot be a field.

14. $\{-1, 1\}$

15. $\{(1,1), (-1,1), (1,-1), (-1,-1)\}$

16. $\{1, 2, 3, 4\}$

17. $\{\frac{a}{b} \mid a, b \in \mathbb{Z} \backslash 0\}$

19. $\{1, 3\}$

20. (a) The order of $M_2(\mathbb{Z}_2)$ is 16

    (b)

$$M_2(\mathbb{Z}_2) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

22. Det is not a ring homomorphism because $det(A + B) \neq det(A) + det(B)$ for all $A, B \in M_n(\mathbb{R})$

23. $\phi(n) = 0$ and $\phi(n) = n$

24. $\phi(n) = (0,0)$, $\phi(n) = (n,0)$, $\phi(n) = (0,n)$, $\phi(n) = (n,n)$

25. $\phi((n,m)) = 0$, $\phi((n,m)) = n$, $\phi((n,m)) = m$

27. This reasoning is not correct because more X exist such that $X^2 = I_3$. For example $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 = I_3$

28.

$$\begin{aligned} x^2 + x - 6 &= 0 \\ (x+3)(x-2) &= 0 \\ x &= 2, 4, 9, 11 \in \mathbb{Z}_{14} \end{aligned}$$

33. (a) True
    (b) False
    (c) False
    (d) False
    (e) True
    (f) False
    (g) False
    (h) True
    (i) False
    (j) True

37. *Proof.*

$$\begin{aligned} &\text{Consider } U \text{ is the collection of units of a ring } \langle R, +, \cdot \rangle \\ &\text{Then } \forall u \in U,\ u' \in U \text{ and any } a, b \in U \text{ is } (ab)(b'a') = 1, \text{ so } ab \in U \\ &\text{Since each element and its inverse forms a unit, inverses hold in } U \\ &\text{We also know that the multiplicative identity is always a unit, so } U \text{ has identity} \\ &\text{Then } \langle U, \cdot \rangle \text{ is a group} \end{aligned}$$

□

38. *Proof.*

$$\begin{aligned} (a+b)(a-b) &= a^2 - ab + ba - b^2 \\ \text{Thus } (a+b)(a-b) = a^2 - b^2 &\text{ iff } ab = ba \text{ for all } a, b \in R \\ \text{This is the definition of commutativity} \end{aligned}$$

□

Section 19

1.

$$\begin{aligned} x^3 - 2x^2 - 3x &= 0 \\ x(x^2 - 2x - 3) &= 0 \\ x(x-3)(x+1) &= 0 \\ x &= 0, 3, 5, 8, 9, 11 \in \mathbb{Z}_{12} \end{aligned}$$

2.

$$\begin{aligned} 3x &= 2 \\ x &= 3 \in \mathbb{Z}_7 \\ x &= 16 \in \mathbb{Z}_{23} \end{aligned}$$

$$\begin{aligned} x^3 - 2x^2 - 3x &= 0 \\ x(x^2 - 2x - 3) &= 0 \\ x(x-3)(x+1) &= 0 \\ x &= 0, 3, 5, 8, 9, 11 \in \mathbb{Z}_{12} \end{aligned}$$

3. No solutions for $x^2+2x+2=0$ in $\mathbb{Z}_6$

4.

$$\begin{aligned} x^2+2x+4&=0\\ x&=2\in\mathbb{Z}_6 \end{aligned}$$

Section 24

4.

$$\begin{aligned} (i+3j)(4+2j+k)&=i4+i2j+ik+3j4+6j^2+3jk\\ &=4i+2k-j+12j-6+3i\\ &=-6+7i+11j+2k \end{aligned}$$

5.

$$\begin{aligned} i^2j^3kji^5&=(-1)(-j)(k)(j)(i)\\ &=jkji\\ &=iji\\ &=ki\\ &=j \end{aligned}$$

6.

$$\begin{aligned} (i+j)^{-1}&=\frac{1}{i+j}\\ &=\frac{1}{i+j}*\frac{i+j}{i+j}\\ &=\frac{i+j}{-1+ij+ji-1}\\ &=\frac{i+j}{-2+k-k}\\ &=\frac{i+j}{-2}\\ &=-\frac{1}{2}i-\frac{1}{2}j \end{aligned}$$

7.

$$\begin{aligned} [(1+3i)(4j+3k)]^{-1}&=[4j+3k+12ij+9ik]^{-1}\\ &=[4j+3k+12k-9j]^{-1}\\ &=[-5j+15k]^{-1}\\ &=\frac{1}{-5j+15k}*\frac{-5j+15k}{-5j+15k}\\ &=\frac{-5j+15k}{-25-225}\\ &=\frac{-5j+15k}{-250}\\ &=\frac{1}{50}j-\frac{3}{50}k \end{aligned}$$