Homework 2

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A. Textbook Problems

Section 1.3

1.

Suppose we have a string undergoing small transverse waves defined by displacement $u(x,t)$
Let ρ be the density of the string and a resistance R proportional to the string's velocity u_t
Consider the string between x_0, x_1 with tension T and the slope of the string at any point is u_x
Using Newton's law $F = ma$ and summing the forces, we derive two equations:

$$\text{Longitudinal waves: } \left.\frac{T}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = 0$$

$$\text{Transverse waves: } \left.\frac{Tu_x}{\sqrt{1+u_x^2}}\right|_{x_0}^{x_1} = \int_{x_0}^{x_1} [\rho u_{tt} + Ru_t] dx$$

Since we assume the waves are small, we have that $|u_x|$ is very small, so $\sqrt{1+u_x^2} = 1$
Since we are only dealing with transverse waves, the longitudinal waves are a non-factor
Differentiating the transverse equation yields $(Tu_x)_x = \rho u_{tt} + Ru_t$
With some cleanup, we have a modified wave equation: $0 = u_{tt} - \frac{T}{\rho} u_{xx} + \frac{R}{\rho} u_t$
Letting $c = \sqrt{\frac{T}{\rho}}$ and $r = \frac{R}{\rho}$ since r, ρ are constants makes $0 = u_{tt} - c^2 u_{xx} + ru_t$

4.

First notice this is a diffusion problem on the z axis
We have an external force due to gravity proportional to the position of the particles
So, add this extra term, we end up with a modified wave equation: $u_t = ku_{zz} + vu_z$

Section 1.5

1.

$$u_{xx} + u = 0 \qquad u(0) = 0 \qquad u(L) = 0$$
$$r^2 + 1 = 0$$
$$r = \pm i$$

Then, from ODE's, we know the solution is $u(x) = c_1 \sin(x) + c_2 \cos(x)$
Since $u(0) = 0$, we can eliminate the first term, leaving $u(x) = c_2 \cos(x)$
So, $u(L) = c_2 \cos(L)$
It is easy to see that $\forall k \in \mathbb{Z}(u(L) = 0)$ where $L = k\pi$
So, the solution $u = 0$ is unique for every L except when $L = k\pi$

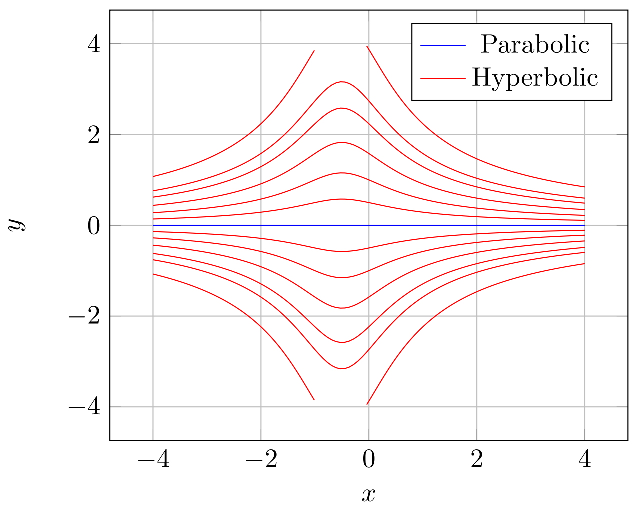
Section 1.6

1. (a) $u_{xx} - u_{xy} - 3u_{yx} + u_{yy} + 2u_y + 4u = 0$
   $a_{11} = 1$, $a_{12} = (-1-3)/2 = -2$, $a_{22} = 1$
   $D = (-2)^2 - 1 * 1 = 4 - 1 = 3 > 0$
   Hyperbolic

   (b) $9u_{xx} + 6u_{xy} + u_{yy} + 2u_x = 0$
   $a_{11} = 9$, $a_{12} = 6/2 = 3$, $a_{22} = 1$
   $D = (3)^2 - (9 * 1) = 9 - 9 = 0$
   Parabolic

2. $(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$
   $a_{11} = 1 + x$, $a_{12} = xy$, $a_{22} = -y^2$
   $D = (xy)^2 - (1+x)(-y^2) = x^2y^2 + xy^2 + y^2$
   $D = y^2(x^2 + x + 1)$
   $y^2 = \frac{D}{x^2+x+1}$

   Parabolic: $D = 0$
   $y^2 = 0 \Rightarrow y = 0$ for any x since $x^2 + x + 1$ has complex roots
   Hyperbolic: $D > 0$
   $y = \pm\sqrt{\frac{D}{x^2+x+1}}$ for all x since $x^2 + x + 1$ has complex roots
   Elliptic: $D < 0$
   Not possible since y^2 cannot be negative



Section 2.1

1.

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} \qquad u(x,0) = e^x \qquad u_t(x,0) = \sin x \\
u(x,y) &= \frac{1}{2}[e^{x+ct} + e^{x-ct}] + \frac{1}{2c}\int_{x-ct}^{x+ct} \sin s \; ds \\
&= \frac{1}{2}[e^{x+ct} + e^{x-ct}] + \frac{1}{2c}[-\cos s|_{x-ct}^{x+ct} \\
&= \frac{1}{2}[e^{x+ct} + e^{x-ct}] - \frac{1}{2c}[\cos(x+ct) - \cos(x-ct)] \\
&= \frac{1}{2}[e^{x+ct} + e^{x-ct}] - \frac{1}{2c}[-2\sin x \sin(ct)] \\
&= \frac{1}{2}e^x[e^{ct} + e^{-ct}] + \frac{1}{c}\sin x \sin(ct) \\
&= e^x \cosh(ct) + \frac{1}{c}\sin x \sin(ct)
\end{aligned}$$

8. $u_{tt} = c^2(u_{rr} + \frac{2}{r}u_r)$

(a)

$$\begin{aligned}
v &= ru \Rightarrow u = \frac{1}{r}v \\
u_t &= \frac{\partial u}{\partial v}\frac{\partial v}{\partial t} = \frac{1}{r}v_t \qquad u_{tt} = \frac{\partial}{\partial t}\left[\frac{1}{r}v_t\right] = \frac{1}{r}v_{tt} \\
u_r &= \frac{\partial}{\partial r}\left[\frac{1}{r}v\right] = -\frac{1}{r^2}v + \frac{1}{r}v_r \\
u_{rr} &= \frac{\partial}{\partial r}\left[-\frac{1}{r^2}v + \frac{1}{r}v_r\right] = \frac{2}{r^3}v - \frac{1}{r^2}v_r + \frac{1}{r}v_{rr} - \frac{1}{r^2}v_r \\
&= \frac{2}{r^3}v - \frac{2}{r^2}v_r + \frac{1}{r}v_{rr} \\
\frac{1}{r}v_{tt} &= c^2\left(\frac{2}{r^3}v - \frac{2}{r^2}v_r + \frac{1}{r}v_{rr} + \frac{2}{r}\left[-\frac{1}{r^2}v + \frac{1}{r}v_r\right]\right) \\
\frac{1}{r}v_{tt} &= c^2\left(\frac{2}{r^3}v - \frac{2}{r^2}v_r + \frac{1}{r}v_{rr} - \frac{2}{r^3}v + \frac{2}{r^2}v_r\right) \\
\frac{1}{r}v_{tt} &= c^2\left(\frac{1}{r}v_{rr}\right) \\
v_{tt} &= c^2 v_{rr}
\end{aligned}$$

(b)

$$\begin{aligned}
v(r,t) &= f(r+ct) + g(r-ct) \\
u &= \frac{1}{r}v \\
u(r,t) &= \frac{1}{r}[f(r+ct) + g(r-ct)]
\end{aligned}$$

(c)

$$\begin{aligned}
u(r,0) &= \phi(r) \qquad u_t(r,0) = \psi(r) \\
v(r,0) &= r\phi(r) \qquad v_t(r,0) = r\psi(r) \\
v(r,t) &= \frac{1}{2}[(r+ct)\phi(r+ct) + (r+ct)\phi(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} s\psi(s)ds \\
u(r,t) &= \frac{1}{2r}[(r+ct)\phi(r+ct) + (r+ct)\phi(r-ct)] + \frac{1}{2cr}\int_{r-ct}^{r+ct} s\psi(s)ds
\end{aligned}$$

9.

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0 \qquad u(x,0) = x^2 \qquad u_t(x,0) = e^x$$
$$(\partial_x - 4\partial_t)(\partial_x + \partial_t)u = 0$$

Change of variables: $\partial_\xi = \partial_x - 4\partial_t \qquad \partial_\eta = \partial_x + \partial_t$

$$\Rightarrow x = \xi + \eta \qquad t = -4\xi + \eta$$
$$\Rightarrow \xi = x - \eta = -\frac{t-\eta}{4} \Rightarrow x = \frac{5\eta - t}{4} \Rightarrow \eta = \frac{4x+t}{5}$$
$$\xi = x - \frac{4x+t}{5} = \frac{x-t}{5}$$

Change of variables: $\dfrac{dx}{dt} = -\dfrac{1}{4} \qquad \dfrac{dx}{dt} = 1$

$$\Rightarrow x = -\frac{1}{4}t + \xi \qquad x = t + \eta$$
$$\Rightarrow \xi = x + \frac{1}{4}t \qquad \eta = x - t$$
$$\Rightarrow \partial_x = \partial_\xi + \partial_\eta \qquad \partial_t = \frac{1}{4}\partial_\xi - \partial_\eta$$
$$\Rightarrow \partial_x - 4\partial_t = 5\partial_\eta \qquad \partial_x + \partial_t = \frac{5}{4}\partial_\xi$$

So we have $u = f(\xi) + g(\eta) = f(x + \frac{1}{4}t) + g(x-t)$

$$u(x,0) = f(x) + g(x) = x^2 \Rightarrow u_t(x,0) = f'(x) + g'(x) = 2x$$
$$u_t(x,0) = \frac{1}{4}f'(x) - g'(x) = e^x$$
$$\frac{5}{4}f'(x) = 2x + e^x \Rightarrow f(s) = \frac{4}{5}(e^s + s^2)$$
$$5g'(x) = 2x - 4e^x \Rightarrow g(s) = \frac{1}{5}(s^2 - 4e^s)$$
$$u(x,t) = \frac{4}{5}\left[e^{(x+\frac{1}{4}t)} + \left(x + \frac{1}{4}t\right)^2\right] + \frac{1}{5}\left[(x-t)^2 - 4e^{x-t}\right]$$
$$= \frac{4}{5}[e^{x+\frac{1}{4}t} - e^{x-t}] + \frac{1}{5}\left[4x^2 + 2xt + \frac{1}{4}t^2 + x^2 - 2xt + t^2\right]$$
$$= \frac{4}{5}[e^{x+\frac{1}{4}t} - e^{x-t}] + \frac{1}{5}\left[5x^2 + \frac{5}{4}t^2\right]$$
$$= \frac{4}{5}[e^{x+\frac{1}{4}t} - e^{x-t}] + x^2 + \frac{1}{4}t^2$$

B. Homework Exercises

1.) (IVP for Wave Equation)

$$
\begin{aligned}
u_{tt} - 10u_{xt} + 3u_{xx} &= 0 \qquad u(x,0) = e^{-x^2} \qquad u_t(x,0) = s\sin(x) \\
D = (-5)^2 - 3 &= 22 > 0 \\
(\partial_t - 5\partial_x)^2 u - (\sqrt{22}\partial_x)^2 u &= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Change of variables: } \partial_\xi &= \partial_t - 5\partial_x \qquad \partial_\eta = \sqrt{22}\partial_x \\
\Rightarrow x &= -5\xi + \sqrt{22}\eta \qquad t = \xi \\
\Rightarrow \xi &= t \qquad \eta = \frac{x+5t}{\sqrt{22}}
\end{aligned}
$$

$$
\begin{aligned}
\text{So we have a simplified PDE: } u_{\xi\xi} - u_{\eta\eta} &= 0 \\
\text{The general solution yields: } u(\xi,\eta) &= f(\xi+\eta) + g(\xi+\eta) \\
\text{Following d'Alembert's formula with IVP: } u(\xi,\eta) &= \frac{1}{2}\left[e^{-(\xi+\eta)^2} + e^{-(\xi-\eta)^2}\right] + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} 2\sin(s)ds \\
&= \frac{1}{2}\left[e^{-(\xi+\eta)^2} + e^{-(\xi-\eta)^2}\right] + (-\cos(s)\Big|_{\xi-\eta}^{\xi+\eta} \\
&= \frac{1}{2}\left[e^{-(\xi+\eta)^2} + e^{-(\xi-\eta)^2}\right] + (\cos(\xi-\eta) - \cos(\xi+\eta)) \\
&= \frac{1}{2}\left[e^{-(\xi+\eta)^2} + e^{-(\xi-\eta)^2}\right] - 2\sin(\xi)\sin(-\eta)
\end{aligned}
$$

$$
\begin{aligned}
&\text{Substituting } x \text{ and } t \text{ back in:} \\
&u(x,t) = \frac{1}{2}\left[e^{-\left(t+\frac{x+5t}{\sqrt{22}}\right)^2} + e^{-\left(t-\frac{x+5t}{\sqrt{22}}\right)^2}\right] - 2\sin(t)\sin\!\left(-\frac{x+5t}{\sqrt{22}}\right)
\end{aligned}
$$

3.) (BSM to Diffusion Transformation)

Proof.

$$0 = \frac{\partial V}{\partial t} + rs\frac{\partial V}{\partial s} + \frac{\sigma^2 s^2}{2}\frac{\partial^2 V}{\partial s^2} - rV$$

$$\text{Given } V(s,t) = e^{-r\tau}G(x,y) \qquad y(t) = \frac{\sigma^2\tau}{2} \qquad x(s,t) = ln(s) + \gamma\tau$$

$$\tau = T - t \qquad \gamma = r - \frac{\sigma^2}{2}$$

$$\frac{\partial V}{\partial t} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial t} = -\gamma e^{-r\tau}G_x - \frac{\sigma^2}{2}e^{-r\tau}G_y$$

$$\frac{\partial V}{\partial s} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial V}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{s}e^{-r\tau}G_x$$

$$\frac{\partial^2 V}{\partial s^2} = \frac{\partial}{\partial s}\left[\frac{1}{s}e^{-r\tau}G_x\right] = \frac{1}{s^2}e^{-r\tau}G_{xx} - \frac{1}{s^2}e^{-r\tau}G_x$$

Substituting back into BSM:

$$\begin{aligned}
0 &= -\gamma e^{-r\tau}G_x - \frac{\sigma^2}{2}e^{-r\tau}G_y + rs\left[\frac{1}{s}e^{-r\tau}G_x\right] + \frac{\sigma^2 s^2}{2}\left(\frac{1}{s^2}\left[e^{-r\tau}G_{xx} - e^{-r\tau}G_x\right]\right) \\
&= -\gamma G_x - \frac{\sigma^2}{2}G_y + rs\left[\frac{1}{s}G_x\right] + \frac{\sigma^2 s^2}{2}\left(\frac{1}{s^2}\left[G_{xx} - G_x\right]\right) \\
&= -\gamma G_x - \frac{\sigma^2}{2}G_y + rG_x + \frac{\sigma^2}{2}\left[G_{xx} - G_x\right] \\
&= -\gamma G_x - \frac{\sigma^2}{2}G_y + \gamma G_x + \frac{\sigma^2}{2}G_{xx} \\
&= -\frac{\sigma^2}{2}G_y + \frac{\sigma^2}{2}G_{xx} \\
&= -G_y + G_{xx} \\
G_y - G_{xx} &= 0
\end{aligned}$$

□