Homework 2

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Section 2

2. $(a * b) * c = b * c = a$
   $a * (b * c) = a * a = a$
   We cannot say whether $*$ is associative based solely on this computation since this case is associative, but there are many more computations that can be made under $*$

3. $(b * d) * c = e * c = a$
   $b * (d * c) = b * b = c$
   We can say that $*$ is not associative since $(b * d) * c \neq b * (d * c)$

4. $*$ is not commutative because $b*e = c$ and $e*b = b$. This also makes the table defining $*$ asymmetrical.

5.

$*$	a	b	c	d
a	a	b	c	d
b	b	d	a	c
c	c	a	d	b
d	d	c	b	a

6.

$*$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	c	d
d	d	d	d	d

7. Not commutative: $3 - 1 \neq 1 - 3$
   Not associative: $3 - (1 - 2) = 4 \neq 0 = (3 - 1) - 2$

8. Commutative: $\forall a, b \in \mathbb{Q}$, $ab = ba$, so $ab + 1 = ba + 1$
   Not associative: $1 * (3 * 2) = 1 * 7 = 8 \neq 9 = 4 * 2 = (1 * 3) * 2$

12. $|S| = 1 \to 1$ operation
    $|S| = 2 \to 16$ operations
    $|S| = 3 \to 3^9$ operations
    $|S| = n \to n^{n^2}$ operations

14. A binary operation $*$ is commutative if and only if $a * b = b * a$ for all $a, b \in S$.

15. A binary operation $*$ on a set S is associative if and only if, for all $a, b, c \in S$, we have $(a*b)*c = a*(b*c)$

16. A subset H of a set S is closed under a binary operation $*$ on S if and only if $(a*b) \in H$ for all $a, b \in H$

17. $a * b = a - b$ fails condition 2 because $2 - 4 = -2 \notin \mathbb{Z}^+$ so it is not a binary operation.

18. $a * b = a^b$ meets both conditions and is therefore a binary operation.

23. (a) H is closed under matrix addition

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} + \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} a+c & -(b+d) \\ b+d & a+c \end{bmatrix}$$

(b) H is closed under matrix multiplication

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \cdot \begin{bmatrix} c & -d \\ d & c \end{bmatrix} = \begin{bmatrix} ac - bd & -(bc + ad) \\ bc + ad & ac - bd \end{bmatrix}$$

24. (a) False, $a * a$ could equal another element of S
    (b) True, $b * c$ is just another element of S so commutativity holds
    (c) False, $a * (b * c) = (b * c) * a$ uses the commutative property so this is not necessarily true
    (d) False, we can define binary operations on sets of nearly anything
    (e) False, commutativity must hold for all elements of S
    (f) True, if $S = \{a\}$ then $a * a = a$ is the only binary operation which is clearly commutative and associative
    (g) True, a binary operation assigns one but no more than one element of S to each ordered pair
    (h) True, a binary operation assigns one but no fewer than one element of S to each ordered pair
    (i) True, a binary operation assigns exactly one element of S to each ordered pair
    (j) False, no more than one element of S can be assigned to each ordered pair

Section 3

1. We must check that ϕ is one to one, ϕ is onto S' and the homomorphism property ($\phi(x * y) = \phi(x) *' \phi(y)\ \forall x, y \in S$)

2. Is an isomorphism.
   ϕ is one to one because it is strictly decreasing
   ϕ is onto because the inverse is just $\phi'(n) = -n$
   $\phi(a + b) = -a - b = \phi(a) + \phi(b)$

3. Not an isomorphism.
   ϕ is not onto since there is no way to multiply an integer by 2 and get an odd integer

4. Not an isomorphism.
   $\phi(a + b) = a + b + 1 \neq a + b + 2 = \phi(a) + \phi(b)$ so homomorphism property fails

6. Not an isomorphism.
   ϕ is not onto since a square can never produce a negative rational number

7. Is an isomorphism.
   ϕ is one to one since it is strictly increasing
   ϕ is onto because the inverse is $\phi'(x) = \sqrt[3]{n}$
   $\phi(a \cdot b) = (a \cdot b)^3 = a^3 \cdot b^3 = \phi(a) \cdot \phi(b)$

17. $\phi(n) = n + 1$
    (a) $\phi(a \cdot b) = a \cdot b + 1 = \phi(a) * \phi(b) = (a + 1) * (b + 1) = (a + 1) \cdot (b + 1) - (a + 1) - (b + 1) + 2$
        $a * b = ab - a - b + 2$
        identity element $= 2$
    (b) $\phi(a * b) = a * b + 1 = \phi(a) \cdot \phi(b) = (a + 1) \cdot (b + 1) = (ab + a + b) + 1$
        $a * b = ab + a + b$
        identity element $= 0$

20. An isomorphism is necessarily commutative, so $a * b = b * a$, which applied to the homomorphism condition means that $\phi(a) *^{'} \phi(b) = \phi(b) *^{'} \phi(a)$

21. A function $\phi : S \to S'$ is an isomorphism if and only if ϕ is one to one and onto, and $\phi(a*b) = \phi(a)*'\phi(b)$

22. Let $*$ be a binary operation on a set S. An element e of S with the property $s * e = s = e * s$ is the identity element for $*$ for all $s \in S$

23. Considering two different identity elements of a set, $e, \bar{e}$, their property as identity elements means that $e * \bar{e} = e, \bar{e}$. They must then be equal.

26. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$, then ϕ^{-1} is an isomorphism of $\langle S', *' \rangle$ with $\langle S, * \rangle$

Proof.

By nature of being an isomorphism, ϕ must be one to one and onto S and S'
Then ϕ is a bijection and we can say that ϕ' is also a bijection
To show ϕ' has the homomorphism property, we let $\phi(a) = a'$ and $\phi(b) = b'$
We already know that $\phi(a * b) = a' *' b'$
Applying ϕ^{-1} to both sides, we get $\phi^{-1}(\phi(a * b)) = \phi^{-1}(a' *' b')$
$\phi{-}1(\phi(a * b))$ simplifies, leaving $a * b = \phi^{-1}(a' *' b')$
Applying this same logic on our mapping means that $a = \phi^{-1}(a')$ and $b = \phi^{-1}(b')$
Substituting those, we end with the homomorphism property on ϕ^{-1}, $\phi^{-1}(a') * \phi^{-1}(b') = \phi^{-1}(a' *' b')$

□

27. Prove that if $\phi : S \to S'$ is an isomorphism of $\langle S, * \rangle$ with $\langle S', *' \rangle$ and $\psi : S \to S'$ is an isomorphism of $\langle S', *' \rangle$ with $\langle S'', *'' \rangle$, then $\psi \circ \phi$ is an isomorphism of $\langle S, * \rangle$ with $\langle S'', *'' \rangle$

Proof.

Since ψ, ϕ are both isomorphisms, they are bijections
So, $\forall a \in S$, a must have a corresponding element in S' and from there, S''
So, $\psi \circ \phi$ is also bijective
Lets define $\phi(a) = a'$ and $\psi(a') = a''$ along with $\phi(b) = b'$ and $\psi(b') = b''$
So, $\psi(\phi(a * b)) = \psi(\phi(a) *' \phi(b)) = \psi(a' *' b') = \psi(a') *'' \psi(b') = a'' *'' b''$
Then $\psi \circ \phi$ has the homomorphism property and is an isomorphism

□

28. Prove that the relation of being isomorphic is an equivalence relation on any set of binary structures

Proof.

Any binary structure must be isomorphic with itself since ϕ would just map elements of S to themselves
Using our proof from (26), for binary operations $*, *'$ we know that, $* \sim *'$ and $*' \sim *$
Using our proof from (27), for binary operations $*, *', *''$ we know that given $* \sim *'$ and $*' \sim *''$, $* \sim *''$
Then isomorphism covers the reflexive, symmetric and transitive properties of an equivalence relation

□

29. Prove that $*$ is commutative is a structural property of a binary operation

Proof.

Consider an isomorphism ϕ from $\langle S, * \rangle$ to $\langle S', *' \rangle$
We know S maps one to one and onto S' and $\langle S, * \rangle$ is commutative
Then we can say for $a, b \in S$, $a * b = b * a$ and therefore $\phi(a * b) = \phi(b * a)$
Since ϕ is an isomorphism, letting $\phi(a) = a'$ and $\phi(b) = b'$, we get $\phi(a * b) = \phi(a) *' \phi(b) = a' *' b'$
But we also have $\phi(b * a) = \phi(b) *' \phi(a) = b' *' a'$
Then $a' *' b' = b' *' a'$, making $\langle S', *' \rangle$ commutative

□