Homework 3

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D. Homework Exercises

(1) Put-Call-Parity

(i)

$$\begin{aligned}
V_0^{ECO} &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right] \\
\tilde{p} &= \frac{(1.05 - 0.8)}{1.2 - 0.8} = \frac{0.25}{0.4} = 0.625 \\
\tilde{q} &= \frac{1.2 - 1.05}{1.2 - 0.8} = \frac{0.15}{0.4} = 0.375 \\
V_0^{ECO} &= \frac{1}{1.05}[0.625 \cdot 10 + 0.375 \cdot 0] = \frac{6.25}{1.05} = \$5.95
\end{aligned}$$

(ii)

$$\begin{aligned}
V_0^{EPO} &= \frac{1}{1.05}[0.625 \cdot 0 + 0.375 \cdot 30] \\
&= \frac{11.25}{1.05} = \$10.71
\end{aligned}$$

(iii)

$$PV(K) = \frac{110}{1.05} = \$104.76$$

(iv)

$$\begin{aligned}
100 + 10.71 - 5.95 &= 104.76 \\
104.76 &= 104.76
\end{aligned}$$

(2) Put-Call-Parity

$$\begin{aligned}
u &= \frac{S_1(H)}{S_0} \\
d &= \frac{S_1(T)}{S_0} \\
u - d &= \frac{S_1(H) - S_1(T)}{S_0} \\
V_0^{ECO} &= \frac{1}{1+r}\tilde{p}V_1(H) = \frac{1}{1+r}\left[\frac{(1+r) - d}{u - d}(S_1(H) - K)\right] \\
&= \frac{(S_1(H) - K)((1+r) - d)}{(1+r)(u-d)} \\
V_0^{EPO} &= \frac{1}{1+r}\tilde{q}V_1(T) = \frac{1}{1+r}\left[\frac{u - (1+r)}{u - d}(K - S_1(T))\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(K - S_1(T))(u - (1+r))}{(1+r)(u-d)} \\
V_0^{EPO} - V_0^{ECO} &= \frac{(K - S_1(T))(u - (1+r)) - (S_1(H) - K)((1+r) - d)}{(1+r)(u-d)} \\
&= \frac{(Ku - S_1(T)u - (1+r)K + (1+r)S_1(T)) - ((1+r)S_1(H) - (1+r)K - S_1(H)d + Kd)}{(1+r)(u-d)} \\
&= \frac{Ku - Kd + S_1(H)d - S_1(T)u + (1+r)S_1(T) - (1+r)S_1(H)}{(1+r)(u-d)} \\
&= \frac{K(u-d) + S_1(H)d - S_1(T)u + (1+r)(S_1(T) - S_1(H))}{(1+r)(u-d)} \\
&= \frac{K}{1+r} + \frac{S_1(H)d - S_1(T)u}{(1+r)(u-d)} + \frac{S_1(T) - S_1(H)}{u-d} \\
PV(K) &= \frac{K}{1+r}
\end{aligned}
$$

$$
\begin{aligned}
PV(K) &= S_0 + V_0^{EPO} - V_0^{ECO} \\
\frac{K}{1+r} &= S_0 + \frac{K}{1+r} + \frac{S_1(H)d - S_1(T)u}{(1+r)(u-d)} + \frac{S_1(T) - S_1(H)}{u-d} \\
0 &= S_0 + \frac{S_1(H)d - S_1(T)u}{(1+r)(u-d)} + \frac{S_1(T) - S_1(H)}{u-d} \\
&= S_0(u-d) + \frac{S_1(H)d - S_1(T)u}{(1+r)} + S_1(T) - S_1(H) \\
&= S_1(H) - S_1(T) + \frac{S_1(H)d - S_1(T)u}{(1+r)} + S_1(T) - S_1(H) \\
&= \frac{S_1(H)d - S_1(T)u}{(1+r)} = S_1(H)d - S_1(T)u \\
&= S_1(H)\frac{S_1(T)}{S_0} - S_1(T)\frac{S_1(H)}{S_0} = 0
\end{aligned}
$$

(3a) Exotic Option in a 3-Binomial Model

$$
\begin{aligned}
u &= \frac{S_1(H)}{S_0} = \frac{220}{110} = 2 \\
d &= \frac{S_1(T)}{S_0} = \frac{80}{110} = 0.727272 \\
\tilde{p} &= \frac{1.05 - 0.727272}{2 - 0.727272} = \frac{0.322727}{1.272727} = 0.25357 \\
\tilde{q} &= 1 - \tilde{p} = 0.74643
\end{aligned}
$$

- $S_0 = 110$, $V_0 = 129.31$, $X_0 = 129.31$, $\Delta_0 = 1.5755$
  - $S_1(H) = 220$, $V_1(H) = 300.42$, $X_1(H) = 300.42$, $\Delta_1(H) = 1.34039$
    - $S_2(HH) = 440$, $V_2(HH) = 595.58$, $X_2(HH) = 595.58$, $\Delta_2(HH) = 0.17857$
      - $S_3(HHH) = 880$, $V_3(HHH) = 700$, $X_3(HHH) = 700$
      - $S_3(HHT) = 320$, $V_3(HHT) = 600$, $X_3(HHT) = 600$
    - $S_2(HT) = 160$, $V_2(HT) = 220.27$, $X_2(HT) = 220.27$, $\Delta_2(HT) = 1.76783$
      - $S_3(HTH) = 320$, $V_3(HTH) = 500$, $X_3(HTH) = 500$
      - $S_3(HTT) = 116.36$, $V_3(HTT) = 140$, $X_3(HTT) = 140$
  - $S_1(T) = 80$, $V_1(T) = 79.85$, $X_1(T) = 79.85$, $\Delta_1(T) = 0.93538$
    - $S_2(TH) = 160$, $V_2(TH) = 154.93$, $X_2(TH) = 154.93$, $\Delta_2(TH) = 0.24553$
      - $S_3(THH) = 320$, $V_3(THH) = 200$, $X_3(THH) = 200$
      - $S_3(THT) = 116.36$, $V_3(THT) = 150$, $X_3(THT) = 150$
    - $S_2(TT) = 58.18$, $V_2(TT) = 59.69$, $X_2(TT) = 59.69$, $\Delta_2(TT) = 0.67522$
      - $S_3(TTH) = 116.36$, $V_3(TTH) = 100$, $X_3(TTH) = 100$
      - $S_3(TTT) = 42.31$, $V_3(TTT) = 50$, $X_3(TTT) = 50$

(3b) European Call Option

$$\begin{aligned}
V_2(HH) &= \frac{1}{1.05}[0.25357 \cdot (880 - 380)] = \frac{126.785}{1.05} = 120.75 \\
V_2(HT) &= \frac{1}{1.05}[0] = 0 \\
V_2(TH) &= \frac{1}{1.05}[0] = 0 \\
V_2(TT) &= \frac{1}{1.05}[0] = 0 \\
V_1(H) &= \frac{1}{1.05}[0.25357 \cdot 120.75] = \frac{30.62}{1.05} = 29.16 \\
V_1(T) &= 0 \\
V_0 &= \frac{1}{1.05}[0.25357 \cdot 29.16] = \frac{7.39}{1.05} = \$7.04
\end{aligned}$$

(3c) European Put Option

$$\begin{aligned}
V_1(H) &= \frac{1}{1.05}[0] = 0 \\
V_1(T) &= \frac{1}{1.05}[0.74643 \cdot (120 - 58.18)] = \frac{46.14}{1.05} = 43.95
\end{aligned}$$

$$V_0 = \frac{1}{1.05}[0.74643 \cdot 43.95] = \frac{32.80}{1.05} = \$31.24$$

(4) Buy Low or Sell High Option

In the case the stock price becomes $S_1(H)$ we will buy it at $S_1(T)$ and sell it for $S_1(H)$, yielding a profit of $S_1(H) - S_1(T)$. In the case the stock price becomes $S_1(T)$, we will once again buy it at $S_1(T)$ and sell it for $S_1(H)$, so that in either case, we profit $S_1(H) - S_1(T)$. Then the price of the option is this profit adjusted for interest, that is, $V_0 = \frac{S_1(H)-S_1(T)}{1+r}$.

(5) Risk Hedging With Puts

(i)

$$V_0^{EPO} = \frac{1}{1.02}[0.4 \cdot (500 - 150)] = \frac{140}{1.02} = \$137.25$$
$$100 \cdot 137.25 = \$13,725.49$$

(ii)

$$V_0^{EPO} = \frac{1}{1.02}[0.4 \cdot (400 - 150)] = \frac{100}{1.02} = \$98.04$$
$$100 \cdot 98.04 = \$9,803.92$$