Homework 7

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1. (a) *Proof.* Assume there is a TM R, which decides whether an input TM M attempts to move its head past the left end of the tape on input string w. Let S be a "universal decider" that implements R to decide A_{TM}.
   $S =$ on input $\langle M, r\rangle$:
   i. Convert $\langle M, r\rangle$ into a program P
   ii. Run R on P
   iii. If R accepts, then accept and if R rejects, then reject

   So we have a universal decider S, which we know cannot exist, so it is undecidable whether M attempts to move its head past the left end of the tape on input string w. □

   (b) *Proof.* Consider a Turing machine R that decides whether on input w to TM M, whether or not the head of M moves past the right end of w. Refer to the following implementation of R:
   i. Start R with the input string w
   ii. Move the head to the end of w and append a special symbol, call it #
   iii. Move the head back to the start of w
   iv. Simulate the rules of M
   v. If the head of R reads # at any point during the simulation, then accept
   vi. If the simulation runs through without reading #, then reject

   □

2. (a) *Proof.* Consider a Turing machine R that decides L_2. Refer to the following implementation of R on input $\langle M, \langle M\rangle\rangle$:
   i. If C is the TM that decides whether a TM is 10-compliant, simulate C
   ii. If C rejects, then reject
   iii. If C accepts, move the head of R to the start of $\langle M\rangle$
   iv. Simulate M on the remainder of the input string
   v. If M accepts, then reject
   vi. If M rejects, then accept

   □

   (b) *Proof.* Suppose our TM R from part (a) decides L_2. Then R should be able to decide if $\langle R\rangle$ is in L_2, so let's run R on the input $\langle R, \langle R\rangle\rangle$. In the first three steps of R, we run into two cases:
   i. If R is not 10-compliant then we are done
   ii. If R is 10-compliant, then we move to the next three steps of R.

   Again, we have two cases, either R accepts $\langle R\rangle$ or it doesn't. However, if R accepts $\langle R\rangle$, then it rejects $\langle R\rangle$ and if it rejects $\langle R\rangle$, then it accepts $\langle R\rangle$. Here we have a contradiction, so we know that R cannot be 10-compliant since R decides L_2. □

   (c) Steps 5 and 6 reverse the acceptance/rejection of M, which when paired with an input of R and $\langle R\rangle$ causes a contradiction since R cannot both accept and reject the same input.

3. (a) *Proof.* Let the property P described by Rice's theorem be recognizing Σ^*. Then we can see that P is nontrivial because a TM could recognize some language $L \neq \Sigma^*$ or it could recognize Σ^*, so it is neither always true nor always false. Then by Rice's theorem, it is undecidable that a TM recognizes Σ^*. $\square$

   (b) *Proof.* Both conditions of Rice's theorem are necessary:

   i. If P is trivial, that is, P is always false or P is always true, then we can construct a TM M that decides P quite easily:

      T. If P is always true, then on any input to M, accept.

      F. If P is always false, then on any input to M, reject.

   ii. An example of a property of TM's that is decidable is the example of 10-compliance from question (2). Indeed, it is decidable whether a TM is 10-compliant.

   $\square$