Homework 2

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Section 2.1

2.1.5

Suppose that $\{a_n\}$ is a convergent sequence. Prove by definition that $\{a_n^2\}$ is a convergent sequence. Provide an example showing that the converse is not necessarily true.

Proof. If $a_n \to a$ then let $|a_n - a| < \frac{\epsilon}{2(|a|+1)}$ for $n > N$. This also means that $|a_n| \leq |a| + 1$ Then let's assume that $a_n^2 \to a^2$.

$$\begin{aligned}
|a_n^2 - a^2| &= |a_n^2 - a^2 + a_n a - a_n a| \\
|a_n(a_n - a) + a(a_n - a)| &\leq |a_n||a_n - a| + |a||a_n - a| \\
&< (|a| + 1)\frac{\epsilon}{2(|a| + 1)} + |a|\frac{\epsilon}{2(|a| + 1)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Then we have show that $a_n^2 \to a^2$ and thus it converges. □

The converse of this statement is if $\{a_n^2\}$ is a convergent sequence then $\{a_n\}$ is a convergent sequence. Consider $a_n = (-1)^n$ and $a_n^2 = 1$. All terms in a_n^2 are 1 and therefore it converges, however we obviously know the alternating sequence $a_n = (-1)^n$ does not.

2.1.8

Let $\{a_n\}$ be a sequence of positive numbers. Then, $\{a_n\}$ diverges to ∞ if and only if $\{\frac{1}{a_n}\}$ converges to zero.

Proof. We will prove both directions, first that if $\{a_n\}$ diverges to ∞ then $\{\frac{1}{a_n}\}$ converges to zero. Since $\{a_n\}$ diverges to ∞, by definition, for any M, $\exists N$ s.t. $a_n > M$ for $n > N$. Then we can divide both terms over to say $\frac{1}{M} > \frac{1}{a_n}$. So $|\frac{1}{a_n} - 0| = |\frac{1}{a_n}| = \frac{1}{a_n} < \frac{1}{M}$. Then, if we take $\epsilon = \frac{1}{M}$, we have shown that $\{\frac{1}{a_n}\}$ converges to zero.

Now we need to prove the other direction, that if $\{\frac{1}{a_n}\}$ converges to zero then $\{a_n\}$ diverges to ∞. By definition, $\frac{1}{a_n} \to 0 \Rightarrow |\frac{1}{a_n} - 0| < \epsilon \Rightarrow |\frac{1}{a_n}| < \epsilon$ for $n > N$. Then $\frac{1}{\epsilon} < a_n$ and we take $M = \frac{1}{\epsilon}$ to satisfy the definition of divergence to ∞. □

2.1.9

Construct sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to 0$, $b_n \to 0$, and...

(a)

... $\frac{a_n}{b_n} \to \alpha$ for some α

Taking $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n}$ it is trivial to see that both converge to 0. But $\frac{a_n}{b_n} = \frac{\frac{1}{n}}{\frac{1}{n}} = \frac{n}{n} = 1$, so $\frac{a_n}{b_n} \to \alpha$ where $\alpha = 1$.

(b)

... $\frac{a_n}{b_n} \to \infty$

If we take $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$ it is trivial to see that both converge to 0. $\frac{a_n}{b_n} = \frac{\frac{1}{n}}{\frac{1}{n^2}} = \frac{n^2}{n} = n$, which diverges to ∞.

(c)

... $\frac{a_n}{b_n} \to -\infty$

Similar to (b), If we take $a_n = -\frac{1}{n}$ and $b_n = \frac{1}{n^2}$ it is trivial to see that both converge to 0. $\frac{a_n}{b_n} = \frac{-\frac{1}{n}}{\frac{1}{n^2}} = -\frac{n^2}{n} = -n$, which diverges to $-\infty$.

2.1.10

Construct sequences $\{a_n\}$ and $\{b_n\}$ such that $a_n \to \infty$, $b_n \to \infty$, and...

(a)

... $a_n - b_n \to \alpha$ for some α

If we take $a_n = n+1$ and $b_n = n$, both clearly diverge to ∞, but $a_n - b_n = n + 1 - n = 1$, so $a_n - b_n \to \alpha$ where $\alpha = 1$.

(b)

... $a_n - b_n \to \infty$

If we take $a_n = 2n$ and $b_n = n$, both clearly diverge to ∞, but $a_n - b_n = 2n - n = n$, so $a_n - b_n$ still diverges to ∞.

(c)

... $a_n - b_n \to -\infty$

Similarly to (b), If we take $a_n = n$ and $b_n = 2n$, both clearly diverge to ∞, but $a_n - b_n = n - 2n = -n$, so $a_n - b_n$ diverges to $-\infty$.

(d)

... $a_n - b_n$ does not converge nor diverge to $\pm\infty$

If we take $a_n = n(\sin^2 n + 1)$ and $b_n = n$, we know b_n diverges to ∞ and $a_n \geq n$ so it also diverges to ∞. However, $a_n - b_n = n\sin^2 n + n - n = n \sin^2 n$ which oscillates between 0 and n, thus never converging nor diverging to $\pm\infty$.

Section 2.2

2.2.1

Let $\{a_n\}$ be a monotone sequence of real numbers. Assume a subsequence of $\{a_n\}$ converges to a. Prove that $\{a_n\}$ converges to a.

Proof. Suppose we have a subsequence of a_n, a_{n_k} which consists of terms $a_{n_1}, a_{n_2}, \ldots a_{n_k}$ and converges to a. Then if we add all a_n terms s.t. $n < n_1$, the new subsequence still converges to a. If we then add all a_n terms s.t. $n_1 < n < n_2$, because the sequence is monotone, all of these terms are between a_{n_1} and a_{n_2} and thus the new subsequence still converges to a. We can continue this process until $a_{n_k} = a_n$ and thus $a_n \to a$. $\square$

2.2.2

Let c be a positive number. Define $a_1 = \sqrt{c}$ and for $n \geq 1$, $a_{n+1} = \sqrt{c + a_n}$. Prove that $\{a_n\}$ converges and find its limit. (Hint: Prove that $\{a_n\}$ is increasing and bounded above.)

Proof. Since $c > 0$, all $a_n > 0$. Let's prove a_n is increasing by induction.

Base case: $n = 2$. Since $a_1 = \sqrt{c}$ and $a_2 = \sqrt{c + \sqrt{c}}$, we can see that $a_1^2 = c \leq c + \sqrt{c} = a_2^2$ which holds since $\sqrt{c} > 0$.
Inductive step: $n = k + 1$. Suppose $a_k \geq a_{k-1}$, then we know that $a_{k+1} = \sqrt{c + a_k}$ and $a_k = \sqrt{c + a_{k-1}}$. So $a_{k+1}^2 = c + a_k \geq c + a_{k-1} = a_k^2$. The c's cancel and we know this holds by induction since $a_k \geq a_{k-1}$. So a_n is an increasing sequence.

Since a_n is monotonically increasing, let's suppose it converges to a. Then the subsequence a_{n+1} must converge to a as well. So, $a_{n+1} = \sqrt{c + a_n} \Rightarrow a = \sqrt{c + a} \Rightarrow a^2 - a - c = 0$. The roots of this equation are $\frac{1 \pm \sqrt{1+4c}}{2}$. But we know the sequence is positive and $\sqrt{1+4c} > 1$, so $a = \frac{1+\sqrt{1+4c}}{2}$. Notice that $\frac{1+\sqrt{1+4c}}{2} < \frac{1+\sqrt{9c}}{2} \leq \frac{\sqrt{c}+\sqrt{9c}}{2} = 2\sqrt{c}$. Let's prove that a_n is bounded above by this value by induction.

Base case: $n = 1$. $a_1 = \sqrt{c} < 2\sqrt{c}$.
Inductive step: $n = k+1$. Assume a_n is bounded above by $1 + \frac{3}{2}\sqrt{c}$ for all $n \leq k$. Then $a_{k+1} = \sqrt{c + a_k}$. By inductive hypothesis, $\sqrt{c + 2\sqrt{c}} = \sqrt{\sqrt{c}\sqrt{c} + 2\sqrt{c}} = \sqrt{\sqrt{c}(\sqrt{c}+2)} \leq \sqrt{\sqrt{c}(\sqrt{c} + 2\sqrt{c})} = \sqrt{3c} = \sqrt{3}\sqrt{c} < 2\sqrt{c}$ since $\sqrt{3} \approx 1.732 < 2$.

So a_n is monotonically increasing and bounded above, thereby proving the existence of $a = \frac{1+\sqrt{1+4c}}{2}$ as its limit. $\square$

2.2.3

Let c be a positive number. Define $a_1 = c$ and for $n \geq 1$, $a_{n+1} = \frac{a_n^2 + c}{2a_n}$. Prove that $\{a_n\}$ converges and find its limit.

Proof. Since $c > 0$, all $a_n > 0$. Let's prove a_n is decreasing by induction.

Base case: $n = 2$. $a_1 = c$ and $a_2 = \frac{c^2+c}{2c} = \frac{c+1}{2} = \frac{c}{2} + \frac{1}{2} \leq c$ since $c \geq 1$.
Inductive step: $n = k+1$. Assume $a_k \leq a_{k-1}$. Then $a_{k+1} = \frac{a_k^2+c}{2a_k} = \frac{a_k}{2} + \frac{c}{2a_k}$ and $a_k = \frac{a_{k-1}^2+c}{2a_{k-1}} = \frac{a_{k-1}}{2} + \frac{c}{2a_{k-1}}$. So $a_{k+1} \leq a_k \Rightarrow \frac{a_k}{2} + \frac{c}{2a_k} \leq \frac{a_{k-1}}{2} + \frac{c}{2a_{k-1}} \Rightarrow \frac{c}{2a_k} - \frac{c}{2a_{k-1}} \leq \frac{a_{k-1}}{2} - \frac{a_k}{2} \Rightarrow c(a_k - a_{k-1}) \leq a_{k-1} - a_k$. Since $a_k \leq a_{k-1}$, by our inductive hypothesis, $c(a_k - a_{k-1}) \leq 0 \leq a_{k-1} - a_k$. So a_n is monotonically decreasing.

Since a_n is monotonically decreasing, let's suppose it converges to a. Then the subsequence a_{n+1} must converge to a as well. So, $a_{n+1} = \frac{a_n^2+c}{2a_n} \Rightarrow a = \frac{a^2+c}{2a} \Rightarrow 2a^2 = a^2 + c \Rightarrow a^2 = c \Rightarrow a = \sqrt{c}$. Since c is positive and a_n is monotonically decreasing, lets prove that a_n is bounded below by $\sqrt{c}$ by induction.

Base case: $n = 1$. $a_1 = c \geq \sqrt{c}$.
Inductive step: $n = k + 1$. Assume $a_k \geq \sqrt{c}$. Then $a_{k+1} = \frac{a_k^2+c}{2a_k}$. By inductive hypothesis $\frac{\sqrt{c}^2+c}{2\sqrt{c}} = \frac{2c}{2\sqrt{c}} = \sqrt{c} \geq \sqrt{c}$. So a_n is bounded below by $\sqrt{c}$ and thus by MCT, we have proved that the limit exists, so a_n converges to $a = \sqrt{c}$. $\square$

2.2.4

Let c be a fixed number with $0 < c < 1$. Prove by the monotone convergence theorem that $\lim_{n\to\infty} nc^n = 0$.

Proof. Observe that $(n+1)c^{n+1} = ncc^n + cc^n = (nc + c)c^n < nc^n$ when $nc + c < n$ so nc^n is monotone decreasing for large enough n. Since nc^n begins decreasing when $nc + c < n \Rightarrow c(n+1) < n \Rightarrow c < \frac{n}{n+1}$, then nc^n will be maximized when $c = \frac{n}{n+1}$. It is clear that nc^n is bounded below by 0 (since $n > 0$ and $c > 0$) so by MCT we know it converges. Suppose the limit of nc^n is l, then $a_{n+1} = (n+1)c^{n+1} = ncc^n + c^{n+1} = lc + c^{n+1} = l$. So, $c^{n+1} = (1-c)l \Rightarrow l = \frac{c^{n+1}}{1-c}$ and when we let $n \to \infty$, $l = \frac{0}{1-c} = 0$, so $nc^n \to 0$. $\square$