Homework 4

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E. Textbook Exercises

Exercise 8

$$
\begin{aligned}
d_1 &= \frac{\ln \frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln \frac{70}{80} + (0.06 + \frac{1}{2}0.3^2)2}{0.3\sqrt{2}} \\
&= \frac{\ln \frac{7}{8} + 0.21}{0.3\sqrt{2}} = 0.180238 \\
d_2 &= d_1 - \sigma\sqrt{t} = 0.180238 - 0.3\sqrt{2} = -0.24403 \\
V_0^{ECO} &= S_0 N(d_1) - e^{-rt} K N(d_2) \\
&= 70 \cdot 0.57151714 - 80e^{-0.12} \cdot 0.40360379 \\
&= \$11.37
\end{aligned}
$$

Exercise 15

$$
\begin{aligned}
V_0 &= e^{-rT}E(V_T) = e^{-rT}E(0.1S_T^2) \\
&= 0.1e^{-rT}E([S_0 e^{(r-\frac{1}{2}\sigma^2)T} e^{\sigma\sqrt{T}Z}]^2) \\
&= 0.1e^{-rT}E(S_0^2 e^{2(r-\frac{1}{2}\sigma^2)T} e^{2\sigma\sqrt{T}Z}) \\
&= 0.1e^{-rT}S_0^2 e^{2(r-\frac{1}{2}\sigma^2)T} E(e^{2\sigma\sqrt{T}Z}) \\
&= 0.1e^{-rT}S_0^2 e^{2(r-\frac{1}{2}\sigma^2)T} e^{2\sigma^2 T} \\
&= 0.1S_0^2 e^{(r+\sigma^2)T}
\end{aligned}
$$

F. Homework Exercises

(1) Options Pricing

(i)

$$
\begin{aligned}
V_0 &= e^{-rT}E(V_T) = e^{-rT}E(3S_T + 40) = e^{-rt}[3E(S_T) + E(40)] \\
&= e^{-rT}[3S_0 e^{(r-\frac{1}{2}\sigma^2)T} E(e^{\sigma\sqrt{T}Z}) + 40] \\
&= e^{-rT}[3S_0 e^{(r-\frac{1}{2}\sigma^2)T} e^{\frac{1}{2}\sigma T} + 40] \\
&= e^{-rt}[3S_0 e^{rT} + 40] = 3S_0 + 40e^{-rT}
\end{aligned}
$$

(ii)

$$\begin{aligned}
V_0 &= e^{-rT}E(V_T) = e^{-rT}E(S_T^3)\\
&= e^{-rT}E([S_0e^{(r-\frac{1}{2}\sigma^2)T}e^{\sigma\sqrt{T}Z}]^3)\\
&= e^{-rT}E(S_0^3e^{3(r-\frac{1}{2}\sigma^2)T}e^{3\sigma\sqrt{T}Z})\\
&= e^{-rT}S_0^3e^{3(r-\frac{1}{2}\sigma^2)T}E(e^{3\sigma\sqrt{T}Z})\\
&= e^{-rT}S_0^3e^{3(r-\frac{1}{2}\sigma^2)T}e^{\frac{9}{2}\sigma^2T}\\
&= S_0^3e^{(2r+3\sigma^2)T}
\end{aligned}$$

(iii)

$$\begin{aligned}
V_0 &= e^{-rT}E(V_T) = e^{-0.03\cdot 1.5}E(200)\\
&= e^{-0.045}\cdot 200 = 0.95599748183\cdot 200\\
&= \$191.20
\end{aligned}$$

(iv)

$$\begin{aligned}
d_1 &= \frac{\ln\frac{S_0}{K}+(r+\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{100}{110}+(0.03+\frac{1}{2}0.4^2)0.9}{0.4\sqrt{0.9}}\\
&= \frac{\ln\frac{10}{11}+(0.03+0.08)0.9}{0.4\sqrt{0.9}} = \frac{\ln\frac{10}{11}+0.099}{0.4\sqrt{0.9}}\\
&= \frac{0.00368982019}{0.379473} = 0.0097\\
d_2 &= d_1-\sigma\sqrt{T} = 0.0097-0.4\sqrt{0.9} = 0.0097-0.379473 = -0.37\\
V_0^{ECO} &= S_0N(d_1)-e^{-rT}KN(d_2)\\
&= 100N(0.0097)-e^{-0.03\cdot 0.9}110N(-0.37)\\
&= 100\cdot 0.5039-110e^{-0.027}\cdot 0.3558\\
&= 50.39-38.10 = \$12.29
\end{aligned}$$

(v)

$$\begin{aligned}
V_0^{ECO}+e^{-rT}K &= V_0^{EPO}+S_0\\
V_0^{EPO} &= V_0^{ECO}+e^{-rT}K-S_0\\
&= 12.29+110e^{-0.027}-100\\
&= 12.29+107.07-100\\
&= \$19.36
\end{aligned}$$

(2) Derivation of Δ

$$\begin{aligned}
\Delta &= \frac{\partial V_0}{\partial S_0}\\
V_0 &= S_0N(d_1)-e^{-rt}KN(d_2)\\
\Delta &= N(d_1)+S_0\frac{\partial N}{\partial d_1}\frac{\partial d_1}{\partial S_0}-Ke^{-rt}\frac{\partial N}{\partial d_2}\frac{\partial d_2}{\partial S_0}\\
\frac{\partial N}{\partial d_1} &= \frac{\partial}{\partial d_1}\int_0^x\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}dy = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}
\end{aligned}$$

$$\frac{\partial N}{\partial d_2} = \frac{\partial}{\partial d_2}\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}d_2^2}$$

$$\frac{\partial d_1}{\partial S_0} = \frac{\partial d_2}{\partial S_0} = \frac{\partial}{\partial S_0}\left[\frac{\ln\frac{S_0}{K} + (r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right] = \frac{\partial}{\partial S_0}\left[\frac{\ln\frac{S_0}{K}}{\sigma\sqrt{t}} + \frac{(r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}\right]$$

$$= \frac{1}{\sigma\sqrt{t}}\frac{\partial}{\partial S_0}\ln\frac{S_0}{K} = \frac{1}{\sigma\sqrt{t}}\frac{\partial}{\partial S_0}[\ln S_0 - \ln K] = \frac{1}{\sigma\sqrt{t}}\frac{1}{S_0} = \frac{1}{S_0\sigma\sqrt{t}}$$

$$\Delta = N(d_1) + S_0\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_1^2}\right]\left[\frac{1}{S_0\sigma\sqrt{t}}\right] - Ke^{-rt}\left[\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_2^2}\right]\left[\frac{1}{S_0\sigma\sqrt{t}}\right]$$

$$= N(d_1) + \frac{1}{S_0\sigma\sqrt{2\pi t}}\left[S_0 e^{-\frac{1}{2}d_1^2} - Ke^{-rt}e^{-\frac{1}{2}d_2^2}\right]$$

$$= N(d_1) + \frac{e^{-\frac{1}{2}d_1^2}}{S_0\sigma\sqrt{2\pi t}}\left[S_0 - Ke^{-rt+\frac{1}{2}(d_1^2-d_2^2)}\right]$$

$$d_1^2 - d_2^2 = 2\ln\frac{S_0e^{rt}}{K}$$

$$\Delta = N(d_1) + \frac{e^{-\frac{1}{2}d_1^2}}{S_0\sigma\sqrt{2\pi t}}\left[S_0 - Ke^{-rt+\frac{1}{2}\cdot 2\ln\frac{S_0e^{rt}}{K}}\right]$$

$$= N(d_1) + \frac{e^{-\frac{1}{2}d_1^2}}{S_0\sigma\sqrt{2\pi t}}\left[S_0 - Ke^{-rt}e^{\ln\frac{S_0e^{rt}}{K}}\right] = N(d_1) + \frac{e^{-\frac{1}{2}d_1^2}}{S_0\sigma\sqrt{2\pi t}}\left[S_0 - Ke^{-rt}\frac{S_0e^{rt}}{K}\right]$$

$$= N(d_1) + \frac{e^{-\frac{1}{2}d_1^2}}{S_0\sigma\sqrt{2\pi t}}[S_0 - S_0] = N(d_1)$$