Homework 1

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1. Prove or disprove that $\log(x^2+1)$ and $\log(x)$ are of the same order (Θ).

   Proof. Assume $\log(x^2+1)$ and $\log(x)$ are of the same order. Then $\log(x^2+1) \in O(\log(x))$ and $\log(x) \in O(\log(x^2+1))$.

   To show $\log(x^2+1) \in O(\log(x))$, we can start by rewriting $\log(x^2+1) = \log(x^2(1+\frac{1}{x^2}))$. And since $\frac{1}{x^2} \leq 1$ for $x \geq 1$, we know that $1+\frac{1}{x^2} \leq 2$. So we have that $\log(x^2+1) \leq \log(2x^2) = \log(x^2)+\log(2) = 2\log(x)+\log(2)$. And for $x \geq 2$, $\log(2) \leq \log(x)$, so $\log(x^2+1) \leq 2\log(x)+\log(2) \leq 2\log(x)+\log(x) = 3\log(x)$ in this domain. So for $x \geq 2$ and $c = 3$ we have that $\log(x^2+1) \in O(\log(x))$.

   To show $\log(x) \in O(\log(x^2+1))$, we simply follow the idea that $\log(n) \leq \log(m)$ when $n \leq m$ since log is always increasing. Considering x and x^2+1, we know that $x \leq x^2+1$ for $x \in \mathbb{R}$. So, $\log(x) \leq \log(x^2+1)$ for all x. Taking $c = 1$ leaves $\log(x) \in O(\log(x^2+1))$. $\square$

2. a. $\lg^k(n) \in O(n^\epsilon)$, $\lg^k(n) \notin \Omega(n^\epsilon)$, $\lg^k(n) \notin \Theta(n^\epsilon)$

   b. $n^k \in O(c^n)$, $n^k \notin \Omega(c^n)$, $n^k \notin \Theta(c^n)$

   f. $\lg(n!) \in O(\lg(n^n))$, $\lg(n!) \notin \Omega(\lg(n^n))$, $\lg(n!) \notin \Theta(\lg(n^n))$

3. $(n+1)!$: This is equivalent to $(n+1)n!$
   $n!$: Factorials grow faster than exponents
   $2^{2^{n+1}}$: This is equivalent to $2^{2\cdot 2^n} = 4^{2^n}$
   2^{2^n}
   $n \cdot 2^n$: Multiply 2^n by an increasing amount
   2^n
   $(3/2)^n$: $2^n = (4/3)^n \cdot (3/2)^n$
   n^3
   n^2
   $n\lg(n)$
   $\lg(n!)$
   $(\lg(n))^{\lg(n)}$, $n^{\lg(\lg(n))}$: The two logarithms causes these to grow much slower
   n
   $(\lg(n))!$
   $4^{\lg(n)}$
   $2^{\lg(n)}$
   $\lg^2(n)$
   $n^{1/\lg(n)}$
   1

4. Closed form solution for Tower of Hanoi problem with forwards substitution.

$$H_i = \begin{cases} 2H_{i-1} + 1 & i > 1 \\ 1 & i = 1 \end{cases}$$

$$\begin{aligned}
H_2 &= 2 * H_1 + 1 = 2 * 1 + 1 = 2^1 + 2^0 \\
H_3 &= 2 * H_2 + 1 = 2(2^1 + 2^0) + 2^0 = 2^2 + 2^1 + 2^0 \\
H_4 &= 2 * H_3 + 1 = 2(2^2 + 2^1 + 2^0) + 2^0 = 2^3 + 2^2 + 2^1 + 2^0 \\
H_n &= \sum_{i=0}^{n-1} 2^i = \frac{1 * 2^{n-1+1} - 1}{2 - 1} = \frac{2^n - 1}{1} \\
&= 2^n - 1
\end{aligned}$$

5. Prove for a positive integer n that $f_0 f_1 + f_1 f_2 + \ldots + f_{2n-1} f_{2n} = (f_{2n})^2$

$$\text{Base case: } n = 1$$
$$f_0 f_1 + f_1 f_2 = 0 * 1 + 1 * 1 = 1 = 1^2 = (f_2)^2$$

$$\begin{aligned}
\text{Inductive Step: assume that } f_0 f_1 + f_1 f_2 + \ldots + f_{2n-1} f_{2n} &= (f_{2n})^2 \\
f_0 f_1 + \ldots + f_{2n-1} f_{2n} + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} &= (f_0 f_1 + \ldots + f_{2n-1} f_{2n}) + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} \\
&= (f_{2n})^2 + f_{2n} f_{2n+1} + f_{2n+1} f_{2n+2} \\
&= f_{2n}(f_{2n} + f_{2n+1}) + f_{2n+1} f_{2n+2} \\
&= f_{2n} f_{2n+2} + f_{2n+1} f_{2n+2} \\
&= f_{2n+2}(f_{2n} + f_{2n+1}) \\
&= f_{2n+2} f_{2n+2} \\
&= (f_{2n+2})^2
\end{aligned}$$