Exam 1 A2 Rewrite

Walker Bagley

October 22, 2023

A. Take Home

2.

$$
\begin{aligned}
u_{tt} - 8u_{xt} + 12uxx = 0 \qquad & -\infty < x < \infty \qquad t > 0 \\
u(x,0) = e^{-x^4} \qquad & u_t(x,0) = \cos(x) \\
(\partial_t - 6\partial_x)(\partial_t - 2\partial_x)u = 0 &
\end{aligned}
$$

$$
\begin{aligned}
\text{Change variables: } & \begin{cases} x = -2t + \xi \\ x = -6t + \eta \end{cases} \quad \begin{cases} \xi = x + 2t \\ \eta = x + 6t \end{cases} \\
& \begin{cases} \partial_x = \partial_\xi + \partial_\eta \\ \partial_t = 2\partial_\xi + 6\partial_\eta \end{cases} \quad \begin{cases} -4\partial_\xi = \partial_t - 6\partial_x \\ 4\partial_\eta = \partial_t - 2\partial_x \end{cases} \\
-16\partial_{\xi\eta}u = 0 & \quad \Rightarrow \quad \partial_{\xi\eta}u = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Modifying D'Alambert's: } u(\xi,\eta) &= f(x+2t) + g(x+6t) \\
u(x,0) &= \phi(x) = f(x) + g(x) \Rightarrow \phi'(x) = f'(x) + g'(x) \\
u_t(x,0) &= \psi(x) = 2f'(x) + 6g'(x) \\
\text{So, 2 equations and 2 unknowns: } & \begin{cases} \phi'(x) = f'(x) + g'(x) & (1) \\ \psi(x) = 2f'(x) + 6g'(x) & (2) \end{cases} \\
6 * (\text{eqn } 1) - (\text{eqn } 2)\text{: } 4f'(x) &= 6\phi'(x) - \psi(x) \\
(\text{eqn } 2) - 2 * (\text{eqn } 1)\text{: } 4g'(x) &= \psi(x) - 2\phi'(x) \\
\text{Solving for } f' \text{ and } g' \text{ leaves } & \begin{cases} f'(x) = \frac{3}{2}\phi'(x) - \frac{1}{4}\psi(x) \\ g'(x) = \frac{1}{4}\psi(x) - \frac{1}{2}\phi'(x) \end{cases} \\
\text{Integrating both sides from 0 to } s\text{: } & \begin{cases} f(s) = \frac{3}{2}\int_0^s \phi'(y)dy - \frac{1}{4}\int_0^s \psi(y)dy \\ g(s) = -\frac{1}{2}\int_0^s \phi'(y)dy + \frac{1}{4}\int_0^s \psi(y)dy \end{cases} \\
& \begin{cases} f(s) = \frac{3}{2}\phi(s) + \frac{1}{4}\int_s^0 \psi(y)dy \\ g(s) = -\frac{1}{2}\phi(s) + \frac{1}{4}\int_0^s \psi(y)dy \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\text{Since } u(\xi,\eta) = f(\xi) + g(\eta) \text{ we have: } u(\xi,\eta) &= \frac{3}{2}\phi(\xi) + \frac{1}{4}\int_\xi^0 \psi(y)dy - \frac{1}{2}\phi(\eta) + \frac{1}{4}\int_0^\eta \psi(y)dy \\
&= \frac{1}{2}[3\phi(\xi) - \phi(\eta)] + \frac{1}{4}\int_\xi^\eta \psi(y)dy
\end{aligned}
$$

$$
\text{Applying our new solution, } u(x,t) = \frac{1}{2}\left[3e^{-(x+2t)^4} - e^{-(x+6t)^4}\right] + \frac{1}{4}[\sin(x+6t) - \sin(x+2t)]
$$