PDE Final Project

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1 History of the Burgers Equation

In 1915, Harry Bateman, an English mathematician, introduced a partial differential equation with corresponding initial and boundary conditions:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = v\frac{\partial^2 u}{\partial x^2}, 0 < x < L, 0 < t < T \tag{1.1}$$

$$u(x, 0) = f(x), 0 < x < L \tag{1.2}$$

$$u(0, t) = c_1(t), u(L, t) = c_2(t), 0 < t < T \tag{1.3}$$

In this equation u, x, t and v are the velocity, spatial coordinate, time and kinematic viscosity, respectively. The functions f, c_1, and c_2 depend on the specific conditions for the problem to be solved.
33 years later, in 1948, Johannes Marinus Burgers, a Dutch physicist, explained the mathematical modelling of turbulent fluid motion with the help of Bateman's equation. Burgers went on to become a leading figure in the study of fluid dynamics and contributted much of what we know today in his many papers.
In 1950 and 1951 respectively, Eberhard Hopf and Julian David Cole discovered a method to transform Burgers' equation into a linear heat equation and solved for an arbitrary initial condition. The Cole-Hopf transformation is given by:

$$w(u) = e^{-\frac{b}{a}u} \tag{1.4}$$

Since then, several important developments have been made in the field of viscous flow and turbulence describing aerodynamic flow at standard temperatures and pressures. The Burgers Equation, having much in common with the Navier–Stokes Equation, plays a vital role in analyzing fluid turbulence. Murray found that there is an ultimate steady turbulent state.

2 Burgers Equation

The Burgers Equation that we will examine was first introduced by Bateman. Later, Burgers applied this equation to turbulence, specifically to explain how the opposite forces of convection and diffusion interact in fluid mechanics.

The Burgers Equation in one dimensional space x and one dimensional time t as we will examine it is given by

$$u_t + uu_x = \epsilon u_{xx} \tag{2.1}$$

Where

(1) $u(x, t)$ is the velocity of the fluid at position x and time t

(2) ϵ is the kinematic viscosity or diffusion coefficient given by the fluid

Considering the following two cases with regards to the viscosity ϵ:

(1) $\epsilon = 0 \Rightarrow u_t + uu_x = 0$

Then, eliminating the viscosity, we yield a much simpler equation, regarded as the Inviscid Burgers Equation, which we solve as a nonlinear Transport Equation and discover the effect of shocks to a fluid system.

(2) $\epsilon \neq 0 \Rightarrow u_t + uu_x = \epsilon u_{xx}$

Then the equation remains the same and is regarded as the Viscous Burgers Equation, which relates heavily to the Navier-Stokes Equations through the damping caused by accounting for viscosity. This damping allows the Burgers equation to converge to a specific form of the Navier-Stokes Equations. We solve this equation by changing variables to reduce the Viscous Burgers Equation to the Heat Diffusion Equation and reverting the change of variables upon solving.

3 Inviscid Burgers Equation

3.1 Derivation

The derivation of the Inviscid Burgers Equation comes from modeling the motion of particles along the x-axis where every particle moves with constant speed for all $t > 0$ and having initial velocity, $u_0(x)$. Starting with the fact that the speed is constant we get

$$\frac{dx}{dt} = u(x,t) = constant \tag{3.1}$$

This implies

$$\frac{d}{dt}u(x,t) = 0 \tag{3.2}$$

Finally using the chain rule

$$u_t(1) + u_x\frac{dx}{dt} = 0 \tag{3.3}$$

$$u_t + uu_x = 0 \tag{3.4}$$

The derivation of the Inviscid Burgers Equation results in a nonlinear version of the transport equation with wave speed $c = u$. The speed of the wave now depends on the size of the disturbance u, rather than the position x. We assume a few things when solving the IVP that are important pieces in conceptualizing the derivation. First x is over the reals, while $t > 0$. Additionally, we assume the initial position is known. This gives us the nonlinear IVP

$$u_t + uu_x = 0, x \in \mathbb{R}, t > 0 \tag{3.5}$$

$$u(x,0) = u_0(x) \tag{3.6}$$

$$x = X(y,t), X(0) = y \tag{3.7}$$

This equation without finding solutions implies a few things. Larger waves will move faster, and overtake smaller, slower moving waves. Waves of elevation, where $u > 0$, move to the right, while waves of depression, where $u < 0$, move to the left. It and its multi-dimensional and multi-component generalizations play a crucial role in the modeling of gas dynamics, acoustics, shock waves in pipes, flood waves in rivers, chromatography, chemical reactions, traffic flow, and many other areas. [4]

3.2 Solution to IVP

The initial value problem of interest is

$$u_t + uu_x = 0 \tag{3.8}$$

$$u(x,0) = u_0(x) \tag{3.9}$$

Using the method of characteristics we have

$$\frac{dx}{dt} = u \tag{3.10}$$

Solving for the general solution to the Inviscid Burgers Equation is equivalent to solving the above ordinary differential equation. Thus,

$$dx = udt \quad \Rightarrow \quad x = ut + c \quad \Rightarrow \quad c = x - ut \tag{3.11}$$

To obtain the arbitrary function that solves the PDE, we can substitute the above equation into $u(x,t)$, resulting in

$$u(x,t) = u(ut + c, t) \tag{3.12}$$

By letting $t = 0$

$$u(u(0) + c, 0) = u(c, 0) = u(x - ut, 0) \tag{3.13}$$

Therefore, $u(x,t) = f(x - ut)$ for some arbitrary function f.

To solve the initial value problem with $u(x,0) = u_0(x)$, we set $t = 0$ in the general solution such that

$$u(x,0) = f(x) = u_0(x) \tag{3.14}$$

Thus, the solution to the IVP is $u(x,t) = u_0(x - ut)$.

However, since our solution is an implicit equation, the solution's features, dynamical behavior, and whether $u(x,t)$ is even a defined function are not clear.

Another important aspect to the solutions of the Inviscid Burgers IVP equation is the characteristic curves of the solutions. Starting with our initial condition of the model

$$\frac{dx}{dt} = u \tag{3.15}$$

We integrate both sides of the equation with respect to t and use our assumptions about the initial positions

$$\int_0^t \frac{dx}{dt}dt = \int_0^t u dt \tag{3.16}$$

$$X(y,t) - X(y,0) = \int_0^t u(X(y,t'),t')dt' \tag{3.17}$$

$$X(y,t) = y + \int_0^t u(X(y,t'),t')dt' \tag{3.18}$$

Utilizing the fact that the particles move with a fixed velocity, we see that u doesn't change with time because it is equal to the velocity.

$$\frac{d}{dt}[u(X(y,t),t)] = 0 \tag{3.19}$$

$$u(X(y,t),t) = u(X(y,0),0) = u_0(y) \tag{3.20}$$

Combing these equations we get:

$$X(y,t) = y + \int_0^t u_0(y)dt' \tag{3.21}$$

$$x = y + u_0(y)t \tag{3.22}$$

The following graph shows the characteristic curves, a straight line starting at the point $x = y$ with a slope of $u_0(y)$

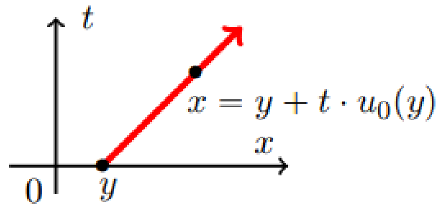

Figure 3.1: Characteristic line

Following the method of characteristics, we have

$$u(y + t \cdot u_0(y), t) = u_0(y) \tag{3.23}$$

and we can solve the characteristic line for y as follows

$$y = x - t \cdot u_0(y) \tag{3.24}$$

We can then substitute into the relation

$$u(y + t \cdot u_0(y), t) = u_0(x - t \cdot u_0(y)) \tag{3.25}$$

$$= u_0(x - t \cdot u(y + t \cdot u_0(y), t)) \tag{3.26}$$

By substituting the characteristic line equation back into the above equation, we obtain the implicit formula for the Burgers equation IVP. [1]

$$u(x,t) = u_0(x - t \cdot u(x,t)) \tag{3.27}$$

$$u = u_0(x - ut) \tag{3.28}$$

Example 1: Let $u_0(x) = 1$. Then, the characteristic line shown in the figure is

$$x = y + t \cdot u_0(y) \tag{3.29}$$
$$= y + 1 \cdot t \tag{3.30}$$

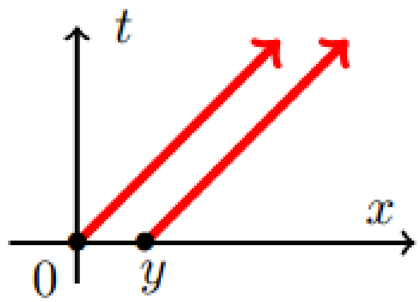

Figure 3.2: Graph of characteristic solutions, with initial condition $u_0(x) = 1$

and since the solution is a constant, we have

$$u = u_0(x - tu) = 1 \tag{3.31}$$

Example 2: Let $u(x, 0) = f(x) = ax + b$ for some real constants a, b. Then

$$u = a(x - ut) + b \tag{3.32}$$

$$u(x,t) = \frac{(ax + b)}{(1 + at)} \tag{3.33}$$

is the solution to the Inviscid Burgers Equation.

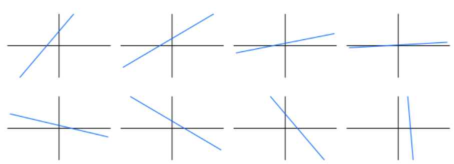

Figure 3.3: Two Solutions to IVP

The figure shows two solutions to the Inviscid Burgers Equation IVP, with each successive graph in the same row representing a different time of the same solution. We see that at each fixed t, the graph of the solution is a straight line. If $a > 0$ the solution flattens out or $u(x, t)$ goes to 0 as t approaches infinity. However if $a < 0$, the solution line steepens vertically quickly as t approaches $-1/a$, where the solution ceases to exist. This gives some insight on the lifespan of solutions for the inviscid Burgers equation. [4]

Example 3: Let $f(x) = u(0, x) = \frac{1}{6}\pi - \frac{1}{3}\arctan(x)$. Thus, we have the characteristic equation

$$x = y + u_0(y) \cdot t = y + \left(\frac{1}{6}\pi - \frac{1}{3}\arctan(y)\right) \cdot t \tag{3.34}$$

and the implicit solution

$$u = u_o(x - ut) = \frac{1}{6}\pi - \frac{1}{3}\arctan(x - ut) \tag{3.35}$$

The following graph are the characteristic lines for $u_0(x) = \frac{1}{6}\pi - \frac{1}{3}\arctan(x)$

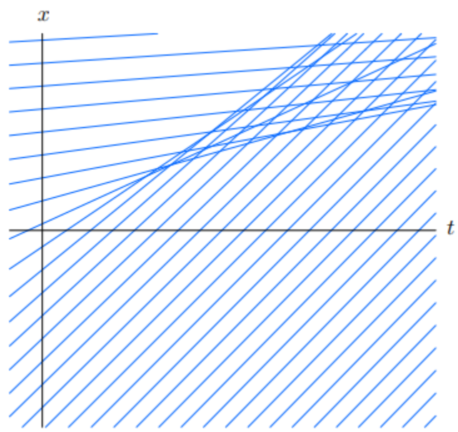

Figure 3.4: Graph of characteristic solutions, with initial condition $u_0(x) = \frac{1}{6}\pi - \frac{1}{3}\arctan(x)$

This graph has characteristic lines that are not parallel and, thus, intersect. The value of the solution at a point is supposed to be equal to the slope of the characteristic line passing through the point. However, at an intersection, the solution has to assume different values depending on the characteristic line. We can further examine this paradoxical issue.

There are three basic scenarios:

1) The characteristic lines are parallel:

   These lines have the same slope, c. Therefore, $u(x, t) = c$ is the trivial constant solution.

2) The initial data is non-decreasing everywhere (i.e. $f(x) \leq f(y)$ for all $x \leq y$ so $f'(x) \geq 0$):

   This scenario has characteristics lines such as the following figure. The characteristic lines never cross for

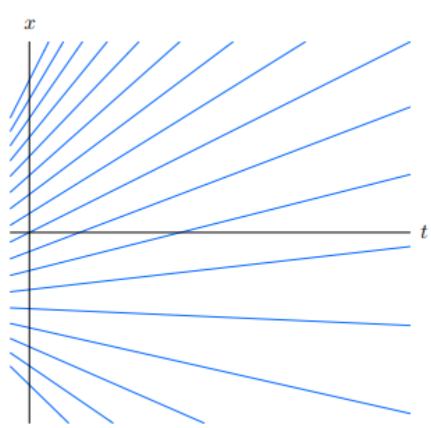

Figure 3.5: Graph of characteristic lines for a rarefaction wave

   $t > 0$. Thus, the solution $u(x, t)$ is well-defined for $t > 0$. These solutions represent rarefaction waves which spread out as time progresses.

3) The initial data is a decreasing function (i.e. $f'(x) < 0$):

   For this type of initial data, the characteristic lines cross and the value of the solution is no longer uniquely determined. Since u(x,t) is supposed to represent a measurable quantity a unique value should exist at each point.

We will use this example to confront the difficulty of the third set of initial value problems. At a critical time $t_* > 0$, the first two characteristic lines cross each other. In addition, there exists a wedge shaped region consisting of points that lie on the intersection of three distinct characteristic lines. The solution has one value outside of this wedge region and two unique solutions on the boundary of the region.

The following is a series of graphs of multiple valued solutions plotted at different times. The graph of the solution is vertical at the critical time t_* when the first two characteristic lines cross.

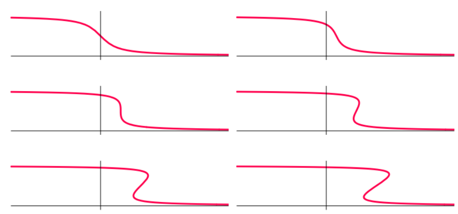

Figure 3.6: Multi Valued Solutions for IVP

We can find the critical time t_* from the implicit solution formula. By differentiating the solution $u = f(\zeta)$ for $\zeta = x - tu$ with respect to x. We get

$$\frac{\partial u}{\partial x} = f'(\zeta)\left(1 - t\frac{\partial u}{\partial x}\right) = \frac{f'(\zeta)}{1 + tf'(\zeta)} \tag{3.36}$$

The earliest critical time is

$$t_* = min\left(-\frac{1}{f'(x)}\right)\Bigg|\ f'(x) < 0 \tag{3.37}$$

Thus, in our example, we have $f(x) = \frac{\pi}{6} - \frac{1}{3}\arctan(x)$ and $f'(x) = \frac{-1}{3(1+x^2)}$. The critical time is

$$t_* = min(3(1 + x^2)) = 3 \tag{3.38}$$

and the position of the vertical point on the graph is

$$x_* = f(0)t_* = \frac{1}{2}\pi \tag{3.39}$$

The multiple valued solution is mathematically plausible but physically untenable. At the critical time, one needs to decide if any of the possible solution values is physically appropriate. This is dependent of the phenomenon that is being modeled. This critical time is also known as the breaking time. [4]

3.3 Shocks

When two characteristics lines intersect, the solution to the inviscid Burgers equation IVP,

$$u_t + uu_x = 0, \quad x \in \boldsymbol{R}, t > 0 \tag{3.40}$$

$$u(x,0) = u_0(x) \tag{3.41}$$

forms a shock and the solution becomes multivalued. In the following graph of intersecting characteristic lines, the lines intersect at T_b, the breaking time.

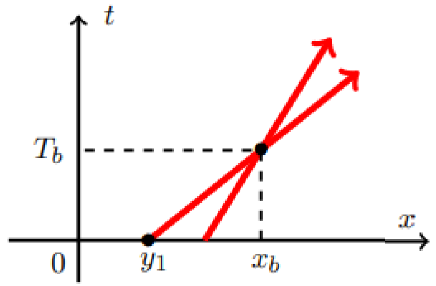

Figure 3.7: Intersecting Characteristic Lines

The figure shows two characteristic lines, one starting at point y_1 and the other starting at y_2, which intersect at time T_b and point x_b. The equations for the two characteristic lines are as follows

$$x = y_1 + t \cdot u_0(y_1) \tag{3.42}$$

$$x = y_2 + t \cdot u_0(y_2) \tag{3.43}$$

The solution is constant on each characteristic line and equal to the slope of the line. Thus, we have

$$u_0(y_1) = u(x_b, T_b) = u_0(y_2) \tag{3.44}$$

However, the slopes of these lines are not the same, and we have a the problem of multivalue solutions at the breaking time T_b. We define breaking time, formally, as the minimal time t where two characteristic curves intersect.

Assuming $y_1 < y_2$, we can solve for the breaking time by setting the two equations equal to each other. We then have

$$y_1 + t \cdot u_0(y_1) = y_2 + t \cdot u_0(y_2) \tag{3.45}$$

Solving for t, we get

$$t = -\frac{y_2 - y_1}{u_0(y_2) - u_0(y_1)} \tag{3.46}$$

By applying the Mean Value Theorem, we have for some y^* with $y_1 < y^* < y_2$

$$u_0'(y^*) = \frac{u_0(y_2) - u_0(y_1)}{y_2 - y_1} \tag{3.47}$$

Thus,

$$t = -\frac{1}{u_0'(y^*)} \quad y_1 < y^* < y_2 \tag{3.48}$$

Combining the above equations, we can solve for the breaking time

$$T_b = -\frac{1}{min(u_0'(x))} \tag{3.49}$$

Example: Compute the breaking time for the solution of the following ivp

$$u_t + uu_x = 0 \quad x \in \mathbb{R},\ t > 0 \tag{3.50}$$

$$u(x, 0) = e^{-(x-1)^2} \tag{3.51}$$

The following is a graph of $u_0(x)$

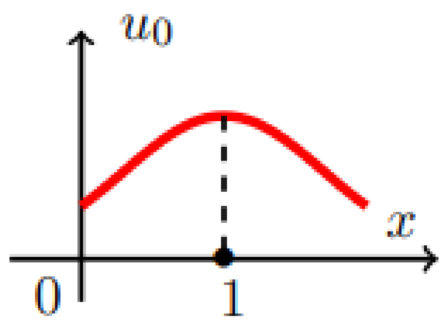

Figure 3.8: Graph of $u_0(x)$

In order to solve for the breaking time, we must first find the derivative of the initial data $u_0(x)$.

$$u_0'(x) = -2(x-1)e^{-(x-1)^2} \tag{3.52}$$

We can find the minimum by finding when the second derivative of $u_0'(x)$ is equal to 0

$$u_0''(x) = [-2 + 4(x-1)^2]e^{-(x-1)^2} \tag{3.53}$$

Setting $u_0''(x) = 0$, we get

$$(x-1)^2 = \frac{1}{2}x = 1 + \frac{1}{\sqrt{2}} \tag{3.54}$$

Thus, the breaking time is

$$T_b = -\frac{1}{u_0'\left(\frac{1}{\sqrt{2}}\right)} = -\frac{1}{-\frac{2}{\sqrt{2}e^{\frac{1}{2}}}} = \frac{\sqrt{2e}}{2} \tag{3.55}$$

The following graph shows the multivalued solution

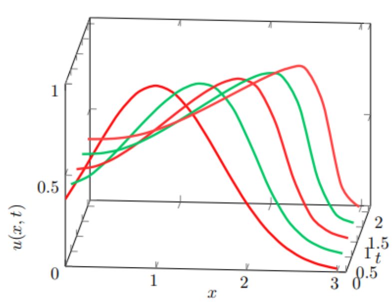

Figure 3.9: Graph of Solutions

The breaking time is also referred to as the lifespan of the solution. [1]

3.3.1 Lifespan of Solution

The lifespan of a nonlinear partial differential equation gives information into how the nonlinearity and initial condition(s) affect large-time behaviors of the solution.

Calculating the lifespan of such solutions requires two key formulas:

1. Characteristic Equation: $x = y + t * u_0(y)$.
2. Implicit Solution: $u(x,t) = u_0(x - t * u(x,t))$, or $u = u_0(x - tu)$.

To examine the lifespan of solutions we will look at some further examples.

Example 3: let $u_0(x) = x$. Then the characteristic is:

$$x = y + (1 * t) \tag{3.56}$$

And the solution $u(x,t)$ is given by:

$$u(x,t) = x - t * u \tag{3.57}$$

Solving for $u(x,t)$ yields:

$$u(x,t) = \frac{x}{(1+t)} \tag{3.58}$$

Because $t > 0$, there is no value of t for which this solution is undefined. Therefore, the lifespan of the solution is $T = \infty$. Additionally, the solution $u(x,t)$ is global.

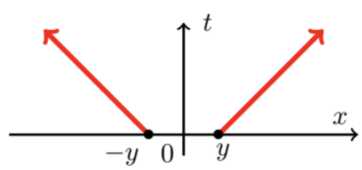

Figure 3.10: Graph of characteristic solutions, with initial condition $u_0(x) = x$.

Example 4: Let $u_0(x) = 1 - x$. Then the characteristic is:

$$x = y + (1 - y) * t \tag{3.59}$$

And the solution $u(x,t)$ is given by:

$$u(x,t) = 1 - (x - t * u) \tag{3.60}$$

Solving for $u(x,t)$ yields:

$$u(x,t) = \frac{1-x}{(1-t)} \tag{3.61}$$

$u(x,t)$ becomes undefined when $t = 1$. Therefore, the lifespan of the solution is $T = 1$, i.e. the solution is not global.

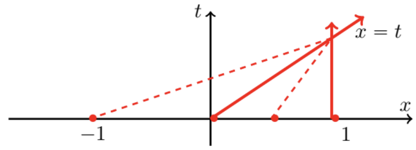

Figure 3.11: Graph of characteristic solutions, with initial condition $u_0(x) = 1 - x$.

The graph (with various chosen y-values) provides the same lifespan as the analytical solution, i.e. there is an intersection at $t = 1$.

3.4 Inviscid Burgers Equation in Concrete Situations

3.4.1 Shock Waves

The Inviscid Burger's equation is directly related to the propagation of shock waves, for the wave speed, c = u.

(1) $u > 0$: waves move to the right (elevation waves)

(2) $u < 0$: waves move to the left (depression waves)

In this situation, u is called the wave disturbance and has direct effects on the size and speed of the wave. This means that larger waves move faster than smaller, slower-moving waves. For the solutions of equation with different characteristics, the beginning of the shock occurs at the location and time of intersection. [5]

Taking the implicit differentiation of

$$u(x,t) = u_0(x - ut) = u_0(\xi) \tag{3.62}$$

(With respect to x) yields:

$$u_x = \frac{\partial}{\partial x} u_0(\xi) = u_0'(\xi)\frac{\partial \xi}{\partial x} = u_0'(\xi)[1 - tu_x] = \frac{u_0'(\xi)}{1 + tu_0'(\xi)} \tag{3.63}$$

The shock forms at the earliest time that $u(x,t)$ has a vertical tangent, i.e. u_x is undefined. To find the point in time of the shock, we examine when the denominator equals 0 to cause u_x to become undefined:

$$1 + tu_0'(\xi) = 0 \quad \Rightarrow \quad t = -\frac{1}{u_0'(\xi)} \tag{3.64}$$

To find the x-coordinate of this shock, calculate the value of ξ for which u_x becomes undefined: [5]

$$1 + t_0 u_0'(x_0) = 0 \tag{3.65}$$

Different types of waves can propagate depending on initial conditions:

(1) $u_0'(x) > 0$: rarefaction waves → flatten over time because $1 + u_0(x) > 0$

(2) $u_0'(x) < 0$: compression waves → steepen and forms a shock over time because $1 + u_0(x) < 0$

The example of $u_0(x) = sin(x)$ shows the wave propagation for a shock wave over time:

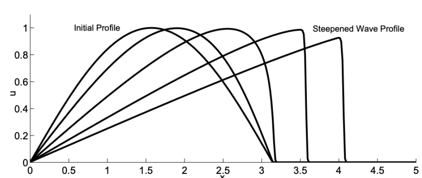

Figure 3.12: Propagating sine wave, via "Inviscid Burgers Equations and Its Numerical Solutions", [5]

3.5 Relation to Euler Equations

Consider the system of Euler Equations for conservation of mass, momentum and energy in expanded convective form for a thermodynamic system. Such a system contains convective variables for mass density, flow velocity and pressure and conserved variables for mass density, momentum density and energy density. Consider this system of equations for the conservation system described with

$$\frac{\partial \rho}{\partial t} + \boldsymbol{u} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{u} = 0 \tag{3.66}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \frac{\nabla p}{\rho} = \boldsymbol{g} \tag{3.67}$$

$$\frac{\partial e}{\partial t} + \boldsymbol{u} \cdot \nabla e + \frac{p}{\rho} \nabla \cdot \boldsymbol{u} = 0 \tag{3.68}$$

Where

(1) $\boldsymbol{u}$ represents the vector flow velocities

(2) ρ represents the fluid mass density function governing the system

(3) p represents the pressure function of the system

(4) $\boldsymbol{g}$ represents the external accelerations enacted on the system by gravity, electric or magnetic fields, or other imposed forces

(5) e represents the internal fluid energy per unit mass

In a case where there are no external accelerations and constant pressure, we find

$$\boldsymbol{g} = 0 \tag{3.69}$$

$$\nabla p = 0 \tag{3.70}$$

Substituting these constraints into equation (3.67), the conservation of momentum, yields

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = 0 \tag{3.71}$$

Considering the modified equation in (3.71) in a one dimensional space along an axis x, we reduce this Euler Equation to the Inviscid Burgers Equation

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla_x \boldsymbol{u} = 0 \tag{3.72}$$

$$\boldsymbol{u}_t + \boldsymbol{u}\boldsymbol{u}_x = 0 \tag{3.73}$$

It should be noted that this equation, following from the conservation of momentum given by the Euler Equations, describes the motion of waves in a one dimensional characteristic form in terms of x and a one dimensional time space t.

4 Viscous Burgers Equation

4.1 Solving the IVP for the Viscous Burgers Equation [2]

We consider the initial value problem for the Viscous Burgers Equation

$$u_t + uu_x - \epsilon u_{xx} = 0 \tag{4.1}$$

$$u(x, 0) = u_0(x) \tag{4.2}$$

We will first prove that the IVP has a solution, and then transform it to the heat equation to solve the problem. Using the following transformations

$$v(x,t) = \int_{-\infty}^{x} u(y,t)dy \tag{4.3}$$
$$v_0(x) = \int_{-\infty}^{x} u_0(y)dy \tag{4.4}$$

We can then consider the PDE

$$v_t - \epsilon v_{xx} + \frac{1}{2}v_x^2 = 0 \tag{4.5}$$
$$v(x,0) = v_0(x) \tag{4.6}$$

By solving the above PDE, we can obtain a solution to the Viscous Burgers Equation by reversing our change of variables since $u = \partial_x v$. So we generalize the PDE by letting $a = \epsilon$ and $b = \frac{1}{2}$ to

$$v_t - av_{xx} + bv_x^2 = 0 \tag{4.7}$$
$$v(x,0) = v_0(x) \tag{4.8}$$

Through the following change of variables we can reduce the above PDE to a linear PDE

$$w = \varphi(v) \tag{4.9}$$
$$w(x,0) = w_0(x) = \varphi(v_0(x)) \tag{4.10}$$

Assuming that φ is smooth and applying the chain rule, we calculate the partial derivatives

$$w_t = \varphi'(v)v_t \tag{4.11}$$
$$w_x = \varphi'(v)v_x \tag{4.12}$$
$$w_{xx} = \varphi'(v)v_{xx} + \varphi''(v)v_x^2 \tag{4.13}$$

Manipulating w_t, we have

$$\begin{aligned} w_t &= \varphi'(v)v_t && (4.14)\\ &= \varphi'(v)[av_{xx} - bv_x^2] && (4.15)\\ &= a\varphi'(v)v_{xx} - b\varphi'(v)v_x^2 && (4.16)\\ &= a[w_{xx} - \varphi''(v)v_x^2] - b\varphi'(v)v_x^2 && (4.17)\\ &= aw_{xx} - [a\varphi''(v) + b\varphi'(v)]v_x^2 && (4.18) \end{aligned}$$

Letting $\varphi = e^{-\frac{b}{a}z}$, then

$$w = e^{-\frac{b}{a}v} \tag{4.19}$$
$$w_0(x) = e^{-\frac{b}{a}v_0(x)} \tag{4.20}$$

So, using our choice of φ

$$a\varphi'' + b\varphi' = a \cdot \frac{b^2}{a^2}\varphi - b \cdot \frac{b}{a}\varphi \tag{4.21}$$

$$= \frac{b^2}{a}\varphi - \frac{b^2}{a}\varphi = 0 \tag{4.22}$$

Therefore, $w_t = aw_{xx}$ and w satisfies the following heat IVP

$$w_t - aw_{xx} = 0 \tag{4.23}$$

$$w(x,0) = w_0(x) = e^{-\frac{b}{a}v_0(x)} \tag{4.24}$$

Theorem 4.1 (Solution to heat IVP with conductivity a)**.** *Given $w_0 \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we define the solution to the heat IVP as*

$$w(x,t) = \frac{1}{(4\pi at)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4at}} w_0(y)dy \tag{4.25}$$

Then

(1) $w \in C^\infty(\mathbb{R} \times (0,\infty))$

(2) $w_t - aw_{xx} = 0 \quad x \in \mathbb{R},\ t > 0$

(3) $\lim_{(x,t)\to(x^0,0)} = w_0(x^0)$ *for each* $x^0 \in \mathbb{R}$

Using this theorem, we can solve the earlier generalization of the PDE by observing that

$$v = -\frac{a}{b}\ln v \tag{4.26}$$

Thus

$$v(x,t) = -\frac{a}{b}\ln\left(\frac{1}{(4\pi at)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4at}} w_0(y)dy\right) \tag{4.27}$$

By letting $a = \epsilon$ and $b = \frac{1}{2}$, we obtain the solution to the original PDE using the above formula.

$$u(x,t) = -2\epsilon\partial_x \ln\left(\frac{1}{(4\pi\epsilon t)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\epsilon t} - \frac{v_0(y)}{2\epsilon}} dy\right) \tag{4.28}$$

$$= \frac{\int_{\mathbb{R}} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\epsilon t} - \frac{v_0(y)}{2\epsilon}} dy}{\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4\epsilon t} - \frac{v_0(y)}{2\epsilon}} dy} \tag{4.29}$$

The above solution holds with the following constraint on the initial data

$$\int_{-\infty}^{x} u_0(y)dy \in C(\mathbb{R}) \cap L^\infty(\mathbb{R}) \tag{4.30}$$

We will then fix the initial data u_0. For each $\epsilon > 0$, we have a solution u^ϵ to the corresponding initial value problem given. We, thus, aim to prove that pointwise

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = u(x,t) \tag{4.31}$$

To determine the behavior of $u(x,t)$, we need the following lemmas:

Lemma 4.1. *Suppose that $k, l : \mathbb{R} \to \mathbb{R}$ are continuous functions such that l grows at most linearly and k grows at least quadratically. If there exists a unique point $y_0 \in \mathbb{R}$ such that*

$$k(y_0) = min_{y\in\mathbb{R}}(k(y)) \tag{4.32}$$

Then

$$\lim_{\epsilon\to 0}\frac{\int_{\mathbb{R}} l(y)e^{-\frac{k(y)}{\epsilon}}dy}{\int_{\mathbb{R}} e^{-\frac{k(y)}{\epsilon}}dy} = l(y_0) \tag{4.33}$$

Proof.
Let $k_0 = k(y_0)$. Then

$$\mu_\epsilon(y) = \frac{e^{\frac{k_0-k(y)}{\epsilon}}}{\int_{\mathbb{R}} e^{\frac{k_0-k(z)}{\epsilon}}dz} \tag{4.34}$$

has the following properties

(1) $\mu_\epsilon \geq 0$, $\int_{\mathbb{R}} \mu_\epsilon(y)dy = 1$

(2) If $y \neq y_0$, $\mu_\epsilon \to 0$ as $\epsilon \to 0$

Then

$$\lim_{\epsilon\to 0}\frac{\int_{\mathbb{R}} l(y)e^{\frac{-k(y)}{\epsilon}}dy}{\int_{\mathbb{R}} e^{\frac{-k(y)}{\epsilon}}dy} = \lim_{\epsilon\to 0}\int_{\mathbb{R}} l(y)\mu_\epsilon(y)dy \tag{4.35}$$

$$= l(y_0) \tag{4.36}$$

□

By defining K as

$$K(x,y,t) = \frac{(x-y)^2}{4t} + \frac{v_0(y)}{2} \tag{4.37}$$

we then observe

$$u^\epsilon(x,t) = \frac{\int_{\mathbb{R}} \frac{x-y}{t}e^{-\frac{K(x,y,t)}{\epsilon}}dy}{\int_{\mathbb{R}} e^{-\frac{K(x,y,t)}{\epsilon}}dy} \tag{4.38}$$

By letting

$$l(y) = \frac{x-y}{t} \tag{4.39}$$

$$k(y) = -K(x,y,t) \tag{4.40}$$

we satisfy the hypotheses of the above lemma.

Lemma 4.2. *For each time $t > 0$, the mapping $y \to K(x,y,t)$ has a minimum at a unique $y = y(x,t)$ for all but at most countably many points x*

Since there exists a unique minimum, the first lemma is satisfied, and we have

$$\lim_{\epsilon\to 0} u^\epsilon(x,t) = \frac{x-y(x,t)}{t} \tag{4.41}$$

Theorem 4.2. *Assume $F : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex, and $u_0 \in L^\infty(\mathbb{R}$. Then*

(1) For each time $t > 0$ and for all but at most countably many values of $x \in \mathbb{R}$, there exists a unique $y(x,t)$ such that

$$min_{y\in\mathbb{R}}\left(tL\left(\frac{x-y}{t}\right)\right) + h(y) \tag{4.42}$$

(2) The mapping $x \to y(x,t)$ is nondecreasing

(3) For each time $t > 0$, the function u which is a solution to the PDE $u_t + F(u)_x = 0$ is given as

$$u(x,t) = G\left(\frac{x - y(x,t)}{t}\right) \tag{4.43}$$

where $G = (F')^{-1}$

Using this theorem, we obtain a solution to the inviscid burgers equation by letting $F(X) = \frac{z^2}{2}$. Thus, $G(Z) = (F'(z)^{-1} = (z)^{-1} = z$. We then obtain the solution

$$u(x,t) = G\left(\frac{x - y(x,t)}{t}\right) = \frac{x - y(x,t)}{t} \tag{4.44}$$

Since we concluded that

$$\lim_{\epsilon \to 0} u^{\epsilon}(x,t) = \frac{x - y(x,t)}{t} \tag{4.45}$$

we have that

$$\lim_{\epsilon \to 0} u^{\epsilon} = u \tag{4.46}$$

4.2 Viscous Burgers IVP Examples

4.2.1 Traffic Flow

Consider a highway along an axis x, with $\rho(x,t)$ describing the density of vehicles at a point along the highway at any given moment and $v(x,t)$ describing the speed of traffic. Defining v as follows [7]

$$v(x,t) = v_{max}\left(1 - \frac{\rho(x,t)}{\rho_{max}(x,t)}\right) - C\frac{\rho_x(x,t)}{\rho(x,t)} \tag{4.47}$$

We have that the traffic speed is proportional to the density of vehicles at that point, accounting for the change in density with ρ_x, the magnitude of which depends on an arbitrary constant C. Applying basic calculus, over some span of time h and space on the highway on an interval $[a,b]$, we have

$$\int_a^b \frac{\rho(x,t+h) - \rho(x,t)}{h}\ dx = \rho(a,t)v(a,t) - \rho(b,t+h)v(b,t+h) \tag{4.48}$$

And thus we are left with the following system of equations if we take the limit as $h \to 0$

$$\int_a^b \rho_t(x,t)\ dx = \rho(a,t)v(a,t) - \rho(b,t)v(b,t) \tag{4.49}$$

$$\rho(a,t)v(a,t) - \rho(b,t)v(b,t) = -\int_a^b (\rho(x,t)v(x,t))_x\ dx \tag{4.50}$$

Taking a linear combination to remove the non-integral terms [7], we have

$$\int_a^b \rho_t + (\rho v)_x\ dx = 0 \tag{4.51}$$

And therefore, since a, b are just an arbitrary range

$$\rho_t + (\rho v)_x = 0 \tag{4.52}$$

Substituting in our equation for v yields

$$\rho_t + \left[\rho \cdot \left(v_{max}\left(1 - \frac{\rho}{\rho_{max}}\right) - C\frac{\rho_x}{\rho}\right)\right]_x = \rho_t + \left[\rho v_{max} - \frac{\rho^2 v_{max}}{\rho_{max}} - C\rho_x\right]_x \tag{4.53}$$

$$= \rho_t + \rho_x v_{max} - 2\frac{v_{max}}{\rho_{max}}\rho\rho_x - C\rho_{xx} \tag{4.54}$$

$$\rho_t + \rho_x v_{max} - 2\frac{v_{max}}{\rho_{max}}\rho\rho_x - C\rho_{xx} = 0 \tag{4.55}$$

Changing variables [7]

$$\begin{cases} x' = -x + v_{max}t \\ t' = t \end{cases} \tag{4.56}$$

Applying chain rule gives us the following partial derivatives

$$\rho_t(x', t') = \rho_{x'} v_{max} + \rho_{t'} \tag{4.57}$$

$$\rho_x(x', t') = -\rho_{x'} \tag{4.58}$$

$$\rho_{xx}(x', t') = \rho_{x'x'} \tag{4.59}$$

Substituting these new partial derivatives into the form of our PDE above, the equation now looks like

$$\rho_{t'} + 2\frac{v_{max}}{\rho_{max}}\rho\rho_{x'} - C\rho_{x'x'} = 0 \tag{4.60}$$

This equation looks oddly familiar! This is just a slightly modified version of the Viscous Burgers equation. Through the above reduction, we were able to model traffic flow via the viscous form of the Burgers equation.

4.3 Relation to Navier-Stokes Equation

The Navier-Stokes Equation is used for modeling currents, airflow, and water flow. More particularly, this equation describes the relationship between velocity, pressure, temperature, and density of moving fluids. [7]

The Burger's Equation can be considered a simplification of the Navier-Stokes Equation (with the pressure gradient terms removed).

Begin with the incompressible Navier-Stokes Equation:

$$\rho(v_t + v\nabla v) = -\nabla p + \nu\Delta v + F \tag{4.61}$$

$$\nabla v = 0 \tag{4.62}$$

$$v(x, 0) = v_0(x) \tag{4.63}$$

In this equation:

(1) $\rho =$ density

(2) $p =$ pressure

(3) $\nu =$ fluid viscosity

(4) $F =$ external force

Assuming that there are no external forces ($F = 0$) and the pressure term is negligible, the Navier-Stokes Equation becomes the viscous Burgers Equation [7]

$$u_t + uu_x = \frac{\nu}{\rho} u_{xx} \tag{4.64}$$

$$u(x, 0) = u_0(x) \tag{4.65}$$

The term $\frac{\nu}{\rho}$ is called the kinematic viscosity. If this term is equal to 0, the equation is further transformed into the Inviscid Burgers Equation.

4.4 Relation to Korteweg–De Vries Equation

By combining essential elements from the Korteweg-De Vries equation and the Viscous Burgers equation [6], we have a general form of the KdV-Burgers equation. This is useful in modeling a number of physical phenomena including the propagation of large waves in shallow water, the flow of liquids containing gas bubbles, the propagation of waves in an elastic tube containing a viscous fluid and much more. [6]

$$u_t + \alpha uu_x + \beta u_{xx} + su_{xxx} = 0 \tag{4.66}$$

Taking $\beta = -\epsilon$ and $s = 0$ gives us the Viscous Burgers equation, and allowing $\beta = 0$ yields the standard form of the KdV Equation. Assuming this equation has a travelling wave solution of the form $u(x, t) = u(\xi) = u(x - vt + \xi_0)$ with convergent (nice) boundary conditions, we can obtain an approximate series solution to the KdV-Burgers Equation through Adomian decompositions. This solution also converges due to the boundary conditions previously mentioned and can be confirmed by inverting the transformations applied in the Adomian decomposition method. [6]

5 References

[1] Himonas, Alex. *Nonlinear PDE: The (Inviscid) Burgers Equation.* University of Notre Dame Mathematics Department.

[2] Himonas, Alex and Holliman, Curtis. *The Burgers Equation with Viscosity.*

[3] Martinez Nunez, Oscar. *Solution of Burger Equation with Viscosity Applying the Boundary Layer Theory.* The University of Cartagena. www.arxiv.org/pdf/2009.05715.pdf. Accessed 12 Nov. 2023.

[4] Olver, Peter. *Introduction to Partial Differential Equations.* Springer International Publishing. 8 Nov. 2013.

[5] Oyar, Nazmi. *Inviscid Burgers Equations and Its Numerical Solutions.* The Graduate Schools of Natural and Applied Sciences of Middle East Technical University. Sept. 2017.

[6] Wang, Xiaohui. *The Korteweg–de Vries–Burgers equation and its approximate solution.* University of Texas - Pan American and Virginia Tech. University Departments of Mathematics. 25 Mar. 2007. https://www.researchgate.net/publication/220661267_The_Korteweg-de_Vries-Burgers_equation_and_its_approximate_solution

[7] Zhenhe Zhang, *Burgers Equation.* McGill University Department of Mathematics and Statistics. 21 Apr. 2019. www.math.mcgill.ca/gantumur/math581w19/BurgersEquation_ZhenheZhang.pdf.