Homework 4

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Section 2.3

2.3.1

Find the upper and lower limits of each of the following sequences.

(a)

$a_n = 3 + (-1)^n$

Upper limit: Suppose $\overline{\lim}\ a_n = 4$. Proving (a), take some $\epsilon > 0$, then $\forall n$, $a_n \leq \overline{\lim}\ a_n + \epsilon$ since $a_n \leq 4$ so we can take $n > N = 1$. Proving (b), we again take some $\epsilon > 0$, then we take the subsequence a_{n_k} where n_k is even so that all $a_{n_k} = 4$. Then we can see that $a_{n_k} = 4 \geq \overline{\lim}\ a_n - \epsilon$.

Lower limit: Suppose $\underline{\lim}\ a_n = 2$. Proving (a), take some $\epsilon > 0$, then $\forall n$, $a_n \geq \underline{\lim}\ a_n - \epsilon$ since $a_n \geq 2$ so we can take $n > N = 1$. Proving (b), we again take some $\epsilon > 0$, then we take the subsequence a_{n_k} where n_k is odd so that all $a_{n_k} = 2$. Then we can see that $a_{n_k} = 2 \leq \underline{\lim}\ a_n + \epsilon$.

(b)

$a_n = 3 + (-2)^n$

Upper limit: Suppose $\overline{\lim}\ a_n = \infty$. Then we need to show that a_n is not bounded above. TAC, suppose a_n were bounded above. Then $\exists M$ s.t. $\forall n$, $a_n < M$. However, $a_n < M \Rightarrow 3 + (-2)^n < M \Rightarrow (-2)^n < M - 3$ However, if we take $n > 2\log_2(M-3)$, we get $(-2)^{2\log_2(M-3)} = 2^{2\log_2(M-3)} = 2^{\log_2(M-3)^2} = (M-3)^2 > M - 3$. So a_n is bounded above and $\overline{\lim}\ a_n = \infty$.

Lower limit: Suppose $\underline{\lim}\ a_n = -\infty$. Then we need to show that a_n is not bounded below. TAC, suppose a_n were bounded below. Then $\exists M$ s.t. $\forall n$, $a_n > M$. However, $a_n > M \Rightarrow 3 + (-2)^n > M \Rightarrow (-2)^n > M - 3$ However, if we take $n > 2\log_2(M-3)+1$, we get $(-2)^{2\log_2(M-3)-1} = (-2)\cdot 2^{2\log_2(M-3)} = (-2)\cdot 2^{\log_2(M-3)^2} = (-2)\cdot(M-3)^2 < M-3$. So a_n is bounded below and $\underline{\lim}\ a_n = \infty$.

(c)

$a_n = 3 + \frac{1}{n}\sin n$

Upper limit: Suppose $\overline{\lim}\ a_n = 3 + \sin 1$. Proving (a), take some $\epsilon > 0$, then $\forall n$, $a_n \leq \overline{\lim}\ a_n + \epsilon$ since $a_n = 3 + \frac{\sin n}{n} \leq 3 + \sin 1$ so we can take $n > N = 1$. This holds because $\sin 1 > \frac{1}{2}$, so even if $\sin n = 1$ for some n, $\frac{1}{n} < \sin 1$ for all $n > 1$. Proving (b), we again take some $\epsilon > 0$, then we take the subsequence $\{a_1\}$ so that all $a_{n_k} = 3 + \sin 1$. Then we can see that $a_{n_k} = 3 + \sin 1 \geq \overline{\lim}\ a_n - \epsilon$.

Lower limit: Suppose $\underline{\lim}\ a_n = 3 + \frac{\sin 5}{5}$. Proving (a), take some $\epsilon > 0$, then $\forall n$, $a_n \geq \underline{\lim}\ a_n - \epsilon$ since $a_n = 3 + \frac{\sin n}{n} \geq 3 + \frac{\sin 5}{5}$ so we can take $n > N = 5$. This holds because $\sin 5 < -\frac{19}{20}$, so even if $\sin n = -1$ for some n, $-\frac{1}{n} > -\frac{19}{100} > 3 + \frac{\sin 5}{5}$ for all $n > 5$. Proving (b), we again take some $\epsilon > 0$, then we take the subsequence $\{a_5\}$ so that all $a_{n_k} = 3 + \frac{\sin 5}{5}$. Then we can see that $a_{n_k} = 3 + \frac{\sin 5}{5} \leq \overline{\lim}\ a_n + \epsilon$.

2.3.2

Let $\{a_n\}$ be a sequence of real numbers. Then show $\underline{\lim}\ a_n \leq \overline{\lim}\ a_n$.

Proof. By definition $\underline{\lim}\ a_n$ means that for some $\epsilon > 0$, we can find some N_l s.t. $a_n \geq \underline{\lim}\ a_n - \epsilon$ for all $n > N$. Similarly, $\overline{\lim}\ a_n$ means that for some $\epsilon > 0$, we can find some N_u s.t. $a_n \leq \overline{\lim}\ a_n + \epsilon$ for all $n > N$. Then we can take the same $\epsilon > 0$ for both the upper and lower limits and $N = \max(N_l, N_u)$ so that both of these hold. Then combining inequalities, we have that $\underline{\lim}\ a_n - \epsilon \leq a_n \leq \overline{\lim}\ a_n + \epsilon$, so $\underline{\lim}\ a_n - \epsilon \leq \overline{\lim}\ a_n + \epsilon$. If we take the limit of this as $\epsilon \to 0$, we get $\underline{\lim}\ a_n \leq \overline{\lim}\ a_n$. ☐

2.3.3

Let $\{a_n\}$ be a sequence of real numbers and α be a real number.

(a)

If $\alpha > 0$, prove $\overline{\lim}(\alpha a_n) = \alpha \overline{\lim}\ a_n$ and $\underline{\lim}(\alpha a_n) = \alpha \underline{\lim}\ a_n$.

Proof. By definition, $\overline{\lim}(\alpha a_n)$ means that for some $\epsilon_1 > 0$ we can find N s.t. $\alpha a_n \leq \overline{\lim}(\alpha a_n) + \epsilon_1$ for all $n > N$. Similarly, $\overline{\lim}\ a_n$ means that for some $\epsilon_2 > 0$ we can find N s.t. $a_n \leq \overline{\lim}\ a_n + \epsilon_2$ for all $n > N$. If we multiply both sides by α, then $\alpha a_n \leq \alpha \overline{\lim}\ a_n + \alpha\epsilon_2$ for all $n > N$. Then if we take $\epsilon_1 = \alpha\epsilon_2$, both of these are the upper limit of αa_n, so $\overline{\lim}(\alpha a_n) = \alpha \overline{\lim}\ a_n$.

The same principle holds for the lower limit: for some $\epsilon_1 > 0$ we can find N s.t. $\alpha a_n \geq \underline{\lim}(\alpha a_n) - \epsilon_1$ for all $n > N$. Similarly, $\underline{\lim}\ a_n$ means that for some $\epsilon_2 > 0$ we can find N s.t. $a_n \geq \overline{\lim}\ a_n - \epsilon_2$ for all $n > N$. If we multiply both sides by α, then $\alpha a_n \geq \alpha \underline{\lim}\ a_n - \alpha\epsilon_2$ for all $n > N$. Then if we take $\epsilon_1 = \alpha\epsilon_2$, both of these are the lower limit of αa_n, so $\underline{\lim}(\alpha a_n) = \alpha \underline{\lim}\ a_n$. ☐

(b)

If $\alpha < 0$, prove $\overline{\lim}(\alpha a_n) = \alpha \underline{\lim}\ a_n$ and $\underline{\lim}(\alpha a_n) = \alpha \overline{\lim}\ a_n$.

Proof. By definition, $\overline{\lim}(\alpha a_n)$ means that for some $\epsilon_1 > 0$ we can find N s.t. $\alpha a_n \leq \overline{\lim}(\alpha a_n) + \epsilon_1$ for all $n > N$. Similarly, $\underline{\lim}\ a_n$ means that for some $\epsilon_2 > 0$ we can find N s.t. $a_n \geq \underline{\lim}\ a_n - \epsilon_2$ for all $n > N$. If we multiply both sides by α, then $\alpha a_n \leq \alpha \underline{\lim}\ a_n - \alpha\epsilon_2$ for all $n > N$ since $\alpha < 0$. Then if we take $\epsilon_1 = -\alpha\epsilon_2$, both of these are the upper limit of αa_n, so $\overline{\lim}(\alpha a_n) = \alpha \underline{\lim}\ a_n$.

The same principle holds for the lower limit: for some $\epsilon_1 > 0$ we can find N s.t. $\alpha a_n \geq \underline{\lim}(\alpha a_n) - \epsilon_1$ for all $n > N$. Similarly, $\overline{\lim}\ a_n$ means that for some $\epsilon_2 > 0$ we can find N s.t. $a_n \leq \overline{\lim}\ a_n + \epsilon_2$ for all $n > N$. If we multiply both sides by α, then $\alpha a_n \geq \alpha \overline{\lim}\ a_n + \alpha\epsilon_2$ for all $n > N$ since $\alpha < 0$. Then if we take $\epsilon_1 = -\alpha\epsilon_2$, both of these are the lower limit of αa_n, so $\underline{\lim}(\alpha a_n) = \alpha \overline{\lim}\ a_n$. ☐

2.3.4

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Assume $a_n \leq b_n$ for all n. Prove $\underline{\lim}\ a_n \leq \underline{\lim}\ b_n$ and $\overline{\lim}\ a_n \leq \overline{\lim}\ b_n$.

Proof. First, $\underline{\lim}\ a_n$ means that for some $\epsilon_a > 0$ we can find N_a s.t. $a_n \geq \underline{\lim}\ a_n - \epsilon_a$ for all $n > N_a$. Similarly, $\underline{\lim}\ b_n$ means that for some $\epsilon_b > 0$ we can find N s.t. $b_n \geq \underline{\lim}\ b_n - \epsilon_b$ for all $n > N$. Then we can add these two inequalities to get $a_n + b_n \geq \underline{\lim}\ a_n + \underline{\lim}\ b_n - (\epsilon_a + \epsilon_b)$. We then know that $2b_n \geq a_n + b_n \geq \underline{\lim}\ a_n + \underline{\lim}\ b_n - (\epsilon_a + \epsilon_b)$ and taking the lower limit of each side, we get $2\underline{\lim}\ b_n \geq \underline{\lim} b_n + \underline{\lim}\ a_n - (\epsilon_a + \epsilon_b)$ since lower limits and both ϵ are constant. So we get $\underline{\lim}\ b_n \geq \underline{\lim}\ a_n - (\epsilon_a + \epsilon_b)$ and letting both $\epsilon_a, \epsilon_b \to 0$, $\underline{\lim}\ b_n \geq \underline{\lim}\ a_n$.

Similarly for the upper limits, $\overline{\lim}\ a_n$ means that for some $\epsilon_a > 0$ we can find N_a s.t. $a_n \leq \overline{\lim}\ a_n + \epsilon_a$ for all $n > N_a$. Similarly, $\overline{\lim}\ b_n$ means that for some $\epsilon_b > 0$ we can find N s.t. $b_n \leq \overline{\lim}\ b_n + \epsilon_b$ for all $n > N$. Then we can add these two inequalities to get $a_n + b_n \leq \overline{\lim}\ a_n + \overline{\lim}\ b_n + (\epsilon_a + \epsilon_b)$. We then know that $2a_n \leq a_n + b_n \leq \overline{\lim}\ a_n + \overline{\lim}\ b_n + (\epsilon_a + \epsilon_b)$ and taking the upper limit of each side, we get $2\overline{\lim}\ a_n \leq \overline{\lim} b_n + \overline{\lim}\ a_n + (\epsilon_a + \epsilon_b)$ since upper limits and both ϵ are constant. So we get $\overline{\lim}\ a_n \leq \overline{\lim}\ b_n + (\epsilon_a + \epsilon_b)$ and letting both $\epsilon_a, \epsilon_b \to 0$, $\overline{\lim}\ a_n \leq \overline{\lim}\ b_n$. ☐

Section 3.1

3.1.1

Let f be a function defined on a set S and c be a cluster point of S. Suppose $f(x) \to l$ as $x \to c$. Prove $|f(x)| \to |l|$ as $x \to c$.

Proof. Since $f(x) \to l$, we know that for some $\epsilon > 0$, $|f(x) - l| < \epsilon$ when $|x - c| < \delta$. By the inverse triangle inequality, $||f(x)| - |l|| \leq |f(x) - l| < \epsilon$, so $||f(x)| - |l|| < \epsilon$ when $|x - c| < \delta$. Thus $|f(x)| \to |l|$ when $x \to c$. $\square$

3.1.2

Let $f(x) = 2x - 1$ and let ϵ be a given positive number. Find the best δ so that $|x - 1| < \delta$ implies $|f(x) - 1| < \epsilon$?

For $|f(x) - 1| < \epsilon$ let's substitute to get $|2x - 1 - 1| = |2x - 2| < \epsilon$. Then $1 - \delta < x < 1 + \delta$ so WLOG let's assume x is the maximum, $1 + \delta$. Then we have $|2(1 + \delta) - 2| = |2 + 2\delta - 2| = |2\delta| < \epsilon$, so we choose $\delta < \frac{\epsilon}{2}$. It is trivial to see that this works when x is the minimum, or $1 - \delta$.

3.1.3

Let $f(x) = x^2$ and let ϵ be a given positive number.

(a)

Find a δ so that $|x - 1| < \delta$ implies $|f(x) - 1| < \epsilon$.

For $|f(x) - 1| < \epsilon$ let's substitute to get $|x^2 - 1| < \epsilon$. Then $1 - \delta < x < 1 + \delta$ so WLOG let's assume x is the maximum, $1 + \delta$. Then we have $|(1 + \delta)^2 - 1| = |1 + 2\delta + \delta^2 - 1| = |(2 + \delta)\delta| < |3\delta| < \epsilon$ since δ is small. So we choose $\delta < \frac{\epsilon}{3}$. It is trivial to see that this works when x is the minimum, or $1 - \delta$.

(b)

For each $c > 0$, find a δ so that $|x - c| < \delta$ implies $|f(x) - c^2| < \epsilon$.

For $|f(x) - c^2| < \epsilon$ let's substitute to get $|x^2 - c^2| < \epsilon$. Then $c - \delta < x < c + \delta$ so WLOG let's assume x is the maximum, $c + \delta$. Then we have $|(c + \delta)^2 - c^2| = |c^2 + 2c\delta + \delta^2 - c^2| = |(2c + \delta)\delta| < |3c\delta| < \epsilon$ since δ is small. So we choose $\delta < \frac{\epsilon}{3c}$. It is trivial to see that this works when x is the minimum, or $c - \delta$.

3.1.4

Consider $f(x) = \sqrt{x}$ in $[0, \infty)$ and let $\epsilon > 0$ be a given positive number.

(a)

For each $c > 0$, find δ such that $|x - c| < \delta$ implies $|\sqrt{x} - \sqrt{c}| < \epsilon$.

We know $c - \delta < x < c + \delta$ so WLOG let's assume x is the maximum, $c + \delta$. Then we have $|\sqrt{c + \delta} - \sqrt{c}| = \sqrt{c + \delta} - \sqrt{c} < \epsilon$ since $c, \delta > 0$. Then $\sqrt{c + \delta} - \sqrt{c} < \epsilon \Rightarrow \sqrt{c + \delta} < \epsilon + \sqrt{c} \Rightarrow c + \delta < (\epsilon + \sqrt{c})^2 \Rightarrow \delta < (\epsilon + \sqrt{c})^2 - c = \epsilon^2 + 2\epsilon\sqrt{c} = \epsilon(\epsilon + 2\sqrt{c})$ So we choose $\delta < \epsilon(\epsilon + 2\sqrt{c})$. It is trivial to see that this works when x is the minimum, or $c - \delta$.

(b)

Find δ such that $0 \leq x < \delta$ implies $|\sqrt{x} - 0| < \epsilon$.

$|\sqrt{\delta}| < \epsilon \Rightarrow \sqrt{\delta} < \epsilon \Rightarrow \delta < \epsilon^2$ so take any $\delta < \epsilon^2$.