Final Exam

Walker Bagley

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1.

$$\begin{cases} u_{tt} - 25u_{xx} = 2t - 3t^2 \\ u(x,0) = e^{-|x|} \\ u_t(x,0) = \cos(4x) \end{cases}$$

$$\text{Let } v \text{ be the solution to } \begin{cases} u_{tt} - 25u_{xx} = 0 \\ u(x,0) = e^{-|x|} \\ u_t(x,0) = \cos(4x) \end{cases}$$

$$\text{Let } w \text{ be the solution to } \begin{cases} u_{tt} - 25u_{xx} = 2t - 3t^2 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

$$\text{Notice that } u = v + w$$

$$\begin{aligned}
\text{Using d'Alembert's we see } v(x,t) &= \frac{1}{2}\left[e^{-|x+5t|} + e^{-|x-5t|}\right] + \frac{1}{10}\int_{x-5t}^{x+5t} \cos(4s)ds \\
&= \frac{1}{2}\left[e^{-|x+5t|} + e^{-|x-5t|}\right] + \frac{1}{40}\left(\sin(4s)|_{x-5t}^{x+5t}\right. \\
&= \frac{1}{2}\left[e^{-|x+5t|} + e^{-|x-5t|}\right] + \frac{1}{40}\left[\sin(4x+20t) - \sin(4x-20t)\right] \\
&= \frac{1}{2}\left[e^{-|x+5t|} + e^{-|x-5t|}\right] + \frac{1}{20}\left[\sin(20t)\cos(4x)\right] \\
&= \frac{1}{2}\left[e^{-|x+5t|} + e^{-|x-5t|} + \frac{1}{10}\sin(20t)\cos(4x)\right]
\end{aligned}$$

$$\begin{aligned}
w(x,t) &= \frac{1}{10}\int_0^t \int_{x-5(t-s)}^{x+5(t-s)} 2s - 3s^2 dyds \\
&= \frac{1}{10}\int_0^t \left[y(2s-3s^2)|_{x-5(t-s)}^{x+5(t-s)}\right] ds = \frac{1}{10}\int_0^t 10(t-s)(2s-3s^2)ds \\
&= \int_0^t 2ts - 3ts^2 - 2s^2 + 3s^3 ds = \left(ts^2 - ts^3 - \frac{2}{3}s^3 + \frac{3}{4}s^4\right|_0^t \\
&= t^3 - t^4 - \frac{2}{3}t^3 + \frac{3}{4}t^4 = \frac{1}{3}t^3 - \frac{1}{4}t^4
\end{aligned}$$

$$u(x,t) = v(x,t) + w(x,t) = \frac{1}{2}\left[e^{-|x+5t|} + e^{-|x-5t|} + \frac{1}{10}\sin(20t)\cos(4x)\right] + \frac{1}{3}t^3 - \frac{1}{4}t^4$$

2.

$$\begin{cases} u_t - u_{xx} = -u & -\infty < x < \infty,\ t > 0 \\ u(x,0) = e^{-x^2} & -\infty < x < \infty \end{cases}$$

Make the transformation $u(x,t) = e^{-t}v(x,t)$

$$u_t = -e^{-t}v + e^{-t}v_t$$

$$u_{xx} = e^{-t}v_{xx}$$

Then the equation is reduced to $v_t - v_{xx} = 0$

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-y)^2} e^{-y^2} dy \\ &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}[(x-y)^2 + 4ty^2]} dy \end{aligned}$$

$$(x-y)^2 + 4ty^2 = x^2 - 2xy + y^2 + 4ty^2 = \left[\frac{1}{\sqrt{1+4t}}x - \sqrt{1+4t}y\right]^2 + \frac{4t}{1+4t}x^2$$

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}\left(\left[\frac{1}{\sqrt{1+4t}}x - \sqrt{1+4t}y\right]^2 + \frac{4t}{1+4t}x^2\right)} dy \\ &= \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{1+4t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}\left(\frac{1}{\sqrt{1+4t}}x - \sqrt{1+4t}y\right)^2} dy \end{aligned}$$

$$\text{Letting } z = \frac{1}{\sqrt{4t+16t^2}}x - \frac{\sqrt{1+4t}}{\sqrt{4t}}y$$

$$dz = -\frac{\sqrt{1+4t}}{\sqrt{4t}}dy \quad \Rightarrow \quad dy = -\frac{\sqrt{4t}}{\sqrt{1+4t}}dz$$

$$\begin{aligned} v(x,t) &= -\frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{1+4t}} \frac{\sqrt{4t}}{\sqrt{1+4t}} \int_{-\infty}^{\infty} e^{-z^2} dz \\ &= -\frac{1}{\sqrt{\pi(1+4t)}} e^{-\frac{x^2}{1+4t}} \sqrt{\pi} \\ &= -\frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+4t}x^2} \end{aligned}$$

$$u(x,t) = -\frac{1}{\sqrt{1+4t}} e^{-\frac{1}{1+4t}x^2 - t}$$

3.

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < \pi,\ t > 0 \\ u_x(0,t) = u_x(\pi,t) = 0 & t > 0 \\ u(x,0) = f(x) - \cos(4x) & 0 < x < \pi \end{cases}$$

$$f(x) = \begin{cases} \frac{\pi}{2} - x & 0 \le x \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < x < \pi \end{cases}$$

$$f(x) = \begin{cases} \frac{\pi}{2} - x - \cos(4x) & 0 \le x \le \frac{\pi}{2} \\ -\cos(4x) & \frac{\pi}{2} < x < \pi \end{cases}$$

$$A_0 = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{\pi}{2} - x dx\right] = \frac{2}{\pi}\left(\frac{\pi}{2}x - \frac{1}{2}x^2\bigg|_0^{\frac{\pi}{2}}\right) = \frac{2}{\pi}\left(\frac{\pi^2}{4} - \frac{\pi^2}{8}\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$
\begin{aligned}
A_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)\cos(nx)dx\right] \\
&= \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{\pi}{2}\cos(nx) - x\cos(nx)dx\right] \\
&= \frac{2}{\pi}\left[\left(\frac{\pi}{2n}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} x\cos(nx)dx\right]\right. \\
&= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\Big(\frac{n\pi}{2}\Big) - \left(\left[\frac{x}{n}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\frac{1}{n}\sin(nx)dx\right)\right]\right. \\
&= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\Big(\frac{n\pi}{2}\Big) - \frac{\pi}{2n}\sin\Big(\frac{n\pi}{2}\Big) - \left(\frac{1}{n^2}\cos(nx)\Big|_0^{\frac{\pi}{2}}\right]\right. = \frac{2}{n^2\pi}\left(1-\cos\Big(\frac{n\pi}{2}\Big)\right)
\end{aligned}
$$

$$
u(x,t) = \frac{\pi}{8} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}\left(1-\cos\Big(\frac{n\pi}{2}\Big)\right)\cos(nx)e^{-n^2kt} - e^{-16kt}\cos(4x)
$$

4.

$$
\begin{cases} u_t - \frac{1}{2}u_{xx} = 1 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = -x^2 \end{cases}
$$

$$
\text{Let } v \text{ be the solution to } \begin{cases} u_t - \frac{1}{2}u_{xx} = 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = -x^2 \end{cases}
$$

$$
\text{Let } w \text{ be the solution to } \begin{cases} u_t - \frac{1}{2}u_{xx} = 1 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 0 \end{cases}
$$

Notice that $u = v + w$

$$
\begin{aligned}
FS(-x^2) &= -FS(x^2) \\
&= -\left(\frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4(-1)^n\frac{\cos(nx)}{n^2}\right) \\
&= -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4(-1)^{n+1}\frac{\cos(nx)}{n^2} \\
v(x,t) &= -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4(-1)^{n+1}\frac{\cos(nx)}{n^2}e^{-\frac{1}{2}n^2t}
\end{aligned}
$$

$$
w(x,t) = \int_0^t \frac{1}{\sqrt{2\pi(t-\tau)}}\int_0^{\pi} e^{-\frac{1}{2(t-\tau)}(x-y)^2}dyd\tau
$$

$$
\text{Changing variables } z = \frac{x-y}{\sqrt{2(t-\tau)}} \quad \Rightarrow \quad dz = -\frac{1}{\sqrt{2(t-\tau)}}dy
$$

$$
\text{Changing bounds for } z \begin{cases} y = 0 \quad z = \frac{x}{\sqrt{2\pi(t-\tau)}} \\ y = \pi \quad z = \frac{x-\pi}{\sqrt{2\pi(t-\tau)}} \end{cases}
$$

$$
\begin{aligned}
w(x,t) &= \int_0^t -\frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{2\pi(t-\tau)}}}^{\frac{x-\pi}{\sqrt{2\pi(t-\tau)}}} e^{-z^2}\,dz d\tau \\
&= \int_0^t -\frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{x-\pi}{\sqrt{2\pi(t-\tau)}}\right) - \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2\pi(t-\tau)}}\right)\right] d\tau \\
&= \int_0^t \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2\pi(t-\tau)}}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{x-\pi}{\sqrt{2\pi(t-\tau)}}\right) d\tau
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) = v(x,t) + w(x,t) &= -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} 4(-1)^{n+1} \frac{\cos(nx)}{n^2} e^{-\frac{1}{2}n^2 t} \\
&+ \int_0^t \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2\pi(t-\tau)}}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{x-\pi}{\sqrt{2\pi(t-\tau)}}\right) d\tau
\end{aligned}
$$

5.

(a)

$$
\begin{aligned}
u_{xx} + u_{yy} &> 0 \quad (x,y) \in D = \mathbb{R}^2 \\
\text{We say } \delta D &= \text{the boundary of } D \\
\text{Let } M &= \max(u(x,y)) \quad (x,y) \in \delta D \text{ where } (x_m, y_m) \text{ denotes the point } M = u(x_m, y_m) \\
\text{Define } v_\epsilon(x,y) &= u(x,y) + \epsilon(x^2+y^2) \text{ where } \epsilon > 0 \text{ and } (x,y) \in D \\
\Delta v_\epsilon &= \Delta u + \epsilon(4) \\
\text{Since } \Delta u &= 0 \text{ by definition of Laplacian} \\
\Delta v_\epsilon &= 4\epsilon > 0 \\
\text{So, } v_\epsilon(x,y) &< v_\epsilon(x_m, y_m) \quad (x,y) \in D \\
\text{Then for } (x,y) \in D,\ u(x,y) \le v_\epsilon(x,y) &< v_\epsilon(x_m,y_m) = u(x_m,y_m) + \epsilon|(x_m,y_m)|^2 \le M + \epsilon R^2 \\
&\text{where } R \text{ is the radius of a circle enclosing } D
\end{aligned}
$$

So, for $\epsilon > 0$ we have $u(x,y) < M + \epsilon R^2$

As $\epsilon \to 0$, we get $u(x,y) < M$

(b)

$$
\begin{cases}
u_{xx} + u_{yy} = 0 & -\infty < x < \infty,\ y > 0 \\
u(x,0) = 0 & -\infty < x < \infty
\end{cases}
$$

The solutions u then consist of the sum of solutions for DP2, DP3 and DP4

Note that each of these solutions is a variation of the solution for DP1 with different constraints

Let l be our upper bound in the x direction

Let L be our upper bound in the y direction

Case 1: DP2

$$
\begin{aligned}
u(x,L) &= f_2(x) \quad 0 < x < l \\
u_2(x,y) &= \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{l}\right) \frac{\sinh\left(\frac{n\pi}{l} y\right)}{\sinh\left(\frac{n\pi L}{l}\right)} \\
B_n &= \frac{2}{l} \int_0^l f_2(x) \sin\left(\frac{n\pi x}{l}\right) dx
\end{aligned}
$$

Case 2: DP3

$$u(0,y) = g_1(x) \quad 0 < y < L$$
$$v_1(x,y) = \sum_{n=1}^{\infty} B_n \sin\Big(\frac{n\pi y}{L}\Big) \frac{\sinh\big(\frac{n\pi}{L}(l-x)\big)}{\sinh\big(\frac{n\pi l}{L}\big)}$$
$$B_n = \frac{2}{L}\int_0^L g_1(x) \sin\Big(\frac{n\pi y}{L}\Big) dy$$

Case 3: DP4

$$u(l,y) = g_2(x) \quad 0 < y < L$$
$$v_2(x,y) = \sum_{n=1}^{\infty} B_n \sin\Big(\frac{n\pi y}{L}\Big) \frac{\sinh\big(\frac{n\pi}{L}x\big)}{\sinh\big(\frac{n\pi l}{L}\big)}$$
$$B_n = \frac{2}{L}\int_0^L g_2(x) \sin\Big(\frac{n\pi y}{L}\Big) dy$$

6.

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 \le r < a,\ 0 \le \theta \le 2\pi \\ u(a,\theta) = f(\theta) & 0 \le \theta \le 2\pi \end{cases}$$

$$u(r,\theta) = R(r)\Theta(\theta)$$
$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$
$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$
$$\begin{cases} r^2R'' + rR' - \lambda R = 0 \\ \Theta'' + \lambda\Theta = 0 & \Theta(0) = \Theta(2\pi),\ \Theta'(0) = \Theta'(2\pi) \end{cases}$$
$$\lambda = n^2$$
$$\Theta_n(\theta) = a_n\cos(n\theta) + b_n\sin(n\theta)$$
$$R = r^\alpha$$
$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - n^2r^\alpha = 0$$
$$\alpha^2 - n^2 = 0$$

Since R must be smooth, we have $R_n(r) = r^n$

$$u_n(r,\theta) = r^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$
$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} r^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$
$$u(a,\theta) = f(\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$
$$a^na_n = A_n$$
$$a^nb_n = B_n$$
$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n(A_n\cos(n\theta) + B_n\sin(n\theta))$$

7.

$$\text{Disc: } \{(r,\theta) : r < 7 \text{ and } -\pi < \theta \leq \pi\}$$

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(7,\theta) = f(\theta) \end{cases}$$

$$f(\theta) = \begin{cases} \frac{\pi}{2} - |\theta| & |\theta| \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < |\theta| < \pi \end{cases}$$

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos(n\theta) + B_n \sin(n\theta))$$

Since $f(\theta)$ is even, $B_n = 0$

$$A_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta) d\theta = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} \frac{\pi}{2} - \theta d\theta + \int_{\frac{\pi}{2}}^{\pi} 0 d\theta\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\theta - \frac{1}{2}\theta^2\Big|_0^{\frac{\pi}{2}}\right] = \frac{2}{\pi}\left[\frac{\pi^2}{4} - \frac{\pi^2}{8}\right] = \frac{\pi}{4}$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(\theta)\cos(n\theta) d\theta$$

Applying the same integration as (3), $A_n = \dfrac{2}{n^2\pi}\left[1 - \cos\!\left(\dfrac{n\pi}{2}\right)\right]$

$$u(r,\theta) = \frac{\pi}{8} + \sum_{n=1}^{\infty}\left[\left(\frac{r}{7}\right)^2 \frac{2}{n^2\pi}\left(1 - \cos\!\left(\frac{n\pi}{2}\right)\right)\cos(n\theta)\right]$$

8.

$$\text{Wedge: } \left\{(r,\theta) : 0 < r < 7 \text{ and } 0 < \theta \leq \frac{\pi}{2}\right\}$$

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(r,0) = u(r,\frac{\pi}{2}) = 0 \\ u(3,\theta) = f(\theta) \end{cases}$$

$$f(\theta) = \begin{cases} \theta & 0 \leq \theta \leq \frac{\pi}{4} \\ \frac{\pi}{2} - \theta & \frac{\pi}{4} \leq \theta \leq \frac{\pi}{2} \end{cases}$$

$$u(r,\theta) = \sum_{n=1}^{\infty} B_n \left(\frac{r}{a}\right)^{\frac{n\pi}{\alpha}} \sin\!\left(\frac{n\pi\theta}{\alpha}\right)$$

$$B_n = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)\sin\!\left(\frac{n\pi\theta}{\alpha}\right) d\theta$$

$$= \frac{4}{\pi}\left[\int_0^{\frac{\pi}{4}} \theta\sin(2n\theta) d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{\pi}{2} - \theta\right)\sin(2n\theta) d\theta\right]$$

$$\int_0^{\frac{\pi}{4}} \theta\sin(2n\theta) d\theta = \left(-\frac{\theta}{2n}\cos(2n\theta)\Big|_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{2n}\cos(2n\theta) d\theta\right.$$

$$= -\frac{\pi}{8n}\cos\!\left(\frac{n\pi}{2}\right) + \left(\frac{1}{4n^2}\sin(2n\theta)\Big|_0^{\frac{\pi}{4}}\right.$$

$$= -\frac{\pi}{8n}\cos\!\left(\frac{n\pi}{2}\right) + \frac{1}{4n^2}\sin\!\left(\frac{n\pi}{2}\right)$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{\pi}{2} - \theta\right) \sin(2n\theta) d\theta = \left(-\frac{1}{2n}\left(\frac{\pi}{2} - \theta\right)\cos(2n\theta)\bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2n}\cos(2n\theta)d\theta\right.$$

$$= \frac{\pi}{8n}\cos\left(\frac{n\pi}{2}\right) - \left(\frac{1}{4n^2}\sin(2n\theta)\bigg|_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right.$$

$$= \frac{\pi}{8n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{4n^2}\sin\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{4}{\pi}\left[-\frac{\pi}{8n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{4n^2}\sin\left(\frac{n\pi}{2}\right) + \frac{\pi}{8n}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{4n^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{4}{\pi}\left[\frac{1}{2n^2}\sin\left(\frac{n\pi}{2}\right)\right] = \frac{2}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left(\frac{r}{3}\right)^{2n}\sin\left(\frac{n\pi}{2}\right)\sin(2n\theta)$$

9.

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in D = (0,1)\times(0,1) \\ u(x,0) = u(x,1) = 0 \\ u(1,y) = 0 \\ u(0,y) = y - y^2 \end{cases}$$

Since we only have one side with data, $u(x,y) = v_1(x,y) =$ solution to DP3

$$v_1(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{L}\right)\frac{\sinh\left(\frac{n\pi}{L}(l-x)\right)}{\sinh\left(\frac{n\pi l}{L}\right)}$$

$$B_n = \frac{2}{L}\int_0^L g_1(y)\sin\left(\frac{n\pi y}{L}\right)dy = 2\int_0^1 (y-y^2)\sin(n\pi y)dy$$

$$= 2\left[\int_0^1 y\sin(n\pi y)dy - \int_0^1 y^2\sin(n\pi y)dy\right]$$

$$\int_0^1 y\sin(n\pi y)dy = \left(-\frac{y}{n\pi}\cos(n\pi y)\bigg|_0^1 + \int_0^1 \frac{1}{n\pi}\cos(n\pi y)dy\right.$$

$$= -\frac{1}{n\pi}\cos(n\pi) + \left(\frac{1}{n^2\pi^2}\sin(n\pi y)\bigg|_0^1 = \frac{1}{n\pi}(-1)^{n+1}\right.$$

$$\int_0^1 y^2\sin(n\pi y)dy = \left(-\frac{y^2}{n\pi}\cos(n\pi y)\bigg|_0^1 + \int_0^1 \frac{2y}{n\pi}\cos(n\pi y)dy\right.$$

$$= \frac{1}{n\pi}(-1)^{n+1} + \frac{1}{n\pi}\left[\left(\frac{2y}{n\pi}\sin(n\pi y)\bigg|_0^1 - \int_0^1 \frac{2}{n\pi}\sin(n\pi y)dy\right.\right]$$

$$= \frac{1}{n\pi}(-1)^{n+1} + \frac{1}{n\pi}\left(\frac{2}{n^2\pi^2}\cos(n\pi y)\bigg|_0^1\right.$$

$$= \frac{1}{n\pi}(-1)^{n+1} + \frac{2}{(n\pi)^3}(-1)^n - \frac{2}{(n\pi)^3}$$

$$= \frac{1}{n\pi}(-1)^{n+1} + \frac{2}{(n\pi)^3}((-1)^n - 1)$$

$$B_n = \frac{4}{(n\pi)^3}(1-(1)^n)$$

$$u(x,y) = v_1(x,y) = \sum_{n=1}^{\infty} \frac{4}{(n\pi)^3}(1-(1)^n)\sin(n\pi y)\frac{\sinh(n\pi(1-x))}{\sinh(n\pi)}$$

12.

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = x^2 \end{cases}$$

(i)

$$\begin{aligned} \frac{dx}{dt} &= u \\ x &= y + u_0 t \\ &= y + y^2 t \end{aligned}$$

(ii)

$$u = u_0(x - ut)$$

(iii)

$$\begin{aligned} u &= (x-ut)^2 \\ u &= x^2 - 2xtu + t^2u^2 \\ -x^2 &= t^2u^2 - (2xt+1)u \\ -x^2 &= \left(ut - \left[\frac{2xt+1}{2t}\right]\right)^2 - \left[\frac{2xt+1}{2t}\right]^2 \\ \left[\frac{2xt+1}{2t}\right]^2 - x^2 &= \left(ut - \left[\frac{2xt+1}{2t}\right]\right)^2 \\ \frac{4x^2t^2 - 4xt + 1 - 4x^2t^2}{4t^2} &= \left(ut - \left[\frac{2xt+1}{2t}\right]\right)^2 \\ \pm\sqrt{\frac{1-4xt}{4t^2}} &= ut - \frac{2xt+1}{2t} \\ ut &= \frac{\pm\sqrt{1-4xt}}{2t} + \frac{2xt+1}{2t} \\ u &= \frac{\pm\sqrt{1-4xt} + 2xt + 1}{2t^2} \end{aligned}$$

(iv)

$$\begin{aligned} 2t^2 &\neq 0 \quad \Rightarrow \quad t \neq 0 \\ 1 - 4xt &> 0 \\ \frac{1}{4} &> xt \end{aligned}$$