Homework 3

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(2)

Find all integer solutions of the equation $2023x + 1001y = 21$.

$$\begin{aligned} 2023 &= 1001 * 2 + 21 \\ 1001 &= 21 * 47 + 14 \\ 21 &= 14 * 1 + 7 \\ 14 &= 7 * 2 + 0 \\ (2023, 1001) &= 7 \end{aligned} \qquad \begin{aligned} 14 - 2 * 7 &= 0 \\ 14 - 2 * (21 - 14) &= 0 \\ 3 * (1001 - 47 * 21) - 2 * 21 &= 0 \\ 3 * 1001 - 143 * (2023 - 2 * 1001) &= 0 \\ 289 * 1001 - 143 * 2023 &= 0 \end{aligned}$$

Then taking the trivial solution $x = 1, y = -2 \Rightarrow 2023 * 1 + 1001 * (-2) = 21$ and the homogeneous solution $x = -143k, y = 289k \Rightarrow 2023 * (-143k) + 1001 * (289k) = 0$ where $k \in \mathbb{Z}$ we have $x = 1 - 143k$ and $y = -2 + 289k$ as all integer solutions.

(3)

Determine two positive integers x, y such that $17x + 12y = 101$.

$$\begin{aligned} 17 &= 12 * 1 + 5 \\ 12 &= 5 * 2 + 2 \\ 5 &= 2 * 2 + 1 \\ (17, 12) &= 1 \end{aligned} \qquad \begin{aligned} 5 - 2 * 2 &= 1 \\ 5 - 2 * (12 - 2 * 5) &= 1 \\ 5 * (17 - 12) - 2 * 12 &= 1 \\ 5 * 17 - 7 * 12 &= 1 \\ 101[5 * 17 - 7 * 12] &= 101 \\ 505 * 17 - 707 * 12 &= 101 \end{aligned}$$

Then we have $x = 505$ and $y = -707$ as a solution.

(4)

Solve explicitly the equation $2x + 3y + 5z = 7$ in the integers. [Hint: First solve $2x + 3y = N$ for an arbitrary N.]

We know that $(-1) * 2 + (1) * 3 = 1$, so $(-N) * 2 + (N) * 3 = N$ taking $x = -N$ and $y = N$. So, our equation then becomes $N + 5z = 7 \Rightarrow N = 7 - 5z$. So, taking any $z \in \mathbb{Z}$, we have a solution at $x = 5z - 7$ and $y = 7 - 5z$.

(5)

Find two integers x, y such that $455x + 1235y = (455, 1235)$.

$$\begin{aligned} 1235 &= 455 * 2 + 325 \\ 455 &= 325 * 1 + 130 \\ 325 &= 130 * 2 + 65 \\ 130 &= 65 * 2 + 0 \\ (455, 1235) &= 65 \end{aligned}$$

$$\begin{aligned} 325 - 2 * 130 &= 65 \\ 325 - 2 * (455 - 325) &= 65 \\ 3 * (1235 - 2 * 455) - 2 * 455 &= 65 \\ 3 * 1235 - 8 * 455 &= 65 \end{aligned}$$

Then we have $x = -8$ and $y = 3$ as a solution.

(8)

Suppose you divide the positive integers m and n with remainder, and get $m = nq + r$ with $0 \leq r < n$. Find the integer quotient Q such that $2^m - 1 = (2^n - 1)Q + 2^r - 1$, is the division with remainder of $2^m - 1$ by $2^n - 1$ yielding the remainder $0 \leq 2^r - 1 < 2^n - 1$. [Of course, you can always solve for Q in this equation, the point of the problem is to show that Q is an integer.]

We need a 2^m, so lets start with $Q = 2^{m-n}$. Then we have $(2^n - 1)2^{m-n} + 2^r - 1 = 2^m - 2^{m-n} + 2^r - 1$. Then we need to get rid of the 2^{m-n} term, which we can do by adding another term to Q. Then $Q = 2^{m-n} + 2^{m-2n}$ means that $(2^n - 1)(2^{m-n} + 2^{m-2n}) + 2^r - 1 = 2^m - 2^{m-2n} + 2^r - 1$. We continue this process until we have $m - kn = r$ for some $k \in \mathbb{Z}$ so that $(2^n - 1)Q + 2^r - 1 = 2^n + (2^{m-n} - 2^{m-n}) + (2^{m-2n} - 2^{m-2n}) + \ldots + (2^r - 2^r) - 1 = 2^n - 1$. So $Q = 2^{m-n} + 2^{m-2n} + \ldots + 2^r$. Since $0 \leq r < n$ and n, m, r are positive integers, all of these terms must be integers. Then Q must be an integer.