Homework 1

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Section 0

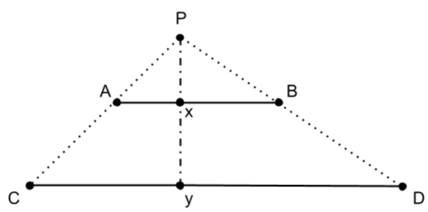
3. $\{m \in \mathbb{Z} | mn = 60 \text{ for some } n \in \mathbb{Z}\}$
$= \{-1, 1, -2, 2, -3, 3, -4, 4, -5, 5, -6, 6, -10, 10, -12, 12, -15, 15, -20, 20, -30, 30, -60, 60\}$

5. $\{n \in \mathbb{Z}^+ | n \text{ is a large number}\}$
This set is not well defined because it is unclear what defines a large number.

7. $\{n \in \mathbb{Z} | 39 < n^3 < 57\}$
$= \varnothing$

11. $\{a, b, c\} \times \{1, 2, c\}$
$= \{(a, 1), (a, 2), (a, c), (b, 1), (b, 2), (b, c), (c, 1), (c, 2), (c, c)\}$

12. $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$
   (a) $\{(1, 4), (2, 4), (3, 6)\}$ is a function
   (b) $\{(1, 4), (2, 6), (3, 4)\}$ is a function
   (c) $\{(1, 6), (1, 2), (1, 4)\}$ is not a function
   (d) $\{(2, 2), (1, 6), (3, 4)\}$ is a function that is one to one and onto
   (e) $\{(1, 6), (2, 6), (3, 6)\}$ is a function
   (f) $\{(1, 2), (2, 6), (2, 4)\}$ is not a function

13. 

15. Prove that for $S = \{x \in \mathbb{R} | 0 < x < 1\}$, $|S| = |\mathbb{R}|$

Proof.

Consider a function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ s.t. $f(x) = \tan x$, which maps $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ one to one onto $\mathbb{R}$

We know $\tan x$ has an inverse $\arctan x$, so f must be a bijection and therefore $\left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right| = |\mathbb{R}|$

Let's define $g : (0, 1) \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by $g(x) = \pi x - \frac{\pi}{2}$, which is one to one

Then, we can say that g^{-1} is defined by $x = \dfrac{g(x) + \frac{\pi}{2}}{\pi}$

This makes g a bijection and therefore $|(0, 1)| = \left|\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right|$

By composing f and g and then using transitivity, we know that $|S| = |\mathbb{R}|$

$\square$

16. (a) $\mathscr{P}(\varnothing) = \{\varnothing\}$
    (b) $\mathscr{P}(\{a\}) = \{\varnothing, \{a\}\}$
    (c) $\mathscr{P}(\{a, b\}) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$
    (d) $\mathscr{P}(\{a, b, c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

17. For some finite set A s.t. $|A| = s$, $|\mathscr{P}(A)| = 2^s$

    Proof.

    $$\text{Base case: } |A| = 0. \text{ Then } A = \varnothing \text{ and } \mathscr{P}(A) = \{\varnothing\}. \text{ Then } |\mathscr{P}(A)| = 1 = 2^0$$
    $$\text{Assume that for some } A \text{ s.t. } |A| = s, |\mathscr{P}(A)| = 2^s$$
    $$\text{WTS that for } |A| = s+1, |\mathscr{P}(A)| = 2^{s+1}$$
    $$\text{When adding an element to } A, \mathscr{P}(A_{s+1}) \text{ contains all elements of } \mathscr{P}(A_s)$$
    $$\text{It also contains } \mathscr{P}(A_s) \bigcup \{a_{s+1}\}$$
    $$\text{This means that } \mathscr{P}(A_{s+1}) = 2 * |\mathscr{P}(A_s)|$$
    $$\text{Substituting the inductive hypothesis, } \mathscr{P}(A_{s+1}) = 2 * 2^s = 2^{s+1}$$

    □

25. Given a set with 3 elements $A = \{a, b, c\}$, there are 5 partitions as follows:
    - $\{\{a, b, c\}, \varnothing\}$
    - $\{\{a, b\}, \{c\}\}$
    - $\{\{a\}, \{b, c\}\}$
    - $\{\{a, c\}, \{b\}\}$
    - $\{\{a\}, \{b\}, \{c\}\}$

28. To illustrate why the relation $\mathscr{R}$ corresponding to a partition of S satisfies the reflexive condition, we need only observe that if x is in a cell, it must then be in the same cell as itself, meaning $x \sim x$.

    To illustrate why the relation $\mathscr{R}$ corresponding to a partition of S satisfies the transitive condition, we need only observe that if x, y are in the same cell ($x \sim y$), and y, z are in the same cell ($y \sim z$), then x, z must be in the same cell, meaning $x \sim z$.

31. $x \sim y$ in R if $|x| = |y|$ is an equivalence relation because the reflexive, symmetric and transitive properties hold.
    $\bar{0} = \{0\}$ and $\bar{a} = \{a, -a\}$ for any nonzero $a \in \mathbb{R}$

32. $x \sim y$ in R if $|x - y| \leq 3$ is not an equivalence relation because it does not satisfy the transitive property. Consider $x = 4, y = 2, z = 0$. $x \sim y$ and $y \sim z$, but $|4 - 0| = 4 > 3$, so $x \not\sim z$.

Section 1

3. $i^{23} = i^{4*5+3} = i^3 * i^{4*5} = i^3 * 1 = -1 * i = -i$

7. $(2 - 3i)(4 + i) + (6 - 5i) = (8 - 10i + 3) + (6 - 5i) = 17 - 15i$

11. $|6 + 4i| = \sqrt{6^2 + 4^2} = \sqrt{36 + 16} = \sqrt{52} = 2\sqrt{13}$

13. $|-1 + i| = \sqrt{2}$ and $\theta = 3\pi/4$, so $-1 + i = \sqrt{2}(cos(3\pi/4) + isin(3\pi/4)) = \sqrt{2}(-\sqrt{2}/2 + i\sqrt{2}/2)$

19. $z^3 = -27i \Rightarrow |z|^3 = 27 \Rightarrow |z| = 3$
    3 solutions separated by $2\pi/3$
    $-27i \Rightarrow \theta = 3\pi/2$ so solution 1 is at $(3\pi/2)/3 = \pi/2$
    solutions are at $\pi/2, 7\pi/6, 11\pi/6$ which equates to $i, \pm\sqrt{3}/2 - i/2$ on the unit circle
    $|z| = 3 \Rightarrow 3i, \pm 3\sqrt{3}/2 - i3/2$

23. $8 +_{10} 6 = 4$

29. $x +_{15} 7 = 3 \Rightarrow x = 11$

33. $x +_{12} x = 2 \Rightarrow x = 1, 7$

35. $\zeta^0 \leftrightarrow 0, \zeta^3 \leftrightarrow 7, \zeta^4 \leftrightarrow 4, \zeta^5 \leftrightarrow 1, \zeta^6 \leftrightarrow 6, \zeta^7 \leftrightarrow 3$

37. Because with modular arithmetic in $\mathbb{Z}_6$, ζ^n will always be even if $\zeta = 4$, making an isomorphism impossible.

39.

$$\begin{aligned}
z_1 &= |z_1|(\cos\theta_1 + i\sin\theta_1) \\
z_2 &= |z_2|(\cos\theta_2 + i\sin\theta_2) \\
z_1 z_2 &= |z_1||z_2|(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\
z_1 z_2 &= |z_1||z_2|[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1)] \\
z_1 z_2 &= |z_1||z_2|(cos(\theta_1 + \theta_2) + i(sin(\theta_1 + \theta_2))
\end{aligned}$$

40. (a)

$$\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
e^{3i\theta} &= \cos 3\theta + i\sin 3\theta \\
(e^{i\theta})^3 &= \cos 3\theta + i\sin 3\theta \\
(\cos\theta + i\sin\theta)^3 &= \cos 3\theta + i\sin 3\theta \\
(\cos\theta + i\sin\theta)(\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta) &= \cos 3\theta + i\sin 3\theta \\
\cos^3\theta - 3\sin^2\theta\cos\theta + 3i\sin\theta\cos^2\theta - i\sin^3\theta &= \cos 3\theta + i\sin 3\theta \\
\cos^3\theta - 3\cos\theta\sin^2\theta &= \cos 3\theta
\end{aligned}$$

(b)

$$\begin{aligned}
\cos^3\theta - 3\cos\theta\sin^2\theta &= \cos 3\theta \\
1 - \cos^2\theta &= \sin^2\theta \\
\cos^3\theta - 3\cos\theta(1 - \cos^2\theta) &= \cos 3\theta \\
\cos^3\theta - 3\cos\theta + 3\cos^3\theta &= \cos 3\theta \\
4\cos^3\theta - 3\cos\theta &= \cos 3\theta
\end{aligned}$$