Homework 8

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1. *Proof.* $\sum_{i=1}^{n}(i+1)2^i = n2^{n+1}$ for all $n \in \mathbb{N}\backslash\{0\}$

$$\text{Base case: } \sum_{i=1}^{1}(i+1)2^i = (1+1)2^1 = (2)2 = 4$$
$$1 * 2^{1+1} = 2^2 = 4$$
$$\text{Inductive Step: let } k \in \mathbb{N}\backslash\{0\} \text{ and assume } \sum_{i=1}^{k}(i+1)2^i = k2^{k+1}$$
$$\text{Observe } \sum_{i=1}^{k+1}(i+1)2^i = \left(\sum_{i=1}^{k}(i+1)2^i\right) + (k+2)2^{k+1}$$
$$\begin{aligned}\text{So, substituting the IH, } \sum_{i=1}^{k+1}(i+1)2^i &= k2^{k+1} + (k+2)2^{k+1} \\ &= k2^{k+1} + k2^{k+1} + 2 * 2^{k+1} \\ &= 2k2^{k+1} + 2^{k+2} \\ &= k2^{k+2} + 2^{k+2} \\ &= (k+1)2^{k+2}\end{aligned}$$
$$\text{Therefore, } \sum_{i=1}^{k+1}(i+1)2^i = (k+1)2^{k+2}$$

□

2. *Proof.* $(\forall n \in \mathbb{N})(n \text{ is transitive})$

Let $a \in \mathbb{N}$

Recall that by def, $a = \{a-1, a-2, a-3, ... \varnothing\}$

Let $b \in a$

So then $(\exists k \in \mathbb{N}\backslash\{0\})(b = a-k = \{a-k-1, a-k-2, ...\varnothing\})$

Then by def, $(\forall x \in b)(x \in a)$

So by def, $b \subseteq a$

Then $(\forall a \in \mathbb{N})(a \text{ is transitive})$

□

3. *Proof.* $\mathscr{F}(n) = \frac{1}{\sqrt{5}}(\varphi_1^n - \varphi_2^n)$ for all $n \in \mathbb{N}$, where $\varphi_1 := \frac{1+\sqrt{5}}{2}$ and $\varphi_2 := \frac{1-\sqrt{5}}{2}$

$$\text{Base Case 1: } \mathscr{F}(0) = 0 = \frac{1}{\sqrt{5}}(\varphi_1^0 - \varphi_2^0) = \frac{1}{\sqrt{5}}(1-1) = 0$$

$$\begin{aligned}\text{Base Case 2: } \mathscr{F}(1) = 1 = \frac{1}{\sqrt{5}}(\varphi_1^1 - \varphi_2^1) &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) \\ &= \frac{1}{2\sqrt{5}}(1+\sqrt{5}-1+\sqrt{5}) \\ &= \frac{1}{2\sqrt{5}}(2\sqrt{5}) = 1\end{aligned}$$

$$\text{Inductive Case: Assume } k \in \mathbb{N} \text{ and } (\forall l \in \mathbb{N})\left(l < k+2 \to \mathscr{F}(l) = \frac{1}{\sqrt{5}}(\varphi_1^l - \varphi_2^l)\right)$$

$$\mathscr{F}(k+2) = \mathscr{F}(k) + \mathscr{F}(k+1)$$

$$\begin{aligned}\text{Then, using the I.H., } \mathscr{F}(k+2) &= \frac{1}{\sqrt{5}}(\varphi_1^k - \varphi_2^k) + \frac{1}{\sqrt{5}}(\varphi_1^{k+1} - \varphi_2^{k+1}) \\ &= \frac{1}{\sqrt{5}}(\varphi_1^k - \varphi_2^k + \varphi_1^{k+1} - \varphi_2^{k+1}) \\ &= \frac{1}{\sqrt{5}}(\varphi_1^k + \varphi_1\varphi_1^k - (\varphi_2^k + \varphi_2\varphi_2^k)) \\ &= \frac{1}{\sqrt{5}}(\varphi_1^k(1+\varphi_1) - (\varphi_2^k(1+\varphi_2))) \\ \text{Using the hint, } &= \frac{1}{\sqrt{5}}(\varphi_1^k(\varphi_1^2) - (\varphi_2^k(\varphi_2^2))) \\ &= \frac{1}{\sqrt{5}}(\varphi_1^{k+2} - \varphi_2^{k+2})\end{aligned}$$

$$\text{Therefore, } (\forall n \in \mathbb{N})\left(\mathscr{F}(n) = \frac{1}{\sqrt{5}}(\varphi_1^n - \varphi_2^n)\right)$$

□

4. $R := \{(x,y) \in \mathbb{R} \times \mathbb{R} | x^2 + y^2 = 1\}$

(a) *Proof.* Not Reflexive

$$\text{Let } x = 2$$
$$\text{Since } x = y,\ x^2 + x^2 = 1, \text{ but } (2)^2 + (2)^2 = 8 \neq 1$$
$$\text{So, } (x,x) \notin R$$

□

(b) *Proof.* Not Irreflexive

$$\text{Let } x = \frac{1}{\sqrt{2}}$$
$$\text{Towards a contradiction, assume } x = y$$
$$\text{So, } x^2 + x^2 = 1, \text{ and } \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + \frac{1}{2} = 1$$
$$\text{So, } (x,x) \in R$$

□

(c) *Proof.* Symmetric

$$\text{Let } a, b \in \mathbb{R}$$
$$\text{Assume } (a,b) \in R$$
$$\text{By def, } a^2 + b^2 = 1$$
$$a^2 + b^2 = 1 = b^2 + a^2$$
$$\text{So, } (b,a) \in R$$

□

(d) *Proof.* Not Antisymmetric

$$\text{Let } a = 1 \text{ and } b = 0$$
$$(1)^2 + (0)^2 = 1, \text{ so } (a,b) \in R$$
$$\text{But also, } (0)^2 + (1)^2 = 1, \text{ so } (b,a) \in R$$
$$\text{However, } a \neq b$$

□

(e) *Proof.* Not Transitive

$$\text{Consider } a = 0,\ b = 1,\ c = 0$$
$$(0)^2 + (1)^2 = 1 \text{ and } (1)^2 + (0)^2 = 1 \text{ so } (a, b) \in R \land (b, c) \in R$$
$$\text{However, } (0)^2 + (0)^2 \neq 1, \text{ so } (a, c) \notin R$$

□

5. $R := \{(x, y) \in \mathbb{Z} \times \mathbb{Z} | x | (y + 1)\}$

(a) *Proof.* Not Reflexive

$$\text{Let } a = 5 \in \mathbb{Z}$$
$$\text{Observe, } 5 \nmid (5 + 1) \text{ because } 5 * \frac{6}{5} = 6 \text{ and } \frac{6}{5} \notin \mathbb{Z}$$
$$\text{So, } (a, a) \notin R$$

□

(b) *Proof.* Not Irreflexive

$$\text{Let } a = 1 \in \mathbb{Z}$$
$$\text{Observe, } 1 | (1 + 1) \text{ because } 1 * 2 = 2 \text{ and } 2 \in \mathbb{Z}$$
$$\text{So, } (a, a) \in R$$

□

(c) *Proof.* Not Symmetric

$$\text{Let } a = 5 \in \mathbb{Z} \text{ and } b = 9 \in \mathbb{Z}$$
$$\text{Observe, } 5 | (9 + 1) \text{ because } 5 * 2 = 10 \text{ and } 2 \in \mathbb{Z}$$
$$\text{So, } (a, b) \in R$$
$$\text{However, } 9 \nmid (5 + 1) \text{ because } 9 * \frac{2}{3} = 6 \text{ and } \frac{2}{3} \notin \mathbb{Z}$$
$$\text{So, } (b, a) \notin R$$

□

(d) *Proof.* Not Antisymmetric

$$\text{Let } a = 0 \in \mathbb{Z} \text{ and } b = -1 \in \mathbb{Z}$$
$$0 | (-1 + 1) \text{ because } (\forall k \in \mathbb{Z})(0 * k = 0), \text{ so } (a, b) \in R$$
$$\text{But also, } -1 | (0 + 1) \text{ because } (-1) * (-1) = 1 \text{ and } -1 \in \mathbb{Z}, \text{ so } (b, a) \in R$$

□

(e) *Proof.* Not Transitive

$$\text{Let } a = 2 \in \mathbb{Z},\ b = 5 \in \mathbb{Z},\ c = 14 \in \mathbb{Z}$$
$$\text{Observe } 2 | (5 + 1) \text{ because } 2 * 3 = 6 \text{ and } 3 \in \mathbb{Z}$$
$$\text{So, } (a, b) \in R$$
$$\text{Also observe } 5 | (14 + 1) \text{ because } 5 * 3 = 15 \text{ and } 3 \in \mathbb{Z}$$
$$\text{So, } (b, c) \in R$$
$$\text{However, consider } 2 | (14 + 1)$$
$$2 * \frac{15}{2} = 15 \land 15 \notin \mathbb{Z}$$
$$\text{So, } (a, c) \notin R$$

□