Homework 9

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Section 14

1. Order of $\mathbb{Z}_6/\langle 3\rangle = \frac{|G|}{|H|} = \frac{6}{2} = 3$
2. Order of $(\mathbb{Z}_4 \times \mathbb{Z}_{12})/(\langle 2\rangle \times \langle 2\rangle) = \frac{|G|}{|H|} = \frac{48}{12} = 4$
7. Order of $(\mathbb{Z}_2 \times S_3)/\langle(1, \rho_1)\rangle = \frac{|G|}{|H|} = \frac{12}{6} = 2$
9. $|\langle 5 + \langle 4\rangle\rangle| = |\{5, 9, 1\}, \{10, 2, 6\}, \{3, 7, 11\}, \{8, 0, 4\}| = 4$
10. $|\langle 26+\langle 12\rangle\rangle| = |\{26, 38, 50, 2, 14\}, \{52, 4, 16, 28, 40\}, \{18, 30, 42, 54, 6\}, \{44, 56, 8, 20, 32\}, \{10, 22, 34, 46, 58\}, \{36, 48, 0, 12, 2$
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17. A normal subgroup H of G is one satisfying $gH = Hg$ for all $g \in G$
18. A normal subgroup H of G is one satisfying $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$
20. Normal subgroups are important because their cosets form the factor groups of a group G.
23. (a) False
    (b) True
    (c) True
    (d) True
24. Since $|S_n| = 2|A_n|$ and therefore $\frac{|S_n|}{|A_n|} = 2$, so S_n/A_n is isomorphic to $\mathbb{Z}_2$. Further, we know that A_n contains all rotations of a geometry of size n, then for all $g \in S_n$, $gA_n = A_ng$, so A_n is normal.
30. *Proof.*

$$\begin{aligned} \text{First, we know that } (G : H) = |G/H| = m \\ \text{This means that every element of } G/H \text{ has an order which divides } m \\ a^mH = eH \text{ for all } a \in G \end{aligned}$$

$\square$

Section 15

1. $\langle(0, 1)\rangle = \{(0, 0), (0, 1), (0, 2), (0, 3)\}$
   $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0, 1)\rangle = \mathbb{Z}_2$
2. $\langle(0, 2)\rangle = \{(0, 0), (0, 2)\}$
   $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(0, 1)\rangle = \mathbb{Z}_4$
3. $\langle(1, 2)\rangle = \{(0, 0), (1, 2)\}$
   $(\mathbb{Z}_2 \times \mathbb{Z}_4)/\langle(1, 2)\rangle = \mathbb{Z}_4$
4. $\langle(1, 2)\rangle = \{(0, 0), (1, 2), (2, 4), (3, 6)\}$
   $(\mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1, 2)\rangle = \mathbb{Z}_4 \times \mathbb{Z}_2 = \mathbb{Z}_8$

5. $\langle(1,2,4)\rangle = \{(0,0,0),(1,2,4),(2,0,0),(3,2,4)\}$
$(\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_8)/\langle(1,2,4)\rangle = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_4 = \mathbb{Z}_4 \times \mathbb{Z}_8$

19. (a) True
    (b) False
    (c) False
    (d) False
    (e) False
    (f) True
    (g) False
    (h) True
    (i) True
    (j) False

28. $G = \mathbb{Z} \times \mathbb{Z}_2$ where $H = \langle(1,1)\rangle$, there are infinitely many subgroups in the factor group G/H of the form $(n,0) + \langle(1,1)\rangle$, but they all have elements of order 2

30. (a) The whole group since G is abelian
    (b) $\{e\}$ since $ee = e = ee$ is the only abelian member of a nonabelian group

31. (a) $\{e\}$ since $aba^{-1}b^{-1} = aa^{-1}bb^{-1} = ee = e$ for all $a, b \in G$
    (b) The whole group

34. *Proof.*

$$\begin{aligned} \text{Let } G \text{ be a group and let } H \text{ be a subgroup of } G \text{ with index } 2 \\ \text{Thus, for all } a \in G,\ a \in H \text{ or } a \text{ is in the other coset} \\ aH = eH = H \text{ iff } a \in H \text{ and } aH = \text{other coset iff } a \notin H \\ \text{But this also applies with the right cosets} \\ \text{Thus, } H \text{ is a nontrivial normal subgroup and } G \text{ is not simple} \end{aligned}$$

□