Homework 2

Walker Bagley

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(1)

Suppose $2^N + 1$ is a prime number. Show that N is a power of 2.

Proof. Suppose N is not a power of 2, then $N = c \cdot 2^m$ for some odd c (if it wasn't there would be another 2 to factor out and $N = \frac{c}{2}2^{m+1}$. So we have $2^N + 1 = 2^{c2^m} + 1 = (2^{2^m})^c + 1$. Then we know that $x + 1 | x^n + 1$ when n is odd, so we can apply this here to say $2^{2^m} + 1 | (2^{2^m})^c + 1 = 2^N + 1$. We can conclude that $2^N + 1$ is not prime so we have a contradiction, and N must be a power of 2. $\square$

(3)

Find a 4-digit perfect square whose first two digits are equal and whose last two digits are equal. Explain your reasoning.

Writing out a 4 digit number with these constraints: $aabb = 1000a + 100a + 10b + b = 1100a + 11b$

I'm going to claim that the square is a 2 digit palindrome as $10^2 < aabb < 100^2$. We can write it like this: $(cc)^2 = (10c + c)^2 = 121c^2 = (11c)^2$

Then we know these are equal, so we can deduce that $100a + b = 11c^2$ by dividing by 11. Since a, b, c are single digits then the left must be just past a multiple of 100, limiting our search to a handful of numbers. We will start with $c = 4$ and increase it until we find the right number.
$11 \cdot 4^2 = 176$
$11 \cdot 5^2 = 275$
$11 \cdot 6^2 = 396$
$11 \cdot 7^2 = 539$
$11 \cdot 8^2 = 704$

When $c = 8$, our equation holds, so 88^2 must be what we are looking for. A simple check says $88^2 = 7744$.

(4)

Find number bases a, b, c, d such that $100_{(a)} = 121_{(b)} = 144_{(c)} = 169_{(d)}$.

$$
\begin{aligned}
a^2 &= b^2 + 2b + 1 = c^2 + 4c + 4 = d^2 + 6d + 9 \\
a^2 &= (b+1)^2 = (c+2)^2 = (d+3)^2 \\
a &= b + 1 = c + 2 = d + 3
\end{aligned}
$$

Then we pick any combination of a, b, c, d that satisfies this such that $d \geq 10$: $a = 13, b = 12, c = 11, d = 10$.

(8)

(a)

Proof. Suppose $a^n | b^n$, then $ka^n = b^n$ for some $k \in \mathbb{Z}$ and $a | b^n$. From here we know that $a | bbbb \dots b$ which means that $a|b \vee a|b \vee a|b \dots \vee a|b$. So, $a|b$. $\square$

(b)

Proof. Suppose $p|a^k$, then $cp = a^k$ for some $c \in \mathbb{Z}$ so $p|a \vee p|a \vee \ldots \vee p|a$ so we can say $p|a$. It follows then that $cp = a$ for some $c \in \mathbb{Z}$. Then $a^k = c^k p^k$ and therefore $p^k|a^k$. $\square$

(10)

Find an infinite list of bases b with the property that $121_{(b)}$ is a 3-digit palindrome in another base as well.

$$\begin{aligned} 121_{(b)} &= b^2 + 2b + 1 = (b+1)^2 \\ nmn_{(c)} &= n(c^2+1) + mc \\ (b+1)^2 &= n(c^2+1) + mc \end{aligned}$$

Then $121_{(b)}$ when converted to base c, will have 3 digits if $c^2 \leq (b+1)^2 < c^3$ so we can say that $c < b$.

Some examples:

$$\begin{aligned} 121_{(9)} &= 100 = 202_{(7)} \\ 121_{(10)} &= 121 = 232_{(7)} \\ 121_{(18)} &= 361 = 191_{(15)} \\ 121_{(19)} &= 400 = 484_{(9)} \\ 121_{(21)} &= 484 = 484_{(10)} \\ 121_{(23)} &= 576 = 484_{(11)} \\ 121_{(25)} &= 676 = 484_{(12)} \\ 121_{(27)} &= 784 = 484_{(13)} \\ 121_{(28)} &= 841 = 1j1_{(21)} \\ 121_{(29)} &= 900 = 484_{(14)} \\ 121_{(30)} &= 961 = 1g1_{(24)} \\ 121_{(31)} &= 1024 = 484_{(15)} \\ 121_{(37)} &= 1444 = 484_{(18)} \\ 121_{(38)} &= 1521 = 333_{(22)} = 6b6_{(15)} \\ 121_{(39)} &= 1600 = 484_{(19)} \\ 121_{(40)} &= 1681 = 1q1_{(30)} \end{aligned}$$

I claim that all odd $b \geq 19$ are equivalent to 484 in base $c = \frac{b-1}{2}$ since $c \geq 9$ and 8 does not exist as a digit in bases lower than 9. So from earlier, we have...

$$\begin{aligned} (b+1)^2 &= 4\left(\frac{(b-1)^2}{4} + 1\right) + 8\frac{b-1}{2} \\ &= (b-1)^2 + 4 + 4(b-1) \\ &= [(b-1)+2]^2 \\ &= (b+1)^2 \end{aligned}$$

Thus, we have found infinitely many bases with 3 digit palindromes in another base.