Homework 10

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1. $x^2 + 2x + 5 \in \mathcal{O}(x^2)$

   Proof.

$$\begin{aligned} &\text{Let } r = 2 \text{ and } k = 4 \\ &\text{Let } x \in \mathbb{R} \text{ s.t. } x \geq 2 \\ &\text{Observe, } x^2 + 2x + 5 \leq 4 * x^2 \\ &x^2 + 2x + 5 \leq x^2 + 3 * x^2 \\ &2x + 5 \leq 3 * x^2 \\ &2x + 5 \leq x^2 + 2 * x^2 \\ &\text{We know that since } x \geq 2,\ x^2 \geq 2x \\ &\text{Using this fact, we can reduce to } 5 \leq 2 * x^2 \\ &\text{This is true for all } x \geq 2 \end{aligned}$$

   □

2. $2^x \in \mathcal{O}(x!)$

   Proof.

$$\begin{aligned} &\text{Let } r = 1 \text{ and } k = 2 \\ &\text{Let } x \in \mathbb{N} \text{ s.t. } x \geq 1 \\ &\text{Observe, } 2^x \leq 2 * x! \\ &2_1 * 2_2 * ...\ 2_x \leq 2x * 2(x-1) * ...\ 2 \\ &\text{Same number of terms per side, but each term on the right is greater than each term on the left} \end{aligned}$$

   □

3. $x \log x \notin \mathcal{O}(x)$

   Proof.

$$\begin{aligned} &\text{Let } r, k \in \mathbb{N} \\ &\text{Towards a contradiction, assume } (\forall x \geq r)(x \log x \leq k * x) \\ &\text{Since } x = x, \text{ we can reduce to } \log x \leq k \\ &\text{However, } (\forall k \in \mathbb{N})(\exists x \in \mathbb{N})(\log x > k) \\ &\text{Further, } \log 10^k = k, \text{ so } x = 10^k + 1 \text{ satisfies the equation} \\ &\text{Therefore, } x \log x \notin \mathcal{O}(x) \end{aligned}$$

   □

4. $(\forall a, b \in \mathbb{R}_+)(a, b \neq 1 \to \log_a x \in \mathcal{O}(\log_b x))$

   Proof.

$$\begin{aligned} &\text{Let } a, b \in \mathbb{R}_+ \backslash \{1\},\ r = 1, \text{ and } k > \log_a b \\ &\text{Then } \log_a x \leq k * \log_b x \\ &\text{Using change of base, } \log_a x \leq k * \frac{\log_a x}{\log_a b} \\ &\text{Cancelling out the } \log_a x \text{ leaves } 1 \leq \frac{k}{\log_a b} \\ &\log_a b \leq k \\ &\text{Since } k > \log_a b, \text{ this is always true} \\ &\text{Therefore } \log_a x \in \mathcal{O}(\log_b x) \end{aligned}$$

   □

5. $f_6 \preccurlyeq f_1 \asymp f_3 \preccurlyeq f_5 \asymp f_7 \preccurlyeq f_4 \preccurlyeq f_9 \preccurlyeq f_{10} \preccurlyeq f_2 \preccurlyeq f_8$