Exam 1

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A. Take Home

2.

$$u_{tt} - 8u_{xt} + 12uxx = 0 \qquad -\infty < x < \infty \qquad t > 0$$
$$u(x,0) = e^{-x^4} \qquad u_t(x,0) = \cos(x)$$
$$(\partial_t - 6\partial_x)(\partial_t - 2\partial_x)u = 0$$

Change variables: $\begin{matrix} x = -2t + \xi \\ x = -6t + \eta \end{matrix} \quad \begin{cases} \xi = x + 2t \\ \eta = x + 6t \end{cases} \quad \begin{cases} \partial_x = \partial_\xi + \partial_\eta \\ \partial_t = 2\partial_\xi + 6\partial_\eta \end{cases} \quad \begin{cases} -4\partial_\xi = \partial_t - 6\partial_x \\ 4\partial_\eta = \partial_t - 2\partial_x \end{cases}$

$$-16\partial_{\xi\eta}u = 0 \quad \Rightarrow \quad \partial_{\xi\eta}u = 0$$

Modifying D'Alambert's: $u(\xi,\eta) = f(x+2t) + g(x+6t) \quad \Rightarrow \quad u(x,0) = \phi(x) = f(x) + g(x)$

$$\phi'(x) = f'(x) + g'(x)$$
$$u_t(x,0) = \psi(x) = 2f'(x) + 6g'(x)$$
$$\begin{matrix} f'(x) = \frac{1}{4}[\phi'(x) - \psi(x)] \\ g'(x) = \frac{1}{4}[\psi(x) - 2\phi'(x)] \end{matrix} \quad \begin{cases} f(s) = \frac{3}{2}\int_0^s \phi'(y)dy - \frac{1}{4}\int_0^s \psi(y)dy \\ g(s) = -\frac{1}{2}\int_0^s \phi'(y)dy + \frac{1}{4}\int_0^s \psi(y)dy \end{cases} \quad \begin{cases} f(s) = \frac{3}{2}\phi(s) + \frac{1}{4}\int_s^0 \psi(y)dy \\ g(s) = -\frac{1}{2}\phi(s) + \frac{1}{4}\int_0^s \psi(y)dy \end{cases}$$
$$u(\xi,\eta) = \frac{1}{2}[3\phi(\xi) - \phi(\eta)] + \frac{1}{4}\int_\xi^\eta \psi(y)dy$$
$$u(x,t) = \frac{1}{2}\left[3e^{-(x+2t)^4} - e^{-(x+6t)^4}\right] + \frac{1}{4}[\sin(x+6t) - \sin(x+2t)]$$

3.

$$\begin{aligned} u_t - u_{xx} + 2tu &= 0 \qquad -\infty < x < \infty \qquad t > 0 \\ u(x,0) &= e^{-x} \end{aligned}$$

$$\begin{aligned}
\text{Let } u \text{ be of the form: } u(x,t) &= c(t)v(x,t) \\
\text{Then we have the equation: } 0 &= c'v + cv_t - cv_{xx} + 2tcv \\
&= c(v_t - v_{xx}) + v(c' + 2tc) \\
\text{This yields two equations: } &\begin{cases} v_t - v_{xx} = 0 \\ c' + 2tc = 0 \end{cases} \\
c(t) &= e^{-t^2} \\
v(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x-y)^2} e^{-y} dy = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{1}{4t}(x^2-2xy+y^2+4ty)} dy \\
x^2 - 2xy + y^2 + 4ty &= (y + [2t - x])^2 + x^2 - (2t - x)^2 \\
&= (y + [2t - x])^2 + 4tx - 4t^2 = (y + [2t - x])^2 + 4t(x - t) \\
\text{Change variables: } z &= \frac{y + [2t - x]}{\sqrt{4t}} \quad \Rightarrow \quad dy = \sqrt{4t}\; dz \\
v(x,t) &= \frac{1}{\sqrt{4\pi t}} \cdot \sqrt{4t} \cdot e^{-(x-t)} \int_{-\infty}^{\infty} e^{-z^2} dz = \frac{1}{\sqrt{\pi}} \cdot e^{t-x} \cdot \sqrt{\pi} \\
&= e^{t-x} \\
u(x,t) &= c(t)v(x,t) = e^{-t^2} e^{t-x} = e^{t-t^2-x}
\end{aligned}$$