Homework 8

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Section 11

1. $(0,0)$ order 0 $\quad$ $(1,0)$ order 2
   $(0,1)$ order 4 $\quad$ $(1,1)$ order 4
   $(0,2)$ order 2 $\quad$ $(1,2)$ order 2
   $(0,3)$ order 4 $\quad$ $(1,3)$ order 4

2. $(0,0)$ order 0 $\quad$ $(1,0)$ order 3 $\quad$ $(2,0)$ order 3
   $(0,1)$ order 4 $\quad$ $(1,1)$ order 12 $\quad$ $(2,1)$ order 12
   $(0,2)$ order 2 $\quad$ $(1,2)$ order 6 $\quad$ $(2,2)$ order 6
   $(0,3)$ order 4 $\quad$ $(1,3)$ order 12 $\quad$ $(2,3)$ order 12

7. Order is 60

8. Largest order for $\mathbb{Z}_6 \times \mathbb{Z}_8 = 24$
   Largest order for $\mathbb{Z}_{12} \times \mathbb{Z}_{15} = 60$

9. $\{(0,0),(1,1)\}$
   $\{(0,0),(0,1)\}$
   $\{(0,0),(1,0)\}$

13. $\mathbb{Z}_{20} \times \mathbb{Z}_3$
    $\mathbb{Z}_{15} \times \mathbb{Z}_4$
    $\mathbb{Z}_{12} \times \mathbb{Z}_5$
    $\mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_3$

14. (a) 4
    (b) 12
    (c) 12
    (d) $\mathbb{Z}_2 \times \mathbb{Z}_2$
    (e) 8

18. $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ since $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ is isomorphic to $\mathbb{Z}_{1920}$ but $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ does not have a relatively prime factorization within $\mathbb{Z}$

20. $\mathbb{Z}_4 \times \mathbb{Z}_{18} \times \mathbb{Z}_{15} \cong \mathbb{Z}_4 \times (\mathbb{Z}_2 \times \mathbb{Z}_9) \times (\mathbb{Z}_3 \times \mathbb{Z}_5) \cong \mathbb{Z}_3 \times (\mathbb{Z}_4 \times \mathbb{Z}_9) \times (\mathbb{Z}_2 \times \mathbb{Z}_5) \cong \mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$
    This is an isomorphism since $gcd(2,9) = 1$, $gcd(3,5) = 1$, $gcd(4,9) = 1$, $gcd(2,5) = 1$

24. $720 = 2^4 * 3^2 * 5$
    $\mathbb{Z}_{16} \times \mathbb{Z}_9 \times \mathbb{Z}_5$
    $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
    $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
    $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
    $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
    $\mathbb{Z}_{16} \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
    $\mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
    $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$

32. (a) True
    (b) True
    (c) False
    (d) True
    (e) False
    (f) False
    (g) False
    (h) False
    (i) True
    (j) True

36. (a) True
    (b) True
    (c) False
    (d) True
    (e) True
    (f) False
    (g) True
    (h) False
    (i) False
    (j) True

46. *Proof.*

$$\begin{aligned} &\text{Let } G_1, G_2 \dots G_n \text{ be abelian groups} \\ &\text{Then } \forall a_n, b_n \in G_n\ \forall n \in \mathbb{Z},\ a_n b_n = b_n a_n \text{ by definition} \\ &\text{So } \forall a, b,\ G_a \times G_b = (a_1, \dots a_n)(b_1, \dots b_n) \\ &= (a_1 b_1, \dots a_n b_n) = (b_1 a_1, \dots b_n a_n) \\ &= (b_1, \dots b_n)(a_1, \dots a_n) = G_b \times G_a \end{aligned}$$

□

47. *Proof.*

$$\begin{aligned} &e \in H \text{ by assumption} \\ &\text{Closed under group operation:} \\ &\text{For any } a, b \in H,\ abab = aabb = e, \text{ so } ab \in H \text{ since it is order 2} \\ &\text{Closed under inverses:} \\ &\text{For elements of order 2, each element is its own inverse} \end{aligned}$$

□

Section 13

1. Is a homomorphism because $\phi(ab) = ab = \phi(a)\phi(b)$
2. Not a homomorphism because when $x = 1.5$ and $y = 1.6$, $\phi(x + y) = 3 \neq 2 = \phi(x) + \phi(y)$
3. Is a homomorphism because $\phi(ab) = |ab| = |a||b| = \phi(a)\phi(b)$

4. Is a homomorphism because $\phi(1^6) = 0 = \phi(1)^6$
5. Not a homomorphism because $\phi(1^9) = \phi(0) = 0 \neq 1 = 1^9 = \phi(1)^9$
9. Is a homomorphism because derivatives are closed under addition
10. Not a homomorphism because $\phi(ab) = \int_0^4 a(b(x))dx \neq \int_0^4 a(x)dx \int_0^4 b(x)dx = \phi(a)\phi(b)$
16. $ker(\phi) = A_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$
17. $ker(\phi) = 7\mathbb{Z}$
    $\phi(1) = 4 \Rightarrow \phi(25) = \phi(25 * 1) = 25 * 4 = 100 = 2$
22. $ker(\phi) = \{(x, y) \mid x = 3n, y = -5m \text{ for } n, m \in \mathbb{Z}\}$
    $\phi(-3, 2) = -19$
24. $ker(\phi) = \{(x, y) \mid x = 2n, y = 4m \text{ for } n, m \in \mathbb{Z}\}$
    $\phi(3, 10) = \phi(1, 0)^3\phi(0, 1)^{10} = (1, 7)(3, 5)(2, 4)(6, 8)(10, 9)$
25. $\mathbb{Z}$ under addition with $\phi(n) = n$
    $\mathbb{Z}$ under addition with $\phi(n) = -n$
28. We need $\phi_g(ab) = g(ab) = (ga)(gb) = \phi_g(a)\phi_g(b)$
    $g(ab) = (ga)(gb)$ when $g = e$
29. Let $x, y \in G$
    Then $\phi_g(xy) = gxyg^{-1} = gxeyg^{-1} = gxg^{-1}gyg^{-1} = \phi_g(x)\phi_g(y)$
32. (a) True
    (b) False
    (c) False
    (d) True
    (e) False
    (f) False
    (g) True
    (h) False
    (i) False
    (j) True
33. There is no nontrivial homomorphism for $\phi : \mathbb{Z}_{12} \to \mathbb{Z}_5$ since 5 does not divide 12
34. Let $\phi(m) = r$ for $m = 4q + r \in \mathbb{Z}_{12}$
39. $\phi(n, m) = 2n$
42. Let $\phi(\sigma) = \sigma(4)$ for $\sigma \in S_3$ where σ is a product of cycles
47. *Proof.*

$$\begin{aligned}
\text{We know } ker(\phi) \text{ is a subgroup so } |ker(\phi)| \text{ divides } |G| \\
\text{Thus } |ker(\phi)| = 1 \text{ or } |ker(\phi)| = |G| \\
\text{If } |ker(\phi)| = 1, \text{ then } ker(\phi) = e \text{ and } \phi \text{ is one to one} \\
\text{If } |ker(\phi)| = |G| \text{ then we have the trivial homomorphism}
\end{aligned}$$

$\square$

48. The kernel is the even permutations of S_n since $1 = e \in \{1, -1\}$

49. *Proof.*

$$\text{Take } x, y \in G$$
$$\gamma\phi(xy) = \gamma(\phi(xy)) = \gamma(\phi(x)\phi(y)) = \gamma(\phi(x))\gamma\phi(y))$$

□

52. *Proof.*

$$\text{Let } H \text{ be the kernel of the homomorphism } \phi : G \to G'$$
$$\text{Then } H = \{h \in G \mid \phi(h) = e'\}$$
$$\text{So } Ha = \{ha \in G \mid \phi(ha) = e'\phi(a) \text{ and } h \in H\}$$
$$\text{Then, } Ha = \{x \in G \mid \phi(x) = \phi(a)\}$$

□