Homework 3

Walker Bagley

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A. Textbook Problems

Section 2.4

1.

$$
\begin{aligned}
u_t - ku_{xx} &= 0 \\
u(x,0) &= \phi(x) = 1 \qquad |x| < l \\
u(x,0) &= \phi(x) = 0 \qquad |x| > l \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2} \phi(y) dy \\
&= \frac{1}{\sqrt{4\pi kt}} \left[\int_{-l}^{l} e^{-\frac{1}{4kt}(x-y)^2} dy + \int_{-\infty}^{-l} 0 dy + \int_{l}^{\infty} 0 dy \right] \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-1}^{l} e^{-\frac{1}{4kt}(x-y)^2} dy \\
\text{Let } z &= \frac{x-y}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\ dz \\
\text{Then our bounds become } \frac{x+l}{\sqrt{4kt}} &\to \frac{x-l}{\sqrt{4kt}} \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \cdot \sqrt{4kt} \int_{\frac{x+l}{\sqrt{4kt}}}^{\frac{x-l}{\sqrt{4kt}}} e^{-z^2} dz \\
&= \frac{1}{\sqrt{\pi}} \left[\int_0^{\frac{x-l}{\sqrt{4kt}}} e^{-z^2} dz - \int_0^{\frac{x+l}{\sqrt{4kt}}} e^{-z^2} dz \right] \\
\\
\text{Recall } \mathscr{E}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp \\
u(x,t) &= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} \mathscr{E}\left(\frac{x-l}{\sqrt{4kt}} \right) - \frac{\sqrt{\pi}}{2} \mathscr{E}\left(\frac{x+l}{\sqrt{4kt}} \right) \right] \\
&= \frac{1}{2} \left[\mathscr{E}\left(\frac{x-l}{\sqrt{4kt}} \right) - \mathscr{E}\left(\frac{x+l}{\sqrt{4kt}} \right) \right]
\end{aligned}
$$

2.

$$\begin{aligned}
u_t - ku_{xx} &= 0 \\
u(x,0) &= \phi(x) = 1 \qquad x > 0 \\
u(x,0) &= \phi(x) = 3 \qquad x < 0 \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2} \phi(y) dy \\
&= \frac{1}{\sqrt{4\pi kt}} \left[\int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2} dy + 3 \int_{-\infty}^{0} e^{-\frac{1}{4kt}(x-y)^2} dy \right] \\
\text{Let } z &= \frac{x-y}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\ dz
\end{aligned}$$

Then bound at 0 becomes $0 \to \dfrac{x}{\sqrt{4kt}}$

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \cdot \sqrt{4kt} \left[\int_{\frac{x}{\sqrt{4kt}}}^{\infty} e^{-z^2} dz + 3 \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\int_0^{\infty} e^{-z^2} dz + \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz + 3 \int_{-\infty}^{0} e^{-z^2} dz - 3 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-z^2} dz \right] \\
&= \frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \mathscr{E}\left(\frac{x}{\sqrt{4kt}} \right) + \frac{3\sqrt{\pi}}{2} - \frac{3\sqrt{\pi}}{2} \mathscr{E}\left(\frac{x}{\sqrt{4kt}} \right) \right] \\
u(x,t) &= 2 - \mathscr{E}\left(\frac{x}{\sqrt{4kt}} \right)
\end{aligned}$$

3.

$$\begin{aligned}
u_t - ku_{xx} = 0 \qquad u(x,0) &= \phi(x) = e^{3x} \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2} e^{3y} dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2 - 12kty} dy \\
(x-y)^2 - 12kty &= x^2 - 2xy + y^2 - 12kty = x^2 - 2(x+6kt)y + y^2 \\
&= (y - [x+6kt])^2 + x^2 - (x+6kt)^2 \\
&= (y - [x+6kt])^2 + x^2 - x^2 - 12ktx - 36k^2t^2 \\
&= (y - [x+6kt])^2 - 12kt(x+3kt)
\end{aligned}$$

$$\begin{aligned}
\text{Let } z &= \frac{y-(x+6kt)}{\sqrt{4kt}} \Rightarrow dz = \frac{1}{\sqrt{4kt}} dy \Rightarrow dy = \sqrt{4kt} dz \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \cdot \sqrt{4kt} \cdot e^{3x+9kt} \int_{-\infty}^{\infty} e^{-z^2} dz \\
&= \frac{1}{\sqrt{\pi}} \cdot e^{3x+9kt} \cdot \sqrt{\pi} = e^{3x+9kt}
\end{aligned}$$

4.

$$
\begin{aligned}
u_t - ku_{xx} &= 0 \\
u(x,0) &= \phi(x) = e^{-x} \qquad x > 0 \\
u(x,0) &= \phi(x) = 0 \qquad x < 0 \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2} \phi(y) dy \\
&= \frac{1}{\sqrt{4\pi kt}} \left[\int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2} e^{-y} dy + \int_{-\infty}^{0} 0 dy \right] \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2 + 4kty} dy \\
(x-y)^2 + 4kty &= x^2 - 2xy + y^2 + 4kty = y^2 - 2(x-2kt)y + x^2 \\
&= (y - [x-2kt])^2 + x^2 - (x-2kt)^2 = (y - [x-2kt])^2 + 4kt(x-kt) \\
\int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2 + 4kty} dy &= \int_0^{\infty} e^{-\frac{1}{4kt}(y-[x-2kt])^2} e^{-(x-kt)} dy \\
&= e^{-(x-kt)} \int_0^{\infty} e^{-\frac{1}{4kt}(y-[x-2t])^2} dy \\
\text{Letting } z &= \frac{y-(x-2kt)}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\ dz \\
\int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2 + 4kty} dy &= \sqrt{4kt} \cdot e^{-(x-kt)} \int_{\frac{-(x-2kt)}{\sqrt{4kt}}}^{\infty} e^{-z^2} dz \\
&= \sqrt{4kt} \cdot e^{-(x-kt)} \left[\int_0^{\infty} e^{-z^2} dz - \int_0^{\frac{-(x-2kt)}{\sqrt{4kt}}} e^{-z^2} dz \right] \\
&= \sqrt{4kt} \cdot e^{-(x-kt)} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right] \\
&= \frac{\sqrt{4\pi kt}}{2} \cdot e^{-(x-kt)} \left[1 - \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right] \\
\text{Substituting back in, } u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \cdot \frac{\sqrt{4\pi kt}}{2} \cdot e^{-(x-kt)} \left[1 - \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right] \\
&= \frac{1}{2} e^{-(x-kt)} \left[1 - \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right]
\end{aligned}
$$

6.

$$
\begin{aligned}
P &= \int_0^{\infty} e^{-x^2} dx \\
P^2 &= \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy = \int_0^{\infty} e^{-y^2} \int_0^{\infty} e^{-x^2} dx dy \\
&= \int_0^{\infty} \int_0^{\infty} e^{-y^2} e^{-x^2} dx dy = \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} dx dy
\end{aligned}
$$

Switch to polar coordinates

$$
\begin{aligned}
P^2 &= \int_0^{\infty} \int_0^{\frac{\pi}{2}} e^{-r^2} r d\theta dr = \frac{\pi}{2} \int_0^{\infty} e^{-r^2} r dr \\
s &= r^2 \Rightarrow ds = 2rdr \\
P^2 &= \frac{\pi}{4} \int_0^{\infty} e^{-s} ds = \frac{\pi}{4} (-e^{-s}|_0^{\infty} = \frac{\pi}{4} \\
P &= \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}
\end{aligned}
$$

14.

$$|\phi(x)| \le Ce^{ax^2}$$

$$|e^{-\frac{1}{4kt}(x-y)^2}\phi(y)| \le Ce^{-\frac{1}{4kt}(x-y)^2+ay^2} = Ce^{(a-\frac{1}{4kt})y^2+\frac{xy}{2kt}-\frac{x^2}{4kt}}$$

To complete the square, $a - \dfrac{1}{4kt} < 0$

$$a < \frac{1}{4kt}$$

$$0 < t < \frac{1}{4ak}$$

Section 3.1

1.

$$\begin{aligned}
u_t - ku_{xx} = 0 \qquad u(x,0) &= e^{-x} \qquad u(0,t) = 0 \qquad 0 < x < \infty \\
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[e^{-\frac{1}{4kt}(x-y)^2} - e^{-\frac{1}{4kt}(x+y)^2} \right] e^{-y} dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{1}{4kt}(x-y)^2 + 4kty} - e^{-\frac{1}{4kt}(x+y)^2+4kty} dy
\end{aligned}$$

$$\begin{aligned}
(x-y)^2 + 4kty &= x^2 - 2xy + y^2 + 4kty = y^2 - 2(x-2kt)y + x^2 \\
&= (y - [x-2kt])^2 + x^2 - (x-2kt)^2 = (y-[x-2kt])^2 + 4kt(x-kt) \\
\int_0^\infty e^{-\frac{1}{4kt}(x-y)^2+4kty} dy &= \int_0^\infty e^{-\frac{1}{4kt}(y-[x-k2t])^2} e^{-(x-kt)} dy \\
&= e^{-(x-kt)} \int_0^\infty e^{-\frac{1}{4kt}(y-[x-2kt])^2} dy \\
\text{Letting } z &= \frac{y-(x-2kt)}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\ dz \\
\int_0^\infty e^{-\frac{1}{4kt}(x-y)^2+4kty} dy &= \sqrt{4kt} \cdot e^{-(x-kt)} \int_{\frac{-(x-2kt)}{\sqrt{4kt}}}^\infty e^{-z^2} dz \\
&= \sqrt{4kt} \cdot e^{-(x-kt)} \left[\int_0^\infty e^{-z^2} dz - \int_0^{\frac{-(x-2kt)}{\sqrt{4kt}}} e^{-z^2} dz \right] \\
&= \sqrt{4kt} \cdot e^{-(x-kt)} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right] \\
&= \frac{\sqrt{4\pi kt}}{2} \cdot e^{-(x-kt)} \left[1 - \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right]
\end{aligned}$$

$$\begin{aligned}
(x+y)^2 + 4kty &= x^2 + 2xy + y^2 + 4kty = y^2 + 2(x+2kt)y + x^2 \\
&= (y + [x+2kt])^2 + x^2 - (x+2kt)^2 = (y+[x+2kt])^2 - 4kt(x+kt) \\
\int_0^\infty e^{-\frac{1}{4kt}(x+y)^2+4kty} dy &= \int_0^\infty e^{-\frac{1}{4kt}(y+[x+k2t])^2} e^{x+kt} dy \\
&= e^{x+kt} \int_0^\infty e^{-\frac{1}{4kt}(y+[x+2kt])^2} dy \\
\text{Letting } z &= \frac{y+(x+2kt)}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\ dz \\
\int_0^\infty e^{-\frac{1}{4kt}(x+y)^2+4kty} dy &= \sqrt{4kt} \cdot e^{x+kt} \int_{\frac{x+2kt}{\sqrt{4kt}}}^\infty e^{-z^2} dz \\
&= \sqrt{4kt} \cdot e^{x+kt} \left[\int_0^\infty e^{-z^2} dz - \int_0^{\frac{x+2kt}{\sqrt{4kt}}} e^{-z^2} dz \right] \\
&= \sqrt{4kt} \cdot e^{x+kt} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \mathscr{E} \left(\frac{x+2kt}{\sqrt{4kt}} \right) \right] \\
&= \frac{\sqrt{4\pi kt}}{2} \cdot e^{x+kt} \left[1 - \mathscr{E} \left(\frac{x+2kt}{\sqrt{4kt}} \right) \right]
\end{aligned}$$

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\left[\frac{\sqrt{4\pi kt}}{2}\cdot e^{-(x-kt)}\left[1-\mathscr{E}\left(\frac{-(x-2kt)}{\sqrt{4kt}}\right)\right]-\frac{\sqrt{4\pi kt}}{2}\cdot e^{x+kt}\left[1-\mathscr{E}\left(\frac{x+2kt}{\sqrt{4kt}}\right)\right]\right]\\
&= \frac{1}{2}e^{kt}\left[e^{-x}-e^{x}+\mathscr{E}\left(\frac{x+2kt}{\sqrt{4kt}}\right)-\mathscr{E}\left(\frac{-(x-2kt)}{\sqrt{4kt}}\right)\right]\\
&= \frac{1}{2}e^{kt}\left[-2\sinh(x)+\mathscr{E}\left(\frac{x+2kt}{\sqrt{4kt}}\right)-\mathscr{E}\left(\frac{-(x-2kt)}{\sqrt{4kt}}\right)\right]
\end{aligned}
$$

2.

$$
\begin{aligned}
&u_t - ku_{xx} = 0 \qquad u(x,0)=0 \qquad u(0,t)=1 \qquad 0<x<\infty\\
&\quad \text{Let } v = u-1 \qquad v(x,0)=u(x,0)-1=-1 \qquad v(0,t)=u(0,t)-1=0\\
&v(x,t) = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty e^{-\frac{1}{4kt}(x+y)^2}-e^{-\frac{1}{4kt}(x-y)^2}dy\\
&p = \frac{x+y}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}dp\\
&q = \frac{x-y}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}dp\\
&v(x,t) = \frac{1}{\sqrt{\pi}}\left[\int_{\frac{x}{\sqrt{4kt}}}^\infty e^{-p^2}dp - \int_{\frac{x}{\sqrt{4kt}}}^{-\infty} e^{-q^2}dq\right]\\
&\quad = \frac{1}{\sqrt{\pi}}\left[\int_0^\infty e^{-p^2}dp - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2}dp + \int_{-\infty}^0 e^{-q^2}dq - \int_0^{\frac{x}{\sqrt{4kt}}} e^{-q^2}dq\right]\\
&\quad = \frac{1}{\sqrt{\pi}}\left[\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}\mathscr{E}\left(\frac{x}{\sqrt{4kt}}\right)+\frac{\sqrt{\pi}}{2}-\frac{\sqrt{\pi}}{2}\mathscr{E}\left(\frac{x}{\sqrt{4kt}}\right)\right]\\
&\quad = 1-\mathscr{E}\left(\frac{x}{\sqrt{4kt}}\right)
\end{aligned}
$$

B. Homework Exercises

1.) (IVP for Heat Equation with Peakon Data)

i.

Fourier Transform of $f(x) = e^{-|x|}$

$$
\begin{aligned}
\hat{f}(\xi) &= \int_{-\infty}^\infty e^{-i\xi x}e^{-|x|}dx\\
&= \int_0^\infty e^{-i\xi x}e^{-x}dx + \int_{-\infty}^0 e^{-i\xi x}e^{x}dx\\
&= \int_0^\infty e^{(-i\xi-1)x}dx + \int_{-\infty}^0 e^{(-i\xi+1)x}e^{x}dx\\
&= \left(\frac{1}{-i\xi-1}e^{(-i\xi-1)x}\right|_0^\infty + \left(\frac{1}{-i\xi+1}e^{(-i\xi+1)x}\right|_{-\infty}^0\\
&= \left[0-\frac{1}{-i\xi-1}\right]+\left[\frac{1}{-i\xi+1}-0\right]\\
&= \frac{1}{-i\xi+1}-\frac{1}{-i\xi-1} = \frac{(-i\xi-1)-(-i\xi+1)}{(-i\xi+1)(-i\xi-1)}\\
\hat{f}(\xi) &= \frac{-2}{-\xi^2-1} = \frac{2}{\xi^2+1}
\end{aligned}
$$

ii.

$$\begin{aligned} u_t - u_{xx} &= 0 \qquad -\infty < x < \infty \qquad t > 0 \\ u(x,0) &= e^{-|x|} \end{aligned}$$

$$\begin{aligned} \text{Spectral solution: } u(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} e^{-\xi^2 t}\hat{f}(\xi)d\xi \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} e^{-\xi^2 t}\frac{2}{\xi^2+1}d\xi \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\xi x-\xi^2 t}}{\xi^2+1}d\xi \end{aligned}$$

$$\begin{aligned} \text{Physical solution: } u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{1}{4kt}(x-y)^2}e^{-|y|}dy \\ &= \frac{1}{\sqrt{4\pi kt}}\left[\int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2+4kty}dy + \int_{-\infty}^{0} e^{-\frac{1}{4kt}(x-y)^2-4kty}dy\right] \end{aligned}$$

$$\begin{aligned} (x-y)^2+4kty &= x^2-2xy+y^2+4kty = y^2-2(x-2kt)y+x^2 \\ &= (y-[x-2kt])^2+x^2-(x-2kt)^2 = (y-[x-2kt])^2+4kt(x-kt) \\ \int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2+4kty}dy &= \int_0^{\infty} e^{-\frac{1}{4kt}(y-[x-2kt])^2}e^{-(x-kt)}dy \\ &= e^{-(x-kt)}\int_0^{\infty} e^{-\frac{1}{4kt}(y-[x-2t])^2}dy \\ \text{Letting } z &= \frac{y-(x-2kt)}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\ dz \\ \int_0^{\infty} e^{-\frac{1}{4kt}(x-y)^2+4kty}dy &= \sqrt{4kt}\cdot e^{-(x-kt)}\int_{\frac{-(x-2kt)}{\sqrt{4kt}}}^{\infty} e^{-z^2}dz \\ &= \sqrt{4kt}\cdot e^{-(x-kt)}\left[\int_0^{\infty} e^{-z^2}dz - \int_0^{\frac{-(x-2kt)}{\sqrt{4kt}}} e^{-z^2}dz\right] \\ &= \sqrt{4kt}\cdot e^{-(x-kt)}\left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\mathscr{E}\left(\frac{-(x-2kt)}{\sqrt{4kt}}\right)\right] \\ &= \frac{\sqrt{4\pi kt}}{2}\cdot e^{-(x-kt)}\left[1-\mathscr{E}\left(\frac{-(x-2kt)}{\sqrt{4kt}}\right)\right] \end{aligned}$$

$$\begin{aligned} (x-y)^2-4kty &= x^2-2xy+y^2-4kty = y^2-2(x+2kt)y+x^2 \\ &= (y-[x+2kt])^2+x^2-(x+2kt)^2 = (y-[x+2kt])^2-4kt(x+kt) \\ \int_{-\infty}^{0} e^{-\frac{1}{4kt}(x-y)^2-4kty}dy &= \int_{-\infty}^{0} e^{-\frac{1}{4kt}(y-[x+2t])^2}e^{x+kt}dy \\ &= e^{x+t}\int_{-\infty}^{0} e^{-\frac{1}{4kt}(y-[x+2kt])^2}dy \\ \text{Letting } z &= \frac{y-(x+2kt)}{\sqrt{4kt}} \Rightarrow dy = \sqrt{4kt}\ dz \\ \int_{-\infty}^{0} e^{-\frac{1}{4kt}(x-y)^2-4kty}dy &= \sqrt{4kt}\cdot e^{x+kt}\int_{-\infty}^{\frac{-(x+2kt)}{\sqrt{4kt}}} e^{-z^2}dz \\ &= \sqrt{4kt}\cdot e^{x+kt}\left[\int_{-\infty}^{0} e^{-z^2}dz + \int_0^{\frac{-(x+2kt)}{\sqrt{4kt}}} e^{-z^2}dz\right] \end{aligned}$$

$$
\begin{aligned}
&= \sqrt{4kt} \cdot e^{x+kt} \left[\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \mathscr{E} \left(\frac{-(x+2kt)}{\sqrt{4kt}} \right) \right] \\
&= \frac{\sqrt{4\pi kt}}{2} \cdot e^{x+kt} \left[1 + \mathscr{E} \left(\frac{-(x+2kt)}{\sqrt{4kt}} \right) \right]
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \left[\frac{\sqrt{4\pi kt}}{2} \cdot e^{-(x-kt)} \left[1 - \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right] + \frac{\sqrt{4\pi kt}}{2} \cdot e^{x+kt} \left[1 + \mathscr{E} \left(\frac{-(x+2kt)}{\sqrt{4kt}} \right) \right] \right] \\
&= \frac{1}{2} \left[e^{-(x-kt)} \left[1 - \mathscr{E} \left(\frac{-(x-2kt)}{\sqrt{4kt}} \right) \right] + e^{x+kt} \left[1 + \mathscr{E} \left(\frac{-(x+2kt)}{\sqrt{4kt}} \right) \right] \right]
\end{aligned}
$$

5.) (IBVP for Diffusion Equation)

$$\begin{aligned}
u_t - u_{xx} &= 0 \qquad 0 < x < \infty \qquad t > 0 \\
u(x,0) &= e^{-x} \qquad u(0,t) = 0 \\
u(x,t) &= \frac{1}{\sqrt{4\pi t}} \int_0^\infty [e^{-\frac{1}{4t}(x-y)^2} - e^{-\frac{1}{4t}(x+y)^2}] e^{-y} dy \\
&= \frac{1}{\sqrt{4\pi t}} \left[\int_0^\infty e^{-\frac{1}{4t}(x-y)^2 + 4ty} dy - \int_0^\infty e^{-\frac{1}{4t}(x+y)^2 + 4ty} dy \right]
\end{aligned}$$

$$\begin{aligned}
(x-y)^2 + 4ty &= x^2 - 2xy + y^2 + 4ty = y^2 - 2(x-2t)y + x^2 \\
&= (y - [x-2t])^2 + x^2 - (x-2t)^2 = (y - [x-2t])^2 + 4t(x-t) \\
\int_0^\infty e^{-\frac{1}{4t}(x-y)^2 + 4ty} dy &= \int_0^\infty e^{-\frac{1}{4t}(y-[x-2t])^2} e^{-(x-t)} dy \\
&= e^{-(x-t)} \int_0^\infty e^{-\frac{1}{4t}(y-[x-2t])^2} dy \\
\text{Letting } z &= \frac{y - (x-2t)}{\sqrt{4t}} \Rightarrow dy = \sqrt{4t}\ dz \\
\int_0^\infty e^{-\frac{1}{4t}(x-y)^2 + 4ty} dy &= \sqrt{4t} \cdot e^{-(x-t)} \int_{\frac{-(x-2t)}{\sqrt{4t}}}^\infty e^{-z^2} dz = \sqrt{4t} \cdot e^{t-x} \left[\int_0^\infty e^{-z^2} dz - \int_0^{\frac{-(x-2t)}{\sqrt{4t}}} e^{-z^2} dz \right] \\
&= \sqrt{4t} \cdot e^{t-x} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \mathscr{E} \left(\frac{-(x-2t)}{\sqrt{4t}} \right) \right] \\
&= \frac{\sqrt{4\pi t}}{2} \cdot e^{t-x} \left[1 - \mathscr{E} \left(\frac{-(x-2t)}{\sqrt{4t}} \right) \right]
\end{aligned}$$

$$\begin{aligned}
(x+y)^2 + 4ty &= x^2 + 2xy + y^2 + 4ty = y^2 + 2(x+2t)y + x^2 \\
&= (y + [x+2t])^2 + x^2 - (x+2t)^2 = (y + [x+2t])^2 - 4t(x+t) \\
\int_0^\infty e^{-\frac{1}{4t}(x+y)^2 + 4ty} dy &= \int_0^\infty e^{-\frac{1}{4t}(y+[x+2t])^2} e^{x+t} dy \\
&= e^{x+t} \int_0^\infty e^{-\frac{1}{4t}(y+[x+2t])^2} dy \\
\text{Letting } z &= \frac{y + (x+2t)}{\sqrt{4t}} \Rightarrow dy = \sqrt{4t}\ dz \\
\int_0^\infty e^{-\frac{1}{4t}(x+y)^2 + 4ty} dy &= \sqrt{4t} \cdot e^{x+t} \int_{\frac{x+2t}{\sqrt{4t}}}^\infty e^{-z^2} dz = \sqrt{4t} \cdot e^{t+x} \left[\int_0^\infty e^{-z^2} dz - \int_0^{\frac{x+2t}{\sqrt{4t}}} e^{-z^2} dz \right] \\
&= \sqrt{4t} \cdot e^{t+x} \left[\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} \mathscr{E} \left(\frac{x+2t}{\sqrt{4t}} \right) \right] \\
&= \frac{\sqrt{4\pi t}}{2} \cdot e^{t+x} \left[1 - \mathscr{E} \left(\frac{x+2t}{\sqrt{4t}} \right) \right]
\end{aligned}$$

$$\begin{aligned}
\text{Substituting back in, } u(x,t) &= \frac{1}{\sqrt{4\pi t}} \left[\frac{\sqrt{4\pi t}}{2} \cdot e^{t-x} \left[1 - \mathscr{E} \left(\frac{-(x-2t)}{\sqrt{4t}} \right) \right] - \frac{\sqrt{4\pi t}}{2} \cdot e^{t+x} \left[1 - \mathscr{E} \left(\frac{x+2t}{\sqrt{4t}} \right) \right] \right] \\
&= \frac{1}{2} \left[e^{t-x} \left[1 - \mathscr{E} \left(\frac{-(x-2t)}{\sqrt{4t}} \right) \right] - e^{t+x} \left[1 - \mathscr{E} \left(\frac{x+2t}{\sqrt{4t}} \right) \right] \right]
\end{aligned}$$