Homework 5

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Section 7

1. $\langle gcd(2,3)\rangle = \langle 1\rangle = \{2,3,4,5,6,7,8,9,10,11,0,1\}$

6. $\langle gcd(18,24,39)\rangle = \langle 3\rangle = \{\ldots,-9,-6,-3,0,3,6,9,\ldots\}$

7. (a) $(a^2b)a^3 = (a^2b)aaa = (ab)aa = baa = ba = a^3b$
   (b) $(ab)(a^3b) = (ab)aaab = baab = (a^3b)ab = (a^2b)b = a^2$
   (c) $b(a^2b) = baab = (a^3b)ab = (a^2b)b = a^2$

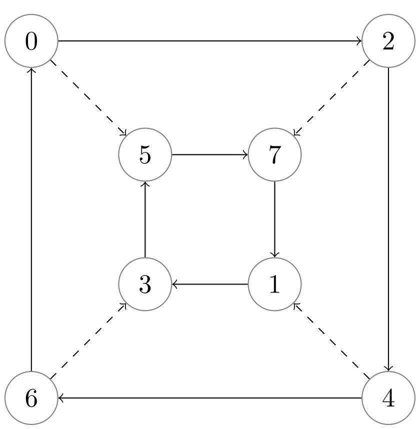
9.

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	c	f	b	c
b	b	c	e	a	f	d
c	c	f	a	d	e	b
d	d	b	f	e	c	a
f	f	c	d	b	a	e

10.

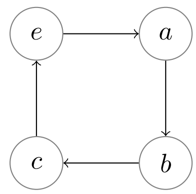
	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	c	f	e	b	d
b	b	d	e	f	a	c
c	c	e	d	a	f	b
d	d	f	c	b	e	a
f	f	b	a	d	c	e

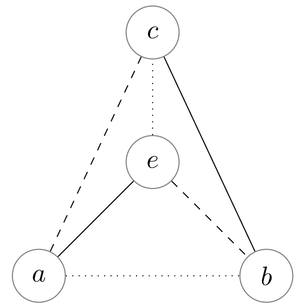
16. Let $x - y$ represent $x+2=y$ and $x - - y$ represent $x+5=y$



17. (a) Starting from any vertex in the graph, travel until reaching the same vertex. This operation must then yield the identity and give a relation.
    (b) $a^4 = e$, $b^2 = e$, $(ab)^2 = e$

18.





19. Take the dihedral group D_4 which has 8 elements. This can be generated by μ_1 and δ_1, which are both order 2.

Section 8

1. $\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix}$

5. $\sigma^{-1}\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 5 & 4 & 3 \end{pmatrix}$

6. $|\langle\sigma\rangle| = 0$ since there are no cycles in σ

8. $\sigma^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix}$
   $\sigma^4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}$
   $\sigma^8 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 2 & 1 \end{pmatrix} = \sigma^2$
   $\sigma^{32} = (\sigma^{16})^2 = ((\sigma^8)^2)^2) = ((\sigma^2)^2)^2 = (\sigma^4)^2) = \sigma^8 = \sigma^2$
   $\sigma^{64} = (\sigma^{32})^2 = (\sigma^2)^2 = \sigma^4$
   $\sigma^{96} = \sigma^{64}\sigma^{32} = \sigma^4\sigma^2 = \sigma^6$
   $\sigma^{98} = \sigma^{96}\sigma^2 = \sigma^6\sigma^2 = \sigma^8 = \sigma^2$
   $\sigma^{100} = \sigma^{98}\sigma^2 = \sigma^2\sigma^2 = \sigma^4$
   $\sigma^{100} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}$

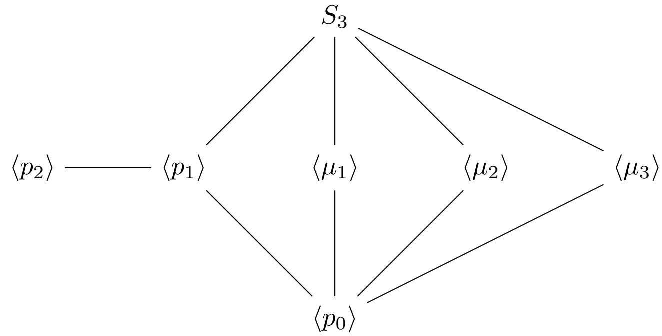
10. 1: $\mathbb{Z}$ under addition, $17\mathbb{Z}$ under addition, $3\mathbb{Z}$ under addition, $\langle\pi\rangle$ under multiplication
    2: S_2, $\mathbb{Z}_2$
    3: $\mathbb{Z}_6$, $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}$
    4: $\mathbb{R}$ under addition, $\mathbb{R}^+$ under multiplication
    Each remaining subcollection contains one of the remaining groups

11. $O_{1.\sigma} = \{\sigma^n(1) | n \in \mathbb{Z}\} = \{1, 2, 3, 4, 5, 6\}$

17. $m = \{\sigma \in S_5 | \sigma(2) = 5\}$ means 5 elements with 4 degrees of freedom, so $|m| = 4! = 24$

18. (a) $\langle p_1\rangle = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$
    $\langle p_2\rangle = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$
    $\langle \mu_1\rangle = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$
    (b) remaining subgroups
    $\langle p_0\rangle = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$

$$\langle \mu_2 \rangle = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$$
$$\langle \mu_3 \rangle = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \right\}$$



23. $\mathbb{Z}_2$
24. Klein group
30. Is a permutation because all elements of $\mathbb{R}$ are simply shifted up by 1
31. No, not one to one
35. (a) True
    (b) False
    (c) True
    (d) True
    (e) True
    (f) True
    (g) False
    (h) False
    (i) False
    (j) True
39. *Proof.*

    Define $\phi : G \to S_G$ such that $\phi(x) = \lambda x$ where $\lambda_x(g) = xg$ for all $g \in G$
    This means that λ_x is one to one and a permutation of G
    This means that ϕ is one to one and homomorphic, so it must be onto
    Then $\phi[G]$ is a subgroup of S_G, a group of permutations

    $\square$
41. $\{\sigma \in S_A \mid \sigma(b) \in B\}$ is not a subgroup since it is not closed
43. This is a subgroup. Since $\phi[B] = B$, it is a permutation of B, which must be in S_A
47. *Proof.*

    Suppose $\sigma \neq \iota$
    Then for some a, b with $a \neq b$ and $1 \leq a, b \leq n$, $\sigma(a) = b$
    Since $n \geq 3$, there exists some other element $c \neq a$ and $c \neq b$ s.t. $\gamma(b) = c$ and $\gamma(a) = a$
    Then $(\sigma\gamma)(a) = \sigma(\gamma(a)) = \sigma(a) = b$
    And $(\gamma\sigma)(a) = \gamma(\sigma(a)) = \gamma(b) = c$
    Thus if $\sigma \neq c$ then there exists $\gamma \in S$ s.t. $\sigma\gamma \neq \gamma\sigma$
    This is a contradiction and therefore $\sigma = \iota$ if $\sigma\gamma = \gamma\sigma$

    $\square$