Homework 1

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1. (a) Conversion to statement-reason proof.

   Proof.

y is a substring of s	Given	(1)
v is a substring of y	Given	(2)
$s = xyz$ for some x, z	Definition of substring on (1)	(3)
$v = uvw$ for some u, w	Definition of substring on (2)	(4)
$s = xuvwz$	Substring combination on (3) and (4)	(5)
v is a substring of s	Definition of substring on (5)	(6)

   □

   (b) Conversion to paragraph proof.

   Proof. Let v be a suffix of w, that is, $w = xv$ such for some x; and let y be a prefix of v, that is, $v = yz$ for some z. Then $w = xyz$, making xy a prefix of w and thus y must be a suffix of xy. □

2. Prove that if φ is a string homomorphism, then for any $w = w_1 \dots w_n$ (where $n \geq 0$ and $w_j \in \Sigma$ for $1 \leq j \leq n$), we have $\varphi(w) = \varphi(w_1) \dots \varphi(w_n)$.

   Proof.

   $$\begin{aligned}
   \text{Base case: } n = 0.\ w = \varepsilon \text{ so } \varphi(\varepsilon) = \varepsilon \\
   \text{Assume that for } n = i,\ \varphi(w) = \varphi(w_1) \dots \varphi(w_i) \\
   \text{WTS this holds for } n = i + 1 \\
   \text{Lets split } w \text{ into two substrings: } w = vw_{i+1} \text{ s.t. } v = w_1 \dots w_i \\
   \text{By the definition of a homomorphism, } \varphi(w) = \varphi(vw_{i+1}) = \varphi(w_1 \dots w_i)\varphi(w_{i+1}) \\
   \text{Substituting the IH yields } \varphi(w) = \varphi(w_1) \dots \varphi(w_i)\varphi(w_{i+1})
   \end{aligned}$$

   □

3. (a) There are no languages over Σ in FINITE$\cap$coFINITE

   Proof. Consider an arbitrary finite language $L \in$ FINITE. Take a word $w \notin L$ ($w \in \bar{L}$) s.t. w is the concatenation of every string in L. We know that $w \notin L$ since it is a concatenation of every element and if it was in L, it would be part of the concatenation. Using this, we know that $\{w\}^*$ is infinite and also a subset of $\bar{L}$. Because of this, we know that $\bar{L}$ is infinite and thus cannot exist in FINITE. Since we took an arbitrary finite language L, we can apply this to all elements of FINITE and coFINITE. This means that there are no infinite languages contained in FINITE, but also no finite languages contained in coFINITE. As a result, there cannot be any languages that exist in both FINITE and coFINITE. □

   (b) Yes, there exists a language over Σ that is not in FINITE$\cup$coFINITE

Proof. Consider the language $L = \{a\}^*$. L by definition must then have infinite elements since an a can always be appended to the longest string. Then $L \notin$ FINITE. Towards a contradiction, assume $L \in$ coFINITE. To be in coFINITE, $\bar{L} \in$ FINITE where $\bar{L} = \Sigma^* \backslash L$. Because both $\{a\}^*$ and $\{b\}^*$ are subsets of Σ^*, $\{b\}^* \in \bar{L}$. We know that $\{b\}^*$ is infinite, so $\bar{L} \notin$ FINITE and we have our contradiction. $\square$