ETHz AFEA Cheat Sheet

FEM & Element

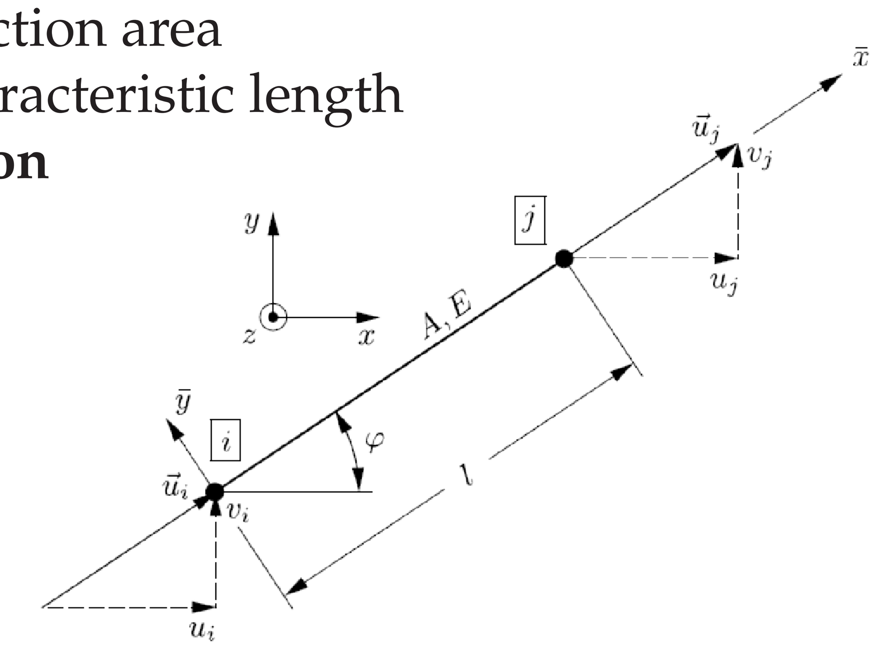
$$Ku = f$$

K: stiffness matrix
u: displacement
f: external force

- Stiffness Matrix

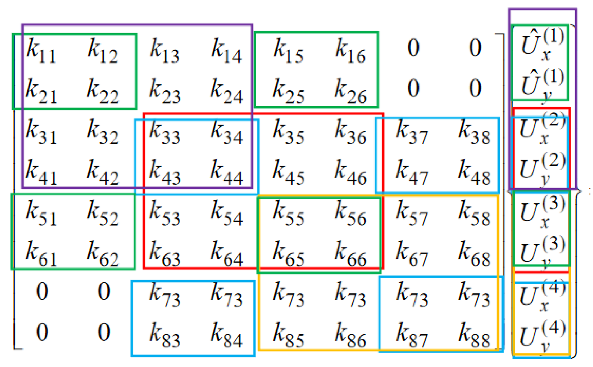
$$K = \frac{EA}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

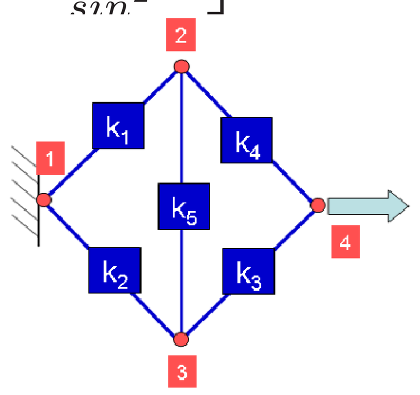
E : Young's Module
A : section area
l : characteristic length
rotation



$$\hat{K} = \begin{bmatrix} cos\theta & sin\theta & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \end{bmatrix}^\top K \begin{bmatrix} cos\theta & sin\theta & 0 & 0 \\ 0 & 0 & cos\theta & sin\theta \end{bmatrix}$$

$$= \frac{EA}{l}\begin{bmatrix} cos^2 & sin\ cos & -cos^2 & -sin\ cos \\ sin\ cos & sin^2 & -sin\ cos & -sin^2 \\ -cos^2 & -sin\ cos & cos^2 & sin\ cos \\ -sin\ cos & -sin^2 & sin\ cos & sin^2 \end{bmatrix}$$





- Continumm Elements

Purpose : Given the position of a point r_j in local coordinate, compute the strain tensor $\varepsilon_{ii}, \varepsilon_{i_1 i_2}$ in global coordinate.

$$J_{ji} = \frac{\partial x_i(r_j)}{\partial r_j} = \frac{\partial \sum_k h^k(r_j)\hat{x}_i^k}{\partial r_j}$$

Interpolate(B matrix)

$$\varepsilon_{ii} = \sum_k \frac{\partial h^k(r_j)}{\partial r_j} J_{ji}^{-1}\hat{u}_i^k$$

$$2\varepsilon_{i_1 i_2} = \sum_k \frac{\partial h^k(r_j)}{\partial r_j} J_{ji_1}^{-1}\hat{u}_{i_1} + \frac{\partial h^k(r_j)}{\partial r_j} J_{ji_2}^{-1}\hat{u}_{i_2}$$

$\hat{u}_i^k$: nodal k displacement in global coordinate
$\hat{x}_i^k$: nodal k position in global coordinate
x_i: random point position in global coordinate
r_j: random point position in local coordinate
$h^k(r_j)$: nodal k interpolate weight $\sum_k h^k(r_j) = 1$
ε_{ii}: diagnal of strain tensor ε
ε_{i_1,i_2} : triangle part of strain tensor ε

Element	Triangular	Tetrahedral	Hexahedral (Brick)
Dimension	2D	3D	3D
Node	3	4	6

Element

- Continumm Elements

Continuity

- C-0: element boundaries displacements
- C-1:Continuous 1st derivatives

Failed

If the element is distorted or folded there will be no 1-to-1 relationship between natural and global coordinates and the Jacobian will be singular

Quadrilateral

$$\hat{u} = \begin{bmatrix} \hat{u}_x^1 & \hat{u}_y^1 & \hat{u}_x^2 & \hat{u}_y^2 & \hat{u}_x^3 & \hat{u}_y^3 & \hat{u}_x^4 & \hat{u}_y^4 \end{bmatrix}^\top$$

$$h^1 = 1/4(1-r_1)(1-r_2) \quad h^2 = 1/4(1+r_1)(1-r_2)$$

$$h^3 = 1/4(1+r_1)(1+r_2) \quad h^4 = 1/4(1-r_1)(1+r_2)$$

$$\frac{\partial h^k}{\partial x_i} = \frac{\partial h^k}{\partial r_j} J_{ji}^{-1}$$

$$B = \begin{bmatrix} \frac{\partial h^1}{\partial x_1} & 0 & \frac{\partial h^2}{\partial x_1} & 0 & \frac{\partial h^3}{\partial x_1} & 0 & \frac{\partial h^4}{\partial x_1} & 0 \\ 0 & \frac{\partial h^1}{\partial x_2} & 0 & \frac{\partial h^2}{\partial x_2} & 0 & \frac{\partial h^3}{\partial x_2} & 0 & \frac{\partial h^4}{\partial x_2} \\ \frac{\partial h^1}{\partial x_1} & \frac{\partial h^1}{\partial x_2} & \frac{\partial h^2}{\partial x_1} & \frac{\partial h^2}{\partial x_2} & \frac{\partial h^3}{\partial x_1} & \frac{\partial h^3}{\partial x_2} & \frac{\partial h^4}{\partial x_1} & \frac{\partial h^4}{\partial x_2} \end{bmatrix}$$

$$\begin{bmatrix} \epsilon_{11} & \epsilon_{22} & \epsilon_{12}+\epsilon_{21} \end{bmatrix}^\top = B\hat{u}$$

- Structual Elements

Truss:Stresses only in the axes direction No bending resistance
Beam:With bending resistance
Plate & Shell

- thickness is 0
- perpendicular to the mid-surface

Kirchhof Shell

- section perpendicular to the mid-surface
- infinite shear-stiffness: for thin shell it's allowed, for thick shell , deviations to the real behavior occur
- $C1$ continuity required
- Displacements and rotations are coupled

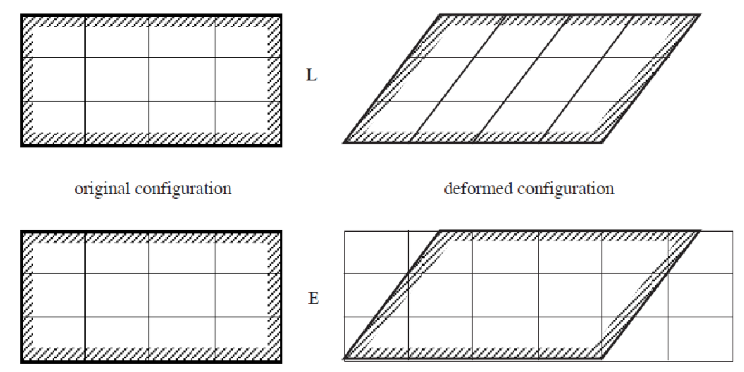
Mindlin Shell

- section may rotate, but still straight
- could be used for thick structures
- strong bending: Compression-strains go to infinity(aspect-ratio matters)
- $C0$ continuity required
- Displacements and rotations are independent of each other

- Numerical Issues

Volumetric Locking : high order elements + reduced integration(reduce computation)
Shear Locking : Reduced integration avoid shear-locking effects, may cause Hourglassing, controlled by adding artificial shear stiffness
Hourglass Control : avoid unphysical deformation of the structure (avoiding zero energy modes)

Continuum Mechanics



lagrangian(T) and eulerian(B) configuartion

$$x = X + u \quad dx = FdX$$

X: is the origin position
x: transform position
u: displacement
F: deformation gradient (not symmetric), $F = RU = VR$
Right Cauchy Green Tensor :$C = F^\top F = U^T U$
Left Cachy Green Tensor :$B = FF^\top = VV^\top$
Engineering stress: $\sigma = \frac{F_t}{A_0}$
Cauchy stress: $\sigma = \frac{F_t}{A_t}$

Green Langrange Strain Tensor

$$E = \frac{1}{2}(F^\top F - I)$$

$$E_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$$

first Piola-Kirchhoff stress

$$P = J\sigma F^{-\top} \quad J = det(F)$$

- not symmetric
- seldom used in constitutive models
- axial stress same as Engineering stress

second Piola-Kirchhoff stress

$$S = JF^{-1}\sigma F^{-\top}$$

- symmetric
- frequently used in constitutive models
- not a clear physical meaning
- Corotational stress(objective)
- energy conjugate to Green Lagrangian strain tensor(E)

Spin tensor

L: velocity gradient, $L = \frac{dv}{dx} = \dot{F}F^{-1}$
W: asymmetric part of L, $W = \frac{1}{2}(L - L^\top)$, $W' = QWQ^\top + \dot{Q}Q^{-1}$
objective: $\sigma' = Q\sigma Q^\top$, Q is orthogonal
Jaumann stress rate

$$\check{\sigma} = \dot{\sigma} + \sigma W - W\sigma$$

take $\dot{\sigma}'$ and use $\dot{Q} = W'Q - QW$ and $\dot{Q}^\top = -Q^\top W' + QW$
Truesdell stress rate

$$\check{\sigma} = \dot{\sigma} - L\sigma - \sigma L^\top + tr(L)\sigma$$

Elasto-Plastic

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \quad d\boldsymbol{\sigma} = \mathbb{C}^e d\varepsilon^e$$

ε: strain tensor, composed of elastic strain ε^e and plastic strain ε^p
$\mathbb{C}^e$: 4-th order elasticity tensor
in uniaxial case, $\bar{\sigma}$ is equivalent stress
$\phi = \bar{\sigma}(\boldsymbol{\sigma}) - \sigma_y < 0$ elastic
$\phi = \bar{\sigma}(\boldsymbol{\sigma}) - \sigma_y = 0$ plastic
Deviatoric Stress:$s = \boldsymbol{\sigma} - \frac{1}{3}tr(\boldsymbol{\sigma})I$

$$J_1 = tr(\boldsymbol{s}) = 0 \quad J_2 = \frac{1}{2}\boldsymbol{s} : \boldsymbol{s} \quad J_3 = det(\boldsymbol{s})$$

Von Mises stress

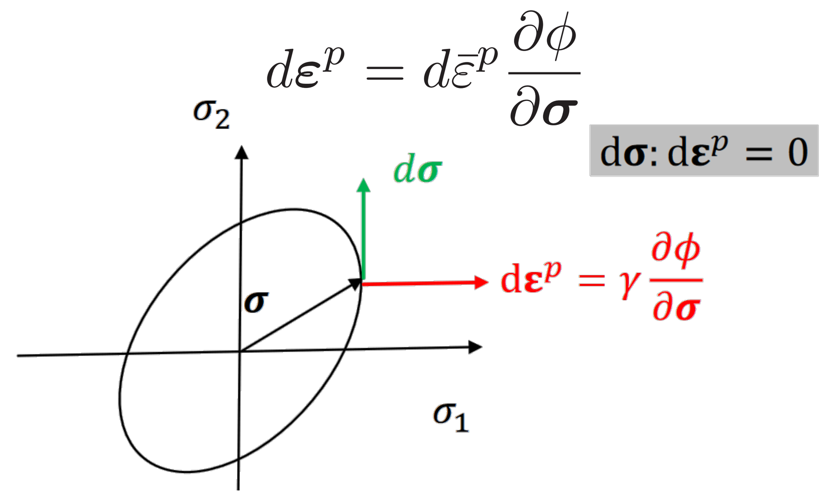
$$\bar{\sigma} = \sqrt{3J_2} = \sqrt{\frac{1}{2}\sum_{i \neq j}[(\sigma_{ii} - \sigma_{jj})^2 + 6\sigma_{ij}^2]}$$

stress tensor only depend on the elastic strain
Equivalent Plastic Strain

$$d\bar{\varepsilon}^p = \sqrt{\frac{2}{3}d\varepsilon^p : d\varepsilon^p}$$

Normality Rule

$$d\varepsilon^p = d\bar{\varepsilon}^p \frac{\partial \phi}{\partial \boldsymbol{\sigma}}$$



plastic strain tensor is proportional to the normal of the yield surface

Hardening

$$\phi = \bar{\sigma}(\boldsymbol{\sigma}) - \sigma_y(\bar{\varepsilon}^p) = 0$$

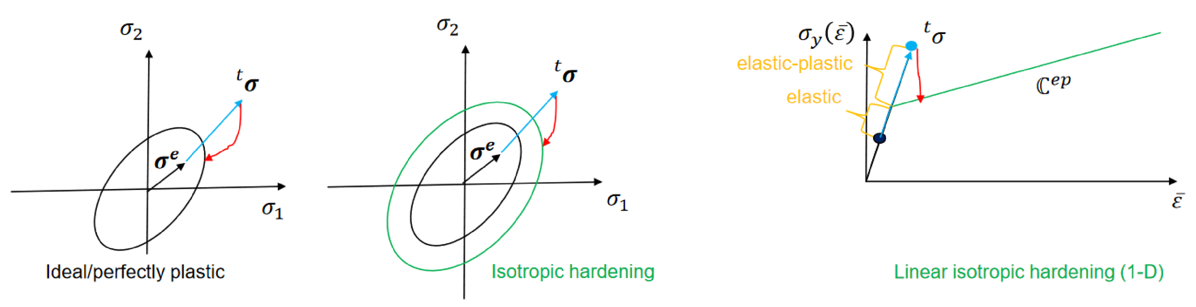
Consistency Condition

$$\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}d\boldsymbol{\sigma} - \frac{\partial \sigma_y}{\partial \bar{\varepsilon}^p}d\bar{\varepsilon}^p = 0$$

elastic-plastic masterial modulus

$$\mathbb{C}^{ep} = \mathbb{C}^e\left(1 - \frac{\frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}}{\frac{\partial \sigma_y}{\partial \bar{\varepsilon}^p} + \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}} : \mathbb{C}^e : \frac{\partial \bar{\sigma}}{\partial \boldsymbol{\sigma}}}\right)$$

return mapping algorithm



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Hyper-Elastic

incompressible material

$$\varepsilon = \varepsilon' + \frac{1}{3}\varepsilon_V \boldsymbol{I}$$

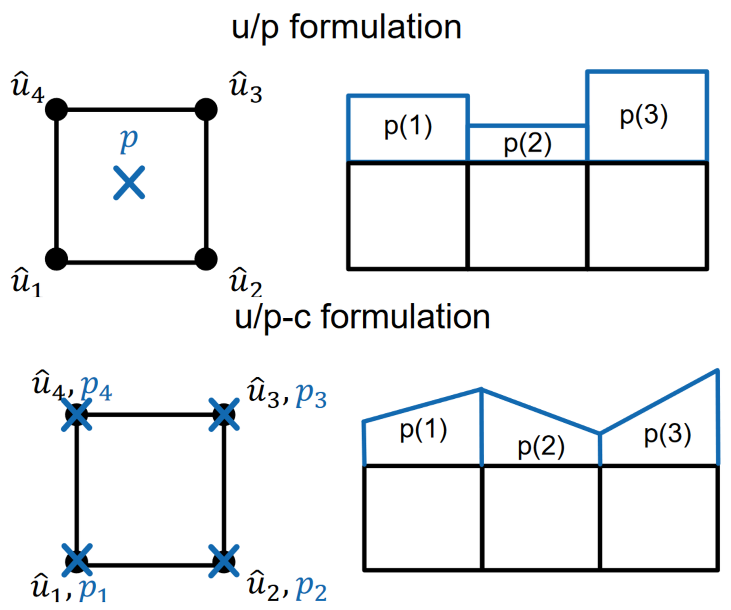
$$\boldsymbol{\sigma} = \kappa\varepsilon_V \boldsymbol{I} + 2G\varepsilon' = -p\boldsymbol{I} + 2G\varepsilon'$$

ε': devitoric strain tensor
ε_V: volumetric strain
κ: Bulk modulus, $\kappa = \frac{E}{3(1-2\nu)}$
G: Shear modulus, $G = \frac{E}{1+\nu}$
p: pressure, $p = -\frac{1}{3}Tr(\boldsymbol{\sigma}) = -\kappa\varepsilon_V$

$$\begin{bmatrix} K_{uu} & K_{up} \\ K_{pu} & K_{pp} \end{bmatrix} \begin{bmatrix} \hat{u} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} f^{ext} \\ 0 \end{bmatrix}$$

if material total incompressible $\kappa \to \infty, K_{pp} \to 0$

Mixed Formulations



Right Cauchy stress tensor : $C = F^\top F = diag(\lambda_i^2)$

$$I_1 = Tr(C) = \sum_i \lambda_i^2$$

$$I_2 = \frac{1}{2}(Tr(C)^2 - Tr(C^2))$$

$$I_3 = det(C)$$

Neo-Hook Model

$$W = \frac{1}{2}G(I_1 - 3)$$

- tensile test same as first Piola-Kirchhof stress
- accurate only for small strain
- only one parameter G

Mooney Model

$$W = C_1(I_1 - 3) + C_2\left(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2} + \frac{1}{\lambda_3^2} - 3\right)$$

assumption

- incompressible and isotropic
- obeys Hooke's law under simple shear

Other Model

Mooney Rivlin Model: $W = C_1(I_1 - 3) + C_2(I_2 - 3)$
Ogden Model: $W = \sum_{k=1}^{N} \frac{\mu_k}{\alpha_k}(\lambda_1^{\alpha_k} + \lambda_2^{\alpha_k} + \lambda_3^{\alpha_k - 3})$

Implicit Quasi-Static FEM

$$K(u)\Delta u = \Delta F$$

Full Newton-Raphson

$$K^i \Delta u^{i+1} = \Delta f^i$$

quadratic convergence rate.

Modified Newton-Raphson

$$K^0 \Delta u^{i+1} = \Delta f^i$$

Quasi Newton-Methods

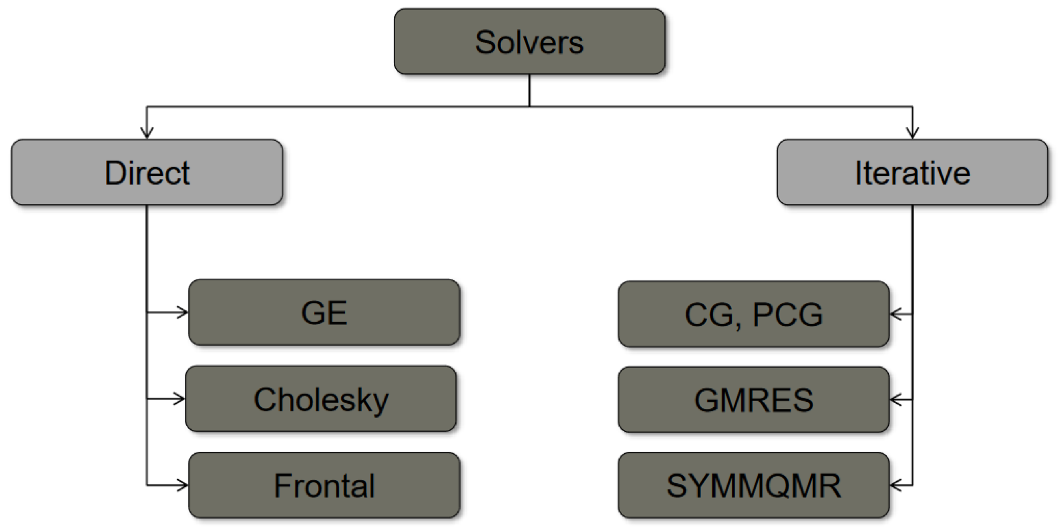
secant approximation
BFGS method

Convergence Criteria

convergence check for each iteration

- Displacement: $\frac{\|\Delta u^{i+1}\|_2}{\|^{t+\Delta t}u\|_2} \leq TOL_{dsp}$
- Residual: $\frac{\|^{t+\Delta t}f_{ext} - {}^{t+\Delta t}f_{int}\|_2}{\|^{t+\Delta t}f_{ext} - {}^{t}f_{int}\|} \leq TOL_{res}$

Linear System Solver



CG solver should be definite

Direct Solver

based on Gauss – Jordan elimination algorithm
Advantage

- predicable number of operations and acc
- fast for small to midsize systems

Disadvantage

- large storage requirements
- slow performance for large system
- hard to parallel

Iterative Solver

Advantage

- smaller storage
- few operations
- found up to desired accuracy
- easy to parallel

Disadvantage

- Convergence not gauranteed
- badly conditioned system slow convergence

Sparse Storage

- Banded Storage: along diagonal
- Skyline Storage: along column near diagonal
- CSR and CSC (no 0 value)

Dynamic FEM

time dependent problem (time integration)

- implicit dynamic FEM
- explicit dynamic FEM

$$M\ddot{u} + C\dot{u} + Ku = F$$

Explicit Dynamic FEM

$$u^{t+\Delta t} = \hat{M}^{-1}\hat{f}$$

$$\ddot{u} = \frac{u^{t+\Delta t} - 2u^t + u^{t-\Delta t}}{\Delta t^2}$$

$$\dot{u} = \frac{u^{t+\Delta t} - u^{t-\Delta t}}{2\Delta t}$$

$\hat{M}$: effective mass(diagonal)
$\hat{f}$: effective load

$$\Delta t_{crit} = \frac{L_{char}}{c}$$

$$T_{tot}^{CPU} \propto \frac{t_{process}}{\Delta t_{critic}}$$

init: $u^{-\Delta t} = u^0 - \dot{u}^0\Delta t + \ddot{u}^0\frac{\Delta t^2}{2}$
accelerate computation

- **time scaling**: increase velocity
- **mass scaling**: increase density

Bar	$c = \sqrt{\frac{E}{\rho}}$	$l_{char} = L$
Shell	$c = \sqrt{\frac{E}{(1-\nu^2)\rho}}$	quadraliteral $l_{char} = \frac{A}{min(l_{max}, d_{max})}$ triangluera $l_{char} = fracAh_{min}$
3D Solid	$c = \sqrt{\frac{E(1-\nu)}{(1+\nu)(1-2\nu)\rho}}$	Hexadral $l_{char} = \frac{V}{A_{max}}$ Tetrahedral $l_{char} = h_{min}$

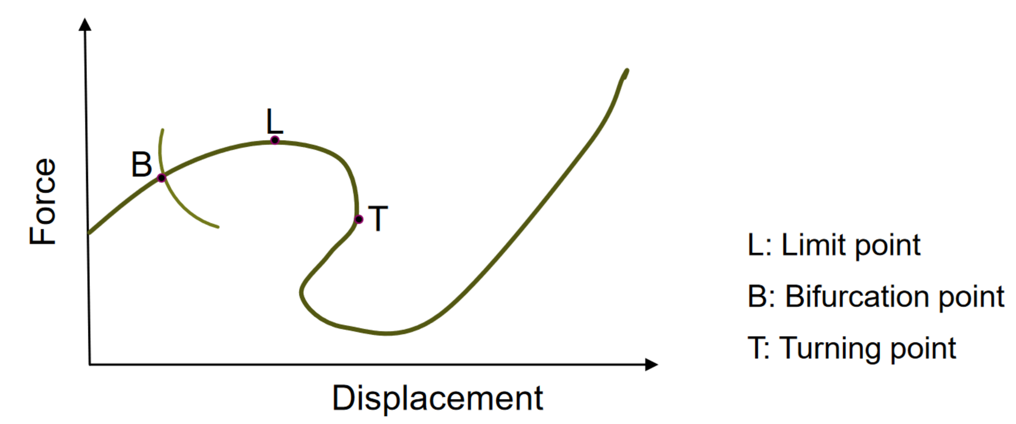
explicit method not suitable for

- slow processed
- spring back analysis
- incompressible material
- small plastic zone but large deformation

	QS Implicit	QS Explicit
Equilibrium	✓	×
Time step	arbitrary	$\Delta t < \Delta t_{char}$
Convergence	stable	conditionally stable
Boundary	RB suppressed	free
Contact	iterated	not iterated
Rezoning	$t_{fine} = t_{course}$	$\Delta t_{critic}^{new} = \frac{1}{2}\Delta t_{cirtic}^{old}$
Stiffness matrix	evaluated	not computed
Parallelization	complex	easy

Instability & Boundary & Cou

Instablility



Arc-Length method

$$\boldsymbol{r}(\boldsymbol{u}, \lambda) := \hat{f}^{int}(\boldsymbol{u}(t)) - \lambda(t)\boldsymbol{f}_0^{ext} = 0$$

$$g(\boldsymbol{u}(t), \lambda(t)) := \sqrt{\|\dot{\boldsymbol{u}}\|^2 + |\dot{\lambda}|^2} - \dot{s} = 0$$

$\dot{s}$ is the chosen value
$n + 1$ unkowns

Contact

Lagrange Multiplier Method

$$\begin{bmatrix} K & e_i \\ e_i^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ \lambda \end{bmatrix} = \begin{bmatrix} R \\ U_i* \end{bmatrix}$$

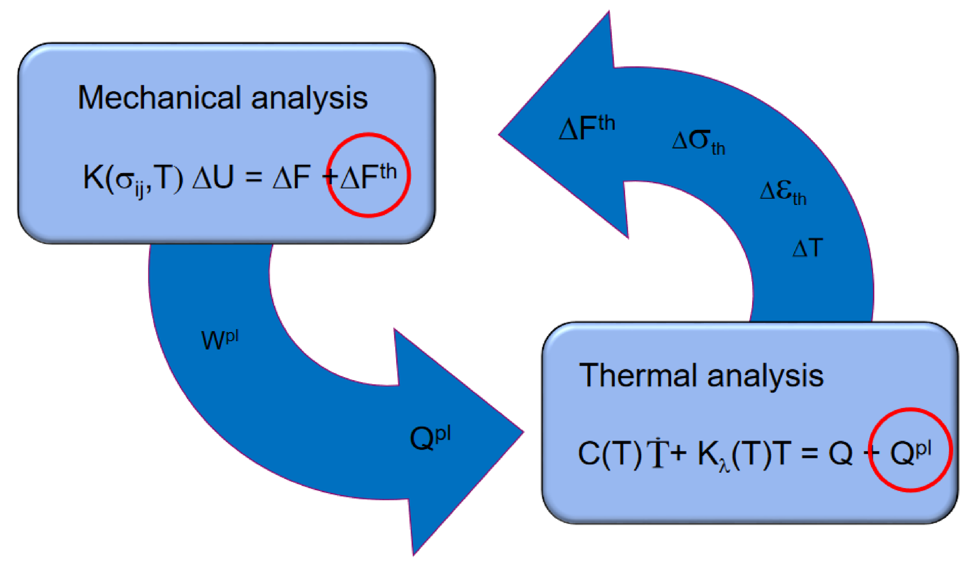
loss band structure of K
λ is additional variable

Penalty method

$$(K + \alpha Z^\top Z)U = R + \alpha Z^\top V$$

boundary condition : $ZU = V$
U is nodal displacement

Thermo coupled



Heating source

- plastic work
- friction
- heat transfer