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Final Exam **January 26th, 2017**

Dynamic Programming & Optimal Control (151-0563-01) **Prof. R. D'Andrea**

Solutions

Exam Duration:	**150 minutes**
Number of Problems:	**4**
Permitted aids:	One A4 sheet of paper. No calculators allowed.

Problem 1 [21 points]

Usually, the later you buy a train or plane ticket, the more you will have to pay. The practice of adjusting prices in order to optimize revenue is called *yield management* or *revenue management*. This practice is commonly seen among capacity-fixed, time-limited products (e.g. transportation tickets or hotel rooms). One way to find the optimal strategy is by using dynamic programming. We explore a simplified setup below.

You are the manager of a small transportation company. You have a bus that travels from Zürich to Geneva. This bus has n seats. The bus tickets are to be sold in N days. The selling price on day i, $i = 1, ..., N$, is denoted by p_i, and is fixed and known. On day i, you put u_i tickets up for sale. There is a corresponding demand D_i for day i, which has probabilistic characteristics (e.g. see Eq. (1)).

Your objective is to maximize your revenue and find the corresponding optimal strategy.

a) Formulate this problem as a dynamic programming problem. Define the stage and stage number, state(s) and state space, control input(s) and control space, disturbance(s) and disturbance space, system dynamics, stage costs, and terminal cost. **[11 points]**

b) Can you solve this problem using forward dynamic programming? Explain why or why not. **[2 points]**

c) Now assume $n = 6$, $N = 2$, p_1= 20 CHF and p_2 =40 CHF. Furthermore, on day 1, you are sure to sell all your tickets. On day 2, the demand D_2 for the tickets has the following probability distribution function:

$$\Pr(D_2) = \begin{cases} 0.25, & \text{for } D_2 = 3 \\ 0.5, & \text{for } D_2 = 4 \\ 0.25, & \text{for } D_2 = 5 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Use the dynamic programming algorithm to find the optimal number of tickets at 20 CHF you will put up for sale on day 1 and your expected revenue. **[8 points]**

Solution 1

a)

- Stage index k: the index k indicates day k on which we are selling the tickets at the k^{th} price level. E.g., $k = 1$ corresponds to day 1 on which we are selling tickets at price p_1. There are N stages. **[1 Point]**

- State space : $x_k \in S$ is the number of remaining tickets at stage k. $x_k \in S = \{0, 1, ..., n\},\ k = 1, ..., N, N + 1$. Note that $x_1 = n$. **[2 Points]**

- Control space : $u_k \in U_k(x_k)$ is the number of tickets we intend to sell on day k. $U_k(x_k) = \{0, 1, ..., x_k\},\ k = 1, ..., N$. **[2 Points]**

- Disturbance : $v_k = D_k$ is the demand at stage k. $D_k \sim \Pr(D_k),\ k = 1, ..., N$. **[2 Points]**

- System dynamics: $x_{k+1} = f(x_k, u_k, v_k) = x_k - \min(u_k, v_k)$. **[2 Points]**

- The cost function of this system is: $G(x_0, x, u, v) = \sum_{k=1}^{N}(-p_k \cdot (x_{k+1} - x_k)) = \sum_{k=1}^{N}(-p_k \cdot \min(u_k, v_k))$. The stage cost is $(-p_k \cdot \min(u_k, v_k))$. There is no terminal cost. **[2 Points]**

b) No. The state evolves in a stochastic manner, therefore it cannot be solved using forward dynamic programming. **[2 Points]**

c) This problem has 2 stages.

There is no terminal cost: we can initialize the expected optimal cost to $J_3(x_3) = 0$, $\forall x_3 \in S$. **[1 Point]**

Stage 2:

$$J_2(x_2) = \min_{u_2 \in U_2(x_2)} \mathop{\mathrm{E}}_{v_2}\{-p_2 \min(u_2, v_2)\}$$

Since $-p_2 < 0 \Rightarrow \mu_2^*(x_2) = x_2$.

Therefore, by evaluating the expectation with respect to v_2 (e.g. if $x_2 = 4$, then $u_2^* = 4, J_2(x_2 = 4) = -(0.25 \cdot 3 + 0.5 \cdot 4 + 0.25 \cdot 4) \cdot 40 = -150$, we have

$$J_2(x_2) = \begin{cases} -40x_2 & \text{, if } x_2 \leq 3 \\ -150 & \text{, if } x_2 = 4 \\ -160 & \text{, if } x_2 \geq 5 \end{cases}$$

[1 Point for setup, 2 Points for correct result]

Stage 1:

v_1 is deterministic and is always equal to n; $x_1 = 6$.

Therefore, we have

$$\begin{aligned} J_1(x_1) &= \min_{u_1 \leq 6}\{-p_1 u_1 + J_2(x_1 - u_1)\} \\ &= \min\{0 + J_2(6), -20 + J_2(5), -40 + J_2(4), -60 + J_2(3), -80 + J_2(2), \\ &- 100 + J_2(1), -120 + J_2(0)\} \\ &= \min\{-160, -180, -190, -180, -160, -140, -120\} \end{aligned}$$

[2 Points for correct formula and calculation]

We have $\mu_1^*(x_1 = 6) = 2$ and $J_1(x_1 = 6) = -190$ (i.e. the expected revenue is 190 CHF).

[2 Points for correct results]

Problem 2 **[13 points]**

You are currently in Zurich and wish to travel to Amsterdam by train. After looking into various routes, you come up with the map shown in Figure 1. Note that Deutsche Bahn has a special offer where they pay you to take their new train from Berlin to Amsterdam.

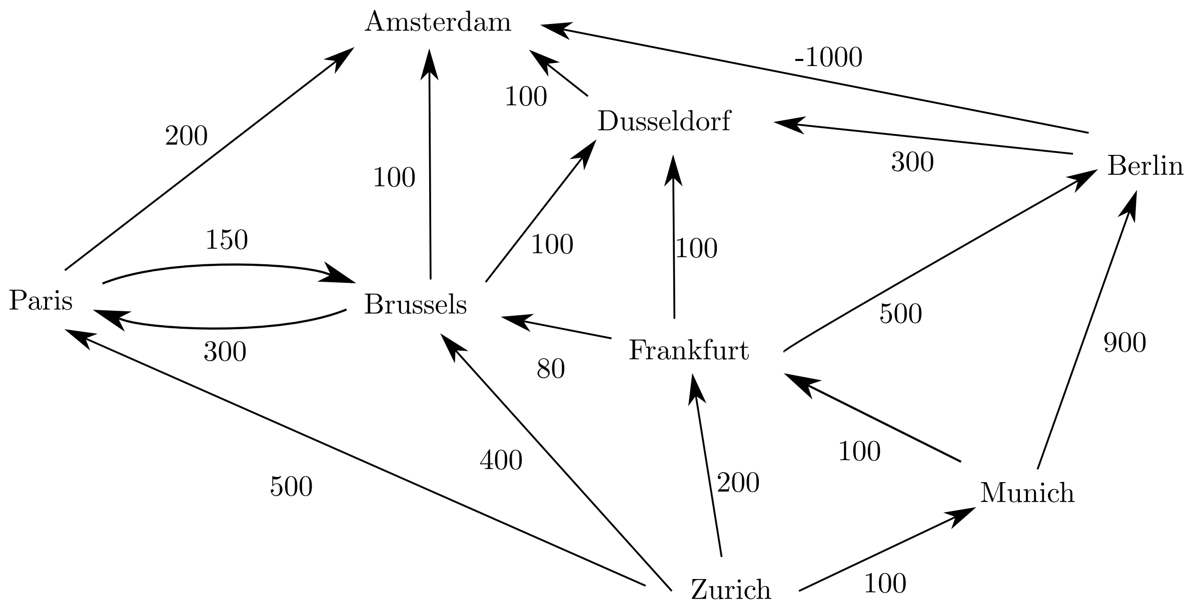

Figure 1: Shortest Path Problem. Edges represent train ticket cost.

a) All you care about is the cheapest route from Zurich to Amsterdam. If you had to choose between the Dynamic Programming Algorithm or the Label Correcting Algorithm to solve the problem, which one would you use and why? **[2 points]**

b) Use the Dynamic Programming Algorithm to find the lowest cost and associated path(s) from any city to Amsterdam. **[11 points]**

Solution 2

a) Use DPA since there are edges with negative cost which violates an assumption of the LCA. **[2 points for correct answer]**

b) Set of nodes $S = \{Z, P, Br, F, M, D, Be\}$, $N = 7$ **[1 point for correct setup]**

Destination node $t = A$.

Start DP Algorithm:
Initialize with one move to the end: $J_{N-1}(i) = a_{it}$, $i \in S$ **[1 point]**

$$J_6(Z) = \infty \tag{2}$$
$$J_6(P) = 200 \tag{3}$$
$$J_6(Br) = 100 \tag{4}$$
$$J_6(F) = \infty \tag{5}$$
$$J_6(M) = \infty \tag{6}$$
$$J_6(D) = 100 \tag{7}$$
$$J_6(Be) = -1000 \tag{8}$$

Recursion:
$k = 5$, i.e. 2 moves to the end, **[2 points]**

$$J_5(Z) = \min\{a_{ZZ} + J_6(Z), a_{ZP} + J_6(P), a_{ZBr} + J_6(Br), a_{ZF} + J_6(F), \tag{9}$$
$$a_{ZM} + J_6(M), a_{ZD} + J_6(D), a_{ZBe} + J_6(Be)\} \tag{10}$$
$$= 500 \tag{11}$$
$$\mu_5^*(Z) = Br \tag{12}$$

$$J_5(P) = \min\{a_{PZ} + J_6(Z), a_{PP} + J_6(P), a_{PBr} + J_6(Br), a_{PF} + J_6(F), \tag{13}$$
$$a_{PM} + J_6(M), a_{PD} + J_6(D), a_{PBe} + J_6(Be)\} \tag{14}$$
$$= 200 \tag{15}$$
$$\mu_5^*(P) = P \tag{16}$$

$$J_5(Br) = \min\{a_{BrZ} + J_6(Z), a_{BrP} + J_6(P), a_{BrBr} + J_6(Br), a_{BrF} + J_6(F), \tag{17}$$
$$a_{BrM} + J_6(M), a_{BrD} + J_6(D), a_{BrBe} + J_6(Be)\} \tag{18}$$
$$= 100 \tag{19}$$
$$\mu_5^*(Br) = Br \tag{20}$$

$$J_5(F) = \min\{a_{FZ} + J_6(Z), a_{FP} + J_6(P), a_{FBr} + J_6(Br), a_{FF} + J_6(F), \tag{21}$$
$$a_{FM} + J_6(M), a_{FD} + J_6(D), a_{FBe} + J_6(Be)\} \tag{22}$$
$$= -500 \tag{23}$$
$$\mu_5^*(F) = Be \tag{24}$$

$$J_5(M) = \min\{a_{MZ} + J_6(Z), a_{MP} + J_6(P), a_{MBr} + J_6(Br), a_{MF} + J_6(F), \tag{25}$$
$$a_{MM} + J_6(M), a_{MD} + J_6(D), a_{MBe} + J_6(Be)\} \tag{26}$$
$$= -100 \tag{27}$$
$$\mu_5^*(M) = Be \tag{28}$$

$$J_5(D) = \min\{a_{DZ} + J_6(Z), a_{DP} + J_6(P), a_{DBr} + J_6(Br), a_{DF} + J_6(F), \quad (29)$$
$$a_{DM} + J_6(M), a_{DD} + J_6(D), a_{DBe} + J_6(Be)\} \quad (30)$$
$$= 100 \quad (31)$$
$$\mu_5^*(D) = D \quad (32)$$

$$J_5(Be) = \min\{a_{BeZ} + J_6(Z), a_{BeP} + J_6(P), a_{BeBr} + J_6(Br), a_{BeF} + J_6(F), \quad (33)$$
$$a_{BeM} + J_6(M), a_{BeD} + J_6(D), a_{BeBe} + J_6(Be)\} \quad (34)$$
$$= -1000 \quad (35)$$
$$\mu_5^*(Be) = Be \quad (36)$$

Continuing on in tabular format and directly omitting paths that do not exist (i.e. $a_{ij} = \infty$) for brevity:

$k = 4$, i.e. 3 moves to the end **[2 points]**

i	$J_4(i)$	$\mu_4^*(i)$	$\leftarrow \min_{j \in S}(a_{ij} + J_4(j))$
Z	-300	F	$\min\{0 + J_5(Z), 500 + J_5(P), 400 + J_5(Br), 200 + J_5(F), 100 + J_5(M)\}$
P	200	P	$\min\{0 + J_5(P), 150 + J_5(Br)\}$
Br	100	Br	$\min\{0 + J_5(Br), 300 + J_5(P), 100 + J_5(D)\}$
F	-500	F	$\min\{0 + J_5(F), 80 + J_5(Br), 100 + J_5(D), 500 + J_5(Be)\}$
M	-400	F	$\min\{0 + J_5(M), 100 + J_5(F), 900 + J_5(Be)\}$
D	100	D	$\min\{0 + J_5(D)\}$
Be	-1000	Be	$\min\{0 + J_5(Be), 300 + J_5(D)\}$

$k = 3$, i.e. 4 moves to the end **[2 points]**

i	$J_3(i)$	$\mu_4^*(i)$	$\leftarrow \min_{j \in S}(a_{ij} + J_4(j))$
Z	-300	M or F	$\min\{0 + J_4(Z), 500 + J_4(P), 400 + J_4(Br), 200 + J_4(F), 100 + J_4(M)\}$
P	200	P	$\min\{0 + J_4(P), 150 + J_4(Br)\}$
Br	100	Br	$\min\{0 + J_4(Br), 300 + J_4(P), 100 + J_4(D)\}$
F	-500	F	$\min\{0 + J_4(F), 80 + J_4(Br), 100 + J_4(D), 500 + J_4(Be)\}$
M	-400	F	$\min\{0 + J_4(M), 100 + J_4(F), 900 + J_4(Be)\}$
D	100	D	$\min\{0 + J_4(D)\}$
Be	-1000	Be	$\min\{0 + J_4(Be), 300 + J_4(D)\}$

$J_3(i) = J_4(i)\ \forall\ i \in S$, therefore can terminate algorithm. **[1 point]**

From the above tables, the shortest paths from each city and corresponding costs are as follows: **[2 points]**
$Z \to M \to F \to Be \to A$ and
$Z \to F \to Be \to A$, cost=-300.
$P \to A$, cost = 200.
$Br \to A$, cost = 100.
$F \to Be \to A$, cost = -500.
$M \to F \to Be \to A$, cost = -400.
$D \to A$, cost = 100.
$Be \to A$, cost = -1000.

Problem 3 [23 points]

a) Three players are playing basketball against three defenders. At the start, one of the players (not the defenders) has the ball. Their goal is to shoot the ball into the basket, which will win them the game and the game ends. Each player can either shoot the ball, or pass the ball to another teammate (note that they cannot pass to themselves). If they choose to pass the ball, there is a probability that the ball gets stolen by a defender (i.e. a defender intercepts the pass). The game ends if the ball gets stolen. If they choose to shoot, there is a probability that they miss. However, when they miss, there is a probability that the shooter or one of his teammates gets the rebound (i.e. obtain the ball again) and the game continues. Otherwise, the game ends. S_{ij} is used to denote the pass success rate from player i to player j.

Player 1 (a typical Point Guard) is good at passing ($S_{1i} = 0.9, i = 2, 3$) and is a mediocre shooter (40 percent success rate).

Player 2 (a typical Shooting Guard) never passes and is good at shooting (70 percent success rate).

Player 3 (a typical Center) is bad at passing ($S_{3i} = 0.6, i = 1, 2$) and bad at shooting (20 percent success rate).

If a shot is missed, the probability of player 1 getting the rebound is 10 %, the probability of player 2 getting the rebound is 20 % and the probability of player 3 getting the rebound is 40 %.

Cast this problem as a stochastic shortest path problem:

i) Define the state(s) and the state space. **[3 points]**

ii) Define the control input(s) and control space. **[3 points]**

iii) Define **every** state transition probability. You can write down the state transition probabilities in matrix form for brevity. Make sure you clearly define the meaning of each entry. **[9 points]**

iv) Define **every** stage cost (i.e. the expected cost $g(i, u)$ using control u at state i). **[3 points]**

v) State the Bellman equation for this problem. Substitute the numbers from the previous questions. You **do not** have to solve the Bellman equation. **[3 points]**

vi) When applying policy iteration algorithm on this problem, will it always converge irrespective of the initialization? Shortly explain why or why not. **[2 points]**

Solution 3

a) Basketball player problem

i) State: the player that has the ball.

State space: { ball is with player 1, ball is with player 2, ball is with player 3, termination state } (denote by $\{x_1, x_2, x_3, t\}$). The termination state is defined as one of the following situations:

- the ball gets stolen
- a successful shot is made
- the offensive team does not get the rebound

[1 Point for correct definition of the states except the termination state, 1 Point for correct definition of the state space, 1 Point for correct definition of the termination state]

ii) Control input: pass to other teammates or shoot the ball.

Control space (dependent on the state): $U(x_1)$ = { pass to player 2, pass to player 3, shoot }, $U(x_2)$ = { shoot }, $U(x_3)$ = { pass to player 1, pass to player 2, shoot }, **[1 Point for correct definition of the control input, 1 Point for pointing out that the control space is dependent on the state, 1 Point for correct definition of the control space]**

iii) Under control action "shoot":

$$P(u = \text{shoot}) = \begin{array}{c} \begin{array}{cccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle t \end{array} \\ \begin{pmatrix} 0.06 & 0.12 & 0.24 & 0.58 \\ 0.03 & 0.06 & 0.12 & 0.79 \\ 0.08 & 0.16 & 0.32 & 0.44 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 2 \\ \scriptstyle 3 \\ \scriptstyle t \end{array} \tag{37}$$

where $P_{ij}(u = \text{shoot})$ denotes the transition probability from state i to j under control action "Shoot the ball", where i and j are labeled outside of the matrix $P(u = \text{shoot})$. E.g. P_{it} is the entry in the first row and fourth column. **[3 Points]**

Under control action "Pass the ball to player 1":

$$P(u = \text{pass to player 1}) = \begin{array}{c} \begin{array}{cccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle t \end{array} \\ \begin{pmatrix} 0.6 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \scriptstyle 3 \\ \scriptstyle t \end{array} \tag{38}$$

the meaning of the entry is similar to the previous matrix $P(u = \text{shoot})$. Note that player 1 cannot pass to himself and player 2 never passes. **[1 Point]**

Under control action "pass to player 2":

$$P(u = \text{pass to player 2}) = \begin{array}{c} \begin{array}{cccc} \scriptstyle 1 & \scriptstyle 2 & \scriptstyle 3 & \scriptstyle t \end{array} \\ \begin{pmatrix} 0 & 0.9 & 0 & 0.1 \\ 0 & 0.6 & 0 & 0.4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \scriptstyle 1 \\ \scriptstyle 3 \\ \scriptstyle t \end{array} \tag{39}$$

the meaning of the entry is similar to the previous matrix $P(u = \text{shoot})$. Note that player 2 never passes. **[2 Points]**

Under control action "pass to player 3":

$$P(u = \text{pass to player 3}) = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & t \end{array} \\ & \begin{pmatrix} 0 & 0 & 0.9 & 0.1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} 1 \\ t \end{array} \end{array} \tag{40}$$

the meaning of the entry is similar to the previous matrix $P(u = \text{shoot})$. Note that player 3 cannot pass to himself and player 2 never passes. **[1 Point]**

[1 Point for pointing out that there exists a probability P_{it} and $\sum_j P_{ij} = 1$. 1 Point for pointing out that the transition probability is dependent on the control action.]

iv)

$$g(i, u) = 0, \quad \forall i = 1, 2, 3, t \quad \forall u = \text{pass to player 1, 2, 3} \tag{41}$$

[1 Point]

$$g(i, u = \text{shoot}) = -\Pr(\text{successful shot})$$

The stage cost is negative since we're minimizing the cost, and therefore **[1 Point]**

$$\begin{aligned} g(1, u = \text{shoot}) &= -0.4, \quad g(2, u = \text{shoot}) = -0.7, \\ g(3, u = \text{shoot}) &= -0.2, \quad g(t, u = \text{shoot}) = 0. \end{aligned}$$

[1 Point]

v) Bellman equation

$$\begin{aligned} J^*(x_1) &= \min_{u \in U(x_1)} \mathrm{E}\{g(1, u) + \sum_{j=1}^{n} p_{1j}(u) J^*(j)\} \\ &= \min\{-0.4 + 0.06J^*(x_1) + 0.12J^*(x_2) + 0.24J^*(x_3), 0.9J^*(x_2), 0.9J^*(x_3)\} \\ J^*(x_2) &= -0.7 + 0.03J^*(x_1) + 0.06J^*(x_2) + 0.12J^*(x_3) \\ J^*(x_3) &= \min_{u \in U(x_3)} \{-0.2 + 0.08J^*(x_1) + 0.16J^*(x_2) + 0.32J^*(x_3), 0.6J^*(x_2), 0.6J^*(x_1)\} \end{aligned}$$

[1 Point for each equation]

vi) In this case, it will always converge with every possible initialization, since every stationary policy will lead to the termination state with a positive probability, regardless of the starting state.

[2 Points]

Problem 4 **[29 points]**

A vehicle with unit mass has velocity v. A force u can be applied and a linear drag term acts on the vehicle, such that the vehicle dynamics is

$$\dot{v}(t) = u(t) - v(t).$$

Initially the vehicle is at rest, i.e. $v(0) = 0$. The goal is to drive the vehicle to a final velocity of $v(T) = 1$.

a) Suppose $u(t) \in [-0.1, 0.1]$ for all t. Is the control problem feasible for any $T > 0$? Show why or why not. **[4 points]**

b) Suppose $u(t) \in \mathbb{R}$ for all t. Additionally we are to minimize the following cost:

$$\frac{1}{2}\int_0^T u(t)^2 \mathrm{d}t$$

where $T = \ln(2)$. Solve for the input $u(t)$ and state $v(t)$ that minimize this cost using Pontryagin's Minimum Principle. **[12 points]**

c) Suppose $u(t) \in [-\bar{u}, \bar{u}]$, $\bar{u} = \frac{1}{1-e^{-10}}$ for all t. Solve the time-optimal control problem, that is, solve for the end time T, input $u(t)$ and state $v(t)$ that minimize the cost

$$\int_0^T \mathrm{d}t$$

using Pontryagin's Minimum Principle. **[13 points]**

Solution 4

a) No, the largest attainable velocity (the terminal velocity) is $v = 0.1 \neq 1$. **[2 points]**
One way to show this is by reasoning that since $v(0) = 0$, and the target is $v(T) = 1.0$, we eventually must hit a $v(t) = 0.1$ for some t (note: $v(t)$ must be continuous). However, $\dot{v}$ will at most be 0 at this point due to our constrained input set and we can no longer increase v. **[2 points for correct reasoning]**

Another way is to solve for the state trajectory if we apply the maximum force; the dynamics become:

$$\dot{v} = 0.1 - v. \tag{42}$$

With $v(0) = 0$, there are a number of ways to solve this initial value problem. Let's solve using Laplace transform:

$$sV - v(0) = \frac{0.1}{s} - V \tag{43}$$

$$=> V(s+1) = \frac{0.1}{s(s+1)} \tag{44}$$

$$=> v(t) = 0.1 \int_0^t e^{-\tau} \mathrm{d}\tau \tag{45}$$

$$=> v(t) = 0.1(1 - e^{-t}) \tag{46}$$

Taking the supremum,

$$\sup_{t \geq 0} v(t) = 0.1 < 1. \tag{47}$$

Therefore, the control problem is infeasible.

b) As usual, let's define the various ingredients necessary to solve the Minimum Principle.

The Hamiltonian: **[1 point]**

$$H(v, u, \lambda) = \frac{1}{2}u^2 + \lambda(u - v) \tag{48}$$

The Hamiltonian is quadratic in u with Hessian > 0. Therefore, take the gradient and set to zero to get the optimal u^*: **[1 point]**

$$u^* = -\lambda \tag{49}$$

Co-state: **[2 points]**

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = \lambda \tag{50}$$

$$=> \lambda(t) = c_1 e^t, \ c_1 = \text{constant}. \tag{51}$$

Substituing u^* into the system dynamics, solve the BVP: **[1 point]**

$$\dot{v} = -c_1 e^t - v \tag{52}$$

$$=> \dot{v} + v = -c_1 e^t \tag{53}$$

This is a linear ODE. One can solve this using Laplace, homogeneous & particular solutions, or using integrating factors. Lets solve using homogeneous & particular solutions. Homogeneous:

$$\dot{v} + v = 0 \tag{54}$$

$$=> v_H(t) = c_2 e^{-t}, \ c_2 = \text{constant} \tag{55}$$

Particular:

$$v_P(t) = Ac_1e^t,\ A = \text{constant} \tag{56}$$
$$=> \dot{v}_P(t) + v_P(t) = -c_1e^t \tag{57}$$
$$=> Ac_1e^t + Ac_1e^t = -c_1e^t \tag{58}$$
$$=> A = -\frac{1}{2} \tag{59}$$

Thus, **[3 points for solving the ODE]**

$$v(t) = v_H(t) + v_P(t) = c_2e^{-t} - \frac{1}{2}c_1e^t. \tag{60}$$

Now use the boundary constraints to resolve the constants $c_1,\ c_2$: **[2 points]**

$$v(0) = c_2 - \frac{1}{2}c_1 = 0 \tag{61}$$
$$v(T = \ln(2)) = c_2\frac{1}{2} - c_1 = 1 \tag{62}$$
$$=> c_1 = -\frac{4}{3}, \quad c_2 = -\frac{2}{3} \tag{63}$$

Finally, **[2 points]**

$$u^*(t) = \frac{4}{3}e^t \tag{64}$$
$$v^*(t) = \frac{2}{3}e^t - \frac{2}{3}e^{-t} \tag{65}$$

c)

$$H(u, v, \lambda) = 1 + \lambda(u - x) \tag{66}$$

[1 point]

$$\dot{\lambda} = -\frac{\partial H}{\partial v} = \lambda \tag{67}$$
$$\Rightarrow \lambda(t) = c_1e^t,\ c_1 = \text{constant} \tag{68}$$

[2 points]

$$u^* = \arg\min_{u\in[-\bar{u},\bar{u}]}(u, v, \lambda) = \begin{cases} \bar{u}: & \lambda < 0 \\ -\bar{u}: & \lambda > 0 \\ ?: & \lambda = 0 \end{cases} \tag{69}$$

[2 points]

The third case is a singular arc. If $\lambda(t) = 0$ for a non-trivial amount of time, then $c_1 = 0$. If this is the case then $H = 1 \neq 0$, a contradiction since T is variable $\Rightarrow H = 0$. **[2 points]**

The second case cannot occur, since $\lambda(t)$ doesn't switch sign, if $c_1 > 0$, then $-\bar{u}$ would always be applied forever and we would only have a negative velocity, violating the terminal state constraint. **[1 point]**

Thus $c_1 < 0 \Rightarrow \lambda(t) < 0\ \forall\ t \geq 0$.

Substituing $u^* = \bar{u}$ into the system dynamics, solve the BVP: **[1 point]**

$$\dot{v} = \bar{u} - v \tag{70}$$

$$v^*(t) = \bar{u}(1 - e^{-t}) \text{ ode already solved for in part a)} \tag{71}$$

$$v^*(t) = \frac{1 - e^{-t}}{1 - e^{-10}} \tag{72}$$

$$v^*(T) = 1 \Rightarrow T = 10. \tag{73}$$

[4 points]