Dynamic Programming & Optimal Control

Lecture 11
Pontryagin's Minimum Principle (cont'd)

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Learning Objectives

Topic: Pontryagin's Minimum Principle (cont'd)

Objectives

- You know how the *Pontryagin's Minimum Principle* can be adapted to:
  - Fixed terminal state
  - Free initial state
  - Free terminal time
  - Time-varying system and cost
- You know how to handle *singular problems* with the Minimum Principle.

Outline

Fixed Terminal State (1/3)

Suppose that in addition to the initial state $x(0) = \mathrm{x}_0$, the final state $x(T) = \mathrm{x}_T$ is also given.

The terminal cost $h(x(T))$ is then not useful as $x(T)$ is fixed.

In effect, we have

$$J(T, \mathrm{x}) = \begin{cases} \text{some constant} & \text{if } \mathrm{x} = \mathrm{x}_T \\ \infty & \text{otherwise} \end{cases}.$$

Thus $J(T, \mathrm{x})$ is not differentiable in x, and the terminal boundary condition $p(T) = \left.\frac{\partial h(\mathrm{x})}{\partial \mathrm{x}}\right|_{x(T)}^{\top}$ for the co-state equation does not hold.

Fixed Terminal State (2/3)

However, as compensation, we have n extra boundary conditions from $x(T) = \mathrm{x}_T$.

We thus have $2n$ ordinary differential equations (ODEs) and $2n$ boundary conditions:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), & \qquad x(0) = \mathrm{x}_0,\ x(T) = \mathrm{x}_T, \\ \dot{p}(t) &= - \left. \frac{\partial H(\mathrm{x}, \mathrm{u}, \mathrm{p})}{\partial \mathrm{x}} \right|_{\substack{x(t)\\u(t)\\p(t)}}^{\top} . \end{aligned}$$

Fixed Terminal State (3/3)

Suppose instead that only some of the terminal states are fixed, that is:

$$x_i(T) = \mathrm{x}_{T,i}, \quad \forall i \in I,$$

where $x_i(T)$ and $\mathrm{x}_{T,i}$ denote the i^{th} entry of $x(T)$ and x_T, respectively, and I is an index set containing the indices of the fixed terminal states. In this case, we have the partial boundary condition

$$p_j(T) = \left.\frac{\partial h(\mathrm{x})}{\partial \mathrm{x}_j}\right|_{x(T)}, \quad \forall j \notin I,$$

for the co-state equation.

We still have $2n$ ODEs and $2n$ boundary conditions.

Example 1 (1/3)

Consider the system

$$\dot{x}(t) = u(t), \quad u(t) \in \mathbb{R},$$
$$0 \le t \le 1, \quad x(0) = 0, \quad x(1) = 1,$$

with cost function

$$\frac{1}{2}\int_0^1 \left(x(t)^2 + u(t)^2\right) \mathrm{d}t.$$

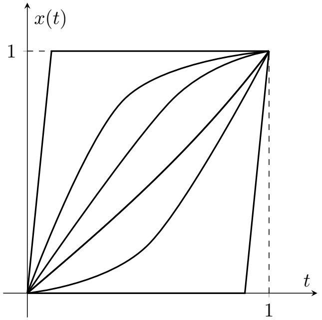


Figure: Possible state trajectories for Example 1. It is not obvious which one minimizes the cost function.

We want $x(t)$ to be small except at the very end to meet the terminal state condition. But this requires a large $u(t)$ towards the end which will incur a large cost (squared). This indicates a trade-off.

Example 1 (2/3)

We apply the Minimum Principle to solve the problem:

$$
\begin{aligned}
H(\mathrm{x},\mathrm{u},\mathrm{p}) &= \frac{1}{2}(\mathrm{x}^2+\mathrm{u}^2)+\mathrm{pu},\\
\dot{x}(t) &= u(t),\\
\dot{p}(t) &= -\left.\frac{\partial H(\mathrm{x},\mathrm{u},\mathrm{p})}{\partial \mathrm{x}}\right|^{\top}_{\substack{x(t)\\u(t)\\p(t)}} = -x(t),\\
u(t) &\in \underset{\mathrm{u}\in\mathcal{U}}{\arg\min}\, H(x(t),\mathrm{u},p(t)) = \underset{\mathrm{u}\in\mathcal{U}}{\arg\min}\left(\frac{1}{2}(x(t)^2+\mathrm{u}^2)+p(t)\mathrm{u}\right)\\
\Rightarrow u(t) &= -p(t).
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\dot{x}(t) &= -p(t),\\
\ddot{x}(t) &= -\dot{p}(t) = x(t).
\end{aligned}
$$

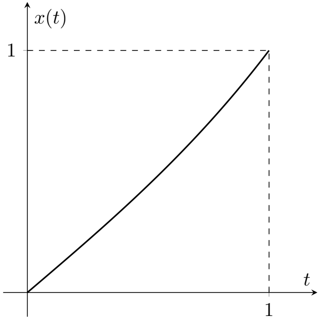
Example 1 (3/3)

The general solution to the ODE $\ddot{x}(t) = x(t)$ is $x(t) = Ae^t + Be^{-t}$, where A and B are constants to be determined from the boundary conditions:

$$x(0) = 0 \Rightarrow A + B = 0,$$
$$x(1) = 1 \Rightarrow Ae^1 + Be^{-1} = 1.$$

Thus:

$$A = \frac{1}{e - e^{-1}},\ B = -\frac{1}{e - e^{-1}},$$
$$x(t) = \frac{e^t - e^{-t}}{e - e^{-1}}.$$



Outline

Free Initial State (1/2)

Recall $J(t, \mathrm{x})$ is the cost-to-go at time t and state x.

Consider a problem where the initial state $x(0)$ is free and subject to optimization and where there is an additional term, $l(x(0))$, in the cost function:

$$\text{total cost} = l(x(0)) + J(0, x(0))\,.$$

A necessary condition for the optimal initial state $x(0) = \mathrm{x}_0$ that attains the minimum of the above is:

$$\begin{aligned} \left(\frac{\partial l(\mathrm{x})}{\partial \mathrm{x}} + \frac{\partial J(0, \mathrm{x})}{\partial \mathrm{x}} \right) \bigg|_{\mathrm{x}_0} &= 0 \\ \Rightarrow \frac{\partial J(0, \mathrm{x})}{\partial \mathrm{x}} \bigg|_{\mathrm{x}_0} &= -\frac{\partial l(\mathrm{x})}{\partial \mathrm{x}} \bigg|_{\mathrm{x}_0}\,. \end{aligned}$$

Free Initial State (2/2)

A necessary condition for the optimal initial state $x(0) = \mathrm{x}_0$ that attains the minimum of the above is:

$$\left(\frac{\partial l(\mathrm{x})}{\partial \mathrm{x}} + \frac{\partial J(0, \mathrm{x})}{\partial \mathrm{x}} \right) \bigg|_{\mathrm{x}_0} = 0$$

$$\Rightarrow \frac{\partial J(0, \mathrm{x})}{\partial \mathrm{x}} \bigg|_{\mathrm{x}_0} = - \frac{\partial l(\mathrm{x})}{\partial \mathrm{x}} \bigg|_{\mathrm{x}_0} .$$

Since $p(0) := \left. \frac{\partial J(0, \mathrm{x})}{\partial \mathrm{x}} \right|_{x(0)}^{\top}$, with the loss of the initial state boundary condition, we gain the adjoint boundary condition:

$$p(0) = - \left. \frac{\partial l(\mathrm{x})}{\partial \mathrm{x}} \right|_{x(0)}^{\top} .$$

Example 2 (1/3)

Consider the same system as before, but with free initial state

$$\dot{x}(t) = u(t), \quad u(t) \in \mathbb{R},$$
$$0 \leq t \leq 1, \; x(1) = 1,$$

and with cost function

$$\frac{1}{2} \int_0^1 \left(x(t)^2 + u(t)^2 \right) \mathrm{d}t.$$

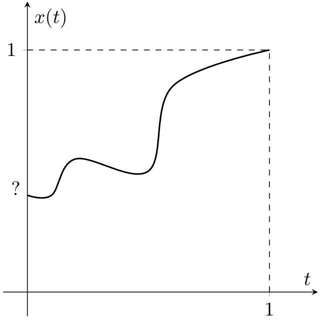


Figure: A possible state trajectory for Example 2.

Here, picking $x(0) = 1$ would result in $u(t) = 0$ for all t, but we accumulate cost due to $x(t)$. If we set $x(0) = 0$, $x(t)$ will contribute to a lower cost but $u(t)$ will add a large cost as the input needs to drive the state to 1 by $t = 1$. Thus in this case there is also a trade-off.

Example 2 (2/3)

We apply the Minimum Principle to solve the problem:

$$H(\mathrm{x},\mathrm{u},\mathrm{p}) = \frac{1}{2}(\mathrm{x}^2+\mathrm{u}^2)+\mathrm{pu},$$

$$\dot{x}(t) = u(t),$$

$$\dot{p}(t) = -\left.\frac{\partial H(\mathrm{x},\mathrm{u},\mathrm{p})}{\partial \mathrm{x}}\right|^{\top}_{\substack{x(t)\\u(t)\\p(t)}} = -x(t),$$

$$u(t) \in \arg\min_{\mathrm{u}\in\mathcal{U}} H(x(t),\mathrm{u},p(t)) = \arg\min_{\mathrm{u}\in\mathcal{U}}\left(\frac{1}{2}(x(t)^2+\mathrm{u}^2)+p(t)\mathrm{u}\right)$$

$$\Rightarrow u(t) = -p(t).$$

Thus:

$$\dot{x}(t) = -p(t), \quad \ddot{x}(t) = -\dot{p}(t) = x(t).$$

Furthermore, $l(\mathrm{x}) = 0$ for all x, thus:

$$p(0) = -\left.\frac{\partial l(\mathrm{x})}{\partial \mathrm{x}}\right|^{\top}_{x(0)} = 0.$$

Example 2 (3/3)

The general solution to the ODE $\ddot{x}(t) = x(t)$ is $x(t) = Ae^t + Be^{-t}$, and

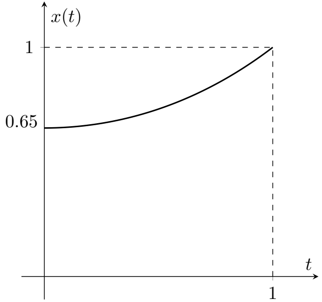
$$p(t) = -\dot{x}(t) = -Ae^t + Be^{-t},$$

where A and B are constants to be determined from the boundary conditions:

$$p(0) = 0 \Rightarrow -A + B = 0,$$
$$x(1) = 1 \Rightarrow Ae^1 + Be^{-1} = 1.$$

Thus:

$$A = B = \frac{1}{e + e^{-1}},$$
$$x(t) = \frac{e^t + e^{-t}}{e + e^{-1}}$$
$$\Rightarrow x(0) \approx 0.65.$$



Outline

Pontryagin's Minimum Principle (cont'd)

- Fixed Terminal State
- Free Initial State
- Free Terminal Time
- Time Varying System and Cost
- Singular Problems
- Additional reading material

Free Terminal Time

Suppose the initial state and/or the terminal state are given, but the terminal time T is free and subject to optimization.

We can compute the total cost for optimal trajectories for various terminal times T, and look for the T that minimizes this cost. This implies that if T is the optimal terminal time, then:

$$\left.\frac{\partial J(t,\mathrm{x})}{\partial t}\right|_{\substack{t=0\\x(0)}} = 0.$$

Recall that, on the optimal trajectory:

$$H(x(t),u(t),p(t)) = -\left.\frac{\partial J(t,\mathrm{x})}{\partial t}\right|_{x(t)} = \text{ constant }, \forall t.$$

Hence, this constant must be 0. In this case we gain an extra degree of freedom with free T, but lose a degree of freedom by the constraint

$$H(x(t),u(t),p(t)) = 0 \quad , \quad \forall t \in [0,T].$$

Example 3 (1/4)

Consider the following problem setup,

$$\dot{x}(t) = u(t), \qquad u(t) \in \mathbb{R},$$
$$x(0) = 0,\ x(T) = 1,$$

with free terminal time T and cost:

$$\int_0^T \left(1 + \frac{1}{2}\left(x(t)^2 + u(t)^2\right)\right) \mathrm{d}t$$
$$\Rightarrow\ g(\mathrm{x}, \mathrm{u}) = 1 + \frac{1}{2}(\mathrm{x}^2 + \mathrm{u}^2).$$

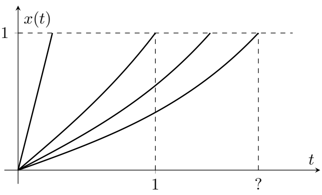


Figure: Possible state trajectories for Example 3. It is not obvious which one minimizes the cost function.

Similarly to Example 1, we tradeoff larger/smaller states with smaller/larger input along the trajectory. However, we can also reach $x(T) = 1$ in a shorter time at the price of larger input, or in a longer time with smaller input.

Example 3 (2/4)

We apply the Minimum Principle to solve the problem:

$$
\begin{aligned}
H(\mathrm{x},\mathrm{u},\mathrm{p}) &= 1 + \frac{1}{2}(\mathrm{x}^2 + \mathrm{u}^2) + \mathrm{pu}, \\
\dot{x}(t) &= u(t), \\
\dot{p}(t) &= -\left.\frac{\partial H(\mathrm{x},\mathrm{u},\mathrm{p})}{\partial \mathrm{x}}\right|^{\top}_{\substack{x(t)\\u(t)\\p(t)}} = -x(t), \\
u(t) &\in \arg\min_{\mathrm{u}\in\mathcal{U}} H(x(t),\mathrm{u},p(t)) = \arg\min_{\mathrm{u}\in\mathcal{U}} \left(1 + \frac{1}{2}(x(t)^2 + \mathrm{u}^2) + p(t)\mathrm{u}\right) \\
\Rightarrow u(t) &= -p(t).
\end{aligned}
$$

Thus:

$$
\begin{aligned}
\dot{x}(t) &= -p(t), \\
\ddot{x}(t) &= -\dot{p}(t) = x(t).
\end{aligned}
$$

Example 3 (3/4)

The general solution to the ODE $\ddot{x}(t) = x(t)$ is $x(t) = Ae^t + Be^{-t}$. The boundary conditions provide:

$$\begin{aligned} x(0) = 0 &\Rightarrow A + B = 0, \\ x(T) = 1 &\Rightarrow Ae^T + Be^{-T} = 1. \end{aligned}$$

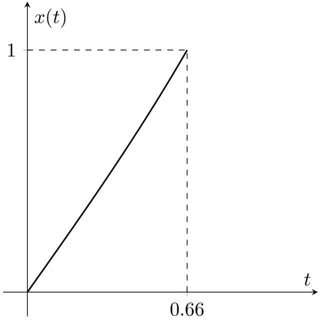
Thus:

$$\begin{aligned} A &= \frac{1}{e^T - e^{-T}}, \quad B = -\frac{1}{e^T - e^{-T}}, \\ x(t) &= \frac{e^t - e^{-t}}{e^T - e^{-T}}, \\ p(t) &= -\dot{x}(t) = -\frac{e^t + e^{-t}}{e^T - e^{-T}}. \end{aligned}$$

Example 3 (3/3)

We use the additional constraint on the Hamiltonian to find T:

$$\begin{aligned}
0 &= H(x(t), u(t), p(t)) \\
&= 1 + \frac{1}{2}(x(t)^2 + u(t)^2) + p(t)u(t) \\
&= 1 + \frac{1}{2}(x(t)^2 - p(t)^2) \\
&= 1 + \frac{1}{2}\left(\frac{(e^t - e^{-t})^2 - (e^t + e^{-t})^2}{(e^T - e^{-T})^2}\right) \\
&= 1 - \frac{2}{(e^T - e^{-T})^2}
\end{aligned}$$



$\Rightarrow T \approx 0.66.$

Note that, if we did not include the 1 in $g(\cdot,\cdot)$ (i.e. use the same $g(\cdot,\cdot)$ as in the previous example), we would get:

$$\begin{aligned}
0 &= H(x(t), u(t), p(t)) \\
&= -\frac{2}{(e^T - e^{-T})^2}
\end{aligned} \qquad \Rightarrow \qquad T \to \infty.$$

Outline

Pontryagin's Minimum Principle (cont'd)

- Fixed Terminal State
- Free Initial State
- Free Terminal Time
- Time Varying System and Cost
- Singular Problems
- Additional reading material

Time Varying System and Cost (1/4)

Consider the case when the system and stage cost vary with time:

$$\dot{x}(t) = f(x(t), u(t), t),$$
$$\text{cost} = h(x(T)) + \int_0^T g(x(\tau), u(\tau), \tau)\mathrm{d}\tau.$$

We can convert the problem to one involving a time-independent system and cost by introducing an extra state variable $y(t)$ representing time:

$$\dot{y}(t) := 1, \quad y(0) = 0$$
$$\Rightarrow y(t) = t.$$

Now we augment the system:

$$z(t) := (x(t), y(t)),$$
$$\dot{z}(t) = \begin{bmatrix} f(x(t), u(t), y(t)), \\ 1 \end{bmatrix} =: \bar{f}(z(t), u(t)),$$
$$\text{cost} = \underbrace{\bar{h}(z(T))}_{:=h(x(T))} + \int_0^T \underbrace{\bar{g}(z(\tau), u(\tau))}_{:=g(x(\tau), u(\tau), y(\tau))} \mathrm{d}\tau.$$

Time Varying System and Cost (2/4)

The Hamiltonian for the augmented system is:

$$\begin{aligned}\bar{H}(\mathrm{z}, \mathrm{u}, \bar{\mathrm{p}}) &= \bar{g}(\mathrm{z}, \mathrm{u}) + \bar{\mathrm{p}}^\top \bar{f}(\mathrm{z}, \mathrm{u}) \\ &= \underbrace{g(\mathrm{x}, \mathrm{u}, \mathrm{y}) + \mathrm{p}^\top f(\mathrm{x}, \mathrm{u}, \mathrm{y})}_{=:H(\mathrm{x},\mathrm{u},\mathrm{p},\mathrm{y})} + \mathrm{q} \\ &= H(\mathrm{x}, \mathrm{u}, \mathrm{p}, \mathrm{y}) + \mathrm{q},\end{aligned}$$

where $\mathrm{z} := (\mathrm{x}, \mathrm{y})$, and $\bar{\mathrm{p}} := (\mathrm{p}, \mathrm{q})$. Let $\bar{p}(t) := (p(t), q(t))$.

Time Varying System and Cost (3/4)

The corresponding conditions of the Minimum Principle are:

$$\dot{p}(t) = - \left. \frac{\partial \bar{H}(\mathrm{z},\mathrm{u},\bar{\mathrm{p}})}{\partial \mathrm{x}} \right|^{\top}_{\substack{z(t)\\u(t)\\\bar{p}(t)}} \qquad \dot{q}(t) = - \left. \frac{\partial \bar{H}(\mathrm{z},\mathrm{u},\bar{\mathrm{p}})}{\partial \mathrm{y}} \right|^{\top}_{\substack{z(t)\\u(t)\\\bar{p}(t)}}$$

$$= - \left. \frac{\partial H(\mathrm{x},\mathrm{u},\mathrm{p},t)}{\partial \mathrm{x}} \right|^{\top}_{\substack{x(t)\\u(t)\\p(t)}}, \qquad = - \left. \frac{\partial H(\mathrm{x},\mathrm{u},\mathrm{p},t)}{\partial t} \right|^{\top}_{\substack{x(t)\\u(t)\\p(t)}},$$

$$\begin{aligned} u(t) &\in \arg\min_{\mathrm{u}\in\mathcal{U}} \bar{H}(z(t),\mathrm{u},\bar{p}(t)) \\ &= \arg\min_{\mathrm{u}\in\mathcal{U}} H(x(t),\mathrm{u},p(t),t) + q(t) \\ &= \arg\min_{\mathrm{u}\in\mathcal{U}} H(x(t),\mathrm{u},p(t),t), \end{aligned}$$

$$\begin{aligned} &\bar{H}(z(t),u(t),\bar{p}(t)) = \text{ constant} \\ &\qquad \Rightarrow H(x(t),u(t),p(t),t) + q(t) = \text{ constant.} \end{aligned}$$

Time Varying System and Cost (4/4)

Thus, with the Hamiltonian of the original system

$$H(\mathrm{x}, \mathrm{u}, \mathrm{p}, t) = g(\mathrm{x}, \mathrm{u}, t) + \mathrm{p}^\top f(\mathrm{x}, \mathrm{u}, t),$$

the conditions of the Minimum Principle are:

$$\dot{p}(t) = - \left. \frac{\partial H(\mathrm{x}, \mathrm{u}, \mathrm{p}, t)}{\partial \mathrm{x}} \right|^\top_{\substack{x(t) \\ u(t) \\ p(t)}}, \quad p(T) = \left. \frac{\partial h(\mathrm{x})}{\partial \mathrm{x}} \right|^\top_{x(T)},$$

$$u(t) \in \arg\min_{\mathrm{u} \in \mathcal{U}} H(x(t), \mathrm{u}, p(t), t).$$

Note that the only difference between the time-invariant and the time-varying cases is that the Hamiltonian in the latter one need not be constant along the optimal trajectory.

Outline

Pontryagin's Minimum Principle (cont'd)

- Fixed Terminal State
- Free Initial State
- Free Terminal Time
- Time Varying System and Cost
- Singular Problems
- Additional reading material

Singular Problems

In some cases, the minimum principle condition

$$u(t) \in \arg\min_{\mathrm{u} \in \mathcal{U}} H(x(t), \mathrm{u}, p(t), t)$$

is insufficient to determine $u(t)$ for all t, because the values of $x(t)$ and $p(t)$ are such that $H(x(t), \mathrm{u}, p(t))$ is independent of u over a nontrivial interval of time.

Such problems are called *singular*.

Their optimal trajectories consist of portions, called *regular arcs*, where $u(t)$ can be determined from the minimum principle condition, and other portions, called *singular arcs*, which can be determined from the condition that the Hamiltonian is independent of u.

Example 4 – Tracking Problem (1/6)

Consider the system:

$$\dot{x}(t) = u(t), \quad 0 \leq t \leq 1,$$

where $|u(t)| \leq 1$ and $x(0)$ and $x(1)$ are free. The time-varying cost function is

$$\frac{1}{2}\int_0^1 \left(x(t) - z(t)\right)^2 \mathrm{d}t,$$

where $z(t) = 1 - t^2,\ 0 \leq t \leq 1$.

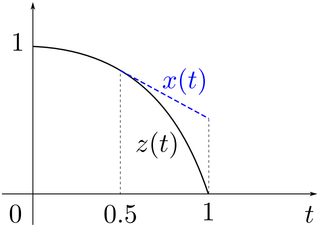


Figure: An example of a feasible state trajectory.

Example 4 – Tracking Problem (2/6)

We can write down the Hamiltonian

$$H(\mathrm{x}, \mathrm{u}, \mathrm{p}, t) = \frac{1}{2}(\mathrm{x} - z(t))^2 + \mathrm{pu},$$

and the co-state equation

$$\dot{p}(t) = - \left. \frac{\partial H(\mathrm{x}, \mathrm{u}, \mathrm{p}, t)}{\partial \mathrm{x}} \right|_{\substack{x(t)\\u(t)\\p(t)}}^{\top} = -(x(t) - z(t)),$$

with boundary conditions:

- free initial state, no initial cost: $p(0) = 0$;
- free terminal state, no terminal cost: $p(1) = 0$.

Furthermore

$$u(t) \in \underset{|\mathrm{u}| \leq 1}{\arg\min}\, H(x(t), \mathrm{u}, p(t), t) = \underset{|\mathrm{u}| \leq 1}{\arg\min} \left[\frac{1}{2}(x(t) - z(t))^2 + p(t)\mathrm{u} \right]$$

$$\Rightarrow u(t) = \begin{cases} -1 & \text{if } p(t) > 0 \\ 1 & \text{if } p(t) < 0 \\ \text{undetermined} & \text{if } p(t) = 0 \end{cases}.$$

Example 4 – Tracking Problem (3/6)

Now we determine the conditions for a singular arc to exist, namely, when $p(t)$ is zero for a non-trivial time interval. This implies that $\dot{p}(t) = 0$ over this time interval. Thus, by

$$\dot{p}(t) = -\left.\frac{\partial H(\mathrm{x}, \mathrm{u}, \mathrm{p}, t)}{\partial \mathrm{x}}\right|^{\top}_{\substack{x(t)\\u(t)\\p(t)}} = -(x(t) - z(t)),$$

we have

$$\begin{aligned} x(t) &= z(t), \\ \dot{x}(t) &= \dot{z}(t) = -2t. \end{aligned}$$

Invoking the system equation $\dot{x}(t) = u(t)$, we obtain the control input for the singular arc

$$u(t) = -2t.$$

Thus

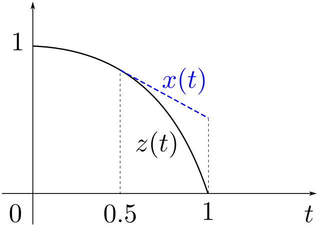
$$u(t) = \begin{cases} -1 & \text{if } p(t) > 0 \\ 1 & \text{if } p(t) < 0 \\ -2t & \text{if } p(t) = 0 \end{cases}.$$

Example 4 – Tracking Problem (4/6)

Since $p(0) = 0$, we assume the state trajectory starts with a singular arc until time t_s.

Note that t_s is at most $\frac{1}{2}$, since for $t > \frac{1}{2}$ the control constraint will be violated. Thus, the switch to the regular arc must happen at or before time $\frac{1}{2}$.

From the figure we can see that the control input should switch to $u(t) = -1$, otherwise the solution would clearly be suboptimal (larger $(x(t) - z(t))^2$).



Example 4 – Tracking Problem (5/6)

Thus, for $0 \leq t \leq t_s \leq \frac{1}{2}$,

$$\begin{aligned} x(t) &= z(t), \\ p(t) &= 0, \end{aligned}$$

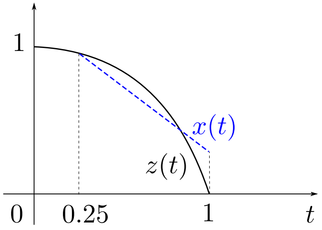
and for $t_s < t \leq 1$,

$$\begin{aligned} &\dot{x}(t) = -1 \\ \implies\ &x(t) = z(t_s) - (t - t_s) = 1 - t_s^2 - t + t_s, \\ &\dot{p}(t) = -(x(t) - z(t)) = -(1 - t_s^2 - t + t_s - 1 + t^2) = t_s^2 - t_s - t^2 + t. \end{aligned}$$

Example 4 – Tracking Problem (6/6)

From $p(1) = p(t_s) = 0$, we get:

$$
\begin{aligned}
0 &= p(1) - p(t_s) \\
&= \int_{t_s}^{1} \dot{p}(t)\mathrm{d}t \\
&= \int_{t_s}^{1} (t_s^2 - t_s - t^2 + t)\,\mathrm{d}t \\
&= \left[t_s^2 t - t_s t - \frac{t^3}{3} + \frac{t^2}{2} \right]_{t_s}^{1} \\
&= (t_s - 1)^2 (1 - 4t_s).
\end{aligned}
$$



Clearly $t_s = 1$ is not possible, whereas $t_s = \frac{1}{4}$ satisfies the switch time constraint.

We can verify that with $t_s = \frac{1}{4}$, $p(t) > 0$ for $t_s < t \leq 1$, which is consistent with $u(t) = -1$ for $t_s < t \leq 1$.

Outline

Pontryagin's Minimum Principle (cont'd)

- Fixed Terminal State
- Free Initial State
- Free Terminal Time
- Time Varying System and Cost
- Singular Problems
- Additional reading material

Additional reading material

Necessary optimality conditions represent a hammer widely adopted in the context of optimization: If something necessarily holds true at optimality, we can study that something to learn properties of the solution, or even the solution itself!

An important concept related to constrained optimization (i.e., the optimization of a function subject to constraints) is the one of Lagrange multipliers.

It turns out that Lagrange multipliers are intimately related to the co-state in the Minimum Principle. If you are interested in learning more, have a look at the following material on optimal control of systems from an optimization perspective: `https://ethz.ch/content/dam/ethz/special-interest/mavt/dynamic-systems-n-control/idsc-dam/Lectures/Optimal-Control/Additional%20Material/obc-optimal_04Jan10.pdf`