Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
Institut für Dynamische Systeme und Regelungstechnik

Final Exam **January 25th, 2018**

Dynamic Programming & Optimal Control (151-0563-01) Prof. R. D'Andrea

Solutions

Exam Duration: **150 minutes**

Number of Problems: **4**

Permitted aids: One A4 sheet of paper.
No calculators allowed.

Problem 1 [29 points]

a) Consider the system

$$x_{k+1} = \left(\mathbb{1}^\top u_k\right) x_k + u_k^\top R u_k, \quad k = 0, 1$$

where

$$\mathbb{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Furthermore, the state $x_k \in \mathbb{R}$ and the control input $u_k \in \mathbb{R}^2$. The cost function is given by

$$\sum_{k=0}^{2} x_k.$$

Calculate an optimal policy $\mu_1^*(x_1)$ using the dynamic programming algorithm and show that it is indeed optimal. Simplify the expression as much as possible. [**4 points**]

b) Consider the system

$$x_{k+1} = f_k(x_k, u_k, w_k), \quad k = 0, 1, ..., N-1.$$

The cost function is given by

$$\sum_{k=0}^{N-1} x_k^2.$$

At the beginning of each period k, we receive a prediction y_k that w_{k+2} will attain a probability distribution out of a given finite collection of distributions $\{p_{w_{k+2}|y_k}(\cdot|1), p_{w_{k+2}|y_k}(\cdot|2), ..., p_{w_{k+2}|y_k}(\cdot|m)\}$. In particular, we receive a forecast that $y_k = i$ and thus $p_{w_{k+2}|y_k}(\cdot|i)$ is used to generate w_{k+2}. Furthermore, the forecast itself has a given *a-priori* probability distribution, namely,

$$y_{k+1} = \xi_k,$$

where ξ_k is a random variable taking value $i \in \{1, 2, ..., m\}$ with probability $p_{\xi_k}(i)$. Given y_{k-2}, w_k is independent of all variables before time k; ξ_k is independent of all variables before time k.

Convert the above problem into the standard problem formulation of dynamic programming. In particular, write down the state vector, the system dynamics, and the disturbance vector with its probability density function (PDF) expressed as a function of the given PDFs. The dimension of the state space should be as small as possible. **You do not have to solve the dynamic programming problem** [**4 points**]

c) Consider a discrete random variable x which is defined by the set of all its possible outcomes $\mathcal{X}$ with $\mathcal{X} = \{1, 2, 3\}$, and a PDF p_x. Each element of $\mathcal{X}$ can only occur with probability 0, 0.5, or 1. The objective is to find the PDF that attains the minimum expected value of x:

$$\underset{p_x \in \mathcal{P}}{\text{minimize}} \quad \underset{x}{\mathrm{E}}[x] \tag{1}$$

where $\mathcal{P}$ is the set of all possible PDFs of x. Define the state y_k as $\sum_{i=1}^{k} p_x(i)$ for all $k \geq 1$. [**11 points**]

i) Formulate an equivalent problem that matches the standard form to which the dynamic programming algorithm can directly be applied, that is, explicitly state the

- dynamics $f_k(y_k, u_k, w_k)$ such that $y_{k+1} = f_k(y_k, u_k, w_k)$, initial condition y_0, and what u_k and w_k correspond to in the original problem.
- number of stages N such that $k = 0, ..., N-1$.
- state-space $\mathcal{S}_k$ such that $y_k \in \mathcal{S}_k$.
- control-space $\mathcal{U}_k(y_k)$ such that $u_k \in \mathcal{U}_k(y_k)$.
- stage costs $g_k(y_k, u_k, w_k)$ and terminal cost $g_N(y_N)$, such that the total cost is $\sum_{k=0}^{N-1} g_k(y_k, u_k, w_k) + g_N(y_N)$.

You do not have to solve the dynamic programming problem

ii) Is it possible to convert the above problem to a deterministic shortest path problem? If so, draw the corresponding graph. In particular, draw all the vertices including the starting node and the terminal node, and all the edges with the associated arc lengths. If not, explain why.

d) Consider a rectangular box with side lengths $l > 0, w > 0$, and $h > 0$, as shown in Fig. 1. The problem is to determine the side lengths that maximize the volume of the box, subject to the constraint $l + w + h = 1$. [**10 points**]

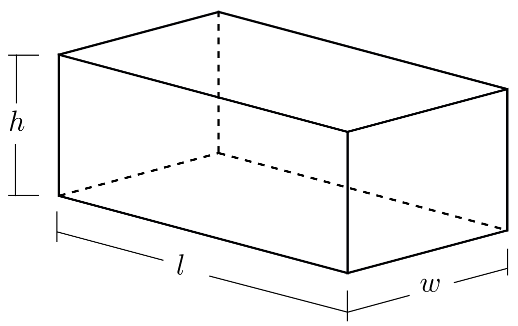

Figure 1: A rectangular box, also known as a right rectangular prism, or a cuboid.

i) Formulate an equivalent problem that matches the standard form to which the dynamic programming algorithm[1] can directly be applied, that is, explicitly state the

- dynamics $f_k(x_k, u_k)$ such that $x_{k+1} = f_k(x_k, u_k)$, initial condition x_0, and what x_k and u_k correspond to in the original problem.
- number of stages N such that $k = 0, ..., N-1$.
- state-space $\mathcal{S}_k$ such that $x_k \in \mathcal{S}_k$.
- control-space $\mathcal{U}_k(x_k)$ such that $u_k \in \mathcal{U}_k(x_k)$.
- stage costs $g_k(x_k, u_k)$ and terminal cost $g_N(x_N)$, such that the total cost is $\sum_{k=0}^{N-1} g_k(x_k, u_k) + g_N(x_N)$.

You do not have to solve the dynamic programming problem

ii) Is it possible to convert the above problem to a deterministic shortest path problem? If so, draw the corresponding graph. In particular, draw all the vertices including the starting node and the terminal node, and all the edges with the associated arc lengths. If not, explain why.

[1] You may replace min with max, and correspondingly the $g_k(\cdot)$ are rewards.

Solution 1

a) $\underline{k = N = 2:}$

$$J_2(x_2) = x_2$$

$\underline{k = 1:}$

$$\begin{aligned} J_1(x_1) &= \min_{u_1 \in \mathbb{R}^2} \left(x_1 + J_2(f_1(x_1, u_1))\right) \\ &= \min_{u_1} \left(x_1 + \left(\mathbb{1}^\top u_1\right) x_1 + u_1^\top R\, u_1\right) \\ \frac{\partial J_1(x_1)}{\partial u_1} &:= 0 \\ \Rightarrow \mathbb{1}^\top x_1 + u_1^\top (2R) &= 0 \\ 2Ru_1 &= -\mathbb{1} x_1 \\ u_1 &= \frac{1}{2} R^{-1}(-\mathbb{1}) x_1 \\ &= -\frac{1}{4} \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 \end{aligned}$$

Furthermore,

$$\frac{\partial^2 J_1(x_1)}{\partial u_1^2} = 2R$$

Its eigenvalues are 4 and 2, which are positive, and thus the matrix is positive definite. The sufficient condition for optimality is therefore satisfied.

b)

- Let $s_k := y_{k-2}$, $r_k := y_{k-1}$, then the augmented state vector $\tilde{x}_k := (x_k, y_k, r_k, s_k)$. Since the forecasts y_{k-2}, y_{k-1}, y_k are known at time k, we still have perfect state information.
- We define our new disturbance as $\tilde{w}_k := (w_k, \xi_k)$, with probability distribution

$$\begin{aligned} p(\tilde{w}_k | \tilde{x}_k, u_k) &= p(w_k, \xi_k | x_k, y_k, r_k, s_k, u_k) \\ &= p(w_k | x_k, y_k, r_k, s_k, u_k)\, p(\xi_k | x_k, y_k, r_k, s_k, u_k) \\ &= p(w_k | s_k)\, p(\xi_k)\,. \end{aligned}$$

  Note that w_k depends only on $\tilde{x}_k$ (in particular s_k), and ξ_k does not depend on anything.
- The dynamics therefore become

$$\tilde{x}_{k+1} = \begin{bmatrix} x_{k+1} \\ y_{k+1} \\ r_{k+1} \\ s_{k+1} \end{bmatrix} = \begin{bmatrix} f_k\left(x_k, u_k, w_k\right) \\ \xi_k \\ y_k \\ r_k \end{bmatrix} =: \tilde{f}_k\left(\tilde{x}_k, u_k, \tilde{w}_k\right),$$

  which now match the standard form.

c) i)

- Stage index: there are 3 elements in the outcome set $\mathcal{X}$. There are 3 stages.
- State: $y_k = \sum_{i=1}^{k} p_x(i)$ for $k = 1, 2, 3$, $y_0 = 0$.
- State space: $\mathcal{S}_0 = \{0\}$; $\mathcal{S}_k = \{0, \frac{1}{2}, 1\}$, $k = 1, 2$; $\mathcal{S}_3 = \{1\}$.
- Control input: $u_k = p_x(k+1)$, $j = 0, 1, 2$.
- Dynamics: $y_{k+1} = y_k + u_k$.

- Control space: $\mathcal{U}_2(y_2) = 1 - y_2$; $\mathcal{U}_k(y_k) = \{u \in \mathbb{R} | u = \frac{1}{2}n, n \in \mathbb{N}, 0 \leq u \leq 1 - y_k\}$, $k = 0, 1$.

- Disturbance: there are no disturbances.

- Stage cost: $u_k(k+1)$, $k = 0, 1, 2$

- Terminal cost: there is no terminal cost.

ii) Yes.

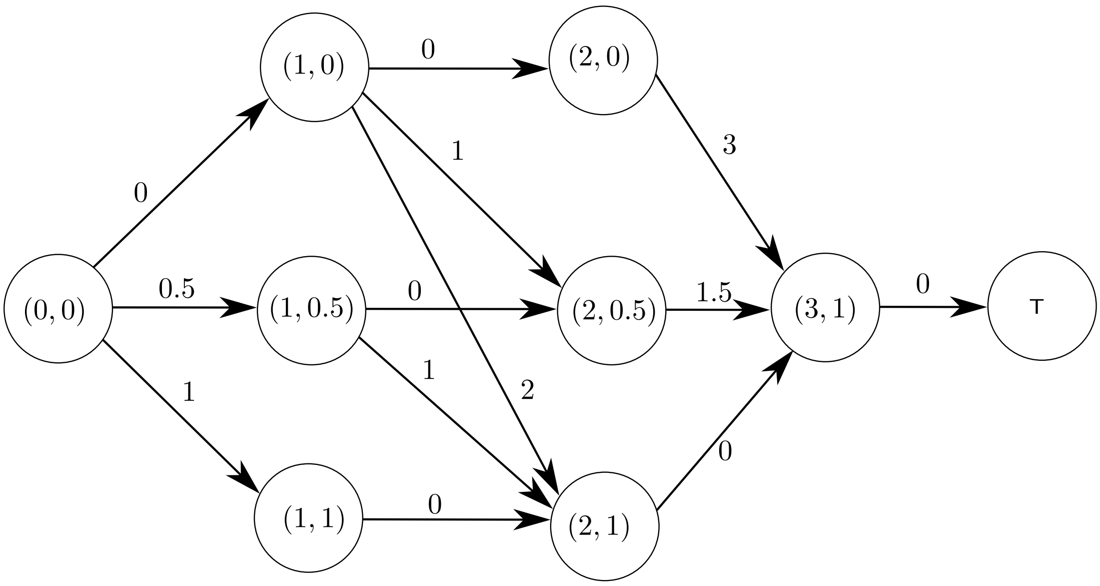

Figure 2

d) The original problem can be written as follows:

$$\begin{aligned} \text{maximize} \quad & lwh \\ \text{s.t.} \quad & l + w + h = 1 \\ & l, w, h > 0 \end{aligned}$$

This is equivalent to

$$\begin{aligned} \text{maximize} \quad & \ln(lwh) = \ln(l) + \ln(w) + \ln(h) \\ \text{s.t.} \quad & l + w + h = 1 \\ & l, w, h > 0 \end{aligned}$$

since $\ln(\cdot)$ is a monotonic increasing function. We have seen its use in the Viterbi Algorithm.

i)

- Let u_k represent the length of side k, $k = 0, 1, 2$, and x_k represent the sum of the lengths of sides 0 to $k-1$ (inclusive). Thus we start with $x_0 = 0$ and the dynamics are

$$x_{k+1} = x_k + u_k$$

- Since there are three sides, $k = 0, 1, 2$, and thus $N = 3$.
- $\mathcal{S}_0 = \{0\},\ \mathcal{S}_k = (0, 1], k = 1, 2,\ \mathcal{S}_3 = \{1\}$.
- $\mathcal{U}_k(x_k) = (0, 1 - x_k], k = 0, 1, \mathcal{U}_2(x_2) = \{1 - x_2\}$.
- $g_k(x_k, u_k) = \ln(u_k),\ g_N(x_N) = 0$.

Method 2 (Defining it as one stage problem)
Something like the following.

- Let $u_0 = (l, w, h)$ be a touple of the edge lengths. Dynamics x_k can be anything, doesn't matter.
- $k = 0, 1$, so $N = 1$.
- State space should be consistent with the dynamics.
- $\mathcal{U}_0 = \{u \, |||u||_1 = 1\}$
- $g_0(x_0, u_0) = -lwh$. $g_N(x_N) = 0$.

ii) No, the state space is not finite.

Problem 2 **[13 points]**

For problems marked with *: Answers left blank are worth 0 points. Each wrong answer is worth -1 point. You do **not** have to explain your answer. Each correct answer is worth 1 point. The minimum score of **Problem 2** is 0.

a) True or False questions. You do **not** have to explain your answer. [**3 points**]

i)* In dynamic programming, every finite state problem can be converted to a deterministic shortest path problem.

ii)* In the Viterbi algorithm, we are given a measurement sequence $Z_N = (z_1, \dots, z_N)$, and we want to find the "most likely" state trajectory $X_N = (x_0, \dots, x_N)$. In particular, we solve for a maximum a-posteriori estimate $\hat{X}_N := (\hat{x}_0, \dots, \hat{x}_N)$ where

$$\hat{X}_N = \arg\max_{X_N} p(X_N|Z_N)\,.$$

Let $Z_k := (z_1, \dots, z_k)$ and $X_k := (x_1, \dots, x_k)$ for some time $k < N$. The estimate $\hat{X}_k$ that maximizes $p(X_k|Z_k)$ can always (in theory) be computed at the end of time k.

iii)* Consider any deterministic shortest path problem, and $-C$ is the smallest arc length, where C is positive. If we add C to every arc length such that now the smallest arc length has length 0, and is thus non-negative, such that the smallest arc length is 0, then we can always apply the label correcting algorithm to find the shortest path.

b) Suppose the label correcting algorithm was applied to a shortest path problem, producing the following table:

Table 1

#	Remove	OPEN	d_S	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9	d_{10}	d_{11}	d_T
0	–	S	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	S	1,2,3	0	3	2	1	∞	∞	∞	∞	∞	∞	∞	∞	∞
2	3	1,2,4,6	0	3	2	1	3	∞	4	∞	∞	∞	∞	∞	∞
3	2	1,4,6,5	0	3	2	1	3	7	3	∞	∞	∞	∞	∞	∞
4	5	1,4,6,7,9	0	3	2	1	3	7	3	17	∞	9	∞	∞	∞
5	4	1,6,7,9,5	0	3	2	1	3	6	3	17	∞	9	∞	∞	∞
6	9	1,6,7,5,10,11	0	3	2	1	3	6	3	17	∞	9	10	10	∞
7	11	1,6,7,5,10	0	3	2	1	3	6	3	17	∞	9	10	10	12
8	10	1,6,7,5	0	3	2	1	3	6	3	17	∞	9	10	10	11
9	5	1,6,7,9	0	3	2	1	3	6	3	17	∞	8	10	10	11
10	6	1,7,9	0	3	2	1	3	6	3	8	∞	8	10	10	11
11	7	1,9,8	0	3	2	1	3	6	3	8	9	8	10	10	11
12	8	1,9	0	3	2	1	3	6	3	8	9	8	10	10	10
13	1	9	0	3	2	1	3	6	3	8	9	7	10	10	10
14	9	10,11	0	3	2	1	3	6	3	8	9	7	8	8	10
15	11	10	0	3	2	1	3	6	3	8	9	7	8	8	10
16	10	-	0	3	2	1	3	6	3	8	9	7	8	8	9

Answer the following questions pertaining to the above table. You do **not** have to explain your answer. [**10 points**]

i) What is the shortest path from S to T.

ii) Find three paths from S to node 8.

iii)* True or False: the associated graph can have more than 4 distinct paths from S to node 8.

iv)* True or False: The Depth-First Search method is used to remove nodes from OPEN.

v)* True or False: $c_{4,5} \geq c_{2,5}$. Recall $c_{i,j}$ is the edge length from node i to node j.

vi)* Which of the following is correct:

- $c_{1,8} < \infty$
- $c_{1,8} = \infty$
- There is insufficient data to determine $c_{1,8}$.

vii)* Which of the following is correct:

- $c_{1,4} < 1$
- $c_{1,4} \geq 1$
- There is insufficient data to determine $c_{1,4}$.

viii)* Which of the following is correct:

- $c_{6,7} = 5$
- $c_{6,7} \neq 5$
- There is insufficient data to determine $c_{6,7}$.

Solution 2

a)
- i) False
- ii) True
- iii) False

b)
- i) $(\mathsf{s}, 1, 9, 10, \mathsf{T})$.
- ii) From the table we can deduce 4 possible paths:
  - $(\mathsf{s}, 2, 6, 7, 8)$
  - $(\mathsf{s}, 3, 4, 5, 7, 8)$
  - $(\mathsf{s}, 3, 6, 7, 8)$
  - $(\mathsf{s}, 2, 5, 7, 8)$.
- iii) True.
- iv) False.
- v) False.
- vi) There is insufficient data to determine this.
- vii) There is insufficient data to determine this.
- viii) $c_{6,7} = 5$.

Problem 3 [17 points]

For problems marked with *: Each correct answer is worth 1 point. Answers left blank are worth 0 points. Each wrong answer is worth -1 point. You do **not** have to explain your answer. The minimum score of **Problem 3** is 0.

a) True of False questions. You do **not** have to explain your answer. [**5 points**]

i)* In stochastic shortest path problems, the value iteration algorithm always converges after a finite number of iterations.

ii)* In stochastic shortest path problems, the value iteration algorithm involves solving a system of linear equations.

iii)* In stochastic shortest path problems, the policy iteration algorithm in discounted problems can be initialized with an arbitrary admissible policy.

iv)* In stochastic shortest path problems, the policy iteration algorithm involves solving a system of linear equations.

v)* In stochastic shortest path problems, let the state space be $\mathcal{S} = \{0, 1, ..., n\}$ with the termination state 0, and $P_\mu \in \mathbb{R}^{n \times n}$ be the probability transition matrix associated with a policy μ, whose $(i, j)^{\text{th}}$ entry is $P_{ij}(\mu(i))$ with $i, j \in \mathcal{S} \backslash \{0\}$. The invertibility of the matrix $(I - P_\mu)$ for the policy μ is equivalent to the properness of that policy.

b) You are implementing the policy iteration algorithm for a stochastic shortest path problem on the computer. You printed out the cost vectors you solved at each iteration and the cost vectors at the second and third iteration are shown in Table 2:

Table 2

	second iteration	third iteration
state 1	5.03	4.93
state 2	4.67	4.32
state 3	2.87	2.89
state 4	1.50	1.28

Which of the following is correct? **Explain your answer**. [**2 points**]

- The implementation is definitely correct
- The implementation is definitely wrong
- Nothing can be deduced from Table 2

c) Consider the stochastic shortest path problem represented in Figure 3, where at state $i \in \{0, 1, 3\}$, the control action u can either be A or B, and at state $i = 2$, the control action u can only be A. [**10 points**]

i)* True or False: The policy $\mu(1) = A,\ \mu(2) = A,\ \mu(3) = B$ is proper.

ii)* True or False: The policy $\mu(1) = A,\ \mu(2) = A,\ \mu(3) = A$ is proper.

iii)* True or False: When solving this problem using the policy iteration algorithm, we can initialize the algorithm with the policy $\mu(1) = B,\ \mu(2) = A,\ \mu(3) = B$.

iv) For the policy in part i), construct the transition probability matrix $P_\mu \in \mathbb{R}^{3\times 3}$, whose $(i, j)^{\text{th}}$ entry is $P_{ij}(\mu(i))$ with $i, j \in \{1, 2, 3\}$. Is this matrix invertible or not?

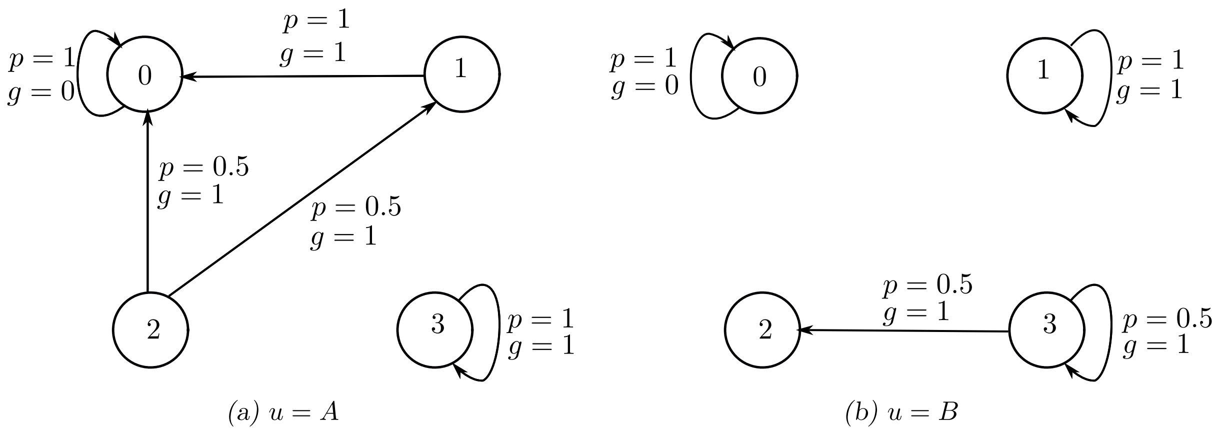

Figure 3: Probability transition graph, with the associated probabilities p, and stage costs g, denoted on each arc, that is,

$$
\begin{aligned}
P_{00}(A) &= 1 \\
P_{10}(A) &= 1 \\
P_{21}(A) &= 0.5 \\
P_{20}(A) &= 0.5 \\
P_{33}(A) &= 1 \\
g(i, A, j) &= 1 \ \forall i \neq 0, \forall j \\
g(0, A, 0) &= 0
\end{aligned}
$$

$$
\begin{aligned}
P_{00}(B) &= 1 \\
P_{11}(B) &= 1 \\
P_{32}(B) &= 0.5 \\
P_{33}(B) &= 0.5 \\
g(i, B, j) &= 1 \ \forall i \in \{1, 3\}, \forall j \\
g(0, B, 0) &= 0
\end{aligned}
$$

v) The optimal cost vector to the above stochastic shortest path problem can be obtained by solving a linear program of the generic form

$$
\begin{aligned}
\underset{V}{\text{minimize}} \quad & f^\top V \\
\text{subject to} \quad & MV \leq h
\end{aligned}
$$

where V, f and h are vectors, and M is a matrix. Write down a choice for f, h, and M such that the optimal cost vector is obtained by solving the above linear program.

Solution 3

a)
- i) False
- ii) False
- iii) True
- iv) True
- v) True

b) The implementation is definitely wrong, since the cost associated with state 3 increases after iteration 2. In policy iteration, the cost stays the same or decreases for any state after each iteration.

c)
- i) True
- ii) False
- iii) False
- iv)

$$P_\mu = \begin{bmatrix} 0 & 0 & 0 \\ 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

No, it is not.[2]

- v) The optimization problem for the stochastic shortest path problem has the form

$$\begin{aligned} &\underset{V}{\text{minimize}} \quad \sum_{i=1}^{3} f_i V(i) \\ &\text{subject to} \quad V(i) \leq \left(q(i,u) + \sum_{j=1}^{3} P_{ij}(u) V(j) \right), \quad \forall u \in \mathcal{U}(i), \forall i \in \mathcal{S} \backslash \{0\} \end{aligned}$$

Write out the inequalities:

$$\begin{aligned} V(1) &\leq q(1,A) \\ V(1) &\leq q(1,B) + V(1) \\ V(2) &\leq q(2,A) + 0.5V(1) \\ V(3) &\leq q(3,A) + V(3) \\ V(3) &\leq q(3,B) + 0.5V(2) + 0.5V(3) \end{aligned}$$

Since every stage cost is 1, the expected stage cost $q(i,u)$ is equal to 1 for all $i \in \mathcal{S}$ and $u \in \mathcal{U}(i)$. Hence we get the inequalities in matrix form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -0.5 & 0.5 \end{bmatrix} V \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

[2] Note that $(I - P_\mu)$ is invertible.

therefore

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -0.5 & 0.5 \end{bmatrix}, \quad h = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

and

$$f = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}.$$

Problem 4 **[22 points]**

Consider a ground vehicle traveling on a horizontal plane at a constant speed:

$$\dot{x}(t) = \cos(\theta(t))$$
$$\dot{y}(t) = \sin(\theta(t))$$
$$\dot{\theta}(t) = u$$

where $(x(t), y(t))$ is the vehicle's position on the plane at time t, $\theta(t)$ is its heading (see Fig. 4), and $u(t) \in [-1, 1]$, for all t, is the control input.

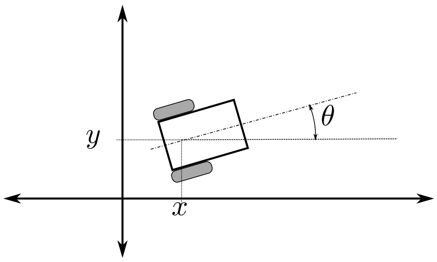

Figure 4: A ground vehicle.

The vehicle starts off at position $(0, 0)$ with a heading of 0 at $t = 0$. The objective is to determine the time-optimal trajectory that transfers the vehicle to position $(0, 3)$.

a) Compute Pontryagin's necessary conditions for optimality, including any singular arc conditions. [**9 points**]

b) In a couple of sentences, motivate why the optimal trajectory must end in a singular arc, and that the optimal $u(0) = 1$. [**1 point**]

c) Compute the optimal state and input trajectories, and the optimal terminal time T, using the hints from part b). Show that your solution satisfies the conditions from part a). [**12 points**]

Table 3: Trigonometry table for reference.

ϕ	$\sin(\phi)$	$\cos(\phi)$	$\tan(\phi)$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	∞
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
π	0	-1	0

Solution 4

a) Let $x_1 := x$, $x_2 := y$, $x_3 := \theta$.

$$\begin{aligned}
H(\mathrm{x},\mathrm{u},\mathrm{p}) &:= 1 + \mathrm{p}_1\cos(\mathrm{x}_3) + \mathrm{p}_2\sin(\mathrm{x}_3) + \mathrm{p}_3\mathrm{u}\\
\dot{x}_1(t) &= \left.\frac{\partial H}{\partial \mathrm{p}_1}\right|_{x(t),u(t),p(t)} = \cos(x_3(t))\\
\dot{x}_2(t) &= \left.\frac{\partial H}{\partial \mathrm{p}_2}\right|_{x(t),u(t),p(t)} = \sin(x_3(t))\\
\dot{x}_3(t) &= \left.\frac{\partial H}{\partial \mathrm{p}_2}\right|_{x(t),u(t),p(t)} = u(t)\\
\dot{p}_1(t) &= -\left.\frac{\partial H}{\partial \mathrm{x}_1}\right|_{x(t),u(t),p(t)} = 0\\
&\Leftrightarrow p_1(t) = c_1\\
\dot{p}_2(t) &= -\left.\frac{\partial H}{\partial \mathrm{x}_2}\right|_{x(t),u(t),p(t)} = 0\\
&\Leftrightarrow p_2(t) = c_2\\
\dot{p}_3(t) &= -\left.\frac{\partial H}{\partial \mathrm{x}_3}\right|_{x(t),u(t),p(t)} = c_1\sin(x_3(t)) - c_2\cos(x_3(t)) \qquad (2)
\end{aligned}$$

Boundary conditions:

$$\begin{aligned}
x(0) &= (0,0,0)\\
x(T) &= (0,3,\text{free})\\
p_3(T) &= 0
\end{aligned}$$

Since the problem data is time-invariant and T is free,

$$H(x(t),u(t),p(t)) = 1 + p_1(t)\cos(x_3(t)) + p_2(t)\sin(x_3(t)) + p_3(t)u(t) = 0 \ \forall t \qquad (3)$$

Optimal input:

$$\begin{aligned}
u(t) &= \underset{\mathrm{u}\in[-1,1]}{\arg\min}\, H(x(t),\mathrm{u},p(t))\\
&= \underset{\mathrm{u}\in[-1,1]}{\arg\min}\, p_3(t)\mathrm{u} = \begin{cases} -1 & p_3(t) > 0\\ 1 & p_3(t) < 0\\ ? & p_3(t) = 0\end{cases}
\end{aligned}$$

where the last case corresponds to a potential singular arc. Check if $p_3(t) = 0$ can occur non-trivially:

$$\begin{aligned}
\dot{p}_3(t) &= 0\\
&\Leftrightarrow c_1\sin(x_3(t)) = c_2\cos(x_3(t))\\
&\Leftrightarrow x_3(t) = \arctan 2\,(c_2, c_1)\\
&\Rightarrow u(t) = 0
\end{aligned}$$

The optimal input trajectory is then

$$u(t) = \begin{cases} -1 & p_3(t) > 0 \\ 1 & p_3(t) < 0 \\ 0 & p_3(t) = 0, x_3(t) = \arctan 2\,(c_2, c_1) =: c_3 \end{cases}$$

b) Note that $p_3(T) = 0$, so we *can* end with a singular arc. The intuition is this: apply $u(t) = 1$ until the vehicle heading faces the target position at some time $t = \tilde{t}$, at which point we switch to the singular arc, stop turning, and drive into the target position. Any other solution would take more time.

c) For $T \geq t \geq \tilde{t}$ - singular arc, $\tilde{t}$ is a switching time:

$$\begin{aligned} x_3(t) &= c_3 \\ \dot{x}_1 &= \cos(c_3) \Rightarrow x_1(t) = (t-T)\cos(c_3) \\ \dot{x}_2 &= \sin(c_3) \Rightarrow x_2(t) = (t-T)\sin(c_3) + 3 \\ p_3(t) &= 0. \\ u(t) &= 0 \qquad (4) \\ H(x(t), u(t), p(t)) &= 1 + c_1\cos(c_3) + c_2\sin(c_3) = 0 \qquad (5) \end{aligned}$$

For $\tilde{t} > t \geq 0$ - regular arc:
Assume $p_3(t) < 0$ so that $u(t) = 1$ (we will verify this later). Then,

$$\begin{aligned} u(t) &= 1 \\ x_3(t) &= t \\ x_1(t) &= \sin(t) \\ x_2(t) &= 1 - \cos(t) \end{aligned}$$

At the switch, we must have:

$$\begin{aligned} x_3(\tilde{t}) &= \tilde{t} = c_3 = \arctan 2\,(c_2, c_1) \qquad (6) \\ x_1(\tilde{t}) &= \sin(\tilde{t}) = (\tilde{t} - T)\cos(\tilde{t}) \qquad (7) \\ x_2(\tilde{t}) &= 1 - \cos(\tilde{t}) = (\tilde{t} - T)\sin(\tilde{t}) + 3 \\ &\Rightarrow \cos(\tilde{t}) + 2 = -(\tilde{t} - T)\sin(\tilde{t}). \qquad (8) \\ p_3(\tilde{t}) &= 0 \qquad (9) \end{aligned}$$

Now sum eq.(7)$*\sin(\tilde{t})$ and eq.(8)$*\cos(\tilde{t})$:

$$\begin{aligned} 1 + 2\cos(\tilde{t}) = 0 \Rightarrow \tilde{t} &= \frac{2\pi}{3}. \\ \Rightarrow T &= \frac{2\pi}{3} + \sqrt{3}. \end{aligned}$$

(note we reject the solution $\tilde{t} = \frac{4\pi}{3}$ since this will not get us to the terminal position).
Now we must determine c_1 and c_2 and show that $p_3(t) < 0$ up to $t = \tilde{t} = \frac{2\pi}{3}$. From (5),

$$1 - c_1\frac{1}{2} + c_2\frac{\sqrt{3}}{2} = 0$$

and from (6)

$$\tan\left(\frac{2\pi}{3}\right) = \frac{c_2}{c_1} = -\sqrt{3}$$

and thus $c_1 = \frac{1}{2},\ c_2 = -\frac{\sqrt{3}}{2}$.

From (2) and (9):

$$\begin{aligned} p_3(t) &= \int \frac{1}{2}\sin(t) + \frac{\sqrt{3}}{2}\cos(t)\mathrm{d}t + c_4 \\ &= -\frac{1}{2}\cos(t) + \frac{\sqrt{3}}{2}\sin(t) + c_4 \\ p_3\left(\frac{2\pi}{3}\right) &= 0 \Rightarrow c_4 = -1 \end{aligned}$$

and indeed $p_3(t) < 0$ for $t \in [0, \frac{2\pi}{3})$ since $p_3(0) = -1.5$ and $p_3(t)$ is at most 0 (can maximize to show this, or can plug in values from the table) which first happens after $t = 0$ at $t = \frac{2\pi}{3}$.

Finally, we must show (3) during the regular arc as well:

$$\begin{aligned} H(x(t), u(t), p(t)) &= 1 + \frac{1}{2}\cos(t) - \frac{\sqrt{3}}{2}\sin(t) + p_3(t) \\ &= 0 \end{aligned}$$