ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
IDSC
Institut für Dynamische Systeme und Regelungstechnik

Final Exam **August 23rd, 2022**

Dynamic Programming & Optimal Control (151-0563-01)

Prof. R. D'Andrea

Exam

Exam Duration: **150 minutes**

Number of Problems: **4**

Permitted aids: One A4 sheet of paper.

No calculators allowed.

Problem 1 **[25 points]**

Consider the dynamics

$$x_{k+1} = x_k + u_k + w_k, \quad k = 0, 1$$

with $x_k, u_k \in \mathbb{R}$, $w_k \in W_k = \{-1, 1\}$ and $p(w_k = -1) = p(w_k = 1)$. We want to minimize the following cost function:

$$E_{w_0, w_1}\{x_2^2 + u_1^2 + u_0^2\}$$

a) What is the terminal cost $g(x_2)$ and the stage cost $g(x_k, u_k, w_k),\ k = 0, 1$? *[2 points]*

b) Solve this problem with the Dynamic Programming Algorithm and provide the optimal policy $u_k^* = \mu_k(x_k),\ k = 0, 1$ and the corresponding optimal cost-to-go $J_k(x_k),\ k = 0, 1, 2$. *[11 points]*

c) Now, assume that we receive a forecast $y_k \in \{0, 1\}$ at each time-step. A priori, the forecast takes the value 0 or 1 with equal probability. Given the forecast, the probability distribution of the disturbance is the following:

$$p(w_k = -1 | y_k = 0) = \frac{3}{4} \qquad p(w_k = 1 | y_k = 0) = \frac{1}{4}$$
$$p(w_k = -1 | y_k = 1) = \frac{1}{10} \qquad p(w_k = 1 | y_k = 1) = \frac{9}{10}$$

Calculate the optimal policy $u_0^* = \mu_0(x_0, y_0)$.
To do that, use $J_1(x_1, y_1) = x_1^2 - y_1 x_1$ *[12 points]*

Solution 1

1.

$$\begin{aligned}
g_2(x_2) &= x_2^2 \\
g(x_1, u_1, w_1) &= u_1^2 \\
g(x_0, u_0, w_0) &= u_0^2
\end{aligned}$$

2. - $\underline{k = N = 2}$

$$J(x2) = x_2^2$$

- $\underline{k = 1}$

$$\begin{aligned}
J(x_1) &= \min_{u_1} E_{w_1}\{u_1^2 + J(x_2)\} \\
&= \min_{u_1} E_{w_1}\{u_1^2 + J(x_1 + u_1 + w_1)\} \\
&= \min_{u_1} E_{w_1}\{u_1^2 + (x_1 + u_1 + w_1)^2\} \\
&= \min_{u_1} E_{w_1}\{u_1^2 + (x_1 + u_1)^2 + w_1^2 + 2w_1(x_1 + u_1)\} \\
&= \min_{u_1}\{u_1^2 + (x_1 + u_1)^2 + 1\} \\
\frac{\partial J_1(x_1)}{\partial u_1} &= 2u_1 + 2(x_1 + u_1) = 0 \\
\Rightarrow \quad u_1^* &= \mu(x_1)^* = -\frac{x_1}{2} \\
J^*(x_1) &= \frac{x_1^2}{2} + 1
\end{aligned}$$

- $\underline{k = 0}$

$$\begin{aligned}
J(x_0) &= \min_{u_0} E_{w_0}\{u_0^2 + J(x_1)\} \\
&= \min_{u_0} E_{w_0}\{u_0^2 + J(x_0 + u_0 + w_0)\} \\
&= \min_{u_0} E_{w_0}\{u_0^2 + \frac{(x_0 + u_0 + w_0)^2}{2} + 1\} \\
&= \min_{u_0} E_{w_0}\{u_0^2 + \frac{(x_0 + u_0)^2}{2} + w_0(x_0 + u_0) + \frac{w_0^2}{2} + 1\} \\
&= \min_{u_0}\{u_0^2 + \frac{(x_0 + u_0)^2}{2} + \frac{3}{2}\} \\
\frac{\partial J_0(x_0)}{\partial u_0} &= 2u_0 + x_0 + u_0 = 0 \\
\Rightarrow \quad u_0^* &= \mu(x_0)^* = -\frac{x_0}{3} \\
J^*(x_0) &= \frac{x_0^2}{3} + \frac{3}{2}
\end{aligned}$$

3.

$$
\begin{aligned}
J(x_0, y_0) &= \min_{u_0} E_{w_0}\{u_0^2 + \frac{1}{2}J(x_1, y_1 = 0) + \frac{1}{2}J(x_1, y_1 = 1)\} \\
&= \min_{u_0} E_{w_0|y_0}\{u_0^2 + \frac{1}{2}x_1^2 + \frac{1}{2}(x_1^2 - x_1)\} \\
&= \min_{u_0} E_{w_0|y_0}\{u_0^2 + (x_0 + u_0 + w_0)^2 - \frac{1}{2}(x_0 + u_0 + w_0)\} \\
&= \min_{u_0} E_{w_0|y_0}\{u_0^2 + (x_0 + u_0)^2 + w_0^2 + 2w_0(x_0 + u_0) - \frac{1}{2}(x_0 + u_0 + w_0)\} \\
&= \min_{u_0}\{u_0^2 + (x_0 + u_0)^2 - \frac{1}{2}(x_0 + u_0) + E_{w_0|y_0}\{w_0^2 + 2w_0(x_0 + u_0) - \frac{1}{2}w_0)\}\}
\end{aligned}
$$

$$
\begin{aligned}
E_{w_0|y_0=0}\{w_0\} &= -\frac{3}{4} + \frac{1}{4} = -\frac{1}{2} & \qquad E_{w_0|y_0=0}\{w_0^2\} &= \frac{3}{4} + \frac{1}{4} = 1 \\
E_{w_0|y_0=1}\{w_0\} &= -\frac{1}{10} + \frac{9}{10} = \frac{4}{5} & \qquad E_{w_0|y_0=1}\{w_0^2\} &= \frac{1}{10} + \frac{9}{10} = 1
\end{aligned}
$$

$$
\begin{aligned}
J(x_0, y_0 = 0) &= \min_{u_0}\{u_0^2 + (x_0 + u_0)^2 - \frac{1}{2}(x_0 + u_0) + 1 - (x_0 + u_0) + \frac{1}{4}\} \\
&= \min_{u_0}\{u_0^2 + (x_0 + u_0)^2 - \frac{3}{2}(x_0 + u_0) + \frac{5}{4}\} \\
\frac{\partial J_0(x_0, y_0 = 0)}{\partial u_0} &= 2u_0 + 2x_0 + 2u_0 - \frac{3}{2} = 0 \\
\Rightarrow \quad u_0^* &= \mu(x_0, y_0 = 0)^* = -\frac{x_0}{2} + \frac{3}{8}
\end{aligned}
$$

$$
\begin{aligned}
J(x_0, y_0 = 1) &= \min_{u_0}\{u_0^2 + (x_0 + u_0)^2 - \frac{1}{2}(x_0 + u_0) + 1 + \frac{8}{5}(x_0 + u_0) - \frac{2}{5}\} \\
&= \min_{u_0}\{u_0^2 + (x_0 + u_0)^2 + \frac{11}{10}(x_0 + u_0) + \frac{3}{5}\} \\
\frac{\partial J_0(x_0, y_0 = 1)}{\partial u_0} &= 2u_0 + 2x_0 + 2u_0 + \frac{11}{10} = 0 \\
\Rightarrow \quad u_0^* &= \mu(x_0, y_0 = 1)^* = -\frac{x_0}{2} - \frac{11}{40}
\end{aligned}
$$

Problem 2 [25 points]

In Gotham City, Bruce Wayne, a wealthy playboy, swore vengeance against criminals after witnessing the murder of his parents. He trains himself physically and intellectually, crafts a bat-inspired persona and monitors Gotham's streets at night as a masked vendetta against crime: Batman.

With his high-tech Batsuit, Bruce can fight against criminals, but the suit has limited durability. The condition of the Batsuit can be categorized as 4 (new), 3 (slightly damaged), 2 (badly damaged), 1 (completely damaged). Every night, Bruce can choose to fight against dangerous criminals (D), punish weak criminals (W) or repair his Batsuit (R). If his Batsuit is completely damaged (1), he must repair it. If his Batsuit is new (4), he cannot repair it. The transition of the Batsuit condition depends on Bruce's actions and can be described as:

$$\begin{aligned} x_{k+1} &= w_k, \\ x_k &\in \{4, 3, 2, 1\} \\ u_k &\in \{D, W, R\} \end{aligned}$$

The risk of damaging the Batsuit increases for a more dangerous action. The transition probability $P_{i,j}(u_k) := P(w_k = j | x_k = i, u_k)$ can be described as follows (the unlisted transitions have zero probability):

$$\begin{aligned} P_{ij}(D) &= \frac{i - j + 1}{\sum_{s=1}^{i} s}, \quad \forall i \in \{2, 3, 4\},\ j \in \{1, 2, 3, 4\},\ j \leq i, \\ P_{ij}(W) &= \frac{j}{\sum_{s=1}^{i} s}, \quad \forall i \in \{2, 3, 4\},\ j \in \{1, 2, 3, 4\},\ j \leq i, \\ P_{i4}(R) &= 1, \quad \forall i \in \{1, 2, 3\}. \end{aligned}$$

Each action Bruce takes comes with a cost. Since criminals can harm Gotham's citizens during nights when Bruce chooses to repair his Batsuit, this action incurs the highest cost. The costs are defined as:

$$\begin{aligned} q(i, D) &= 4, \quad \forall i \in \{2, 3, 4\}, \\ q(i, W) &= 6, \quad \forall i \in \{2, 3, 4\}, \\ q(i, R) &= 15, \quad \forall i \in \{1, 2, 3\}. \end{aligned}$$

The total cost function that Bruce wants to **minimize** is the following:

$$\lim_{N \to \infty} \mathbb{E} \sum_{k=0}^{N-1} \alpha^k q(x_k, u_k),$$

where $\alpha = \frac{15}{16}$, which is the discount factor.

a) Using $J^*(x)$ as the optimal expected cost for a certain state x in this Infinite Horizon Problem, write down the Bellman equation for each state $x \in \{1, 2, 3, 4\}$. *[6 points]*

b) Perform one iteration of the **Value Iteration Algorithm** for the Infinite Horizon Problem and write down the corresponding policy. Consider $J^0(i) = 20 - 4i, \forall i \in \{1, 2, 3, 4\}$ as the initial guess. *[6 points]*

c) Answer the following True or False questions relating to the **Policy Iteration Algorithm**. You need to briefly explain your answers.

i) In this problem, the policy iteration algorithm can be initialized with an arbitrary admissible policy.

ii) There are in total 27 admissible policies for this problem.

iii) The systems of linear equations you need to solve when applying policy iteration always have a solution for this problem.

iv) You have implemented the policy iteration algorithm for this problem and printed out the cost for each value in the following table at two non-consecutive iterations. Your implementation is correct.

state	iteration 4	iteration 8
4	2.56	2.12
3	4.24	3.34
2	5.12	5.78
1	6.34	5.67

[8 points]

d) Denote the optimal value vector as $\mathbf{x} = [V(4), V(3), V(2), V(1)]^T$. The optimal value vector can be obtained by solving a **Linear Program** of the generic form

$$\begin{aligned} &\textit{minimize} \quad \mathbf{f}^T\mathbf{x} \\ &\textit{subject to} \quad \mathbf{A}\mathbf{x} \le \mathbf{b} \end{aligned}$$

where $\mathbf{f}$ and $\mathbf{b}$ are column vectors, and $\mathbf{A}$ is a matrix. Write down a choice of $\mathbf{f}$, $\mathbf{b}$ and $\mathbf{A}$ such that the optimal value vector $\mathbf{x}$ is obtained by solving the linear program. (Use numerical values for $\mathbf{f}$ and $\mathbf{b}$, but leave $\mathbf{A}$ in symbolic form). *[5 points]*

Solution 2

a) The Bellman equation is

$$J^*(x) = \min_{u\in\{D,W,R\}} [$$
$$q(x,u) + \alpha P_{x,4}(u)J^*(4) + \alpha P_{x,3}(u)J^*(3) + \alpha P_{x,2}(u)J^*(2) + \alpha P_{x,1}(u)J^*(1)]$$
$$\forall x \in \{4,3,2,1\}$$

The stage costs for all states are:

$$q(\cdot, D) = 4$$
$$q(\cdot, W) = 6$$
$$q(\cdot, R) = 15$$

The transition probabilities are:

$$\begin{array}{llll}
P_{4,4}(D) = 1/10 & P_{4,3}(D) = 2/10 & P_{4,2}(D) = 3/10 & P_{4,1}(D) = 4/10 \\
P_{3,4}(D) = 0 & P_{3,3}(D) = 1/6 & P_{3,2}(D) = 2/6 & P_{3,1}(D) = 3/6 \\
P_{2,4}(D) = 0 & P_{2,3}(D) = 0 & P_{2,2}(D) = 1/3 & P_{2,1}(D) = 2/3 \\
\\
P_{4,4}(W) = 4/10 & P_{4,3}(W) = 3/10 & P_{4,2}(W) = 2/10 & P_{4,1}(W) = 1/10 \\
P_{3,4}(W) = 0 & P_{3,3}(W) = 3/6 & P_{3,2}(W) = 2/6 & P_{3,1}(W) = 1/6 \\
P_{2,4}(W) = 0 & P_{2,3}(W) = 0 & P_{2,2}(W) = 2/3 & P_{2,1}(W) = 1/3 \\
\\
P_{3,4}(R) = 1 & P_{3,3}(R) = 0 & P_{3,2}(R) = 0 & P_{3,1}(R) = 0 \\
P_{2,4}(R) = 1 & P_{2,3}(R) = 0 & P_{2,2}(R) = 0 & P_{2,1}(R) = 0 \\
P_{1,4}(R) = 1 & P_{1,3}(R) = 0 & P_{1,2}(R) = 0 & P_{1,1}(R) = 0
\end{array}$$

The Bellman Equations are:

$$\begin{aligned}
J^*(4) = \min_{u\in\{D,W\}} &[4 + \frac{3}{32}\cdot J^*(4) + \frac{3}{16}\cdot J^*(3) + \frac{9}{32}\cdot J^*(2) + \frac{3}{8}\cdot J^*(1), \\
&6 + \frac{3}{8}\cdot J^*(4) + \frac{9}{32}\cdot J^*(3) + \frac{3}{16}\cdot J^*(2) + \frac{3}{32}\cdot J^*(1)] \\
J^*(3) = \min_{u\in\{D,W,R\}} &[4 + \frac{5}{32}\cdot J^*(3) + \frac{5}{16}\cdot J^*(2) + \frac{15}{32}\cdot J^*(1), \\
&6 + \frac{15}{32}\cdot J^*(3) + \frac{5}{16}\cdot J^*(2) + \frac{5}{32}\cdot J^*(1), \\
&15 + \frac{15}{16}\cdot J^*(4)] \\
J^*(2) = \min_{u\in\{D,W,R\}} &[4 + \frac{5}{16}\cdot J^*(2) + \frac{5}{8}\cdot J^*(1), \\
&6 + \frac{5}{8}\cdot J^*(2) + \frac{5}{16}\cdot J^*(1), \\
&15 + \frac{15}{16}\cdot J^*(4)] \\
J^*(1) = &15 + \frac{15}{16}\cdot J^*(4)
\end{aligned}$$

b)

$$J^0(4) = 4$$
$$J^0(3) = 8$$
$$J^0(2) = 12$$
$$J^0(1) = 16$$

$$\begin{aligned}
J^1(4) = &\min_{u\in\{D,W\}}[4 + \frac{3}{32}\cdot 4 + \frac{3}{16}\cdot 8 + \frac{9}{32}\cdot 12 + \frac{3}{8}\cdot 16,\\
&6 + \frac{3}{8}\cdot 4 + \frac{9}{32}\cdot 8 + \frac{3}{16}\cdot 12 + \frac{3}{32}\cdot 16] = \min_{u\in\{D,W\}}[\frac{61}{4}, \frac{27}{2}] = \frac{27}{2}\\
J^1(3) = &\min_{u\in\{D,W,R\}}[4 + \frac{5}{32}\cdot 8 + \frac{5}{16}\cdot 12 + \frac{15}{32}\cdot 16,\\
&6 + \frac{15}{32}\cdot 8 + \frac{5}{16}\cdot 12 + \frac{5}{32}\cdot 16,\\
&15 + \frac{15}{16}\cdot 4] = \min_{u\in\{D,W,R\}}[\frac{33}{2}, 16, \frac{75}{4}] = 16\\
J^1(2) = &\min_{u\in\{D,W,R\}}[4 + \frac{5}{16}\cdot 12 + \frac{5}{8}\cdot 16,\\
&6 + \frac{5}{8}\cdot 12 + \frac{5}{16}\cdot 16,\\
&15 + \frac{15}{16}\cdot 4] = \min_{u\in\{D,W,R\}}[\frac{71}{4}, \frac{37}{2}, \frac{75}{4}] = \frac{71}{4}\\
J^1(1) = &15 + \frac{15}{16}\cdot 4 = \frac{75}{4}
\end{aligned}$$

The policy after one iteration of VI is

$$u^1(4) = W$$
$$u^1(3) = W$$
$$u^1(2) = D$$
$$u^1(1) = R$$

c)
- i) True. This problem is a discounted problem.
- ii) False. There are in total $2\cdot 3\cdot 3\cdot 1 = 18$ admissible policies.
- iii) True. The problem is discounted and every policy is proper, so the matrix $I - \alpha P$ is always invertible.
- iv) False. For PI, the corresponding values of each state at each iteration should decrease or remain the same, so the value for state 3 increases incorrectly.

d) Be careful about the signs and inequalities.

$$\begin{aligned}
&\textit{maximize} \quad V(4) + V(3) + V(2) + V(1)\\
&\textit{subject to} \quad V(i) \le q(i,u) + \alpha\sum_{j=1}^{4} P_{ij}(u)V(j), \forall u\in\{D,W,R\}, i\in\{1,2,3,4\}
\end{aligned}$$

Since it is minimization in the standard form, we have

$$\mathbf{f} = \begin{bmatrix} -1 & -1 & -1 & -1 \end{bmatrix}^T$$

Constraints are:

$$\begin{aligned}
&V(4) \le 4 + \alpha P_{44}(D)V(4) + \alpha P_{43}(D)V(3) + \alpha P_{42}(D)V(2) + \alpha P_{41}(D)V(1) \\
&V(4) \le 6 + \alpha P_{44}(W)V(4) + \alpha P_{43}(W)V(3) + \alpha P_{42}(W)V(2) + \alpha P_{41}(W)V(1) \\
&V(3) \le 4 + \alpha P_{33}(D)V(3) + \alpha P_{32}(D)V(2) + \alpha P_{31}(D)V(1) \\
&V(3) \le 6 + \alpha P_{33}(W)V(3) + \alpha P_{32}(W)V(2) + \alpha P_{31}(W)V(1) \\
&V(3) \le 15 + \alpha P_{34}(R)V(4) \\
&V(2) \le 4 + \alpha P_{22}(D)V(2) + \alpha P_{21}(D)V(1) \\
&V(2) \le 6 + \alpha P_{22}(W)V(2) + \alpha P_{21}(W)V(1) \\
&V(2) \le 15 + \alpha P_{24}(R)V(4) \\
&V(1) \le 15 + \alpha P_{14}(R)V(4)
\end{aligned}$$

reorganize it:

$$\mathbf{A} = \begin{bmatrix}
1 - \alpha P_{44}(D) & -\alpha P_{43}(D) & -\alpha P_{42}(D) & -\alpha P_{41}(D) \\
1 - \alpha P_{44}(W) & -\alpha P_{43}(W) & -\alpha P_{42}(W) & -\alpha P_{41}(W) \\
0 & 1 - \alpha P_{33}(D) & -\alpha P_{32}(D) & -\alpha P_{31}(D) \\
0 & 1 - \alpha P_{33}(W) & -\alpha P_{32}(W) & -\alpha P_{31}(W) \\
0 & 0 & 1 - \alpha P_{22}(D) & -\alpha P_{21}(D) \\
0 & 0 & 1 - \alpha P_{22}(W) & -\alpha P_{21}(W) \\
-\alpha P_{34}(R) & 1 & 0 & 0 \\
-\alpha P_{24}(R) & 0 & 1 & 0 \\
-\alpha P_{14}(R) & 0 & 0 & 1
\end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 4 & 6 & 4 & 6 & 4 & 6 & 15 & 15 & 15 \end{bmatrix}^T$$

Problem 3 [25 points]

Consider the transition graph shown in Figure 1.

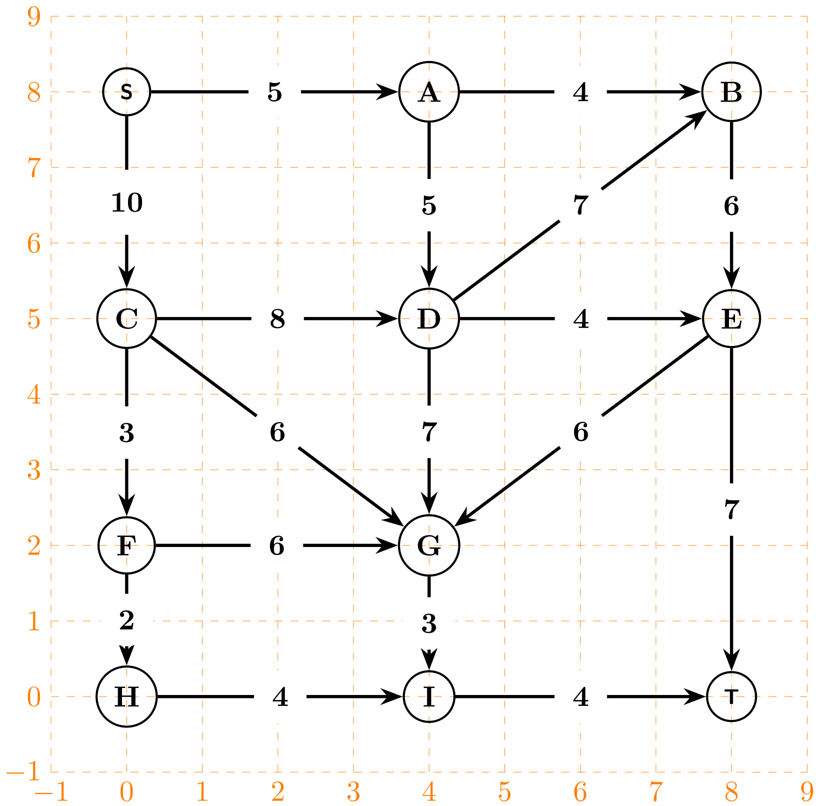

Figure 1: Rows are on the ordinate, columns are on the abscissa

a) $\mathcal{V}$ is the vertex space of the graph and $|\mathcal{V}|$ is the number of elements in the space. How many elements are in the space? What is the time horizon N of the equivalent deterministic finite state system? *[2 points]*

b) Calculate the shortest path from node S to node T and the corresponding optimal cost using the Label Correcting Algorithm. Use the depth-first (last-in/first-out) method to determine at each iteration which node exits the OPEN bin. Solve the problem by populating table 1 with all iterations and find the optimal path. *[9 points]*

Instructions: If a node that is already in the OPEN bin enters the OPEN bin again, remove the one that has already been in the OPEN bin. If two nodes enter the OPEN bin in the same iteration, add the nodes in the inverted alphabet sequence, i.e. add the one that comes first in the alphabet last. Example: OPEN bin: C, B, F; Node exiting OPEN: C (nodes entering OPEN: E, F); new OPEN bin: E, F, B; Node exiting OPEN: E.

Iteration	Remove	OPEN	d_s	d_A	d_B	d_C	d_D	d_E	d_F	d_G	d_H	d_I	d_t
0	-	s	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	s												

Table 1: Label Correcting Algorithm Table for question b) (Depth-first search)

Optimal Path and Distance: ....

c) Can this problem also be solved using the backward Dynamic Programming Algorithm? Give a short explanation. *[2 points]*

d) Can this problem also be solved using the forward Dynamic Programming Algorithm? Give a short explanation. *[2 points]*

e) Assume all nodes are on a grid as indicated in Figure 1. The problem is again to find the shortest path from node S to node T. The distance $d_{i\mathsf{T}}$ between a node i and the terminal node T satisfies the following equation:

$$d_{i\mathsf{T}} \geq |r_i - 0| + |c_i - 8|,$$

where r_i is the number of the row of node i and c_i is the number of the column of node i. Use this information as a lower bound to strengthen the condition on whether a node enters the OPEN bin of the algorithm that was given in b) (this is known as the A^* algorithm). Solve the problem by populating a similar table and find the optimal path. *[10 points]*

Iteration	Remove	OPEN	d_s	d_A	d_B	d_C	d_D	d_E	d_F	d_G	d_H	d_I	d_t
0	-	s	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	s												

Table 2: Label Correcting Algorithm Table for question e) (A^*)

Optimal Path and Distance: ....

Solution 3

a) Cardinality is 11, time horizon is 10

b) The completed table is the following:

Iteration	Remove	OPEN	d_s	d_A	d_B	d_C	d_D	d_E	d_F	d_G	d_H	d_I	d_t
0	-	s	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	s	A,C	0	5	∞	10	∞	∞	∞	∞	∞	∞	∞
2	A	B,D,C	0	5	9	10	10	∞	∞	∞	∞	∞	∞
3	B	E,D,C	0	5	9	10	10	15	∞	∞	∞	∞	∞
4	E	G,D,C	0	5	9	10	10	15	∞	21	∞	∞	22
5	G	D,C	0	5	9	10	10	15	∞	21	∞	∞	22
6	D	E,G,C	0	5	9	10	10	14	∞	17	∞	∞	22
7	E	G,C	0	5	9	10	10	14	∞	17	∞	∞	21
8	G	I,C	0	5	9	10	10	14	∞	17	∞	20	21
9	I	C	0	5	9	10	10	14	∞	17	∞	20	21
10	C	F,G	0	5	9	10	10	14	13	16	∞	20	21
11	F	H,G	0	5	9	10	10	14	13	16	15	20	21
12	H	I,G	0	5	9	10	10	14	13	16	15	19	21
13	I	G	0	5	9	10	10	14	13	16	15	19	21
14	G	-	0	5	9	10	10	14	13	16	15	19	21

Table 3: Label Correcting Algorithm Table for question b). Dept-first search

Best Path: s - A - D - E - T.
Distance: 21.

c) Yes, a shortest path problem can be converted to a deterministic Dynamic Programming problem.

d) Yes, since the optimal control problem is deterministic.

e) The completed table is the following:

Iteration	Remove	OPEN	d_s	d_A	d_B	d_C	d_D	d_E	d_F	d_G	d_H	d_I	d_t
0	-	s	0	∞	∞	∞	∞	∞	∞	∞	∞	∞	∞
1	s	A,C	0	5	∞	10	∞	∞	∞	∞	∞	∞	∞
2	A	B,D,C	0	5	9	10	10	∞	∞	∞	∞	∞	∞
3	B	E,D,C	0	5	9	10	10	15	∞	∞	∞	∞	∞
4	E	D,C	0	5	9	10	10	15	∞	∞	∞	∞	22
5	D	E,C	0	5	9	10	10	14	∞	∞	∞	∞	22
6	E	C	0	5	9	10	10	14	∞	∞	∞	∞	21
7	C	-	0	5	9	10	10	14	∞	∞	∞	∞	21

Table 4: Label Correcting Algorithm Table for question e). Dept-first search with A^*

Adding G at iteration 4 in the open bin can be accepted as well.
Best Path: s - A - D - E - T.
Distance: 21.

Problem 4 [25 points]

A go-kart with unit mass is driving in a straight track with velocity $v(t)$. A force $u(t)$ can be applied to either accelerate or brake the go-kart, and a linear drag term acts on the vehicle ($b > 0$ is the drag coefficient), such that the vehicle dynamics are:

$$\dot{v}(t) = u(t) - bv(t),$$

At time 0, the go-kart is at rest, i.e., $v(0) = 0$.

a) Suppose $u(t) \in [-0.5, 0.5]$ for all t; the goal of the go-kart is to reach a final velocity of $v(T) = 2$. For which b values is the control problem feasible for any $T > 0$? *[4 points]*

b) Suppose $u(t) \in R$ for all t. We want to minimise the following cost:

$$\int_0^T u(t)^2 dt$$

where $T = ln(2)$. Solve for the input $u(t)$ and state $v(t)$ that minimize this cost using Pontryagin's Minimum Principle. Compute $u^*(t)$ and $v(t)$ as a function of b. *[12 points]*

c) Suppose $u(t) \in [-\bar{u}, \bar{u}]$, $\bar{u} = \dfrac{1}{1 - e^{-5b}}$ for all t. Solve the time-optimal control problem, that is, solve for the end time T, input $u(t)$ and state $v(t)$ that minimize the cost

$$\int_0^T 1dt$$

using Pontryagin's Minimum Principle. For which values of b and T is the final speed achieved? *[9 points]*

Solution 4

a) The idea is to solve for the state trajectory if we apply the maximum force; the dynamics become:

$$\dot{V} = 0.5 - bV$$

by applying Laplace transform, and considering $V(0) = 0$, we get:

$$\begin{aligned} sV - v(0) &= \frac{0.5}{s} - bV \\ \Rightarrow V &= \frac{0.5}{s(s+b)} \\ \Rightarrow V &= \frac{A}{s} + \frac{B}{s+b} \\ \Rightarrow v(t) &= A + Be^{-bt} \end{aligned}$$

By taking the partial fraction, we get $A = \frac{0.5}{b}$ and $B = -\frac{0.5}{b}$

$$v(t) = \frac{0.5}{b} - \frac{0.5}{b}e^{-bt}$$

Taking the supremum

$$\sup_{t\geq 0} v(t) = \frac{0.5}{b}$$

$\frac{0.5}{b} > 2$ hence b must be less than 0.25 to have a feasible problem, i.e. $0 < b < 0.25$.

b) The Hamiltonian is

$$H(v, u, p) = u^2 + p(u - bv)$$

The Hamiltonian is quadratic in u with Hessian > 0. Therefore, tabe the gradient and set to zero to get the optimal u^*:

$$u^* = -p/2$$

The co-state derivative is defined by

$$\dot{p}(t) = -\frac{\partial H(v, u, p)}{\partial v},$$

hence, by substituting $v(t)$, $p(t)$, $u(t)$, we have $\dot{p}(t) = +bp(t)$.
From the co-state we get: $p(t) = c_1e^{bt}$ with $c_1 =$ constant. Then, Substituting u^* into the system dynamics, solve the BVP:

$$\begin{aligned} \dot{v}(t) &= -c_1e^{bt}/2 - bv(t) \\ \longrightarrow \dot{v}(t) + bv(t) &= -c_1e^{bt}/2 \end{aligned}$$

This is a linear ODE. One can solve this using Laplace, homogeneous and particular solutions, or using integrating factors. Lets solve using homogeneous and particular solutions. Homogeneous:

$$\begin{aligned} \dot{v}(t) + bv(t) &= 0 \\ \longrightarrow v_H(t) &= c_2e^{-bt}. \end{aligned}$$

with c_2 constant. Particular, with A as a constant:

$$\begin{aligned}
v_P(t) &= Ac_1e^{bt} \\
\longrightarrow \dot{v}_P(t) + bv_P(t) &= -c_1e^{bt}/2. \\
\longrightarrow bAc_1e^{bt} + bAc_1e^{bt} &= -c_1e^{bt}/2. \\
\longrightarrow bA + bA &= -1/2. \\
\longrightarrow A &= -\frac{1}{4b}.
\end{aligned}$$

Thus,

$$v(t) = v_H(t) + v_P(t) = c_2e^{-bt} + \frac{-c_1e^{bt}}{4b}$$

Now use the boundary constraints to resolve the constants c_1, c_2:

$$\begin{aligned}
v(0) = c_2 - \frac{c_1}{4b} &= 0 \longrightarrow c_2 = \frac{c_1}{4b} \\
v(T = ln2) = c_2 2^{-b} + \frac{-c_1 2^b}{4b} &= 2. \\
v(T = ln2) = \frac{c_1 2^{-b}}{4b} - \frac{c_1 2^b}{4b} &= 2. \\
v(T = ln2) = c_1(2^{-b} - 2^b) &= 8b. \\
c_1 &= \frac{8b}{2^{-b} - 2^b}. \\
c_2 &= \frac{2}{2^{-b} - 2^b}.
\end{aligned}$$

c) The Hamiltonian is

$$H(v, u, p) = 1 + p(u - bv)$$

The co-state derivative is defined by

$$\dot{p}(t) = -\frac{\partial H(v, u, p)}{\partial v},$$

hence, by substituting $v(t)$, $p(t)$, $u(t)$, we have $\dot{p}(t) = +bp(t)$.
From the co-state we get: $p(t) = c_3e^{bt}$ with $c_3 =$ constant.

$$u^* = \underset{u(t)\in[-\bar{u},\bar{u}]}{\arg\min}\ H(v, u, p) = \begin{cases} \bar{u} & p < 0 \\ -\bar{u} & p > 0 \\ ? & p = 0 \end{cases}$$

The third case is a singular arc. If $p(t) = 0$ for a non-trivial amount of time, then $c_3 = 0$. If this is the case, then $H(v, u, p) = 1 \neq 0$, which is a contradiction, since T is variable $\longrightarrow H = 0$.
The second case cannot occur, since $p(t)$ does not switch sign, if $c_3 > 0$, then $-\bar{u}$ would always be applied forever and we would only have a negative velocity, violating the terminal

state constraint. Thus $c_3 < 0 \implies p(t) < 0 \quad \forall t \geq 0$. Substituting $u^* = \bar{u}$ into the system dynamics, solve the BVP:

$$
\begin{aligned}
\dot{v}(t) &= \bar{u} - bv(t) \\
\longrightarrow v^*(t) &= \frac{\bar{u}}{b}(1 - e^{-bt}) \\
\longrightarrow v^*(t) &= \frac{1 - e^{-bt}}{b(1 - e^{-5b})} \\
v^*(T) &= \frac{1 - e^{-bT}}{b(1 - e^{-5b})} = 2 \\
b &= 1/2, \\
T &= 5
\end{aligned}
$$