Introduction to Computational Physics

Lecture Fractals

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https://moodle-app2.let.ethz.ch/course/view.php?id=18025

Plan for today

1. Fractals
   - Fractals & Self-Similarity
     - Fractal Dimension: Mathematical Definition
     - The Sandbox Method
     - The Box Counting Method
   - Fractals & Percolation
     - Fractal Dimension in Percolation

5 Fractals & Self-Similarity I

- fractal dimension is a concept introduced in the field of fractal geometry
- find a measure to describe how well a given *fractal object* fills a certain space
- a related and simpler concept is self-similarity

In a nut-shell, one could define an object to be 'self-similar' if it is built up of smaller copies of itself. Such objects occur both in mathematics and in nature.

Figure: The Sierpinski triangle - a self-similar mathematical object, which is created iteratively

5 Fractals & Self-Similarity II

The Sierpinski-triangle (Polish mathematician Wacław Franciszek Sierpiński) is a mathematical object constructed by an iterative application of a simple algotithm:

1. start with an equilateral triangle
2. divided the triangle into 4 sub-triangles & delet centre triangle
3. repeat step 2 each of the remaining smaller triangles infinitely

This obviously produces an object that is built up of elements that are almost the same as the complete object, which can be referred to as **approximate self-similarity**.

- it is only in the limit of infinite iterations that the object becomes exactly self-similar: In this case the building blocks of the object are exactly the scaled object
- $\infty-$iteration $\Rightarrow$ fractal

Self-Similarity in Nature I

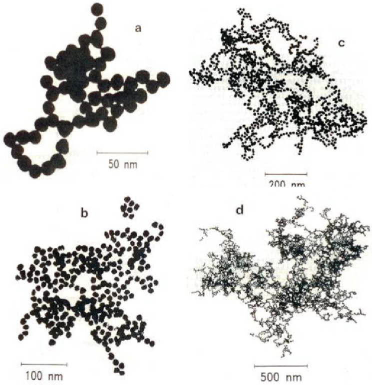


Figure: Barnsley fern & Gold Colloids at different scales

Self-Similarity in Nature II

- Naturally occurring self-similar objects are usually only approximately self-similar. As an illustration of this point, consider a tree; a tree has different branches, and the whole tree looks similar to a branch connected to the tree trunk. The branch itself resembles a smaller branch attached to it and so on. Evidently, this breaks down after a few iterations, when the leaves of the tree are reached.
- A rather beautiful example of a fractal is the fern which can be modelled using affine transformations (see Barnsley fern). The algorithm recursively produces a fern that resembles the natural fern very closely and illustrates self-similarity.
- Another example are gold colloids, which were shown to arrange in fractals of fractal dimension 1.70 by David Weitz in 1984. Colloidal gold is a suspension of sub-micrometer-sized gold particles in a fluid (for example water). These gold colloids arrange in fractals.

5.1.1 Fractal Dimension: Mathematical Definition I

- consider all coverings of the object with spheres of radius $r_i \leq \varepsilon$, where ε is an arbitrary infinitesimal and d denotes the dimension
- let $N_\varepsilon(c)$ be the number of spheres used in the covering c

Then, the volume $V_\varepsilon(c) \in \mathbb{R}^d$ of the covering is

$$V_\varepsilon(c) = \sum_{i=1}^{N_\varepsilon(c)} r_i^d$$

We define V_ε^* as the volume of the covering that uses as few spheres as possible and has minimal volume

$$V_\varepsilon^* = \min_{V_\varepsilon(c)} \left(\min_{N_\varepsilon(c)} \left(V_\varepsilon(c) \right) \right)$$

The fractal dimension of the object with lenght L can then be defined as

$$d_f := \lim_{\varepsilon \to 0} \frac{\log\left(V_\varepsilon^* / \varepsilon^d\right)}{\log\left(L/\varepsilon\right)} \tag{1}$$

Interpretation of 'Fractal Dimensions'

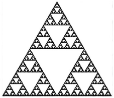
The definition of the fractal dimension (1) can be interpreted in the following way:

- when the length of the object is stretched by a factor of a,
- its volume (or 'mass') grows by a factor of a^{d_f}.
- we obtain this interpretation by rewriting equation 1 with $\varepsilon \to 0$:

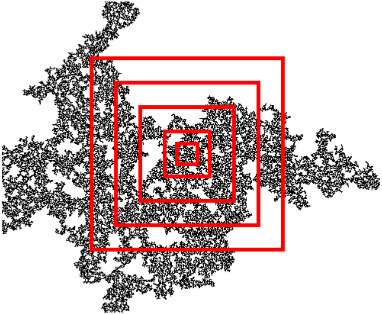
$$\frac{V_\varepsilon^*}{\varepsilon^d} = \left(\frac{L}{\varepsilon}\right)^{d_f}$$

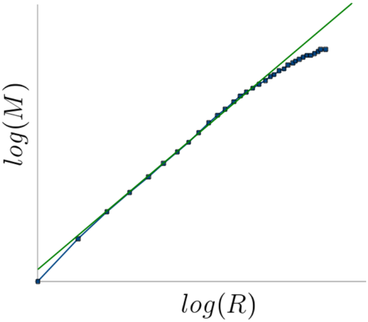
Example Sierpinski triangle:

- stretching its sides $\times 2 \Rightarrow$ increases its volume by a factor of 3
- with equation (1) $\Rightarrow$ the Sierpinski triangle has the fractal dimension $\log(3)/\log(2) \approx 1.585$.



5.1.2 The Sandbox Method I



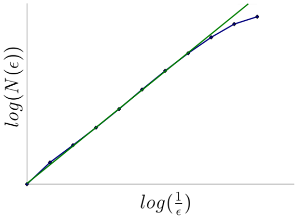


- small box of size R in the center of the picture and count the number of occupied sites (or pixels) in the box $N(R)$
- increase the box size R in small steps until we cover the whole picture
- plot $N(R)$ vs. R in a log-log plot : the fractal dimension is the slope

5.1.3 The Box Counting Method I

The box counting method is a method of numerically determining the fractal dimension of an object. It is conceptually easy, since it is close to the mathematical definition.

- we start with a picture of the fractal
- superimpose a lattice with lattice constant ε
- count the number of boxes in the lattice that are not empty (contain a part of the fractal) as $N(\varepsilon)$
- repeat this for a large range of ε and plot $N(\varepsilon)$ vs. ε in a log-log plot.

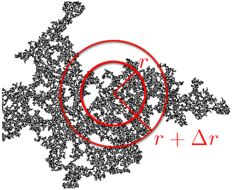


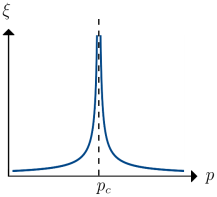
5.1.3 The Box Counting Method II

- region where the slope is constant in the plot; it is in this region that the slope equals the fractal dimension of the object
- the object is only self-similar in this region
- outside this self-similar regime the finite resolution and finite size of the picture disturb the self-similarity

As an illustration of this, it is useful to remember the previously mentioned interpretation of the definition of the fractal dimension; recatll that ε is proportional to the length scale, while $N(\varepsilon)$ is proportional to the volume of the fractal object.

5.2 Fractals & Percolation I





- percolating cluster at p_c being a fractal lies again at the heart of the theory of phase transitions.
- in the infinite system we saw that we can clearly distinguish the disordered phase (non-percolating) ($p < p_c$) from the ordered phase (percolating) ($p > p_c$).
- this is not exactly possible at $p = p_c$

5.2 Fractals & Percolation II

The correlation function $c(r)$ can be written as

$$c(r) = \frac{\Gamma(\frac{d}{2})}{2\pi^{d/2} r^{d-1} \Delta r} \left[M(r + \Delta r) - M(r)\right]$$

- $c(r)$ essentially counts the number of filled sites within a band of size Δr at a distance r from the center and normalizes this expression with the surface area of the sphere in d dimensions at radius r.
- When we compute $c(r)$ for a given cluster, we find that typically the correlation function decreases exponentially with the radius r (neglecting an offset C) where the constant C vanishes in the sub-critical regime (i.e. $p < p_c$).

$$c(r) \propto C + exp\left(-\frac{r}{\xi}\right)$$

5.2 Fractals & Percolation III

- The newly introduced quantity ξ **is called the correlation length**. It describes the typical length scale over which the correlation function of a given system decays. In the sub-critical regime, the correlation length ξ is proportional to the radius of a typical cluster.

$\rightarrow \xi$ is singular at the critical occupation probability p_c,

$$\xi \propto |p - p_c|^{-\nu} \quad \text{where} \quad \nu = \begin{cases} \frac{4}{3} & \text{in 2 dimensions;} \\ 0.88 & \text{in 3 dimensions.} \end{cases}$$

Apparently the assumption of exponential behavior is no longer valid at p_c. Measurements give a correlation function at p_c that behaves like

$$c(r) \propto r^{-(d-2+\eta)} \quad \text{where} \quad \begin{cases} \eta = \frac{5}{24} & \text{in 2 dimensions;} \\ \eta \approx -0.05 & \text{in 3 dimensions.} \end{cases}$$

We see that at p_c, the correlation functions decays like a power law with an exponent that contains a new parameter η which takes different values depending on the dimension.

5.2.1 Fractal Dimension in Percolation I

The fraction of sites P in the spanning cluster ("order parameter") is

$$P \propto (p - p_c)^{\beta}$$

At p_c, not only the order parameter is system size dependent (Eq. 4.11),

$$P \propto L^{-\frac{\beta}{\nu}}$$

but that the number of sites is also system size dependent

$$M \propto L^{d_f}$$

When we combine all of this, we obtain

$$M \propto P L^{d} \propto L^{\left(-\frac{\beta}{\nu}+d\right)} \propto L^{d_f}$$

and we have thus found the fractal dimension

$$d_f = d - \frac{\beta}{\nu}.$$