ETH zürich

Department of Mathematics
Computational Science Engineering

Semester Project Kick off meeting

Mingyuan Chi

19-09

Outline

Outline

1. Introduction
2. Background
3. Methodology
4. Experiment
5. Future work

Finite Element Analysis

Strong Form : $\nabla \cdot \sigma(\varepsilon) + f = \rho \frac{\partial^2 u}{\partial t^2}$

Weak form : $\int_\Omega \sigma : \nabla v + b \cdot v \mathrm{d}\Omega - \int_{\partial\Omega} (\sigma \cdot n) \cdot v \mathrm{d}S = \rho \int_\Omega \frac{\partial^2 u}{\partial t^2} \cdot v \mathrm{d}\Omega$

Galerkin discretilization : $K_{ij} = \int_\Omega \sigma(\sum u_i \phi_i) : \nabla \psi_j + b \cdot \psi_j \mathrm{d}\Omega \quad \phi_i = N_i, \psi_j = N_j$

- σ stress, it's a function of strain $\sigma(\varepsilon)$
  Lame's approximation:
  $\sigma = \lambda \mathrm{Tr}(\varepsilon) I + 2\mu\varepsilon \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \mu = \frac{E}{1+2\nu}$
- ε strain, $\varepsilon = \nabla u$
- : double contraction, $\sigma : \nabla v = \nabla v^\top \cdot \sigma \cdot \nabla v \quad \sigma \in \mathbb{R}^{d \times d}, \nabla v \in \mathbb{R}^d$
- bilinear form : $a(u, v) = \int_\Omega \sigma(\varepsilon) : \nabla v + b \cdot v \mathrm{d}\Omega$ and linear form : $\ell(v) = \int_{\partial\Omega} (\sigma \cdot n) \cdot \mathrm{d}S$

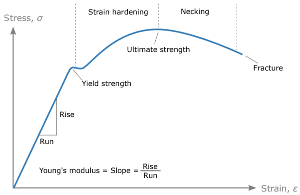


Figure: stress-strain relation

Linear Case

When **Dirichlet Boundary Condition** applied to mesh $\mathcal{M}$, $u_B, K, f \rightarrow u_I$

$$\begin{bmatrix} K_{II} & K_{IB} \\ K_{IB}^\top & K_{BB} \end{bmatrix} \begin{bmatrix} u_I \\ u_B \end{bmatrix} = \begin{bmatrix} f_I \\ f_B \end{bmatrix} \rightarrow K_{II}x_I = f_I - K_{IB}x_B$$

- $A_{\{II,BB,IB\}}$: Galerkin matrix for interior/boundary/interior-boundary connection
  - element matrix : $K_e = \int_\Omega B_e^\top D_e B_e \mathrm{d}\Omega$
  - B : Strain-Displacement Matrix, $\varepsilon = Bu = \frac{1}{2}(\nabla u + \nabla u^\top)$
    - Triangle Element : $B = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \frac{\partial N_3}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & 0 & \frac{\partial N_3}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial y} & \frac{\partial N_3}{\partial x} \end{bmatrix} \quad \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ 2\varepsilon_{xy} \end{bmatrix}$
  - D : Material's Constitutive (Elasticity) Matrix, $\sigma = D\varepsilon = DBu$
    - Triangle Element : $D = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$
- $u_{\{I,B\}}$: displacement for interior/boundary nodes
- $f_{\{I,B\}}$: applied force for interior/ boundary nodes

GNN view

Traditional Case:

$$Ku = f \Leftrightarrow u = \text{GNN}(A, f) \quad A_{ij} = K_{ij} > 0$$

considering **Dirichlet Boundary Condition**

$$K_{II}u_I = f_I - K_{IB}u_B \Leftrightarrow u_I = \text{GNN}(A_{II}, f_I - A_{IB}u_B)$$

Neumann Boundary Condition and Radiation Boundary can be coupled in the PDE form, which is simulated by the GNN

Chanllege and Purpose

1. Theoretical proof of boundary condition GNN for FEM
2. End2end GNN framework applied to Dirichlet Boundary Condition
3. Better precision
4. Faster speed, larger scale
5. Non-Lame's approximation

Outline

Graph Neural Network

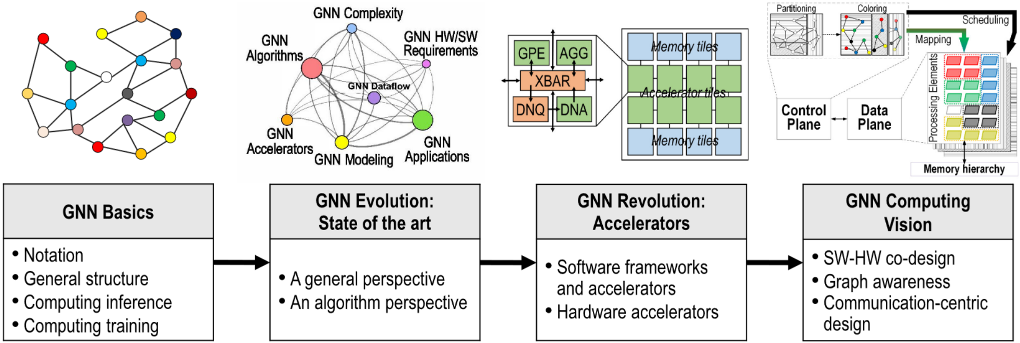


Figure: Graph Neural Network Survey: Algorithms, Application, Modeling, Accelerators Abadal et al. [2021]

GNN for FEM DBC

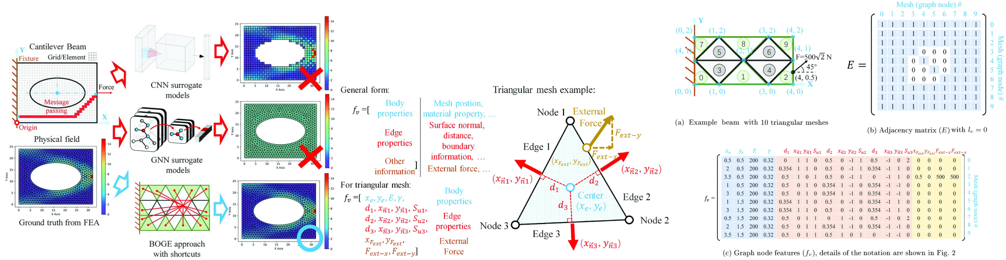


Figure: Fully Connected Solution: each $(v_i, v_j) \in \hat{\mathcal{E}} \quad \forall v_i \in \mathcal{M}_I \quad \forall v_j \in \mathcal{M}_B$, interior nodes and boundary nodes fully connected, element as node in graph Fu et al. [2023]

GNN for FEM DBC

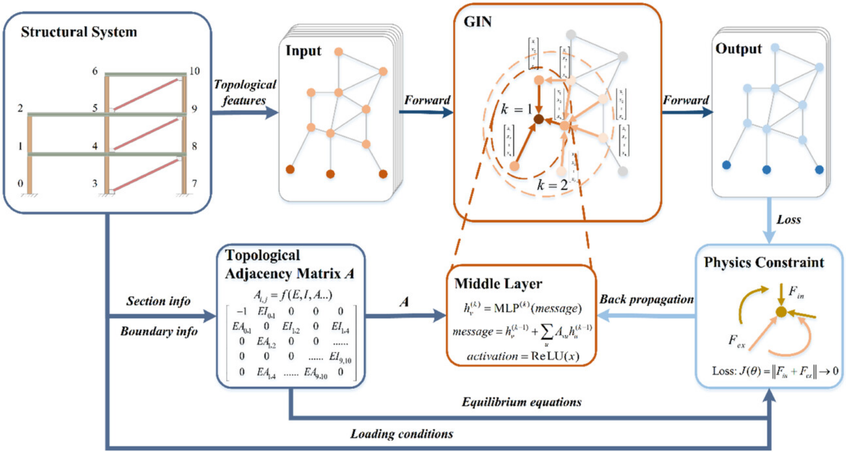
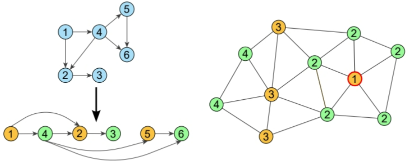


Figure: Physics Informed: $\|f_I + f_B\| = 0$, node as node in graph Song et al. [2023]

Graph Pooling

Survey Liu et al. [2022]



(a) Bi-partition of a DAG (b) Bi-stride of a mesh

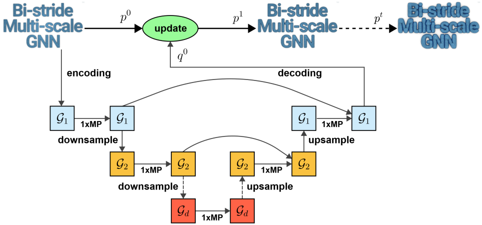


Figure: 2-Strided Pooling Cao et al. [2023]

Outline

Methodology

- How GNN work
- How FEM different boundary condition be tackled
- How our method work

Outline

Experiment

- Strain-Stress precision and baseline comparison
- Case study on airfoil example
- Parameter Sensitivity
- Speed/Memory Comparation
- Topology Optimization? $z = \underset{z}{\text{argmin}} \sum_{e=1}^{N} (z_e)^p u_e^\top A_e u_e \quad z_e > 0$

Outline

- FEM for Structure Engineering Benchmark / Leaderboard?
- FEM for torch

ETH zürich

Sergi Abadal, Akshay Jain, Robert Guirado, Jorge López-Alonso, and Eduard Alarcón. Computing graph neural networks: A survey from algorithms to accelerators. *ACM Comput. Surv.*, 54(9), oct 2021. ISSN 0360-0300. doi: 10.1145/3477141. URL https://doi.org/10.1145/3477141.

Xingyu Fu, Fengfeng Zhou, Dheeraj Peddireddy, Zhengyang Kang, Martin Byung-Guk Jun, and Vaneet Aggarwal. An finite element analysis surrogate model with boundary oriented graph embedding approach for rapid design. *Journal of Computational Design and Engineering*, 10(3):1026–1046, 03 2023. ISSN 2288-5048. doi: 10.1093/jcde/qwad025. URL https://doi.org/10.1093/jcde/qwad025.

Ling-Han Song, Chen Wang, Jian-Sheng Fan, and Hong-Ming Lu. Elastic structural analysis based on graph neural network without labeled data. *Computer-Aided Civil and Infrastructure Engineering*, 38(10):1307–1323, 2023. doi: https://doi.org/10.1111/mice.12944. URL https://onlinelibrary.wiley.com/doi/abs/10.1111/mice.12944.

Chuang Liu, Yibing Zhan, Jia Wu, Chang Li, Bo Du, Wenbin Hu, Tongliang Liu, and Dacheng Tao. Graph pooling for graph neural networks: Progress, challenges, and opportunities. *arXiv preprint arXiv:2204.07321*, 2022.

ETH zürich

The current project description is available at walkerchi.github.io/ETHz-SP/kickoff

D MATH

Mingyuan Chi
minchi@student.ethz.ch