torch-sla: Differentiable Sparse Linear Algebra with Adjoint Solvers and Sparse Tensor Parallelism for PyTorch

Mingyuan Chi

walker.chi.000@gmail.com
https://github.com/walkerchi/torch-sla

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Abstract

We present `torch-sla`, an open-source PyTorch library for differentiable sparse linear algebra. The library addresses two fundamental challenges: (1) **Adjoint-based differentiation** for linear and nonlinear sparse solvers, achieving $\mathcal{O}(1)$ memory complexity for computational graphs regardless of solver iterations; and (2) **Sparse Tensor Parallel** computing via domain decomposition with halo exchange, enabling distributed sparse operations across multiple GPUs. `torch-sla` supports multiple backends (SciPy, cuDSS, PyTorch-native) and scales to 169 million degrees of freedom. Benchmarks demonstrate $12\times$ speedup on multi-GPU configurations and correct gradient computation verified against finite differences.

1 Introduction

Sparse linear systems $\mathbf{A}\mathbf{x} = \mathbf{b}$ are fundamental to scientific computing. They arise in finite element analysis [Hughes, 2012], graph neural networks [Kipf and Welling, 2017], physics-informed machine learning [Raissi et al., 2019], and computational fluid dynamics [Jasak et al., 2007]. With the rise of differentiable programming, there is growing demand for sparse solvers that integrate with automatic differentiation frameworks like PyTorch [Paszke et al., 2019].

The challenge of differentiating through solvers. Consider a loss function $\mathcal{L}(\mathbf{x})$ where $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$ is obtained by solving a linear system. To train neural networks that interact with such solvers, we need gradients $\partial\mathcal{L}/\partial\mathbf{A}$ and $\partial\mathcal{L}/\partial\mathbf{b}$. A naive approach—differentiating through each iteration of an iterative solver—creates $\mathcal{O}(k)$ nodes in the computational graph, where k may be thousands of iterations. This leads to memory explosion and slow backward passes.

The challenge of scale. Industrial problems often exceed single-GPU memory. A 3D finite element mesh with 10 million nodes produces a sparse matrix requiring tens of gigabytes. Distributed computing is essential, but sparse matrix operations require careful communication patterns (halo exchange) that are non-trivial to implement correctly with gradient support.

This paper introduces `torch-sla`, addressing both challenges with two key innovations:

- **Adjoint-based differentiation** (§3.2): We implement the adjoint method for linear solves, eigenvalue problems, and nonlinear systems. This achieves $\mathcal{O}(1)$ computational graph nodes and $\mathcal{O}(\text{nnz})$ memory, independent of solver iterations.
- **Sparse tensor parallelism** (§3.3): We implement domain decomposition with automatic halo exchange following industrial CFD/FEM practices. This enables multi-GPU computing with gradient support.

2 Related Work

Differentiable linear algebra. JAX [Bradbury et al., 2018] provides differentiable sparse CG, but naive differentiation through iterations creates $\mathcal{O}(k)$ graph nodes. Blondel et al. [2022] formalize implicit differentiation for optimization layers. OptNet [Amos and Kolter, 2017] and cvxpylayers [Agrawal et al., 2019] handle convex optimization but not general sparse systems.

Sparse solvers. SciPy [Virtanen et al., 2020] provides SuperLU/UMFPACK for CPU. NVIDIA cuDSS [NVIDIA Corporation, 2024] provides GPU direct solvers. None natively support PyTorch autograd.

Distributed sparse computing. PETSc [Balay et al., 2019], Trilinos [Heroux et al., 2005], and hypre [Falgout and Yang, 2002] implement distributed sparse linear algebra. OpenFOAM [Jasak et al., 2007] uses domain decomposition for CFD. We bring these patterns to PyTorch with automatic differentiation.

3 Methodology

We first introduce the background on sparse linear systems (§3.1), then present our adjoint-based differentiation approach (§3.2), and finally describe our distributed computing implementation (§3.3).

3.1 Preliminaries: Sparse Linear Systems

A sparse linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ with nnz $\ll n^2$ non-zero entries. We store matrices in coordinate (COO) format: three arrays for row indices, column indices, and values.

Direct solvers (LU, Cholesky factorization) compute exact solutions but require $\mathcal{O}(n^{1.5})$ memory for 2D problems due to fill-in—new non-zeros created during factorization [George, 1973]. For a 2D Poisson problem with n unknowns, the factored matrix has $\mathcal{O}(n \log n)$ non-zeros instead of the original $\mathcal{O}(n)$.

Iterative solvers (Conjugate Gradient, BiCGStab, GMRES) maintain $\mathcal{O}(\text{nnz})$ memory by never forming the factorization. However, they require $\mathcal{O}(\sqrt{\kappa})$ iterations for CG where κ is the condition number [Saad, 2003]. Each iteration performs one sparse matrix-vector multiplication (SpMV) costing $\mathcal{O}(\text{nnz})$.

The differentiation problem. Given loss $\mathcal{L}(\mathbf{x})$ where $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$, we need gradients with respect to both $\mathbf{b}$ and the non-zero values of $\mathbf{A}$. If we differentiate through k iterations of CG, the computational graph has $\mathcal{O}(k)$ nodes, each storing intermediate vectors. For $k = 1000$ iterations and $n = 10^6$, this requires $\sim$80 GB just for the graph. Our adjoint approach reduces this to $\mathcal{O}(1)$ nodes.

3.2 Adjoint-Based Differentiation

The adjoint method computes gradients through implicit functions without storing intermediate solver states. We present it for linear systems, eigenvalue problems, and nonlinear systems.

3.2.1 Linear Systems: The Core Idea

Consider $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$. Rather than differentiating through the solver iterations, we use the *implicit function theorem*. The solution satisfies $\mathbf{A}\mathbf{x} - \mathbf{b} = \mathbf{0}$. Differentiating implicitly:

$$d\mathbf{A} \cdot \mathbf{x} + \mathbf{A} \cdot d\mathbf{x} = d\mathbf{b} \implies d\mathbf{x} = \mathbf{A}^{-1}(d\mathbf{b} - d\mathbf{A} \cdot \mathbf{x}) \tag{1}$$

For a scalar loss $\mathcal{L}(\mathbf{x})$, the chain rule gives:

$$d\mathcal{L} = \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)^\top d\mathbf{x} = \left(\frac{\partial \mathcal{L}}{\partial \mathbf{x}}\right)^\top \mathbf{A}^{-1}(d\mathbf{b} - d\mathbf{A} \cdot \mathbf{x}) \tag{2}$$

Define the **adjoint variable** $\boldsymbol{\lambda} = \mathbf{A}^{-\top} \frac{\partial \mathcal{L}}{\partial \mathbf{x}}$, solved via $\mathbf{A}^\top \boldsymbol{\lambda} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}}$. Then the gradients are:

$$\boxed{\frac{\partial \mathcal{L}}{\partial \mathbf{b}} = \boldsymbol{\lambda}, \quad \frac{\partial \mathcal{L}}{\partial \mathbf{A}_{ij}} = -\boldsymbol{\lambda}_i \cdot \mathbf{x}_j} \tag{3}$$

Table 1: Complexity comparison: naive vs. adjoint differentiation through iterative solver with k iterations, n unknowns, and nnz non-zeros.

	Naive (through iterations)	**Adjoint (ours)**
Computational graph nodes	$\mathcal{O}(k)$	$\mathcal{O}(1)$
Memory for graph	$\mathcal{O}(k \cdot n)$	$\mathcal{O}(n + \text{nnz})$
Backward pass time	$\mathcal{O}(k \cdot \text{nnz})$	$\mathcal{O}(T_{\text{solve}} + \text{nnz})$

Complexity analysis. Table 1 compares naive differentiation (through iterations) with our adjoint approach.

The forward solve computes $\mathbf{x}$ in time T_{solve}. The backward pass:

1. Solves $\mathbf{A}^\top \boldsymbol{\lambda} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}}$: $\mathcal{O}(T_{\text{solve}})$ time
2. Computes $\frac{\partial \mathcal{L}}{\partial \mathbf{A}_{ij}} = -\boldsymbol{\lambda}_i \mathbf{x}_j$ for each non-zero: $\mathcal{O}(\text{nnz})$ time

Algorithm. The complete procedure is shown below:

Algorithm 1: Adjoint Linear Solve

Input: Sparse matrix $\mathbf{A}$ (values, rows, cols), RHS $\mathbf{b}$
Output: Solution $\mathbf{x}$, gradients in backward pass

Forward pass:

1. $\mathbf{x} \leftarrow \text{solve}(\mathbf{A}, \mathbf{b})$ *// Any solver: CG, LU, etc.*
2. Store $(\mathbf{A}, \mathbf{x})$ for backward

Backward pass: (given $\partial \mathcal{L}/\partial \mathbf{x}$)

1. $\boldsymbol{\lambda} \leftarrow \text{solve}(\mathbf{A}^\top, \partial \mathcal{L}/\partial \mathbf{x})$ *// One adjoint solve*
2. $\partial \mathcal{L}/\partial \mathbf{b} \leftarrow \boldsymbol{\lambda}$
3. For each non-zero (i, j): $\partial \mathcal{L}/\partial \mathbf{A}_{ij} \leftarrow -\boldsymbol{\lambda}_i \cdot \mathbf{x}_j$ *// $\mathcal{O}(nnz)$ total*

3.2.2 Eigenvalue Problems

For symmetric eigenvalue problem $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$ with normalized eigenvector $\|\mathbf{v}\| = 1$, the gradient of eigenvalue λ with respect to matrix entries is remarkably simple [Magnus, 1985]:

$$\frac{\partial \lambda}{\partial \mathbf{A}_{ij}} = v_i \cdot v_j \tag{4}$$

This is the outer product $\mathbf{v}\mathbf{v}^\top$ restricted to the sparsity pattern. For k eigenvalues, the total cost is $\mathcal{O}(k \cdot \text{nnz})$.

3.2.3 Nonlinear Systems

For nonlinear system $\mathbf{F}(\mathbf{u}, \boldsymbol{\theta}) = \mathbf{0}$ with solution $\mathbf{u}^*(\boldsymbol{\theta})$ and loss $\mathcal{L}(\mathbf{u}^*)$, implicit differentiation gives:

$$\mathbf{J}^\top \boldsymbol{\lambda} = \frac{\partial \mathcal{L}}{\partial \mathbf{u}}, \quad \frac{\partial \mathcal{L}}{\partial \boldsymbol{\theta}} = -\boldsymbol{\lambda}^\top \frac{\partial \mathbf{F}}{\partial \boldsymbol{\theta}} \tag{5}$$

where $\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{u}}$ is the Jacobian at the solution.

Algorithm 2: Newton-Raphson with Adjoint Gradients

Input: Residual function $\mathbf{F}(\mathbf{u}, \boldsymbol{\theta})$, initial guess $\mathbf{u}_0$
Output: Solution $\mathbf{u}^*$

Forward (Newton iteration):

1. $\mathbf{u} \leftarrow \mathbf{u}_0$
2. **while** $\|\mathbf{F}(\mathbf{u}, \boldsymbol{\theta})\| > \epsilon$ **do**:
   (a) $\mathbf{J} \leftarrow \partial \mathbf{F} / \partial \mathbf{u}$ *// Jacobian via autograd*
   (b) $\Delta \mathbf{u} \leftarrow \text{solve}(\mathbf{J}, -\mathbf{F})$
   (c) $\mathbf{u} \leftarrow \mathbf{u} + \alpha \Delta \mathbf{u}$ *// α from line search*
3. Store $(\mathbf{u}^*, \mathbf{J}, \boldsymbol{\theta})$

Backward: (given $\partial \mathcal{L} / \partial \mathbf{u}^*$)

1. $\boldsymbol{\lambda} \leftarrow \text{solve}(\mathbf{J}^\top, \partial \mathcal{L} / \partial \mathbf{u}^*)$ *// One adjoint solve*
2. $\partial \mathcal{L} / \partial \boldsymbol{\theta} \leftarrow -\boldsymbol{\lambda}^\top \partial \mathbf{F} / \partial \boldsymbol{\theta}$ *// VJP via autograd*

Memory analysis. Forward: $\mathcal{O}(n + \text{nnz})$ for solution and Jacobian. Backward: $\mathcal{O}(n)$ for adjoint variable. Total: $\mathcal{O}(n + \text{nnz})$, independent of Newton iterations.

3.3 Sparse Tensor Parallel Computing

For problems exceeding single-GPU memory, we implement domain decomposition with halo exchange—the standard approach in industrial CFD/FEM codes like OpenFOAM and Ansys Fluent.

3.3.1 Domain Decomposition

Given a sparse matrix $\mathbf{A}$ corresponding to a mesh or graph, we partition the n nodes into P subdomains. Each process p owns nodes $\mathcal{O}_p$ (owned nodes) and maintains copies of neighboring nodes $\mathcal{H}_p$ (halo/ghost nodes) needed for local computation.

Partitioning strategies:

- **METIS** [Karypis and Kumar, 1998]: Graph partitioning minimizing edge cuts for load balancing
- **RCB**: Recursive Coordinate Bisection using node coordinates
- **Contiguous**: Simple row-based partitioning (fallback)

3.3.2 Halo Exchange

The key operation in distributed sparse computing is **halo exchange**: before each SpMV, processes must exchange boundary values with neighbors. Figure 1 illustrates this concept.

Algorithm 3: Distributed SpMV with Halo Exchange

Input: Local vector $\mathbf{x}_{\text{local}}$, neighbor map, local matrix $\mathbf{A}_{\text{local}}$
Output: Result $\mathbf{y}_{\text{owned}}$

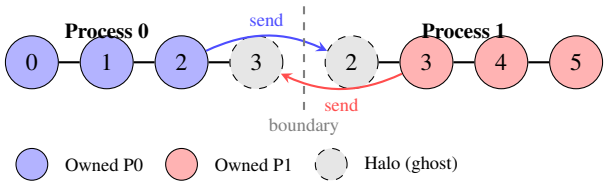

Figure 1: Halo exchange in domain decomposition. Each process owns a subset of nodes (solid colored) and maintains halo copies of boundary neighbors (dashed). Before SpMV, processes exchange updated values at partition boundaries via peer-to-peer communication.

Phase 1: Initiate async communication

1. **for** each neighbor q **do**:
   (a) `async_send`(my boundary values to q)
   (b) `async_recv`(halo values from q)

Phase 2: Wait for completion

1. `synchronize_all`()

Phase 3: Local computation

1. $\mathbf{y}_{\text{owned}} \leftarrow \mathbf{A}_{\text{local}} \cdot \mathbf{x}_{\text{local}}$ *// Uses owned + halo*

3.3.3 Distributed Conjugate Gradient

Building on distributed SpMV, we implement distributed CG. Each iteration requires:

- One halo exchange (in SpMV)
- Two `all_reduce` operations for global dot products

Algorithm 4: Distributed Conjugate Gradient

Input: Distributed $\mathbf{A}$, local RHS $\mathbf{b}_{\text{owned}}$, tolerance ϵ
Output: Solution $\mathbf{x}_{\text{owned}}$

1. $\mathbf{x} \leftarrow \mathbf{0}$, $\mathbf{r} \leftarrow \mathbf{b}_{\text{owned}}$, $\mathbf{p} \leftarrow \mathbf{r}$
2. $\rho \leftarrow$ `all_reduce`$(\mathbf{r}^\top \mathbf{r}, \text{SUM})$ *// Global dot product*
3. **while** $\sqrt{\rho} > \epsilon$ **do**:
   (a) $\mathbf{Ap} \leftarrow \text{DistSpMV}(\mathbf{A}, \mathbf{p})$ *// Algorithm 3*
   (b) $\alpha \leftarrow \rho /$ `all_reduce`$(\mathbf{p}^\top \mathbf{Ap}, \text{SUM})$
   (c) $\mathbf{x} \leftarrow \mathbf{x} + \alpha \mathbf{p}$
   (d) $\mathbf{r} \leftarrow \mathbf{r} - \alpha \mathbf{Ap}$
   (e) $\rho_{\text{new}} \leftarrow$ `all_reduce`$(\mathbf{r}^\top \mathbf{r}, \text{SUM})$
   (f) $\mathbf{p} \leftarrow \mathbf{r} + (\rho_{\text{new}}/\rho)\mathbf{p}$
   (g) $\rho \leftarrow \rho_{\text{new}}$

Table 2: Single-GPU benchmark on 2D Poisson, H200, float64.

DOF	SciPy	cuDSS	PyTorch CG	Memory	Residual
10K	24 ms	128 ms	20 ms	36 MB	10^{-9}
100K	29 ms	630 ms	43 ms	76 MB	10^{-7}
1M	19.4 s	7.3 s	190 ms	474 MB	10^{-7}
2M	52.9 s	15.6 s	418 ms	916 MB	10^{-7}
16M	OOM	OOM	7.3 s	7.1 GB	10^{-6}
169M	OOM	OOM	**224 s**	**74.8 GB**	10^{-6}

Table 3: Distributed solve on 4× H200 GPUs.

DOF	Time	Memory/GPU	Speedup vs. CPU
10K	0.18 s	0.03 GB	2×
100K	0.61 s	0.05 GB	12×
1M	2.82 s	0.27 GB	–
2M	6.02 s	0.50 GB	–

Communication complexity. Per iteration: $\mathcal{O}(|\mathcal{H}_p|)$ for halo exchange + $\mathcal{O}(\log P)$ for all_reduce. Total per solve: $\mathcal{O}(k \cdot (|\mathcal{H}_p| + \log P))$ where k is iterations.

Gradient support. Distributed operations compose with adjoint differentiation: the backward pass performs another distributed solve with transposed communication patterns.

4 Experiments

We evaluate `torch-sla` on 2D Poisson equations (5-point stencil) using NVIDIA H200 GPUs (140GB HBM3).

4.1 Single-GPU Scalability

Table 2 compares solver backends. Key findings:

- **Iterative solvers scale**: PyTorch CG reaches 169M DOF with $\mathcal{O}(n^{1.1})$ time complexity.
- **Direct solvers limited**: cuDSS/SciPy hit OOM at ∼2M DOF due to fill-in.
- **Memory efficient**: 443 bytes/DOF for CG (theoretical minimum: 144 bytes/DOF).

4.2 Multi-GPU Performance

Table 3 shows distributed CG on 4 GPUs with NCCL backend.

With 4 GPUs × 140GB, theoretical maximum is ∼1.3B DOF.

4.3 Gradient Verification

We verify gradient correctness against finite differences. All operations achieve relative error $< 10^{-5}$, confirming correct adjoint implementation.

5 Conclusion

We presented `torch-sla`, a differentiable sparse linear algebra library with two innovations: (1) adjoint-based differentiation achieving $\mathcal{O}(1)$ graph complexity, and (2) distributed sparse computing with halo exchange. The library scales to 169M DOF on single GPU and provides 12× multi-GPU speedup.

5.1 Future Work

- **Learned preconditioners**: Neural networks for adaptive preconditioning.
- **Mixed precision**: FP16/BF16 with FP64 accumulation.
- **Sparse-sparse products**: Efficient SpGEMM with gradients.
- **Integration**: Combining with neural operators [Li et al., 2020] for hybrid solvers.

Availability. MIT license: `https://github.com/walkerchi/torch-sla`

References

Akshay Agrawal, Brandon Amos, Shane Barratt, Stephen Boyd, Steven Diamond, and J Zico Kolter. Differentiable convex optimization layers. In *Advances in Neural Information Processing Systems*, volume 32, 2019.

Brandon Amos and J Zico Kolter. Optnet: Differentiable optimization as a layer in neural networks. In *International Conference on Machine Learning*, pages 136–145, 2017.

Satish Balay, Shrirang Abhyankar, Mark F Adams, et al. Petsc users manual. Technical report, Argonne National Laboratory, 2019.

Mathieu Blondel, Quentin Berthet, Marco Cuturi, et al. Efficient and modular implicit differentiation. *Advances in Neural Information Processing Systems*, 35:5230–5242, 2022.

James Bradbury, Roy Frostig, Peter Hawkins, Matthew James Johnson, Chris Leary, Dougal Maclaurin, George Necula, Adam Paszke, Jake VanderPlas, Skye Wanderman-Milne, and Qiao Zhang. JAX: composable transformations of Python+NumPy programs, 2018. URL `http://github.com/google/jax`.

Robert D Falgout and Ulrike Meier Yang. hypre: A library of high performance preconditioners. In *International Conference on Computational Science*, pages 632–641, 2002.

Alan George. Nested dissection of a regular finite element mesh. *SIAM Journal on Numerical Analysis*, 10(2):345–363, 1973.

Michael A Heroux, Roscoe A Bartlett, et al. An overview of the trilinos project. *ACM Transactions on Mathematical Software*, 31(3):397–423, 2005.

Thomas JR Hughes. *The finite element method: linear static and dynamic finite element analysis*. Courier Corporation, 2012.

Hrvoje Jasak, Aleksandar Jemcov, and Zeljko Tukovic. Openfoam: A c++ library for complex physics simulations. *International Workshop on Coupled Methods in Numerical Dynamics*, 1000:1–20, 2007.

George Karypis and Vipin Kumar. A fast and high quality multilevel scheme for partitioning irregular graphs. *SIAM Journal on Scientific Computing*, 20(1):359–392, 1998.

Thomas N Kipf and Max Welling. Semi-supervised classification with graph convolutional networks. In *International Conference on Learning Representations*, 2017.

Zongyi Li, Nikola Kovachki, Kamyar Azizzadenesheli, et al. Fourier neural operator for parametric partial differential equations. *arXiv preprint arXiv:2010.08895*, 2020.

Jan R Magnus. On differentiating eigenvalues and eigenvectors. *Econometric Theory*, 1(2):179–191, 1985.

NVIDIA Corporation. cuDSS: NVIDIA CUDA direct sparse solver library. `https://developer.nvidia.com/cudss`, 2024.

Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, et al. Pytorch: An imperative style, high-performance deep learning library. *Advances in Neural Information Processing Systems*, 32, 2019.

Maziar Raissi, Paris Perdikaris, and George E Karniadakis. Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *Journal of Computational Physics*, 378:686–707, 2019.

Yousef Saad. Iterative methods for sparse linear systems. *SIAM*, 2003.

Pauli Virtanen, Ralf Gommers, Travis E Oliphant, et al. Scipy 1.0: fundamental algorithms for scientific computing in python. *Nature Methods*, 17(3):261–272, 2020.

A Implementation Details

A.1 Backend Selection

`torch-sla` automatically selects backends based on device and problem size:

- CPU, any size: `scipy` with SuperLU
- CUDA, <2M DOF: `cudss` with Cholesky (if SPD) or LU
- CUDA, >2M DOF: `pytorch` with CG+Jacobi preconditioner

A.2 API Examples

Listing 1: Basic usage with gradient support.

```python
import torch
from torch_sla import SparseTensor

# Create sparse matrix (COO format)
A = SparseTensor(val, row, col, shape)
b = torch.randn(n, requires_grad=True)

# Solve with automatic differentiation
x = A.solve(b)  # Forward: any backend
loss = x.sum()
loss.backward()  # Backward: adjoint method
print(b.grad)    # Gradient w.r.t. RHS
```

Listing 2: Distributed multi-GPU solve.

```python
import torch.distributed as dist
from torch_sla import DSparseMatrix

dist.init_process_group(backend='nccl')
rank = dist.get_rank()

# Each process loads its partition
A = DSparseMatrix.from_global(
    val, row, col, shape,
    num_partitions=4, my_partition=rank
)

# Distributed CG with halo exchange
x = A.solve(b_local, atol=1e-10)
```

B Complexity Proofs

Theorem 1 (Adjoint memory complexity). *The adjoint method for linear solve requires $\mathcal{O}(n + nnz)$ memory, independent of solver iterations.*

Proof. Forward pass stores: solution $\mathbf{x} \in \mathbb{R}^n$, matrix indices ($\mathcal{O}(\text{nnz})$), matrix values ($\mathcal{O}(\text{nnz})$). Backward pass computes adjoint $\boldsymbol{\lambda} \in \mathbb{R}^n$ and gradients $\partial\mathcal{L}/\partial\mathbf{A} \in \mathbb{R}^{\text{nnz}}$. No intermediate solver states are stored. Total: $\mathcal{O}(n + \text{nnz})$. □