Optimal Unconventional Policy in a New Keynesian Preferred Habitat Model*

Rupal Kamdar
Indiana University, Bloomington

Walker Ray
Chicago Federal Reserve, LSE, CEPR

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Abstract

We study the transmission mechanisms of monetary policy in a general equilibrium model featuring heterogeneous households, nominal rigidities, and limits to arbitrage. The interaction of bond market segmentation and household heterogeneity introduces novel rebalancing transmission mechanisms of monetary policy, rationalizing a number of empirical puzzles regarding the reactions of term premia to short-term interest rates. We show that the induced fluctuations in term premia may be suboptimal due to imperfect risk sharing across households. Reducing the volatility of short-rate fluctuations can improve welfare, but necessarily comes at the cost of increased macroeconomic volatility. With balance sheet tools, we derive a separation principle for optimal policy: conventional policy stabilizes aggregate fluctuations, while unconventional policy stabilizes term premia. Balance sheets are necessary but imperfect tools for macroeconomic stabilization when short rates are constrained. Our model provides a justification and implementation details for a central bank "triple mandate" which seeks to stabilize output, inflation, and long-term rates.

Keywords: quantitative easing, market segmentation, optimal policy, term structure

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1 Introduction

Central banks responded aggressively to worsening financial conditions and growing recessionary pressure during the global financial crisis of 2007-8. In addition to steep cuts in policy rates, central banks undertook various unconventional policy actions; the most salient of these were the quantitative easing (QE) programs carried out by the Federal Reserve. With a binding lower bound on the federal funds rate, the purpose of these policies was to reduce long-term rates for households and firms in order to stimulate the economy. The Fed continued to utilize QE programs during the onset of COVID-19, and then began implementing quantitative tightening (QT) in response to growing inflationary pressure starting in 2022.

The purpose of this paper is to study the positive and normative consequences of unconventional monetary policy, and how the design of balance sheet policies interacts with the conduct of more conventional interest rate policies. We develop a tractable general equilibrium model with market segmentation, financial frictions, nominal frictions, and household heterogeneity. We start with a conventional New Keynesian model, where producers face nominal rigidities when setting prices. However, we relax the assumption of frictionless financial markets: households and firms face imperfect bond market access.

We explicitly model bond markets and the determination of the entire term structure; in particular, we embed within our dynamic general equilibrium model a segmented bond market which builds on the seminal paper of Vayanos and Vila (2021). Households and firms have differentiated access to financial assets of different maturities. The positions these agents take in bond markets induce "preferred habitat" market segmentation forces across the term structure. Specialized bond intermediaries re-integrate markets; but when arbitrageur risk-bearing capacity is imperfect, this integration is only partial.

Relative to representative agent models, our setup introduces the possibility of imperfect risk-sharing and consumption dispersion across households. Household consumption and savings decisions now take place across the entire term structure of bond returns. If bond arbitrageurs have perfect risk-bearing capacity, this friction is immaterial in equilibrium, but outside of this special case, borrowing rates differ across differentiated households, inducing differential consumption and saving decisions.

We show that these bond market frictions have important implications for how monetary policy transmits to households and the aggregate economy. First, consider the key mechanisms of conventional monetary policy. Changes in the policy (short-term) interest rate are transmitted to households only via segmented bond markets. In particular, household and firm demand for long-term bonds implies portfolio rebalancing on the part of bond arbitrageurs; the risk exposure of arbitrageurs therefore changes in response to short rate movements. With imperfect risk-bearing capacity, this implies fluctuations in term premia. Thus, a change in the policy rate is not transmitted one-for-one to all borrowing rates, but instead differs across the term structure and therefore across the portfolios of differentiated households.

Next, consider unconventional (QE or QT) policies. Central bank asset purchases and sales directly induce portfolio rebalancing among bond market investors. Once again, when risk-bearing capacity of bond markets is imperfect, such rebalancing implies changes in term premia, even when the policy rate is unchanged. Because households borrow across the term structure, these policies also affect household consumption decisions.

Our general equilibrium model uncovers novel transmission mechanisms of monetary policy. Through market clearing, the bond market positions across heterogeneous households are crucial for understanding movements in term premia. But movements in term premia themselves induce relative changes in bond market positions across the entire distribution of households. Nevertheless, we characterize the dynamics of the model as a function of a small number of "sufficient statistics" of the household asset distribution. Intuitively, term premia are a function of the equilibrium exposure of bond arbitrageurs to the sources of aggregate risk in the economy. Market clearing implies that the entire distribution of bond positions across firms and households is necessary to pin down arbitrageur bond holdings; however, it is only the risk-weighted distribution across the term structure which is relevant for the pricing of risk in equilibrium. Because there are only a small number of aggregate risk factors, we show that the dynamics of the risk-weighted distribution itself is summarized by a small number of factors. We hope that our solution methods have wider applicability for heterogeneous agent models with financial frictions.

The model makes clear predictions for the reaction of term premia to movements in short-term interest rates over different time horizons, and as a function of the conduct of monetary policy. First, consider a recessionary shock which induces the central bank to cut the short-term rate. We show that if firm borrowing demand falls substantially (which occurs under reasonable parameterizations), then the rebalancing mechanism described above implies a large reduction of term premia on impact. However, this induces additional borrowing motives for households exposed to long-duration assets. In particular, slow-moving portfolio rebalancing of households implies that intermediaries become more exposed to long-duration assets. Thus, over longer horizons, the reaction of term premia becomes small or even positive. We further show that when the short rate is constrained and monetary policy is conducted using balance sheet tools, the reaction of term premia is larger.

Now suppose instead that the initial fall in the short rate was due to a monetary policy *shock* (rather than an endogenous reaction to aggregate conditions). Such a shock is modestly expansionary and so (under the same conditions described above) implies modest increases in firm borrowing demand. Thus, while our model predicts large reductions in term premia following declines in short-term rates *unconditionally*, the *conditional* reaction of term premia to monetary policy shocks is smaller, or may even reverse sign. Following the empirical methodology of Hanson et al. (2021), we find strong support for our model predictions in US Treasury markets: (i) unconditionally, increases in short-term rates are associated with large increases in long-dated forward rates over short-horizons; (ii) this reaction is substantially larger during the QE period; (iii) but over longer horizons, increases in short-term rates are associated with small increases or even decreases in long-dated forward rates; (iv) the conditional reaction of long-dated forward rates to high-frequency-identified monetary policy shocks are substantially smaller or negative.

Next we turn to the welfare implications of our model. From the perspective of an "aggregate Euler equation" channel of monetary policy, our model thus implies that conventional and unconventional policies are somewhat substitutable: either policy can be used to target borrowing rates faced by households. In fact, conditional on the dynamics of an appropriately defined "effective rate" composed of the weighted average of bond returns across the term structure, the first-order aggregate dynamics of output and inflation in our model are the same as a model in which a representative household borrows at this effective rate. This implies that if the central bank loss function only depends on the volatility of output and inflation, both conventional and unconventional policies can achieve identical outcomes. In particular, if "divine coincidence" holds, then either policy tool can achieve first-best.

However, we show that such a policy loss function is *suboptimal* from a social welfare perspective. Both policies lead to variation in term premia, and excess fluctuations in term premia implies excess dispersion in borrowing rates. Fluctuations across the term structure imply differentiated consumption and savings decisions across households. This dispersion results in utility losses relative to the first-best because of imperfect risk-sharing. Even when "divine coincidence" holds, we show that neither tool alone can achieve first-best.

We therefore derive the optimal mix of policy rules when the central bank is maximizing social welfare. When short-rate and balance policy tools are unconstrained, we derive an optimal separation result: conventional policy targets the stability of inflation and the output gap, while unconventional policy targets stability of term premia. When divine coincidence holds, this policy achieves first-best.

However, when policy constraints bind, policy must balance tradeoffs. First, if the central bank faces balance sheet constraints, we show that optimal policy implies that the short rate must be less reactive to aggregate shocks in order to minimize financial disruptions. However, this necessarily comes at the cost of increased macroeconomic instability. Second, if the central bank faces constraints on short-rate policy, then QE must be used to offset macroeconomic shocks. However, this policy has welfare consequences: term premia are more volatile than in the first-best, and thus consumption dispersion causes social welfare losses.

Thus, policy constraints imply tradeoffs between macroeconomic stabilization motives and risk-sharing motives. Our findings apply when the central bank pursues simple time-consistent policy rules (where policy tools are functions of the natural state variables only). We also derive optimal policy results when the central bank has full commitment and can choose policy tools freely as a function of current and past realizations of the economy. Such policies are welfare-improving over simple policy rules. For instance, when the central bank can only utilize conventional policy, optimal policy under full commitment implies interest rate changes are smoothed out relative to the optimal time-consistent short rate rule. The ability to commit to further smoothing of interest rate changes implies a reduction in short rate volatility; in equilibrium, term premia are smaller and less volatile. Relative to simple time-consistent policy rules, the entire expected path of short rates can be utilized to keep output gaps small. However, such policies still cannot achieve first-best (unless both short rate and balance sheet policies can be utilized without frictions). Moreover, such optimal policies are complicated, and from a practical perspective the implementation of the simpler optimal policy rules discussed above may be preferred by policymakers.

A general message of our model is that implementation matters for welfare. Consumption dispersion depends directly on the entire term structure of bond premia, which in turn are intimately related to the expected path and volatility of the short-term interest rate. Thus, it is not possible to characterize optimal policy purely as a function of the path of aggregate variables like the output gap and inflation, as these variables alone are not sufficient to characterize the term structure of interest rates.

Finally, our model implies that the goal of optimal policy is characterized by a "triple mandate" of stable inflation, output, and long-term bond returns. This seems to conflict with the commonly understood "dual mandate" of the Federal Reserve. However, the Congressional mandate of the Federal Reserve is to "promote effectively the goals of maximum employment, stable prices, and moderate long-term interest rates." Standard interpretations of both academics, practitioners, and policymakers of this legal mandate is in terms only of the first two objectives; the theoretical reasoning is that stabilizing the macroeconomy necessarily implies moderate long-term yields.[1] Our model shows that macroeconomic stabilization is of course a key policy objective of the central bank, but is not always sufficient for stabilizing long-term rates. Thus, our paper provides a theoretical justification for why the triple mandate of the Federal Reserve is optimal,[2] and provides guidance for the implementation of a policy which three distinct mandates.

Our paper builds on the seminal preferred habitat model of Vayanos and Vila (2021), which formalizes the original concept as described in Modigliani and Sutch (1966).[3] The main insight of preferred habitat models is that the interaction of clientele investors implies important departures from the expectations hypothesis in the determination of the term structure of interest rates. More concretely, demand and supply shocks in these markets induce changing risk exposure on the part of marginal bond investors; when risk-bearing capacity is imperfect, this implies fluctuations in

[1]For further discussion, see Mishkin (2007), "Monetary Policy and the Dual Mandate."

[2]This is true under the interpretation that "moderate long-term rates" implies stabilizing excessive fluctuations in term premia. An alternative view of the third mandate is that the central bank should seek to keep long-term borrowing rates "low" for reasons related to the sustainability of fiscal debt. Such an interpretation is not valid in our model, and indeed would be suboptimal.

[3]Other important theoretical contributions to preferred habitat models are Greenwood and Vayanos (2014), Greenwood et al. (2016), King (2019b), King (2019a), Kekre et al. (2024).

risk premia.[4] These models are typically partial equilibrium models. One exception is Ray et al. (2024), which uses a quantitative version of the model in this paper to study the positive implications of QE policies. That paper considers a richer risk factor structure and a wider set of assets (both riskless and risky bonds), but households are representative; hence, the dynamics in this paper induced by household heterogeneity are not present, and the normative implications explored are absent.

Our work more generally contributes to a large literature studying the effects of QE in general equilibrium models. These models feature various forms of financial frictions such as bank balance sheet constraints which break the textbook "QE neutrality" results (e.g. see Gertler and Karadi 2011, Gertler and Karadi 2013, Cúrdia and Woodford 2011, Chen et al. 2012, Sims and Wu 2020, Karadi and Nakov 2020, Iovino and Sergeyev 2023, Carlstrom et al. 2017, Ippolito et al. 2018.) Our paper also relates to models of market segmentation which study various forms of macroeconomic stabilization or macro-prudential policy (e.g. see Andrés et al. 2004, Cui and Sterk 2021, Auclert 2019, Angeletos et al. 2023, Debortoli and Galí 2017).

Finally, our model overlaps with similar work in an international setting. Most closely related to our optimal policy results is Itskhoki and Mukhin (2023), who study the optimal mix of conventional policy and FX interventions in an open economy setting. Theoretically more closely linked to our framework, Gourinchas et al. (2022) and Greenwood et al. (2023) also study the determination of bond yields and exchange rates in a similar preferred habitat model, though these papers do not study general equilibrium effects.

Our paper is structured as follows. Section 2 describes the private agents in the model, characterizes general equilibrium, and derives the social loss function which the policymaker seeks to minimize. Section 3 studies the aggregate dynamics analytically in a simple version of the model with ad-hoc policy rules. Section 4 empirically tests the key predictions of the model regarding the dynamics of short-term rates and term premia. Section 5 characterizes optimal policy as a function of different assumptions regarding constraints on balance sheet tools and short-term rates, while Section 6

[4] Empirically, there is strong evidence of the existence of demand and supply "preferred habitat" frictions considered in this paper, and that these frictions are important for understanding the transmission of large-scale asset purchases (e.g. see Krishnamurthy and Vissing-Jorgensen 2011, Hamilton and Wu (2012), D'Amico and King 2013, Li and Wei 2013, Krishnamurthy and Vissing-Jorgensen 2012, Cahill et al. 2013, King 2019b, Fieldhouse et al. 2018, Di Maggio et al. 2020, Debortoli et al. 2020).

extends these results to the case of full commitment. Section 7 discusses additional extensions and tests of our model, and Section 8 concludes.

2 Model

Time is continuous and denoted by $t \in (0, \infty)$. The model is made up of the following set of agents. A household sector is formally comprised of differentiated households making labor and consumption decisions. Intermediate goods are produced by monopolistically competitive firms using labor; these firms set prices but face nominal rigidities in the form of Rotemberg pricing frictions. Differentiated goods are aggregated by a perfectly competitive final goods retail sector.

There are a continuum of zero-coupon bonds with maturity $\tau \in (0, T)$: a τ-maturity bond has price $P_t^{(\tau)}$ at time t and pays one dollar at maturity date $t + \tau$. Both firms and households face financial frictions when trading bonds. Each household makes borrowing and savings decisions, but must trade with a bond fund which has fixed portfolio weights across bonds. This friction captures a number of realistic features of household portfolio decisions. The different portfolio weights across households represent a type of preferred habitat for households (for instance, due to demographics such as age) which we summarize parsimoniously as heterogeneity in duration exposure. In reality, this reflects for example some households who own property which is financed with long-duration mortgage securities; while for others, saving decisions are predominantly in short-duration securities (such as money market mutual funds or simple bank deposits). Further, our assumption also reflects the fac that households typically do not directly buy and sell financial assets but instead interact with pension funds and mutual funds, which tend to allocate their portfolio using fixed weights and rebalance sluggishly (e.g., Koijen and Yogo (2019), Koijen and Yogo (2022), Bretscher et al. (2022), Peng and Wang (2022)). In the spirit of the "preferred habitat" literature, we capture these frictions with the parsimonious but stark assumption that each household borrows and saves using bonds of a specific maturity. Firms also face a financial friction in the form of a "working capital" constraint: each firm must borrow a portion of its wage bill in advance of production. Analogous to the household portfolio problem, we also assume that the financing costs for firms depend on a portfolio of bonds with fixed portfolio weights.

Because neither households nor firms can freely trade bonds of different maturities, specialized bond traders intermediate bond markets. Bond arbitrageurs trade bonds across the entire term structure, but have limited risk-bearing capacity: formally, these agents solve a mean-variance portfolio problem. These agents are owned by the household sector, but due to financial frictions do not price bonds using the household stochastic discount factor (SDF). We represent the arbitrageur portfolio problem as a function of risk aversion (formally, preferences of the arbitrageurs); however, we treat this risk aversion parameter as a proxy for risk-bearing capacity of financial intermediaries. This is a friction which hinders arbitrageurs' ability to trade assets perfectly on behalf of households as a whole.

The monetary authority sets the short-term nominal interest rate i_t, and conducts balance sheet policies. A fiscal authority provides optimal production subsidies but is otherwise passive and balances the budget period by period via lump-sum taxes.

We study versions of the model where the monetary authority chooses policy according to an ad-hoc Taylor rule, and compare this to policy rules which are chosen to maximize social welfare (possibly subject to implementation frictions). We focus on a linearized equilibrium and second-order welfare approximations. Our linear-quadratic approximation is around a deterministic first-best steady state, where our approximation method still allows for non-zero bond term premia which affect first-order macroeconomic dynamics.

2.1 Households

Denote the households sector by $\mathcal{H}$ and let $i \in \mathcal{H}$ represent an individual household. Each period, household i chooses a consumption bundle $C_t(i)$ and supplies labor $N_t(i)$. The per-period flow utility (which is identical across households) is

$$u\left(C_t(i), N_t(i)\right) = \frac{C_t(i)^{1-\varsigma} - 1}{1-\varsigma} - \frac{N_t(i)^{1+\varphi}}{1+\varphi}, \tag{1}$$

where ς is the inverse intertemporal elasticity of substitution and φ is the inverse Frisch labor elasticity.

As discussed above, households face financial frictions and so cannot trade all bonds freely. Instead, they save and borrow through a bond fund which has fixed portfolio weights. We make a stark assumption that each household can only access bond markets of a specific maturity.[5] A household i who trades τ bonds faces the following budget constraint

$$\mathrm{d}\mathcal{B}_t(i) = [\mathcal{W}_t N_t(i) - \mathcal{P}_t C_t(i)]\,\mathrm{d}t + \mathcal{B}_t(i)\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} + \mathrm{d}\mathcal{F}_t\,, \tag{2}$$

where $\mathcal{W}_t$ is the nominal wage and $\mathcal{P}_t$ is the price index of consumption bundle (both of which are taken as given and identical across households). Household i has nominal wealth $\mathcal{B}_t(i)$ (with $\mathcal{B}_0(i)$ given), earning the realized instantaneous return on the τ-maturity bond. The term $\mathrm{d}\mathcal{F}_t$ represents nominal flow transfers (described below).

Households have a time discount factor of ϱ, so the resulting value function of household i at time $t = 0$ is

$$V_0(i) \equiv \max_{\{C_t(i),N_t(i)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\varrho t} u\left(C_t(i), N_t(i)\right)\mathrm{d}t\,. \tag{3}$$

The household problem is to maximize (3) subject to (2).

Define real wealth $b_t(i) \equiv \frac{\mathcal{B}_t(i)}{\bar{C}(i)\mathcal{P}_t}$ (relative to steady state consumption $\bar{C}(i)$, defined below). Let $V_{b,t}(i)$ represent the gradient of the value function with respect to real wealth. Then the optimality conditions (see the Appendix for details) are relatively standard. The intratemporal conditions

$$C_t(i)^{-\varsigma} = \frac{V_{b,t}(i)}{\bar{C}(i)}, \quad W_t = C_t(i)^{\varsigma} N_t(i)^{\varphi} \tag{4}$$

characterize the consumption and labor decision as a function of the real wage $W_t = \frac{\mathcal{W}_t}{\mathcal{P}_t}$. Combined with the following intertemporal condition

$$\mathrm{E}_t\,\mathrm{d}V_{b,t}(i) = V_{b,t}(i)\left[\varrho + \pi_t - \mu_t(\tau) + \xi_t(i)\right]\mathrm{d}t - V_{b,t}(i)\,\mathrm{E}_t\,\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}, \tag{5}$$

we see that the consumption/saving decision is a function of expected real borrowing rates (and where $\xi_t(i)$ represents higher order terms, defined in the Appendix). However, unlike a textbook model, in equilibrium the borrowing rate of different households will not necessarily be fully characterized by the monetary policy rate,

[5] In the Appendix, we relax this assumption and instead assume that households may trade bonds of different maturities, but face rebalancing costs. Our model is the limit when such "sluggish rebalancing" costs grow, and initial portfolio positions take the form discussed in the baseline model.

since $\mathrm{E}_t \frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} \neq i_t \,\mathrm{d}t$. Thus, the dynamics of aggregate consumption will depend on the equilibrium determination of all borrowing rate across the term structure.

Importantly, when borrowing rates differ across households, then in equilibrium consumption, labor, and wealth will also differ across households (that is, $C_t(i) \neq C_t(i'), N_t(i) \neq N_t(i'), \mathcal{B}_t(i) \neq \mathcal{B}_t(i')$ for some $i, i' \in \mathcal{H}$), which implies that marginal utility is also not equalized.

2.2 Intermediate Firms

A continuum of intermediate goods producers index by $j \in [0,1]$ produce differentiated goods $Y_t(j)$ and set prices $\mathcal{P}_t(j)$. Final output Y_t is produced by a competitive retail sector, which aggregates according to $Y_t \equiv \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \,\mathrm{d}j\right]^{\frac{\epsilon}{\epsilon-1}}$, where the elasticity of substitution between goods is ϵ. This implies the follow demand and price index for differentiated goods

$$Y_t(j) = \left(\frac{\mathcal{P}_t(j)}{\mathcal{P}_t}\right)^{-\epsilon} Y_t, \;\; \mathcal{P}_t = \left[\int_0^1 \mathcal{P}_t(j)^{1-\epsilon} \,\mathrm{d}j\right]^{\frac{1}{1-\epsilon}}. \tag{6}$$

Each firm j faces the same production function $Y_t(j) = Z_t L_t(j)$, where Z_t is aggregate technology and $L_t(j)$ represents the labor inputs of firm j.

The per-period flow nominal profit of firm j is given by

$$\mathrm{d}\Pi_t(j) = \left[(1+\tau^y)\mathcal{P}_t(j)Y_t(j) - \mathcal{W}_t L_t(j) - \mathcal{T}_t^y\right] \mathrm{d}t - \mathrm{d}\Theta_t(j)\,, \tag{7}$$

where τ^y is a production subsidy, financed by lump-sum transfers $\mathcal{T}_t^y$ (discussed below). In addition to the wage bill $\mathcal{W}_t L_t(j)$, firms face the following flow costs of production:

$$\mathrm{d}\Theta_t(j) = \frac{\vartheta}{2} \left(\pi_t(j) - \varpi_t\right)^2 \mathcal{P}_t Y_t \,\mathrm{d}t + \mathcal{W}_t L_t(j) \left(\int_0^T \theta_t(\tau) \frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} \,\mathrm{d}\tau\right). \tag{8}$$

The first term captures nominal rigidities. Firms choose $\pi_t(j)$, the rate of change of their prices ($\mathrm{d}\mathcal{P}_t(j) = \mathcal{P}_t(j)\pi_t(j) \,\mathrm{d}t$). Whenever $\vartheta > 0$, firms face Rotemberg costs of choosing $\pi_t(j)$ which differs from some exogenous target ϖ_t. The second term captures a "working capital" financing friction. When $\theta_t(\tau) > 0$ for some τ, firms must finance a fraction of their wage bill by borrowing in bond markets.[6] For simplicity, we assume the firm portfolio weights may vary over time according to

$$\theta_t(\tau) \equiv \theta(\tau)(\beta_t - \bar{\beta}), \tag{9}$$

for some exogenous financing shock β_t, where $0 < \bar{\beta} < 1$ and $\int_0^T \theta(\tau)\,\mathrm{d}\tau = 1$. However, it is conceptually straightforward to allow for endogenous fluctuations in β_t, or to allow for a richer factor structure of the firm portfolio weights $\theta_t(\tau)$ over time.[7]

Taking as given CES demand and the costs of production, the firm problem at time $t = 0$ is therefore

$$U_0(j) \equiv \max_{\{\pi_t(j)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\varrho t} Q_t^{\mathcal{H}} \frac{\mathrm{d}\Pi_t(j)}{\mathcal{P}_t}. \tag{10}$$

Firms transfer profits to households; we assume that profits are discounted by $Q_t^{\mathcal{H}} = \int_{i\in\mathcal{H}} V_{b,t}(i)\,\mathrm{d}i$, the average SDF across households.

The firm optimality conditions are given by

$$U_{p,t}(j)P_t(j) = \vartheta Q_t^{\mathcal{H}} Y_t \left(\pi_t(j) - \varpi_t\right), \tag{11}$$

$$\begin{aligned} \mathrm{E}_t\,\mathrm{d}U_{p,t}(j) = \big[&\left(\varrho - (\pi_t(j) - \pi)\right) U_{p,t}(j) \\ &- Q_t^{\mathcal{H}} Y_t \left((1+\tau^y)(1-\epsilon) + \epsilon P_t(j)^{-1} M_t\right) + \xi_t(j)\big]\,\mathrm{d}t\,, \end{aligned} \tag{12}$$

$$M_t\,\mathrm{d}t \equiv \frac{W_t}{Z_t}\,\mathrm{d}t + \frac{W_t}{Z_t}\int_0^T \theta_t(\tau)\,\mathrm{E}_t\,\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}\,\mathrm{d}\tau\,, \tag{13}$$

where $\xi_t(j)$ represents higher order terms (defined in the Appendix). Eq. (13) defines M_t, the real marginal costs of the firm (which are identical across firms and taken as given), and which depend on the wage bill as well as financing costs.

[6] For simplicity, we assume that portfolio weights are the same across each firm. Allowing for heterogeneity across firm portfolios as in the household sector would imply additional price dispersion; this would not affect the first-order aggregate dynamics but would have implications for social welfare.

[7] These can be defined analogously to the preferred habitat demand shocks in Vayanos and Vila (2021) or Ray et al. (2024).

2.3 Arbitrageurs

In addition to the demand for bonds arising from households and firms, bonds are also traded by arbitrageurs. Arbitrageurs actively choose holdings of short- and long-maturity bonds. The representative arbitrageur solves the following mean-variance problem:

$$\max_{\{\mathcal{X}_t(\tau)\}_{\tau=0}^{T}} \mathrm{E}_t\,\mathrm{d}\mathcal{X}_t - \frac{a_t}{2}\,\mathrm{Var}_t\,\mathrm{d}\mathcal{X}_t \tag{14}$$

$$\text{s.t.}\;\; \mathrm{d}\mathcal{X}_t = \mathcal{X}_t i_t\,\mathrm{d}t + \int_0^T \mathcal{X}_t(\tau)\left(\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} - i_t\,\mathrm{d}t\right)\mathrm{d}\tau\,. \tag{15}$$

That is, they choose nominal bond holdings $\mathcal{X}_t(\tau)$ across all maturities τ. Arbitrageurs choose to engage in carry trades across the term structure in order to optimally satisfy the tradeoff between higher expected returns and the volatility on their balance sheet. The risk-return tradeoff is governed by parameter a_t. Formally this is a risk aversion parameter, but more generally this is a proxy for capital constraints, Value-at-Risk constraints, or any frictions which imply that bond returns are not determined as in a model with a representative household with perfect access to financial securities. We assume that arbitrageurs transfer all gains and losses in their wealth $\mathrm{d}\mathcal{X}_t$ to households each period.

The arbitrageur optimality conditions are given by

$$\mathrm{E}_t\,\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} - i_t\,\mathrm{d}t = a_t \int_0^T \mathcal{X}_t(\tau')\,\mathrm{Cov}_t\left(\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}, \frac{\mathrm{d}P_t^{(\tau')}}{P_t^{(\tau')}}\right)\mathrm{d}\tau'\,. \tag{16}$$

Hence, arbitrageurs ensure that no risk-free arbitrage opportunities exist. Equation (16) shows that the expected excess return of any τ-maturity bond is a function of the covariance of bond returns across the entire term structure, scaled by arbitrageur risk aversion a_t and weighted by holdings $\mathcal{X}_t(\tau)$ (which in equilibrium will be determined by market clearing).

We assume that $a_t = \frac{a}{\mathcal{P}_t \bar{Y}}$; that is, risk aversion is constant relative to prices (scaled by steady state output, defined below). Analogous to household wealth, define real arbitrageur bond positions as $x_t(\tau) = \frac{\mathcal{X}_t(\tau)}{\mathcal{P}_t \bar{Y}}$.

2.4 Government

The fiscal authority sets the production subsidy τ^y, which is self-financed through lump-sum taxes on firms. The central bank sets the nominal interest rate i_t, and the fiscal authority pays this interest i_t on short-term bonds (reserves). Besides the production subsidy, the fiscal authority is passive: it levies lump-sum taxes or transfers $\mathcal{P}_t T_t$ on households in order to balance the budget each period, so that $\mathcal{P}_t T_t = -\mathcal{S}_t^0 i_t$ where $\mathcal{S}_t^0$ is the aggregate demand for short-term bonds (reserves). The central bank may also conduct balance sheet operations by taking non-zero positions in bonds $s_t(\tau) \equiv \frac{\mathcal{S}_t(\tau)}{\mathcal{P}_t Y}$. Any proceeds from the central bank bond holdings are renumerated lump-sum to the households.

As a benchmark, we assume that the central bank can utilize balance sheet tools and the short rate without any constraints. More generally we assume that the central bank is subject to constraints. First, the central bank may face real deadweight costs of adjustment when setting the short-term interest rate:

$$Y_t \Psi_t^i \equiv Y_t \frac{\psi^i}{2} \left(i_t - \bar{i}_t\right)^2 , \tag{17}$$

where $\bar{i}_t$ represents a potentially time-varying (exogenous) policy target. We interpret (17) as capturing in reduced-form constraints such as the effective lower bound; however, to maintain our linear-quadratic approximations, we assume a symmetric loss function. Thus, whenever $\psi^i > 0$, setting the short-term interest rate away from $\bar{i}_t$ will imply deadweight losses.

Second, bond holdings may be subject to real deadweight holding costs (measured in terms of output) which take the form:

$$Y_t \Psi_t^S \equiv Y_t \int_0^T \frac{\psi^{(\tau)}}{2} \left(s_t(\tau) - \bar{s}(\tau)\right)^2 \mathrm{d}\tau \, . \tag{18}$$

Thus, whenever $\psi^{(\tau)} > 0$, central bank holdings away from steady state will imply deadweight losses. These costs represent holding costs which the central bank faces (but which do not impact private bond arbitrageurs). These costs can also be thought of as capturing in reduced-form the fiscal costs associated with gains and losses of the central bank balance sheet.[8]

[8]If the fiscal authority can only utilize distortionary taxation, then the gains and losses of the central bank balance sheets will induce fiscal costs; see d'Avernas et al. (2024). Finally, we assume

2.5 Aggregation and Market Clearing

We study a symmetric equilibrium in which all firms make the same decisions and so $Y_t(j) = Y_t$, $\mathcal{P}_t(j) = \mathcal{P}_t$, $\pi_t(j) = \pi_t$, $L_t(j) = L_t$.

Assume a mass $\eta(\tau)$ (in measure $\mathrm{d}\tau$) of $i \in \mathcal{H}$ who access τ bond markets; and $\int_0^T \eta(\tau)\,\mathrm{d}\tau = 1$. Since any household $i, i' \in \mathcal{H}$ with access to τ-bonds is otherwise identical (assuming initial wealth $\mathcal{B}_0(i) = \mathcal{B}_0(i')$), we have $C_t(i) \equiv C_t(\tau)$, $N_t(i) \equiv N_t(\tau)$, and $\mathcal{B}_t(i) \equiv \mathcal{B}_t(\tau)$. Aggregate household consumption is therefore

$$C_t = \int_{i\in\mathcal{H}} C_t(i)\,\mathrm{d}i = \int_0^T \eta(\tau)C_t(\tau)\,\mathrm{d}\tau\,. \tag{19}$$

Labor market clearing implies that aggregate labor demand L_t equals aggregate labor supply $N_t \equiv \int_{i\in\mathcal{H}} N_t(i)\,\mathrm{d}i = \int_0^T \eta(\tau)N_t(\tau)\,\mathrm{d}\tau$.

Market clearing in production and consumption is given by

$$Y_t = Z_t N_t, \;\; C_t = Y_t\left(1 - \frac{\vartheta}{2}\pi_t^2 - \Psi_t^i - \Psi_t^S\right), \tag{20}$$

where aggregate consumption differs from aggregate output due to deadweight loss from price changes when $\vartheta > 0$, or policy frictions when $\psi^i > 0$ or $\psi^{(\tau)} > 0$.

Bond market clearing implies

$$\mathcal{X}_t(\tau) - \theta_t(\tau)\mathcal{W}_t N_t + \eta(\tau)\mathcal{B}_t(\tau) + \mathcal{S}_t(\tau) = 0 \tag{21}$$

for all maturities $\tau > 0$ (and the fiscal authority ensures that the short-term reserve clears).

2.6 First-Best and Approximations

Our model will always include at least one source of aggregate uncertainty: aggregate technology $Z_t = \bar{Z}e^{z_t}$ evolves according to

$$\mathrm{d}z_t = -\kappa_z z_t\,\mathrm{d}t + \sigma_z\,\mathrm{d}B_{z,t}\,, \tag{22}$$

where $B_{z,t}$ is a standard Brownian motion and σ_z governs the volatility of innovations to (log) technology.

that the real costs of these policy frictions are financed via lump-sum taxation of households.

In addition, we also have shocks to firm financing (β_t in (9)) as well as exogenous-cost fluctuations in the Rotemberg pricing frictions (ϖ_t in (8)). We thus consider a generic set of additional exogenous risk factors: $\mathbf{z}_t$ is an $N_z \times 1$ vector and associated vector of Brownian motions $\mathbf{B}_t$ (where we always have $z_t \in \mathbf{z}_t, B_{z,t} \in \mathbf{B}_t$); and an $N_z \times N_z$ diffusion matrix $\boldsymbol{\sigma}$; that is,

$$\mathrm{Var}_t \, \mathrm{d}\mathbf{z}_t = \boldsymbol{\sigma}\boldsymbol{\sigma}^\top \, \mathrm{d}t \,. \tag{23}$$

Given this set of risk factors, write the instantaneous return of a τ bond as

$$\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t(\tau) \, \mathrm{d}t + \boldsymbol{\sigma}_t(\tau) \, \mathrm{d}\mathbf{B}_t \,, \tag{24}$$

where $\mu_t(\tau)$ represents the expected short-horizon return of a τ bond, and $\boldsymbol{\sigma}_t(\tau)$ represents how shocks to risk factors lead to fluctuations in realized returns.

Rewriting the arbitrageur optimality condition (16) gives

$$\mu_t(\tau) - i_t = a_t \boldsymbol{\sigma}_t(\tau) \boldsymbol{\Lambda}_t, \quad \text{where } \boldsymbol{\Lambda}_t^\top = \int_0^T \mathcal{X}_t(\tau) \boldsymbol{\sigma}_t(\tau) \, \mathrm{d}\tau \,.$$

We see that the expected excess return of any τ-maturity bond is a function of bond-specific risk loadings $\boldsymbol{\sigma}_t(\tau)$ and a global set of risk prices $\boldsymbol{\Lambda}_t$. The market price of risk depends on arbitrageur risk aversion a_t as well as holdings $\mathcal{X}_t(\tau)$ (which in equilibrium is determined by market clearing).

We study the dynamics of the model around a riskless limiting case, formalized as follows. Parameterize the diffusion matrix from (23) as $\boldsymbol{\sigma}(\xi)$ and the arbitrageur risk aversion coefficient $a(\xi)$ such that $\boldsymbol{\sigma}(\xi) \to \mathbf{0}$ but $a(\xi)\boldsymbol{\sigma}(\xi) \to \boldsymbol{\sigma}$ as $\xi \to 0$. That is, the product of risk aversion and physical risk remains non-zero and bounded even as risk becomes small. This assumption can also be formalized in terms of the size of the intermediary sector relative to the size of asset markets (e.g. see Itskhoki and Mukhin 2023), an captures explicitly that the arbitrageur "risk aversion" parameter a is a measure of imperfect arbitrage. Our approximation method allows for tractable first- and second-order approximations of the model, while still allowing for first-order variation in risk premia.

The following Proposition characterizes the optimal allocation in the riskless limit.

Proposition 1 (First-best allocation)**.** *Consider the limiting case where $\xi \to 0$ and* *suppose that wealth is initially equalized across households* ($\mathcal{B}_0(i) \equiv \mathcal{B}_0$)*. Then with perfect arbitrage* ($a = 0$)*, the model is equivalent to a representative agent model. With the optimal production subsidy* τ^y *given by* (A1)*, the first-best allocation is obtained with flexible prices* ($\vartheta = 0$)*.*

Prop. 1 shows that, so long as the production subsidy is chosen optimally to remove steady-state distortions from imperfect competition and the working capital frictions, there is no room for policy improvement in the flexible price, perfect arbitrage model. Further, in the proof of Prop. 1, we show that the deterministic steady state of the model features zero inflation ($\bar{\pi} = 0$) and zero arbitrageur positions ($\bar{x}(\tau) = 0$), which implies that consumption and labor supply across households is also equalized ($\bar{C}(i) = \bar{C}, \bar{N}(i) = \bar{N}$). Thus, the steady state in the first-best equilibrium and in the sticky price, imperfect arbitrage equilibrium coincide.

Denote aggregate output in the first-best equilibrium by Y_t^n, and define the output gap $X_t \equiv \frac{Y_t}{Y_t^n}$. Aggregating (5) across households, the dynamics of the (log) output gap x_t are governed by a modified aggregate Euler equation:

$$\mathrm{E}_t \, \mathrm{d}x_t = \varsigma^{-1} \left[\tilde{\mu}_t - \pi_t - v_t\right] \mathrm{d}t \,. \tag{25}$$

The term $v_t \equiv -\varsigma \kappa_z \frac{1+\varphi}{\varsigma+\varphi} z_t$ is the "natural" rate, which is the real borrowing rate in the first-best allocation from Prop. 1. The "household effective borrowing rate" is given by

$$\tilde{\mu}_t = \int_0^T \eta(\tau) \left(\mu_t(\tau) - \varrho\right) \mathrm{d}\tau \,, \tag{26}$$

which is the average borrowing rate faced by the household, weighted by the household member weights $\eta(\tau)$.

Re-writing the firm optimality condition in terms of output gaps gives a New Keynesian Phillips curve:

$$\mathrm{E}_t \, \mathrm{d}\pi_t = \left[\rho \pi_t - \kappa \left(x_t + \bar{\beta} \hat{\mu}_t + \varrho \beta_t\right) - u_t\right] \mathrm{d}t \,, \tag{27}$$

where κ measures the aggregate degree of price rigidity. The exogenous cost-push u_t is a function of the inflation target ϖ_t in (8) (defined in the Appendix). The "firm effective borrowing rate" is

$$\hat{\mu}_t = \int_0^T \theta(\tau)(\mu_t(\tau) - \varrho)\,\mathrm{d}\tau\,, \tag{28}$$

and the term β_t captures firm financing shocks which arise from exogenous fluctuations in firm portfolio weights (defined in (9)).

Market clearing for bonds is given by

$$x_t(\tau) = -(s_t(\tau) - s(\tau)) + \frac{\bar{W}}{\bar{Z}}\theta(\tau)\left[\bar{\beta}(w_t + n_t) + \beta_t\right] - \eta(\tau)(b_t(\tau) - \bar{b}),$$

and the household budget constraint is

$$\mathrm{d}b_t(\tau) = \left[-\left(1 + \frac{\varsigma}{\varphi}\frac{\bar{W}}{\bar{Z}}\right)\check{c}_t(\tau) + \bar{b}(\check{\mu}_t(\tau) - \pi_t) + \varrho\check{b}_t(\tau)\right]\mathrm{d}t + \bar{b}\check{\boldsymbol{\sigma}}(\tau)\,\mathrm{d}\mathbf{B}_t\,, \tag{29}$$

$$\text{where}\;\; \check{b}_t(\tau) \equiv b_t(\tau) - \int_0^T \eta(\tau)b_t(\tau')\,\mathrm{d}\tau' \equiv b_t(\tau) - \tilde{b}_t, \tag{30}$$

$$\check{c}_t(\tau) \equiv c_t(\tau) - \int_0^T \eta(\tau)c_t(\tau')\,\mathrm{d}\tau' \equiv c_t(\tau) - \tilde{c}_t, \tag{31}$$

$$\check{\boldsymbol{\sigma}}(\tau) \equiv \boldsymbol{\sigma}(\tau) - \int_0^T \eta(\tau)\boldsymbol{\sigma}(\tau')\,\mathrm{d}\tau' \equiv \boldsymbol{\sigma}(\tau) - \tilde{\boldsymbol{\sigma}}. \tag{32}$$

The terms $\check{c}_t(\tau)$ and $\check{b}_t(\tau)$ represent consumption and wealth deviations across households, and $\bar{b}$ is the steady state level of household wealth (which in our setting identical across households).

2.7 Social Welfare

The dynamics of the output gap x_t and inflation π_t in (25) and (27) imply that, to a first-order, our model is similar to the representative agent case considered in Ray et al. (2024). However, even to a first-order, our model will feature more complicated dynamics due to the endogenous time-variation in demand for bonds which arises from heterogeneous households.

More importantly, the welfare consequences differ due to the inefficiencies described above. A second-order expansion of social welfare relative to the first-best allocation gives the per-period social loss

$$\mathcal{L}_t \equiv (\varsigma + \varphi) x_t^2 + \vartheta \pi_t^2 \tag{33}$$
$$+ \frac{\varsigma}{\varphi} (\varphi + \varsigma) \operatorname{Var}_\tau \check{c}_t(\tau) \tag{34}$$
$$+ \int_0^T \psi^{(\tau)} \left(s_t^{(\tau)} - \bar{s}(\tau) \right)^2 \mathrm{d}\tau + \psi^i \left(i_t - \bar{i}_t \right)^2. \tag{35}$$

The first two terms in line (33) capture the welfare losses associated with the nominal rigidities in the model; these terms arise in representative agent New Keynesian (RANK) models. Compared to a standard RANK model, line (34) shows that social welfare loss also depends on the term

$$\operatorname{Var}_\tau \check{c}_t(\tau) \equiv \int_0^T \eta(\tau) \left(c_t(\tau) \right)^2 \mathrm{d}\tau - \left[\int_0^T \eta(\tau) c_t(\tau)\, \mathrm{d}\tau \right]^2.$$

Thus, increased consumption dispersion across households implies welfare losses, due to imperfect risk-sharing.

The final terms in line (35) represent losses associated with the central bank balance sheet and short rate policies from equations (18) and (17).

2.8 Equilibrium

Intuitively, what does general equilibrium look like in this model? From the perspective of households, the key factor is how sensitive their borrowing rates are to the short rate (and movements in asset demand, including balance sheet policies). The model reduces to a benchmark New Keynesian model when these rates move one-for-one, but in general $\mu_t(\tau) \neq i_t$. Suppose that long-term borrowing rates are highly responsive to the policy rate. Then household borrowing is also highly sensitive to the policy rate, and therefore the growth rate of consumption will also react strongly to the policy rate. On the other hand, when long-term rates are insensitive to the policy rate, the pass-through of changes in the policy rate to households is weakened. Through the borrowing decisions of the household, the growth rate of consumption is less responsive to the policy rate.

However, the sensitivity of the effective borrowing rate to the policy rate is an equilibrium object, which also depends on financial markets. Bond prices will adjust in order to achieve equilibrium in bond markets, such that arbitrageurs' portfolio allocation satisfies their mean-variance tradeoff while also clearing the market given the demand from households and firms. In this model, arbitrage is imperfect and the term structure will not be characterized by the expectations hypothesis except under special circumstances. Therefore, it is the risk-adjusted dynamics of the macroeconomy which determine bond prices in financial markets, rather than the actual dynamics of the short rate only.

In general, the term structure will be determined by complicated interactions between the different classes of investors in bond markets (arbitrageurs, firms, and households). Because differentiated households also make consumption and savings decisions as a function of different borrowing rates, the general equilibrium dynamics of the macroeconomy will also depend on these interactions. General equilibrium is obtained when these two forces balance. Thus, characterizing equilibrium involves two key steps: first, understanding the differences between the actual and risk-adjusted dynamics of the economy; and second, linking household savings and consumption choices with the bond prices determined in imperfect financial markets.

We characterize the first-order dynamics and second-order social welfare function for a general set of risk factors and dynamics, and then apply these results to specific versions of our model. The key difficulty is to appropriately define the relevant set of state variables which summarize the dynamics of the model. In principle, the entire wealth distribution $\{b_t(\tau)\}_{\tau=0}^{T}$ across households may matter for term premia (through arbitrageur optimality conditions and market clearing). However, we show that a much smaller set of summary statistics of the wealth distribution are all that is necessary for characterizing the first- and second-order properties of the model.

We first make use of the following Proposition to characterizing the behavior of asset prices, conditional on some arbitrary set of state variables.

Proposition 2 (Asset prices)**.** *Suppose that (log) bond prices are affine functions of a set of state variables $\mathbf{x}_t$, given by (endogenous) coefficient functions:*

$$-\log P_t^{(\tau)} = \mathbf{A}(\tau)^\top \mathbf{x}_t + C(\tau), \tag{36}$$

where $\mathbf{x}_t$ evolves according to

$$\mathrm{d}\mathbf{x}_t = -\boldsymbol{\Gamma}\mathbf{x}_t\,\mathrm{d}t + \boldsymbol{\sigma}_x\,\mathrm{d}\mathbf{B}_t\,. \tag{37}$$

Then the affine coefficients in equation (36) are given by

$$\mathbf{A}(\tau) = \left[\mathbf{I} - e^{-\mathbf{M}\tau}\right]\mathbf{M}^{-1}\mathbf{e}_i \tag{38}$$

where $\mathbf{e}_i$ *is a vector such that* $\mathbf{e}_i^\top \mathbf{x}_t = i_t$. *The matrix* $\mathbf{M}$ *solves the fixed point problem:*

$$\mathbf{M} = \mathbf{\Gamma}^\top - \int_0^T \mathbf{\Theta}(\tau)\mathbf{A}(\tau)^\top \,\mathrm{d}\tau\, \tilde{\mathbf{\Sigma}}, \tag{39}$$

where $\tilde{\mathbf{\Sigma}} \equiv a \cdot \boldsymbol{\sigma}_x \boldsymbol{\sigma}_x^\top$ *and* $x_t(\tau) = \mathbf{\Theta}(\tau)^\top \mathbf{x}_t$ *maps the state variables into arbitrageur bond positions.*

The matrix $\mathbf{M}$ can be thought of as the risk-adjusted dynamics of the state. In the first-best, arbitrageurs are perfectly risk-neutral ($a = 0 \implies \tilde{\mathbf{\Sigma}} = \mathbf{0}$), so we have $\mathbf{M} = \mathbf{\Gamma}^\top$. However, when $\tilde{\mathbf{\Sigma}} \neq \mathbf{0}$, $\mathbf{M}$ appears on both sides of equation (39) through the affine coefficients $\mathbf{A}(\tau)$.

Note the similarity with Vayanos and Vila (2021). Conditional on the equilibrium dynamics of the model, the only difference for bond pricing comes from the more general formulation of the dynamics of the short-term interest rate and from the market clearing conditions in bond markets.

Next, in order to define the set of aggregate state variables which summarize the dynamics of the model, from eqs. (30) and (31) define the risk-weighted household wealth dispersion, consumption dispersion, and balance sheet tools as

$$\tilde{\mathbf{b}}_t \equiv \int_0^T \boldsymbol{\sigma}(\tau)^\top \eta(\tau) \check{b}_t(\tau)\,\mathrm{d}\tau \tag{40}$$

$$\tilde{\mathbf{c}}_t \equiv \int_0^T \boldsymbol{\sigma}(\tau)^\top \eta(\tau) \check{c}_t(\tau)\,\mathrm{d}\tau\,, \tag{41}$$

$$\tilde{\mathbf{s}}_t \equiv \int_0^T \boldsymbol{\sigma}(\tau)^\top (s_t(\tau) - s(\tau))\,\mathrm{d}\tau\,, \tag{42}$$

which are all vectors with the same dimension as the number of aggregate risk factors

$\mathbf{z}_t$. Household optimality conditions and market clearing imply that

$$\mathrm{d}\tilde{\mathbf{b}}_t = \left[-\left(1 + \frac{\varsigma}{\varphi}\frac{\bar{W}}{\bar{Z}}\right)\tilde{\mathbf{c}}_t + \bar{b}\tilde{\mathbf{\Sigma}}\mathbf{\Lambda}_t + \bar{i}\tilde{\mathbf{b}}_t\right]\mathrm{d}t + \bar{b}\tilde{\mathbf{\Sigma}}\,\mathrm{d}\mathbf{B}_t\,, \tag{43}$$

$$\mathrm{E}_t\,\mathrm{d}\tilde{\mathbf{c}}_t = \varsigma^{-1}\tilde{\mathbf{\Sigma}}\mathbf{\Lambda}_t\,\mathrm{d}t\,, \tag{44}$$

$$\mathbf{\Lambda}_t = -\tilde{\mathbf{s}}_t + \hat{\boldsymbol{\sigma}}^\top\frac{\bar{W}}{\bar{Z}}\left(\bar{\beta}(w_t + n_t) + \beta_t\right) - \tilde{\mathbf{b}}_t - \tilde{\boldsymbol{\sigma}}^\top(\tilde{b}_t - \bar{b}), \tag{45}$$

where $\mathbf{\Lambda}_t$ are the risk prices from the arbitrageur optimality conditions; the matrix $\tilde{\mathbf{\Sigma}} \equiv \int_0^T \eta(\tau)\boldsymbol{\sigma}(\tau)^\top\check{\boldsymbol{\sigma}}(\tau)\,\mathrm{d}\tau$ represents the dispersion of bond risk across the distribution of households; the vector $\tilde{\boldsymbol{\sigma}} \equiv \int_0^T \eta(\tau)\boldsymbol{\sigma}(\tau)\,\mathrm{d}\tau$ represents the diffusion terms of the household effective borrowing rate; and $\hat{\boldsymbol{\sigma}} \equiv \int_0^T \theta(\tau)\boldsymbol{\sigma}(\tau)\,\mathrm{d}\tau$ represents the diffusion terms of the firm effective borrowing rate.

The following Proposition shows that, conditional on a finite set of endogenous risk terms, these objects are sufficient to characterize the dynamics of the model.

Proposition 3 (Aggregate dynamics)**.** *Collect all risk factors $\mathbf{z}_t$ as well as $\tilde{\mathbf{b}}_t$ defined in (40) into the vector $\mathbf{x}_t$. Collect the output gap x_t, inflation π_t, and $\tilde{\mathbf{c}}_t$ defined in (41) into the vector $\mathbf{y}_t$. Assume that the short rate i_t and balance sheet tools $s_t(\tau)$ are functions of these objects.*

Then given the following objects $\tilde{\mathbf{\Sigma}} \equiv \int_0^T \eta(\tau)\boldsymbol{\sigma}(\tau)^\top\check{\boldsymbol{\sigma}}(\tau)\,\mathrm{d}\tau$, $\tilde{\boldsymbol{\sigma}} \equiv \int_0^T \eta(\tau)\boldsymbol{\sigma}(\tau)\,\mathrm{d}\tau$ and $\hat{\boldsymbol{\sigma}} \equiv \int_0^T \theta(\tau)\boldsymbol{\sigma}(\tau)\,\mathrm{d}\tau$, the linear rational expectations equilibrium is given by

$$\mathrm{d}\mathbf{x}_t = -\mathbf{\Gamma}\mathbf{x}_t\,\mathrm{d}t + \boldsymbol{\sigma}_x\,\mathrm{d}\mathbf{B}_t\,, \quad \mathbf{y}_t = \mathbf{\Omega}\mathbf{x}_t, \tag{46}$$

where the matrices $\mathbf{\Gamma}, \mathbf{\Omega}$, and $\boldsymbol{\sigma}_x$ are given in the Appendix.

With the results in Propositions 2 and 3, we can characterize the equilibrium of the model.

Theorem 1 (Existence and uniqueness)**.** *An affine equilibrium is one in which the state and jump variables evolve according to equations (46), and asset prices are* *determined by the solution to the expressions* (38) *and* (39)*. In this case, we have*

$$\begin{aligned}
\tilde{\boldsymbol{\sigma}} &= -\int_0^T \eta(\tau)\mathbf{A}(\tau)^\top \, \mathrm{d}\tau \, \boldsymbol{\sigma}_x, \\
\hat{\boldsymbol{\sigma}} &= -\int_0^T \theta(\tau)\mathbf{A}(\tau)^\top \, \mathrm{d}\tau \, \boldsymbol{\sigma}_x, \\
\tilde{\boldsymbol{\Sigma}} &= \boldsymbol{\sigma}_x^\top \int_0^T \eta(\tau)\mathbf{A}(\tau)\mathbf{A}(\tau)^\top \, \mathrm{d}\tau \, \boldsymbol{\sigma}_x - \tilde{\boldsymbol{\sigma}}^\top \tilde{\boldsymbol{\sigma}}.
\end{aligned}$$

The per-period loss function can be written

$$\mathcal{L}_t = \mathbf{x}_t^\top \Psi \mathbf{x}_t.$$

In a neighborhood of risk-neutrality ($a \approx 0$), the equilibrium exists and is (locally) unique.

Note that the dynamics matrix of the state $\boldsymbol{\Gamma}$ depends on the affine coefficients $\mathbf{A}(\tau)$ through the effective borrowing rates of the households, which itself is a function of the risk-adjusted dynamics matrix $\mathbf{M}$. Thus, equilibrium is determined as a fixed point that produces asset price dynamics consistent with equilibrium dynamics of the macroeconomy and vice versa. In general, an affine equilibrium of this type may not exist, or there may be multiple solutions to this fixed point problem. However, when $a = 0$, the model reduces to a standard New Keynesian model. The result in Theorem 1 shows that this equilibrium persists and is locally unique as we depart from risk neutrality.

Although Theorem 1 characterizes the first-order dynamics and second-order welfare properties of the model, in general it provides little guidance on how either differ from textbook models. In the next section, we focus on simplified versions of the model and to derive clear predictions regarding dynamics.

3 Aggregate Dynamics: Analytical Results

Before studying welfare consequences, we explore the first-order dynamics of the model. To do so, we consider versions of the model where the central bank follows ad-hoc policy rules.

We focus on a version of the model with only one source of aggregate uncertainty:

natural rate shocks v_t (equivalently, technology shocks z_t). We simplify further for most of this section and study the case of fully rigid prices ($\vartheta \to \infty$), and where household wealth is zero in steady state ($\bar{b} = 0$). In this case, the dynamics are summarized by the dynamics of technology (22) and the following system of equations

$$
\begin{aligned}
\mathrm{d}\tilde{b}_t &= \left[-\left(1 + \frac{\varsigma}{\varphi}\right) \tilde{c}_t - \varrho \tilde{b}_t \right] \mathrm{d}t \,, \\
\mathrm{E}_t \, \mathrm{d}x_t &= -\varsigma^{-1} \left[i_t + \tilde{\sigma}\lambda_t - v_t \right] \mathrm{d}t \,, \\
\mathrm{E}_t \, \mathrm{d}\tilde{c}_t &= -\varsigma^{-1} \tilde{\Sigma} \tilde{\sigma} \lambda_t \, \mathrm{d}t \,, \\
\lambda_t &= -\tilde{s}_t + \bar{\beta}\hat{\sigma} \left((1 + \varsigma + \varphi) x_t - \frac{1}{\varsigma \kappa_z} v_t \right) - \tilde{b}_t
\end{aligned}
$$

where risk prices λ_t as well as the risk objects $\tilde{\sigma}, \hat{\sigma}, \tilde{\Sigma}$ now scalars. Note that in this version of the model, the financing frictions which firms face does not impact their pricing decision (as by assumption, prices are fully rigid). However, through market clearing, firm demand for bonds implies that aggregate fluctuations lead to fluctuations in risk premia through market clearing.

Throughout, we assume that the central bank follows Taylor-type of rules for its policy tools given by

$$
\begin{aligned}
i_t &= \phi_x x_t + \epsilon_{i,t}, \\
s_t(\tau) - \bar{s}(\tau) &= \phi_x(\tau) x_t + \epsilon_{s,t}(\tau), \\
\implies \tilde{s}_t &\equiv \int_0^T \eta(\tau)\sigma(\tau) \left(s_t(\tau) - \bar{s}(\tau) \right) \mathrm{d}\tau = \tilde{\phi}_x x_t + \tilde{\epsilon}_{s,t},
\end{aligned}
$$

where $\epsilon_{i,t}$ and $\tilde{\epsilon}_{s,t}$ represent zero-probability monetary policy shocks, which evolve deterministically according to

$$
\mathrm{d}\epsilon_{i,t} = -\kappa_i \epsilon_{i,t} \, \mathrm{d}t \,, \quad \mathrm{d}\tilde{\epsilon}_{s,t} = -\kappa_s \tilde{\epsilon}_{s,t} \, \mathrm{d}t \,.
$$

We first consider a benchmark case where there is no working capital friction and so firms do not borrow in bond markets, and the central bank does not systematically use balance sheet tools. The following Proposition derives results regarding dynamics for conventional and unconventional monetary policy.

Proposition 4 (Rigid price dynamics, no firm borrowing)**.** *Suppose that* $\bar{\beta} = 0$ *so* *that $\hat{\sigma} = 0$, and $\phi_x(\tau) = 0$. If household wealth is initially equalized at steady state (so $\tilde{b}_t = 0$), then the equilibrium dynamics of the model are equivalent to a representative agent model, and $\lambda_t = 0$. We have*

$$\frac{\partial x_t}{\partial v_t} > 0, \quad \frac{\partial \lambda_t}{\partial v_t} = 0, \quad \frac{\partial x_t}{\partial \epsilon_{i,t}} < 0, \quad \frac{\partial \lambda_t}{\partial \epsilon_{i,t}} = 0.$$

However, following an unconventional monetary policy shock, we have

$$\frac{\partial x_t}{\partial \tilde{\epsilon}_{s,t}} > 0, \quad \frac{\partial \lambda_t}{\partial \tilde{\epsilon}_{s,t}} < 0,$$

and the model is no longer equivalent to a representative agent model, as $\tilde{b}_{t+k} \neq 0$ following the shock.

Prop. 4 shows that without firm borrowing and initial equalization of asset positions across households, the model which only features conventional monetary policy is identical to a representative agent model. If household wealth is initially at steady state, then arbitrageurs initially take zero positions in bond markets and hence do not require any risk compensation. Following movements in the short-term interest rate (either endogenously or due to policy shocks), while the macroeconomy moves away from steady state, household wealth does not change. Without firm borrowing, this implies that arbitrageurs continue to maintain zero positions, and thus risk prices do not move. Thus, the dynamics are equivalent to a representative agent model.

However, even without firm borrowing and with an equalized wealth distribution, the predictions of the model differ from that of a representative agent model when considering unconventional policy. First, because an increase in asset purchases by the central bank reduces the holdings of arbitrageurs on impact, required risk compensation falls. Thus, holding fixed the short-term interest rate, the "effective" borrowing rate faced by households falls as well, which is stimulative. But the decline in borrowing rates is larger for households who borrow with long-maturity assets. Thus, the induced increase in consumption is larger for these households than for households with access to short-maturity bonds. Thus, consumption dispersion $\tilde{c}_t$ becomes non-zero, and the risk-weighted asset positions of households $\tilde{b}_t$ begins to move away form steady state.

Prop. 4 shows that conventional and unconventional policies may be substitutes in the sense that shocks to either tool can boost output, but the transmission mechanisms and implied dynamics following the shocks differ in meaningful ways. The welfare implications will therefore differ as well (which we return to in later sections).

The next Proposition shows that when firms also borrow as part of their production process, the dynamics of the model differ more significantly from a representative agent model.

Proposition 5 (Rigid price dynamics, general case)**.** *Suppose that* $\bar{\beta} > 0$ *so that* $\hat{\sigma} > 0$*. If* $0 < \phi_x < \bar{\phi}_x$ *for some upper bound* $\bar{\phi}_x$*, then following a technology shock which decreases the output gap, both the short-term rate* i_t *and term premia* λ_t *decline on impact. However, for some* $t > \bar{t}$*, term premia* λ_t *eventually turn positive while the short rate remains below steady state. These effects are larger when the central bank also utilizes balance sheet tools (*$\phi_x(\tau) > 0$*).*

The intuition for the results of Prop. 5 can be seen when thinking about the dynamics of the model near risk neutrality. Under risk neutrality, we have that the output gap is an increasing function of natural rates: $x_t \propto v_t$. When $\phi_x < \bar{\phi}_x$, the reaction of firm borrowing in a recession caused by a decline in the natural rate is to reduce borrowing. When $\hat{\sigma} > 0$, this implies that λ_t declines. However, the decline in term premia induces households with long-duration investments to reduce savings (or increase borrowing). This must be funded by arbitrageurs, and eventually implies that arbitrageurs are taking long positions in the bond carry trade, and that these positions are larger for longer maturities.

Thus, our model predicts that unconditionally, decreases in the short-term interest rate are associated with decreases in term premia over short-horizons; but over longer horizons, the reaction of term premia becomes insubstantial or even negative. Moreover, when the central bank utilizes balance sheet tools, the unconditional reaction of term premia over short horizons is larger.

It is important to note, however, that this prediction is an unconditional one: aggregate shocks which induce the central bank to ease policy are associated with initial declines in term premia, but over longer horizons this effect flips. But our model makes different predictions for the *conditional* reaction of term premia to monetary policy *shocks*, as shown in the following Corollary.

Corollary 5.1 (Rigid price dynamics, monetary shocks)**.** *Following a conventional monetary policy shock* $\epsilon_{i,t}$ *which decreases the policy rate* i_t*, the output gap* x_t *increases. If* $0 < \phi_x < \bar{\phi}_x$ *for some upper bound* $\bar{\phi}_x$*, then term premia* λ_t *increase on* *impact.*

The difference between the unconditional and conditional reaction of term premia to movements in the short rate arise because of the endogenous movements of quantities in asset markets. Unconditionally, in this version of the model, recessions imply a reduction in firm borrowing and therefore a decline in risk compensation required by arbitrageurs on impact. Simultaneously, the short-term interest rate declines due to the endogenous reaction of the policy rule. Because in this one risk-factor model, all stochastic dynamics of the model are due to natural rate (technology) shocks, the unconditional correlation between the policy rate and term premia is positive. However, because an unexpected decline in the short-term interest rate is expansionary, the correlation between term premia and the short-term interest rate *conditional on a monetary policy shock* is negative.

Further, the endogenous dynamics of the model induced by slow-moving changes in the relative wealth distribution across households implies that these predictions change when looking over longer horizons. Finally, the quantitative predictions change in version of the model where the central bank is also pursuing balance sheet policies.

4 Empirical Evidence

In order to test our predictions from Section 3, we utilize a similar empirical methodology to Hanson et al. (2021). Rather than attempting to decompose the observed movements in the yield curve into expectation and term premia components (which involves imposing additional structure on the observable data), we analyze the dynamics of the term structure of forward rates. After documenting a number of stylized facts about the dynamics of forward rates, in Section 4.3 we relate these to our model predictions.

We run regressions of the form

$$f_{t+h}^{(\tau)} - f_{t-1}^{(\tau)} = \alpha(\tau) + \beta(\tau) D_t + \epsilon_t(\tau), \tag{47}$$

where $f_t^{(\tau)}$ are τ-year ahead forward rates. Our time frequency is daily, and so $f_{t+h}^{(\tau)} - f_{t-1}^{(\tau)}$ is the h-day-ahead horizon change in forward rates on date t. We explore different choices of right-hand-side variable D_t, as well as maturity τ and horizon h.

Our first set of *unconditional* regressions sets the right-hand-side variable as $D_t =$

$y_t^{(1)} - y_{t-1}^{(1)}$, where $y_t^{(1)}$ is the 1-year spot yield on day t. Thus, our right-hand-size variable is the one-day change in short-term (1-year) yields. We estimate (47) on all days in our sample.[9]

The second set of *conditional* regressions uses high-frequency measures of monetary policy shocks in place of D_t. Our baseline uses the shock series from Nakamura and Steinsson (2018), but results are similar for other choices.

We compare the estimate for $h = 0$ and $h > 0$ (our baseline uses 90-day-ahead regressions, but results are very similar for other longer horizon choices). We use the daily nominal forward and spot yields from Gürkaynak et al. (2007) in all our regressions.[10] In addition to estimating (47) across our entire sample, we also conduct rolling regressions using 5-year windows.

4.1 Forwards and Spot Yields, Unconditional

Figure 1 reports our unconditional results using daily changes in the 1-year spot yield. We report the unconditional estimates of (47) over the entire term structure; the x-axis ranges over maturities $\tau = 2$ to $\tau = 20$ years, and hence each point reports a different estimated slope coefficient. The line labeled "short-horizon" reports estimates using one-day ($h = 0$) changes in forward rates, while the "long-horizon" estimates are for three-month ($h = 90$) changes.

First, we see that over short horizons, the slope coefficient is economically large and significant across the entire term structure. Even for very long-maturity forward rates, a 100 basis point increase in the 1-year spot yield is associated with over 40 basis point increase in the $\tau = 20$-year forward rate.

However, the pattern changes significantly when moving towards longer-horizon differences. We still find that the daily change in 1-year spot yields is associated with large increases in short-horizon forward rates even three months later (if anything, the

[9]Our sample begins in 1982, the date at which high-quality estimates of longer maturity spot and forward yields (up to 20 years) are available. We end the sample in 2020 to avoid including the COVID period in our estimates, although results are similar if we include the COVID period.

[10]We use nominal yield curve data because high-quality estimates are available for both short- and long-maturity spot and forward yields starting since the early 1980s (and even earlier for shorter maturities). In contrast, Treasury Inflation-Protected Securities (TIPS) were only introduced in the late 1990s, and have significantly lower trading volume than nominal Treasury securities; high-quality real yield curve estimates are at best available starting in the mid-2000s, but this creates further issues as the majority of the estimation sample will take place during the zero lower bound period.

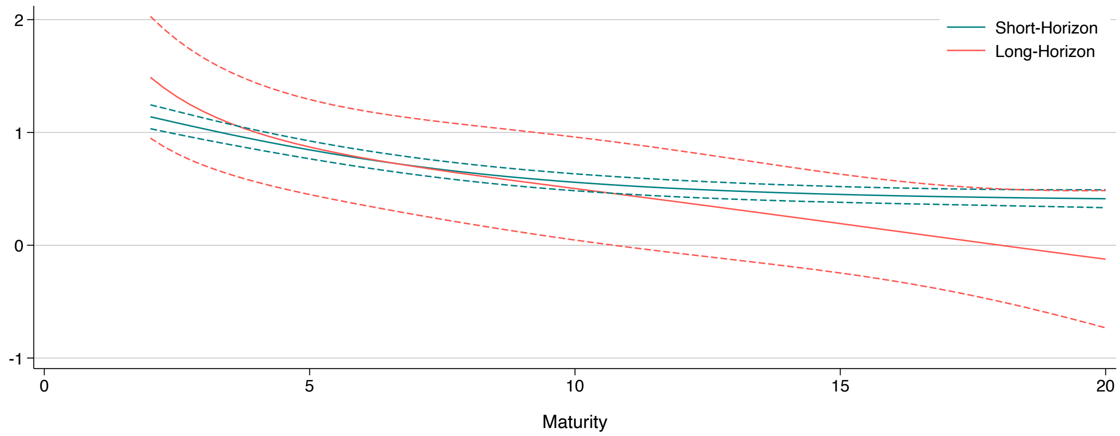

Figure 1: Forward Rates (Unconditional, Varying Maturities)

Notes: estimates from (47) across the term structure; unconditional specification. The estimates labeled short-horizon use $h = 0$ (one day changes), while the estimates labeled long-horizon use $h = 90$ (three month changes). The x-axis ranges over maturities $\tau = 2$ to $\tau = 20$ years. 95% Newey-West confidence intervals are included.

point estimate is larger for short maturities when using three-month-ahead changes); but the associated change in long-maturity forward rates ($\tau = 10$ or greater) becomes statistically insignificant, and the point estimates become smaller. For very long maturity forwards, the point estimate becomes negative (though statistically indistinguishable from zero).

Figure 2 studies the behavior of 2-year and 20-year forward rates (the estimates labeled short-maturity and long-maturity, respectively) in more detail across different horizons. The x-axis varies from $h = 0$ to $h = 150$, which corresponds to estimates of (47) using 1-day changes to roughly 5-month changes. We see that unconditionally, a daily increase in 1-year spot yields is associated with a roughly one-to-one increase in 2-year forward rates over all horizons. However, 20-year forward rates respond significantly only in the first month, and point estimates become negative after about two months (though not significantly so).

Figures 3 and 4 repeat the unconditional regressions for $\tau = 20$-year forwards, but over rolling (5-year) windows. Figure 3 uses short-horizon ($h = 0$) changes, while Figure 4 uses long-horizon ($h = 90$) changes. Consistent with our findings over the full sample, we see that over short horizons, unconditionally forward rates move in the same direction as short-term yields, and the co-movement between the two is

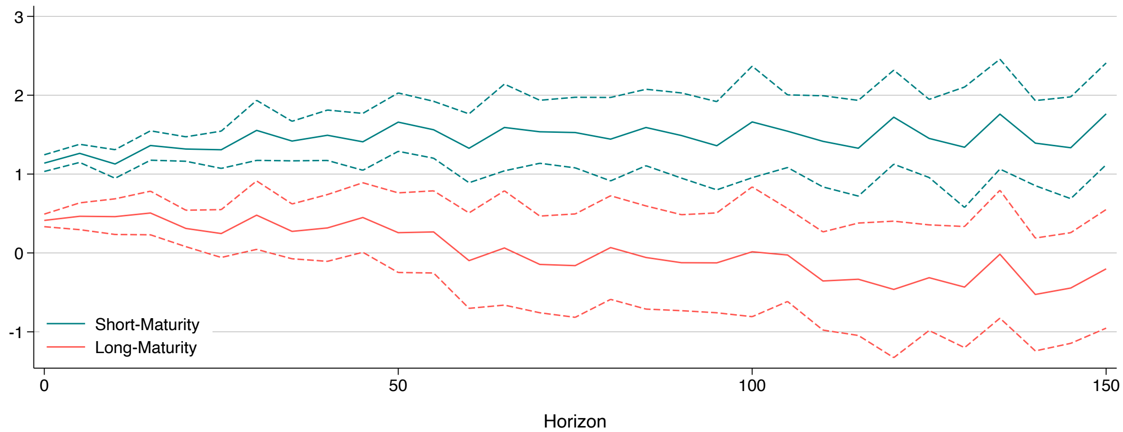

Figure 2: Forward Rates (Unconditional, Varying Horizon)

Notes: estimates from (47) across the different horizons; unconditional specification. The estimates labeled short-maturity use $\tau = 2$ (2-year forwards), while the estimates labeled long-maturity use $\tau = 20$ (20-year forwards). The x-axis ranges over horizons $h = 0$ to $h = 150$ (1-day to 5 months). 95% Newey-West confidence intervals are included.

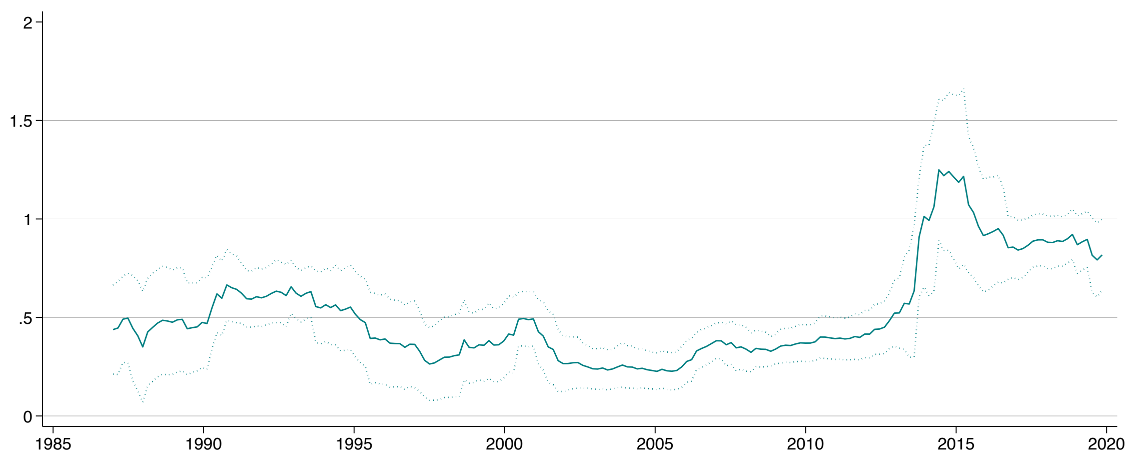

Figure 3: 20-year Forwards (Unconditional, Rolling Short Horizon)

Notes: rolling estimates from (47) for $\tau = 20$-year forward rates; unconditional specification, using short-horizon ($h = 0$) changes. The rolling window is five years; the x-axis includes the end date of each rolling estimate. 95% Newey-West confidence intervals are included.

strong. In addition, the time-variation in our estimates observed in Figure 3 also reveals interesting properties. The unconditional relationship between long-maturity forward rates and short-maturity spot yields becomes the strongest in the end of

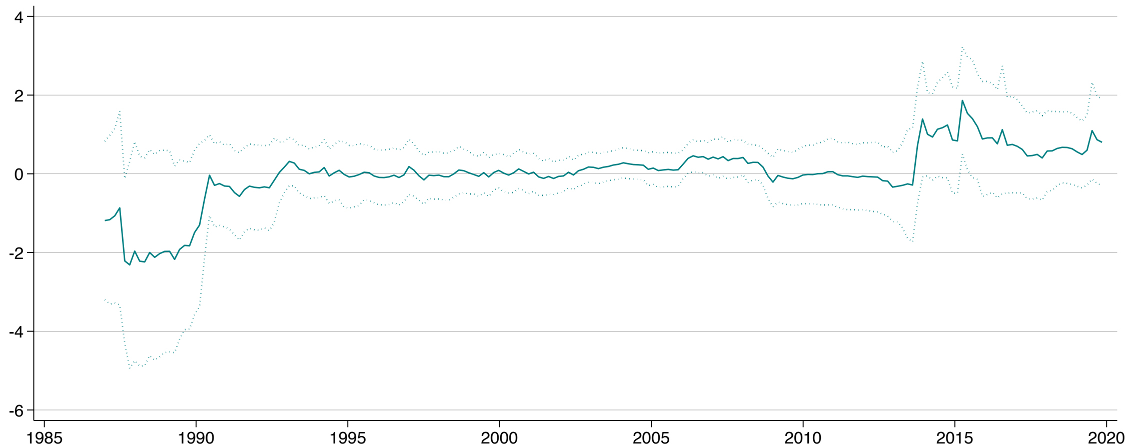

Figure 4: 20-year Forwards (Unconditional, Rolling Long Horizon)

Notes: rolling estimates from (47) for τ = 20-year forward rates; unconditional specification, using long-horizon ($h = 90$) changes. The rolling window is five years; the x-axis includes the end date of each rolling estimate. 95% Newey-West confidence intervals are included.

the sample. The point estimates jump sharply when the sample includes the years 2009-2014, and even become greater than one (though the confidence intervals include values less than one as well).

Figure 4 confirms that over all different sub-samples, long horizon movements in forward rates are much less tied to short-horizon movements in short rates. The estimates are virtually always statistically insignificant, and besides the sample period starting in 2009, the point estimates are economically small and frequently turn negative as well.

4.2 Forwards and Spot Yields, Monetary Shocks

We now turn to our conditional regressions, where we estimate (47) using identified monetary policy shocks. Figure 5 reports our estimates over the entire sample using short-horizon ($h = 0$) changes.[11] Under the assumption that the high-frequency approach to identifying monetary shocks is valid, we can interpret the estimates in this section causally. We see that a contractionary monetary policy shock causes an

[11]Results are similar but significantly more noisy for long-horizon regressions. Note that because the conditional specification requires identified monetary shocks, we have significantly fewer observations than the unconditional specification.

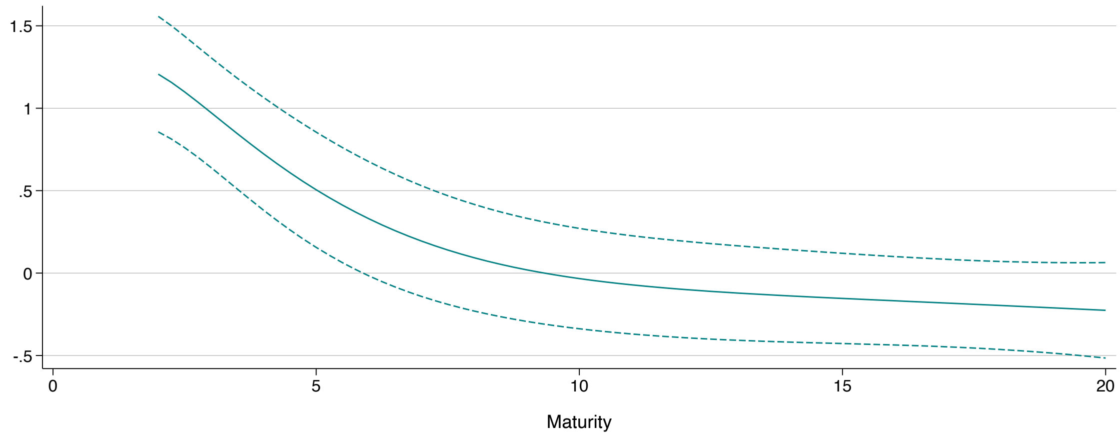

Figure 5: Forwards Rates (Shocks, Varying Maturity)

Notes: estimates from (47) across the term structure; conditional monetary shock specification. We report estimates using $h = 0$ (one day changes) across the term structure. The x-axis ranges over maturities $\tau = 2$ to $\tau = 20$ years. 95% Newey-West confidence intervals are included.

increase in short-maturity forward rates on impact. However, in contrast with our unconditional findings, the transmission of monetary shocks to intermediate maturity forwards is insignificant (for forward rates with maturity of roughly 5 years onward). Moreover, the point estimate becomes negative for 10-year forward rates, and continues to decline as we move further out in the term structure. The point estimate for $\tau = 20$-year maturities is roughly -0.25, and significantly different from zero at the 90% level (though not at the 95% level).

Finally, Figure 6 repeats the analysis from Figure 5, but across rolling windows. We focus on long-maturity ($\tau = 20$-year) forward rates. Across all sub-samples, the point estimate is virtually always statistically insignificant. Additionally, point estimate are economically small, and if anything are typically negative (although using rolling regressions with identified monetary shocks implies that each sub-sample has only a modest number of observations).

4.3 Comparison with Model Predictions

In order to link our model-based predictions with our empirical findings, we need to translate our results regarding forward rates to those regarding term premia. Rather than attempting to extract components corresponding to expected path and risk com-

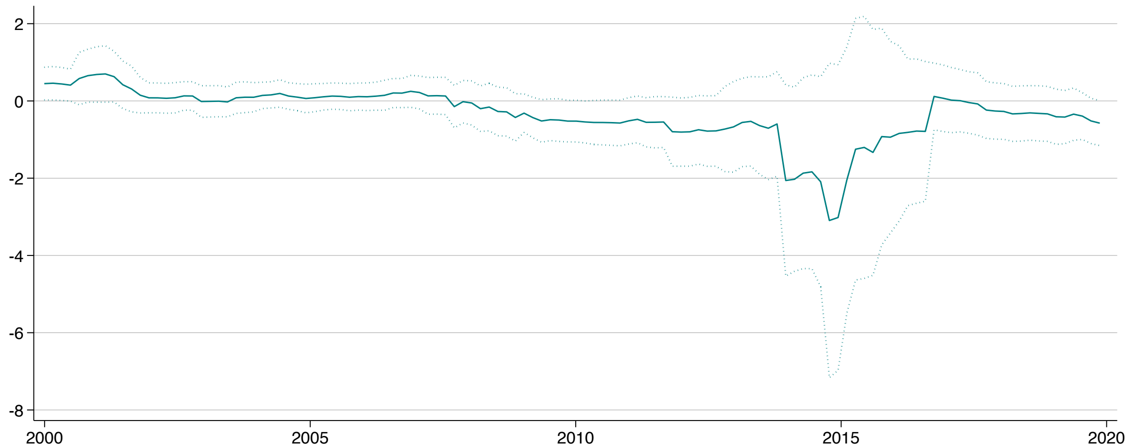

Figure 6: 20-year Forwards (Shocks, Rolling)

Notes: rolling estimates from (47) for τ = 20-year forward rates; conditional monetary shock specification, using short-horizon ($h = 0$) changes. The rolling window is five years; the x-axis includes the end date of each rolling estimate. 95% Newey-West confidence intervals are included.

pensation, we take a more qualitative approach. Under the assumption that market participants expect the daily changes in short-term rates to have largely reverted to steady state after 10 to 20 years, the movement in long-maturity forward rates mostly reflect fluctuations in term premia. With this interpretation in mind, we discuss our empirical results and how they relate to the predictions of the model.

First, the results regarding short-horizon movements in forward rates in Figure 1 confirms a key prediction of our model: over short horizons, term premia and short rates move together unconditionally. Next, the time-variation in our estimates observed in Figure 3 is consistent with another one of our predictions. During periods when the central bank utilizes balance sheet tools more than the short rate, the unconditional relationship between term premia and short rates becomes stronger. Third, the results regarding longer-horizon movements in forward rates in Figures 1, 2, and 4 confirms another prediction of our model: term premia movements over longer horizons are less reactive to short-horizon movements in short rates unconditionally, and the reaction may turn negative. Finally, Figures 5 and 6 confirm another important and sharp prediction of our model: *conditional on monetary shocks*, the reaction of forward rates is weak, and if anything implies that term premia move in the opposite direction of monetary policy *shocks*.

The stark qualitative difference between the unconditional and conditional reaction of term premia to movements in short-term yields, as well as the state dependence during periods of balance sheet tool usage, is a critical feature of our model. Our empirical findings show that the data is consistent with these key theoretical mechanisms. We next turn to considering counterfactual policy rules and the optimal design of conventional and unconventional policy.

5 Welfare Effects: Simple Policy Rules

Using the results of the previous section, we explore the welfare consequences and optimal design of simple policy rules. For now, we continue as in Section 3 to assume that the only aggregate source of uncertainty is natural rate shocks v_t, and that prices are fully rigid.

We first study simple policy rules which (in equilibrium) are only a function of the "natural" state variables; under the simplifying assumptions considered thus far, these are natural rate shocks v_t (equivalently, technology shocks z_t) and the risk-weighted household wealth distribution $\tilde{b}_t$. Thus, we study policy rules which implement

$$i_t = \chi_{i,v} v_t + \chi_{i,b} \tilde{b}_t, \tag{48}$$

$$s_t(\tau) = \chi_{s,v}(\tau) v_t + \chi_{s,b}(\tau) \tilde{b}_t \implies \tilde{s}_t = \tilde{\chi}_{s,v} v_t + \tilde{\chi}_{s,b} \tilde{b}_t, \tag{49}$$

for some choice of policy parameters $\chi_{i,v}, \chi_{i,b}$ and $\{\chi_{s,v}(\tau), \chi_{s,b}(\tau)\}_{\tau=0}^{T}$. We assume that such a policy is implementable; determinacy conditions can be satisfied through a Taylor rule like those considered in Section 3. Note that in this context with a single priced risk factor, for the same reasons as discussed in Section 2 the impact of balance sheet rules across the term structure can be fully summarized by the risk-weighted balance sheet object $\tilde{s}_t$, up to considerations of policy frictions $\psi^{(\tau)}$; we return to this point below.

The policymaker chooses policy parameters in order to minimize unconditional social loss

$$\mathcal{W} \equiv \frac{1}{2} \mathbb{E} \int_0^\infty e^{-\rho t} \mathcal{L}_t \, \mathrm{d}t \,,$$

where per-period social welfare loss $\mathcal{L}_t$ is derived in (33)-(35). Thus, we consider a policymaker who is able to commit to simple policy rules, which are functions of the natural state variables of the economy. In the next section, we explore optimal policy for a policymaker who conducts (history-dependent) policy under full commitment.

5.1 Optimal Policy: Short Rate Only

First, consider the benchmark case of a risk neutral arbitrageurs: $a = 0$. Then the expectations hypothesis holds, so regardless of the asset positions of households and firms (or central bank balance sheet policies), borrowing rates are equalized: $\mu_t(\tau) = i_t$. Moreover, as long as initial household wealth is equalized, the model collapses to the standard RANK case. In particular, consumption and wealth dispersion disappear:

$$\tilde{b}_t = 0, \quad \mathrm{Var}_\tau\, \check{c}_t(\tau) = 0 \;\; \forall t.$$

Because inflation is zero due to rigid prices (or more generally, when divine coincidence holds), the conventional policy rule with $\chi_{i,v} = 1$ which implements $i_t = v_t$ achieves first-best: $x_t = 0$ for all periods t. We further recover the standard QE neutrality result: balance sheet policies do not affect bond prices (and therefore have no aggregate effects).

Under risk-neutrality, so long as the central bank faces no costs to setting the policy rate, first-best is achievable with only conventional policy. The next Proposition shows that these results fail whenever arbitrage is imperfect ($a > 0$); this is true even if absent the consumption risk-sharing motive. We derive optimal short-rate policy when the central bank does not have access to balance sheet tools (formally, when balance sheet costs $\psi^{(\tau)} \to \infty$).

Proposition 6 (Optimal short rate policy rule)**.** *Assume risk aversion $a > 0$, $\tilde{\sigma} > 0$, and $\bar{\beta}\hat{\sigma} > 0$. If bond risk dispersion across households $\tilde{\Sigma} = 0$, then some $\chi^n_{i,v} \lesseqgtr 1$ and $\chi_{i,b} = 0$ in (48) guarantees $x_t = 0$ each period. The sign of $\chi^n_{i,v} - 1$ is determined by the endogenous reaction of firm borrowing to natural rate shocks.*

With $\tilde{\Sigma} > 0$, the optimal short rate policy rule is given by $i_t = \chi^*_{i,v} i_t + \chi_{b,i}\tilde{b}_t$ where $\chi_{b,i} \neq 0$ and $\chi^*_{i,v} < \chi^n_{i,v}$. The optimal short rate policy implies:

1. *Bond carry trade returns $\mu_t(\tau) - i_t$ move in the same direction as i_t iff firm borrowing declines in response to natural rate shocks.*

2. *Output gaps* x_t *are not identically zero.*

3. *Consumption dispersion is non-zero:* $\text{Var}_\tau\, \check{c}_t(\tau) \neq 0$.

We can understand this result using the intuition derived in the previous section. Consider a fall in the natural rate inducing a cut in the policy rate. If firm borrowing declines, then arbitrageurs reduce their position in the bond carry trade, which requires a smaller bond carry trade return. On the other hand, if instead firm borrowing increase, through market clearing, bond arbitrageurs increase their position in the bond carry trade, which requires a larger bond carry trade return.

Because of this, risk premia vary over time. Thus, a simple short rate policy is unable to equalize all borrowing rates. If the risk-weighted dynamics of household wealth $\tilde{b}_t$ are negligible, then the central bank can choose $i_t = \chi^n_{i,v} v_t$, which implies that $\tilde{\mu}_t = v_t$ (and therefore closes output gaps). However, fluctuations in the natural rate induce volatile fluctuations in the policy rate. Ito's lemma implies

$$\mathrm{d}i_t = -\kappa_z i_t \,\mathrm{d}t + \chi^n_{i,v}\sigma_r \,\mathrm{d}B_{z,t}\,,$$

hence the volatility of short rate changes is increasing in the responsiveness of conventional policy to natural rates. More volatile short rates implies greater variation in term premia. Individual Euler equations differ, which implies that consumption choices across the τ households $c_t(\tau) \neq c_t(\tau')$. Therefore, consumption dispersion $\text{Var}_\tau\, \check{c}_t(\tau) \neq 0$. Thus, from a risk-sharing perspective, choosing a policy rule which is less reactive to natural rate fluctuations is welfare-improving.

Of course, if the short rate becomes less responsive to natural rates, output gap volatility increases. For very small choices of $\chi_{i,v}$, consumption dispersion becomes negligible. However, output gap volatility increases substantially. The optimal choice of $\chi^*_{i,v}$ balances these forces. At the optimum, shocks to the natural rate do not fully pass through to the effective borrowing rate: $\tilde{\mu}_t \neq v_t$. Thus, aggregate borrowing demand changes, and hence the output gap $x_t \neq 0$.[12]

Finally, when the risk-weighted dynamics of household wealth $\tilde{b}_t$ are non-negligible, it is not possible for the central bank to choose any $i_t = \chi^n_{i,v} v_t$, which implies that $\tilde{\mu}_t = v_t$. Because fluctuations in $\tilde{b}_t$ affect term premia, then even conditional on the

[12]Also, note that whenever prices are not fully rigid, this induces fluctuations in inflation through the Phillips curve, and so $\pi_t \neq 0$ as well.

level of the natural rate, term premia will fluctuate through the endogenous movements in household wealth. Thus, the optimal policy in this case must place some weight on risk-weighted household wealth $\tilde{b}_t$.

5.2 Optimal Short Rate and Balance Sheet Policy

The failure of conventional policy to achieve first-best is driven by three frictions. First, because of imperfect pass-through of the policy rate to household borrowing rates, natural rate shocks are not fully accommodated, implying excessive volatility of output. Second, borrowing rates across households differ, implying suboptimal consumption dispersion. Third, the dispersion in consumption decisions induces additional equilibrium movements in relative bond positions of households, leading to further fluctuations in term premia. Simple conventional policy rules cannot overcome all frictions simultaneously. However, when the central bank has access to frictionless balance sheet policies, we obtain the following result.

Proposition 7 (Optimal policy separation principle)**.** *Assume risk aversion $a > 0$, $\tilde{\sigma} > 0$, and $\bar{\beta}\hat{\sigma} > 0$. Further, assume that household wealth is initially equalized. Suppose the central bank implements short rate and balance sheet policy according to*

$$i_t = v_t, \quad \tilde{s}_t = -\frac{\bar{\beta}\hat{\sigma}}{\varsigma\kappa_z} v_t.$$

If short rate policy is frictionless ($\psi^i = 0$) and the central bank does not face holding costs ($\psi^{(\tau)} = 0$), then first-best is achieved:

1. *Macroeconomic stabilization: $x_t = 0 \; \forall t$.*

2. *Term premia stabilization: $\mu_t(\tau) = \tilde{\mu}_t \; \forall \tau$.*

3. *Consumption equalization: $c_t(\tau) = c_t(\tau') \; \forall \tau, \tau'$.*

The results follow naturally from our findings regarding ad-hoc policy. The balance sheet policy in Prop. 7 stabilizes shocks to bond markets by offsetting all firm borrowing shocks. The policy implies that risk premia are zero and thus equalizes borrowing rates across households: $\mu_t(\tau) = \tilde{\mu}_t$. When this is the case, the model collapses to a standard RANK model. The elimination of risk premia implies that policy shocks are transmitted one-to-one to borrowing rates, so $\tilde{\mu}_t = i_t$. Thus, the policy $i_t = v_t$ implies output gaps $x_t = 0$ which is optimal as inflation is zero by assumption (but will also be optimal in general when divine coincidence holds).

Thus, we derive an optimal *separation principle* for optimal policy: optimal balance sheet policy *stabilizes long-term rates* while optimal short rate policy *stabilizes macroeconomic aggregates.*

5.3 Optimal Policy: Balance Sheet Only

We now derive the optimal use of balance sheet tools when the central bank faces constraints on the short rate.[13] To capture the essence of short rate constraints in a simple way, we assume that $\psi^i \to \infty$ and assume a simple process for the "target rate" $\bar{i}_t$ in (17). The next Proposition shows that balance sheet tools alone are not enough to achieve first-best.

Proposition 8 (Optimal balance sheet rule)**.** *Assume risk aversion $a > 0$, $\tilde{\sigma} > 0$, and $\bar{\beta}\hat{\sigma} > 0$. Further, assume that household wealth is initially equalized. Suppose that $\psi^i \to \infty$ and that $\bar{i}_t = \tilde{\chi}_i v_t$ where $0 < \tilde{\chi}_{i,v} \ll 1$. Consider the balance sheet rule which implements*

$$\tilde{s}_t = \bar{\beta}\hat{\sigma}\left((1+\varsigma+\varphi)x_t - \frac{1}{\varsigma\kappa_z}v_t\right).$$

This policy equalizes borrowing rates ($\mu_t(\tau) = \tilde{\mu}_t\ \forall\tau$), and consumption and wealth dispersion are zero: $\mathrm{Var}_\tau\, \check{c}_t(\tau) = 0$ and $\tilde{b}_t = 0$. However, output gaps x_t are no longer identically zero.

There exists some balance sheet policy parameters $\tilde{\chi}_{s,v} \neq 0, \tilde{\chi}_{s,b} \neq 0$ such that $\tilde{\mu}_t = v_t$ which satisfies macroeconomic stabilization ($x_t = 0\ \forall t$). However, long-term bond return stabilization fails ($\exists\tau : \mu_t(\tau) \neq \tilde{\mu}_t$), and therefore consumption and wealth dispersion are non-zero. The optimal balance sheet rule implies:

1. *Output gaps x_t are not identically zero. Relative to no balance sheet policies, output gap and inflation volatility are lower.*

2. *Consumption dispersion is non-zero: $\mathrm{Var}_\tau\, \check{c}_t(\tau) \neq 0$. Relative to no balance sheet policies, consumption dispersion is lower.*

[13] Recall that we do not model an explicit ZLB in order to utilize our linear-quadratic approximation techniques.

Prop. 8 shows that when the short rate is constrained, the optimal balance sheet policy must sacrifice term premia stabilization in order to (partially) stabilize macroeconomic volatility. While balance sheet tools can continue to equalize borrowing rates, suboptimal short rate policy implies that the effective household borrowing rate $\tilde{\mu}_t \neq v_t$. Thus, this policy does not achieve macroeconomic stabilization: $x_t \neq 0$. On the other hand, balance sheet policies alone can close the output gap, but this is also suboptimal because borrowing rates are no longer equalized.

It may be surprising that balance sheet tools are insufficient to simultaneously achieve macroeconomic stabilization and bond market stabilization. Although we have eliminated the short rate as an active policy tool, we have given the policymaker a continuum of balance sheet tools $\{s_t(\tau)\}_{\tau=0}^{T}$. The usual textbook intuition seems to suggest that the policymaker has enough tools to deal with the two objectives.

The intuition is that balance sheet policy works by affecting term premia through changes in the market price of risk. Although arbitrage is imperfect in this model, arbitrageurs still enforce tight restrictions between between the market price of risk and term premia across maturities. Hence, while in principle the central bank has a continuum of policy tools, in practice it can only manipulate λ_t (which we have already imposed by working with the risk-weighted balance sheet object $\tilde{s}_t$).

This is related to the localization results in Vayanos and Vila (2021) and Ray et al. (2024). In the one-factor model considered here, the effects of QE are fully global. However, even with more complicated risk structure, localization is not strong enough to allow balance sheet policy rules alone to achieve first-best.

6 Welfare Effects: Full Commitment

We now study the case of full commitment, where the policymaker can choose short rate and balance sheet policies as a function of the entire history of shocks. The policymaker seeks to minimize the conditional social loss function

$$\mathcal{W}_t = \mathbb{E}_t \int_t^{\infty} \frac{1}{2} e^{-\rho s} \mathcal{L}_s \, \mathrm{d}s \,, \tag{50}$$

where per-period social welfare loss $\mathcal{L}_s$ is derived in (33)-(35). Our optimal separate result in Prop. 7 show that simple policy rules can achieve first-best when the policymaker faces no short rate or balance sheet frictions. However, when policy frictions are non-negligible, full commitment policies can improve on simple policy rules.

Theorem 2 (Optimal policy under full commitment)**.** *Collect the state variables* $\mathbf{y}_t$ *and jump variables* $\mathbf{x}_t$ *into a vector* $\mathbf{Y}_t$ *and policy tools into the vector* $\mathbf{u}_t$*, so that social loss is given by*

$$\mathcal{W}_0 = \frac{1}{2}\mathbb{E}_0 \int_0^{\infty} e^{-\rho t} \left(\mathbf{Y}_t^{\top} \mathbf{R} \mathbf{Y}_t + \mathbf{u}_t^{\top} \mathbf{Q} \mathbf{u}_t \right) \mathrm{d}t \,, \quad \mathbf{y}_0 \text{ given.} \tag{51}$$

The policymaker chooses $\mathbf{u}_t = \mathbf{F}\mathbf{Y}_t$*, which induce equilibrium dynamics*

$$\mathrm{d}\mathbf{Y}_t = -\boldsymbol{\Upsilon}\mathbf{Y}_t \,\mathrm{d}t + \mathbf{S}\,\mathrm{d}\mathbf{B}_t \,, \tag{52}$$

where the feedback matrix $\boldsymbol{\Upsilon} \equiv \boldsymbol{\Upsilon}(\mathbf{F})$ *either explicitly or implicitly depends on the policymaker reaction function. Necessary conditions for optimal choice of* $\mathbf{F}^*$ *are given by*

$$\mathbf{y}_0^{\top} \left(\partial_i \mathbf{P}_{11} - \partial_i \mathbf{P}_{12} \mathbf{P}_{22}^{-1} \mathbf{P}_{21} - \mathbf{P}_{12} \mathbf{P}_{22}^{-1} \partial_i \mathbf{P}_{21} + \mathbf{P}_{12} \left(\mathbf{P}_{22}^{-1} \partial_i \mathbf{P}_{22} \mathbf{P}_{22}^{-1} \right) \mathbf{P}_{21} \right) \mathbf{y}_0 = 0$$

where $\mathbf{P}$ *solves the Lyapunov equation* $\rho \mathbf{P} = \mathbf{R} + \mathbf{F}^{\top}\mathbf{Q}\mathbf{F} - \mathbf{P}\boldsymbol{\Upsilon} - \boldsymbol{\Upsilon}^{\top}\mathbf{P}$*, and* ∂_i *represents the derivative of partitioned elements of* $\mathbf{P}$ *with respect to the* i *element of* $\mathbf{F}^*$*, which solve the Lyapunov equation*

$$\rho \partial_i \mathbf{P} = \mathbf{Q} \partial_i \mathbf{F} + \partial_i \mathbf{F}^{\top} \mathbf{Q} - \partial_i \mathbf{P} \boldsymbol{\Upsilon} - \boldsymbol{\Upsilon}^{\top} \partial_i \mathbf{P} - \mathbf{P} \partial_i \boldsymbol{\Upsilon} - \partial_i \boldsymbol{\Upsilon}^{\top} \mathbf{P}.$$

The equilibrium dynamics of the model are

$$\begin{aligned} \mathrm{d}\mathbf{q}_t &= - \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \boldsymbol{\Upsilon} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1} \end{bmatrix} \mathbf{q}_t \,\mathrm{d}t + \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \mathbf{S}\,\mathrm{d}\mathbf{B}_t \\ &\equiv -\boldsymbol{\Gamma}\mathbf{q}_t \,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t \,, \end{aligned} \tag{53}$$

where $\mathbf{q}_0 = \begin{bmatrix} \mathbf{y}_0 & \mathbf{0} \end{bmatrix}^{\top}$*.*

Full commitment allows for potential improvements in social welfare because the induced dynamics of the economy in (53) are richer than when the central bank follows simple policy rules. This allows the central bank to reduce volatility of interest rate changes, while keeping stronger control over the entire path of the policy rate. Note that

$$\mathrm{d}i_t = -\mathbf{e}_i^\top \boldsymbol{\Gamma} \mathbf{q}_t \,\mathrm{d}t + \mathbf{e}_i^\top \boldsymbol{\sigma} \,\mathrm{d}\mathbf{B}_t \,, \quad \mathbf{e}_i \equiv \mathbf{e}_1^\top \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1} \end{bmatrix} \mathbf{F}^*,$$

where $i_t \equiv \mathbf{e}_1^\top \mathbf{u}_t$. The term $\mathbf{e}_i^\top \boldsymbol{\sigma}$ can be made smaller (inducing smaller term premia in equilibrium), while still allowing for sufficiently rich expected movements of the short rate $\int_0^t i_s \,\mathrm{d}s = \mathbf{e}_i^\top \int_0^t \mathbf{q}_s \,\mathrm{d}s$.

In general, the necessary conditions in Theorem 2 are complicated. Ongoing work explores the implications in more detail with numerical simulations. However, we can still derive some qualitative predictions between simple policy rules and full commitment. Intuitively, the tradeoff for monetary policy is that greater pass-through to households (through more aggressive policy reactions to shocks) comes at the cost of larger and more volatile term premia. Full commitment partially relaxes this link. Household decisions depend on entire expected path of borrowing rates $\int_0^\infty \mu_t(\tau)\,\mathrm{d}\tau$, whereas arbitrageur risk compensation depends on volatility of short-run fluctuations $\mathrm{d}i_t \,, \mathrm{d}s_t(\tau)$. The optimal policy under full commitment exploits this in equilibrium by conditioning policy not only the set of natural state variables, but also on past promises which allows for smoother policy reactions.

7 Extensions and Tests

We consider extensions of the baseline model studied in the previous section.

Demand Risk. First, we obtain identical results when allowing for shocks to firm bond demand: $\beta_t^{(\tau)}$ is now an additional stochastic demand factor. The optimal separation principle still holds with $\psi^{(\tau)} = 0$, but QE must be more reactive. So long as the optimal balance sheet policy is implemented, the optimal short rate policy still implements $i_t = v_t$.

Exogenous Cost-Push Shocks. Next, we consider the model which exogenous cost-push shocks, which implies that divine coincidence does not hold:

$$\mathrm{E}_t \,\mathrm{d}\pi_t = (\rho\pi_t - \kappa x_t - u_t)\,\mathrm{d}t\,.$$

For now, we abstract from endogenous cost-push shocks induced by firm borrowing. For simple time-consistent policies considered in Section 5, our separation principle still holds when policy frictions are absent. Unfortunately, this implies that the first-best is not achievable. Optimal balance sheet policy still stabilizes term premia, which implies that short rate policy must contend with the output gap and inflation tradeoffs as is standard. The reason is that despite multiple policy instruments, (un)conventional policy only affects aggregate outcomes through changes in the effective borrowing rate $\tilde{\mu}_t$. Formally, taking any feasible path $\{x_t, \pi_t, \tilde{\mu}_t\}_t$ from an implementation implying policies $\left\{\tilde{i}_t, \tilde{S}_t^{(\tau)}\right\}_t$, this can also be achieved with a policy which equalizes borrowing rates. This guarantees $\operatorname{Var}_\tau c_t(\tau) = 0$ and hence strictly dominates.

However, when firms face financing frictions, under the assumption that $\theta(\tau) \neq \eta(\tau)$, a combination of short rate and balance sheet tools can be used to simultaneously offset cost-push shocks and natural rate shocks. However, this policy leads to large and volatile term premia, and so is suboptimal from a welfare perspective.

Non-Zero Term Premia in the First-Best. Our approximation approach thus far implies that in the first-best, expected carry trade returns are zero. This arises endogenously in our model but is based on our riskless approximation method. While this simplifies our analytical results, it is nevertheless a strong assumption. Suppose instead that first-best bond carry trade returns are given by some (exogenous) $\nu^{(\tau)} \neq 0$. We find that our separation principle still holds when $\nu^{(\tau)}$ is achievable, but optimal short rate policy is a function of $\nu^{(\tau)}$. The intuition is a combination of previous results. Aggregate outcomes arise through changes in the effective borrowing rate $\tilde{\mu}_t$ (as before). Optimal balance sheet policy guarantees $\mu_t(\tau) - i_t \equiv \nu^{(\tau)}$ and hence $\tilde{\mu}_t = i_t + \int_0^T \eta(\tau)\nu^{(\tau)}\,\mathrm{d}\tau \equiv i_t + \tilde{\nu}$. Thus, optimal short rate policy implements $i_t = v_t - \tilde{\nu}$.

7.1 Measuring Policy Objectives: Return Predictability

We now consider simple observable tests related to the optimality of balance sheet policies. We consider the bond return predictability regressions of Fama and Bliss

(1987) (FB):

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_t^{(\tau)}}\right) - y_t^{(\Delta\tau)} = a_{FB}^{(\tau)} + b_{FB}^{(\tau)}\left(f_t^{(\tau-\Delta\tau,\tau)} - y_t^{(\Delta\tau)}\right) + \varepsilon_{t+\Delta\tau}.$$

These regression coefficients measure how the slope of the term structure predicts excess returns. Figure 7 shows the well-known result that (in the full sample), Fama-Bliss coefficients are positive, and an increasing function of maturity.

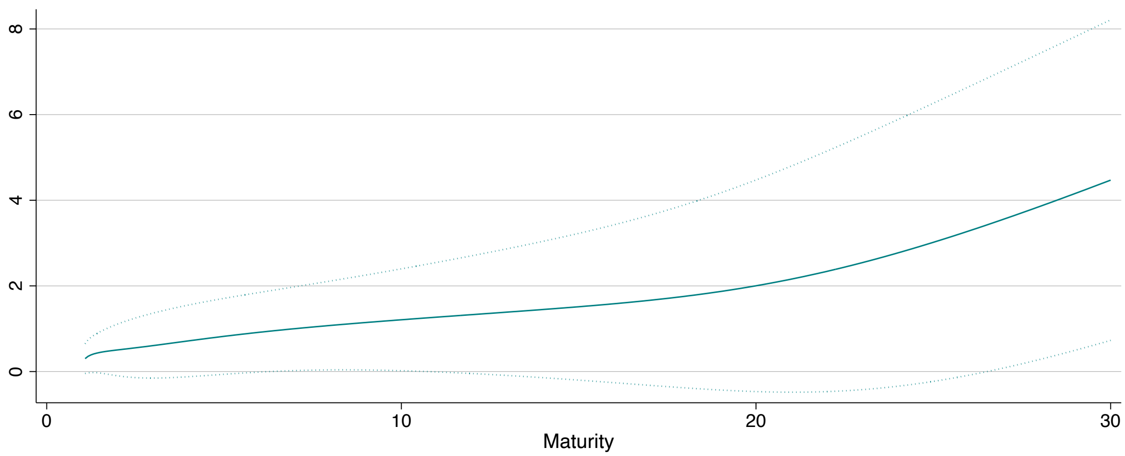

Figure 7: Fama-Bliss Coefficients: Treasuries, Full Sample

In our model, when the central bank does not use balance sheet policies:

$$\bar{b}_{FB}^{(\tau)} > 0.$$

However, if balance sheet policy is successfully pursuing term premia stabilization, then

$$\bar{b}_{FB}^{(\tau)} > b_{FB}^{(\tau)} \to 0.$$

But instead if balance sheet policy is pursuing macroeconomic stabilization, we have

$$b_{FB}^{(\tau)} > \bar{b}_{FB}^{(\tau)}.$$

We examine these stylized predictions by studying how FB regression coefficients vary as a function of different monetary policy regimes.

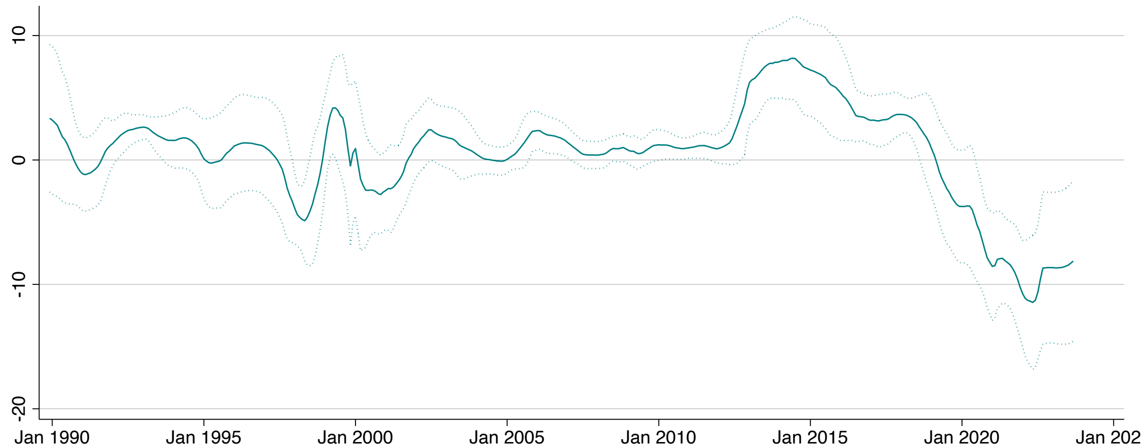

Figure 8: Fama-Bliss Coefficients: 10-year Treasuries, Rolling Sample

Figure 8 estimates rolling Fama-Bliss coefficients (fixing $\tau = 10$ as a baseline maturity). We see that Fama-Bliss coefficients increased substantially during the initial QE regime, but have fallen and even become negative in recent years. This is consistent with the predictions of the model if QE was initially undertaken for purely macroeconomic stabilization purposes. However, these findings imply that in the most recent years since conventional policy has become unconstrained, the mix of conventional and unconventional tools is not optimal.

8 Concluding Remarks

This paper develops a tractable general equilibrium model with market segmentation, nominal frictions, and household heterogeneity. We first derive results for how monetary policy (conventional and unconventional) affects aggregate dynamics. We next show that welfare losses arise from excessive volatility in inflation, output, and excess consumption dispersion across households. The frictions associated with consumption dispersion arise due to market segmentation and excessive volatility of term premia. Optimal short rate and balance sheet policies are characterized by a sharp separation result: conventional policy targets macroeconomic stability, while unconventional policy stabilizes bond returns. Optimal policy removes excess volatility of term premia and hence improves risk-sharing across households, while reducing excess macroeconomic volatility. Policy constraints on either the short rate or balance sheets imply tradeoffs between these two policy objectives.

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A Proofs

Proof of Proposition 1. With $\xi \to 0$, for any τ we have $\boldsymbol{\sigma}_t(\tau) \to \mathbf{0}$, and with perfect arbitrage we have $\mu_t(\tau) = i_t$. Thus $\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} = i_t\,\mathrm{d}t$ and so each household faces the identical budget constraint. Thus, if $\mathcal{B}_0(i) = \mathcal{B}_0\ \forall i \in \mathcal{H}$, each household solves the same problem and chooses $C_t(i) = C_t$ and $N_t(i) = N_t$. Flexible prices imply

$$C_t = Y_t = Z_t N_t, \quad W_t = \frac{Z_t M_t}{1 + i_t}.$$

Suppose the production subsidy is given by

$$\tau^y = \frac{\epsilon}{\epsilon - 1}(1 + \varrho) - 1, \tag{A1}$$

and in the flexible price, perfect arbitrage equilibrium the central bank implements a nominal interest rate peg: $i_t = \varrho$. Then it follows that $W_t = Z_t$ and $Y_t = Z_t^{\frac{1+\varphi}{\varsigma+\varphi}}$, the first-best level of production.

Further, suppose that in steady state we have $\bar{i} = \varrho$ which guarantees $\bar{\pi} = 0$; suppose further that $\bar{s}(\tau) = \theta(\tau) - \eta(\tau)\bar{b}$, which implies $\bar{x}(\tau) = 0$. Then $\bar{W} = \bar{Z}$ and $\bar{\mu}(\tau) = \varrho$, $\bar{Y} = \bar{C} = \bar{C}(\tau)$, and $\bar{b}(\tau) = \bar{b}$ for all τ. Then the steady state values in the first-best and under sticky prices and imperfect arbitrage coincide. $\square$

Proof of Proposition 2. Since asset prices are affine functions of the state, which evolves according to (46), Ito's Lemma implies that the drift and diffusion in (24) are given by $\boldsymbol{\sigma}_t(\tau) = -\mathbf{A}(\tau)^\top \boldsymbol{\sigma}_x$ and

$$\mu_t(\tau) = \mathbf{A}'(\tau)^\top \mathbf{y}_t + C'(\tau) + \mathbf{A}(\tau)^\top \boldsymbol{\Gamma}\left(\mathbf{x}_t - \bar{\mathbf{y}}\right) + \frac{1}{2}\mathbf{A}(\tau)^\top \boldsymbol{\Sigma}\mathbf{A}(\tau), \tag{A2}$$

where $\boldsymbol{\Sigma} \equiv \boldsymbol{\sigma}_x \boldsymbol{\sigma}_x^\top$. The arbitrageur optimality conditions (16) become

$$\mu_t(\tau) - i_t = a\left[\int_0^T x_t(\tau')\mathbf{A}(\tau')\,\mathrm{d}\tau'\right]^\top \boldsymbol{\Sigma}\mathbf{A}(\tau).$$

Substitute market clearing conditions for $x_t(\tau)$ and collect terms that are linear in the state $\mathbf{x}_t$ to get

$$\mathbf{A}'(\tau) + \mathbf{M}\mathbf{A}(\tau) - \mathbf{e}_i = \mathbf{0}, \tag{A3}$$

where $\mathbf{M}$ is given by equation (39). Taking $\mathbf{M}$ as given, this is a linear system of differential equations. To derive initial conditions, note that at maturity, the riskless bonds pay \$1 so the $\tau = 0$ prices are given by $P_t^{(0)} = 1$. Hence, we have $\mathbf{A}(0) = \mathbf{0}$. Then assuming $\mathbf{M}$ is diagonalizable and invertible, the solution is given by (38). $\square$

Proof of Proposition 3. Collect all state and jump variables in a vector $\mathbf{Y}_t = \begin{bmatrix} \mathbf{x}_t^\top & \mathbf{y}_t^\top \end{bmatrix}^\top$. The interest rate process i_t and habitat demand factor processes $\beta_t^{(\tau)}$ are all affine functions of $\mathbf{Y}_t$. Moreover, the (linearized) Phillips curve and IS equation are also affine functions of $\mathbf{Y}_t$, since $\hat{\mu}_t$ and $\tilde{\mu}_t$ are also affine in the state variables. Aggregate dynamics can thus be written

$$\begin{bmatrix} \mathrm{d}\mathbf{x}_t \\ \mathrm{E}_t \, \mathrm{d}\mathbf{y}_t \end{bmatrix} = -\boldsymbol{\Upsilon} \left(\mathbf{Y}_t - \bar{\mathbf{Y}} \right) \mathrm{d}t + \begin{bmatrix} \boldsymbol{\sigma}_x \\ \mathbf{0} \end{bmatrix} \mathrm{d}\mathbf{B}_t \,. \tag{A4}$$

Note that $\boldsymbol{\Upsilon}$ depends on $\hat{\mathbf{A}}$, but which we currently take as given. Then the rational expectations equilibrium is found immediately from Buiter (1984). Partition the eigenvalues and eigenvectors as follows:

$$\boldsymbol{\Upsilon} = \mathbf{Q} \boldsymbol{\Lambda} \mathbf{Q}^{-1}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix},$$

where the partitions correspond to the state $\mathbf{x}_t$ and jump $\mathbf{y}_t$ variables. If the number of "stable" eigenvalues (non-negative real parts) equals the number of state variables, then the rational expectations equilibrium dynamics are given by (46), where

$$\boldsymbol{\Gamma} = \mathbf{Q}_{11} \boldsymbol{\Lambda}_1 \mathbf{Q}_{11}^{-1}, \quad \boldsymbol{\Omega} = \mathbf{Q}_{21} \mathbf{Q}_{11}^{-1}. \tag{A5}$$

$\square$

Proof of Theorem 1. In an affine equilibrium we have that $\tilde{\mu}_t = \int_0^T \eta(\tau) \mu_t(\tau) \, \mathrm{d}\tau$. Substituting equations (A2) and (A3) into this expression and collecting terms which are linear in the state $\mathbf{x}_t$ gives the diffusion vectors defined in the statement of the theorem. Equilibrium is the solution of the fixed point problem implicitly defined by equations (39) and these diffusion vectors. Rewrite these conditions in the following function:

$$f(\hat{\mathbf{A}};\mathbf{M};a) = \begin{bmatrix} \mathbf{e}_i + \left(\boldsymbol{\Gamma}(\hat{\mathbf{A}})^\top - \mathbf{M}\right)\boldsymbol{\nu}(\mathbf{M}) - \hat{\mathbf{A}} \\ \operatorname{vec}\left\{\boldsymbol{\Gamma}(\hat{\mathbf{A}})^\top - a\cdot\boldsymbol{\Lambda}(\mathbf{M}) - \mathbf{M}\right\} \end{bmatrix}, \tag{A6}$$

where $\boldsymbol{\Lambda}(\mathbf{M})$ and $\boldsymbol{\nu}(\mathbf{M})$ are the integral terms from equations (39) and the diffusion terms. In both cases, dependence on $\mathbf{M}$ comes through the affine coefficients $\mathbf{A}(\tau)$. We have also made explicit the dependence of $\boldsymbol{\Gamma}$ on $\hat{\mathbf{A}}$, which can be seen in the proof of Lemma 3. If $J \equiv \dim \mathbf{x}_t$, then $\dim \mathbf{M} = J \times J$ and $\dim \hat{\mathbf{A}} = J$ and the function $f : \mathbb{R}^{J(J+1)+1} \to \mathbb{R}^{J(J+1)}$. For any value of a, equilibrium is defined by $f(\hat{\mathbf{A}};\mathbf{M};a) = \mathbf{0}$.

We now analyze the solution in a neighborhood around $a = 0$. For $a = 0$, clearly $\hat{\mathbf{A}} = \mathbf{e}_i$ and $\mathbf{M} = \boldsymbol{\Gamma}(\mathbf{e}_i)^\top$. The partial derivatives evaluated at this point are given by:

$$\frac{\partial f}{\partial \hat{\mathbf{A}}_j} = \begin{bmatrix} \left[\partial\boldsymbol{\Gamma}^\top \big/ \partial\hat{\mathbf{A}}_j\right]\boldsymbol{\nu}(\mathbf{M}) - \mathbf{e}_j \\ \operatorname{vec}\left[\partial\boldsymbol{\Gamma}^\top \big/ \partial\hat{\mathbf{A}}_j\right] \end{bmatrix}, \qquad \frac{\partial f}{\partial \mathbf{M}_{kl}} = \begin{bmatrix} \mathbf{e}_k\mathbf{e}_l^\top\boldsymbol{\nu}(\mathbf{M}) \\ -\operatorname{vec}\mathbf{e}_k\mathbf{e}_l^\top \end{bmatrix},$$

where $\mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l$ are standard normal basis vectors. The matrix $\partial\boldsymbol{\Gamma}\big/\partial\hat{\mathbf{A}}_j$ is the derivative of the state dynamics matrix $\boldsymbol{\Gamma}$ with respect to the j-element of $\hat{\mathbf{A}}$; because this depends on derivatives of the eigendecomposion defined in the proof of Lemma 3, even in the case of $a = 0$ this is a complicated expression. Nevertheless, from this we can show that the Jacobian of f with respect to $\hat{\mathbf{A}}, \mathbf{M}$ evaluated at the $a = 0$ solution has full rank. In fact, writing this Jacobian in block form, we have

$$Df \equiv \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & -\mathbf{I}_{J^2} \end{bmatrix}, \tag{A7}$$

and $\mathbf{D}_{12} = \begin{bmatrix} \mathbf{I}_J \cdot \nu_1 & \dots & \mathbf{I}_J \cdot \nu_J \end{bmatrix}$, where ν_j is the j-element of $\boldsymbol{\nu}(\mathbf{M})$. Because the elementary row operations which transform $\mathbf{D}_{12}$ into the zero matrix simultaneously transform $\mathbf{D}_{11}$ into $-\mathbf{I}_J$, $\det Df = 1$ and the result follows from the implicit function theorem. $\square$

B Model Details

B.1 Household and Firm Problem Details

Households. Ito's lemma implies that real household wealth $b_t(i)$ evolves according to

$$\mathrm{d}b_t(i) = \left[W_t \frac{N_t(i)}{\bar{C}(i)} - \frac{C_t(i)}{\bar{C}(i)}\right]\mathrm{d}t + b_t(i)\left(\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} - \pi_t \,\mathrm{d}t\right) + \frac{\mathrm{d}\mathcal{F}_t}{\bar{C}(i)\mathcal{P}_t} + O(\xi^2)\,\mathrm{d}t \quad \text{(B1)}$$

$$\equiv \mu_t^b(i)\,\mathrm{d}t + \boldsymbol{\sigma}_t^b(i)\,\mathrm{d}\mathbf{B}_t\,, \quad \text{(B2)}$$

where $O(\xi^2)$ represents higher-order terms.

$$\varrho V_t(i) = \max_{\{C_t(i),N_t(i)\}} u\left(C_t(i), N_t(i)\right) + V_{b,t}(i)\mu_t^b(i) + V_{\mathbf{x},t}(i)\mathbf{h}(\mathbf{x}_t) + O(\xi^2). \quad \text{(B3)}$$

Differentiating (B3) with respect to $C_t(i)$ and $N_t(i)$ and equating gives (4). Differentiating with respect to $b_t(i)$ and applying the envelope theorem gives

$$\varrho V_{b,t}(i) = V_{bb,t}(i)\mu_t^b(i) + V_{b\mathbf{x},t}(i)\mathbf{h}(\mathbf{x}_t) + V_{b,t}(i)\left(\mu_t(\tau) - \pi_t\right) + O(\xi^2).$$

Ito's Lemma implies

$$\mathrm{E}_t\,\mathrm{d}V_{b,t}(i) = \left[V_{bb,t}(i)\mu_t^b(i) + V_{b\mathbf{x},t}(i)\mathbf{h}(\mathbf{x}_t) + O(\xi^2)\right]\mathrm{d}t\,.$$

Combining gives

$$\mathrm{E}_t\,\mathrm{d}V_{b,t}(i) = V_{b,t}(i)\left[\varrho + \pi_t - \mu_t^b(i) + O(\xi^2)\right]\mathrm{d}t\,.$$

Firms. Define the relative price of firm j as $P_t(j) = \frac{\mathcal{P}_t(j)}{\mathcal{P}_t}$. Define

$$M_t = \frac{W_t}{Z_t}\left(1 + \int_0^T \theta_t(\tau)\mu_t(\tau)\,\mathrm{d}\tau\right)$$

We have

$$\mathrm{E}_t\,\frac{\mathrm{d}\Pi_t(j)}{\mathcal{P}_t} = Y_t(j)\left[(1+\tau^y)P_t(j) - M_t\right]\mathrm{d}t - \frac{\vartheta}{2}\left(\pi_t(j) - \varpi_t\right)^2 Y_t\,\mathrm{d}t$$

Under flexible prices, per-period maximization and symmetry yields

$$M_t = \frac{\epsilon - 1}{\epsilon}(1+\tau^y).$$

Firm HJB

$$\varrho U_t(j) = \max_{\pi_t(j)} Q_t^{\mathcal{H}} \left[Y_t(j)\left((1+\tau^y)P_t(j) - M_t\right) - \frac{\vartheta}{2}\left(\pi_t(j) - \varpi_t\right)^2 Y_t \right] \tag{B4}$$
$$+ U_{p,t}(j)P_t(j)(\pi_t(j) - \pi_t) + U_{\mathbf{x},t}\mathbf{h}(\mathbf{x}_t) + O(\xi^2). \tag{B5}$$

Optimality conditions

$$\vartheta Q_t^{\mathcal{H}} Y_t \left(\pi_t(j) - \varpi_t\right) = U_{p,t}(j)P_t(j)$$

Envelope theorem:

$$\varrho U_{p,t}(j) = Q_t^{\mathcal{H}}\left[-Y_t \epsilon P_t(j)^{-\epsilon-1}\left((1+\tau^y)P_t(j) - M_t\right) + (1+\tau^y)Y_t(j)\right]$$
$$+ U_{p,t}(j)(\pi_t(j) - \pi_t) + U_{pp,t}(j)P_t(j)(\pi_t(j) - \pi_t) + U_{p\mathbf{x},t}\mathbf{h}(\mathbf{x}_t) + O(\xi^2)$$

Ito's Lemma

$$\mathrm{E}_t\,\mathrm{d}U_{p,t}(j) = \left[U_{pp,t}(j)P_t(j)(\pi_t(j) - \pi_t) + U_{p\mathbf{x},t}\mathbf{h}(\mathbf{x}_t) + O(\xi^2)\right]\mathrm{d}t$$

Combining

$$\mathrm{E}_t\,\mathrm{d}U_{p,t}(j) = \left[\left(\varrho - (\pi_t(j) - \pi)\right)U_{p,t}(j) - Q_t^{\mathcal{H}} Y_t\left((1+\tau^y)(1-\epsilon) + \epsilon P_t(j)^{-1}M_t\right) + O(\xi^2)\right]\mathrm{d}t$$

Evaluated at a symmetric equilibrium we have

$$\vartheta Q_t^{\mathcal{H}} Y_t\left(\pi_t - \varpi_t\right) = U_{p,t}$$
$$\mathrm{E}_t\,\mathrm{d}U_{p,t} = \left[\varrho U_{p,t} - Q_t^{\mathcal{H}} Y_t\left((1+\tau^y)(1-\epsilon) + \epsilon M_t\right) + O(\xi^2)\right]\mathrm{d}t$$

Aggregate Transfers. The short-term bond market clearing condition is

$$\mathcal{S}_t^0 = \int_0^T \mathcal{X}_t(\tau)\,\mathrm{d}\tau\,.$$

Note: this assumes that HH and firm demand for $\tau = 0$ bonds is in measure $\mathrm{d}\tau$. If there is a mass point in $\eta(0)$ or $\theta(0)$, we instead have

$$\mathcal{S}_t^0 = \int_0^T \mathcal{X}_t(\tau)\,\mathrm{d}\tau - \eta(0)\mathcal{B}_t(0) - \theta(0)\mathcal{W}_t N_t.$$

Transfers from aggregate firm profits:

$$\mathrm{d}\Pi_t = \left[\left(1 - \frac{\vartheta}{2}\pi_t^2\right)\mathcal{P}_t Y_t - \mathcal{W}_t N_t\right]\mathrm{d}t - \mathcal{W}_t N_t \left(\int_0^T \theta(\tau)\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}\,\mathrm{d}\tau\right)$$

Transfers from government

$$\int_0^T \mathcal{S}_t(\tau)\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}\,\mathrm{d}\tau + \mathcal{S}_t^0 i_t\,\mathrm{d}t - \mathcal{P}_t Y_t \Psi_t\,\mathrm{d}t$$

Transfers from arbitrageurs

$$\int_0^T \mathcal{X}_t(\tau)\left(\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} - i_t\,\mathrm{d}t\right)\mathrm{d}\tau$$

Combining and imposing market clearing

$$\mathrm{d}\mathcal{F}_t = [\mathcal{P}_t C_t - \mathcal{W}_t N_t]\,\mathrm{d}t - \int_0^T \eta(\tau)\mathcal{B}_t(\tau)\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}\,\mathrm{d}\tau$$

B.2 Linearization Details

For any variable $a_t(\tau)$, define

$$\begin{aligned}
\tilde{a}_t &= \int_0^T \eta(\tau)a_t(\tau)\,\mathrm{d}\tau \\
\check{a}_t(\tau) &= a_t(\tau) - \tilde{a}_t \\
\tilde{\mathbf{a}}_t &= \int_0^T \boldsymbol{\sigma}(\tau)^\top \eta(\tau)\check{a}_t(\tau)\,\mathrm{d}\tau \\
&= \int_0^T \boldsymbol{\sigma}(\tau)^\top \eta(\tau)a_t(\tau)\,\mathrm{d}\tau - \tilde{\boldsymbol{\sigma}}^\top\left(\int_0^T \eta(\tau)a_t(\tau)\,\mathrm{d}\tau\right)
\end{aligned}$$

where

$$\tilde{\boldsymbol{\sigma}} = \int_0^T \boldsymbol{\sigma}(\tau)\eta(\tau)\,\mathrm{d}\tau$$

Log-linearized intratemporal optimality conditions imply

$$w_t = \varsigma c_t(\tau) + \varphi n_t(\tau) \implies \check{n}_t(\tau) = -\frac{\varsigma}{\varphi}\check{c}_t(\tau)$$

Approximation of the budget constraint, using aggregate transfers and a symmetric steady state

$$\begin{aligned}
\mathrm{d}b_t(\tau) &= \left[\frac{\bar{W}}{\bar{Z}}\check{n}_t(\tau) - \check{c}_t(\tau) + \bar{b}(\check{\mu}_t(\tau) - \pi_t) + \varrho\check{b}_t(\tau)\right]\mathrm{d}t + \bar{b}\check{\boldsymbol{\sigma}}(\tau)\,\mathrm{d}\mathbf{B}_t \\
&= \left[-\left(1 + \frac{\varsigma}{\varphi}\frac{\bar{W}}{\bar{Z}}\right)\check{c}_t(\tau) + \bar{b}(\check{\mu}_t(\tau) - \pi_t) + \varrho\check{b}_t(\tau)\right]\mathrm{d}t + \bar{b}\check{\boldsymbol{\sigma}}(\tau)\,\mathrm{d}\mathbf{B}_t \\
\check{\boldsymbol{\sigma}}(\tau) &\equiv \boldsymbol{\sigma}(\tau) - \tilde{\boldsymbol{\sigma}}
\end{aligned}$$

Note

$$\begin{aligned}
\tilde{b}_t = \int_0^T \eta(\tau)b_t(\tau)\,\mathrm{d}\tau \implies \mathrm{d}\tilde{b}_t = -\bar{b}\pi_t\,\mathrm{d}t \\
\implies \mathrm{d}\check{b}_t(\tau) = \left[-\left(1 + \frac{\varsigma}{\varphi}\frac{\bar{W}}{\bar{Z}}\right)\check{c}_t(\tau) + \bar{b}\check{\mu}_t(\tau) + \varrho\check{b}_t(\tau)\right]\mathrm{d}t + \bar{b}\check{\boldsymbol{\sigma}}(\tau)\,\mathrm{d}\mathbf{B}_t
\end{aligned}$$

From the arbitrageur optimality conditions:

$$\begin{aligned}
\mu_t(\tau) = i_t + \boldsymbol{\sigma}(\tau)\boldsymbol{\Lambda}_t \implies \check{\mu}_t(\tau) = \check{\boldsymbol{\sigma}}(\tau)\boldsymbol{\Lambda}_t \implies \tilde{\boldsymbol{\mu}}_t = \tilde{\boldsymbol{\Sigma}}\boldsymbol{\Lambda}_t \\
\tilde{\mu}_t = \int_0^T \eta(\tau)\mu_t(\tau)\,\mathrm{d}\tau = i_t + \left(\int_0^T \eta(\tau)\boldsymbol{\sigma}(\tau)\,\mathrm{d}\tau\right)\boldsymbol{\Lambda}_t = i_t + \tilde{\boldsymbol{\sigma}}\boldsymbol{\Lambda}_t
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\boldsymbol{\Sigma}} &= \int_0^T \boldsymbol{\sigma}(\tau)^\top\eta(\tau)\check{\boldsymbol{\sigma}}(\tau)\,\mathrm{d}\tau \\
&= \int_0^T \eta(\tau)\boldsymbol{\sigma}(\tau)^\top\boldsymbol{\sigma}(\tau)\,\mathrm{d}\tau - \tilde{\boldsymbol{\sigma}}^\top\tilde{\boldsymbol{\sigma}}
\end{aligned}$$

From the household intertemporal optimality conditions:

$$\begin{aligned}
&\mathrm{E}_t\,\mathrm{d}c_t(\tau) = \varsigma^{-1}\left[(\mu_t(\tau)-\varrho)-\pi_t\right]\mathrm{d}t \implies \mathrm{E}_t\,\mathrm{d}\check{c}_t(\tau) = \varsigma^{-1}\check{\mu}_t(\tau)\,\mathrm{d}t \implies \mathrm{E}_t\,\mathrm{d}\tilde{\mathbf{c}}_t = \varsigma^{-1}\tilde{\boldsymbol{\mu}}_t\,\mathrm{d}t\\
&\quad \mathrm{E}_t\,\mathrm{d}x_t = \varsigma^{-1}\left[(\tilde{\mu}_t-\varrho)-\pi_t-(-\kappa_z z_t)\right]\mathrm{d}t
\end{aligned}$$

Then

$$\begin{aligned}
\tilde{\mathbf{b}}_t &\equiv \int_0^T \boldsymbol{\sigma}(\tau)^\top \eta(\tau)\check{b}_t(\tau)\,\mathrm{d}\tau\\
\mathrm{d}\tilde{\mathbf{b}}_t &= \left[-\left(1+\frac{\varsigma}{\varphi}\frac{\bar{W}}{\bar{Z}}\right)\tilde{\mathbf{c}}_t + \bar{b}\tilde{\boldsymbol{\mu}}_t + \bar{i}\tilde{\mathbf{b}}_t\right]\mathrm{d}t + \bar{b}\tilde{\boldsymbol{\Sigma}}\,\mathrm{d}\mathbf{B}_t
\end{aligned}$$

Firm demand for τ bonds:

$$-\frac{\bar{W}}{\bar{Z}}e^{w_t+n_t}\theta_t(\tau) \approx -\frac{\bar{W}}{\bar{Z}}\left[\theta(\tau)+\theta(\tau)(w_t+n_t)+(\theta_t(\tau)-\theta(\tau))\right]$$

From market clearing, we have (steady state)

$$0 = -s(\tau) + \frac{\bar{W}}{\bar{Z}}\theta(\tau) - \eta(\tau)\bar{b}$$

and over time

$$x_t(\tau) = -(s_t(\tau)-s(\tau)) + \frac{\bar{W}}{\bar{Z}}\left[\theta(\tau)(w_t+n_t)+(\theta_t(\tau)-\theta(\tau))\right] - \eta(\tau)(b_t(\tau)-\bar{b})$$

Thus, risk prices are

$$\begin{aligned}
\boldsymbol{\Lambda}_t &= \int_0^T \boldsymbol{\sigma}(\tau)^\top x_t(\tau)\,\mathrm{d}\tau\\
&= -\int_0^T \boldsymbol{\sigma}(\tau)^\top (s_t(\tau)-s(\tau))\,\mathrm{d}\tau + (w_t+n_t)\frac{\bar{W}}{\bar{Z}}\int_0^T \boldsymbol{\sigma}(\tau)^\top\theta(\tau)\,\mathrm{d}\tau\\
&\quad + \frac{\bar{W}}{\bar{Z}}\int_0^T \boldsymbol{\sigma}(\tau)^\top(\theta_t(\tau)-\theta(\tau))\,\mathrm{d}\tau - \int_0^T \boldsymbol{\sigma}(\tau)^\top\eta(\tau)(b_t(\tau)-\bar{b})\,\mathrm{d}\tau\\
&\equiv -\tilde{\mathbf{s}}_t + \hat{\boldsymbol{\sigma}}^\top\frac{\bar{W}}{\bar{Z}}\left((1+\varsigma+\varphi)x_t + \frac{1+\varphi}{\varsigma+\varphi}z_t\right) + \frac{\bar{W}}{\bar{Z}}\tilde{\boldsymbol{\beta}}_t - \tilde{\mathbf{b}}_t - \tilde{\boldsymbol{\sigma}}^\top(\bar{b}_t-\bar{b})
\end{aligned}$$

B.3 Second-Order Approximations

Log-quadratic approximation of functions of the form:

$$F \equiv g\left[\int_0^T \eta(\tau) f\left(X_t(\tau)\right) \mathrm{d}\tau\right] = g\left[\int_0^T \eta(\tau) f\left(\bar{X} e^{x_t(\tau)}\right) \mathrm{d}\tau\right]$$

We have

$$\begin{aligned}
F\Big|_{SS} &= g(f(\bar{X})) \\
\frac{\partial F}{\partial x_t(\tau')}\Big|_{SS} &= \bar{X} g'(f(\bar{X})) f'(\bar{X}) \eta(\tau') \,\mathrm{d}\tau' \\
\frac{\partial^2 F}{\partial x_t(\tau') \partial x_t(\tau'')}\Big|_{SS} &= g''(f(\bar{X}))(\bar{X} f'(\bar{X}))^2 \eta(\tau') \,\mathrm{d}\tau' \,\eta(\tau'') \,\mathrm{d}\tau'' \\
\frac{\partial^2 F}{\partial x_t(\tau')^2}\Big|_{SS} &= g''(f(\bar{X}))(\bar{X} f'(\bar{X}))^2 (\eta(\tau') \,\mathrm{d}\tau')^2 + g'(f(\bar{X}))\bar{X}(f'(\bar{X}) + \bar{X} f''(\bar{X}))\eta(\tau') \,\mathrm{d}\tau'
\end{aligned}$$

Thus, the second order approximation is

$$\begin{aligned}
F &\approx \Delta_0 + \Delta_1 E_\tau x_t(\tau) + \frac{1}{2}\Delta_2 [E_\tau x_t(\tau)]^2 + \frac{1}{2}\Delta_3 E_\tau[x_t(\tau)^2] \\
&= \Delta_0 + \Delta_1 E_\tau x_t(\tau) + \frac{1}{2}(\Delta_2 + \Delta_3)[E_\tau x_t(\tau)]^2 + \frac{1}{2}\Delta_3 Var_\tau x_t(\tau)
\end{aligned}$$

where

$$\begin{aligned}
\Delta_0 &\equiv g(f(\bar{X})) \\
\Delta_1 &\equiv \bar{X} g'(f(\bar{X})) f'(\bar{X}) \\
\Delta_2 &\equiv g''(f(\bar{X}))(\bar{X} f'(\bar{X}))^2 \\
\Delta_3 &\equiv g'(f(\bar{X}))\bar{X}(f'(\bar{X}) + \bar{X} f''(\bar{X}))
\end{aligned}$$

and

$$\begin{aligned}
E_\tau[h(x_t(\tau))] &\equiv \int_0^T \eta(\tau) h(x_t(\tau)) \,\mathrm{d}\tau \\
Var_\tau[h(x_t(\tau))] &\equiv \int_0^T \eta(\tau) h(x_t(\tau))^2 \,\mathrm{d}\tau - E_\tau[h(x_t(\tau))]^2
\end{aligned}$$

The second-order approximation follows from taking the limit of the finite second-order approximation at grid points $i = 1, \ldots, N$ with Δ_i steps:

$$
\begin{aligned}
F &= g\left[\sum_{i=1}^{N} \eta(i) f\left(\bar{X} e^{x_t(i)}\right) \Delta_i\right] \\
F &\approx \bar{F} + \sum_{i=1}^{N} \left[\left.\frac{\partial F}{\partial x(i)}\right|_{SS}\right] x(i) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[\left.\frac{\partial^2 F}{\partial x(i) \partial x(j)}\right|_{SS}\right] x(i) x(j)
\end{aligned}
$$

and note that the second derivative in the double summation has the additional terms when $i = j$.

B.3.1 Examples

Simple mean: $f(z) = g(z) = z$

$$
\begin{aligned}
&\int_0^T \eta(\tau) \bar{X} e^{x_t(\tau)} \, \mathrm{d}\tau \\
&\approx \bar{X} \left\{1 + E_\tau x_t(\tau) + \frac{1}{2} [E_\tau x_t(\tau)]^2 + \frac{1}{2} Var_\tau x_t(\tau)\right\}
\end{aligned}
$$

Mean of function: $g(z) = z$

$$
\begin{aligned}
&\int_0^T \eta(\tau) f\left(\bar{X} e^{x_t(\tau)}\right) \mathrm{d}\tau \\
&\approx f(\bar{X}) + \bar{X} \left\{f'(\bar{X}) E_\tau x_t(\tau) + \frac{1}{2} (f'(\bar{X}) + \bar{X} f''(\bar{X}))(Var_\tau x_t(\tau) + [E_\tau x_t(\tau)]^2)\right\}
\end{aligned}
$$

Function of mean: $f(z) = z$

$$
\begin{aligned}
&g\left[\int_0^T \eta(\tau) \bar{X} e^{x_t(\tau)} \, \mathrm{d}\tau\right] \\
&\approx g(\bar{X}) + \bar{X} \left\{g'(\bar{X}) E_\tau x_t(\tau) + \frac{1}{2} g'(\bar{X}) Var_\tau x_t(\tau) + \frac{1}{2} (g'(\bar{X}) + \bar{X} g''(\bar{X}))[E_\tau x_t(\tau)]^2\right\}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&f\left[\int_0^T \eta(\tau) \bar{X} e^{x_t(\tau)} \, \mathrm{d}\tau\right] - \int_0^T \eta(\tau) f\left(\bar{X} e^{x_t(\tau)}\right) \mathrm{d}\tau \\
&\approx -\frac{1}{2} \bar{X}^2 f''(\bar{X}) Var_\tau x_t(\tau)
\end{aligned}
$$

Inverse: $g = f^{-1}$

$$f^{-1}\left[\int_0^T \eta(\tau) f\left(\bar{X}e^{x_t(\tau)}\right) \mathrm{d}\tau\right]$$
$$\approx \bar{X}\left\{1 + E_\tau x_t(\tau) + \frac{1}{2}\left[1 + \bar{X}\frac{f''(\bar{X})}{f'(\bar{X})}\right] Var_\tau x_t(\tau) + \frac{1}{2}[E_\tau x_t(\tau)]^2\right\}$$

CES:

$$\left[\int_0^T \eta(\tau)\left(\bar{X}e^{x_t(\tau)}\right)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}\tau\right]^{\frac{\epsilon}{\epsilon-1}}$$
$$\approx \bar{X}\left\{1 + E_\tau x_t(\tau) + \frac{1}{2}\left[\frac{\epsilon-1}{\epsilon}\right] Var_\tau x_t(\tau) + \frac{1}{2}[E_\tau x_t(\tau)]^2\right\}$$

Thus

$$\int_0^T \eta(\tau)\bar{X}e^{x_t(\tau)} \mathrm{d}\tau - \left[\int_0^T \eta(\tau)\left(\bar{X}e^{x_t(\tau)}\right)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}\tau\right]^{\frac{\epsilon}{\epsilon-1}}$$
$$\approx \frac{1}{2}\bar{X}\frac{1}{\epsilon}Var_\tau x_t(\tau)$$

CRRA utility difference from RANK:

$$\frac{1}{1-\varsigma}\left[\int_0^T \eta(\tau)\bar{X}e^{x_t(\tau)} \mathrm{d}\tau\right]^{1-\varsigma} - \frac{1}{1-\varsigma}\int_0^T \eta(\tau)\left(\bar{X}e^{x_t(\tau)}\right)^{1-\varsigma} \mathrm{d}\tau$$
$$\approx \frac{1}{2}\bar{X}^{1-\varsigma}\varsigma Var_\tau x_t(\tau)$$

B.4 Social Welfare: Aggregate Relationships

Equilibrium equations. Consumption:

$$C_t = \int_0^T \eta(\tau) C_t(\tau) \mathrm{d}\tau \tag{B6}$$
$$C_t = Y_t\left[1 - \frac{1}{2}\theta\pi_t^2 - \frac{1}{2}\psi^i(i_t - \bar{i}_t)^2 - \frac{1}{2}\int_0^T \psi^{(\tau)}\left(s_t(\tau)\right)^2 \mathrm{d}\tau\right] \tag{B7}$$

Production:

$$Y_t = Z_t L_t \tag{B8}$$

Aggregate labor/wage index:

$$L_t = \left[\int_0^T \eta(\tau) N_t(\tau)^{\frac{\epsilon_w - 1}{\epsilon_w}} \, \mathrm{d}\tau\right]^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{B9}$$

$$W_t = \left[\int_0^T \eta(\tau) W_t(\tau)^{1-\epsilon_w} \, \mathrm{d}\tau\right]^{1-\epsilon_w} \tag{B10}$$

Labor demand and labor clearing:

$$N_t(\tau) = \left(\frac{W_t(\tau)}{W_t}\right)^{-\epsilon_w} L_t \implies W_t(\tau) = \left(\frac{N_t(\tau)}{L_t}\right)^{-\frac{1}{\epsilon_w}} W_t \tag{B11}$$

Intratemporal wage/consumption/labor:

$$(1+\tau^w)\left(\frac{\epsilon_w - 1}{\epsilon_w}\right) W_t(\tau) = C_t(\tau)^{\varsigma} N_t(\tau)^{\varphi} \tag{B12}$$

$$\implies (1+\tau^w)\left(\frac{\epsilon_w - 1}{\epsilon_w}\right) L_t^{\frac{1}{\epsilon_w}} W_t = C_t(\tau)^{\varsigma} N_t(\tau)^{\varphi + \frac{1}{\epsilon_w}} \tag{B13}$$

Output gap:

$$X_t = \frac{Y_t}{Y_t^n} \tag{B14}$$

B.4.1 Social Welfare Approximation

The log-quadratic approximation of deviations of social welfare from the first-best are given by

$$\begin{aligned}
\mathcal{W}_0 &\equiv \int_0^\infty e^{-\rho t} \mathcal{L}_t \, \mathrm{d}t \\
\mathcal{L}_t &= U\left(C_t^n, N_t^n\right) - \int_0^T \eta(\tau) U\left(C_t(\tau), N_t(\tau)\right) \mathrm{d}\tau \\
&= \frac{1}{2}(\varsigma + \varphi) x_t^2 + \frac{1}{2}\theta \pi_t^2 \\
&\quad + \frac{1}{2}\varsigma Var_\tau c_t(\tau) + \frac{1}{2}\varphi Var_\tau n_t(\tau) + \frac{1}{2}\frac{1}{\epsilon_w} Var_\tau n_t(\tau) \\
&\quad + \frac{1}{2}\psi^i (i_t - \bar{i}_t)^2 + \frac{1}{2}\int_0^T \psi^{(\tau)} \left(s_t(\tau)\right)^2 \mathrm{d}\tau
\end{aligned}$$

We have

$$\varsigma Var_\tau c_t(\tau) + \varphi Var_\tau n_t(\tau) + \frac{1}{\epsilon_w} Var_\tau n_t(\tau) = \left(\varsigma + \frac{\varsigma^2 \varphi}{\left(\frac{1}{\epsilon_w} + \varphi\right)^2} \right) Var_\tau c_t(\tau) + \epsilon_w Var_\tau w_t(\tau)$$

and

$$Var_\tau w_t(\tau) = \left(\frac{\varsigma}{1 + \epsilon_w \varphi} \right)^2 Var_\tau c_t(\tau)$$

The loss associated with $Var_\tau c_t(\tau)$ comes from imperfect risk-sharing. The loss associated with $Var_\tau w_t(\tau)$ comes from labor market inefficiencies.

B.4.2 Equilibrium Approximations: Details

Throughout we make use of big-O properties:

$$\begin{aligned} f = O(h), g = O(k) \implies f + g = O(h + k), f \cdot g = O(h \cdot k) \\ \implies f(\tau) = O(h(\tau)) \implies E_\tau f(\tau) \equiv \int \eta(\tau) f(\tau) \,\mathrm{d}\tau = O\left(\int \eta(\tau) h(\tau) \,\mathrm{d}\tau \right) \equiv O(E_\tau h(\tau)) \end{aligned}$$

We use the variable ξ_t to denote the generic expansion point around the steady state for any variable, so that

$$f(x_t) = O(\xi_t), g(y_t) = O(\xi_t) \implies f(x_t) + g(y_t) = O(\xi_t), f(x_t) \cdot g(y_t) = O(\xi_t^2)$$

Aggregate and HH member consumption from (B6)

$$\begin{aligned} C_t \equiv \bar{C} e^{c_t} = \bar{C} \left\{ 1 + c_t + \frac{1}{2} c_t^2 \right\} + O(\xi_t^3) \\ C_t(\tau) \equiv \bar{C} e^{c_t(\tau)} = \bar{C} \left\{ 1 + c_t(\tau) + \frac{1}{2} c_t(\tau)^2 \right\} + O(\xi_t^3) \\ \implies c_t + \frac{1}{2} c_t^2 + O(\xi_t^3) = E_\tau c_t(\tau) + \frac{1}{2} E_\tau \left[c_t(\tau) \right]^2 + \frac{1}{2} Var_\tau c_t(\tau) + O(\xi_t^3) \end{aligned}$$

Also, aggregate labor supply $N_t \equiv \int_0^T \eta(\tau) N_t(\tau) \,\mathrm{d}\tau$ (which is not equivalent to aggregate labor index L_t):

$$
\begin{aligned}
N_t \equiv \bar{N} e^{n_t} &= \bar{N}\left\{1 + n_t + \frac{1}{2} n_t^2\right\} + O(\xi_t^3) \\
N_t(\tau) \equiv \bar{N} e^{n_t(\tau)} &= \bar{N}\left\{1 + n_t(\tau) + \frac{1}{2} n_t(\tau)^2\right\} + O(\xi_t^3) \\
\implies n_t + \frac{1}{2} n_t^2 + O(\xi_t^3) &= E_\tau n_t(\tau) + \frac{1}{2} E_\tau \left[n_t(\tau)\right]^2 + \frac{1}{2} Var_\tau n_t(\tau) + O(\xi_t^3)
\end{aligned}
$$

From consumption goods market clearing (B7)

$$
\bar{C} e^{c_t} = \bar{Y} e^{y_t} e^{d_t}
$$

where d_t is the log of the deadweight loss terms in (B7) (which in levels is equal to one in SS). Since $\bar{C} = \bar{Y}$, we have

$$
c_t + \frac{1}{2} c_t^2 + O(\xi_t^3) = (y_t + d_t) + \frac{1}{2}(y_t + d_t)^2 + O(\xi_t^3)
$$

Additionally, a second-order expansion of deadweight loss d_t gives

$$
\begin{aligned}
d_t &= -\frac{1}{2}\left[\theta \pi_t^2 + \psi^i (i_t - \bar{i}_t)^2 + \int_0^T \psi^{(\tau)} \left(s_t(\tau)\right)^2 \mathrm{d}\tau\right] + O(\xi_t^3) \\
&\equiv -\frac{1}{2} \tilde{d}_t + O(\xi_t^3)
\end{aligned}
$$

and also

$$
\begin{aligned}
d_t^2 &= O(\xi_t^3) \\
d_t v_t &= O(\xi_t^3)
\end{aligned}
$$

for any variable v_t (in terms of deviations from steady state).

Thus

$$
c_t + \frac{1}{2} c_t^2 + O(\xi_t^3) = y_t + \frac{1}{2} y_t^2 - \frac{1}{2} \tilde{d}_t + O(\xi_t^3)
$$

Production from (B8)

$$
y_t = z_t + \ell_t
$$

Labor and wage indices from (B9)

$$\ell_t + \frac{1}{2}\ell_t^2 + O(\xi_t^3) = E_\tau n_t(\tau) + \frac{1}{2}E_\tau[n_t(\tau)]^2 + \frac{1}{2}\frac{\epsilon_w - 1}{\epsilon_w}Var_\tau n_t(\tau) + O(\xi_t^3)$$
$$w_t + \frac{1}{2}w_t^2 + O(\xi_t^3) = E_\tau w_t(\tau) + \frac{1}{2}E_\tau[w_t(\tau)]^2 + \frac{1}{2}(1-\epsilon_w)Var_\tau w_t(\tau) + O(\xi_t^3)$$

Combining with aggregate labor supply

$$\ell_t + \frac{1}{2}\ell_t^2 + O(\xi_t^3) = n_t + \frac{1}{2}n_t^2 - \frac{1}{2}\frac{1}{\epsilon_w}Var_\tau n_t(\tau) + O(\xi_t^3)$$

Labor demand from (B11)

$$\begin{aligned} n_t(\tau) &= -\epsilon_w(w_t(\tau) - w_t) + \ell_t \\ \implies E_\tau n_t(\tau) &= -\epsilon_w(E_\tau w_t(\tau) - w_t) + \ell_t \\ Var_\tau n_t(\tau) &= \epsilon_w^2 Var_\tau w_t(\tau) \end{aligned}$$

Intratemporal HH conditions from (B12)

$$\begin{aligned} w_t(\tau) &= \varsigma c_t(\tau) + \varphi n_t(\tau) \\ \implies E_\tau w_t(\tau) &= \varsigma E_\tau c_t(\tau) + \varphi E_\tau n_t(\tau) \\ Var_\tau w_t(\tau) &= \varsigma^2 Var_\tau c_t(\tau) + \varphi^2 Var_\tau n_t(\tau) + 2Cov_\tau(c_t(\tau), n_t(\tau)) \end{aligned}$$

Output gap from (B14)

$$x_t = y_t - y_t^n$$

Utility

$$U(C_t(\tau), N_t(\tau)) - \bar{U} = \bar{Z}^\chi \left\{ c_t(\tau) + \frac{1}{2}(1-\varsigma)c_t(\tau)^2 - n_t(\tau) + \frac{1}{2}(1+\varphi)n_t(\tau)^2 \right\} + O(\xi_t^3)$$

where $\bar{U} = U(\bar{C}, \bar{N})$ and $\chi \equiv \frac{(1-\varsigma)(1+\varphi)}{\varsigma+\varphi}$. Thus social welfare

$$\begin{aligned}
\mathcal{U}_t &\equiv \int_0^T \eta(\tau) U(C_t(\tau), N_t(\tau))\,\mathrm{d}\tau \\
\bar{Z}^{-\chi}(\mathcal{U}_t - \bar{U}) &= E_\tau c_t(\tau) + \frac{1}{2}(1-\varsigma)\left(E_\tau[c_t(\tau)]^2 + Var_\tau c_t(\tau)\right) \\
&\quad - E_\tau n_t(\tau) - \frac{1}{2}(1+\varphi)\left(E_\tau[n_t(\tau)]^2 + Var_\tau n_t(\tau)\right) + O(\xi_t^3)
\end{aligned}$$

Second-order approximations of $C_t^{1-\varsigma}/(1-\varsigma)$ and $L_t^{1+\varphi}/(1+\varphi)$ give

$$\begin{aligned}
c_t + \frac{1}{2}(1-\varsigma)c_t^2 + O(\xi_t^3) &= E_\tau c_t(\tau) + \frac{1}{2}(1-\varsigma)E_\tau[c_t(\tau)]^2 + \frac{1}{2}Var_\tau c_t(\tau) + O(\xi_t^3) \\
\ell_t + \frac{1}{2}(1+\varphi)\ell_t^2 + O(\xi_t^3) &= E_\tau n_t(\tau) + \frac{1}{2}(1+\varphi)E_\tau[n_t(\tau)]^2 + \frac{1}{2}\frac{\epsilon_w - 1}{\epsilon_w}Var_\tau n_t(\tau) + O(\xi_t^3)
\end{aligned}$$

Thus social welfare

$$\bar{Z}^{-\chi}(\mathcal{U}_t - \bar{U}) = c_t + \frac{1}{2}(1-\varsigma)c_t^2 - \varsigma Var_\tau c_t(\tau) - \ell_t - \frac{1}{2}(1+\varphi)\ell_t^2 - \left(\varphi + \frac{1}{\epsilon_w}\right)Var_\tau n_t(\tau) + O(\xi_t^3)$$

Combining with the second-order approximations of consumption and production:

$$\begin{aligned}
c_t + \frac{1}{2}(1-\varsigma)c_t^2 &= (y_t + d_t) + \frac{1}{2}(1-\varsigma)\left(y_t + d_t\right)^2 \\
&= y_t - \frac{1}{2}\tilde{d}_t + \frac{1}{2}(1-\varsigma)y_t^2 + O(\xi_t^3) \\
\ell_t + \frac{1}{2}(1+\varphi)\ell_t^2 &= (y_t - z_t) + \frac{1}{2}(1+\varphi)\left(y_t - z_t\right)^2
\end{aligned}$$

In the first-best, we have

$$\begin{aligned}
y_t^n &= \frac{1+\varphi}{\varsigma+\varphi}z_t \\
\implies y_t &= x_t + \frac{1+\varphi}{\varsigma+\varphi}z_t \equiv x_t + \frac{\chi}{1-\varsigma}z_t \\
y_t - z_t &= x_t + \frac{1-\varsigma}{\varsigma+\varphi}z_t \equiv x_t + \frac{\chi}{1+\varphi}z_t
\end{aligned}$$

Thus

$$
\begin{aligned}
& y_t + \frac{1}{2}(1-\varsigma)y_t^2 - (y_t - z_t) - \frac{1}{2}(1+\varphi)\left(y_t - z_t\right)^2 \\
&= z_t + \frac{1}{2}(1-\varsigma)\left(x_t + \frac{\chi}{1-\varsigma}z_t\right)^2 - \frac{1}{2}(1+\varphi)\left(x_t + \frac{\chi}{1+\varphi}z_t\right)^2 \\
&= z_t + \frac{1}{2}\chi z_t^2 - \frac{1}{2}(\varsigma+\varphi)x_t^2
\end{aligned}
$$

And note social welfare at the first-best is

$$
\bar{Z}^{-\chi}(\mathcal{U}_t^n - \bar{U}) = z_t + \frac{1}{2}\chi z_t^2 + O(\xi_t^3)
$$

Combining, we have that social welfare differences from the first-best are

$$
\bar{Z}^{-\chi}(\mathcal{U}_t^n - \mathcal{U}_t) = \frac{1}{2}(\varsigma+\varphi)x_t^2 + \frac{1}{2}\tilde{d}_t + \varsigma Var_\tau c_t(\tau) + \left(\varphi + \frac{1}{\epsilon_w}\right) Var_\tau n_t(\tau) + O(\xi_t^3)
$$

The variance terms are related as follows:

$$
\begin{aligned}
\left(\varphi + \frac{1}{\epsilon_w}\right)^2 Var_\tau n_t(\tau) &= \varsigma^2 Var_\tau c_t(\tau) \\
Var_\tau w_t(\tau) &= \left(\frac{1}{\epsilon_w}\right)^2 Var_\tau n_t(\tau)
\end{aligned}
$$

Thus

$$
\varsigma Var_\tau c_t(\tau) + \left(\varphi + \frac{1}{\epsilon_w}\right) Var_\tau n_t(\tau) = \left(\frac{\varsigma}{\varphi + \frac{1}{\epsilon_w}}\right)\left(\varsigma + \varphi + \frac{1}{\epsilon_w}\right) Var_\tau c_t(\tau)
$$

We can also decompose these terms as follows:

$$
\begin{aligned}
\varsigma Var_\tau c_t(\tau) + \varphi Var_\tau n_t(\tau) &= \frac{\varsigma}{\varphi}\left(\varphi + \varsigma\left[\frac{\varphi\epsilon_w}{1+\varphi\epsilon_w}\right]^2\right) Var_\tau c_t(\tau) \\
\frac{1}{\epsilon_w} Var_\tau n_t(\tau) &= \epsilon_w Var_\tau w_t(\tau)
\end{aligned}
$$