A Preferred-Habitat Model of Term Premia, Exchange Rates, and Monetary Policy Spillovers

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Motivation

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- Textbook international macro:
    - Uncovered Interest Parity (UIP) holds
    - The Expectation Hypothesis (EH) holds

- Empirically:
    1. Strong patterns in FX: currency carry trade is profitable $\implies$ deviations from UIP
       [Fama 1984...]
    2. Strong patterns in FI: bond carry trade is profitable $\implies$ deviations from the EH
       [Fama & Bliss 1987, Campbell & Shiller 1991...]
    3. The two risk premia are deeply connected
       [Lustig et al 2019, Lloyd & Marin 2019, Chernov & Creal 2020...]
    4. Quantitative easing not only reduced domestic yields, but also had strong effects on exchange rates and foreign yields
       [Bhattarai & Neely 2018...]

Motivation

- Making sense of these facts is important:
  - To understand what determines exchange rates (volatility, disconnect...)
  - To understand monetary policy transmission, both domestically (along the yield curve)...
  - ...but also via international spillovers, to exchange rates and foreign yields
- On the theory side:
  - Standard representative agent no-arbitrage models have a hard time
  - Recent literature emphasizes the optimization of financial intermediaries and the constraints they face
    [Gabaix & Maggiori 2015, Itskhoki & Mukhin 2019, Koijen & Yogo 2020]
  - Revives an old literature on portfolio-balance
    [Kouri 1982, Jeanne & Rose 2002...]

Our Model

- This paper: introduce risk averse 'global rate arbitrageur' absorbing supply and demand shocks in bond and currency markets
- Clientele investors introduce a degree of market segmentation
  - FX and bond markets populated by different investor clienteles (pension funds, importers/exporters)
  - Arbitrageurs (hedge funds, fixed income desk of broker-dealer) partly overcome segmentation
- Formally: Two-country version of Vayanos & Vila's (2021) preferred-habitat model
  - Contemporaneous paper by Greenwood et al (2022) in discrete time with two bonds

Findings

1. Can reproduce qualitative and quantitative facts about the joint behavior of bond and currency risk premia
2. Rich transmission of monetary policy shocks via exchange rate and term premia, contrasting with standard models
3. Key mechanisms:
   - Shifts in arbitrageurs' risk exposure lead to changes in required risk compensation
   - Hedging behavior of global arbitrageurs $\Longrightarrow$ tight linkage between bond term premia and currency risk premia
   - In the presence of market segmentation, policy shocks (particularly unconventional) lead to large shifts in risk exposure
4. General message: floating exchange rates provide limited insulation. Failure of Friedman-Obtsfeld-Taylor's Trilemma

Set-Up

Set-Up: Two-Country Vayanos & Vila (2021)

- Continuous time $t \in (0, \infty)$, 2 countries $j = H, F$
- Nominal exchange rate e_t: H price of F (increase $\equiv$ depreciation of H's currency)
- In each country j, continuum of zero coupon bonds in zero net supply with maturity $0 \leq \tau \leq T$, and $T \leq \infty$
- Bond price (in local currency) $P_{jt}^{(\tau)}$, with yield to maturity $y_{jt}^{(\tau)} = -\log P_{jt}^{(\tau)} / \tau$
- Nominal short rate ("monetary policy") $i_{jt} = \lim_{\tau \to 0} y_{jt}^{(\tau)}$ follows (exogenous, stochastic) mean-reverting process

Arbitrageurs and Preferred-Habitat Investors

- Home and foreign preferred-habitat bond investors
  (hold bonds in a specific currency and maturity: $Z_{jt}(\tau)$)
  - Eg, pension funds, money market mutual funds
  - Time-varying demand β_{jt}, downward sloping in terms of bond price (elasticity $\alpha_j(\tau)$)
- Preferred-habitat currency traders
  (hold foreign currency: Z_{et})
  - Eg, importers/exporters
  - Time-varying demand γ_t, downward sloping in terms of exchange rate (elasticity α_e)
- Global rate arbitrageurs
  (can trade in both currencies, in domestic and foreign bonds: $W_{Ft}, X_{jt}(\tau)$)
  - Eg, global hedge funds
  - Mean-variance preferences (risk aversion a)
  - Engage in currency carry trade, domestic and foreign bond carry trade

Global Rate Arbitrageur

- Mean-variance optimization

$$\max \mathbb{E}_t(\mathrm{d}W_t) - \frac{a}{2}\mathbb{V}\mathrm{ar}_t(\mathrm{d}W_t)$$

$$\text{s.t. } \mathrm{d}W_t = W_t i_{Ht}\,\mathrm{d}t + W_{Ft}\left(\frac{\mathrm{d}e_t}{e_t} + (i_{Ft} - i_{Ht})\,\mathrm{d}t\right) + \int_0^T X_{Ht}^{(\tau)}\left(\frac{\mathrm{d}P_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht}\,\mathrm{d}t\right)\mathrm{d}\tau + \int_0^T X_{Ft}^{(\tau)}\left(\frac{\mathrm{d}(P_{Ft}^{(\tau)}e_t)}{P_{Ft}^{(\tau)}e_t} - \frac{\mathrm{d}e_t}{e_t} - i_{Ft}\,\mathrm{d}t\right)\mathrm{d}\tau$$

- Wealth W_t:
  - W_{Ft} invested in country F short rate (*CCT*)
  - $X_{jt}^{(\tau)}$ invested in bond of country j and maturity τ (BCT_j)
  - Remainder in country H short rate

Key Insight: Risk averse arbitrageurs' holdings increase with expected return

Preferred-Habitat Bond and FX Investors

- Demand for bonds in currency j, of maturity τ:

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log P_{jt}^{(\tau)} - \theta_j(\tau)\beta_{jt}$$

  - $\alpha_j(\tau)$: demand elasticity for τ investor in country j
  - $\theta_j(\tau)$: how variations in factor β_{jt} affect demand for τ investor in country j

- Demand for foreign currency (spot):

$$Z_{et} = -\alpha_e \log e_t - \theta_e \gamma_t$$

  - Can accommodate forward demand. Under CIP, equivalent to spot + H and F bond trades

- Exogenous bond and FX demand risk factors:

$$\mathrm{d}\beta_{jt} = -\kappa_{\beta j}\beta_{jt}\,\mathrm{d}t + \sigma_{\beta j}\mathrm{d}B_{\beta jt}, \quad \mathrm{d}\gamma_t = -\kappa_\gamma \gamma_t\,\mathrm{d}t + \sigma_\gamma \mathrm{d}B_{\gamma t}$$

Key Insight: elastic habitat traders. Price movements require portfolio rebalancing

Equilibrium

- Risk factors: short rates ($\mathrm{d}B_{ijt}$), bond demands ($\mathrm{d}B_{\beta jt}$) and currency demand ($\mathrm{d}B_{\gamma t}$)
- Arbitrageurs' optimality conditions imply expected excess returns are given by:

$$\mathbb{E}_t \,\mathrm{d}P_{jt}^{(\tau)} / P_{jt}^{(\tau)} - i_{jt} = \mathbf{A}_j(\tau)^\top \boldsymbol{\Lambda}_t, \quad \mathbb{E}_t \,\mathrm{d}e_t / e_t + i_{Ft} - i_{Ht} = \mathbf{A}_e{}^\top \boldsymbol{\Lambda}_t$$

$$\text{where} \quad \boldsymbol{\Lambda}_t = a\boldsymbol{\Sigma}\left(W_{Ft}\mathbf{A}_e + \sum_{j=H,F}\int_0^T X_{jt}\mathbf{A}_j(\tau)\,\mathrm{d}\tau\right)$$

- Endogenous coefficients $\mathbf{A}_j(\tau), \mathbf{A}_e$ govern sensitivity to market price of risk $\boldsymbol{\Lambda}_t$
- Model is closed through market clearing: $X_{jt}^{(\tau)} + Z_{jt}^{(\tau)} = 0, \; W_{Ft} + Z_{et} = 0$

Key Insight: market price of risk $\boldsymbol{\Lambda}_t$ depends on equilibrium holdings. Bond and currency premia jointly determined

Data Generating Process: Assumptions

- In order to derive analytical results, we assume independent short-rate processes, and non-stochastic demand factors:

$$\mathrm{d}i_{Ht} = \kappa_{iH}(\bar{i}_H - i_{Ht})\,\mathrm{d}t + \sigma_{iH}\mathrm{d}B_{iHt}, \quad \mathrm{d}i_{Ft} = \kappa_{iF}(\bar{i}_F - i_{Ft})\,\mathrm{d}t + \sigma_{iF}\mathrm{d}B_{iFt}$$

- For quantitative results, we can allow for rich demand structure embodied in dynamics of risk factors. DGP:

$$\mathbf{q}_t = \begin{bmatrix} i_{Ht} & i_{Ft} & \beta_{Ht} & \beta_{Ft} & \gamma_t \end{bmatrix}^\top$$

$$\mathrm{d}\mathbf{q}_t = -\boldsymbol{\Gamma}\,(\mathbf{q}_t - \bar{\mathbf{q}})\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t$$

Risk Neutral Global Arbitrageur

1. Benchmark: Risk Neutral Global Rate Arbitrageur ("Standard Model")

Consider the benchmark case of a risk neutral global rate arbitrageur: $a = 0$

- Expectation Hypothesis holds:

$$\mathbb{E}_t \, \mathrm{d}P_{Ht}^{(\tau)} / P_{Ht}^{(\tau)} = i_{Ht}, \quad \mathbb{E}_t \, \mathrm{d}P_{Ft}^{(\tau)} / P_{Ft}^{(\tau)} = i_{Ft}$$

  - No effect of QE on yield curve, at Home or Foreign
  - Yield curve independent from foreign short rate shocks

- Uncovered Interest Parity holds:

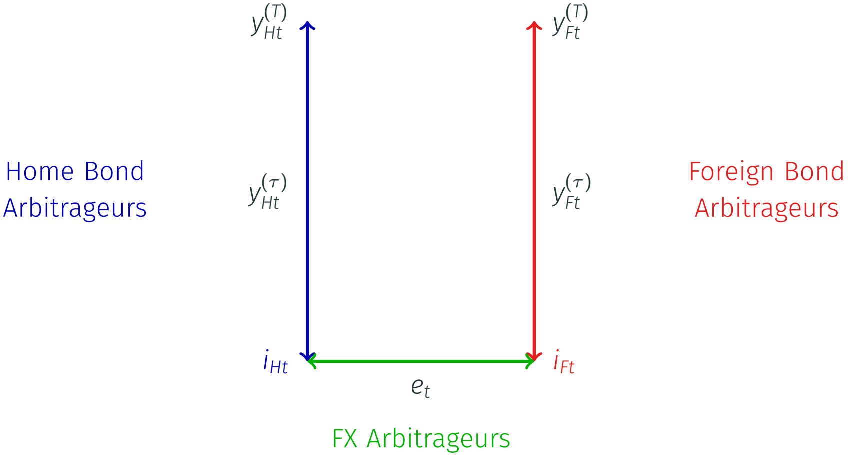
$$\mathbb{E}_t \, \mathrm{d}e_t / e_t = i_{Ht} - i_{Ft}$$

  - 'Mundellian' insulation: shock to short rates 'absorbed' into the exchange rate
  - Classical Trilemma: capital flows and floating exchange rates deliver monetary autonomy

Segmented Arbitrage

2. Intermediate Step: Segmented Arbitrage and No Demand Shocks

Assume foreign currency and bonds traded by three disjoint sets of arbitrageurs



2. Intermediate Step: Segmented Arbitrage and No Demand Shocks

Postulate: $\log P_{jt}^{(\tau)} = -A_{ij}(\tau) i_{jt} - C_j(\tau)$; $\log e_t = A_{iFe} i_{Ft} - A_{iHe} i_{Ht} - C_e$

Proposition (Segmented Arbitrage, Currency Carry Trade CCT and UIP Deviations)

When arbitrage is segmented, risk aversion $a > 0$ and FX price elasticity $\alpha_e > 0$

- Attenuation: $0 < A_{ije} < 1/\kappa_{ij}$
- CCT expected return $\mathbb{E}_t \,\mathrm{d}e_t \,/e_t + i_{Ft} - i_{Ht}$ decreases in i_{Ht} and increases in i_{Ft} (UIP deviation)

Intuition: Similar to Kouri (1982), Gabaix and Maggiori (2015)

- When $i_{Ht} \downarrow$ or $i_{Ft} \uparrow$, FX arbitrageurs want to invest more in the CCT
- Foreign currency appreciates ($e_t \uparrow$)
- As $e_t \uparrow$, price elastic FX traders ($\alpha_e > 0$) reduce holdings: $Z_{et} \downarrow$
- FX arbitrageurs increase their holdings $W_{Ft} \uparrow$, which requires a higher CCT return

2. Intermediate Step: Segmented Arbitrage and No Demand Shocks

Proposition (Segmented Arbitrage and Bond Carry Trade BCT)

When arbitrage is segmented, $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$:

- Attenuation: $A_{ij}(\tau) < (1 - e^{-\kappa_{ij}\tau})/\kappa_{ij}$
- Bond prices in country j only respond to country j short rates (no spillover)
- BCT_j expected return $\mathbb{E}_t \, \mathrm{d}P_{jt}^{(\tau)} / P_{jt}^{(\tau)} - i_{jt}$ decreases in i_{jt}

Intuition: Similar to Vayanos & Vila (2021)

- When $i_{jt} \downarrow$, bond arbitrageurs want to invest more in the BCT
- Bond prices increase ($P_{jt}^{(\tau)} \uparrow$)
- As $P_{jt}^{(\tau)} \uparrow$, price-elastic habitat bond investors ($\alpha_j(\tau) > 0$) reduce their holdings: $Z_{jt}^{(\tau)} \downarrow$
- Bond arbitrageurs increase their holdings $X_{jt}^{(\tau)} \uparrow$, which requires a larger BCT return

Macro Implications of the Segmented Model

Assume $a > 0$, $\theta_j(\tau) > 0$ and $\theta_e > 0$:

- Unexpected increase in bond demand in country j (QE_j) reduces yields in country j
- No effect on bond yields in the other country or on the exchange rate
  - QE purchases: $Z_{jt}^{(\tau)} \uparrow$
  - Bond arbitrageurs reduce holdings $X_{jt}^{(\tau)} \downarrow$, reducing risk exposure and pushing down yields
  - Arbitrageurs in other markets are unaffected

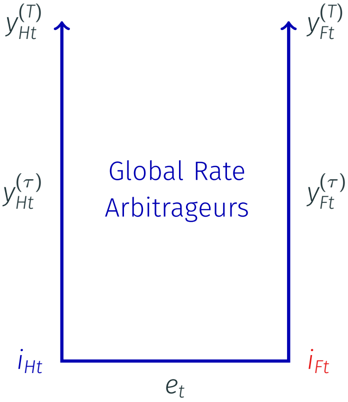
Open Economy Macro Implications:

- Changes in Home monetary conditions (conventional or QE) have no effect on the Foreign yield curve. Full insulation
- Insulation is even stronger in the case of QE: exchange rate is unchanged
- Trilemma? As we will see, this result arises because of markets segmentation (limited capital flows), not because of floating exchange rates

Global Arbitrage

3. Global Rate Arbitrageur and No Demand Shocks

Assume now global rate arbitrageur can invest in bonds (H and F) and FX



3. Global Rate Arbitrageur and No Demand Shocks

Postulate $\log P_{jt}^{(\tau)} = -A_{ijj}(\tau) i_{jt} - A_{ijj'}(\tau) i_{j't} - C_H(\tau)$; $\log e_t = A_{iFe} i_{Ft} - A_{iHe} i_{Ht} - C_e$

Proposition (Global Arbitrage and Carry Trades CCT, BCT)

When arbitrage is global, risk aversion $a > 0$ and price elasticities $\alpha_e, \alpha_j(\tau) > 0$:

- The results of the previous propositions obtain: both CCT and BCT_H return decrease with i_{Ht}, and attenuation is stronger than with segmented markets
- ⚠ In addition, BCT_F increases with i_{Ht}
- The effect of i_{jt} on bond yields is smaller in the other country: $A_{jj'}(\tau) < A_{jj}(\tau)$

Intuition: Bond and FX Premia Cross-Linkages

- When $i_{Ht} \downarrow$ global arbitrageurs want to invest more in CCT and BCT_H
- e_t and $W_{Ft} \uparrow$: increased FX exposure (risk of $i_{Ft} \downarrow$)
- Hedge by investing more in BCT_F since price of foreign bonds increases when i_{Ft} drops: foreign yields decline and BCT_F decreases

Macro Implications of Global Rate Arbitrageur Model

Assume $a > 0$ and $\alpha_e, \alpha_j(\tau) > 0$:

- Unexpected QE_H reduces yields in country H
- ⚠ Also reduces yields in country F, and depreciates the Home currency
  - Arbitrageurs decrease H bond exposure (less exposed to risk of $i_{Ht} \uparrow$)
  - More willing to hold assets exposed to this risk: increase holdings of F bonds and currency, pushing down F yields and depreciating the H currency

Open Economy Macro Implications:

- Changes in Home monetary conditions (conventional or QE) affect both yield curves and the exchange rate: potential spillovers from monetary policy. Imperfect insulation even with floating rates
- QE or FX interventions in one country affect monetary conditions in both countries and depreciate the currency
- Failure of the Classical Trilemma

The Full Model

The Full Model: Adding Demand Shocks

- Now we allow for richer demand structure of risk factors:

$$\mathrm{d}\mathbf{q}_t = -\boldsymbol{\Gamma}\left(\mathbf{q}_t - \overline{\mathbf{q}}\right)\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t$$

  - We assume independent processes for all factors, except shocks to short rates may be correlated, and currency demand γ_t may respond to short rates
- Numerical calibration
  - Data: Zero coupon data: US Treasuries (H) and German Bunds (F); exchange rate data: German mark/euro
  - Targets: second moments of short/long term rates, exchange rates, and volumes
- Return predictability (untargeted)
  - Bond returns and slope of the term structure
  - Currency returns and UIP
  - Cross-country bond and currency returns

Numerical Calibration

- Data: Zero coupon data: US Treasuries (H) and German Bunds (F); exchange rate data: German mark/euro
- Targets: second moments of short/long term rates, exchange rates, and volumes

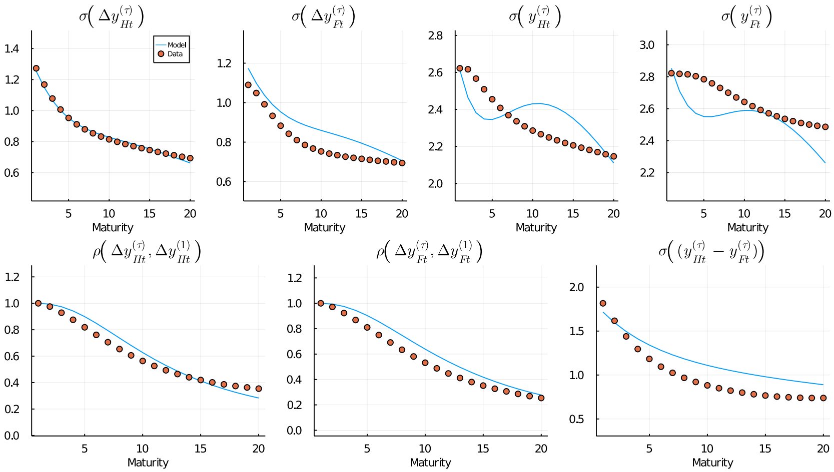
Parameter	Value	Parameter	Value	Parameter	Value
κ_{iH}	0.126	κ_{γ}	0.134	$a\sigma_{\beta}\theta_0$	90.6
κ_{iF}	0.0896	$\kappa_{\gamma,iH}$	-0.267	$a\alpha_e$	73.4
σ_{iH}	1.43	$\kappa_{\gamma,iF}$	0.252	$a\alpha_0$	4.74
σ_{iF}	0.751	$a\sigma_{\gamma}\theta_e$	763.0	α_1	0.144
$\sigma_{iH,iF}$	1.05	κ_{β}	0.0501	θ_1	0.374

- For policy experiments: CRRA $\gamma = 2$ and arbitrageur wealth $\frac{W}{GDP_H} \approx 5\% \implies a = 40$

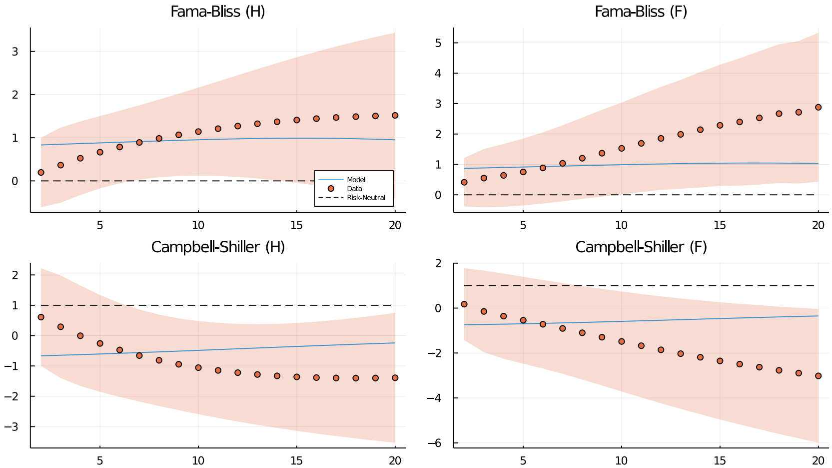
Model Fit: Short Rates and Exchange Rates

Moment	Data	Model	Moment	Data	Model
$\sigma\left(y_{Ht}^{(1)}\right)$	2.622	2.614	$\rho\left(\Delta \log e_t, (y_{Ht}^{(1)} - y_{Ft}^{(1)})\right)$	-0.105	-0.096
$\sigma\left(\Delta y_{Ht}^{(1)}\right)$	1.273	1.254	$\rho\left(\Delta \log e_t, \Delta y_{Ht}^{(1)}\right)$	-0.095	-0.214
$\sigma\left(y_{Ft}^{(1)}\right)$	2.822	2.853	$\rho\left(\Delta \log e_t, \Delta y_{Ft}^{(1)}\right)$	0.048	0.071
$\sigma\left(\Delta y_{Ft}^{(1)}\right)$	1.09	1.174	$\rho\left(\Delta^{(5)} \log e_t, (y_{Ht}^{(5)} - y_{Ft}^{(5)})\right)$	0.12	0.06
$\sigma\left((y_{Ht}^{(1)} - y_{Ft}^{(1)})\right)$	1.816	1.717	$\tilde{V}_H(0 \leq \tau \leq 3)$	0.361	0.378
$\sigma\left(\Delta \log e_t\right)$	10.186	10.183	$\tilde{V}_H(11 \leq \tau \leq 30)$	0.08	0.116

Model Fit: Long Rates

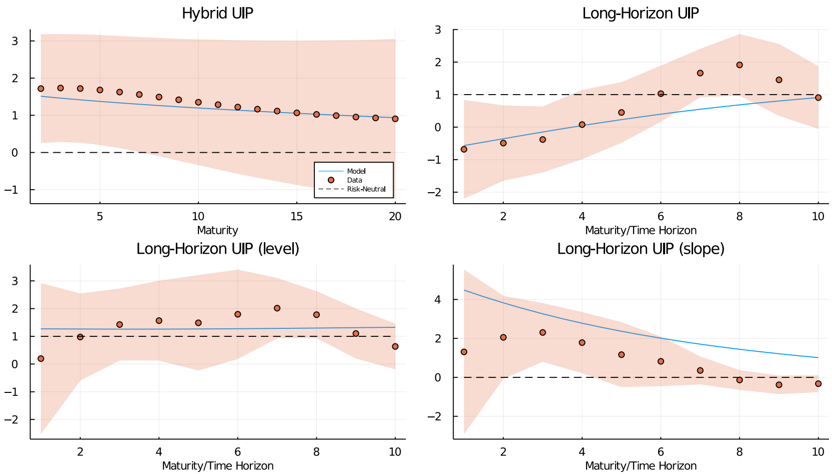


Regression Coefficients: Term Structure



Implications: Positive slope-premia relationship

Regression Coefficients: UIP



Implications: CCT is profitable, but profitability goes to zero if CCT is done with long-term bonds or over long horizon. Slope differential predicts CCT return

Policy Spillovers

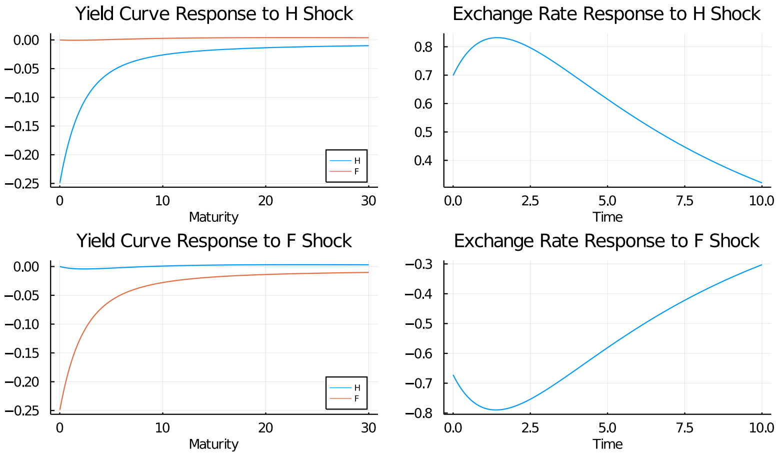
Conduct policy experiments:

- Monetary policy shock: unanticipated and idiosyncratic 25bp decrease in policy rate
- QE shock: unanticipated and idiosyncratic positive demand shock $= 10\%$ of GDP

Examine spillovers:

- Across the yield curves (short and long rates; and across countries)
- To the exchange rate

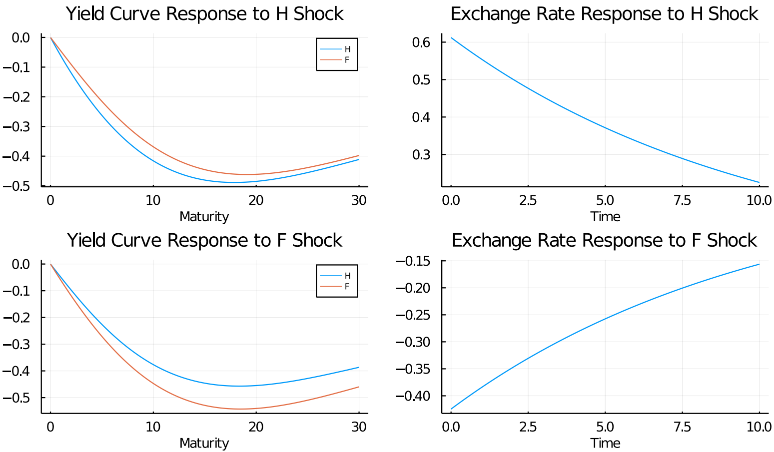
Monetary Shock Spillovers



Implications: small cross-country yield response; exchange rate “delayed overshooting”

- Intuition: correlated short rates, currency demand response

QE Shock Spillovers



Implications: large spillovers of QE, both to foreign yields and exchange rate

- Intuition: correlated short rates, elastic currency traders

Concluding Remarks

- Present an integrated framework to understand term premia and currency risk
- Resulting model ties together
  - Deviations from Uncovered Interest Parity
  - Deviations from Expectation Hypothesis
- Rich transmission of monetary policy domestically and abroad via FX and term premia

Thank You!

Details: Arbitrageur Optimality Conditions

- Ito's Lemma:

$$\frac{\mathrm{d}P_{jt}^{(\tau)}}{P_{jt}^{(\tau)}} = \mu_{jt}^{(\tau)}\,\mathrm{d}t + \boldsymbol{\sigma}_j^{(\tau)}\,\mathrm{d}\mathsf{B}_t$$

$$\frac{\mathrm{d}e_t}{e_t} = \mu_{et}\,\mathrm{d}t + \boldsymbol{\sigma}_e\,\mathrm{d}\mathsf{B}_t$$

where

$$\mu_{jt}^{(\tau)} = \mathsf{q}_t^\top \mathsf{A}_j'(\tau) + C_j'(\tau) + \left[\boldsymbol{\Gamma}(\mathsf{q}_t - \overline{\mathsf{q}})\right]^\top \mathsf{A}_j(\tau) + \frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{\sigma}\mathsf{A}_j(\tau)\mathsf{A}_j(\tau)^\top\boldsymbol{\sigma}\right]$$

$$\mu_e = \left[\boldsymbol{\Gamma}(\mathsf{q}_t - \overline{\mathsf{q}})\right]^\top \mathsf{A}_e + \frac{1}{2}\,\mathrm{Tr}\left[\boldsymbol{\sigma}\mathsf{A}_e\mathsf{A}_e^\top\boldsymbol{\sigma}\right]$$

$$\boldsymbol{\sigma}_j^{(\tau)} = -\mathsf{A}_j(\tau)^\top\boldsymbol{\sigma}$$

$$\boldsymbol{\sigma}_e = -\mathsf{A}_e^\top\boldsymbol{\sigma}$$

Details: Arbitrageur Optimality Conditions

- Arbitrageurs' optimality conditions imply expected excess returns are given by:

$$\mu_{jt}^{(\tau)} - i_{jt} = \mathbf{A}_j(\tau)^\top \mathbf{\Lambda}_t$$
$$\mu_{et} + i_{Ft} - i_{Ht} = \mathbf{A}_e^\top \mathbf{\Lambda}_t$$

- Endogenous coefficients $\mathbf{A}_j(\tau), \mathbf{A}_e$ govern sensitivity to market price of risk $\mathbf{\Lambda}_t$

$$\mathbf{\Lambda}_t = a\mathbf{\Sigma}\left(W_{Ft}\mathbf{A}_e + \sum_{j=H,F}\int_0^T X_{jt}^{(\tau)}\mathbf{A}_j(\tau)\,\mathrm{d}\tau\right)$$

where $\mathbf{\Sigma} \equiv \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$

Details: Preferred-Habitat Bond and FX Investors

- Demand for bonds in currency j, of maturity τ:

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log P_{jt}^{(\tau)} - \theta_j(\tau)\beta_{jt}$$

  - $\alpha_j(\tau)$: demand elasticity for τ investor in country j
  - $\theta_j(\tau)$: how variations in factor β_{jt} affect demand for τ investor in country j
- Demand for foreign currency (spot):

$$Z_{et} = -\alpha_e \log e_t - \theta_e \gamma_t$$

  - Can accommodate forward demand. Under CIP, equivalent to spot + H and F bond trades
- Market clearing and zero net supply: $X_{jt}^{(\tau)} = -Z_{jt}^{(\tau)}$ and $W_{Ft} = -Z_{et}$
  - WLOG: can rewrite intercept terms to include positive supply
- Rewrite using affine functional form:

$$X_{jt}^{(\tau)} = -\alpha_j(\tau)\left[\mathbf{A}_j(\tau)^\top \mathbf{q}_t + C_j(\tau)\right] + \mathbf{\Theta}_j(\tau)^\top \mathbf{q}_t + \zeta_j(\tau)$$
$$W_{Ft} = -\alpha_e\left[\mathbf{A}_e^\top \mathbf{q}_t + C_e\right] + \mathbf{\Theta}_e^\top \mathbf{q}_t + \zeta_e$$

Details: Solution Characterization

- Substitute market clearing into arbitrageur optimality conditions, collect $\mathbf{q}_t$ terms:

$$\mathsf{A}_j'(\tau) + \mathsf{M}\mathsf{A}_j(\tau) - \mathbf{e}_j = \mathbf{0}, \quad \mathsf{M}\mathsf{A}_e - (\mathbf{e}_H - \mathbf{e}_F) = \mathbf{0} \quad (\text{where } \mathbf{e}_j^\top \mathbf{q}_t = i_{jt})$$

- The matrix M is defined as

$$\begin{aligned} \mathsf{M} = \boldsymbol{\Gamma}^\top - a\Big\{ & \int_0^T \left[-\alpha_H(\tau)\mathsf{A}_H(\tau) + \boldsymbol{\Theta}_H(\tau)\right] \mathsf{A}_H(\tau)^\top \,\mathrm{d}\tau \\ & + \int_0^T \left[-\alpha_F(\tau)\mathsf{A}_F(\tau) + \boldsymbol{\Theta}_F(\tau)\right] \mathsf{A}_F(\tau)^\top \,\mathrm{d}\tau \\ & + \left[-\alpha_e \mathsf{A}_e + \boldsymbol{\Theta}_e\right] \mathsf{A}_e^\top \Big\}\boldsymbol{\Sigma} \end{aligned} \tag{1}$$

- Initial conditions $\mathsf{A}_j(0) = \mathbf{0}$. Hence

$$\mathsf{A}_j(\tau) = \left[\mathbf{I} - e^{-\mathsf{M}\tau}\right] \mathsf{M}^{-1}\mathbf{e}_j \tag{2}$$

$$\mathsf{A}_e = \mathsf{M}^{-1}(\mathbf{e}_H - \mathbf{e}_F) \tag{3}$$

Details: Existence and Uniqueness

- Note: $\mathbf{M}$ appears on both sides of equation (1), through the solution of the affine coefficients (2), (3)
    - Interpretation: risk-adjusted dynamics of the risk factors
- In general: system of J^2 nonlinear equations in J^2 unknowns, where $J = \dim \mathbf{q}_t$
- Under risk neutrality ($a = 0$), the solution is simple: $\mathbf{M} = \mathbf{\Gamma}^\top$
- When $a > 0$, the solution may not exist, or there may be multiple equilibria
- Can show (using IFT) that in a neighborhood of $a = 0$, the solution exists and is (locally) unique. Beyond that, very difficult to prove anything analytically in the fully general version of the model

Numerical Solution: Algorithm

- Numerical solution for $\mathbf{M}$ in the general model
- Continuation algorithm:
  1. For $\hat{a} = \hat{a}^{(0)} = 0$, the known solution is $\mathbf{M}^{(0)} = \mathbf{\Gamma}^\top$
  2. Given a solution $\mathbf{M}^{(n)}$ for $\hat{a} = \hat{a}^{(n)}$, use this as the initial value for $\hat{a}^{(n+1)} = \hat{a}^{(n)} + \epsilon$
  3. Solution $\mathbf{M}^{(N)} = \mathbf{M}$ for $\hat{a}^{(N)} = a$
- For our purposes, we use a fine grid (small fixed step size ϵ)
- $\implies$ the algorithm doubles as an equilibrium selection criteria: we trace out the solution which uniquely converges to the risk-neutral benchmark when $a \to 0$

Numerical Solution: Laplace Transformations

- In order to solve the model numerically, we need to parameterize the habitat functions $\alpha_j(\tau), \theta_j(\tau)$. Our approach:

$$\alpha_j(\tau) = \alpha_{j0} e^{-\alpha_{j1}\tau}$$
$$\theta_j(\tau) = \theta_{j0} \tau e^{-\theta_{j1}\tau}$$

  - Implies price elasticities are declining in τ, yield elasticities are single peaked
  - Demand functions are single-peaked
- If we take maximum maturity $T \to \infty$, then we can use properties of Laplace transforms to simplify the fixed point problem characterizing $\mathbf{M}$
- Implies $\mathcal{A}(s) \equiv \mathcal{L}\left\{\mathbf{A}(\tau)\right\}(s)$ given by:

$$s\mathcal{A}(s) + \mathbf{M}\mathcal{A}(s) - \frac{1}{s}\mathbf{e}_i = \mathbf{0} \implies \mathcal{A}(s) = \left[s\mathbf{I} + \mathbf{M}\right]^{-1}\left[\frac{1}{s}\mathbf{e}_i\right]$$