Optimal Macro-Financial Stabilization in a New Keynesian Preferred Habitat Model*

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Abstract

We develop a general equilibrium model featuring heterogeneous households, nominal rigidities, and limits to arbitrage due to segmentation in long-term bond markets. Even when conventional monetary policy can stabilize aggregate fluctuations, the presence of market segmentation implies excessively volatile term premia in long-term yields, imperfect risk sharing, and consumption and labor dispersion. The effectiveness of conventional policy alone is limited; to improve welfare, the central bank must reduce the volatility of short-rate fluctuations, but this implies a degree of macroeconomic volatility. However, when the central bank has access to balance sheet tools, we derive a separation principle for optimal policy: conventional policy stabilizes the output gap while unconventional policy stabilizes risk premia. Only when the short rate is constrained should balance sheet policy be used for macroeconomic stabilization, but this comes at the cost of imperfect financial stabilization.

Keywords: quantitative easing, market segmentation, optimal monetary policy, term structure

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1 Introduction

Central banks responded aggressively to worsening financial conditions and growing recessionary pressure during the global financial crisis of 2007-8. In addition to steep cuts in policy rates, central banks undertook various unconventional policy actions; the most salient of these were the quantitative easing (QE) programs carried out by the Federal Reserve. The Fed continued to utilize QE programs during the onset of COVID-19, and further began implementing quantitative tightening (QT) in response to growing inflationary pressure starting in 2022.

The purpose of this paper is to study the welfare consequences of unconventional monetary policy, and how the design of balance sheet policies interacts with the conduct of more conventional interest rate policies. We develop a tractable general equilibrium model with market segmentation, nominal frictions, and household heterogeneity. We start with a conventional New Keynesian model, where firms produce using differentiated labor but face nominal rigidities when setting prices. However, we depart from the representative household assumption: households in our model have differentiated access to bond markets of different maturities.

We explicitly model bond markets and the determination of the entire term structure; in particular, we embed in our dynamic general equilibrium model a segmented bond market. Households borrow through differentiated bond funds; these "preferred habitat" or "clientele investors" (pension funds, mutual funds) introduce a degree of market segmentation. However, specialized bond arbitrageurs (hedge funds, fixed income broker-dealers) re-integrate markets; but when arbitrageur risk-bearing capacity is imperfect, this integration is only partial.

Our setup introduces the possibility of imperfect risk-sharing, consumption dispersion, and labor dispersion across households. Household consumption and savings decisions now take place across the entire term structure of bond returns. If bond arbitrageurs have perfect risk-bearing capacity, this friction is immaterial in equilibrium; but whenever risk-bearing capacity is imperfect, borrowing rates differ across differentiated households, and therefore optimal decisions do as well.

Bond market frictions have important implications for how monetary policy transmits to households and the aggregate economy. First, consider the key mechanisms of conventional monetary policy. Changes in the policy (short-term) interest rate are transmitted to households only via segmented bond markets. In particular, the interactions of preferred habitat funds and bond arbitrageurs imply portfolio rebalancing; the risk exposure of arbitrageurs therefore changes in response to short rate movements. With imperfect risk-bearing capacity, this implies fluctuations in term premia. Thus, a change in the policy rate is not transmitted one-for-one to households; moreover, the transmission differs across the term structure and therefore across differentiated households.

Next, consider unconventional (QE or QT) policies. Central bank asset purchases and sales directly induce portfolio rebalancing amongst bond market investors. Once again, when risk-bearing capacity of bond markets is imperfect, such rebalancing implies changes in term premia, even when the policy rate is unchanged. Because households borrow at rates across the term structure, these policies also affect household consumption decisions.

From the perspective of an "aggregate Euler equation" channel of monetary policy, our model thus implies that conventional and unconventional policies are somewhat substitutable: either policy can be used to target borrowing rates faced by households. In fact, we show that to a first order, the aggregate dynamics of output and inflation in our model are the same as a model in which a representative household borrows at a weighted average of bond returns across the term structure. This implies that if the central bank loss function only depends on the volatility of output and inflation, both conventional and unconventional policies can achieve identical outcomes. In particular, if "divine coincidence" holds, then either policy tool can achieve first-best.

However, we show that such a policy loss function is not optimal from a social welfare perspective. Both policies lead to variation in term premia, and excess fluctuations in term premia implies excess dispersion in borrowing rates. Fluctuations across the term structure imply differentiated consumption, savings, and labor supply decisions across households. This dispersion results in utility losses relative to the first-best because of imperfect risk-sharing and efficiency losses due to differentiated labor markets. We show that neither tool alone can achieve first-best.

We therefore derive the optimal mix of policy rules when the central bank is maximizing social welfare. When short-rate and balance policy tools are unconstrained, we derive an optimal separation result: conventional policy targets macroeconomic stability (inflation and output gap volatility), while unconventional policy targets financial stability (excess fluctuations in term premia). When divine coincidence holds, this policy achieves first-best.

However, when policy constraints bind, policy must balance tradeoffs. First, if the central bank faces balance sheet constraints (which we model as deadweight holding cost losses), we show that optimal policy implies that the short rate must be less reactive to aggregate shocks in order to minimize financial disruptions. However, this necessarily comes at the cost of increased macroeconomic instability. Second, if the central bank faces constraints on short-rate policy (which we model formally as deadweight costs of short rate changes), then QE must be used to offset macroeconomic shocks. However, this comes at the cost of financial stability: term premia are more volatile than in the first-best, and thus consumption and labor dispersion causes social welfare losses.

Thus, policy constraints imply tradeoffs between macroeconomic and financial stabilization. Our findings apply when the central bank pursues simple time-consistent policy rules (where policy tools are functions of the natural state variables only). We also derive optimal policy results when the central bank has full commitment and can choose policy tools freely as a function of current and past realizations of the economy. Such policies are welfare-improving over simple policy rules. For instance, when the central bank can only utilize conventional policy, optimal policy under full commitment implies interest rate changes are smoothed out relative to the optimal time-consistent short rate rule. By smoothing out interest rate changes, short rate volatility is lowered and hence in equilibrium term premia are smaller. Relative to simple time-consistent policy rules, the entire expected path of short rates can be utilized to keep output gaps small. However, such policies still cannot achieve first-best (unless both short rate and balance sheet policies can be utilized without frictions).

A general message of our model is that implementation matters for welfare. While we present a tractable, stylized model where term premia fluctuations lead to welfare losses through household consumption and labor dispersion, the lessons of our model are transferable to richer models; for instance, if firms face segmented access to bond and asset markets, excessive fluctuations in risk premia across the term structure and asset classes would have similar implications. Our focus is a tractable model which can deliver clear analytical results.

Our paper builds on the seminal preferred habitat model of Vayanos and Vila (2021), which formalizes the original concept as described in Modigliani and Sutch (1966).[1] The main insight of preferred habitat models is that the interaction of clien-

[1] Other important theoretical contributions to preferred habitat models are Greenwood and

tele investors implies important departures from the expectations hypothesis in the determination of the term structure of interest rates. More concretely, demand and supply shocks in these markets induce changing risk exposure on the part of marginal bond investors; when risk-bearing capacity is imperfect, this implies fluctuations in risk premia.[2] These models are typically partial equilibrium models. One exception is Ray et al. (2024), which uses a quantitative version of the model in this paper to study the positive implications of QE policies. That paper considers a richer risk factor structure and a wider set of assets (both riskless and risky bonds), but households are representative; hence, the normative implications explored in this paper are absent.

Our work more generally contributes to a large literature studying the effects of QE in general equilibrium models. These models feature various forms of financial frictions such as bank balance sheet constraints which break the textbook "QE neutrality" results (e.g. see Gertler and Karadi 2011, Gertler and Karadi 2013, Cúrdia and Woodford 2011, Chen et al. 2012, Sims and Wu 2020, Karadi and Nakov 2020, Iovino and Sergeyev 2023, Carlstrom et al. 2017, Ippolito et al. 2018.) Our paper also relates to models of market segmentation which study various forms of macroeconomic stabilization or macro-prudential policy (e.g. see Andrés et al. 2004, Cui and Sterk 2021, Auclert 2019, Angeletos et al. 2023, Debortoli and Galí 2017)

Finally, our model overlaps with similar work in an international setting. Most closely related to our optimal policy results is Itskhoki and Mukhin (2023), who study the optimal mix of conventional policy and FX interventions in an open economy setting. Theoretically more closely linked to our framework, Gourinchas et al. (2022) and Greenwood et al. (2023) also study the determination of bond yields and exchange rates in a similar preferred habitat model, though these papers do not study general equilibrium effects.

Our paper is structured as follows. Section 2 describes the private agents in the model, characterizes general equilibrium, and derives the social loss function which the policymaker seeks to minimize. Section 3 studies the aggregate dynamics of the model

Vayanos (2014), Greenwood et al. (2016), King (2019b), King (2019a), Kekre et al. (2024).

[2]Empirically, there is strong evidence of the existence of demand and supply "preferred habitat" frictions considered in this paper, and that these frictions are important for understanding the transmission of large-scale asset purchases (e.g. see Krishnamurthy and Vissing-Jorgensen 2011, D'Amico and King 2013, Li and Wei 2013, Krishnamurthy and Vissing-Jorgensen 2012, Cahill et al. 2013, King 2019b Fieldhouse et al. 2018, Di Maggio et al. 2020, Debortoli et al. 2020).

under ad-hoc policy rules. Section 4 studies optimal policy for simple time-consistent rules, while Section 5 extends these results to the case of full commitment. Section 6 discusses additional extensions and tests of our model, and Section 7 concludes.

2 Model

Time is continuous and denoted by $t \in (0, \infty)$. The model is made up of the following set of agents. A household (HH) sector is formally comprised of differentiated members making labor and consumption decisions. Each household member supplies differentiated labor to productive firms. Intermediate goods are produced by monopolistically competitive firms using labor; these firms set prices but face nominal rigidities in the form of Rotemberg pricing frictions. Differentiated goods are aggregated by a perfectly competitive final goods retail sector.

Additionally, in our model household members cannot trade a complete set of financial securities. Instead, each household member trades with a maturity-specific bond mutual fund. Bonds of maturity $\tau \in (0, T)$ are traded in financial markets populated by preferred habitat mutual funds and specialized bond arbitrageurs. We assume that households cannot trade bonds directly, but instead make borrowing and lending decisions through preferred habitat funds. These funds trade bonds of specific maturities, both on behalf of their household clients as well as for their proprietary trading desk. Arbitrageurs trade bonds across the entire term structure, but have limited risk-bearing capacity: formally, these agents solve a myopic mean-variance portfolio problem. These agents are owned by the household sector, but due to financial frictions do not price bonds using the household stochastic discount factor (SDF). We represent the arbitrageur portfolio problem as a function of risk aversion (formally, preferences of the arbitrageurs); however, we treat this risk aversion parameter as a proxy for risk-bearing capacity of financial intermediaries. This is a friction which hinders arbitrageurs' ability to trade assets perfectly on behalf of households as a whole.

The monetary authority sets the short term nominal interest rate, and conducts balance sheet policies. A fiscal authority provides optimal production and labor subsidies but is otherwise passive and balances the budget period by period.

We study versions of the model where the monetary authority chooses policy according to an ad-hoc Taylor rule, and compare this to policy rules which are chosen to maximize social welfare (possibly subject to implementation frictions). We focus on a linearized equilibrium and second-order welfare approximations. Our linear-quadratic approximation is around a deterministic first-best steady state, where our approximation method still allows for non-zero bond term premia which affect first-order macroeconomic dynamics.

2.1 Setup

Before describing the set of investors, we start with bond prices as a function of arbitrary sets of risk factors. The price of a τ bond (which pays one dollar at period $t+\tau$) is $P_t^{(\tau)}$, and the return is given by:

$$\frac{\mathrm{d}P_t^{(\tau)}}{P^{(\tau)}} = \mu_t^{(\tau)}\,\mathrm{d}t + \boldsymbol{\sigma}_t^{(\tau)}\,\mathrm{d}\mathbf{B}_t\,, \tag{1}$$

where $\mu_t^{(\tau)}$ represents the mean short-horizon return of a τ bond, and $\boldsymbol{\sigma}_t^{(\tau)}$ represents how shocks to risk factors (the vector of Brownian terms $\mathbf{B}_t$) lead to fluctuations in returns. These objects are endogenous and determined in equilibrium (which we describe in detail below); however, all agents take these objects as given.

Bond prices and yields are related in the usual way: $y_t^{(\tau)} = -\log P_t^{(\tau)}/\tau$. In equilibrium, taking the limit as maturity $\tau \to 0$, we recover the nominal risk-free interest rate: $y_t^{(\tau)} \to i_t$. The short rate is the conventional policy tool and is controlled by the central bank.

When convenient, we work with an arbitrary set of risk factors. However, we also separately consider two baseline models. In the first, we assume one aggregate risk factor: technology follows a (log) drift-diffusion OrnsteinUhlenbeck process:

$$\mathrm{d}z_t = -\kappa_z z_t\,\mathrm{d}t + \sigma_z\,\mathrm{d}B_{z,t}\,. \tag{2}$$

In the second, we consider a version of the model where the only source of risk comes from monetary policy shocks. We consider Taylor-type of rule policies so that the (linearized) dynamics of the policy rate are given by

$$\mathrm{d}i_t = -\kappa_i(i_t - \phi_\pi \pi_t - \phi_x x_t)\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_{i,t}\,, \tag{3}$$

where π_t and x_t are inflation and the output gap (in terms of log-deviations from steady state, discussed below). The coefficients ϕ_π and ϕ_x are standard Taylor rule coefficients, which govern how the central bank changes the policy rate in response to macroeconomic fluctuations. The term κ_i represents the degree of inertia in the policy rule; as $\kappa_i \to \infty$, we recover a Taylor rule which has no persistence. The Brownian term $\sigma_i \, \mathrm{d}B_{i,t}$ represents stochastic shocks to the monetary policy rule, which we take as given.

The first model will be our baseline for studying optimal policy; we use the second to derive analytical results regarding macroeconomic dynamics when taking as given a simple policy rule.

2.2 Intermediate Firms

A continuum of intermediate goods producers index by $j \in [0,1]$ produce differentiated goods $Y_t(j)$ and set prices $P_t(j)$. Final goods Y_t are produced by a competitive retail sector, which aggregates according to $Y_t \equiv \left[\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} \, \mathrm{d}j\right]^{\frac{\epsilon}{\epsilon-1}}$, where the elasticity of substitution between goods is ϵ. This implies the follow demand and price index for differentiated goods

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t, \quad P_t = \left[\int_0^1 P_t(j)^{1-\epsilon} \, \mathrm{d}j\right]^{\frac{1}{1-\epsilon}}. \tag{4}$$

Firms produce according to the production technology $Y_t(j) = Z_t L_t(j)$. Aggregate technology $Z_t \equiv e^{z_t}$ evolves according to (2). The index of labor input $L_t(j)$ is given by

$$L_t(j) \equiv \left[\int_{h \in \mathcal{H}} L_t(j,h)^{\frac{\epsilon_w-1}{\epsilon_w}} \, \mathrm{d}h\right]^{\frac{\epsilon_w}{\epsilon_w-1}}, \tag{5}$$

where $L_t(j,h)$ is firm j's demand for h-type labor supplied by the household sector $\mathcal{H}$ (discussed below), and ϵ_w is the elasticity of substitution between labor varieties. Firms hire type-h labor at the nominal wage $\mathcal{W}_t(h)$ (which is taken as given by firms). Cost minimization implies the following demand and wage index for labor varieties

$$L_t(j,h) = \left(\frac{\mathcal{W}_t(h)}{\mathcal{W}_t}\right)^{-\epsilon_w} L_t(j), \quad \mathcal{W}_t = \left[\int_{h \in \mathcal{H}} \mathcal{W}_t(h)^{1-\epsilon_w} \, \mathrm{d}h\right]^{\frac{1}{1-\epsilon_w}}. \tag{6}$$

When choosing prices $P_t(j)$, intermediate goods producers face the following convex costs of adjustment

$$\Theta(\pi_t(j)) = \frac{\theta}{2}\pi_t(j)^2 P_t Y_t, \tag{7}$$

where $\pi_t(j)$ is the inflation rate of firm j: $\mathrm{d}P_t\,(j) = P_t(j)\pi_t(j)\,\mathrm{d}t$.

Nominal profits of firm j are therefore given by

$$\mathcal{F}_t(P_t(j), Y_t(j), \pi_t(j)) = (1+\tau^y)P_t(j)Y_t(j) - \mathcal{W}_t L_t(j) - \Theta(\pi_t(j)) - \mathcal{T}_t, \tag{8}$$

where τ^y is a production subsidy and $\mathcal{T}_t$ are nominal lump-sum taxes set by the fiscal authority (described below). Combining with equations (4), (6), and (7) and firm production, real profits are given by

$$\frac{\mathcal{F}_t(j)}{P_t} = Y_t\left[(1+\tau^y)\left(\frac{P_t(j)}{P_t}\right)^{1-\epsilon} - \frac{W_t}{Z_t}\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} - \frac{\theta}{2}\pi_t(j)^2\right] - \frac{\mathcal{T}_t}{P_t}, \tag{9}$$

where the real wage index $W_t \equiv \mathcal{W}_t/P_t$. The firm problem at time $t=0$ is therefore

$$U_0 \equiv \max_{\{\pi_t(j)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} Q_t^{\mathcal{H}} \frac{\mathcal{F}_t(j)}{P_t}\,\mathrm{d}t \quad \text{s.t. } \mathrm{d}P_t\,(j) = P_t(j)\pi_t(j)\,\mathrm{d}t\,. \tag{10}$$

Firms transfer profits to households, thus profits are discounted by $e^{-\rho t}Q_t^{\mathcal{H}}$, the real SDF of households (defined below).

Note that in a symmetric equilibrium (which is obtained when initial prices are equalized $P_0(j) \equiv P_0$), we have that the aggregate dynamics of the price index $\mathrm{d}P_t = P_t\pi_t\,\mathrm{d}t$ are (locally) non-stochastic. Additionally, when the production subsidies are self-financing ($\int_0^1 \tau^y P_t(j)Y_t(j)\,\mathrm{d}j = \mathcal{T}_t$), real profit transfers to households are given by

$$\frac{\mathcal{F}_t}{P_t} \equiv \int_0^1 \frac{\mathcal{F}_t(j)}{P_t}\,\mathrm{d}j = Y_t\left(1 - \frac{W_t}{Z_t} - \frac{\theta}{2}\pi_t^2\right). \tag{11}$$

Note that the costs of price adjustments are deadweight loss.

2.3 Households

Households are made up of a "head of household" as well as a continuum of household members, denoted by $h \equiv h(i,\tau) \in \mathcal{H}$. There is a mass $\eta(\tau)$ of each τ group:

$\int_i h(i,\tau)\,\mathrm{d}i = \eta(\tau)$, where $\int_0^T \eta(\tau)\,\mathrm{d}\tau = 1$. Within each τ group, each member is identical. A household member h chooses consumption of the final good $C_t(h)$ and supplies differentiated labor $N_t(h)$. We assume that household members set the nominal wage $\mathcal{W}_t(h)$ frictionlessly, taking as given the demand for differentiated labor in (6). When borrowing or saving, τ-type households face the τ-maturity bond return in (1) (which they take as given).

Each household member faces the same per-period flow utility function

$$u\left(C_t(h), N_t(h)\right) = \frac{C_t(h)^{1-\varsigma} - 1}{1-\varsigma} - \frac{N_t(h)^{1+\varphi}}{1+\varphi}, \tag{12}$$

where ς is the inverse intertemporal elasticity of substitution and φ is the inverse Frisch labor elasticity. Households face a discount factor ρ, so the resulting value function and budget constraint of an h household member at time $t = 0$ is given by:

$$V_0(h) \equiv \max_{\{C_t(h), N_t(h)\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^\infty e^{-\rho t} u\left(C_t(h), N_t(h)\right) \mathrm{d}t \tag{13}$$

$$\text{s.t.}\;\; \mathrm{d}A_t\,(h) = \left[(1+\tau^w)\mathcal{W}_t(h)N_t(h) - P_t C_t(h)\right]\mathrm{d}t + A_t(h)\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} + \mathrm{d}F_t\,(h), \tag{14}$$

where $A_t(h)$ is nominal wealth and $\mathrm{d}F_t\,(h)$ are (flow) nominal transfers (from firms, funds, arbitrageurs, the government, and the head of the household, defined below). The term τ^w is a labor supply subsidy. In equilibrium, transfers follow some (endogenous) process taken as given by household member h

$$\mathrm{d}F_t\,(h) = \mu_t^F(h)\,\mathrm{d}t + \boldsymbol{\sigma}_t^F(h)\,\mathrm{d}\mathbf{B}_t\,.$$

Define real wealth $a_t(h)$, Ito's lemma along with the return process for bonds in (1) implies

$$\begin{aligned} \mathrm{d}a_t\,(h) = & \left[(1+\tau^w)W_t(h)N_t(h) - C_t(h) + a_t(h)(\mu_t^{(\tau)} - \pi_t) + \frac{\mu_t^F(h)}{P_t}\right]\mathrm{d}t \\ & + \left[a_t(h)\boldsymbol{\sigma}_t^{(\tau)} + \boldsymbol{\sigma}_t^F(h)\right]\mathrm{d}\mathbf{B}_t\,, \end{aligned} \tag{15}$$

where $W_t(h)$ is the real wage of household member h (and note we have used the fact that the price level P_t is locally non-stochastic).

2.4 Preferred Habitat Mutual Funds and Arbitrageurs

In addition to the household members, bonds are traded by a continuum of "preferred habitat" mutual funds (who specialize in bonds of specific maturities τ) as well as a set of representative arbitrageurs (who buy and sell bonds across the entire term structure).

A τ-maturity preferred habitat fund desires to trade bonds for proprietary reasons. Their demand is exogenous and takes the same form as in Vayanos and Vila (2021)

$$Z_t^{(\tau)} = -\alpha(\tau) \log P_t^{(\tau)} - \beta_t^{(\tau)}. \tag{16}$$

This implies that habitat bond traders rebalance for two reasons: first, $\alpha(\tau) > 0$ implies that they are price-elastic. Second, $\beta_t^{(\tau)}$ represents a potentially time-varying demand shock which is independent of the demand from households. In the baseline model, we abstract from demand risk. We consider separately the model where stochastic demand risk is an aggregate risk factor.

Letting $\omega_t^{(\tau)}$ represent the wealth of the τ-maturity habitat fund, their budget constraint is given by

$$\mathrm{d}\omega_t^{(\tau)} = \left[\omega_t^{(\tau)} - Z_t^{(\tau)}\right] i_t \,\mathrm{d}t + Z_t^{(\tau)} \frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}. \tag{17}$$

That is, τ-maturity bond funds fund their position (given by proprietary demand (16)) in τ-maturity bonds using the short-term borrowing i_t. Any profits or losses of τ-habitat funds are transferred to the household sector.

Unlike habitat funds, bond arbitrageurs choose holdings of short- and long-maturity bonds. The representative arbitrageur has wealth ω_t and solves the following mean-variance problem:

$$\max \mathbb{E}_t \,\mathrm{d}\omega_t - \frac{a}{2} \mathbb{V}\mathrm{ar}_t \,\mathrm{d}\omega_t \tag{18}$$

$$\text{s.t.} \;\; \mathrm{d}\omega_t = \omega_t i_t \,\mathrm{d}t + \int_0^T X_t^{(\tau)} \left(\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} - i_t \,\mathrm{d}t \right) \mathrm{d}\tau \,. \tag{19}$$

That is, they choose bond holdings $X_t^{(\tau)}$ across all maturities τ. Arbitrageurs choose to engage in carry trades across the term structure in order to optimally satisfy the trade-off between higher expected returns and the volatility on their balance sheet.

This problem is governed by the risk aversion coefficient a. Like the habitat funds, any profits or losses are transferred to the household sector. The risk aversion parameter a is a proxy for capital constraints or Value-at-Risk constraints. More generally, the arbitrageur risk aversion coefficient is a stand-in for the financial frictions which imply that bond returns are not determined as in a model with a representative household with perfect access to financial securities. We return to this point in the discussion of our linear-quadratic approximation method.

2.5 Government

The fiscal authority sets optimal production and labor subsidies τ^y and τ^w such that the steady state price and wage markups are zero: $\tau^y = (\epsilon - 1)^{-1}, \tau^w = (\epsilon^w - 1)^{-1}$. These are self-financed through lump sum taxes on firms and households, respectively. The central bank sets the nominal interest rate i_t, and the fiscal authority pays this interest i_t on short-term bonds (reserves). Besides the production and labor subsidies, the fiscal authority is passive: it levies lump-sum taxes or transfers $P_t T_t$ on households in order to balance the budget each period, so that $P_t T_t = -B_t i_t$ where B_t is the aggregate demand for short-term bonds (reserves). The central bank may also conduct balance sheet operations by taking non-zero positions in bonds $S_t^{(\tau)}$. Any proceeds from the central bank bond holdings are renumerated lump-sum to the households.

In our baseline model, we assume that the central bank can utilize balance sheet tools and the short rate without any constraints. However, we also consider cases where bond holdings are subject to real deadweight holding costs (measured in terms of output) which take the form:

$$Y_t \Psi_t^S \equiv Y_t \int_0^T \frac{\psi^{(\tau)}}{2} \left(S_t(\tau)\right)^2 \mathrm{d}\tau \,. \tag{20}$$

Thus, whenever $\psi^{(\tau)} > 0$, non-zero central bank holdings will imply deadweight losses. Additionally, the central bank may also face real convex costs of adjustment when setting the short-term interest rate:

$$Y_t \Psi_t^i \equiv Y_t \frac{\psi^i}{2} \left(i_t - \bar{i}_t\right)^2 , \tag{21}$$

where $\bar{i}_t$ represents a potentially time-varying (exogenous) policy target. We interpret (21) as capturing in reduced-form constraints such as the effective lower bound; however, to maintain our linear-quadratic approximations, we assume a symmetric loss function. Thus, whenever $\psi^i > 0$, changes in the short-term interest rate away from $\bar{i}_t$ will imply deadweight losses.

The government levies taxes

$$Y_t \Psi_t \equiv Y_t \Psi_t^S + Y_t \Psi_t^i, \tag{22}$$

which are paid lump-sum by households. These taxes fund the costs associated with policy frictions arising from holding costs or short-rate targeting.

2.6 Equilibrium

We first discuss the intuition behind the equilibrium forces in the model. From the arbitrageurs' problem, mean-variance preferences imply that bond holdings increase with expected returns and decrease with volatility of bonds. In other words, these arbitrageurs engage in more aggressive bond carry trades when expected returns are higher (all else equal). But preferred habitat demand also reacts to bond price fluctuations. When prices increase (yields decrease), bond habitat funds reduce their holdings of bonds (all else equal). Thus, in equilibrium, any shocks which affect the term structure of interest rates will lead to equilibrium re-balancing in the bond market.

Turning to the household sector, households make otherwise standard consumption, savings, and labor decisions. But the borrowing rates faced by household members potentially differ whenever expected returns move. Thus, in addition to the forces described above, in equilibrium any shocks which affect the term structure of interest rates will potentially lead to differentiated consumption and labor choices across households.

Therefore, sources of inefficiency arise from the deadweight losses associated with changes in prices (due to nominal rigidities); production inefficiencies from dispersion in labor supply (due to labor market frictions); and imperfect risk-sharing from dispersion in consumption and savings decisions (due to the bond market segmentation). These frictions show up in aggregate dynamics as well: nominal rigidities will have effects on the aggregate price-setting behavior of firms and thus on inflation. Moreover, aggregate household consumption and labor supply will depend on the entire term structure of interest rates as well. Finally, bond prices will depend not only on the portfolio rebalancing channels discussed above, but also on the expected movements of risk factors and the short-term interest rate, which through central bank decisions will depend on aggregate dynamics. When choosing policy, the central bank will attempt to undo any harmful effects of these frictions.

2.7 Aggregation and Market Clearing

In a symmetric equilibrium, all firms make the same decisions and so $Y_t(j) = Y_t$, $P_t(j) = P_t$, $\pi_t(j) = \pi_t$, $L_t(j,h) = L_t(h)$. Each τ-type of household members $h(i,\tau) = h(i',\tau)$ are identical. Thus, $C_t(i,\tau) \equiv C_t(\tau)$, $N_t(i,\tau) \equiv N_t(\tau)$, and $A_t(i,\tau) \equiv A_t(\tau)$. Aggregate household consumption is therefore

$$C_t = \int_{h\in\mathcal{H}} C_t(i,\tau)\,\mathrm{d}h = \int_0^T \eta(\tau)C_t(\tau)\,\mathrm{d}\tau\,. \tag{23}$$

Labor market clearing implies $L_t(h) = N_t(h)$. Additionally, wages are equalized across $h(i,\tau) = h(i',\tau)$ household members: $\mathcal{W}_t(i,\tau)$. Then aggregate demand for the labor index (defined in (6)) is given by

$$L_t = \left[\int_0^T \eta(\tau)N_t(\tau)^{\frac{\epsilon_w - 1}{\epsilon_w}}\,\mathrm{d}\tau\right]^{\frac{\epsilon_w}{\epsilon_w - 1}}. \tag{24}$$

Thus, aggregate labor supply $N_t \equiv \int_{h\in\mathcal{H}} N_t(h)\,\mathrm{d}h = \int_0^T \eta(\tau)N_t(\tau)\,\mathrm{d}\tau$ differs from aggregate labor index demand L_t whenever $N_t(\tau) \neq N_t(\tau')$.

Using (23) and (24), market clearing in goods and production therefore implies

$$Y_t = Z_t L_t, \tag{25}$$

$$C_t = Y_t\left(1 - \frac{\theta}{2}\pi_t^2 - \Psi_t\right), \tag{26}$$

where aggregate consumption differs from aggregate output due to deadweight loss from price changes when $\theta > 0$ (as well as losses from policy frictions when $\psi^{(\tau)} > 0$ or $\psi^i > 0$ in equations (20) or (21)).

The aggregate wealth of the household sector is given by

$$A_t = \int_{h\in\mathcal{H}} A_t(i,\tau)\,\mathrm{d}h = \int_0^T \eta(\tau)A_t(\tau)\,\mathrm{d}\tau\,. \tag{27}$$

Bond market clearing implies

$$X_t^{(\tau)} + Z_t^{(\tau)} + \eta(\tau)A_t(\tau) + S_t^{(\tau)} = 0 \tag{28}$$

for all maturities $\tau > 0$ (and the passive fiscal authority ensures that the short-term bond market clears).

2.7.1 Head-of-Household Transfers

We assume a "head of household" sets inter-member transfers in order to maintain some degree of wealth equality. We consider two such policies. The first, which we make purely for tractability, are such that in equilibrium, wealth is equalized period-by-period: across τ household groups, $A_t(\tau) \equiv A_t$.[3] We make this assumption to place a clear focus on the role market segmentation plays in generating consumption dispersion and imperfect risk-sharing across households. In ongoing work, we relax this assumption and instead assume that the head-of-household levies a "wealth tax" which equalizes wealth across household members in steady state (but not period-by-period). This induces more complicated dynamics in the demand for bonds of different maturities, but does not change our main optimal policy conclusions.

2.8 Optimality Conditions

The arbitrageur optimality conditions are given by

$$\mu_t^{(\tau)} - i_t = \boldsymbol{\sigma}_t^{(\tau)}\boldsymbol{\Lambda}_t, \quad \text{where } \boldsymbol{\Lambda}_t^\top = a\int_0^T X_t^{(\tau)}\boldsymbol{\sigma}_t^{(\tau)}\,\mathrm{d}\tau\,. \tag{29}$$

Hence, arbitrageurs ensure that no risk-free arbitrage opportunities exist. Equation (29) shows that the expected excess return of any τ-maturity bond is a function of bond-specific risk loadings $\boldsymbol{\sigma}_t^{(\tau)}$ (the diffusion terms from (1)) and a global set of

[3] However, consumption is not equalized off-equilibrium because the transfers are conditioned on the maturity type τ of household members. Transfers are not conditional on the specific actions of household member $h(i,\tau)$.

risk prices $\mathbf{\Lambda}_t$. The market price of risk depends on arbitrageur risk aversion a as well as equilibrium holdings $X_t^{(\tau)}$ (which in equilibrium will be determined by market clearing).

The intratemporal optimality conditions for τ-type household members (along with the demand for differentiated labor and the optimal labor subsidy) imply that (log) consumption and labor supply $c_t(\tau), n_t(\tau)$ are related to the differentiated (log) real wage $w_t(\tau)$ according to

$$w_t(\tau) = \varsigma c_t(\tau) + \phi n_t(\tau). \tag{30}$$

We can re-write this in terms of aggregate (log) wage and labor indices w_t, ℓ_t

$$w_t = \varsigma c_t(\tau) + \phi n_t(\tau) + \frac{1}{\epsilon_w}\left(n_t(\tau) - \ell_t\right). \tag{31}$$

The final term arises due to the existence of differentiated labor (which disappears as $\epsilon_w \to \infty$). However, note that even with frictionless labor markets, labor decisions still may differ across households.

Combined with bond price dynamics and household (log-linearized) intertemporal optimality conditions, we find that the (log) consumption $c_t(\tau)$ of τ-type households satisfies a standard Euler equation

$$\mathrm{d}c_t(\tau) = \varsigma^{-1}\left(\mu_t^{(\tau)} - \pi_t - \rho\right)\mathrm{d}t\,, \tag{32}$$

where the only departure from a textbook model is that a τ-type household member makes consumption and savings decisions as a function of the τ-bond return $\mu_t^{(\tau)}$, rather than the short-term policy rate i_t.

Firm (log-linearized) optimality conditions imply that aggregate inflation depends on the real marginal costs faced by firms. In aggregate, firm marginal costs are a function of (log) technology z_t and (log) wage index w_t:

$$\mathrm{d}\pi_t = \left(\rho\pi_t - \delta_w w_t\right)\mathrm{d}t\,, \tag{33}$$

where δ_w is a composite parameter which measures the aggregate degree of price rigidity.

2.9 Social Welfare

We approximate the model around the first-best allocation, where we do so around a "riskless" limiting case: we assume that aggregate risk converges to zero so that $\boldsymbol{\sigma}_t^{(\tau)} \to \mathbf{0}$, but that arbitrageur risk aversion $a \to \infty$, such that the product of risk aversion and risk $a^{1/2} \cdot \boldsymbol{\sigma}_t^{(\tau)} \equiv \tilde{\boldsymbol{\sigma}}_t^{(\tau)}$ remains non-zero and bounded. This assumption captures explicitly that arbitrageur risk aversion a is a measure of imperfect arbitrage. Our approximation method allows for tractable first- and second-order approximations of the model, while still allowing for time-variation in risk premia.

The following Proposition derives the first-best allocation in the riskless limiting case.

Proposition 1 (First-best (natural) allocation)**.** *In the limiting case where $\boldsymbol{\sigma}_t^{(\tau)} \to \mathbf{0}$, the first-best (natural) allocation is obtained when $\theta = 0$ and $a = 0 \iff \tilde{\boldsymbol{\sigma}}_t^{(\tau)} = \mathbf{0}$.*

We denote the first-best as the "natural" equilibrium, in which firm nominal rigidities disappear ($\theta = 0$) and arbitrage is perfect ($a = 0$). We denote aggregate output in the natural equilibrium by Y_t^n, and define the output gap $X_t \equiv \frac{Y_t}{Y_t^n}$ with respect to the natural benchmark.

Aggregating (32) across households, the dynamics of the (log) output gap x_t are governed by a modified aggregate Euler equation:

$$\mathrm{d}x_t = \varsigma^{-1} \left(\tilde{\mu}_t - \pi_t - r_t^n\right) \mathrm{d}t \,. \tag{34}$$

The term $r_t^n \equiv -\kappa_z z_t$ is the "natural" rate, which is the real borrowing rate in the first-best allocation from Prop. 1. The "effective borrowing rate" $\tilde{\mu}_t$ is given by

$$\tilde{\mu}_t = \int_0^T \eta(\tau) \mu_t^{(\tau)} \, \mathrm{d}\tau \,, \tag{35}$$

which is the average borrowing rate faced by the household, weighted by the household member weights $\eta(\tau)$.

Re-writing the firm optimality condition (33) in terms of output gaps gives a New Keynesian Phillips curve:

$$\mathrm{d}\pi_t = \left(\rho \pi_t - \delta x_t\right) \mathrm{d}t \,, \tag{36}$$

where δ measures the aggregate degree of price rigidity.

The dynamics of the output gap x_t and inflation π_t in (34) and (36) imply that, to a first-order, our model is essentially the same as Ray et al. (2024). However, the welfare consequences differ due to the inefficiencies described above. A second-order expansion of social welfare relative to the first-best allocation gives the per-period social loss

$$\mathcal{L}_t \equiv (\varsigma + \varphi)x_t^2 + \theta\pi_t^2 \tag{37}$$

$$+ \frac{\varsigma}{\varphi}\left(\varphi + \varsigma\left[\frac{\varphi\epsilon_w}{1+\varphi\epsilon_w}\right]^2\right)\mathbb{V}\mathrm{ar}_\tau\, c_t^{(\tau)} + \epsilon_w\,\mathbb{V}\mathrm{ar}_\tau\, w_t^{(\tau)} \tag{38}$$

$$+ \int_0^T \psi^{(\tau)}\left(S_t^{(\tau)}\right)^2 \mathrm{d}\tau + \psi^i\left(i_t - \bar{i}_t\right)^2. \tag{39}$$

The first two terms in line (37) capture the welfare losses associated with the nominal rigidities in the model; these terms arise in representative agent New Keynesian (RANK) models. Compared to a standard RANK model, line (38) shows that social welfare loss also depends on terms

$$\mathbb{V}\mathrm{ar}_\tau\, c_t^{(\tau)} \equiv \int_0^T \eta(\tau)\left(c_t^{(\tau)}\right)^2 \mathrm{d}\tau - \left[\int_0^T \eta(\tau)c_t^{(\tau)}\,\mathrm{d}\tau\right]^2,$$

$$\mathbb{V}\mathrm{ar}_\tau\, w_t^{(\tau)} \equiv \int_0^T \eta(\tau)\left(w_t^{(\tau)}\right)^2 \mathrm{d}\tau - \left[\int_0^T \eta(\tau)w_t^{(\tau)}\,\mathrm{d}\tau\right]^2.$$

Thus, increased consumption dispersion across households implies welfare losses, due to imperfect risk-sharing. Additionally, wage dispersion also induces welfare losses, due to production labor market inefficiencies.

The final terms in line (39) represent losses associated with the central bank balance sheet and short rate policies from equations (20) and (21).

3 Aggregate Dynamics

Before studying the welfare consequences of the model, we explore the first-order reactions to monetary policy. To do so, we consider versions of the model where the central bank follows ad-hoc policy rules. In particular, we focus on a version of the model with a simple Taylor rule (subject to shocks) as well as zero-probability QE/QT policies. In the next section, we utilize these insights to explore the optimal design of monetary policy rules.

Intuitively, what does general equilibrium look like in this model? From the perspective of households, the key factor is how sensitive their borrowing rates are to the short rate (and balance sheet policies). The model reduces to a benchmark New Keynesian model when these rates move one-for-one, but in general $\mu_t^{(\tau)} \neq i_t$. Suppose that long-term borrowing rates are highly responsive to the policy rate. Then household borrowing is also highly sensitive to the policy rate, and therefore the growth rate of consumption will also react strongly to the policy rate. On the other hand, when long-term rates are insensitive to the policy rate, the pass-through of changes in the policy rate to households is weakened. Through the borrowing decisions of the household, the growth rate of consumption is less responsive to the policy rate.

However, the sensitivity of the effective borrowing rate to the policy rate is an equilibrium object, which also depends on financial markets. Bond prices will adjust in order to achieve equilibrium in bond markets, such that arbitrageurs' portfolio allocation satisfies their mean-variance tradeoff while also clearing the market given the demand from preferred habitat investors. In this model, arbitrage is imperfect and the term structure will not be characterized by the expectations hypothesis except under special circumstances. Therefore, it is the risk-adjusted dynamics of the macroeconomy which determine bond prices in financial markets, rather than the actual dynamics of the short rate only.

In general, the term structure will be determined by complicated interactions between the different classes of investors in bond markets (arbitrageurs, habitat funds, and households). Because differentiated households also make consumption and savings decisions as a function of different borrowing rates, the general equilibrium dynamics of the macroeconomy will also depend on these interactions. However, two limiting cases can be analyzed immediately. First, if arbitrageurs are risk-neutral ($a = 0$, so they only care about expected returns), then equilibrium can only be achieved if $\mathbb{E}_t\left[\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}}\right] = i_t\,\mathrm{d}t$. And if expected instantaneous returns of all bonds are equalized at the short rate, then risk-neutral arbitrageurs are indifferent between any bond allocation. In this case, they will absorb any demand shifts from preferred habitat or household investors without any equilibrium price changes. In other words, idiosyncratic demand shifts will not affect the term structure of interest rates.

In the other extreme, if arbitrageurs abandon the bond market (allocating the entirety of their wealth to the risk-free short rate), then equilibrium is only satisfied if prices satisfy $\log P_t^{(\tau)} = \beta_t^{(\tau)}/\alpha(\tau)$ (assuming no balance sheet actions of the central bank, so market clearing implies $0 = X_t^{(\tau)} = -Z_t^{(\tau)}$). This would imply that bonds of very close maturity could have very different prices (and would potentially evolve unrelated to the short rate). However, this extreme segmentation does not occur in equilibrium because arbitrageurs will optimally take non-zero positions in long-term bonds. The impact of changes in preferred habitat demand (if any) will depend on how arbitrageurs adjust their portfolio allocations. In turn, this will depend on the equilibrium dynamics of the short rate and other macroeconomics variables.

General equilibrium is obtained when these two forces balance. Thus, characterizing equilibrium involves two key steps: first, understanding the differences between the actual and risk-adjusted dynamics of the economy; and second, linking household savings and consumption choices with the bond prices determined in imperfect financial markets.

The central bank chooses the (nominal) short rate i_t (in terms of deviations from the steady state value) through the following Taylor rule with persistence:

$$\mathrm{d}i_t = -\kappa_i(i_t - \phi_\pi \pi_t - \phi_x x_t)\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_{i,t}\,, \tag{40}$$

where $B_{i,t}$ is a standard Brownian motion and σ_i governs the size of the shocks (relative to arbitrageur risk aversion). The parameters ϕ_π, ϕ_x govern the feedback rule for inflation and output to changes in the policy rate. Inertia in the policy rule is determined by κ_i; if $\kappa_i \to \infty$, (40) simplifies to a Taylor rule with no gradual adjustments in the policy rate. Note that i_t is measured in terms of deviations from the long-run target, which delivers a steady state with zero inflation and zero output gap.

We assume that the central bank also conducts ad-hoc QE/QT policies according to

$$S_t^{(\tau)} = \theta^{QE}(\tau) S_t, \quad \mathrm{d}S_t = -\kappa_S S_t\,\mathrm{d}t\,. \tag{41}$$

A QE (QT) policy is a zero-probability purchase (sale) of τ bonds. We assume that these asset purchases follow a simple factor structure according to S_t, which follow the shock reverts back towards zero. The function $\theta^{QE}(\tau)$ translates movements in S_t into changes in τ-bond holdings of the central bank. For simplicity, we assume that

$\theta^{QE}(\tau) > 0$ so that the central bank purchases or sells bonds across all maturities in the same direction.

We first characterize equilibrium for a general set of risk factors and dynamics, and then apply these results to specific versions of our model. Collect the state variables, jump variables, and Brownian terms into vectors $\mathbf{y}_t$, $\mathbf{x}_t$ and $\mathbf{B}_t$, respectively (all in terms of deviations from steady state). In order to define this set of state variables, we assume that the demand shifter of preferred habitat demand in (16) follows a factor structure

$$\beta_t(\tau) = \sum_{k=1}^{K} \theta^k(\tau) y_{t,k}, \tag{42}$$

where the functions $\theta^k(\tau)$ govern how demand reacts to movements in a state variable $y_{t,k}$.

The following Lemma describes the aggregate dynamics of the model, taking as given the effective borrowing rate. All proofs are in Appendix A.

Lemma 1 (Aggregate dynamics)**.** *Suppose the effective borrowing rate (in terms of deviations from steady state) is given by*

$$\tilde{\mu}_t = \hat{\mathbf{A}}^\top \mathbf{y}_t. \tag{43}$$

Then the linear rational expectations equilibrium is given by

$$\mathrm{d}\mathbf{y}_t = -\mathbf{\Gamma}\mathbf{y}_t \,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t\,, \quad \mathbf{x}_t = \mathbf{\Omega}\mathbf{y}_t, \tag{44}$$

where the matrices $\mathbf{\Gamma}, \mathbf{\Omega}$ are a function of the eigendecomposition of the linearized dynamics of the model (and therefore functions of $\hat{\mathbf{A}}$).

If $\hat{\mathbf{A}} = \mathbf{e}_i$, the vector which "selects" the policy rate $\mathbf{e}_i^\top \mathbf{y}_t = i_t$, then the effective borrowing rate responds one-for-one with the policy rate i_t and the aggregate dynamics of the model reduce to a standard New Keynesian model.

Next, we turn to characterizing the behavior of asset prices. We focus on a solution to the model in which (log) bond prices are affine functions of the state variables, given by (endogenous) coefficient functions:

$$-\log P_t^{(\tau)} = \mathbf{A}(\tau)^\top \mathbf{y}_t + C(\tau). \tag{45}$$

Lemma 2 (Asset prices)**.** *Suppose that the state variables* $\mathbf{y}_t$ *evolve according to equation* (44)*. Then the affine coefficients in equation* (45) *are given by*

$$\mathbf{A}(\tau) = \left[\mathbf{I} - e^{-\mathbf{M}\tau}\right]\mathbf{M}^{-1}\mathbf{e}_i \tag{46}$$

where $\mathbf{e}_i$ *is a vector such that* $\mathbf{e}_i^\top \mathbf{y}_t = i_t$*. The matrix* $\mathbf{M}$ *solves the fixed point problem:*

$$\mathbf{M} = \mathbf{\Gamma}^\top - \int_0^T \left[-\alpha(\tau)\mathbf{A}(\tau) + \mathbf{\Theta}(\tau)\right]\mathbf{A}(\tau)^\top \,\mathrm{d}\tau\, \tilde{\mathbf{\Sigma}}, \tag{47}$$

where $\tilde{\mathbf{\Sigma}} \equiv a \cdot \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$ *and* $\mathbf{\Theta}(\tau)$ *stack the habitat and central bank demand functions* $\theta^k(\tau)$, $\theta^{QE}(\tau)$ *into vectors.*

The matrix $\mathbf{M}$ can be thought of as the risk-adjusted dynamics of the state. In the first-best, arbitrageurs are perfectly risk-neutral ($a = 0 \implies \tilde{\mathbf{\Sigma}} = \mathbf{0}$), so we have $\mathbf{M} = \mathbf{\Gamma}^\top$. However, when $\tilde{\mathbf{\Sigma}} \neq \mathbf{0}$, $\mathbf{M}$ appears on both sides of equation (47) through the affine coefficients $\mathbf{A}(\tau)$.

With the results in Lemmas 1 and 2, we can characterize the equilibrium of the model.

Proposition 2 (Existence and uniqueness)**.** *An affine equilibrium is one in which the state and jump variables evolve according to equations* (44)*, and asset prices are determined by the solution to the expressions* (46) *and* (47)*. In this case, the effective borrowing rate is given by equation* (43)*, where* $\hat{\mathbf{A}}$ *solves the fixed point problem*

$$\hat{\mathbf{A}} = \mathbf{e}_i + \left(\mathbf{\Gamma}^\top - \mathbf{M}\right)\int_0^T \eta(\tau)\mathbf{A}(\tau)\,\mathrm{d}\tau\,. \tag{48}$$

In a neighborhood of risk-neutrality ($a \approx 0$)*, the equilibrium exists and is (locally) unique.*

Note that the dynamics matrix of the state $\mathbf{\Gamma}$ depends on the effective borrowing rate coefficients $\hat{\mathbf{A}}$, which itself is a function of the risk-adjusted dynamics matrix $\mathbf{M}$. Thus, equilibrium is determined as a fixed point that produces asset price dynamics consistent with equilibrium dynamics of the macroeconomy and vice versa. In general, an affine equilibrium of this type may not exist, or there may be multiple solutions to this fixed point problem. However, when $a = 0$, the model reduces to a standard New

Keynesian model. The result in Proposition 2 shows that this equilibrium persists and is locally unique as we depart from risk neutrality.

3.1 Conventional Policy

For now, we assume that monetary policy is the only source of uncertainty, so that the natural rate r_t^n is fixed at its steady sate value. In equilibrium, bond prices therefore only respond to changes in the short rate. In terms of log-deviations from steady state, we thus have

$$-\log P_t^{(\tau)} = A_i(\tau) i_t \implies \tilde{\mu}_t = \hat{A}_i i_t.$$

The follow Proposition characterizes the responses of inflation and the output gap to changes in the policy rate in equilibrium.

Proposition 3 (Conventional monetary policy responses)**.** *Determinacy is satisfied if and only if*

$$\hat{A}_i > \frac{\delta}{\delta\phi_\pi + \rho\phi_x}. \tag{49}$$

When this condition holds, $\hat{A}_i$ *is the unique solution to the following fixed point problems:*

$$\begin{aligned}
\hat{A}_i(\gamma) &\equiv 1 - \left(1 - \frac{\gamma}{m(\gamma)}\right)\left(1 - e^{-m(\gamma)\tau}\right), \\
m(\gamma) &= \gamma + a\sigma_i^2 \int_0^T \alpha(\tau)\left(\frac{1 - e^{-m(\gamma)\tau}}{m(\gamma)}\right)^2 \mathrm{d}\tau\,, \\
\hat{A}_i(\gamma) &= \frac{(\gamma - \kappa_i)(\gamma^2 + \gamma\rho - \varsigma^{-1}\delta)}{\varsigma^{-1}\kappa_i\left(\delta\phi_\pi + \rho\phi_x + \gamma\phi_x\right)},
\end{aligned}$$

and in equilibrium $\hat{A}_i \in (0,1)$, $\gamma > \kappa_i$, *and* $\mathrm{d}i_t = -\gamma i_t\,\mathrm{d}t + \sigma_i\,\mathrm{d}B_{i,t}$. *Inflation and output gap dynamics are given by* $\pi_t = \omega_\pi i_t$ *and* $x_t = \omega_x i_t$, *where*

$$\omega_\pi = -\frac{\delta(\gamma - \kappa_i)}{\kappa_i\left(\delta\phi_\pi + \rho\phi_x + \gamma\phi_x\right)}, \quad \omega_x = -\frac{(\gamma + \rho)(\gamma - \kappa_i)}{\kappa_i\left(\delta\phi_\pi + \rho\phi_x + \gamma\phi_x\right)}.$$

Note that the aggregate dynamics nest the benchmark New Keynesian model, where the affine coefficients are simply $\hat{A}_i = 1$. When this is the case, if the central bank only cares about inflation (so $\phi_x = 0$), the determinacy condition (49) simplifies to the standard condition that $\phi_\pi > 1$. Instead, in our model the reaction of the effective borrowing rate to policy rate changes is a general equilibrium object. In particular, $\hat{A}_i$ depends on the risk aversion of arbitrageurs. The follow Corollary derives comparative statics of the reaction to conventional policy.

Corollary 3.1 (Conventional policy comparative statics). *The general equilibrium responses to conventional policy vary with the underlying parameters of the model:*

1. *Arbitrageur risk aversion:* $\frac{\partial \gamma}{\partial a} < 0$, $\frac{\partial \hat{A}_i}{\partial a} < 0$, $\frac{\partial |\omega_\pi|}{\partial a} < 0$, $\frac{\partial |\omega_x|}{\partial a} < 0$.
2. *Policy rate inertia:* $\frac{\partial \gamma}{\partial \kappa_i} > 0$, $\frac{\partial \hat{A}_i}{\partial \kappa_i} > 0$, $\frac{\partial |\omega_\pi|}{\partial \kappa_i} < 0$, $\frac{\partial |\omega_x|}{\partial \kappa_i} < 0$.
3. *Policy rate reaction to inflation:* $\frac{\partial \gamma}{\partial \phi_\pi} > 0$, $\frac{\partial \hat{A}_i}{\partial \phi_\pi} > 0$, $\frac{\partial |\omega_\pi|}{\partial \phi_\pi} < 0$, $\frac{\partial |\omega_x|}{\partial \phi_\pi} < 0$.
4. *Policy rate reaction to the output gap:* $\frac{\partial \gamma}{\partial \phi_x} > 0$, $\frac{\partial \hat{A}_i}{\partial \phi_x} > 0$, $\frac{\partial |\omega_\pi|}{\partial \phi_x} < 0$, $\frac{\partial |\omega_x|}{\partial \phi_x} < 0$.
5. *Consider two different weighting functions $\eta^s(\tau)$ and $\eta^\ell(\tau)$, such that for some T^*, $\eta^s(\tau) \geq \eta^\ell(\tau) \iff \tau \leq T^*$. Then $\gamma^s > \gamma^\ell$, $\hat{A}_i^s > \hat{A}_i^\ell$, $|\omega_\pi^s| > \left|\omega_\pi^\ell\right|$, $|\omega_x^s| > \left|\omega_x^\ell\right|$ where superscripts denote the equilibrium outcomes under the corresponding weighting functions.*

The first result in Cor. 3.1 says shows that as the risk aversion of arbitrageurs increases, household borrowing becomes less responsive to changes in the policy rate: $\hat{A}_i$ is decreasing in the risk aversion of arbitrageurs. This occurs because long-term borrowing rates under-react to shifts in the short rate, which in turn implies that household borrowing (and hence output and inflation) responds less than it would when financial markets exhibit perfect risk-bearing capacity. This result makes clear that monetary policy is effective only to the extent that policy changes are transmitted through financial markets, and that transmission is muted due to imperfect arbitrage.

The next result relates to the persistence of the central bank's policy rule (governed by the mean reversion in the Taylor rule, κ_i). This governs the level of inertia in the central bank's policy rate (a higher value implies the rate returns to the target rate faster). Recall that γ determines the equilibrium mean reversion behavior of the policy rate. So unsurprisingly, if the central bank reduces the inertia in its policy rule (increases κ_i) then the policy rate in equilibrium reverts more quickly (higher γ); because policy rate gaps persist for less time, term premia are less volatile and so the effective borrowing rate responds more to monetary shocks (higher $\hat{A}_i$).

The intuition regarding the central bank's sensitivity to inflation or output (ϕ_π and ϕ_x) is similar to the inertia parameter. The central bank responds more forcefully to deviations in inflation or output gaps, so in equilibrium the policy rate deviations subside faster (higher γ). Because they are shorter lived, inflation and output react to monetary shocks less.

The final result in Cor. 3.1 shows how the model depends on the weighting function $\eta(\tau)$, which determines the effective borrowing rate. The two weighting functions correspond to two models where the effective borrowing rate is more geared towards short-term rates (η^s) or long-term rates (η^ℓ). The results says that as the effective borrowing rate becomes more dependent on long-term rates, the model is less sensitive to the policy rate. This is because of the under-reaction result discussed above: the degree of under-reaction is increasing in longer-maturity borrowing rates. Thus, the effective borrowing rate reacts less to changes in the policy rate; in equilibrium, this implies that monetary shocks persist longer.

3.2 Quantitative Easing and Tightening

Given that the expectations hypothesis does not hold, purchases by the central bank may have price effects. This section studies QE and QT policies and shows that indeed, QE can push down long-term borrowing rates when arbitrage is imperfect. We now suppose that in addition to setting the short rate, the central bank also directly purchases longer term bonds through open market operations according to (41).

Now the affine functional form of bond prices (in terms of log-deviations from steady state) are given by

$$-\log P_t^{(\tau)} = A_t(\tau)i_t + A_S(\tau)S_t \implies \tilde{\mu}_t = \hat{A}_i i_t + \hat{A}_S S_t,$$

which introduces a new coefficient function $A_S(\tau)$. The next Proposition characterizes the aggregate response of the model to QE or QT shocks.

Proposition 4 (Unconventional monetary policy responses)**.** *Assume the determinacy condition* (49) *is satisfied. Whenever $a > 0$, $\hat{A}_S < 0$ which implies purchases of long-term bonds by the central bank ($S_t > 0$) reduce borrowing rates. Aggregate* *dynamics therefore satisfy*

$$\frac{\partial \pi_t}{\partial S_t} > 0, \quad \frac{\partial x_t}{\partial S_t} > 0.$$

Additionally, $\hat{A}_S \to 0$ *as* $\kappa_S \to \infty$.

Prop. 4 shows that central bank purchases of long-term bonds increase both inflation and the output gap whenever arbitrageurs have imperfect risk-bearing capacity. This follows because QE purchases reduce term premia and therefore reduce household borrowing rates. QE effectively reduces the amount of risk which arbitrageurs are required to hold, which puts downward pressure on returns of all bonds.

Note that this partial equilibrium effect is mitigated by a general equilibrium effect: the expansionary effects of QE put countervailing upward pressure on the expected path of short rates. But since arbitrageurs are risk-averse, this upward pressure is weakened relative to the predictions of the expectations hypothesis. Prop. 4 shows that this indirect effect does not outweigh the direct effect, and it will still be the case that in general equilibrium QE purchases will push down effective borrowing rates, leading to an increase in output.

The second result in Prop. 4 shows that the effect of QE depends critically on the mean reversion properties of purchases. Even when financial markets are highly disrupted, the aggregate effects will be minimal if the purchases are undone very quickly.

4 Welfare Effects: Simple Policy Rules

Using the results of the previous section, we explore the welfare consequences and optimal design of simple policy rules. For now, we assume that the only aggregate source of uncertainty is natural rate shocks r_t^n. From Ito's Lemma, we have

$$\mathrm{d}r_t^n = -\kappa_z r_t^n \,\mathrm{d}t + \sigma_r \,\mathrm{d}B_{z,t}\,,$$

where $\sigma_r \equiv -\kappa_z \sigma_z$.

We study simple policy rules which (in equilibrium) are only a function of the natural state variables of the model, which in this case are only natural rate shocks.

Thus, we study a conventional policy rule which which implements

$$i_t = \chi_i r_t^n, \tag{50}$$

for some choice of policy parameter χ_i. We assume that such a policy is implementable; determinacy conditions can be satisfied through a Taylor rule like those considered in Section 3. Balance sheet policies implement

$$S_t^{(\tau)} = \chi_S^{(\tau)} r_t^n, \tag{51}$$

for choices of policy parameters $\chi_S^{(\tau)}$. We also consider ad-hoc QE or QT policies according to (41).

The policymaker chooses policy parameters in order to minimize unconditional social loss

$$\mathcal{W} \equiv \frac{1}{2}\mathbb{E}\int_0^\infty e^{-\rho t}\mathcal{L}_t \, \mathrm{d}t \,,$$

where per-period social welfare loss $\mathcal{L}_t$ is derived in (37)-(39). Thus, we consider a policymaker who is able to commit to simple policy rules, which are functions of the natural state variables of the economy. In the next section, we explore optimal policy for a policymaker who conducts (history-dependent) policy under full commitment.

4.1 Optimal Policy: Short Rate Only

First, consider the benchmark case of a risk neutral arbitrageurs: $a = 0$. Then the expectations hypothesis holds, so regardless of preferred habitat demand (or central bank balance sheet policies), borrowing rates are equalized: $\mu_t^{(\tau)} = i_t = \chi_i r_t^n$. This implies that the model collapses to the standard RANK case. In particular, consumption and wage dispersion disappear:

$$\mathbb{V}\mathrm{ar}_\tau \, c_t^{(\tau)} = \mathbb{V}\mathrm{ar}_\tau \, w_t^{(\tau)} = 0.$$

Because divine coincidence holds, the conventional policy rule with $\chi_i = 1$ which implements $i_t = r_t^n$ achieves first-best: $x_t = \pi_t = 0$ for all periods t. We further recover the standard QE neutrality result: balance sheet policies do not affect bond prices (and therefore have no aggregate effects).

Under risk-neutrality, so long as the central bank faces no costs to setting the policy rate, first-best is achievable with only conventional policy. The next Proposition shows that these results fail whenever arbitrage is imperfect ($a > 0$). We derive optimal short-rate policy when the central bank does not have access to balance sheet tools (formally, when balance sheet costs $\psi^{(\tau)} \to \infty$).

Proposition 5 (Optimal short rate policy rule)**.** *Assume risk aversion $a > 0$ and price elasticities $\alpha(\tau) > 0$. Then there exists some $\chi_i^n > 1$ such that (50) guarantees $x_t = 0$ and inflation $\pi_t = 0$ each period. However, the optimal short rate policy rule is given by $i_t = \chi_i^* i_t$, where $\chi_i^* < \chi_i^n$. The optimal short rate policy implies:*

1. *Bond carry trade returns $\mu_t^{(\tau)} - i_t$ are decreasing in the short rate i_t.*
2. *Output gaps x_t and inflation π_t are not identically zero.*
3. *Consumption and wage dispersion are non-zero: $\mathbb{V}\mathrm{ar}_\tau\, c_t^{(\tau)} \neq 0$, $\mathbb{V}\mathrm{ar}_\tau\, w_t^{(\tau)} \neq 0$.*

We can understand this result using the intuition derived in the previous section. Consider a fall in the natural rate inducing a cut in the policy rate. As i_t decreases, bond arbitrageurs want to invest more in the bond carry trade. This implies an increase in bond prices $P_t^{(\tau)}$, which induces price-elastic habitat bond investors ($\alpha(\tau) > 0$) to reduce their holdings, and so $Z_t^{(\tau)}$ declines. Through market clearing, bond arbitrageurs increase their holdings $X_t^{(\tau)}$, which requires a larger bond carry trade return.

Because of this, risk premia vary over time. Thus, a simple short rate policy is unable to equalize all borrowing rates. The central bank can choose $i_t = \chi_i^n r_t^n$, which implies that $\tilde{\mu}_t = r_t^n$ (and therefore closes output gaps and keeps inflation at steady state). However, fluctuations in the natural rate induce volatile fluctuations in the policy rate. Ito's lemma implies

$$\mathrm{d}i_t = -\kappa_z i_t \,\mathrm{d}t + \chi_i \sigma_r \,\mathrm{d}B_{z,t}\,,$$

hence the volatility of short rate changes is increasing in the responsiveness of conventional policy to natural rates. More volatile short rates implies greater variation in term premia. Individual Euler equations differ, which implies that consumption choices across the τ households $c_t^{(\tau)} \neq c_t^{(\tau')}$. Therefore, consumption dispersion

$\mathbb{V}\text{ar}_\tau\, c_t^{(\tau)} \neq 0$. Differential consumption choices imply differential labor supply decisions, which additionally imply dispersion in wages $w_t(\tau)$ whenever $\epsilon_w < \infty$. Thus, choosing a policy rule which is less reactive to natural rate fluctuations is welfare-improving.

As the short rate becomes less responsive to natural rates, output gap and inflation volatility increase. For very small choices of χ_i, consumption and wage dispersion become negligible. However, inflation and output gap volatility increases substantially. The optimal choice of χ_i^* balances these forces. At the optimum, shocks to the natural rate do not fully pass through to the effective borrowing rate: $\tilde{\mu}_t \neq r_t^n$. Thus, aggregate borrowing demand changes, and hence the output gap $x_t \neq 0$. Whenever prices are not fully rigid, this induces fluctuations in inflation through the Phillips curve, and so $\pi_t \neq 0$ as well.

4.2 Optimal Short Rate and Balance Sheet Policy

The failure of conventional policy to achieve first-best is driven by three frictions. First, because of imperfect pass-through of the policy rate to household borrowing rates, natural rate shocks are not fully accommodated, implying excessive volatility of output and inflation. Second, borrowing rates across households differ, implying sub-optimal consumption dispersion. Thirds, differentiated wages imply inefficient labor supply dispersion. Simple conventional policy rules cannot overcome all frictions simultaneously. However, when the central bank has access to frictionless balance sheet policies, we obtain the following result:

Proposition 6 (Optimal policy separation principle)**.** *Assume risk aversion $a > 0$ and price elasticities $\alpha(\tau) > 0$. Suppose the central bank implements short rate and balance sheet policy according to*

$$i_t = r_t^n, \quad S_t^{(\tau)} = \alpha(\tau) \log P_t^{(\tau)}.$$

If short rate policy is frictionless ($\psi^i = 0$) and the central bank does not face holding costs ($\psi^{(\tau)} = 0$), then first-best is achieved:

1. *Macroeconomic stabilization: $x_t = \pi_t = 0\ \forall t$.*

2. *Financial stabilization: $\mu_t^{(\tau)} = \tilde{\mu}_t\ \forall \tau$.*

3. Consumption and wage equalization: $c_t^{(\tau)} = c_t^{(\tau')}, w_t(\tau) = w_t(\tau') \; \forall \tau, \tau'$.

The results follow naturally from our findings regarding ad-hoc policy. The balance sheet policy in Prop. 6 stabilizes shocks to bond markets by offsetting all habitat portfolio rebalancing shocks:

$$S_t^{(\tau)} = -Z_t^{(\tau)} \implies \boldsymbol{\sigma}_t^{(\tau)} \boldsymbol{\Lambda}_t = 0.$$

The policy implies that risk premia are zero and thus equalizes borrowing rates across households: $\mu_t^{(\tau)} = \tilde{\mu}_t$. Hence the model collapses to a standard RANK model. The elimination of risk premia implies that policy shocks are transmitted one-to-one to borrowing rates, so $\tilde{\mu}_t = i_t$. Thus, the policy $i_t = r_t^n$ implies output gaps $x_t = 0$; and because divine coincidence holds, this policy is optimal.

Thus, we derive an optimal *separation principle* for optimal policy: optimal balance sheet policy *stabilizes financial markets* while optimal short rate policy *stabilizes macroeconomic aggregates*.

4.3 Optimal Policy: Balance Sheet Only

We now derive the optimal use of balance sheet tools when the central bank faces constraints on the short rate.[4] To capture the essence of short rate constraints in a simple way, we assume that $\psi^i \to \infty$ and that $i_t^* = \tilde{\chi}_i r_t^n$ where $0 < \tilde{\chi}_i < 1$ in (21). The next Proposition shows that balance sheet tools alone are not enough to achieve first-best.

Proposition 7 (Optimal balance sheet rule)**.** *Assume risk aversion $a > 0$ and price elasticities $\alpha(\tau) > 0$. Suppose the short rate in equilibrium evolves according to*

$$i_t = \tilde{\chi}_i r_t^n, \;\; 0 < \tilde{\chi}_i < 1.$$

Then the balance sheet policy which implements $S_t^{(\tau)} = \alpha(\tau) \log P_t^{(\tau)}$ still satisfies financial stabilization ($\mu_t^{(\tau)} = \tilde{\mu}_t \; \forall \tau$). Consumption and wage dispersion are zero; however, output gaps x_t and inflation π_t are not identically zero.

There exists some balance sheet policy parameters $\left\{ \chi_S^{(\tau)} \right\}$ such that $\tilde{\mu}_t = r_t^n$ and therefore satisfies macroeconomic stabilization ($x_t = 0, \pi_t = 0 \forall t$). However, financial

[4] Recall that we do not model an explicit ZLB in order to utilize our linear-quadratic approximation techniques.

stabilization fails ($\mu_t^{(\tau)} \neq \tilde{\mu}_t \, \forall \tau$), and therefore consumption and wage dispersion are non-zero. The optimal balance sheet rule implies:

1. *Output gaps x_t and inflation π_t are not identically zero. Relative to no balance sheet policies, output gap and inflation volatility are lower.*

2. *Consumption and wage dispersion are non-zero: $\mathbb{V}\mathrm{ar}_\tau \, c_t^{(\tau)} \neq 0$, $\mathbb{V}\mathrm{ar}_\tau \, w_t^{(\tau)} \neq 0$. Relative to no balance sheet policies, output gap and inflation volatility are lower.*

Prop. 7 shows that when the short rate is constrained, the optimal balance sheet policy must sacrifice financial stabilization in order to (partially) stabilize macroeconomic volatility. While balance sheet tools can continue to equalize borrowing rates, sub-optimal short rate policy implies that the effective household borrowing rate $\tilde{\mu}_t \neq r_t^n$. Thus, this policy does not achieve macroeconomic stabilization: $x_t \neq 0, \pi_t \neq 0$.

On the other hand, balance sheet policies alone can close the output gap (and stabilize inflation), but this is also sub-optimal because borrowing rates are no longer equalized. From (29), with only natural rate shocks we have that $\mu_t^{(\tau)} = i_t + \sigma_t^{(\tau)} \lambda_t$ where

$$\lambda_t \equiv a \int_0^T \left[\alpha(\tau) \log P_t^{(\tau)} + S_t^{(\tau)} \right] \sigma_t^{(\tau)} \, \mathrm{d}\tau \, .$$

Hence, the policymaker can always guarantee $\tilde{\mu}_t = r_t^n$ by choosing holdings $\left\{ S_t^{(\tau)} \right\}$ such that

$$\lambda_t^* = \frac{r_t^n - i_t}{\int_0^T \eta(\tau) \sigma_t^{(\tau)} \, \mathrm{d}\tau}.$$

However, because $\sigma_t^{(\tau)} \neq \sigma_t^{(\tau')}$ this necessitates

$$\mu_t^{(\tau)} = i_t + \sigma_t^{(\tau)} \left(\frac{r_t^n - i_t}{\int_0^T \eta(\tau') \sigma_t^{(\tau')} \, \mathrm{d}\tau'} \right) \neq r_t^n,$$

for some τ (unless $i_t = r_t^n$).

The intuition is that balance sheet policy works by affecting term premia through changes in the market price of risk. Although arbitrage is imperfect in this model, arbitrageurs still enforce tight restrictions between between the market price of risk and term premia across maturities. Hence, while in principle the central bank has a continuum of policy tools $\left\{S_t^{(\tau)}\right\}$, in practice it can only manipulate λ_t.

This is related to the localization results in Vayanos and Vila (2021) and Ray et al. (2024). In the one-factor model considered here, the effects of QE are fully global. However, even with more complicated risk structure, localization is not strong enough to allow balance sheet policy rules alone to achieve first-best.

5 Welfare Effects: Full Commitment

We now study the case of full commitment, where the policymaker can choose short rate and balance sheet policies as a function of the entire history of shocks. The policymaker seeks to minimize the conditional social loss function

$$\mathcal{W}_t = \mathbb{E}_t \int_t^\infty \frac{1}{2} e^{-\rho s} \mathcal{L}_s \, \mathrm{d}s \,, \tag{52}$$

where per-period social welfare loss $\mathcal{L}_s$ is derived in (37)-(39). Our optimal separate result in Prop. 6 show that simple policy rules can achieve first-best when the policymaker faces no short rate or balance sheet frictions. However, when policy frictions are non-negligible, full commitment policies can improve on simple policy rules.

Theorem 1 (Optimal policy under full commitment)**.** *Collect the state variables $\mathbf{y}_t$ and jump variables $\mathbf{x}_t$ into a vector $\mathbf{Y}_t$ and policy tools into the vector $\mathbf{u}_t$, so that social loss is given by*

$$\mathcal{W}_0 = \frac{1}{2} \mathbb{E}_0 \int_0^\infty e^{-\rho t} \left(\mathbf{Y}_t^\top \mathbf{R} \mathbf{Y}_t + \mathbf{u}_t^\top \mathbf{Q} \mathbf{u}_t \right) \mathrm{d}t \,, \quad \mathbf{y}_0 \text{ given.} \tag{53}$$

The policymaker chooses $\mathbf{u}_t = \mathbf{F}\mathbf{Y}_t$, which induce equilibrium dynamics

$$\mathrm{d}\mathbf{Y}_t = -\boldsymbol{\Upsilon} \mathbf{Y}_t \, \mathrm{d}t + \mathbf{S} \, \mathrm{d}\mathbf{B}_t \,, \tag{54}$$

where the feedback matrix $\boldsymbol{\Upsilon} \equiv \boldsymbol{\Upsilon}(\mathbf{F})$ either explicitly or implicitly depends on the policymaker reaction function. Necessary conditions for optimal choice of $\mathbf{F}^*$ are *given by*

$$\mathbf{y}_0^\top \left(\partial_i \mathbf{P}_{11} - \partial_i \mathbf{P}_{12}\mathbf{P}_{22}^{-1}\mathbf{P}_{21} - \mathbf{P}_{12}\mathbf{P}_{22}^{-1}\partial_i \mathbf{P}_{21} + \mathbf{P}_{12}\left(\mathbf{P}_{22}^{-1}\partial_i \mathbf{P}_{22}\mathbf{P}_{22}^{-1}\right)\mathbf{P}_{21}\right)\mathbf{y}_0 = 0$$

where $\mathbf{P}$ *solves the Lyapunov equation* $\rho\mathbf{P} = \mathbf{R} + \mathbf{F}^\top\mathbf{Q}\mathbf{F} - \mathbf{P}\boldsymbol{\Upsilon} - \boldsymbol{\Upsilon}^\top\mathbf{P}$*, and* ∂_i *represents the derivative of partitioned elements of* $\mathbf{P}$ *with respect to the* i *element of* $\mathbf{F}^*$*, which solve the Lyapunov equation*

$$\rho\partial_i\mathbf{P} = \mathbf{Q}\partial_i\mathbf{F} + \partial_i\mathbf{F}^\top\mathbf{Q} - \partial_i\mathbf{P}\boldsymbol{\Upsilon} - \boldsymbol{\Upsilon}^\top\partial_i\mathbf{P} - \mathbf{P}\partial_i\boldsymbol{\Upsilon} - \partial_i\boldsymbol{\Upsilon}^\top\mathbf{P}.$$

The equilibrium dynamics of the model are

$$\begin{aligned}\mathrm{d}\mathbf{q}_t &= -\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{bmatrix}\boldsymbol{\Upsilon}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1}\end{bmatrix}\mathbf{q}_t\,\mathrm{d}t + \begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{P}_{21} & \mathbf{P}_{22}\end{bmatrix}\mathbf{S}\,\mathrm{d}\mathbf{B}_t\\ &\equiv -\boldsymbol{\Gamma}\mathbf{q}_t\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t\,,\end{aligned} \tag{55}$$

where $\mathbf{q}_0 = \begin{bmatrix}\mathbf{y}_0 & \mathbf{0}\end{bmatrix}^\top$*. Equilibrium in bond markets is given by Prop. 2.*

Full commitment allows for potential improvements in social welfare because the induced dynamics of the economy in (55) are richer than when the central bank follows simple policy rules. This allows the central bank to reduce volatility of interest rate changes, while keeping stronger control over the entire path of the policy rate. Note that

$$\mathrm{d}i_t = -\mathbf{e}_i^\top\boldsymbol{\Gamma}\mathbf{q}_t\,\mathrm{d}t + \mathbf{e}_i^\top\boldsymbol{\sigma}\,\mathrm{d}\mathbf{B}_t\,,\quad \mathbf{e}_i \equiv \mathbf{e}_1^\top\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ -\mathbf{P}_{22}^{-1}\mathbf{P}_{21} & \mathbf{P}_{22}^{-1}\end{bmatrix}\mathbf{F}^*,$$

where $i_t \equiv \mathbf{e}_1^\top\mathbf{u}_t$. The term $\mathbf{e}_i^\top\boldsymbol{\sigma}$ can be made smaller (inducing smaller term premia in equilibrium), while still allowing for sufficiently rich expected movements of the short rate $\int_0^t i_s\,\mathrm{d}s = \mathbf{e}_i^\top\int_0^t \mathbf{q}_s\,\mathrm{d}s$.

In general, the necessary conditions in 1 are complicated. We explore the implications in a stylized numerical example. Suppose that the central bank only has access to short rate policy (formally, $\psi^{(\tau)} \to \infty$). We assume that there are two types of households with equal measure who borrow using short- and long-term bonds τ^s, τ^ℓ.

Formally,

$$\eta(\tau) = \frac{1}{2}\hat{\delta}(\tau - \tau^s) + \frac{1}{2}\hat{\delta}(\tau - \tau^\ell),$$

where $\hat{\delta}(\tau)$ is the dirac delta function. We make further parametric simplifying assumptions: prices are fully rigid ($\theta \to \infty \implies \delta \to 0$ in (36)); labor markets are efficient ($\epsilon_w \to \infty$); and unitary intertemporal and labor supply elasticities ($\varsigma = \varphi = 1$). This implies that the policymaker problem simplifies to

$$\begin{aligned}
\mathcal{W}_t = \max_{\{i_t\}} \ \mathbb{E}_t \int_t^\infty \left(x_t(s)^2 + x_t(\ell)^2\right) \mathrm{d}s \\
\text{subject to: } \ \mathrm{d}x_t(s) &= (\mu_t^{(s)} - r_t^n)\,\mathrm{d}t \\
\mathrm{d}x_t(\ell) &= (\mu_t^{(\ell)} - r_t^n)\,\mathrm{d}t \\
\mathrm{d}r_t^n &= -\kappa_z r_t^n\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{z,t}\,.
\end{aligned}$$

We additionally simplify the bond markets to focus on "short" and "long" maturities by assuming that the habitat elasticity function is also given by $\alpha(\tau) = \alpha\hat{\delta}(\tau - \tau^\ell)$, and take limits $\tau^s \to 0, \tau^\ell \to \infty$. This assumption regarding habitat elasticities implies that the fixed point problem described in Lemma 2 is scalar. Numerical exercises show that, relative to the case of full commitment, the policymaker chooses to react to movements in $x_t(s)$ and $x_t(\ell)$ (in addition to the natural rate r_t^n). This additionally allows for a larger response of the short rate to natural rate shocks (relative to the case without full commitment).

6 Extensions and Tests

We consider extensions of the baseline model studied in the previous section.

6.1 Demand Risk and Financial Shocks

First, we obtain identical results when allowing for shocks to habitat demand: $\beta_t^{(\tau)}$ is now an additional stochastic demand factor. The optimal separation principle still holds with $\psi(\tau) = 0$, but QE must be more reactive:

$$S_t^{(\tau)} = \alpha(\tau)\log P_t^{(\tau)} + \beta_t^{(\tau)}.$$

So long as the optimal balance sheet policy is implemented, the optimal short rate policy still implements $i_t = r_t^n$.

In this case however, we have an additional result: if noise demand dynamics are such that demand falls in response to increases in the natural rate (that is, $\beta_t^{(\tau)}$ increases in response to an increase in r_t^n), then it is optimal to expand the balance sheet $S_t^{(\tau)}$ while simultaneously hiking short rates i_t. Intuitively, suppose during a tightening cycle, in the absence of QE we expect to observe an increase in term premia. Then the optimal balance sheet policy is to conduct additional QE purchases in order to offset the spike in term premia. This suggests that at times, optimal conventional and unconventional policy seem to be at odds with one another.

6.2 Cost-Push Shocks

Next, we consider the model which cost-push shocks, which implies that divine coincidence does not hold:

$$\mathrm{d}\pi_t = (\rho\pi_t - \delta x_t - u_t)\,\mathrm{d}t\,.$$

For simple time-consistent policies considered in Section 4, our separation principle still holds when policy frictions are absent. Unfortunately, this implies that the first-best is not achievable. Optimal balance sheet policy still stabilizes term premia, which implies that short rate policy must contend with the output gap and inflation tradeoffs as is standard. The reason is that despite multiple policy instruments, (un)conventional policy only affects aggregate outcomes through changes in the effective borrowing rate $\tilde{\mu}_t$. Formally, taking any feasible path $\{x_t, \pi_t, \tilde{\mu}_t\}_t$ from an implementation implying policies $\left\{\tilde{i}_t, \tilde{S}_t^{(\tau)}\right\}_t$, this can also be achieved with $i_t = \tilde{\mu}_t, S_t^{(\tau)} = \alpha(\tau)\log P_t^{(\tau)} + \beta_t^{(\tau)}$. This guarantees $\mathbb{V}\mathrm{ar}_\tau\, c_t^{(\tau)} = \mathbb{V}\mathrm{ar}_\tau\, w_t^{(\tau)} = 0$ and hence strictly dominates.

6.3 Non-Zero Term Premia in the First-Best

Our approximation approach thus far implies that in the first-best, expected carry trade returns are zero. This arises endogenously in our model but is based on our simplifying riskless approximation method. While this simplifies our analytical results, it is nevertheless a strong assumption. Suppose instead that first-best bond carry trade returns are given by some (exogenous) $\nu^{(\tau)} \neq 0$. We find that our separation principle still holds when $\nu^{(\tau)}$ is achievable, but optimal short rate policy is a function of $\nu^{(\tau)}$. The intuition is a combination of previous results. Aggregate outcomes arise through changes in the effective borrowing rate $\tilde{\mu}_t$ (as before). Optimal balance sheet policy guarantees $\mu_t^{(\tau)} - i_t \equiv \nu^{(\tau)}$ and hence $\tilde{\mu}_t = i_t + \int_0^T \eta(\tau)\nu^{(\tau)}\,\mathrm{d}\tau \equiv i_t + \tilde{\nu}$. Thus, optimal short rate policy implements $i_t = r_t^n - \tilde{\nu}$.

6.4 Measuring Policy Objectives: Return Predictability

We now consider simple observable tests related to the optimality of balance sheet policies. We consider the bond return predictability regressions of Fama and Bliss (1987) (FB):

$$\frac{1}{\Delta\tau}\log\left(\frac{P_{t+\Delta\tau}^{(\tau-\Delta\tau)}}{P_t^{(\tau)}}\right) - y_t^{(\Delta\tau)} = a_{FB}^{(\tau)} + b_{FB}^{(\tau)}\left(f_t^{(\tau-\Delta\tau,\tau)} - y_t^{(\Delta\tau)}\right) + \varepsilon_{t+\Delta\tau}.$$

These regression coefficients measure how the slope of the term structure predicts excess returns. Figure 1 shows the well-known result that (in the full sample), Fama-Bliss coefficients are positive, and an increasing function of maturity.

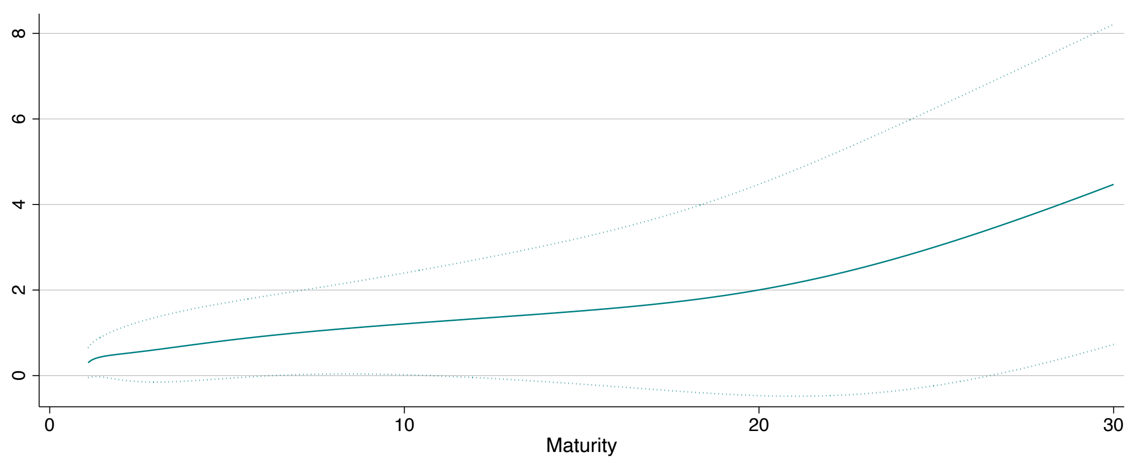

Figure 1: Fama-Bliss Coefficients: Treasuries, Full Sample

In our model, when the central bank does not use balance sheet policies:

$$\bar{b}_{FB}^{(\tau)} > 0.$$

However, if balance sheet policy is successfully pursuing financial stabilization, then

$$\bar{b}_{FB}^{(\tau)} > b_{FB}^{(\tau)} \to 0.$$

But instead if balance sheet policy is pursuing macroeconomic stabilization, we have

$$b_{FB}^{(\tau)} > \bar{b}_{FB}^{(\tau)}.$$

We examine these stylized predictions by studying how FB regression coefficients vary as a function of different monetary policy regimes.

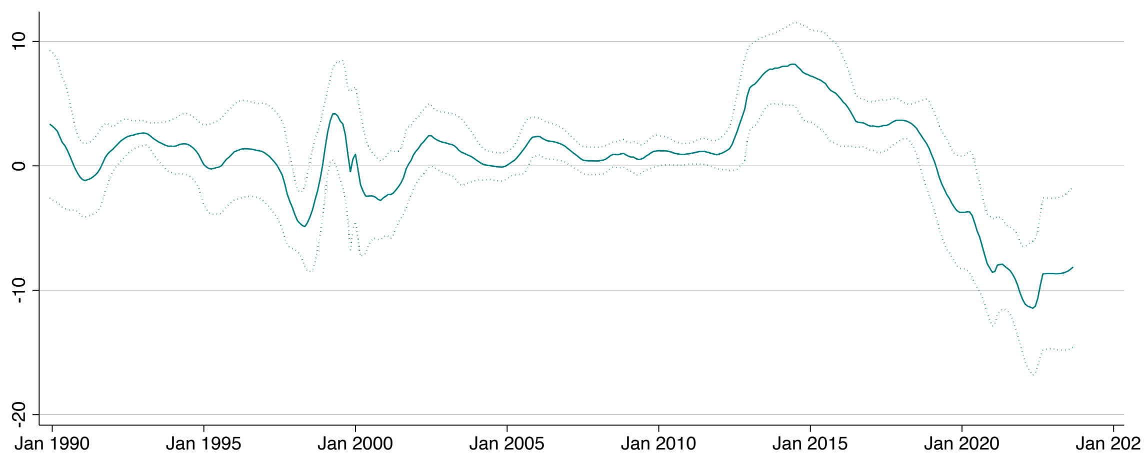

Figure 2: Fama-Bliss Coefficients: 10-year Treasuries, Rolling Sample

Figure 2 estimates rolling Fama-Bliss coefficients (fixing $\tau = 10$ as a baseline maturity). We see that Fama-Bliss coefficients increased substantially during the initial QE regime, but have fallen and even become negative in recent years. This is consistent with the predictions of the model if QE was initially undertaken for purely macroeconomic stabilization purposes, but has shifted in part to deliver financial stabilization.

7 Concluding Remarks

This paper develops a tractable general equilibrium model with market segmentation, nominal frictions, and household heterogeneity. We first derive results for how monetary policy (conventional and unconventional) affects aggregate dynamics. We next show that welfare losses arise from three sources: (i) excessive volatility in inflation and output, as is standard in New Keynesian models; (ii) imperfect risk-sharing and excess consumption dispersion; and (iii) labor market frictions and excess wage dispersion. The frictions associated with consumption and wage dispersion arise due to market segmentation and excessive volatility of term premia. Optimal short rate and balance sheet policies are characterized by a sharp separation result: conventional policy targets macroeconomic stability, while unconventional policy targets financial stability. Optimal policy removes excess volatility of risk premia and hence improves risk-sharing across households, while reducing excess macroeconomic volatility. Policy constraints on either the short rate or balance sheets imply tradeoffs between these two policy objectives.

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A Proofs

Proof of Lemma 1. Collect all state and jump variables in a vector $\mathbf{Y}_t = \begin{bmatrix} \mathbf{y}_t^\top & \mathbf{x}_t^\top \end{bmatrix}^\top$. The interest rate process i_t and habitat demand factor processes $\beta_t^{(\tau)}$ are all affine functions of $\mathbf{Y}_t$. Moreover, the (linearized) Phillips curve and IS equation are also affine functions of $\mathbf{Y}_t$, since from equation (43) $\hat{\mu}_t = \hat{\mathbf{A}}^\top \mathbf{y}_t + \hat{C}$ is affine in the state variables. Aggregate dynamics can thus be written

$$\mathrm{d}\mathbf{Y}_t = -\boldsymbol{\Upsilon}\left(\mathbf{Y}_t - \bar{\mathbf{Y}}\right)\mathrm{d}t + \mathbf{S}\,\mathrm{d}\mathbf{B}_t\,. \tag{A1}$$

Note that $\boldsymbol{\Upsilon}$ depends on $\hat{\mathbf{A}}$, but which we currently take as given. Then the rational expectations equilibrium is found immediately from Buiter (1984). Partition the eigenvalues and eigenvectors as follows:

$$\boldsymbol{\Upsilon} = \mathbf{Q}\boldsymbol{\Lambda}\mathbf{Q}^{-1}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \boldsymbol{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_2 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{bmatrix},$$

where the partitions correspond to the state $\mathbf{y}_t$ and jump $\mathbf{x}_t$ variables. If the number of "stable" eigenvalues (non-negative real parts) equals the number of state variables, then the rational expectations equilibrium dynamics are given by (44), where

$$\boldsymbol{\Gamma} = \mathbf{Q}_{11}\boldsymbol{\Lambda}_1\mathbf{Q}_{11}^{-1}, \quad \boldsymbol{\Omega} = \mathbf{Q}_{21}\mathbf{Q}_{11}^{-1}. \tag{A2}$$

□

Proof of Lemma 2. Since asset prices are affine functions of the state, which evolves according to (44), Ito's Lemma implies that $\frac{\mathrm{d}P_t^{(\tau)}}{P_t^{(\tau)}} = \mu_t^{(\tau)}\,\mathrm{d}t + \boldsymbol{\sigma}^{(\tau)}\,\mathrm{d}\mathbf{B}_t$, with $\boldsymbol{\sigma}^{(\tau)} = -\mathbf{A}(\tau)^\top\boldsymbol{\sigma}$ and

$$\mu_t^{(\tau)} = \mathbf{A}'(\tau)^\top\mathbf{y}_t + C'(\tau) + \mathbf{A}(\tau)^\top\boldsymbol{\Gamma}\left(\mathbf{y}_t - \bar{\mathbf{y}}\right) + \frac{1}{2}\mathbf{A}(\tau)^\top\boldsymbol{\Sigma}\mathbf{A}(\tau), \tag{A3}$$

where $\boldsymbol{\Sigma} \equiv \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$. Differentiating the arbitrageur budget constraint with respect to holdings $X_t^{(\tau)}$ gives the optimality conditions

$$\mu_t^{(\tau)} - i_t = a\left[\int_0^T X_t^{(\tau)}\mathbf{A}(\tau)\,\mathrm{d}\tau\right]^\top \boldsymbol{\Sigma}\mathbf{A}(\tau).$$

Substituting the affine pricing equation into the habitat demand curves gives $Z_t^{(\tau)} = [\alpha(\tau)\mathbf{A}(\tau) - \mathbf{\Theta}(\tau)]^\top \mathbf{y}_t$ where

$$\mathbf{\Theta}(\tau) = \begin{bmatrix} \dots & \theta^k(\tau) & \dots \end{bmatrix}^\top. \tag{A4}$$

Then substitute market clearing conditions $X_t^{(\tau)} = -Z_t^{(\tau)}$ into the optimality conditions and collect terms that are linear in the state $\mathbf{y}_t$ to get:

$$\mathbf{A}'(\tau) + \mathbf{M}\mathbf{A}(\tau) - \mathbf{e}_i = \mathbf{0}, \tag{A5}$$

where $\mathbf{M}$ is given by equation (47). Taking $\mathbf{M}$ as given, this is a linear system of differential equations. To derive initial conditions, note that at maturity, the riskless bonds pay \$1 so the $\tau = 0$ prices are given by $P_t^{(0)} = 1$. Hence, we have $\mathbf{A}(0) = \mathbf{0}$. Then assuming $\mathbf{M}$ is diagonalizable and invertible, the solution is given by equation (46). □

Proof of Proposition 2. In an affine equilibrium we have that $\tilde{\mu}_t = \int_0^T \eta(\tau)\mu_t^{(\tau)}\,\mathrm{d}\tau$. Substituting equations (A3) and (A5) into this expression and collecting terms which are linear in the state $\mathbf{y}_t$ gives equation (48). Equilibrium is the solution of the fixed point problem implicitly defined by equations (47) and (48). Rewrite these conditions in the following function:

$$f(\hat{\mathbf{A}}; \mathbf{M}; a) = \begin{bmatrix} \mathbf{e}_i + \left(\mathbf{\Gamma}(\hat{\mathbf{A}})^\top - \mathbf{M}\right)\boldsymbol{\nu}(\mathbf{M}) - \hat{\mathbf{A}} \\ \mathrm{vec}\left\{\mathbf{\Gamma}(\hat{\mathbf{A}})^\top - a \cdot \mathbf{\Lambda}(\mathbf{M}) - \mathbf{M}\right\} \end{bmatrix}, \tag{A6}$$

where $\mathbf{\Lambda}(\mathbf{M})$ and $\boldsymbol{\nu}(\mathbf{M})$ are the integral terms from equations (47) and (48). In both cases, dependence on $\mathbf{M}$ comes through the affine coefficients $\mathbf{A}(\tau)$. We have also made explicit the dependence of $\mathbf{\Gamma}$ on $\hat{\mathbf{A}}$, which can be seen in the proof of Lemma 1. If $J \equiv \dim \mathbf{y}_t$, then $\dim \mathbf{M} = J \times J$ and $\dim \hat{\mathbf{A}} = J$ and the function $f : \mathbb{R}^{J(J+1)+1} \to \mathbb{R}^{J(J+1)}$. For any value of a, equilibrium is defined by $f(\hat{\mathbf{A}}; \mathbf{M}; a) = \mathbf{0}$.

We now analyze the solution in a neighborhood around $a = 0$. For $a = 0$, clearly $\hat{\mathbf{A}} = \mathbf{e}_i$ and $\mathbf{M} = \mathbf{\Gamma}(\mathbf{e}_i)^\top$. The partial derivatives evaluated at this point are given by:

$$\frac{\partial f}{\partial \hat{\mathbf{A}}_j} = \begin{bmatrix} \left[\partial \mathbf{\Gamma}^\top \middle/ \partial \hat{\mathbf{A}}_j\right] \boldsymbol{\nu}(\mathbf{M}) - \mathbf{e}_j \\ \operatorname{vec}\left[\partial \mathbf{\Gamma}^\top \middle/ \partial \hat{\mathbf{A}}_j\right] \end{bmatrix}, \qquad \frac{\partial f}{\partial \mathbf{M}_{kl}} = \begin{bmatrix} \mathbf{e}_k \mathbf{e}_l^\top \boldsymbol{\nu}(\mathbf{M}) \\ -\operatorname{vec} \mathbf{e}_k \mathbf{e}_l^\top \end{bmatrix},$$

where $\mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l$ are standard normal basis vectors. The matrix $\partial \mathbf{\Gamma} \big/ \partial \hat{\mathbf{A}}_j$ is the derivative of the state dynamics matrix $\mathbf{\Gamma}$ with respect to the j-element of $\hat{\mathbf{A}}$; because this depends on derivatives of the eigendecomposion defined in the proof of Lemma 1, even in the case of $a = 0$ this is a complicated expression. Nevertheless, from this we can show that the Jacobian of f with respect to $\hat{\mathbf{A}}, \mathbf{M}$ evaluated at the $a = 0$ solution has full rank. In fact, writing this Jacobian in block form, we have

$$Df \equiv \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & -\mathbf{I}_{J^2} \end{bmatrix}, \tag{A7}$$

and $\mathbf{D}_{12} = \begin{bmatrix} \mathbf{I}_J \cdot \nu_1 & \dots & \mathbf{I}_J \cdot \nu_J \end{bmatrix}$, where ν_j is the j-element of $\boldsymbol{\nu}(\mathbf{M})$. Because the elementary row operations which transform $\mathbf{D}_{12}$ into the zero matrix simultaneously transform $\mathbf{D}_{11}$ into $-\mathbf{I}_J$, $\det Df = 1$ and the result follows from the implicit function theorem. □

B Model Details

B.1 Second-Order Approximations

Lo-quadratic approximation of functions of the form:

$$F \equiv g\left[\int_0^T \eta(\tau) f\left(X_t(\tau)\right) \mathrm{d}\tau\right] = g\left[\int_0^T \eta(\tau) f\left(\bar{X} e^{x_t(\tau)}\right) \mathrm{d}\tau\right]$$

We have

$$
\begin{aligned}
F\bigg|_{SS} &= g(f(\bar{X})) \\
\frac{\partial F}{\partial x_t(\tau')}\bigg|_{SS} &= \bar{X} g'(f(\bar{X})) f'(\bar{X}) \eta(\tau')\,\mathrm{d}\tau' \\
\frac{\partial^2 F}{\partial x_t(\tau')\partial x_t(\tau'')}\bigg|_{SS} &= g''(f(\bar{X}))(\bar{X} f'(\bar{X}))^2 \eta(\tau')\,\mathrm{d}\tau'\,\eta(\tau'')\,\mathrm{d}\tau'' \\
\frac{\partial^2 F}{\partial x_t(\tau')^2}\bigg|_{SS} &= g''(f(\bar{X}))(\bar{X} f'(\bar{X}))^2 (\eta(\tau')\,\mathrm{d}\tau')^2 + g'(f(\bar{X}))\bar{X}(f'(\bar{X}) + \bar{X} f''(\bar{X}))\eta(\tau')\,\mathrm{d}\tau'
\end{aligned}
$$

Thus, the second order approximation is

$$
\begin{aligned}
F &\approx \Delta_0 + \Delta_1 E_\tau x_t(\tau) + \frac{1}{2}\Delta_2 [E_\tau x_t(\tau)]^2 + \frac{1}{2}\Delta_3 E_\tau [x_t(\tau)^2] \\
&= \Delta_0 + \Delta_1 E_\tau x_t(\tau) + \frac{1}{2}(\Delta_2 + \Delta_3)[E_\tau x_t(\tau)]^2 + \frac{1}{2}\Delta_3 Var_\tau x_t(\tau)
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_0 &\equiv g(f(\bar{X})) \\
\Delta_1 &\equiv \bar{X} g'(f(\bar{X})) f'(\bar{X}) \\
\Delta_2 &\equiv g''(f(\bar{X}))(\bar{X} f'(\bar{X}))^2 \\
\Delta_3 &\equiv g'(f(\bar{X}))\bar{X}(f'(\bar{X}) + \bar{X} f''(\bar{X}))
\end{aligned}
$$

and

$$
\begin{aligned}
E_\tau[h(x_t(\tau))] &\equiv \int_0^T \eta(\tau) h(x_t(\tau))\,\mathrm{d}\tau \\
Var_\tau[h(x_t(\tau))] &\equiv \int_0^T \eta(\tau) h(x_t(\tau))^2\,\mathrm{d}\tau - E_\tau[h(x_t(\tau))]^2
\end{aligned}
$$

The second-order approximation follows from taking the limit of the finite second-order approximation at grid points $i = 1, \ldots, N$ with Δ_i steps:

$$\begin{aligned}
F &= g\left[\sum_{i=1}^{N} \eta(i) f\left(\bar{X} e^{x_t(i)}\right) \Delta_i\right] \\
F &\approx \bar{F} + \sum_{i=1}^{N}\left[\left.\frac{\partial F}{\partial x(i)}\right|_{SS}\right] x(i) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\left[\left.\frac{\partial^2 F}{\partial x(i)\partial x(j)}\right|_{SS}\right] x(i)x(j)
\end{aligned}$$

and note that the second derivative in the double summation has the additional terms when $i = j$.

B.1.1 Examples

Simple mean: $f(z) = g(z) = z$

$$\begin{aligned}
&\int_0^T \eta(\tau)\bar{X} e^{x_t(\tau)}\, \mathrm{d}\tau \\
&\approx \bar{X}\left\{1 + E_\tau x_t(\tau) + \frac{1}{2}[E_\tau x_t(\tau)]^2 + \frac{1}{2} Var_\tau x_t(\tau)\right\}
\end{aligned}$$

Mean of function: $g(z) = z$

$$\begin{aligned}
&\int_0^T \eta(\tau) f\left(\bar{X} e^{x_t(\tau)}\right) \mathrm{d}\tau \\
&\approx f(\bar{X}) + \bar{X}\left\{f'(\bar{X}) E_\tau x_t(\tau) + \frac{1}{2}(f'(\bar{X}) + \bar{X} f''(\bar{X}))(Var_\tau x_t(\tau) + [E_\tau x_t(\tau)]^2)\right\}
\end{aligned}$$

Function of mean: $f(z) = z$

$$\begin{aligned}
&g\left[\int_0^T \eta(\tau)\bar{X} e^{x_t(\tau)}\, \mathrm{d}\tau\right] \\
&\approx g(\bar{X}) + \bar{X}\left\{g'(\bar{X}) E_\tau x_t(\tau) + \frac{1}{2} g'(\bar{X}) Var_\tau x_t(\tau) + \frac{1}{2}(g'(\bar{X}) + \bar{X} g''(\bar{X}))[E_\tau x_t(\tau)]^2\right\}
\end{aligned}$$

Thus

$$\begin{aligned}
&f\left[\int_0^T \eta(\tau)\bar{X} e^{x_t(\tau)}\, \mathrm{d}\tau\right] - \int_0^T \eta(\tau) f\left(\bar{X} e^{x_t(\tau)}\right) \mathrm{d}\tau \\
&\approx -\frac{1}{2}\bar{X}^2 f''(\bar{X}) Var_\tau x_t(\tau)
\end{aligned}$$

Inverse: $g = f^{-1}$

$$f^{-1}\left[\int_0^T \eta(\tau) f\left(\bar{X}e^{x_t(\tau)}\right) \mathrm{d}\tau\right]$$
$$\approx \bar{X}\left\{1 + E_\tau x_t(\tau) + \frac{1}{2}\left[1 + \bar{X}\frac{f''(\bar{X})}{f'(\bar{X})}\right] Var_\tau x_t(\tau) + \frac{1}{2}[E_\tau x_t(\tau)]^2\right\}$$

CES:

$$\left[\int_0^T \eta(\tau)\left(\bar{X}e^{x_t(\tau)}\right)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}\tau\right]^{\frac{\epsilon}{\epsilon-1}}$$
$$\approx \bar{X}\left\{1 + E_\tau x_t(\tau) + \frac{1}{2}\left[\frac{\epsilon-1}{\epsilon}\right] Var_\tau x_t(\tau) + \frac{1}{2}[E_\tau x_t(\tau)]^2\right\}$$

Thus

$$\int_0^T \eta(\tau)\bar{X}e^{x_t(\tau)} \mathrm{d}\tau - \left[\int_0^T \eta(\tau)\left(\bar{X}e^{x_t(\tau)}\right)^{\frac{\epsilon-1}{\epsilon}} \mathrm{d}\tau\right]^{\frac{\epsilon}{\epsilon-1}}$$
$$\approx \frac{1}{2}\bar{X}\frac{1}{\epsilon}Var_\tau x_t(\tau)$$

CRRA utility difference from RANK:

$$\frac{1}{1-\varsigma}\left[\int_0^T \eta(\tau)\bar{X}e^{x_t(\tau)} \mathrm{d}\tau\right]^{1-\varsigma} - \frac{1}{1-\varsigma}\int_0^T \eta(\tau)\left(\bar{X}e^{x_t(\tau)}\right)^{1-\varsigma} \mathrm{d}\tau$$
$$\approx \frac{1}{2}\bar{X}^{1-\varsigma}\varsigma Var_\tau x_t(\tau)$$

B.2 Social Welfare: Aggregate Relationships

Equilibrium equations. Consumption:

$$C_t = \int_0^T \eta(\tau) C_t(\tau) \mathrm{d}\tau \tag{B1}$$
$$C_t = Y_t\left[1 - \frac{1}{2}\theta\pi_t^2 - \frac{1}{2}\psi^i(i_t - \bar{i}_t)^2 - \frac{1}{2}\int_0^T \psi^{(\tau)}\left(S_t^{(\tau)}\right)^2 \mathrm{d}\tau\right] \tag{B2}$$

Production:

$$Y_t = Z_t L_t \tag{B3}$$

Aggregate labor/wage index:

$$L_t = \left[\int_0^T \eta(\tau) N_t(\tau)^{\frac{\epsilon_w - 1}{\epsilon_w}} \,\mathrm{d}\tau\right]^{\frac{\epsilon_w}{\epsilon_w - 1}} \tag{B4}$$

$$W_t = \left[\int_0^T \eta(\tau) W_t(\tau)^{1-\epsilon_w} \,\mathrm{d}\tau\right]^{1-\epsilon_w} \tag{B5}$$

Labor demand and labor clearing:

$$N_t(\tau) = \left(\frac{W_t(\tau)}{W_t}\right)^{-\epsilon_w} L_t \implies W_t(\tau) = \left(\frac{N_t(\tau)}{L_t}\right)^{-\frac{1}{\epsilon_w}} W_t \tag{B6}$$

Intratemporal wage/consumption/labor:

$$(1+\tau^w)\left(\frac{\epsilon_w - 1}{\epsilon_w}\right) W_t(\tau) = C_t(\tau)^{\varsigma} N_t(\tau)^{\varphi} \tag{B7}$$

$$\implies (1+\tau^w)\left(\frac{\epsilon_w - 1}{\epsilon_w}\right) L_t^{\frac{1}{\epsilon_w}} W_t = C_t(\tau)^{\varsigma} N_t(\tau)^{\varphi + \frac{1}{\epsilon_w}} \tag{B8}$$

Output gap:

$$X_t = \frac{Y_t}{Y_t^n} \tag{B9}$$

B.2.1 Social Welfare Approximation

The log-quadratic approximation of deviations of social welfare from the first-best are given by

$$\begin{aligned}
\mathcal{W}_0 &\equiv \int_0^\infty e^{-\rho t} \mathcal{L}_t \,\mathrm{d}t \\
\mathcal{L}_t &= U\left(C_t^n, N_t^n\right) - \int_0^T \eta(\tau) U\left(C_t(\tau), N_t(\tau)\right) \mathrm{d}\tau \\
&= \frac{1}{2}(\varsigma + \varphi) x_t^2 + \frac{1}{2}\theta \pi_t^2 \\
&\quad + \frac{1}{2}\varsigma Var_\tau c_t(\tau) + \frac{1}{2}\varphi Var_\tau n_t(\tau) + \frac{1}{2}\frac{1}{\epsilon_w} Var_\tau n_t(\tau) \\
&\quad + \frac{1}{2}\psi^i (i_t - \bar{i}_t)^2 + \frac{1}{2}\int_0^T \psi^{(\tau)} \left(S_t^{(\tau)}\right)^2 \mathrm{d}\tau
\end{aligned}$$

We have

$$\varsigma Var_\tau c_t(\tau) + \varphi Var_\tau n_t(\tau) + \frac{1}{\epsilon_w} Var_\tau n_t(\tau) = \left(\varsigma + \frac{\varsigma^2 \varphi}{\left(\frac{1}{\epsilon_w} + \varphi\right)^2} \right) Var_\tau c_t(\tau) + \epsilon_w Var_\tau w_t(\tau)$$

and

$$Var_\tau w_t(\tau) = \left(\frac{\varsigma}{1 + \epsilon_w \varphi} \right)^2 Var_\tau c_t(\tau)$$

The loss associated with $Var_\tau c_t(\tau)$ comes from imperfect risk-sharing. The loss associated with $Var_\tau w_t(\tau)$ comes from labor market inefficiencies.

B.2.2 Equilibrium Approximations: Details

Throughout we make use of big-O properties:

$$\begin{aligned} f = O(h), g = O(k) &\implies f + g = O(h + k), f \cdot g = O(h \cdot k) \\ \implies f(\tau) = O(h(\tau)) \implies E_\tau f(\tau) \equiv \int \eta(\tau) f(\tau) \, \mathrm{d}\tau &= O\left(\int \eta(\tau) h(\tau) \, \mathrm{d}\tau \right) \equiv O(E_\tau h(\tau)) \end{aligned}$$

We use the variable ξ_t to denote the generic expansion point around the steady state for any variable, so that

$$f(x_t) = O(\xi_t), g(y_t) = O(\xi_t) \implies f(x_t) + g(y_t) = O(\xi_t), f(x_t) \cdot g(y_t) = O(\xi_t^2)$$

Aggregate and HH member consumption from (B1)

$$\begin{aligned} C_t \equiv \bar{C} e^{c_t} &= \bar{C} \left\{ 1 + c_t + \frac{1}{2} c_t^2 \right\} + O(\xi_t^3) \\ C_t(\tau) \equiv \bar{C} e^{c_t(\tau)} &= \bar{C} \left\{ 1 + c_t(\tau) + \frac{1}{2} c_t(\tau)^2 \right\} + O(\xi_t^3) \\ \implies c_t + \frac{1}{2} c_t^2 + O(\xi_t^3) &= E_\tau c_t(\tau) + \frac{1}{2} E_\tau \left[c_t(\tau) \right]^2 + \frac{1}{2} Var_\tau c_t(\tau) + O(\xi_t^3) \end{aligned}$$

Also, aggregate labor supply $N_t \equiv \int_0^T \eta(\tau) N_t(\tau) \, \mathrm{d}\tau$ (which is not equivalent to aggregate labor index L_t):

$$\begin{aligned} N_t \equiv \bar{N} e^{n_t} &= \bar{N}\left\{1 + n_t + \frac{1}{2} n_t^2\right\} + O(\xi_t^3) \\ N_t(\tau) \equiv \bar{N} e^{n_t(\tau)} &= \bar{N}\left\{1 + n_t(\tau) + \frac{1}{2} n_t(\tau)^2\right\} + O(\xi_t^3) \\ \implies n_t + \frac{1}{2} n_t^2 + O(\xi_t^3) &= E_\tau n_t(\tau) + \frac{1}{2} E_\tau \left[n_t(\tau)\right]^2 + \frac{1}{2} Var_\tau n_t(\tau) + O(\xi_t^3) \end{aligned}$$

From consumption goods market clearing (B2)

$$\bar{C} e^{c_t} = \bar{Y} e^{y_t} e^{d_t}$$

where d_t is the log of the deadweight loss terms in (B2) (which in levels is equal to one in SS). Since $\bar{C} = \bar{Y}$, we have

$$c_t + \frac{1}{2} c_t^2 + O(\xi_t^3) = (y_t + d_t) + \frac{1}{2}(y_t + d_t)^2 + O(\xi_t^3)$$

Additionally, a second-order expansion of deadweight loss d_t gives

$$\begin{aligned} d_t &= -\frac{1}{2}\left[\theta \pi_t^2 + \psi^i (i_t - \bar{i}_t)^2 + \int_0^T \psi^{(\tau)} \left(S_t^{(\tau)}\right)^2 \mathrm{d}\tau\right] + O(\xi_t^3) \\ &\equiv -\frac{1}{2} \tilde{d}_t + O(\xi_t^3) \end{aligned}$$

and also

$$\begin{aligned} d_t^2 &= O(\xi_t^3) \\ d_t v_t &= O(\xi_t^3) \end{aligned}$$

for any variable v_t (in terms of deviations from steady state).

Thus

$$c_t + \frac{1}{2} c_t^2 + O(\xi_t^3) = y_t + \frac{1}{2} y_t^2 - \frac{1}{2} \tilde{d}_t + O(\xi_t^3)$$

Production from (B3)

$$y_t = z_t + \ell_t$$

Labor and wage indices from (B4)

$$\ell_t + \frac{1}{2}\ell_t^2 + O(\xi_t^3) = E_\tau n_t(\tau) + \frac{1}{2}E_\tau[n_t(\tau)]^2 + \frac{1}{2}\frac{\epsilon_w - 1}{\epsilon_w}Var_\tau n_t(\tau) + O(\xi_t^3)$$
$$w_t + \frac{1}{2}w_t^2 + O(\xi_t^3) = E_\tau w_t(\tau) + \frac{1}{2}E_\tau[w_t(\tau)]^2 + \frac{1}{2}(1-\epsilon_w)Var_\tau w_t(\tau) + O(\xi_t^3)$$

Combining with aggregate labor supply

$$\ell_t + \frac{1}{2}\ell_t^2 + O(\xi_t^3) = n_t + \frac{1}{2}n_t^2 - \frac{1}{2}\frac{1}{\epsilon_w}Var_\tau n_t(\tau) + O(\xi_t^3)$$

Labor demand from (B6)

$$\begin{aligned} n_t(\tau) &= -\epsilon_w(w_t(\tau) - w_t) + \ell_t \\ \implies E_\tau n_t(\tau) &= -\epsilon_w(E_\tau w_t(\tau) - w_t) + \ell_t \\ Var_\tau n_t(\tau) &= \epsilon_w^2 Var_\tau w_t(\tau) \end{aligned}$$

Intratemporal HH conditions from (B7)

$$\begin{aligned} w_t(\tau) &= \varsigma c_t(\tau) + \varphi n_t(\tau) \\ \implies E_\tau w_t(\tau) &= \varsigma E_\tau c_t(\tau) + \varphi E_\tau n_t(\tau) \\ Var_\tau w_t(\tau) &= \varsigma^2 Var_\tau c_t(\tau) + \varphi^2 Var_\tau n_t(\tau) + 2Cov_\tau(c_t(\tau), n_t(\tau)) \end{aligned}$$

Output gap from (B9)

$$x_t = y_t - y_t^n$$

Utility

$$U(C_t(\tau), N_t(\tau)) - \bar{U} = \bar{Z}^\chi \left\{ c_t(\tau) + \frac{1}{2}(1-\varsigma)c_t(\tau)^2 - n_t(\tau) + \frac{1}{2}(1+\varphi)n_t(\tau)^2 \right\} + O(\xi_t^3)$$

where $\bar{U} = U(\bar{C}, \bar{N})$ and $\chi \equiv \frac{(1-\varsigma)(1+\varphi)}{\varsigma+\varphi}$. Thus social welfare

$$
\begin{aligned}
\mathcal{U}_t &\equiv \int_0^T \eta(\tau) U(C_t(\tau), N_t(\tau))\,\mathrm{d}\tau \\
\bar{Z}^{-\chi}(\mathcal{U}_t - \bar{U}) &= E_\tau c_t(\tau) + \frac{1}{2}(1-\varsigma)\left(E_\tau[c_t(\tau)]^2 + Var_\tau c_t(\tau)\right) \\
&\quad - E_\tau n_t(\tau) - \frac{1}{2}(1+\varphi)\left(E_\tau[n_t(\tau)]^2 + Var_\tau n_t(\tau)\right) + O(\xi_t^3)
\end{aligned}
$$

Second-order approximations of $C_t^{1-\varsigma}/(1-\varsigma)$ and $L_t^{1+\varphi}/(1+\varphi)$ give

$$
\begin{aligned}
c_t + \frac{1}{2}(1-\varsigma)c_t^2 + O(\xi_t^3) &= E_\tau c_t(\tau) + \frac{1}{2}(1-\varsigma)E_\tau[c_t(\tau)]^2 + \frac{1}{2}Var_\tau c_t(\tau) + O(\xi_t^3) \\
\ell_t + \frac{1}{2}(1+\varphi)\ell_t^2 + O(\xi_t^3) &= E_\tau n_t(\tau) + \frac{1}{2}(1+\varphi)E_\tau[n_t(\tau)]^2 + \frac{1}{2}\frac{\epsilon_w - 1}{\epsilon_w}Var_\tau n_t(\tau) + O(\xi_t^3)
\end{aligned}
$$

Thus social welfare

$$
\bar{Z}^{-\chi}(\mathcal{U}_t - \bar{U}) = c_t + \frac{1}{2}(1-\varsigma)c_t^2 - \varsigma Var_\tau c_t(\tau) - \ell_t - \frac{1}{2}(1+\varphi)\ell_t^2 - \left(\varphi + \frac{1}{\epsilon_w}\right)Var_\tau n_t(\tau) + O(\xi_t^3)
$$

Combining with the second-order approximations of consumption and production:

$$
\begin{aligned}
c_t + \frac{1}{2}(1-\varsigma)c_t^2 &= (y_t + d_t) + \frac{1}{2}(1-\varsigma)\left(y_t + d_t\right)^2 \\
&= y_t - \frac{1}{2}\tilde{d}_t + \frac{1}{2}(1-\varsigma)y_t^2 + O(\xi_t^3) \\
\ell_t + \frac{1}{2}(1+\varphi)\ell_t^2 &= (y_t - z_t) + \frac{1}{2}(1+\varphi)\left(y_t - z_t\right)^2
\end{aligned}
$$

In the first-best, we have

$$
\begin{aligned}
y_t^n &= \frac{1+\varphi}{\varsigma+\varphi}z_t \\
\implies y_t &= x_t + \frac{1+\varphi}{\varsigma+\varphi}z_t \equiv x_t + \frac{\chi}{1-\varsigma}z_t \\
y_t - z_t &= x_t + \frac{1-\varsigma}{\varsigma+\varphi}z_t \equiv x_t + \frac{\chi}{1+\varphi}z_t
\end{aligned}
$$

Thus

$$\begin{aligned}
&y_t + \frac{1}{2}(1-\varsigma)y_t^2 - (y_t - z_t) - \frac{1}{2}(1+\varphi)(y_t - z_t)^2 \\
&= z_t + \frac{1}{2}(1-\varsigma)\left(x_t + \frac{\chi}{1-\varsigma}z_t\right)^2 - \frac{1}{2}(1+\varphi)\left(x_t + \frac{\chi}{1+\varphi}z_t\right)^2 \\
&= z_t + \frac{1}{2}\chi z_t^2 - \frac{1}{2}(\varsigma+\varphi)x_t^2
\end{aligned}$$

And note social welfare at the first-best is

$$\bar{Z}^{-\chi}(\mathcal{U}_t^n - \bar{U}) = z_t + \frac{1}{2}\chi z_t^2 + O(\xi_t^3)$$

Combining, we have that social welfare differences from the first-best are

$$\bar{Z}^{-\chi}(\mathcal{U}_t^n - \mathcal{U}_t) = \frac{1}{2}(\varsigma+\varphi)x_t^2 + \frac{1}{2}\tilde{d}_t + \varsigma Var_\tau c_t(\tau) + \left(\varphi + \frac{1}{\epsilon_w}\right)Var_\tau n_t(\tau) + O(\xi_t^3)$$

The variance terms are related as follows:

$$\begin{aligned}
\left(\varphi + \frac{1}{\epsilon_w}\right)^2 Var_\tau n_t(\tau) &= \varsigma^2 Var_\tau c_t(\tau) \\
Var_\tau w_t(\tau) &= \left(\frac{1}{\epsilon_w}\right)^2 Var_\tau n_t(\tau)
\end{aligned}$$

Thus

$$\varsigma Var_\tau c_t(\tau) + \left(\varphi + \frac{1}{\epsilon_w}\right)Var_\tau n_t(\tau) = \left(\frac{\varsigma}{\varphi + \frac{1}{\epsilon_w}}\right)\left(\varsigma + \varphi + \frac{1}{\epsilon_w}\right)Var_\tau c_t(\tau)$$

We can also decompose these terms as follows:

$$\begin{aligned}
\varsigma Var_\tau c_t(\tau) + \varphi Var_\tau n_t(\tau) &= \frac{\varsigma}{\varphi}\left(\varphi + \varsigma\left[\frac{\varphi\epsilon_w}{1+\varphi\epsilon_w}\right]^2\right)Var_\tau c_t(\tau) \\
\frac{1}{\epsilon_w}Var_\tau n_t(\tau) &= \epsilon_w Var_\tau w_t(\tau)
\end{aligned}$$