

A PREFERRED-HABITAT MODEL OF TERM PREMIA, EXCHANGE RATES, AND MONETARY POLICY SPILLOVERS

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Motivation

Motivation

- Textbook international macro:
 - Uncovered Interest Parity (UIP) holds
 - The Expectation Hypothesis (EH) holds
- Empirically:
 1. Strong patterns in FX: currency carry trade is profitable \implies deviations from UIP
[Fama 1984...]
 2. Strong patterns in FI: bond carry trade is profitable \implies deviations from the EH
[Fama & Bliss 1987, Campbell & Shiller 1991...]
 3. The two risk premia are deeply connected
[Lustig et al 2019, Lloyd & Marin 2019, Chernov & Creal 2020...]
 4. Quantitative easing not only reduced domestic yields, but also had strong effects on exchange rates and foreign yields
[Bhattarai & Neely 2018...]

- Making sense of these facts is important:
 - To understand what determines exchange rates (volatility, disconnect...)
 - To understand [monetary policy transmission](#), both domestically (along the yield curve)...
 - ...but also [via international spillovers](#), to exchange rates and foreign yields
- On the theory side:
 - Standard representative agent no-arbitrage models have a hard time
 - Recent literature emphasizes the optimization of financial intermediaries and the constraints they face
[Gabaix & Maggiori 2015, Itskhoki & Mukhin 2019, Koijen & Yogo 2020]
 - Revives an old literature on portfolio-balance
[Kouri 1982, Jeanne & Rose 2002...]

- [This paper](#): introduce risk averse 'global rate arbitrageur' absorbing supply and demand shocks in bond and currency markets
- Clientele investors introduce a degree of [market segmentation](#)
 - FX and bond markets populated by different investor clienteles (pension funds, importers/exporters)
 - Arbitrageurs (hedge funds, fixed income desk of broker-dealer) partly overcome segmentation
- Formally: Two-country version of Vayanos & Vila's (2021) [preferred-habitat model](#)
 - Contemporaneous paper by Greenwood et al (2022) in discrete time with two bonds

Findings

1. Can reproduce **qualitative** and **quantitative** facts about the joint behavior of bond and currency risk premia
2. Rich transmission of monetary policy shocks via exchange rate and term premia, contrasting with standard models
3. **Key mechanisms:**
 - Shifts in arbitrageurs' risk exposure lead to changes in required risk compensation
 - Hedging behavior of **global** arbitrageurs \implies tight linkage between bond term premia and currency risk premia
 - In the presence of market segmentation, policy shocks (particularly **unconventional**) lead to large shifts in risk exposure
4. General message: **floating exchange rates provide limited insulation.**
Failure of Friedman-Obtsfeld-Taylor's Trilemma

Set-Up

Set-Up: Two-Country Vayanos & Vila (2021)

- Continuous time $t \in (0, \infty)$, 2 countries $j = H, F$
- Nominal exchange rate e_t : H price of F (increase \equiv depreciation of H 's currency)
- In each country j , continuum of zero coupon bonds in zero net supply with maturity $0 \leq \tau \leq T$, and $T \leq \infty$
- Bond price (in local currency) $P_{jt}^{(\tau)}$, with yield to maturity $y_{jt}^{(\tau)} = -\log P_{jt}^{(\tau)} / \tau$
- Nominal short rate ("monetary policy") $i_{jt} = \lim_{\tau \rightarrow 0} y_{jt}^{(\tau)}$ follows (exogenous, stochastic) mean-reverting process

Arbitrageurs and Preferred-Habitat Investors

- Home and foreign preferred-habitat **bond investors**
(hold bonds in a specific currency and maturity: $Z_{jt}(\tau)$)
 - Eg, pension funds, money market mutual funds
 - Time-varying demand β_{jt} , **downward sloping** in terms of bond price (elasticity $\alpha_j(\tau)$)
- Preferred-habitat **currency traders**
(hold foreign currency: Z_{et})
 - Eg, importers/exporters
 - Time-varying demand γ_t , **downward sloping** in terms of exchange rate (elasticity α_e)
- **Global rate arbitrageurs**
(can trade in both currencies, in domestic and foreign bonds: $W_{Ft}, X_{jt}(\tau)$)
 - Eg, global hedge funds
 - Mean-variance preferences (risk aversion a)
 - Engage in **currency carry trade, domestic and foreign bond carry trade**

Global Rate Arbitrageur

- Mean-variance optimization

$$\begin{aligned} & \max \mathbb{E}_t(dW_t) - \frac{a}{2} \text{Var}_t(dW_t) \\ \text{s.t. } & dW_t = W_t i_{Ht} dt + W_{Ft} \left(\frac{de_t}{e_t} + (i_{Ft} - i_{Ht}) dt \right) \\ & + \int_0^T \chi_{Ht}^{(\tau)} \left(\frac{dP_{Ht}^{(\tau)}}{P_{Ht}^{(\tau)}} - i_{Ht} dt \right) d\tau + \int_0^T \chi_{Ft}^{(\tau)} \left(\frac{d(P_{Ft}^{(\tau)} e_t)}{P_{Ft}^{(\tau)} e_t} - \frac{de_t}{e_t} - i_{Ft} dt \right) d\tau \end{aligned}$$

- Wealth W_t :
 - W_{Ft} invested in country F short rate (CCT)
 - $\chi_{jt}^{(\tau)}$ invested in bond of country j and maturity τ (BCT _{j})
 - Remainder in country H short rate

Key Insight: Risk averse arbitrageurs' holdings increase with expected return

Preferred-Habitat Bond and FX Investors

- Demand for bonds in currency j , of maturity τ :

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log P_{jt}^{(\tau)} - \theta_j(\tau) \beta_{jt}$$

- $\alpha_j(\tau)$: demand elasticity for τ investor in country j
- $\theta_j(\tau)$: how variations in factor β_{jt} affect demand for τ investor in country j
- Demand for foreign currency (spot):

$$Z_{et} = -\alpha_e \log e_t - \theta_e \gamma_t$$

- Can accommodate forward demand. Under CIP, equivalent to spot + H and F bond trades
- Exogenous bond and FX demand risk factors:

$$d\beta_{jt} = -\kappa_{\beta j} \beta_{jt} dt + \sigma_{\beta j} dB_{\beta jt}, \quad d\gamma_t = -\kappa_{\gamma} \gamma_t dt + \sigma_{\gamma} dB_{\gamma t}$$

Key Insight: elastic habitat traders. Price movements require portfolio rebalancing

Equilibrium

- Affine solution:

$$-\log p_{jt}^{(\tau)} = \mathbf{A}_j(\tau)^\top \mathbf{q}_t + C_j(\tau), \quad -\log e_t = \mathbf{A}_e^\top \mathbf{q}_t + C_e$$

where \mathbf{q}_t collects risk factors (short rates and demand factors)

- Arbitrageurs' optimality conditions imply expected excess returns are given by:

$$\mathbb{E}_t dP_{jt}^{(\tau)} / P_{jt}^{(\tau)} - i_{jt} = \mathbf{A}_j(\tau)^\top \boldsymbol{\Lambda}_t, \quad \mathbb{E}_t de_t / e_t + i_{Ft} - i_{Ht} = \mathbf{A}_e^\top \boldsymbol{\Lambda}_t$$

$$\text{where } \boldsymbol{\Lambda}_t = a\boldsymbol{\Sigma} \left(W_{Ft}\mathbf{A}_e + \sum_{j=H,F} \int_0^T X_{jt}\mathbf{A}_j(\tau) d\tau \right)$$

- Endogenous coefficients $\mathbf{A}_j(\tau), \mathbf{A}_e$ govern sensitivity to market price of risk $\boldsymbol{\Lambda}_t$
- Model is closed through market clearing: $X_{jt}^{(\tau)} + Z_{jt}^{(\tau)} = 0$, $W_{Ft} + Z_{et} = 0$

Key Insight: market price of risk $\boldsymbol{\Lambda}_t$ depends on equilibrium holdings. Bond and currency premia jointly determined

Data Generating Process: Assumptions

- In order to derive analytical results, we assume **independent** short-rate processes, and non-stochastic demand factors:

$$di_{Ht} = \kappa_{iH}(\bar{i}_H - i_{Ht}) dt + \sigma_{iH} dB_{iHt}, \quad di_{Ft} = \kappa_{iF}(\bar{i}_F - i_{Ft}) dt + \sigma_{iF} dB_{iFt}$$

- For quantitative results, we can allow for **rich demand structure** embodied in dynamics of risk factors. DGP:

$$\mathbf{q}_t = \begin{bmatrix} i_{Ht} & i_{Ft} & \beta_{Ht} & \beta_{Ft} & \gamma_t \end{bmatrix}^\top$$
$$d\mathbf{q}_t = -\mathbf{\Gamma}(\mathbf{q}_t - \bar{\mathbf{q}}) dt + \mathbf{\sigma} d\mathbf{B}_t$$

Risk Neutral Global Arbitrageur

1. Benchmark: Risk Neutral Global Rate Arbitrageur (“Standard Model”)

Consider the benchmark case of a risk neutral global rate arbitrageur: $a = 0$

- Expectation Hypothesis holds:

$$\mathbb{E}_t dP_{Ht}^{(\tau)} / P_{Ht}^{(\tau)} = i_{Ht}, \quad \mathbb{E}_t dP_{Ft}^{(\tau)} / P_{Ft}^{(\tau)} = i_{Ft}$$

- No effect of QE on yield curve, at Home or Foreign
- Yield curve independent from foreign short rate shocks

- Uncovered Interest Parity holds:

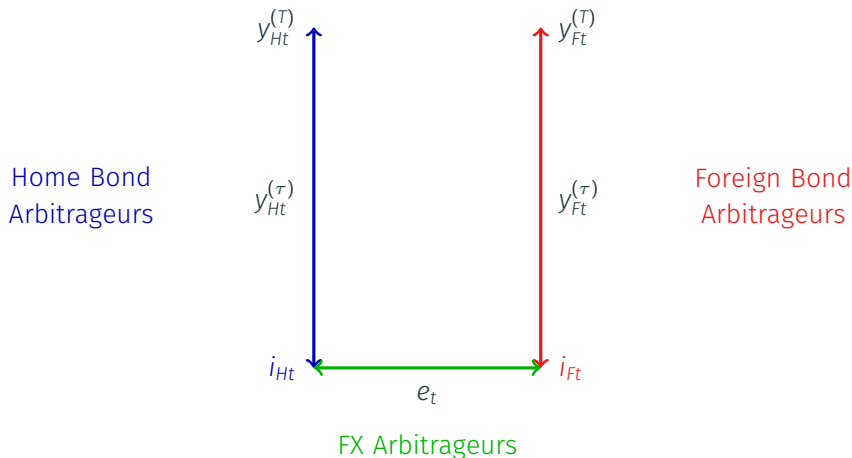
$$\mathbb{E}_t de_t / e_t = i_{Ht} - i_{Ft}$$

- ‘Mundellian’ insulation: shock to short rates ‘absorbed’ into the exchange rate
- Classical Trilemma: capital flows and floating exchange rates deliver monetary autonomy

Segmented Arbitrage

2. Intermediate Step: Segmented Arbitrage and No Demand Shocks

Assume foreign currency and bonds traded by **three disjoint sets of arbitrageurs**



2. Intermediate Step: Segmented Arbitrage and No Demand Shocks

Postulate: $\log P_{jt}^{(\tau)} = -A_{ij}(\tau)i_{jt} - C_j(\tau)$; $\log e_t = A_{iFe}i_{Ft} - A_{iHe}i_{Ht} - C_e$

Proposition (Segmented Arbitrage, Currency Carry Trade CCT and UIP Deviations)

When arbitrage is segmented, risk aversion $a > 0$ and FX price elasticity $\alpha_e > 0$

- Attenuation: $0 < A_{ije} < 1/\kappa_{ij}$
- CCT expected return $\mathbb{E}_t de_t / e_t + i_{Ft} - i_{Ht}$ decreases in i_{Ht} and increases in i_{Ft} (UIP deviation)

Intuition: Similar to Kouri (1982), Gabaix and Maggiori (2015)

- When $i_{Ht} \downarrow$ or $i_{Ft} \uparrow$, FX arbitrageurs want to invest more in the CCT
- Foreign currency appreciates ($e_t \uparrow$)
- As $e_t \uparrow$, price elastic FX traders ($\alpha_e > 0$) reduce holdings: $Z_{et} \downarrow$
- FX arbitrageurs increase their holdings $W_{Ft} \uparrow$, which requires a higher CCT return

2. Intermediate Step: Segmented Arbitrage and No Demand Shocks

Proposition (Segmented Arbitrage and Bond Carry Trade BCT)

When arbitrage is segmented, $a > 0$ and $\alpha(\tau) > 0$ in a positive-measure subset of $(0, T)$:

- Attenuation: $A_{ij}(\tau) < (1 - e^{-\kappa_{ij}\tau})/\kappa_{ij}$
- Bond prices in country j only respond to country j short rates (no spillover)
- BCT_j expected return $\mathbb{E}_t dP_{jt}^{(\tau)} / P_{jt}^{(\tau)} - i_{jt}$ decreases in i_{jt}

Intuition: Similar to Vayanos & Vila (2021)

- When $i_{jt} \downarrow$, bond arbitrageurs want to invest more in the BCT
- Bond prices increase ($P_{jt}^{(\tau)} \uparrow$)
- As $P_{jt}^{(\tau)} \uparrow$, price-elastic habitat bond investors ($\alpha_j(\tau) > 0$) reduce their holdings: $Z_{jt}^{(\tau)} \downarrow$
- Bond arbitrageurs increase their holdings $X_{jt}^{(\tau)} \uparrow$, which requires a larger BCT return

Macro Implications of the Segmented Model

Assume $a > 0$, $\theta_j(\tau) > 0$ and $\theta_e > 0$:

- Unexpected **increase in bond demand** in country j (QE_j) reduces yields in country j
- No effect on bond yields in the other country or on the exchange rate
 - QE purchases: $Z_{jt}^{(\tau)} \uparrow$
 - Bond arbitrageurs reduce holdings $X_{jt}^{(\tau)} \downarrow$, reducing risk exposure and pushing down yields
 - Arbitrageurs in other markets are unaffected

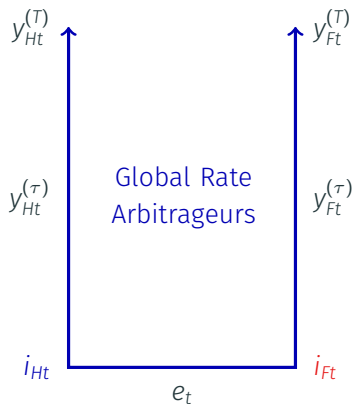
Open Economy Macro Implications:

- Changes in Home monetary conditions (conventional or QE) have no effect on the Foreign yield curve. **Full insulation**
- Insulation is even stronger in the case of QE: exchange rate is unchanged
- **Trilemma?** As we will see, this result arises because of markets segmentation (limited capital flows), not because of floating exchange rates

Global Arbitrage

3. Global Rate Arbitrageur and No Demand Shocks

Assume now [global rate arbitrageur](#) can invest in bonds (H and F) and FX




3. Global Rate Arbitrageur and No Demand Shocks

Postulate $\log P_{jt}^{(\tau)} = -A_{ijj}(\tau)i_{jt} - A_{ijj'}(\tau)i_{j't} - C_H(\tau)$; $\log e_t = A_{iFe}i_{Ft} - A_{iHe}i_{Ht} - C_e$

Proposition (Global Arbitrage and Carry Trades CCT, BCT)

When arbitrage is global, risk aversion $a > 0$ and price elasticities $\alpha_e, \alpha_j(\tau) > 0$:


- The results of the previous propositions obtain: both CCT and BCT_H return decrease with i_{Ht} , and attenuation is stronger than with segmented markets
-  In addition, BCT_F increases with i_{Ht}
- The effect of i_{jt} on bond yields is smaller in the other country: $A_{jj'}(\tau) < A_{jj}(\tau)$

Intuition: Bond and FX Premia Cross-Linkages

- When $i_{Ht} \downarrow$ global arbitrageurs want to invest more in CCT and BCT_H
- e_t and $W_{Ft} \uparrow$: increased FX exposure (risk of $i_{Ft} \downarrow$)
- Hedge by investing more in BCT_F since price of foreign bonds increases when i_{Ft} drops: foreign yields decline and BCT_F decreases

Macro Implications of Global Rate Arbitrageur Model

Assume $a > 0$ and $\alpha_e, \alpha_j(\tau) > 0$:

- Unexpected QE_H reduces yields in country H
-  Also reduces yields in country F , and depreciates the Home currency
 - Arbitrageurs decrease H bond exposure (less exposed to risk of $i_{Ht} \uparrow$)
 - More willing to hold assets exposed to this risk: increase holdings of F bonds and currency, pushing down F yields and depreciating the H currency

Open Economy Macro Implications:

- Changes in Home monetary conditions (conventional or QE) affect both yield curves and the exchange rate: potential spillovers from monetary policy. Imperfect insulation even with floating rates
- QE or FX interventions in one country affect monetary conditions in both countries and depreciate the currency
- Failure of the Classical Trilemma

The Full Model

The Full Model: Adding Demand Shocks

- Now we allow for **richer demand structure** of risk factors:

$$dq_t = -\mathbf{\Gamma}(q_t - \bar{q})dt + \boldsymbol{\sigma}dB_t$$

- We assume independent processes for all factors, except shocks to short rates may be correlated, and currency demand γ_t may respond to short rates
- **Numerical calibration**
 - **Data:** Zero coupon data: US Treasuries (H) and German Bunds (F); exchange rate data: German mark/euro
 - **Targets:** second moments of short/long term rates, exchange rates, and volumes
- **Return predictability** (untargeted)
 - Bond returns and slope of the term structure
 - Currency returns and UIP
 - Cross-country bond and currency returns

Numerical Calibration

- **Data:** Zero coupon data: US Treasuries (H) and German Bunds (F); exchange rate data: German mark/euro
- **Targets:** second moments of short/long term rates, exchange rates, and volumes

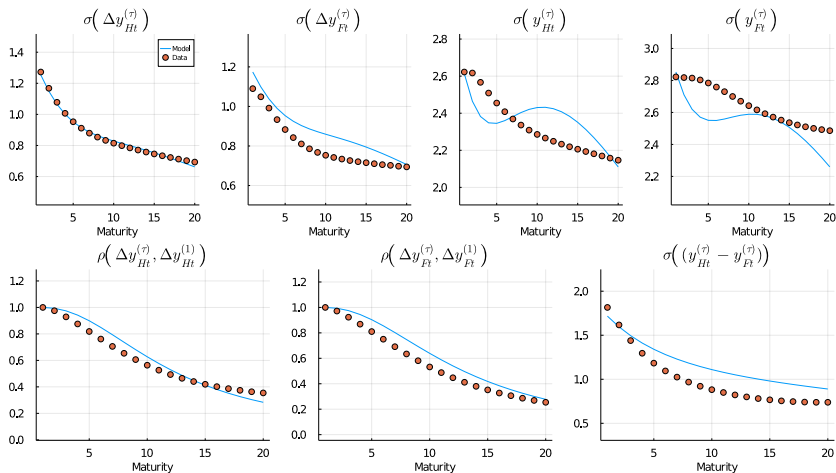
Parameter	Value	Parameter	Value	Parameter	Value
κ_{iH}	0.126	κ_{γ}	0.134	$a\sigma_{\beta}\theta_0$	90.6
κ_{iF}	0.0896	$\kappa_{\gamma,iH}$	-0.267	$a\alpha_e$	73.4
σ_{iH}	1.43	$\kappa_{\gamma,iF}$	0.252	$a\alpha_0$	4.74
σ_{iF}	0.751	$a\sigma_{\gamma}\theta_e$	763.0	α_1	0.144
$\sigma_{iH,iF}$	1.05	κ_{β}	0.0501	θ_1	0.374

- For policy experiments: CRRA $\gamma = 2$ and arbitrageur wealth $\frac{W}{GDP_H} \approx 5\% \implies a = 40$

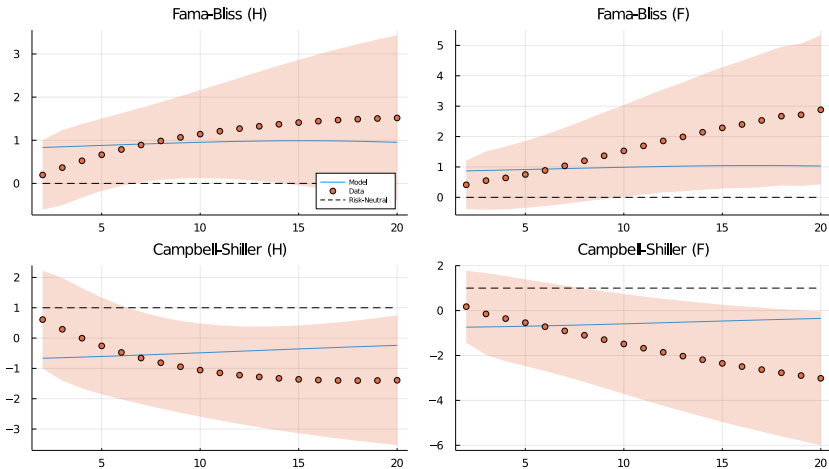
Model Fit: Short Rates and Exchange Rates

Moment	Data	Model	Moment	Data	Model
$\sigma \left(y_{Ht}^{(1)} \right)$	2.622	2.614	$\rho \left(\Delta \log e_t, (y_{Ht}^{(1)} - y_{Ft}^{(1)}) \right)$	-0.105	-0.096
$\sigma \left(\Delta y_{Ht}^{(1)} \right)$	1.273	1.254	$\rho \left(\Delta \log e_t, \Delta y_{Ht}^{(1)} \right)$	-0.095	-0.214
$\sigma \left(y_{Ft}^{(1)} \right)$	2.822	2.853	$\rho \left(\Delta \log e_t, \Delta y_{Ft}^{(1)} \right)$	0.048	0.071
$\sigma \left(\Delta y_{Ft}^{(1)} \right)$	1.09	1.174	$\rho \left(\Delta^{(5)} \log e_t, (y_{Ht}^{(5)} - y_{Ft}^{(5)}) \right)$	0.12	0.06
$\sigma \left((y_{Ht}^{(1)} - y_{Ft}^{(1)}) \right)$	1.816	1.717	$\tilde{V}_H(0 \leq \tau \leq 3)$	0.361	0.378
$\sigma \left(\Delta \log e_t \right)$	10.186	10.183	$\tilde{V}_H(11 \leq \tau \leq 30)$	0.08	0.116

Model Fit: Long Rates

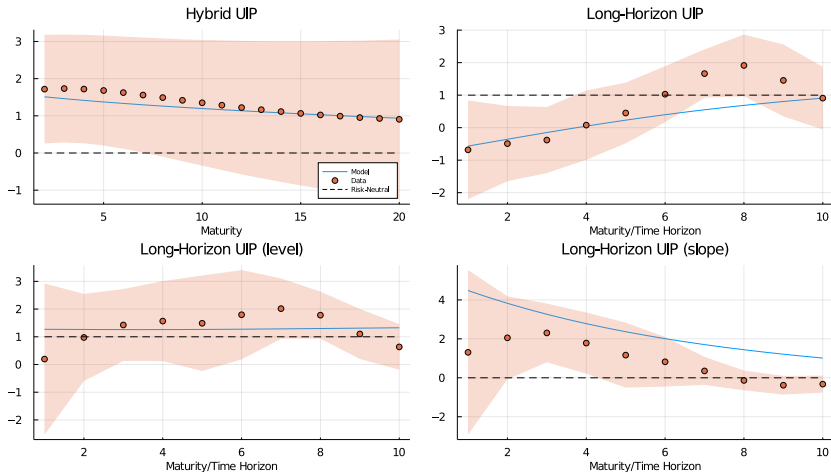


Regression Coefficients: Term Structure



Implications: Positive slope-premia relationship

Regression Coefficients: UIP



Implications: CCT is profitable, but profitability goes to zero if CCT is done with long-term bonds or over long horizon. Slope differential predicts CCT return

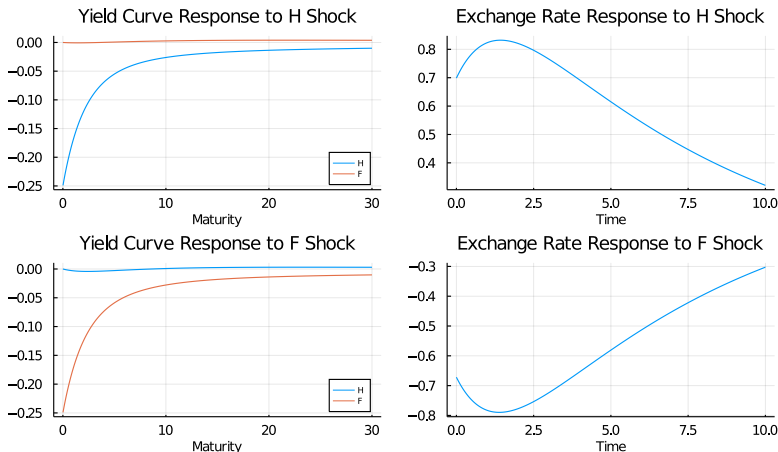
Conduct policy experiments:

- **Monetary policy shock:** unanticipated and idiosyncratic 25bp decrease in policy rate
- **QE shock:** unanticipated and idiosyncratic positive demand shock = 10% of GDP

Examine **spillovers**:

- Across the yield curves (short and long rates; and across countries)
- To the exchange rate

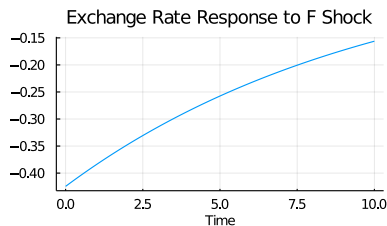
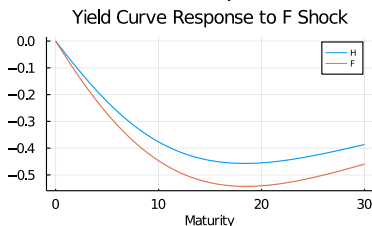
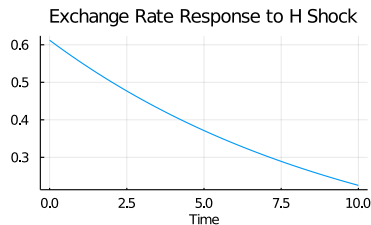
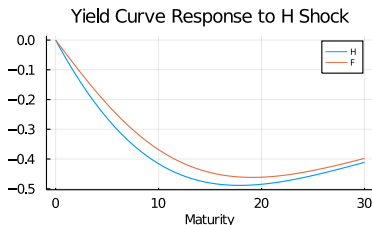
Monetary Shock Spillovers



Implications: small cross-country yield response; exchange rate “delayed overshooting”

- **Intuition:** correlated short rates, currency demand response

QE Shock Spillovers



Implications: large spillovers of QE, both to foreign yields and exchange rate

- **Intuition:** correlated short rates, elastic currency traders

Concluding Remarks

- Present an **integrated framework** to understand term premia and currency risk
- Resulting model ties together
 - Deviations from Uncovered Interest Parity
 - Deviations from Expectation Hypothesis
- **Rich transmission of monetary policy** domestically and abroad via FX and term premia

Thank You!

Details: Arbitrageur Optimality Conditions

- Ito's Lemma:

$$\frac{dP_{jt}^{(\tau)}}{P_{jt}^{(\tau)}} = \mu_{jt}^{(\tau)} dt + \boldsymbol{\sigma}_j^{(\tau)} d\mathbf{B}_t$$
$$\frac{de_t}{e_t} = \mu_{et} dt + \boldsymbol{\sigma}_e d\mathbf{B}_t$$

where

$$\mu_{jt}^{(\tau)} = \mathbf{q}_t^\top \mathbf{A}'_j(\tau) + C'_j(\tau) + [\boldsymbol{\Gamma}(\mathbf{q}_t - \bar{\mathbf{q}})]^\top \mathbf{A}_j(\tau) + \frac{1}{2} \text{Tr} [\boldsymbol{\sigma} \mathbf{A}_j(\tau) \mathbf{A}_j(\tau)^\top \boldsymbol{\sigma}]$$

$$\mu_e = [\boldsymbol{\Gamma}(\mathbf{q}_t - \bar{\mathbf{q}})]^\top \mathbf{A}_e + \frac{1}{2} \text{Tr} [\boldsymbol{\sigma} \mathbf{A}_e \mathbf{A}_e^\top \boldsymbol{\sigma}]$$

$$\boldsymbol{\sigma}_j^{(\tau)} = -\mathbf{A}_j(\tau)^\top \boldsymbol{\sigma}$$

$$\boldsymbol{\sigma}_e = -\mathbf{A}_e^\top \boldsymbol{\sigma}$$

Details: Arbitrageur Optimality Conditions

- Arbitrageurs' optimality conditions imply expected excess returns are given by:

$$\begin{aligned}\mu_{jt}^{(\tau)} - i_{jt} &= \mathbf{A}_j(\tau)^\top \boldsymbol{\Lambda}_t \\ \mu_{et} + i_{Ft} - i_{Ht} &= \mathbf{A}_e^\top \boldsymbol{\Lambda}_t\end{aligned}$$

- Endogenous coefficients $\mathbf{A}_j(\tau)$, \mathbf{A}_e govern sensitivity to market price of risk $\boldsymbol{\Lambda}_t$

$$\boldsymbol{\Lambda}_t = a\boldsymbol{\Sigma} \left(W_{Ft}\mathbf{A}_e + \sum_{j=H,F} \int_0^T X_{jt}^{(\tau)} \mathbf{A}_j(\tau) d\tau \right)$$

where $\boldsymbol{\Sigma} \equiv \boldsymbol{\sigma}\boldsymbol{\sigma}^\top$

Details: Preferred-Habitat Bond and FX Investors

- Demand for bonds in currency j , of maturity τ :

$$Z_{jt}^{(\tau)} = -\alpha_j(\tau) \log P_{jt}^{(\tau)} - \theta_j(\tau) \beta_{jt}$$

- $\alpha_j(\tau)$: demand elasticity for τ investor in country j
- $\theta_j(\tau)$: how variations in factor β_{jt} affect demand for τ investor in country j
- Demand for foreign currency (spot):

$$Z_{et} = -\alpha_e \log e_t - \theta_e \gamma_t$$

- Can accommodate forward demand. Under CIP, equivalent to spot + H and F bond trades
- Market clearing and zero net supply: $X_{jt}^{(\tau)} = -Z_{jt}^{(\tau)}$ and $W_{Ft} = -Z_{et}$
 - WLOG: can rewrite intercept terms to include positive supply
- Rewrite using affine functional form:

$$X_{jt}^{(\tau)} = -\alpha_j(\tau) [\mathbf{A}_j(\tau)^\top \mathbf{q}_t + C_j(\tau)] + \boldsymbol{\Theta}_j(\tau)^\top \mathbf{q}_t + \zeta_j(\tau)$$

$$W_{Ft} = -\alpha_e [\mathbf{A}_e^\top \mathbf{q}_t + C_e] + \boldsymbol{\Theta}_e^\top \mathbf{q}_t + \zeta_e$$

Details: Solution Characterization

- Substitute market clearing into arbitrageur optimality conditions, collect \mathbf{q}_t terms:

$$\mathbf{A}'_j(\tau) + \mathbf{M}\mathbf{A}_j(\tau) - \mathbf{e}_j = \mathbf{0}, \quad \mathbf{M}\mathbf{A}_e - (\mathbf{e}_H - \mathbf{e}_F) = \mathbf{0} \quad (\text{where } \mathbf{e}_j^\top \mathbf{q}_t = i_{jt})$$

- The matrix \mathbf{M} is defined as

$$\begin{aligned} \mathbf{M} = \mathbf{\Gamma}^\top - a \bigg\{ & \int_0^T [-\alpha_H(\tau)\mathbf{A}_H(\tau) + \mathbf{\Theta}_H(\tau)] \mathbf{A}_H(\tau)^\top d\tau \\ & + \int_0^T [-\alpha_F(\tau)\mathbf{A}_F(\tau) + \mathbf{\Theta}_F(\tau)] \mathbf{A}_F(\tau)^\top d\tau \\ & + [-\alpha_e\mathbf{A}_e + \mathbf{\Theta}_e] \mathbf{A}_e^\top \bigg\} \mathbf{\Sigma} \end{aligned} \quad (1)$$

- Initial conditions $\mathbf{A}_j(0) = \mathbf{0}$. Hence

$$\mathbf{A}_j(\tau) = [\mathbf{I} - e^{-\mathbf{M}\tau}] \mathbf{M}^{-1} \mathbf{e}_j \quad (2)$$

$$\mathbf{A}_e = \mathbf{M}^{-1}(\mathbf{e}_H - \mathbf{e}_F) \quad (3)$$

Details: Existence and Uniqueness

- Note: \mathbf{M} appears on both sides of equation (1), through the solution of the affine coefficients (2), (3)
 - Interpretation: risk-adjusted dynamics of the risk factors
- In general: system of J^2 nonlinear equations in J^2 unknowns, where $J = \dim \mathbf{q}_t$
- Under risk neutrality ($a = 0$), the solution is simple: $\mathbf{M} = \mathbf{\Gamma}^\top$
- When $a > 0$, the solution may not exist, or there may be multiple equilibria
- Can show (using IFT) that in a neighborhood of $a = 0$, the solution exists and is (locally) unique. Beyond that, very difficult to prove anything analytically in the fully general version of the model

Numerical Solution: Algorithm

- Numerical solution for \mathbf{M} in the general model
- Continuation algorithm:
 1. For $\hat{a} = \hat{a}^{(0)} = 0$, the known solution is $\mathbf{M}^{(0)} = \mathbf{\Gamma}^\top$
 2. Given a solution $\mathbf{M}^{(n)}$ for $\hat{a} = \hat{a}^{(n)}$, use this as the initial value for $\hat{a}^{(n+1)} = \hat{a}^{(n)} + \epsilon$
 3. Solution $\mathbf{M}^{(N)} = \mathbf{M}$ for $\hat{a}^{(N)} = a$
- For our purposes, we use a fine grid (small fixed step size ϵ)
- \implies the algorithm doubles as an equilibrium selection criteria: we trace out the solution which uniquely converges to the risk-neutral benchmark when $a \rightarrow 0$

Numerical Solution: Laplace Transformations

- In order to solve the model numerically, we need to parameterize the habitat functions $\alpha_j(\tau), \theta_j(\tau)$. Our approach:

$$\alpha_j(\tau) = \alpha_{j0} e^{-\alpha_{j1}\tau}$$

$$\theta_j(\tau) = \theta_{j0} \tau e^{-\theta_{j1}\tau}$$

- Implies price elasticities are declining in τ , yield elasticities are single peaked
 - Demand functions are single-peaked
- If we take maximum maturity $T \rightarrow \infty$, then we can use properties of Laplace transforms to simplify the fixed point problem characterizing \mathbf{M}
- Implies $\mathcal{A}(s) \equiv \mathcal{L}\{\mathbf{A}(\tau)\}(s)$ given by:

$$s\mathcal{A}(s) + \mathbf{M}\mathcal{A}(s) - \frac{1}{s}\mathbf{e}_i = \mathbf{0} \implies \mathcal{A}(s) = [s\mathbf{I} + \mathbf{M}]^{-1} \left[\frac{1}{s}\mathbf{e}_i \right]$$