Monetary Policy and the Limits to Arbitrage: Insights from a New Keynesian Preferred Habitat Model

Walker Ray*

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Abstract

With conventional monetary policy unable to stabilize the economy in the wake of the global financial crisis, central banks turned to unconventional tools. This paper embeds a model of the term structure of interest rates featuring market segmentation and limits to arbitrage within a New Keynesian model to study these policies. Because the transmission of monetary policy depends on private agents with limited risk-bearing capacity, financial market disruptions reduce the efficacy of both conventional policy as well as forward guidance. Conversely, financial crises are precisely when large scale asset purchases are most effective. Policymakers can take advantage of the inability of financial markets to fully absorb these purchases, which can push down long-term interest rates and help stabilize output and inflation.

Keywords: unconventional monetary policy, large scale asset purchases, forward guidance, market segmentation

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*Department of Economics, UC Berkeley. Email: walkerdray@berkeley.edu. I am grateful to Yuriy Gorodnichenko as well as Pierre-Olivier Gourinchas and David Romer for invaluable guidance and support on this project. I thank Martin Beraja, Byoungchan Lee, Rupal Kamdar, Nick Sander, Dmitriy Sergeyev, Mauricio Ulate, and seminar participants at Berkeley for helpful comments on an earlier version of the paper.

1 Introduction

Central banks responded aggressively to worsening financial conditions and growing recessionary pressure during the global financial crisis of 2007-8. After steep cuts in policy rates, central banks found themselves constrained by the zero lower bound, and the crisis was followed by a deep recession. Not content to sit on their hands, policymakers undertook various unconventional policy actions such as forward guidance and large scale asset purchases, the most salient of which was the quantitative easing programs carried out by the Federal Reserve.

What was the purpose of these unconventional policies? With policy rates constrained, the immediate goal was to push down long-term interest rates. But more fundamentally, policymakers believed these actions would stimulate the economy by boosting output and stabilizing inflation. As economic conditions have returned to normal, pivotal questions for macroeconomics remain. The emerging view (though not quite a consensus) in the empirical literature surrounding unconventional monetary policies is that large scale asset purchases (LSAPs) were effective at reducing long-term rates. On the other hand, the economy was not as sensitive to forward guidance as implied by some workhorse models. Why was this? And what were the feedback mechanisms of unconventional policy actions to the broader economy?

The purpose of this paper is to study these monetary tools within a tractable, unified framework. To this end, this paper embeds a model of the term structure featuring market segmentation and limits to arbitrage within a New Keynesian framework. There are two key departures from a benchmark model. First, borrowing depends not only on the policy rate but also on the entire term structure of interest rates. Second, the term structure is determined in financial markets whose participants face limited risk-bearing capacity and are susceptible to demand shocks, as in Vayanos and Vila (2009). I use this model as a laboratory to study conventional and unconventional monetary policy. Crucially, the analysis considers policy both during normal times and over increasing degrees of financial crisis, and studies how policy actions interact with disruptions in financial markets.

The empirical literature has highlighted the importance of financial frictions, and in particular market segmentation, for understanding unconventional policy (for example, D'Amico and King (2013), Hamilton and Wu (2012), Gorodnichenko and Ray (2017)). Assuming that all borrowing takes place frictionlessly at the short (policy) rate is a useful simplification in many settings, but is too strong of an assumption for the purposes of this paper. Adding segmented bond markets to a macroeconomic model allows for more realistic and complicated dynamics in the determination of the term structure of interest rates. This enables the model to accomplish two goals: first, to match the relevant empirical findings regarding the term structure's response to demand shifts; and second, to study how these term structure changes interact with aggregate outcomes in general equilibrium.

The implications of the model are important for understanding the efficacy of monetary policy. When financial markets are healthy, so that marginal investors in financial markets have high risk-bearing capacity, the "expectation hypothesis" holds. That is, long-term rates are (roughly) the average of expected short rates. As a result, both conventional monetary policy and forward guidance are effective at stabilizing the economy. In this situation, household borrowing responds strongly to shifts in the path of the policy rate, leading to movements in output and inflation.

However, the link between expected short rates and the term structure is weakened when financial distress is high. As a result, long-term rates under-react to changes in the policy rate. Therefore, the model predicts that during a financial crisis output and inflation are less responsive to monetary shocks than usual.

Similar logic applies to LSAPs, but the implications are precisely the opposite. Purchases of long-term bonds, as in the various rounds of quantitative easing (QE), will have little to no effect on long-term rates when financial markets are healthy. When the central bank purchases a large amount of debt securities on the secondary market, the purchases change the portfolio allocation of the marginal investors in the debt market. Effectively, QE purchases allow financial investors to offload a source of risk from their portfolios. If financial markets are healthy, these investors are not very concerned with this source of risk to begin with, and so they do not require much excess returns to hold these securities. In this case, policies like QE will have little effect. But as financial markets become unable to bear risk, these purchases may matter a great deal. The shifts in the bond holdings alter the overall riskiness of these portfolios. By changing the riskiness of marginal investors' portfolio allocations, QE leads to changes in equilibrium prices of bonds, which in turn feed back into the household borrowing decision. This general equilibrium channel is akin to the familiar household Euler equation; under the right conditions, QE can boost output and stabilize the economy.

Benchmark models are not amenable to studying unconventional policies. Because of the extreme forward-looking behavior of agents and the lack of any financial frictions in such models, forward guidance policies are highly effective to the point of implausibility. Conversely, LSAP policies are completely ineffective in conventional models. The presence of limited arbitrage in my framework breaks this tight link. But it will still be the case that the term structure is rendered arbitrage-free, so that there are no riskless trades left on the table. Any deviations from the expectations hypothesis are due to the risky portfolio allocations chosen by financial market arbitrageurs (the marginal investors in the model). Hence, the channel through which unconventional policies like QE can have aggregate effects is by changing the market prices of risk.

As always, one fundamental source of risk is the movement in the short (policy) rate that is set by the central bank. All bonds are exposed to this risk, so as long as arbitrageurs are not perfectly risk-neutral there will be deviations from the expectations hypothesis. All else equal, arbitrageurs will require excess expected returns in order to take non-zero positions in long-term debt. This effect weakens the strength of forward guidance, but opens the door for LSAPs. The central bank is able to change the portfolio allocations of arbitrageurs, which through changes in the price of risk lead to changes in interest rates.

The exact impact of LSAP programs depends on how the purchases are structured. The amounts to be purchased, which maturities are targeted, and the duration of the program all affect the interaction with the sources of risk in the economy and the broader feedback mechanisms in the macroeconomy. The model delivers interesting and important interactions in general equilibrium. Because the policy rate responds to shifts in output and inflation, the expected path of short rates is a function of future expected output and inflation. When financial markets exhibit imperfect risk-bearing capacity, there is not a perfect link between longer-term rates and the expected path of short rates. Since these long-term rates affect household borrowing and hence influence output and inflation, the model exhibits rich feedback mechanisms; moreover, the dynamics of the model depend crucially on the health of financial markets. On the other hand, conditional on the term structure dynamics, the aggregate dynamics of the model stay close to benchmark models. The model adds only a handful of additional endogenous parameters which differentiate it from more familiar "three-equation" New Keynesian models. Therefore, the model is amenable to closed-form analysis.

I lay out the main building blocks of my "New Keynesian preferred habitat" framework in Section 2. Section 3 considers the case where prices are fully rigid. This is of course extreme and rules out important dynamics. However, many of the results can still be obtained and this simplifying assumption allows for a clearer focus on the intuition for the results. The main benefit is that this simplification rules out interesting but tricky determinacy issues (to which I return to later). In the most basic setup, the central bank sets the policy rate according to a Taylor-type rule subject to shocks. This is the only source of uncertainty. I then consider two extensions: first, I study forward guidance by assuming the central bank announces a path of policy rates; second, I study QE by allowing the central bank to directly purchase long-term bonds in the secondary market. The analysis confirms the intuition described above: conventional monetary policy and forward guidance become less effective as financial markets become disrupted (in the sense of both moving long-term rates and of impacting output); while at the same time, LSAP policies become more effective.

Next, Section 4 allows for prices to be sticky but not fully fixed. The main results go through here, but only if a determinacy condition is met. This condition is similar to the standard Taylor principle in textbook models, but with a key difference: the determinacy condition depends on the health of financial markets. A novel implication is that as financial markets become more disrupted, the model moves toward the region of indeterminacy. To the extent that model indeterminacy is either a proxy or a cause of excess volatility, this result shows how a purely financial crisis can lead to macroeconomy instability.

The focus of Sections 3 and 4 is delivering analytical results, but this comes at a cost of empirical realism. Section 5 extends the model to allow for many sources of aggregate and financial shocks in order to better match the data. In this section I develop the tools to solve the model numerically and estimate the model using U.S. data from before and during the recent financial crisis. The results confirm the qualitative findings of the more parsimonious models: monetary policy becomes less effective during financial crises; QE becomes more effective. Quantitatively, the model predicts that the aggregate output effects of a policy like the first round of QE were roughly 40% larger than a 50 basis point expansionary monetary shock during a period of relative financial calm. Further, had the zero lower bound not been binding during the financial crisis, additional rounds of rate cuts would have been 20% less effective than rate cuts during normal times.

Section 5 also studies more complicated LSAP programs like Operation Twist (where the Federal Reserve bought long-term debt and sold short-term debt). The model predicts these policies may be effective, depending on which maturities are targeted for purchase and the overall structure of risk in financial markets. In particular, when financial markets are relatively healthy, Operation Twist will have the net effect of pushing down interest rates across the entire term structure. However, the estimated model shows that when financial frictions are very high, Operation Twist-style policies will push down long-term rates but push up short-term rates. To the extent output is most sensitive to intermediate yields, this can have the net effect of raising effective borrowing rates of households, leading to contractionary outcomes.

I also show how determinacy can be restored if the central bank follows an endogenous rule for QE purchases. The main result is that, as financial frictions increase, the standard Taylor rule is less effective at stabilizing the economy. Formally, a Taylor rule eventually becomes unable to guarantee determinacy (for any parameterization). But the same forces that make standard policy ineffective also make a QE rule more effective, hence this carries some of the weight of the determinacy issues and restores stability. Finally, Section 5 also conducts optimal policy experiments and finds that the endogenous policy response to inflation should become more aggressive in financial crises. Section 6 discusses additional extensions of the model and concludes.

This paper makes a number of contributions to the literature. One theoretical contribution is to extend the logic of Vayanos and Vila (2009) to a general equilibrium macroeconomic setting. This setup is amenable to analyzing state-dependent responses to policy changes while maintaining a relatively tractable framework. The paper adds to the large literature exploring the importance of macroeconomic factors in explaining the term structure, such as Ang and Piazzesi (2003). Other papers that tie reduced-form term structure modeling to New Keynesian macroeconomic dynamics include Hördahl et al. (2006) and Rudebusch and Wu (2008). My model contributes to this literature and can be viewed as a microfoundation for an affine term structure in macroeconomic factors.

From a partial equilibrium perspective, preferred habitat models can rationalize the interest rate response of QE but are silent on the aggregate effects of QE on inflation and output. Moreover and crucially, even the term structure response is conditional on the path of the short rate and independent of all other possible macroeconomic determinants of the term structure. My model is able to study both the direct and indirect effects of QE on the term structure and the transmission channels to the aggregate economy. Beyond that, the model makes additional important predictions: the frictions that imply QE is effective also imply that monetary policy conducted through changes in the short rate is less effective, and leads to increased aggregate instability. The model also highlights the centrality of these financial frictions as opposed to the zero lower bound constraint on conventional policy. The zero lower bound is neither necessary nor sufficient: QE is effective even away from the ZLB, but only when financial markets are imperfect; conversely, even at the ZLB, if financial markets are healthy then QE will have no effect.

From a theoretical perspective, this paper also introduces a relatively tractable model in which aggregate demand explicitly depends on long-term rates, and demonstrates how to solve the model with or without the expectations hypothesis. The model is also one in which aggregate dynamics are approximated linearly while still demonstrating sensitivity to risk.

I focus on limits to arbitrage and preferred habitat as important mechanisms for understanding conventional and unconventional monetary policy, based not only on empirical work looking at QE, but also studies of the determinants of the yield curve more generally (e.g. D'Amico and King (2013), Hamilton and Wu (2012), Gorodnichenko and Ray (2017), Greenwood and Vayanos (2014), Beraja et al. (2015)). In this way, my paper adds to the literature studying how market segmentation interacts with unconventional monetary policy (e.g. Alvarez et al. (2002), Gertler and Karadi (2013), Chen et al. (2012), Carlstrom et al. (2017)). This overcomes the irrelevance results derived in Wallace (1981) and contrasts with an alternative view that treats QE as a signalling tool of the central bank (e.g. Bauer and Rudebusch (2014) or Bhattarai et al. (2015)).

In addition, I move beyond studying QE and explore the implications of bond market frictions for monetary policy more broadly defined. Hence, the paper also falls into a broad class of macroeconomic models focusing on financial frictions (for a recent paper see Adrian and Duarte (2018); see Brunnermeier et al. (2012) for a survey). The approach in this paper differs from recent work focusing on borrowing constraints on the part of households (e.g. Kaplan et al. (2018)). The model also has a similar flavor as recent work which focuses on breaking the tight link between the path of future expected policy and current economic responses (e.g. McKay et al. (2016), Farhi and Werning (2017), Gabaix (2016), Angeletos and Lian (2018)). These papers make agents less responsive to expected future shocks. My approach has similar implications, but the degree of under-reaction is governed by the risk-bearing capacity of financial markets. In other words, the key frictions I focus on are those which mitigate the transmission of monetary policy through financial markets to the broader macroeconomy.

2 A New Keynesian Preferred Habitat Framework

I work with continuous time New Keynesian models that are largely characterized by a Phillips curve (relating current inflation π_t to current and future output gaps x_t), an IS curve (relating output growth to the real borrowing rate), and a monetary policy rule that governs how the nominal policy rate r_t reacts to macroeconomic variables. The setup is similar to Werning (2011).

The key difference between my model and a textbook New Keynesian model is that output depends on some "effective" nominal borrowing rate that depends not only on short rates, but also on longer rates. Assume that there is a continuum of zero-coupon nominal bonds with maturities $\tau \in (0, T]$, with time t price $P_{t,\tau}$ and yield given by

$$R_{t,\tau} = -\frac{\log P_{t,\tau}}{\tau}.$$

I assume that the effective nominal rate is

$$\tilde{r}_t \equiv \int_0^T \eta(\tau) R_{t,\tau}\, \mathrm{d}\tau \tag{1}$$

where $\eta(\tau)$ is a positive but otherwise arbitrary weighting function. This is a flexible way to allow for borrowing to depend explicitly on long-term interest rates. Assuming that all borrowing takes place at the short rate is a useful simplification but is too strong an assumption for the purposes of studying unconventional policies like forward guidance and QE. My specification aims to capture aspects of investment and savings decisions that are typically abstracted away in simple models. Although I do not explicitly model durable consumption or housing, in reality these are drivers of household borrowing that depend a great deal on long-term interest rates. Capital investment is also outside the model, but firm investment is similarly sensitive to long-term interest rates.[1]

Formally, these weights may arise for lifecycle borrowing reasons, or due to household's limited access to debt markets. Appendix C presents microfoundations based on the latter setup. I relegate the derivations to the appendix due to the similarity with the derivation of a benchmark New Keynesian model and start the exposition with the familiar linearized aggregate equations governing the dynamics of the macroeconomy. The IS curve is modified such that the output gap evolves according to

$$\mathrm{d}x_t = \varsigma^{-1} \left(\tilde{r}_t - \pi_t - \bar{r}\right) \mathrm{d}t \tag{2}$$

where ς^{-1} is the intertemporal elasticity of substitution and $\bar{r}$ is the "natural" real borrowing rate (assumed constant in this model). As in a benchmark model, changes in the output gap

[1] See Kaplan and Violante (2014) for a discussion of household portfolio allocations across short- and long-term securities in the United States. Note that I focus purely on maturity, rather than differences between liquid and illiquid savings vehicles.

are increasing in the nominal borrowing rate and decreasing in inflation. The only difference is that now the growth rate of output depends explicitly on the entire term structure of interest rates.

The Phillips curve, relating current inflation to current and future output, is unchanged relative to a standard New Keynesian model. The dynamics of inflation are governed by

$$\mathrm{d}\pi_t = (\rho\pi_t - \delta x_t)\,\mathrm{d}t \tag{3}$$

where ρ is the discount rate and δ governs the degree of price stickiness. $\delta \to \infty$ implies fully flexible prices, while $\delta \to 0$ implies fully rigid prices.

Finally, the central bank controls the instantaneous short rate r_t (the policy rate). At first, I will assume that this takes the form of a Taylor-type rule where the policy rate reacts to current levels of output and inflation subject to shocks. When studying forward guidance, I will also consider models in which the central bank announces a path of the policy rate.

For modeling the macroeconomic data generating process, I attempt to stay as close as possible to a benchmark "three equation" New Keynesian model for two reasons. First, even in partial equilibrium solving the term structure model becomes quite complicated. Embedding this setup in a dynamic general equilibrium model adds to the complexity. Keeping the model tractable and deriving analytical results is only possible if the underlying dynamics of the macroeconomy are kept simple. Second, because the mechanisms of simple New Keynesian models are still present in more sophisticated models, it's reasonable to expect that the findings of my model will hold in more detailed extensions.

Before diving into the details, some intuitive results are immediate. The central bank sets only the short rate, while the rest of the term structure of interest rates is an equilibrium object. Since the effective borrowing rate depends on the entire yield curve, specifying only a rule for the policy rate does not necessarily close the model. However, if the expectations hypothesis holds, then long-term yields are fully determined by the expected path of short rates. In this case, without fully solving the model it's possible to see how unconventional policies will work. Forward guidance is powerful: by announcing a path of short rates, long-term rates will react immediately, and therefore the effective borrowing rate will also move sharply. This implies that consumption (and output) will also respond sharply. On the other hand, QE is ineffective: purchases of long-term bonds have no effect on the path of short rates, and hence do not change long-term yields.

But if the expectations hypothesis does not hold, a monetary policy rule no longer closes the model and it becomes necessary to specify how the entire term structure of interest rates are determined. For this purpose, I embed a "preferred habitat" model of the term structure, as in Vayanos and Vila (2009). Interest rates are determined by the interaction of two types of investors: clientele investors, who have idiosyncratic demand for bonds of specific maturities, and arbitrageurs with limited risk-bearing capacity, who integrate bond markets. The preferred habitat view of the term structure has long been of relevance to practitioners, but less so in academic models. Partly this is due to the fact that naïve forms of preferred habitat models conflict with no-arbitrage conditions: if the term structure were determined only by clientele investors with extreme preferences for bonds of specific maturities, bonds that are close to one another in maturity space could have large price differences. By allowing for arbitrageurs to integrate bonds of different maturities, the model avoids the unrealistic outcome of extreme segmentation and ensures that no-arbitrage conditions are satisfied. However, when arbitrageurs do not have perfect risk-bearing capacity, deviations from the expectations hypothesis arise.

I follow Vayanos and Vila (2009) in setting up the preferences for arbitrageurs and idiosyncratic preferred habitat investors. Arbitrageurs choose how much of each bond to hold (denoted by $b_{t,\tau}$) in order to maximize an instantaneous mean-variance trade-off of the change in wealth, subject to their budget constraint:

$$\max_{b_{t,\tau}} \ \mathrm{E}_t \, \mathrm{d}W_t - \frac{a}{2} \mathrm{Var}_t \, \mathrm{d}W_t$$
$$\text{s.t.} \ \ \mathrm{d}W_t = \left(W_t - \int_0^T b_{t,\tau} \, \mathrm{d}\tau \right) r_t \, \mathrm{d}t + \int_0^T b_{t,\tau} \frac{\mathrm{d}P_{t,\tau}}{P_{t,\tau}} \, \mathrm{d}\tau \, .$$

By holding $b_{t,\tau}$ of a τ bond, arbitrageurs receives the instantaneous return $\frac{\mathrm{d}P_{t,\tau}}{P_{t,\tau}}$. The remainder of their wealth not invested in long-term bonds is held at the risk-free rate r_t (the short rate). The risk-aversion parameter $a \geq 0$ is fixed, but should be thought of as a proxy for limited risk-bearing capacity of financial markets.

The other side of the bond market is the demand from idiosyncratic clientele preferred habitat investors, which is given by

$$\tilde{b}_{t,\tau} = \alpha(\tau)\tau(R_{t,\tau} - \beta_{t,\tau}). \tag{4}$$

The function $\alpha(\tau) > 0$ is the semi-elasticity of preferred habitat demand (note $\tau R_{t,\tau} = -\log P_{t,\tau}$), and hence governs how sensitive these investors are to returns for τ bonds. This function is otherwise unrestricted; but the sign restriction implies that demand from preferred habitat investors is downward-sloping (increasing in yields). $\beta_{t,\tau}$ is a demand shifter, and can be thought of as a target yield. When rates are above the target, a τ investor increases their demand for τ bonds; and vice versa for when rates fall below the target. Note that, holding $R_{t,\tau}$ fixed, an increase in $\beta_{t,\tau}$ implies that τ investors *reduce* their holdings of τ bonds. I will consider different forms of this shifter (both deterministic and stochastic) in later sections. In equilibrium, bond prices must adjust so that arbitrageurs absorb the demand from preferred habitat investors ($b_{t,\tau} = -\tilde{b}_{t,\tau}$) while satisfying their mean-variance portfolio allocation problem.

This bond market setup is stylized: arbitrageurs are infinitesimally lived, and a τ-bond preferred habitat investor cares only about a specific slice of maturity space. Nevertheless, the model captures important facets of segmented markets, and how limited arbitrage smooths out idiosyncratic demand shocks. The preferred habitat setup is a natural way to study the affects of LSAP programs. Moreover, private investors such as pension funds and insurance companies often have demand for long-term bonds that arise from the need to match their long-term liabilities; these important sources of demand are not captured by intertemporal consumption substitution decisions that drive the term structure in more standard models. But these investors are not the only participants; otherwise debt markets would exhibit extreme segmentation. Arbitrageurs integrate debt markets and eliminate risk-free arbitrage opportunities, but are risk-averse and face limits to their trading activities. How conventional and unconventional policy affects the entire term structure will depend heavily on the limits to arbitrageurs' risk-bearing capacity.

In general, the term structure will be determined by complicated interactions between these two classes of investors and the general equilibrium dynamics of the macroeconomy. However, two limiting cases can be analyzed immediately. First, if arbitrageurs are risk-neutral ($a = 0$, so they only care about expected returns), then equilibrium can only be achieved if $\mathrm{E}_t\left[\frac{\mathrm{d}P_{t,\tau}}{P_{t,\tau}}\right] = r_t$. And if expected instantaneous returns of all bonds are equalized at the short rate, then risk-neutral arbitrageurs are indifferent between any bond allocation. In this case, they will absorb any demand shifts from preferred habitat investors without any equilibrium price changes. In other words, idiosyncratic demand shifts will not affect the term structure of interest rates.

In the other extreme, if arbitrageurs abandon the bond market (allocating the entirety of their wealth to the risk-free short rate), then equilibrium is only satisfied ($0 = b_{t,\tau} = -\tilde{b}_{t,\tau}$) if yields satisfy $R_{t,\tau} = \beta_{t,\tau}$. This would imply that bonds of very close maturity could have very different yields (and would potentially evolve unrelated to the short rate). Again, note that in this extreme case an increase in the demand shifter $\beta_{t,\tau}$ would push up the τ-bond yield. But this extreme segmentation does not occur in equilibrium because arbitrageurs do take non-zero positions in long-term bonds. The impact of changes in preferred habitat demand (if any) will depend on how arbitrageurs adjust their portfolio allocations. In turn, this will depend on the equilibrium dynamics of the short rate and other macroeconomics variables.

Intuitively, what does general equilibrium look like in this model? From the perspective of households, the key factor is how sensitive their effective borrowing rate is to the short rate. The model reduces to a benchmark New Keynesian model when these rates move one-for-one, but in general $\tilde{r}_t \neq r_t$. Suppose that the effective rate is highly responsive to the policy rate. Then household borrowing is also highly sensitive to the policy rate, and therefore the growth rate of consumption will also react strongly to the policy rate. On the other hand, when the effective rate is insensitive to the policy rate, the pass-through of changes in the policy rate to households is weakened. Through the borrowing decisions of the household, the growth rate of consumption is less responsive to the policy rate. That is,

$$\frac{\partial \, \mathrm{d}x_t}{\partial r_t} \propto \frac{\partial \tilde{r}_t}{\partial r_t}$$

and moreover, the sensitivity of the change in the output gap to the policy rate will determine the equilibrium reversion rate of monetary shocks. The higher the sensitivity, the quicker output gaps revert to steady state. Inflation will follow a similar path since, due to standard Phillips curve dynamics, inflation is the present discounted value of future output gaps. Thus, through the central bank's endogenous reactions to either output or inflation, the policy rate also reverts back to steady state quickly.[2]

However, the sensitivity of the effective borrowing rate to the policy rate is an *equilibrium* object, which also depends on financial markets. Bond prices will adjust in order to achieve equilibrium in bond markets, such that arbitrageurs' portfolio allocation satisfies their mean-variance tradeoff while also clearing the market given the demand from preferred habitat investors. In this model, arbitrage is imperfect and the term structure will not be characterized by the expectations hypothesis except under special circumstances. Therefore, it is the risk-adjusted dynamics of the macroeconomy which determine bond prices in financial markets, rather than the actual dynamics of the short rate only. For example, in response to a monetary shock, the response of the yield of a τ bond will be roughly equal to the risk-adjusted average of the short rate over the life of the bond. Thus, if the short rate has a very high risk-adjusted mean reversion rate, long-term bond yields will not respond much to shocks to the short rate. This force implies that increases in the risk-adjusted reversion rate of monetary shocks lead to decreases in the sensitivity of all bond yields to the policy rate. In particular, the effective borrowing rate also becomes less responsive.

General equilibrium is obtained when these two forces balance. Thus, characterizing equilibrium involves two key steps: first, understanding the differences between the actual and risk-adjusted dynamics of the economy; and second, linking household savings and consumption choices with the bond prices determined in imperfect financial markets.

3 A Rigid Price Model

This section simplifies the model by assuming prices are fully fixed. This is an obviously extreme assumption, but many of the key results can still be obtained in this case. The upside is that the solution is considerably simpler, and I can sidestep determinacy issues that arise when prices can adjust. I return to these questions in Section 4.

As discussed in Section 2, the difficulty in solving the model relative to standard New

[2] The sensitivity of the change in the output gap to the policy rate plays a similar role as the intertemporal elasticity of substitution (ς^{-1}). In this case, the responsiveness of household borrowing to the policy rate is governed by preferences, namely the willingness to tolerate large changes in consumption across short periods of time. But the outcome is the same: when changes in the output gap are sensitive to the policy rate, the equilibrium rate at which shocks dissipate is high.

Keynesian models is the mismatch between the effective borrowing rate and the policy rate. The approach I take to solving the model is as follows. First, start with the conjecture that bond prices are affine functions of the macroeconomic state variables. This allows the macroeconomic dynamics to be transformed into a system of linear differential equations. Next, conditional on the affine coefficients, solving for the rational expectations equilibrium is straightforward. Then, I turn to solving for these affine coefficients by solving the arbitrageur's portfolio problem. Finally, putting both sides of the model together, I characterize the unique general equilibrium solution.

After solving the model, I explore the implications for monetary policy. First I focus on conventional policy, where the central bank sets the policy rate through a Taylor rule. Then I study forward guidance by assuming the central bank instead announces a long-lived interest rate peg. Finally, I study LSAPs by allowing the central bank to purchase longer-term bonds on the secondary market. Throughout, I focus on how these policies affect the macroeconomic dynamics of the model, and in particular how these effects depend on the risk-bearing capacity of investors in financial markets.

3.1 Macroeconomic Dynamics

Prices are fully fixed when the parameter $\delta \to 0$. In this case, eq. (2) is simply

$$\mathrm{d}x_t = \varsigma^{-1} \left(\tilde{r}_t - \bar{r}\right) \mathrm{d}t \,. \tag{5}$$

Again, consumption (and output) growth is increasing in the borrowing rate, but now borrowing depends on some weighted average of the term structure of interest rates, given by eq. (1).

I assume the central bank follows a Taylor rule with persistence:

$$\mathrm{d}r_t = -\kappa_r (r_t - \phi_x x_t - r^*) \, \mathrm{d}t + \sigma_r \, \mathrm{d}B_{r,t} \,, \tag{6}$$

where $B_{r,t}$ is a standard Brownian motion and σ_r governs the size of the shocks. For now, I assume that changes in the short rate are the only source of uncertainty. ϕ_x govern the feedback rule for changes in output to changes in the policy rate. κ_r is a mean-reversion parameter; if $\kappa_r \to \infty$, eq. (6) simplifies to a (non-stochastic) Taylor rule with no gradual adjustments in the policy rate.[3] r^* is the central bank's target policy rate, which it sets in order to deliver a steady state with zero output gap. In a benchmark model where the borrowing rate is the same as the policy rate, this is accomplished by setting $r^* = \bar{r}$, but in this setup the optimal target is more complicated. I return to this in later sections.

Unlike a standard New Keynesian model, the interest rate rule does not close the model;

[3]The inertia term is present because empirically, central banks rarely change the policy rate by large jumps. Rather than assume that the policy rule is subject to long-lasting shocks, I instead assume that the policymaker prefers to smooth out changes over time (see Coibion and Gorodnichenko (2012) for an overview).

it is necessary to specify how the entire term structure of interest rates is determined in equilibrium. I conjecture that the model features an affine term structure (which I will confirm in the next section):

$$-\log P_{t,\tau} = A_r(\tau) r_t + C(\tau).$$

Bond prices are sensitive to changes in the short rate; the sensitivity of a τ-maturity bond is governed by the coefficient function $A_r(\tau)$. Note this also implies the effective borrowing rate can be written as

$$\begin{aligned}\tilde{r}_t &= \left[\int_0^T \frac{\eta(\tau)}{\tau} A_r(\tau)\,\mathrm{d}\tau\right] r_t + \left[\int_0^T \frac{\eta(\tau)}{\tau} C(\tau)\,\mathrm{d}\tau\right] \\ &\equiv \hat{A}_r r_t + \hat{C}\end{aligned}$$

and hence the IS curve given by eq. (5) becomes

$$\mathrm{d}x_t = \varsigma^{-1}\left(\hat{A}_r r_t + \hat{C} - \bar{r}\right)\mathrm{d}t\,. \tag{7}$$

In terms of the macroeconomic dynamics, the coefficients $\hat{A}_r$ and $\hat{C}$ are the only difference between my model and a standard New Keynesian model; setting $\hat{A}_r = 1$ and $\hat{C} = 0$ recovers the standard dynamics. From the perspective of understanding aggregate dynamics, it is seen from eq. (7) that $\hat{A}_r$ is the key determinant of the responsiveness of consumption growth to the policy rate. Additionally, the coefficient functions $A_r(\tau)$ and $C(\tau)$ are equilibrium objects which will depend on the interplay of arbitrageurs and preferred habitat investors in imperfect financial markets. In particular, $A_r(\tau)$ governs the sensitivity of the price of a τ-bond to short-rate. Since the short rate is the only risk factor in the model, $\hat{A}_r$ is the weighted average of risk sensitivity of the entire term structure. In general equilibrium, $\hat{A}_r$ will have to satisfy both of these roles.

The affine functional form implies that the macroeconomic dynamics are governed by a linear stochastic differential equation. I solve for the rational expectations equilibrium following Buiter (1984), the continuous time analogue of Blanchard and Kahn (1980). In general, let $\mathbf{Y}_t = [\mathbf{y}_t \;\; \mathbf{x}_t]^T$ where $\mathbf{x}_t$ are the "jump" variables and $\mathbf{y}_t$ are the state variables. Writing the model in general matrix form (in terms of deviations from steady state) gives

$$\mathrm{d}\mathbf{Y}_t = -\Upsilon\left(\mathbf{Y}_t - \mathbf{Y}^{SS}\right)\mathrm{d}t + \mathbf{S}\,\mathrm{d}\mathbf{B}_t\,. \tag{8}$$

Under certain determinacy conditions, the rational expectations dynamics are given by

$$\mathrm{d}\mathbf{y}_t = -\Gamma\left(\mathbf{y}_t - \mathbf{y}^{SS}\right)\mathrm{d}t + \mathbf{S}\,\mathrm{d}\mathbf{B}_t \tag{9}$$

$$\mathbf{x}_t - \mathbf{x}^{SS} = \Omega\left(\mathbf{y}_t - \mathbf{y}^{SS}\right) \tag{10}$$

where Γ and Ω are given by eqs. (C10) and (C11).[4]

In the current rigid price model, the output gap x_t is the only jump variable, while the interest rate rule implies that r_t is the only state variable. The dynamics matrix Υ is given by

$$\Upsilon = \begin{bmatrix} \kappa_r & -\kappa_r \phi_x \\ -\varsigma^{-1} \hat{A}_r & 0 \end{bmatrix}.$$

Then the rational expectations equilibrium is determinate if and only if Υ has one stable eigenvalue λ_1; the general equilibrium dynamics simplify to

$$\begin{aligned} \mathrm{d}r_t &= -\lambda_1 (r_t - r^{SS})\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} \\ x_t - x^{SS} &= \omega_x (r_t - r^{SS}). \end{aligned}$$

The equilibrium mean-reversion properties of the short rate are governed by λ_1, while the equilibrium response of the output gap to changes in the short rate are characterized by ω_x. The following Lemma characterizes the conditions under which the model is determinate under rational expectations. Additionally, it characterizes the relationship between $\hat{A}_r$, λ_1, and ω_x. All proofs are in Appendix A.

Lemma 1 (Characterizing $\hat{A}_r$, rigid prices)**.** *Consider the rigid price model.*

1. *Υ has exactly one eigenvalue with positive real part if and only if $\hat{A}_r > 0$. Further, this stable root is real: $\lambda_1 > 0$.*
2. *$\hat{A}_r = h(\lambda_1)$ where $h : \mathbb{R}_+ \to \mathbb{R}$:*

$$h(\lambda) \equiv \frac{\lambda(\lambda - \kappa_r)}{\varsigma^{-1} \kappa_r \phi_x}. \tag{11}$$

3. *The output gap dynamics are given by*

$$\omega_x = -\frac{\varsigma^{-1} \hat{A}_r}{\lambda_1} = \frac{\kappa_r - \lambda_1}{\kappa_r \phi_x}.$$

The first result in Lemma 1 says that the model is determinate when consumption growth moves in the same direction as changes in the policy rate. This is a natural conjecture, which I will confirm holds in general equilibrium. Since prices are fully fixed, changes in the policy rate coincide with changes in the real short rate. Therefore, through standard intertemporal decisions of the household, one would expect that a higher real borrowing rate would lead to higher savings and thus increasing consumption into the future. However, this result is not immediate in my model because of the disconnect between the policy

[4] With an abuse of notation, I use the same symbols for the shocks $\mathbf{S}$ and $\mathbf{B}_t$ in the general dynamics eq. (8) and the state space representation eq. (9).

rate and the effective borrowing rate (even when nominal and real rates coincide). To show that this result holds requires explicitly determining the term structure of interest rates.

The second and third results characterize $\hat{A}_r$ and ω_x in terms of the macroeconomic parameters and the equilibrium eigenvalues of the model. Note that when the model is determinate, $\omega_x < 0$; this says that a positive (contractionary) shock to the policy rate leads to an immediate decline in the output gap.

As discussed previously, $\hat{A}_r$ is pulling double duty: it governs both the sensitivity of the term structure to risk, as well as the sensitivity of changes in the output gap to the policy rate. Lemma 1 characterizes $\hat{A}_r$ in terms of the latter interpretation. Thus, the function $h(\cdot)$ should be thought of as a mapping between $\frac{\partial \mathrm{d}x_t}{\partial r_t}$ (how sensitive the output gap is to the policy rate) and λ_1 (the equilibrium mean reversion rate of monetary shocks). This result, together with the results in the next section, are used to solve for $\hat{A}_r$ in general equilibrium.

Focusing on the determinate case, a high degree of sensitivity of consumption growth to the policy rate is associated with high mean reversion of the policy rate. Intuitively, in this case borrowing decisions are highly responsive to the policy rate. An increase in the level of the policy rate implies that households increase savings significantly; this leads to an immediate drop in the level of consumption (hence output) and then a rapid increase in consumption growth leading back to steady state. Through the endogenous response of the monetary rule, this implies that the policy rate also quickly returns to steady state. That is, a high $\hat{A}_r$ is associated with high λ_1 and large (negative) ω_x.

Conversely, as the sensitivity of output growth falls ($\hat{A}_r$ decreases towards zero), so does the sensitivity of the level of the output gap (ω_x approaches zero). Additionally, the equilibrium reversion rate of the short rate λ_1 approaches κ_r, the inertial term of the Taylor rule. Note that $h(\lambda_1)$ is negative when $0 < \lambda_1 < \kappa_r$. That is, model indeterminacy implies that in equilibrium monetary shocks mean revert *slower* than the inertial term of the Taylor rule κ_r.

The general equilibrium value of the sensitivity of output growth and the mean reversion rate of monetary shocks are of course endogenous. However, studying how the dynamics of the model change as these parameters vary exogenously is illuminating. Because the model only features two macroeconomic variables, phase diagrams can be used to study the aggregate dynamics.

Figure 1 plots phase diagrams of the rigid price models for different values $\hat{A}_r$, the sensitivity of changes in the output gap to the policy rate. This leads to variation in the equilibrium mean reversion rate of monetary shocks λ_1. First, I consider a benchmark New Keynesian model where the Euler equation depends only on the short rate. Equivalently, in my model this corresponds to $\hat{A}_r = 1$; with the given parameterization I consider this implies $\lambda_1 \approx 0.9$. This model is shown in Panel A. Note that the stable arm (the solid blue line) slopes downwards; after a contractionary monetary shock (increase in the short rate), the output gap jumps down and then the economy moves along this arm back to steady

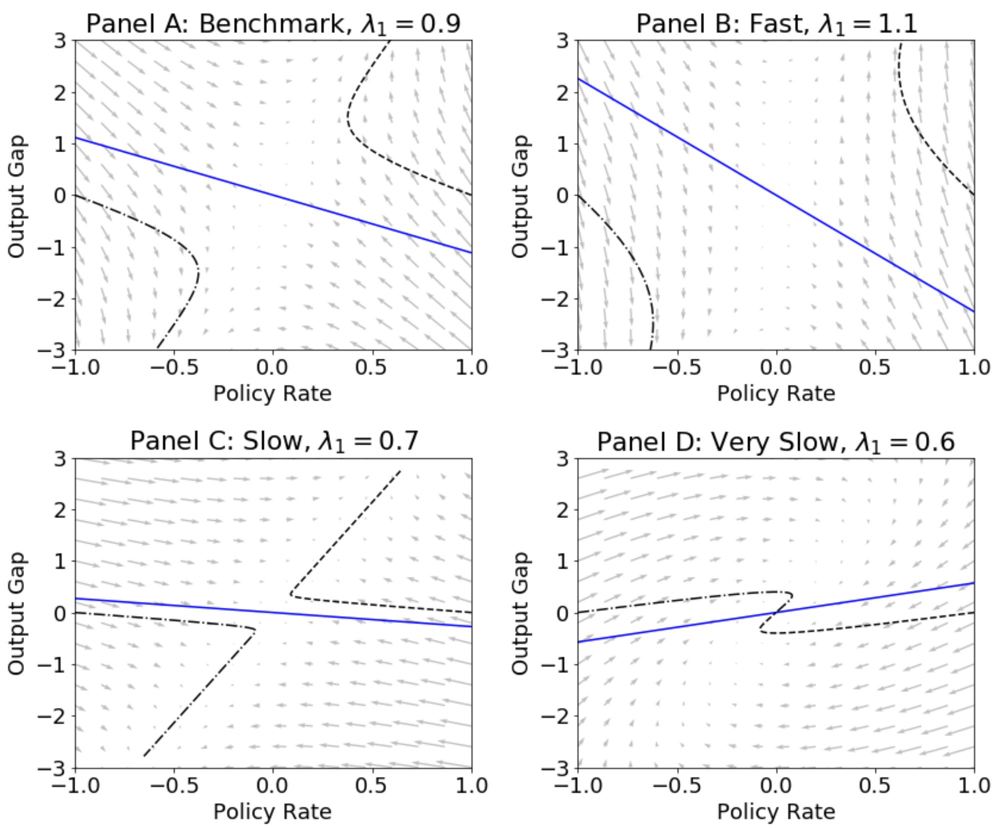

Figure 1: Phase Diagrams, Varying Output Sensitivity to Policy Rate

Notes: phase diagrams of the rigid price model (in terms of deviations from steady state). The solid blue line is the stable arm, while dashed black lines correspond to the trajectory of output and the policy rate given two different initial values. Each panel corresponds to different values of $\hat{A}_r$ and λ_1. Higher sensitivity of output to the policy rate ($\uparrow \hat{A}_r$) implies a faster mean equilibrium reversion of monetary shocks ($\uparrow \lambda_1$, the stable eigenvalue). The other parameters are set to $\kappa_r = 0.7$, $\phi_x = 0.25$, and $\varsigma^{-1} = 1$.

state. The dashed and dot-dashed lines show example (unstable) trajectories for the initial conditions $r = -1, x = 0$ and $r = 0, x = 1$.

What happens if $\hat{A}_r$ is higher than this benchmark case? This would mean that in equilibrium, monetary shocks revert faster towards steady state. In Panel B, I set $\hat{A}_r = 2.5$, which implies from Lemma 1 that $\lambda = 1.1$. Now the stable arm (the solid blue line) is much steeper (more negatively sloped) than the benchmark case, meaning that monetary shocks move output more than the benchmark prediction. The opposite is true when equilibrium monetary shocks mean revert slower: for $\hat{A}_r = 0.2$ so that $\lambda_1 = 0.8$ in Panel C, the stable arm is flattened.

Finally, at this point I cannot rule out the theoretical possibility that the model is in the region of indeterminacy. In Panel D I set $\hat{A}_r = -0.3$; that is, output growth is decreasing as the policy rate increases. This implies that monetary shocks revert towards steady state slowly ($\lambda_1 = 0.6$). Now the "stable" arm is upward-sloping (which would imply a contractionary monetary shock increases output), but note that this is no longer the unique stable path. Any initial level of the short rate and the output gap will return to steady state. The trajectories from initial conditions $r = -1, x = 0$ and $r = 0, x = 1$ shown by the dotted and dash-dotted lines are seen to return to steady state, whereas in the previous examples these were unstable paths. This is because, when $\hat{A}_r$ is negative, household borrowing moves in the opposite direction as the policy rate, leading to model indeterminacy.

Which of these cases will occur in general equilibrium? When I bring in the term structure side of the model, it will turn out that the relevant parameter space is one in which monetary shocks mean revert more slowly than a benchmark model (where the effective rate and policy rate coincide), but will still guarantee determinacy. In order to derive these results, I now turn to explicitly modeling the term structure.

3.2 Term Structure Determination

To study limited arbitrage, I embed a "preferred habitat" model of the term structure based on Vayanos and Vila (2009). Interest rates are determined by the interaction of two types of investors: clientele investors with idiosyncratic demand for bonds, and risk-averse arbitrageurs.

Given the affine functional form conjecture, I derive the optimality conditions of the arbitrageurs. Ito's lemma allows for the calculation of the instantaneous return of a τ bond, and hence the mean and variance of the change of arbitrageur's wealth. Arbitrageurs' expectations are rational and hence in equilibrium they expect the state variables to evolve according to eq. (9). For the case of a scalar state, the dynamics are given by

$$\mathrm{d}r_t = -\lambda(r_t - r^{SS})\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t}\,. \tag{12}$$

In equilibrium it will be the case that $\lambda = \lambda_1$, the (only) positive eigenvalue of the matrix Υ

described in the previous section. However, since arbitrageurs take as given the dynamics of the short rate, it is useful to study how the term structure is determined for arbitrary dynamics of the short rate, governed by any $\lambda > 0$.

Lemma 2 (Arbitrageur optimality conditions, scalar state)**.** *Suppose the short rate is characterized by eq. (12) for some $\lambda > 0$. Then arbitrageurs choose a portfolio allocation such that*

$$\mu_{t,\tau} - r_t = A_r(\tau)\zeta_t \tag{13}$$

$$\zeta_t \equiv a\sigma_r^2 \int_0^T b_{t,\tau} A_r(\tau)\,\mathrm{d}\tau \tag{14}$$

where $\mu_{t,\tau}$ is the expected instantaneous return of a τ-maturity bond, given by eq. (A1).

In this simple rigid price model, conditional on the dynamics of the policy rate the optimality conditions are the same as in Vayanos and Vila (2009). What this says is that a τ bond's expected excess return, $\mu_{t,\tau} - r_t$, is proportional to its sensitivity to the short rate, as measured by $A_r(\tau)$. This measure of proportionality is the same across all bonds, and follows solely from the absence of (risk-free) arbitrage.

No-arbitrage implies that the amount of excess return per unit of risk is the same for all bonds; this is ζ_t, the market price for risk. The expected excess return compensates arbitrageurs for taking on additional risk, which in this case comes from the short rate and so is fully characterized by the coefficient function $A_r(\tau)$. This compensation is higher when risk aversion is high (a), volatility is high (σ_r^2); or their portfolio is already sensitive to risk (the integral term in eq. (14)). The optimality conditions immediately imply that when arbitrageurs are risk neutral ($a = 0$), expected excess returns of all bonds are zero, and arbitrageurs are indifferent between holding any amount of bonds.

Arbitrageurs must hold the opposite of preferred habitat investors, whose demand is given by eq. (4). In this section, I assume that the demand shifter $\beta_{t,\tau}$ is independent of time and deterministic: $\beta_{t,\tau} \equiv \bar{\beta}(\tau)$. I will relax this assumption in later sections.

In equilibrium, prices must adjust such that arbitrageurs absorb the demand from preferred habitat investors and satisfy their mean-variance tradeoff. When arbitrageurs are risk-neutral and only care about expected returns, they will want to buy (sell) any bond with expected return greater than (less than) the risk-free rate. In this case, in equilibrium all bonds will have expected excess returns of zero and arbitrageurs will accomodate any shifts in demand from preferred habitat investors, recovering the results of a standard model of the term structure. But when arbitrageurs are risk averse, arbitrageurs will require non-zero excess expected returns to accomodate preferred habitat investors' demand. In this case, prices will depend on the arbitrageurs' portfolio allocations.

Thus, when $a > 0$, their portofolio is endogenous. The demand for bonds from preferred habitat investors along with the condition $b_{t,\tau} = -\tilde{b}_{t,\tau}$ leads to equilibrium conditions that characterize the coefficient function $A_r(\tau)$ and $C(\tau)$ in terms of the dynamics of the short rate.

Lemma 3 (Affine coefficients, scalar state term structure)**.** *Suppose the short rate is characterized by eq. (12) for some $\lambda > 0$. Then $A_r(\tau)$ is given by*

$$A_r(\tau) = \tau f\left(\nu(\lambda)\tau\right) \tag{15}$$

where $f(x) = \frac{1-e^{-x}}{x}$ and

$$\nu(\lambda) = \lambda + a\sigma_r^2 \int_0^T \alpha(\tau)\tau^2 f\left(\nu(\lambda)\tau\right)^2 \mathrm{d}\tau\,. \tag{16}$$

$C(\tau)$ is given by eq. (A2). Then $\hat{A}_r = g(\lambda)$ where $g : \mathbb{R}_+ \to \mathbb{R}$

$$g(\lambda) = \int_0^T \eta(\tau) f\left(\nu(\lambda)\tau\right) \mathrm{d}\tau\,. \tag{17}$$

The exponential function $f(x) = \frac{1-e^{-x}}{x}$ is a common functional form which occurs as solutions to differential equations such as these. The results from Lemma 3 help to characterize the responsiveness of the term structure to the policy rate. It follows that $A_r(\tau) > 0$, and moreover the affine functional form of bond prices implies the following:

$$\begin{aligned}
\frac{\partial \log P_{t,\tau}}{\partial r_t} &= -A_r(\tau) = \frac{e^{-\nu\tau} - 1}{\nu} \\
\frac{\partial R_{t,\tau}}{\partial r_t} &= \frac{1}{\tau} A_r(\tau) = \frac{1 - e^{-\nu\tau}}{\nu\tau} \\
\mu_{t,\tau} - r_t = A_r(\tau)\zeta_t &= \frac{1 - e^{-\nu\tau}}{\nu}\zeta_t.
\end{aligned}$$

Therefore, taking derivatives with respect to maturity τ gives:

$$\left|\frac{\partial^2 \log P_{t,\tau}}{\partial r_t \partial \tau}\right| > 0, \quad \frac{\partial^2 R_{t,\tau}}{\partial r_t \partial \tau} < 0, \quad \left|\frac{\partial \mu_{t,\tau} - r_t}{\partial \tau}\right| > 0.$$

That is, as maturity increases, bond (log) prices become more sensitive while bond yields become less sensitive to the short rate. And as maturity increases, excess expected returns grow in magnitude (the sign depends on ζ_t, the market price of short-rate risk).

Recall that from the perspective of aggregate dynamics, $\hat{A}_r$ is the key determinant of the sensitivity of output growth to the policy rate (determined by the function $h(\cdot)$ derived in Lemma 1). Additionally, from the perspective of term structure determination, $\hat{A}_r$ also governs the weighted average sensitivity of bond yields to short-rate risk. Lemma 3 shows that this is determined by the function $g(\cdot)$.

The key to characterizing the behavior of $g(\cdot)$ is the parameter ν. Intuitively, what is ν, and how does it compare to λ? λ governs the actual dynamics of the short rate, while ν

governs the dynamics of the short rate under the risk-neutral measure. When risk aversion is non-zero, these parameters do not coincide. Eq. (16), which determines the parameter ν, is a fixed point problem and so will not have a simple solution except in the case when $a = 0$. But the proof of Lemma 3 shows that $\nu \geq \lambda$ and is increasing in λ, with equality if and only if $a = 0$. Since ν is increasing in λ, it follows that $g'(\lambda) < 0$. That is, bond yields become less responsive to policy rate movements when monetary shocks revert faster. Thus, the effective borrowing rate also becomes less sensitive.

The fact that $\nu \geq \lambda$ says that the risk-adjusted average of the short rate over the life of a long-term bond is lower than the expected average short rate over the same period. Thus

$$\frac{1}{\tau} E_t \left[\int_0^\tau \frac{\partial r_{t+u}}{\partial r_t} \, \mathrm{d}u \right] = f(\lambda\tau) \geq f(\nu\tau) = \frac{\partial R_{t,\tau}}{\partial r_t}.$$

The expectations hypothesis implies that these two responses should be identical, which occurs only when arbitrageurs are perfectly risk-neutral. Thus, this result says that long-term yields under-react to changes in the short rate.

Putting everything together, Lemma 3 shows how the interaction of arbitrageurs and preferred habitat investors characterizes the term structure. The arbitrageur's portfolio problem leads to a disconnect between the actual and risk-adjusted dynamics of the model. This leads to the under-reaction of longer-term rates (and therefore the effective borrowing rate) to changes in the policy rate. Further, if monetary shocks mean revert more quickly, then long-term rates are less responsive to the policy rate. In turn, this implies that the response of the effective borrowing rate to the policy rate is muted.

However, these are partial equilibrium results, which characterizes the affine coefficients in terms of the parameters from the term structure side of the model, taking as given the dynamics of the policy rate. The previous section documented a general equilibrium effect: increases in the sensitivity of household borrowing to the policy rate leads to longer-lasting monetary shocks. Formally, the results in these sections imply that increases in λ_1 lead to an increase in $h(\lambda_1)$ but a decrease in $g(\lambda_1)$. In general equilibrium, it must be that $\hat{A}_r = h(\lambda_1) = g(\lambda_1)$, so these forces must balance. The next section characterizes this interaction and solves for the general equilibrium solution.

3.3 General Equilibrium Solution

The results of Lemma 1 and 3 lead to a solution for $\hat{A}_r$, which characterizes general equilibrium. In this model, household borrowing decisions depend on the weighted average of longer-term yields. Hence consumption growth also depends on the weighted average of longer-term yields. Therefore, the sensitivity of output growth to the policy rate must coincide with the sensitivity of the effective borrowing rate to the policy rate. When output growth reacts strongly to the policy rate, monetary shocks mean revert quickly. On the other hand, due to the interaction of arbitrageurs and preferred habitat investors in im-

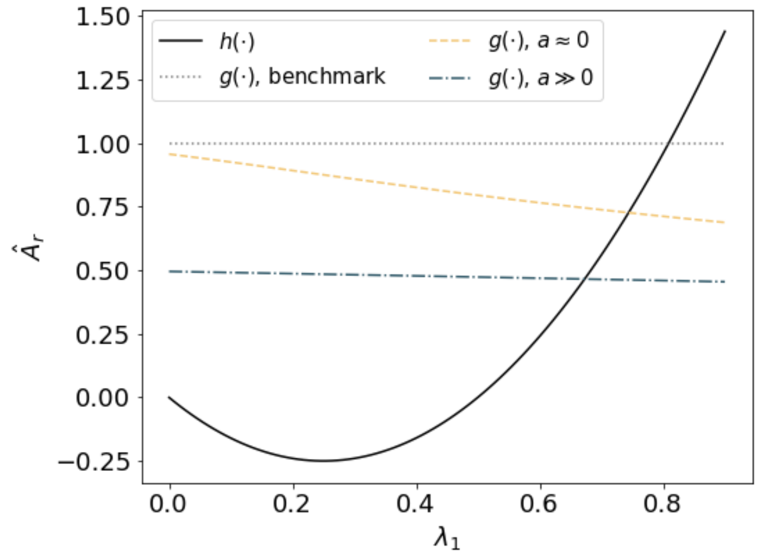

Figure 2: Intersection of $g(\lambda)$ and $h(\lambda)$, rigid prices

Notes: intersection of the functions $g(\lambda)$ and $h(\lambda)$, which determine λ_1 and $\hat{A}_r$ in equilibrium. The black line is $h(\cdot)$; the dashed light orange and dash-dotted dark teal lines are $g(\cdot)$ for low and high levels of risk aversion, respectively. The dotted grey line is the equivalent function $g(\cdot)$ for a benchmark New Keynesian model (fixed at unity). The parameters are set to $\kappa_r = 0.4$, $\phi_x = 0.25$, $\varsigma^{-1} = 1$, $T = 10$, $\alpha(\tau) = e^{-0.1\tau}$, $\sigma_r = 0.1$, $\eta(\tau)$ is the pdf of a (truncated) Gamma distribution, and $a \in \{0, 5\}$.

perfect financial markets, short-lived monetary shocks lead to a low degree of sensitivity of longer-term yields to the policy rate.

Recall $h(\cdot)$ determines the sensitivity of output growth to the policy rate, while $g(\cdot)$ determines the sensitivity of longer-term yields to the policy rate. Both are a function of the equilibrium rate of mean reversion of monetary shocks. In equilibrium, $\hat{A}_r$ will have to satisfy both of these jobs. Prop. 1 characterizes λ_1 and $\hat{A}_r$.

Proposition 1 (General equilibrium, rigid prices)**.** *Consider the rigid price model. There exists a unique positive eigenvalue of Υ $\lambda_1 > 0$ for which $g(\lambda_1) = h(\lambda_1)$, which fully characterizes the model equilibrium. Further, this implies $0 < \hat{A}_r < 1$.*

Figure 2 illustrates the equilibrium obtained in Prop. 1. In a benchmark New Keynesian model, implicitly it is always the case that $\hat{A}_r = 1$. This is because household borrowing takes place entirely at the short rate, so through the lens of my model the effective borrowing rate and the policy rate coincide. Once household borrowing depends on longer-term rates, this one-to-one correspondence breaks down. In particular, since long-term rates respond less than one-to-one with short rates, then $\hat{A}_r < 1$. Figure 2 also shows that in equilibrium, monetary shocks last longer than a benchmark model (lower λ_1). Since monetary shocks move long-term rates (and hence the effective borrowing rate) less than one-for-one, output also moves less. This means that the endogenous response of the central bank is muted

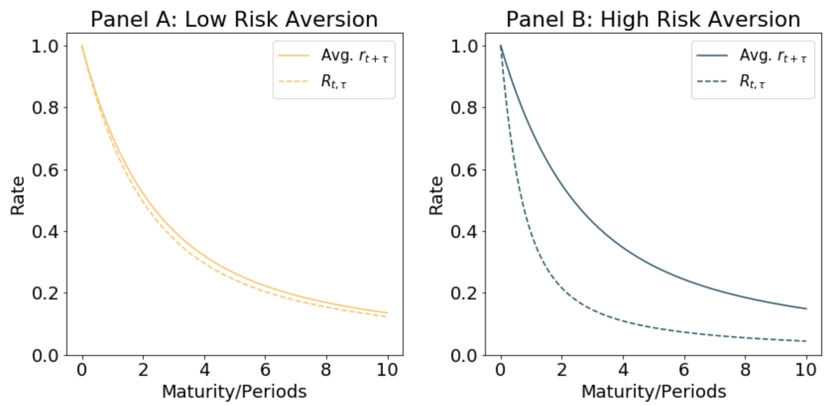

Figure 3: Rate Responses to Monetary Shock
Notes: responses of average short rates (solid lines) and spot rates (dotted lines) in response to a unit policy rate shock. The first panel shows the response when risk aversion is low ($a = 1$), while the second panel plots the responses when risk aversion is high ($a = 150$).

relative to the benchmark, hence the shock mean reverts slower in equilibrium.

How does the sensitivity of long-term rates depend on the health of financial markets? Figure 2 shows that $\hat{A}_r$ (the weighted sensitivity of long-term rates to changes in the short rate) declines as risk aversion increases. In fact, the entire curve $g(\lambda)$ shifts down. Recall from the discussion of Lemma 3, there is a partial equilibrium effect that leads to an under-reaction of long-term rates to changes in the policy rate. Due to financial market imperfections, long-term rates under-react to monetary shocks relative to the predictions of the expectations hypothesis. Moreover, this under-reaction becomes more severe as the risk-bearing capacity of arbitrageurs declines. What are the general equilibrium implications of this under-reaction?

Figure 3 plots of the responses the short rate and long-term rates to a monetary shock, and explores the effects of increasing risk aversion graphically. Panel A corresponds to a low level of risk aversion, while Panel B sets a higher value of risk aversion. In both panels, the solid line is the average change in the average short rate over the course of τ periods, while the dotted line is the immediate change in the yield of a τ bond. Under the expectations hypothesis, the two responses would be identical. Hence as expected, in Panel A when risk aversion is close to zero, the responses are very similar. But in Panel B where risk aversion is high, the responses differ by quite a bit. The immediate response of long-term rates lies well below the expected path of average short rates (which is the response of long-term rates that would occur under the expectations hypothesis).

Interestingly, the response of the term structure under the expectations hypothesis differs between the two experiments. This is due to the fact that the expectations hypothesis implies that the response of long-term rates is solely determined by λ_1, the mean reversion of the short rate. But this is an equilibrium object, which depends on the risk aversion of arbitrageurs. In the parameterizations considered in these two experiments, in equilibrium $\lambda_1 = 0.74$ when risk aversion is low vs. $\lambda_1 = 0.67$ when risk aversion is high. This can be seen comparing the solid lines in each panel: the monetary shock lasts longer when risk aversion is high in the second panel compared to the first.

3.4 Conventional Policy

What are the implications for monetary policy, and how do they differ from benchmark models? Note that (from Lemma 3), the parameters governing the macroeconomic dynamics (for example, κ_r and ϕ_x) only enter through the eigenvalue λ_1. Fixing λ, the term structure side of the model is independent of these parameters. Similarly, from Lemma 1, the term structure parameters (for example, a and σ_r^2) only enter through the coeffcient function $A_r(\tau)$. Fixing $A_r(\tau)$, the aggregate dynamics of the model are independent of these parameters. Hence, for comparative statics, it is possible to make substantial progress despite the complexity of the model.

Corollary 1.1 (Comparative statics, rigid prices)**.** *Consider the rigid price model. In general equilibrium:*

1. $\frac{\partial \lambda_1}{\partial a} < 0$, $\frac{\partial \hat{A}_r}{\partial a} < 0$, $\frac{\partial \omega_x}{\partial a} > 0$. *Moreover,* $\hat{A}_r \to 0$, $\omega_x \to 0$, *and* $\lambda_1 \to \kappa_r$ *as* $a \to \infty$. *Further, if* $a \neq 0$, *the same results hold for* σ_r.
2. $\frac{\partial \lambda_1}{\partial \kappa_r} > 0$, $\frac{\partial \hat{A}_r}{\partial \kappa_r} < 0$, $\frac{\partial \omega_x}{\partial \kappa_r} > 0$.
3. $\frac{\partial \lambda_1}{\partial \phi_x} > 0$, $\frac{\partial \hat{A}_r}{\partial \phi_x} < 0$, $\frac{\partial \omega_x}{\partial \phi_x} > 0$.
4. *Consider two different weighting functions* $\eta^s(\tau)$ *and* $\eta^\ell(\tau)$, *such that for some* T^*, $\eta^s(\tau) \geq \eta^\ell(\tau) \iff \tau \leq T^*$. *Then* $\lambda_1^s > \lambda_1^\ell$, $\hat{A}_r^s > \hat{A}_r^\ell$, $\omega_x^s < \omega_x^\ell$ *where superscripts denote the equilibrium outcomes under the corresponding weighting functions.*

Note that when household borrowing depends on long-term interest rates (so $\eta(\tau) > 0$ for some $\tau > 0$) it will always be the case that $\hat{A}_r < 1$. This is because the effective borrowing rate does not respond one-to-one with the policy rate. Partly this is somewhat mechanical, and will be true even when arbitrageurs are risk neutral. In this case, the response of long-term rates to a change in the policy rate will be equal to the average change of expected future policy rates, and these shocks to short rates are not permanent but instead mean-revert.

The first result in Cor. 1.1 says something more interesting: as the risk aversion of arbitrageurs increases, household borrowing becomes less responsive to changes in the policy rate. That is to say, $\hat{A}_r$ is decreasing in the risk aversion of arbitrageurs. This occurs because long-term rates under-react to shifts in the expected path of short rates. In other words, now the response of long-term rates to a change in the policy rate will be smaller than the average change of expected future policy rates. This implies that current borrowing (and hence output) responds less than it would when financial markets exhibit perfect risk-bearing capacity.

The fact that current macroeconomic outcomes are less responsive to future expected policy changes bears a similarity to recent work that derives a "discounted Euler equation," as in Gabaix (2016) or Farhi and Werning (2017). But the mechanism here is not that, for behavioral reasons, household forecasts deviate from the actual paths of future outcomes. Rather, expected changes in policy are transmitted through imperfect financial markets, which endogenously decreases the responsiveness of current macroeconomic variables to future expected policy changes.

The model makes clear that monetary policy is effective only to the extent that policy changes are transmitted through financial markets. In the model, the health of financial markets is proxied by the risk aversion of arbitrageurs, which is a fixed parameter. But more generally, risk aversion may increase during periods of financial panics (e.g. as in Kyle and Xiong (2001)). This implies that monetary policy becomes less effective during financial crises.

The next result relates to the persistence of the central bank's policy rule (governed by the mean reversion in the Taylor rule, κ_r). This governs the level of inertia in the central bank's policy rate (a higher value implies the rate returns to the target rate faster). Recall that λ_1 determines the equilibrium mean reversion behavior of the policy rate. So unsurprisingly, if the central bank reduces the inertia in its policy rule (increases κ_r) then the policy rate in equilibrium mean reverts faster (higher λ_1); because policy rate gaps persist for less time, the effective borrowing rate responds less to these monetary shocks (lower $\hat{A}_r$).

The intuition regarding the central bank's sensitivity to output (ϕ_x) is somewhat similar to the inertia parameter. The central bank responds more forcefully to output gaps, so in equilibrium the policy rate deviations subside faster (higher λ_1). Because they are shorter lived, output responds less as well.

How does the model depend on the weighting function, $\eta(\tau)$, that determines the effective borrowing rate? The final result in Cor. 1.1 answers this question. The two weighting functions correspond to two models where the effective borrowing rate is more geared towards short-term rates (η^s) or long-term rates (η^ℓ). The results says that as the effective borrowing rate becomes more dependent on long-term rates, the model is less sensitive to the policy rate. Further, in equilibrium this implies that monetary shocks persist longer.

3.5 Optimal Long-Run Monetary Target

Before turning to unconventional policy, I also solve for the "optimal" target in the central bank's policy rule r^*, which guarantees a steady state with zero output gap. In a benchmark model the central bank simply sets $r^* = \bar{r}$, the natural short rate. But when the effective borrowing rate depends on long-term rates, this is no longer the case. The central bank still should set their target to the natural short rate; but the natural short rate differs from the household's natural effective rate. Moreover, as with the transmission of monetary shocks, the optimal target also depends on the risk-bearing capacity of financial markets.

Corollary 1.2 (Optimal long-run target, rigid prices)**.** *Consider the rigid price model. Then the optimal target short rate that delivers $x^{SS} = 0$ is the "natural" short rate, given by*

$$r^* = \frac{\bar{r} - \hat{C}}{\hat{A}_r}. \tag{18}$$

Further, when $a > 0$, a higher level of habitat demand $\bar{\beta}(\tau)$ leads to decreases in the optimal target.

Note that the natural effective borrowing rate (which is the steady state value of the effective borrowing rate) is $\tilde{r}^{SS} = \bar{r}$ and is determined by factors outside of the control of the central bank (assumed to be fixed in this model). The effective borrowing rate is an affine function of the short rate (both in transition and in steady state), and hence the natural short rate is the steady state value of the short rate that delivers the natural effective rate:

$$\tilde{r}^{SS} = \bar{r} = \hat{A}_r r^{SS} + \hat{C}.$$

When arbitrageurs are not perfectly risk-neutral, the constant term is affected by shifts in habitat demand. Cor. 1.2 shows that as overall demand increases, the optimal central bank target falls whenever arbitrageurs are not perfectly risk-neutral.

3.6 Forward Guidance

Given the key under-reaction result when financial markets exhibit limited risk-bearing capacity, it is natural to expect that more explicit forward guidance policies will also prove less effective. This section studies this policy and confirms this intuition.

Instead of following a Taylor rule for setting the short rate, suppose instead that the central bank announces a target peg for interest rates $r^\diamond$, which will last for a set period of time $t^\diamond$ before returning to a standard Taylor rule. That is, the short rate evolves according to

$$\mathrm{d}r_t = \begin{cases} -\kappa_r^\diamond (r_t - r^\diamond)\,\mathrm{d}t + \sigma_r^\diamond\,\mathrm{d}B_{r,t} & \text{if } 0 < t < t^\diamond \\ -\kappa_r (r_t - \phi_x x_t - r^*)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} & \text{if } t \geq t^\diamond \end{cases}$$

and initially, the short rate at $t = 0$ is at the peg: $r_0 = r^\diamond$. Note that this setup implies that the peg is the target, but the policy rate may deviate from this target. The shocks to the short rate are again the only source of uncertainty in order to keep the solution tractable.

The output gap still evolves according to eq. (5), but the dynamics of the effective borrowing rate $\tilde{r}_t$ differ from the previous section. Conjecturing once again that bond prices are affine implies

$$-\log P_{t,\tau} = \begin{cases} A_r^\diamond(\tau) r_t + C^\diamond(\tau) & \text{if } 0 < t < t^\diamond \\ A_r(\tau) r_t + C(\tau) & \text{if } t \geq t^\diamond \end{cases} \implies \tilde{r}_t = \begin{cases} \hat{A}_r^\diamond r_t + \hat{C}^\diamond & \text{if } 0 < t < t^\diamond \\ \hat{A}_r r_t + \hat{C} & \text{if } t \geq t^\diamond \end{cases}.$$

Now there are two sets of affine coefficient functions to solve for, corresponding to the two different monetary regimes. But note that after the peg ends ($t \geq t^\diamond$), the model reduces to the one considered in Section 3. Further, during the peg ($0 < t < t^\diamond$), the results derived previously can be utilized to solve for the term structure coefficients. Lemmas 2 and 3 still apply, with $\lambda = \kappa_r^\diamond$.

With this characterization of the term structure during the two monetary regimes, I now turn to solving for the rational expectations equilibrium dynamics, and in particular the initial level of the output gap x_0.

Proposition 2 (Forward guidance, rigid prices)**.** *Consider the forward guidance rigid price model. In general equilibrium:*

1. $\frac{\partial x_0}{\partial r^\diamond} \leq 0$*, is increasing in* a*, and approaches* 0 *as* $a \to \infty$.
2. $\frac{\partial^2 x_0}{\partial r^\diamond \partial t^\diamond} \leq 0$*, is increasing in* a*, and approaches* 0 *as* $a \to \infty$.

As in a benchmark model of forward guidance, my model predicts that if the central bank sets a very low interest rate peg, then the output gap falls ($\frac{\partial x_0}{\partial r^\diamond} \leq 0$), and that this effect grows as the duration of the peg lengthens ($\frac{\partial^2 x_0}{\partial r^\diamond \partial t^\diamond} \leq 0$). The more interesting results from Prop. 2 is how these forces interact with the level of risk aversion of arbitrageurs. The first result says that the current output gap becomes less sensitive to the size of the forward guidance shock as the risk-bearing capacity of arbitrageurs falls. Moreover, output eventually becomes completely insensitive as arbitrageurs become infinitely risk averse. The second part of Prop. 2 is an analogous finding for the interaction of the duration of an interest rate peg and risk aversion. The effectiveness of lengthening the peg also diminishes as arbitrageur risk aversion increases, eventually becoming completely ineffective.

Figure 4 shows this graphically. The dark "level" line corresponds to $\frac{\partial x_0}{\partial r^\diamond}$, while the lighter "length" line corresponds to $\frac{\partial^2 x_0}{\partial r^\diamond \partial t^\diamond}$. As risk aversion increases, both of this effects are mitigated.

Intuitively, when financial markets are disrupted, the sensitivity of output to forward guidance falls. Note that in this model, households are still very forward-looking: households are still very responsive to far-off changes in their borrowing rates, and so conditional on the expected path of the effective borrowing rate the model delivers similar predictions as a benchmark New Keynesian model. The mitigating effect comes from the mismatch between the policy rate and the effective borrowing rate, which is bridged by imperfect financial markets.

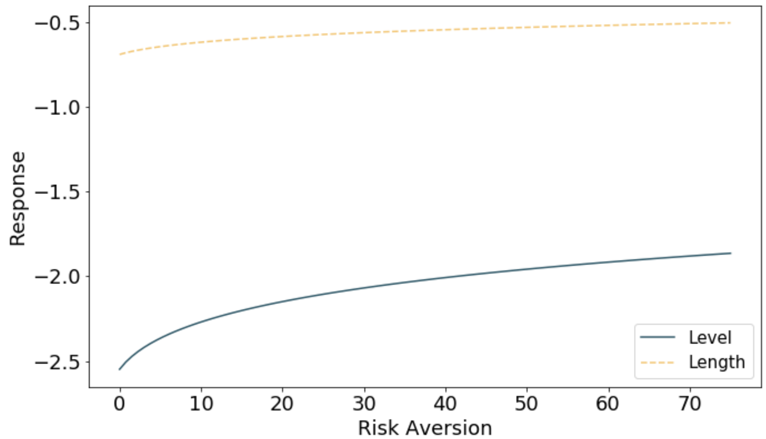

Figure 4: Output Responses to Forward Guidance
Notes: plots of $\frac{\partial x_0}{\partial r^\diamond}$ ("level"; the interaction of the level of the peg and output) and $\frac{\partial^2 x_0}{\partial r^\diamond \partial t^\diamond}$ ("length"; the interaction of the length of the peg and output). These objects are plotted for various levels of risk aversion (x-axis).

3.7 Quantitative Easing

While forward guidance is less effective when financial markets are disrupted, this imperfection opens the door to LSAP policies. Given that the expectations hypothesis does not hold, purchases by the central bank may have price effects; moreover, it's natural to think that QE-type policies would push down long-term rates. This section studies this policy and confirms this result.

I now suppose that in addition to setting the short rate, the central bank also directly purchases longer term bonds through open market operations. In the model, these purchases take place through the demand shifter $\beta_{t,\tau}$ in preferred habitat investor demand given by eq. (4). I assume that

$$\begin{aligned} \beta_{t,\tau} &= \bar{\beta}(\tau) + \theta(\tau)\beta_t \\ \mathrm{d}\beta_t &= -\kappa_\beta \beta_t \, \mathrm{d}t \, . \end{aligned} \tag{19}$$

Note that this formulation treats LSAP programs as a zero-probability shock β_t to preferred habitat investor demand, which returns to zero at a rate according to κ_β. The function $\theta(\tau)$ governs where in maturity space the purchases are targeted. To capture the essence of a QE shock, I assume that $\theta(\tau) \geq 0$ for all maturities (and strictly positive for some maturities). This means that the central bank is only seeking to purchase positive amounts of long-term bonds. This rules out LSAP programs like Operation Twist; I return to this type of policy in an extension of the model in Section 5.[5]

[5]Were the QE programs demand or a supply shocks? By treating the central bank as another "preferred habitat" investor, the model implicitly assumes that the QE purchases are (positive) demand shocks. But if the central bank is thought of as a conglomerate with the fiscal authority, then QE purchases are perhaps more naturally thought of as (negative) supply shocks: by buying bonds, the central bank acts on behalf of the fiscal authority and removes these bonds from the secondary market. Whichever the preferred labeling,

Now the affine functional form of bond prices implies that

$$
\begin{aligned}
-\log P_{t,\tau} &= A_r(\tau) r_t + A_\beta(\tau)\beta_t + C(\tau) \\
\implies \tilde{r}_t &= \hat{A}_r r_t + \hat{A}_\beta \beta_t + \hat{C},
\end{aligned}
$$

which introduces a new coefficient function $A_\beta(\tau)$.

This change complicates the model, because there are now two state variables. However, monetary policy remains the only source of uncertainty, so solving the model is similar to the previous case. First, I solve the macroeconomic dynamics, taking as given the affine coefficients. Write the model in matrix form according to eq. (8), where

$$
\Upsilon = \begin{bmatrix} \kappa_r & 0 & -\kappa_r \phi_x \\ 0 & \kappa_\beta & 0 \\ -\varsigma^{-1}\hat{A}_r & -\varsigma^{-1}\hat{A}_\beta & 0 \end{bmatrix}.
$$

Lemma 4 (Characterizing $\hat{A}_r$ and $\hat{A}_\beta$, rigid prices)**.** *Consider the rigid price QE model.*

1. *Υ has exactly two eigenvalues with positive real part if and only if $\hat{A}_r > 0$. Further, these stable roots are real. One of these eigenvalues is κ_β, the other is $\lambda_1 > 0$.*

2. *$\hat{A}_r$ is given by eq. (11).*

3. *The rational expectations equilibrium dynamic matrices (from eqs. (9) and (10)) are given by*

$$
\Gamma = \begin{bmatrix} \lambda_1 & \frac{\hat{A}_\beta \varsigma^{-1} \kappa_r \phi_x}{\lambda_1 + \kappa_\beta - \kappa_r} \\ 0 & \kappa_\beta \end{bmatrix} \tag{20}
$$

$$
\Omega = \begin{bmatrix} \frac{\kappa_r - \lambda_1}{\kappa_r \phi_x} & \frac{\varsigma^{-1}\hat{A}_\beta}{\kappa_r - \lambda_1 - \kappa_\beta} \end{bmatrix}. \tag{21}
$$

Next, given how the state evolves in general equilibrium, I solve the arbitrageurs' optimality conditions and equilibrium allocations, which solves for the affine term structure coefficient functions.

Lemma 5 (Affine coefficients, demand factor term structure)**.** *Suppose the short rate and demand factor are characterized by*

$$
\begin{aligned}
\mathrm{d}r_t &= -\left(\gamma_1 (r_t - r^{SS}) + \gamma_{12}\beta_t\right)\mathrm{d}t + \sigma_r \,\mathrm{d}B_{r,t} \\
\mathrm{d}\beta_t &= -\gamma_2 \beta_t \,\mathrm{d}t\,.
\end{aligned}
$$

in either case the outcome is the same. As the model makes clear, the effects are all about how the purchases impact the marginal investors, the arbitrageurs. Whether QE is thought of as a demand or supply shock, the result is that QE leads to changes in the portfolio allocation of arbitrageurs.

Then $\hat{A}_r$ is given by eq. (17) *(and ν given by eq.* (16)*). $\hat{A}_\beta$ is given by*

$$\hat{A}_\beta = \frac{\nu_{12}}{\nu - \gamma_2} \int_0^T \eta(\tau)(f(\nu\tau) - f(\gamma_2\tau))\,\mathrm{d}\tau \tag{22}$$

where the coefficient ν_{12} is given by eq. (A3).

Note that, conditional on the equilibrium value of λ_1, $\hat{A}_r$ is the same as the baseline rigid price model. This also implies that $\hat{A}_\beta$, the sensitivity of the effective borrowing rate to QE shocks, does not affect the determinacy of the model. In fact, if $\beta_t = 0$, then the dynamics of the model are identical to the baseline rigid price model. This is unsurprising, as in this case QE is a zero-probability shock and hence the model evolves as if QE will never occur.

Proposition 3 (QE, rigid prices)**.** *Consider the QE rigid price model. Suppose that $\theta(\tau) \geq 0$. In general equilibrium:*

1. *$\hat{A}_\beta \geq 0$ and $\frac{\partial x_t}{\partial \beta_t} \leq 0$, with equality if and only if $a = 0$.*
2. *$\hat{A}_\beta \to 0$ as $\kappa_\beta \to \infty$.*

The coefficient $\hat{A}_\beta$ determines the effects of QE in general equilibrium. The key is under what conditions this coefficient is non-zero, and what sign. In words, Prop. 3 says: if arbitrageurs are perfectly risk-neutral (financial markets exhibit perfect arbitrage), QE has no effect. This is the standard result: the expectations hypothesis holds, and since QE purchases do not change the expected path of short rates, there is no change in long-term rates. More explicitly, with perfect risk-neutrality, arbitrageurs (the marginal investors in bond markets) only care about expected instantaneous returns. In equilibrium it therefore must be the case that these are equalized for all bonds (and equal to the policy rate). When expected excess returns are always zero, arbitrageurs are happy to accomodate any shifts in demand, hence in equilibrium yields are unchanged. Since long-term rates are unaffected by QE purchases, there is no direct effect on household consumption or savings decisions, therefore no indirect effect on the expected path of the policy rate.

But whenever arbitrageurs are risk-averse, LSAPs push down interest rates and boost output. It is still the case that QE purchases do not have a (direct) effect on the expected path of the policy rate. But when arbitrageurs care about risk, expected excess returns of long-term bonds are not necessarily zero. Arbitrageurs demand compensation for taking on risk, and in equilibrium the market price of risk is not zero. The price of risk depends on the portfolio allocations of arbitrageurs, so the more concentrated the arbitrageurs' portfolio is in risky long-term bonds, the higher this compensation is required to be. By purchasing long-term debt, QE effectively reduces the amount of risk arbitrageurs are required to hold, which puts downward pressure on returns of all bonds.

This partial equilibrium effect is mitigated by a general equilibrium effect: when long-term rates fall, so does the effective borrowing rate of households. Through the standard

Euler equation dynamics, this implies that consumption (hence output) will rise. This indirect effect puts countervailing upward pressure on long-term rates, as the expected path of short rates is higher than before. But since arbitrageurs are risk-averse, this upward pressure is weakened relative to the predictions of the expectations hypothesis. Prop. 3 shows that this indirect effect does not outweigh the direct effect, and it will still be the case that in general equilibrium QE purchases will push down effective borrowing rates, leading to an increase in output.

Together, these results show that the effectiveness of the two major unconventional monetary policy tools are mirror images of one another. In either case, passthrough to households only occurs to the extent that arbitrageurs respond to the policy changes. For forward guidance, healthy financial markets are key as arbitrageurs only care about the future path of the policy rate. But as financial markets become disrupted, arbitrageurs become more concerned with risk and less responsive to future changes in the short rate. Then this is precisely the time when LSAPs are most effective: by removing risk from the portfolio of arbitrageurs, these purchases push down long-term interest rates and boost output.

The second result in Prop. 3 shows that the effect of QE depends critically on the mean reversion properties of purchases. Even when financial markets are highly disrupted, the aggregate effects will be minimal if the purchases are undone very quickly. While in the case of QE these purchases were not directly unwound quickly, Greenwood et al. (2016) provide some evidence that Treasury actions had an equivalent effect. While QE was removing long-term debt from private portfolios, the Treasury was extending the average maturity of newly issued debt, perhaps partially offsetting the impact of QE.

4 Allowing for Sticky Prices

This section extends the analysis from Section 3 to allow for inflation. I confirm the results in the case of fully rigid prices go through when prices are sticky but not fully fixed. The main difference between the two models is the conditions for determinacy. In the rigid price model, determinacy in general equilibrium was guaranteed. Once prices are not fixed, determinacy is only guaranteed for some parameterizations of the model.

This result is also present in a benchmark New Keynesian model, and is often stated as the following: the central bank must move the nominal rate more than one-for-one with inflation, in order to move real rates. In benchmark models this is achieved by a simple inequality condition on the Taylor rule coefficients (frequently $\phi_\pi > 1$). But because aggregate dynamics depend on the household's real effective borrowing rate, the determinacy condition involves the entire term structure of interest rates. And since monetary policy is transmitted to the term structure through imperfect financial markets, this condition will implicitly depend on the risk-bearing capacity of arbitrageurs. As I will show, increasing limits to arbitrage moves the model towards the region of indeterminacy.

4.1 Macroeconomic Dynamics

Now I assume that prices are not fully rigid, so $\delta > 0$ and inflation and the output gap evolve according to eqs. (2) and (3). The central bank follows a Taylor rule with persistence:

$$\mathrm{d}r_t = -\kappa_r(r_t - \phi_\pi \pi_t - \phi_x x_t - r^*)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} \tag{23}$$

where $B_{r,t}$ is a standard Brownian motion. ϕ_π and ϕ_x govern the feedback rule for changes in inflation and output to changes in the policy rate, and κ_r is the mean-reversion parameter. r^* is the central bank's target policy rate, which is set to deliver zero inflation and output gap in steady state.

Again start with the conjecture that the model features an affine term structure in the state variables. Both inflation π_t and the output gap x_t are jump variables, while as before the interest rate rule implies that r_t is the only state variable, so the term structure is characterized by two coefficient functions $A_r(\tau)$ and $C(\tau)$. Writing the model in matrix form according to eq. (8) gives

$$\Upsilon = \begin{bmatrix} \kappa_r & -\kappa_r\phi_\pi & -\kappa_r\phi_x \\ 0 & -\rho & \delta \\ -\varsigma^{-1}\hat{A}_r & \varsigma^{-1} & 0 \end{bmatrix}.$$

The rational expectations equilibrium is determinate if and only if Υ has one stable eigenvalue λ_1; under rational expectations the dynamics of the short rate and inflation and the output gap are given by:

$$\begin{aligned} \mathrm{d}r_t &= -\lambda_1(r_t - r^{SS})\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} \\ \pi_t &= \omega_\pi(r_t - r^{SS}), \quad x_t = \omega_x(r_t - r^{SS}). \end{aligned}$$

The following Lemma characterizes the equilibrium object $\hat{A}_r$ in terms of this eigenvalue.

Lemma 6 (Characterizing $\hat{A}_r$, sticky prices)**.** *Consider the sticky price model.*

1. *Υ has exactly one eigenvalue with positive real part if and only if*

$$\hat{A}_r > \frac{\delta}{\delta\phi_\pi + \rho\phi_x}. \tag{24}$$

2. *$\hat{A}_r$ is given by the function $h : \mathbb{R} \to \mathbb{R}$:*

$$h(\lambda_1) = \frac{(\lambda_1 - \kappa_r)(\lambda_1^2 + \lambda_1\rho - \varsigma^{-1}\delta)}{\varsigma^{-1}\kappa_r\left(\delta\phi_\pi + \rho\phi_x + \lambda_1\phi_x\right)}. \tag{25}$$

3. *The inflation and output gap dynamics are given by*

$$\omega_\pi = \frac{\delta(\kappa_r - \lambda_1)}{\kappa_r\left(\delta\phi_\pi + \rho\phi_x + \lambda_1\phi_x\right)}, \quad \omega_x = \frac{(\lambda_1 + \rho)(\kappa_r - \lambda_1)}{\kappa_r\left(\delta\phi_\pi + \rho\phi_x + \lambda_1\phi_x\right)}.$$

The macroeconomic dynamics continue to nest the benchmark New Keynesian model, where the affine coefficients are simply $\hat{A}_r = 1$ and $\hat{C} = 0$. When this is the case, if the central bank only cares about inflation (so $\phi_x = 0$), the determinacy condition eq. (24) simplifies to the standard condition that $\phi_\pi > 1$. But in this model, $\hat{A}_r$ is a general equilibrium object which will depend on the risk aversion of arbitrageurs. Thus, whether the model satisfies determinacy will also depend on the level of risk aversion.

Note that as $\delta \to 0$, all the above results simplify to what was found in the case of fully rigid prices studied in the previous section.

4.2 General Equilibrium Solution

The policy rate is the only state variable; hence, conditional on the equilibrium dynamics of the policy rate, the arbitrageur optimality conditions and the characterization of the affine coefficients is the same as when prices are fully fixed. Thus, the results of Lemma 6 and 3 together solve for $\hat{A}_r$ and λ_1 in general equilibrium. This also allows for a characterization of when the model has a unique equilibrium.

Proposition 4 (General equilibrium, sticky prices)**.** *Consider the sticky price model.*

1. *There exists some positive eigenvalue of* Υ $\lambda_1 > 0$ *for which* $g(\lambda_1) = h(\lambda_1)$*.*

2. *If the model is determinate (the inequality in eq.* (24) *is satisfied), then* $\hat{A}_r < 1$ *and is unique, which fully characterizes the model equilibrium.*

Note that if eq. (24) is violated, there exists another eigenvalue $\lambda_2 > 0$ such that $\hat{A}_r = h(\lambda_2)$. This implies the model is indeterminate. $\hat{A}_r$ may still be unique, but it may also be the case that there is some $0 < \lambda' < \lambda_1$ such that $g(\lambda') = h(\lambda')$. Hence, rather than attempting to define some selection criteria to re-establish determinacy, I choose to only focus on models that satisfy the conditions for determinacy.

Figure 5 illustrates the equilibrium obtained in Prop. 4. The dotted light orange and dark teal lines are $g(\cdot)$ for low and high levels of risk aversion, respectively. Unlike the rigid price model, this plot illustrates that for some parameterizations (in this case, when risk aversion is very high), equilibrium is no longer unique. Finally, I also plots the equivalent function $g(\cdot)$ for a benchmark New Keynesian model (the dotted grey line). Note that the determinacy condition of the benchmark model depends only on the properties of the function $h(\cdot)$ (which is determined by the macroeconomic parameters); financial frictions are not present.

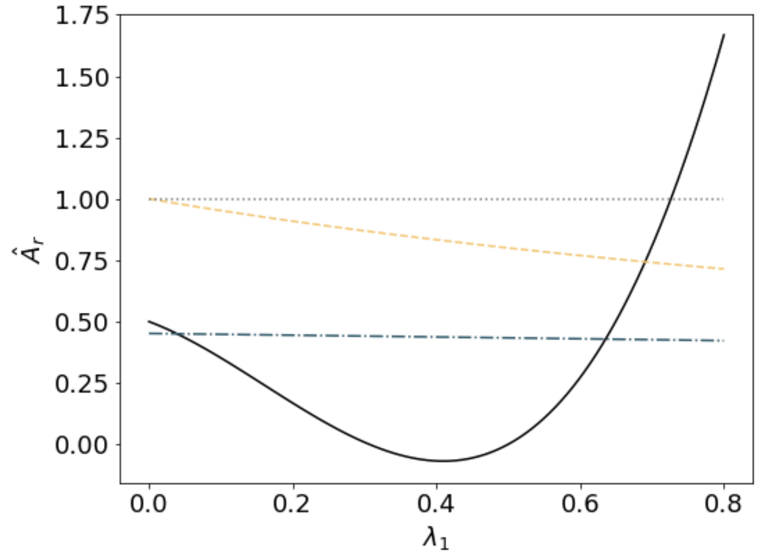

Figure 5: Intersection of $g(\lambda)$ and $h(\lambda)$, sticky prices
Notes: intersection of the functions $g(\lambda)$ and $h(\lambda)$, which determine λ_1 and $\hat{A}_r$ in equilibrium. The black line is $h(\cdot)$; the dotted light orange and dark teal lines are $g(\cdot)$ for low and high levels of risk aversion, respectively. The dotted grey line is the equivalent function $g(\cdot)$ for a benchmark New Keynesian model.

4.3 Conventional Policy

This section studies how conventional monetary policy works in the sticky price model.

Corollary 4.1 (Comparative statics, sticky prices)**.** *Consider the sticky price model. In general equilibrium:*

1. $\frac{\partial \lambda_1}{\partial a} < 0$, $\frac{\partial \hat{A}_r}{\partial a} < 0$, $\frac{\partial \omega_\pi}{\partial a} > 0$, $\frac{\partial \omega_x}{\partial a} > 0$.
2. $\frac{\partial \lambda_1}{\partial \kappa_r} > 0$, $\frac{\partial \hat{A}_r}{\partial \kappa_r} < 0$, $\frac{\partial \omega_\pi}{\partial \kappa_r} > 0$, $\frac{\partial \omega_x}{\partial \kappa_r} > 0$.
3. $\frac{\partial \lambda_1}{\partial \phi_\pi} > 0$, $\frac{\partial \hat{A}_r}{\partial \phi_\pi} < 0$, $\frac{\partial \omega_\pi}{\partial \phi_\pi} > 0$, $\frac{\partial \omega_x}{\partial \phi_\pi} > 0$.
4. $\frac{\partial \lambda_1}{\partial \phi_x} > 0$, $\frac{\partial \hat{A}_r}{\partial \phi_x} < 0$, $\frac{\partial \omega_\pi}{\partial \phi_x} > 0$, $\frac{\partial \omega_x}{\partial \phi_x} > 0$.
5. *Consider two different weighting functions $\eta^s(\tau)$ and $\eta^\ell(\tau)$, such that for some T^*, $\eta^s(\tau) \geq \eta^\ell(\tau) \iff \tau \leq T^*$. Then $\lambda_1^s > \lambda_1^\ell$, $\hat{A}_r^s > \hat{A}_r^\ell$, $\omega_\pi^s < \omega_\pi^\ell$, $\omega_x^s < \omega_x^\ell$ where superscripts denote the equilibrium outcomes under the corresponding weighting functions.*

The results in Cor. 4.1 confirm the results of the rigid price model carry over to the case of sticky prices. The first result shows that the responsiveness of the (nominal) effective borrowing rate to changes in the short rate declines as risk aversion increases. In general equilibrium, when the model is determinate, this will imply that the real effective borrowing rate also becomes less responsive, hence inflation and output also respond less to changes in the short rate as risk aversion increases. This also implies that monetary shocks become less persistent. The final result, which shows what happens when household borrowing becomes more concentrated towards long-term rates, has similar results. Borrowing becomes less responsive to the short rate, and thus inflation and output also respond less to changes in the short rate. In equilibrium, this implies that monetary shocks are not as persistent.

The other set of results relate to the central bank's policy rule. Increases in the mean-reversion parameter (κ_r), the sensitivity to inflation (ϕ_π), and the sensitivity to output (ϕ_x) all cause shocks to the policy rate in equilibrium to return to steady state faster (higher λ_1) and therefore the (nominal) effective borrowing rate responds less to these monetary shocks (lower $\hat{A}_r$). In general equilibrium when the model is determinate, this also leads to smaller changes in inflation and output.

4.4 Determinacy

In the rigid price model, determinacy was guaranteed. This is no longer the case once prices are not fixed. Moreover, the determinacy condition is more complicated than a benchmark New Keynesian model.

Corollary 4.2 (Determinacy, sticky prices)**.** *Consider the determinacy condition of the sticky price model. If $\delta > 0$, there exists some upper bound $\overline{a}$ such that, whenever $a > \overline{a}$, the model is indeterminate. Similarly, there are upper bounds for κ_r and σ_r (if $a \neq 0$) above which the model is indeterminate. For ϕ_π and ϕ_x there exists upper bounds above which the model is determinate.*

From a macroeconomic perspective, the reason for indeterminacy in this model is analogous to a standard model. Suppose inflation increases. The central bank responds by increasing the policy rate, but if the real borrowing rate faced by households does not rise then the policy response is unable to stabilize the macroeconomy. This logic holds in my model, but the relationship between the policy rate and the household effective borrowing rate is complicated by the fact that policy changes are passed through by risk-averse arbitrageurs.

The first result in Cor. 4.2 says that if financial markets become severely disrupted, the model moves into the region of indeterminacy. This is for the same reason that conventional policy becomes less effective: the passthrough of policy rate changes to the (nominal) effective borrowing rate becomes dampened, and this dampening can become severe enough that the real effective borrowing rate no longer moves to stabilize the economy.

On the other hand, the central bank can guarantee determinacy given any level of (finite) risk aversion by responding more aggressively to inflation (or output). Also, if the policy rule does not exhibit enough inertia (that is, the mean reversion rate κ_r is very high), this can also induce model indeterminacy. This is perhaps one theoretical reason why central banks seem to pursue inertial policy rules. A high degree of monetary policy inertia implies that changes in the policy rate are endogenously passed to long-term rates.

When household borrowing depends on a mix of short- and long-term rates, this leads to larger responses of output and inflation.

What does it mean in reality for a macroeconomic model to exhibit indeterminacy? As discussed in e.g. Clarida et al. (2000), one practical way of thinking about indeterminacy is that it induces excess volatility. When policy is such that it gives rise to an indeterminate model, this opens up the possibility of self-fulfilling expectations. This may increase the amount of volatility in the model.

In the model, the parameters governing both the central bank's policy rule as well as the risk aversion of arbitrageurs are fixed across time. But stepping outside the model, what do the results in Cor. 4.2 say? One interpretation is that if financial crises are thought of as large disruptions in financial markets, with big increases in risk aversion of investors (or the risk-bearing capacity of investors), then financial crises can lead to macroeconomic instability, even if the cause of the crises was unrelated to the macroeconomy.

Central bankers can induce stability again, but they can do so by becoming more aggressive. This runs counter to the idea that central banks should become more passive in crises, and make up for it after the crisis passes.

4.5 Forward Guidance

This section extends the previous analysis of forward guidance to the case of sticky prices. As before, the central bank announces a target peg for interest rates $r^\diamond$, which will last for a set period of time $t^\diamond$ before returning to a standard Taylor rule. That is, the short rate is evolves according to

$$\mathrm{d}r_t = \begin{cases} -\kappa_r^\diamond(r_t - r^\diamond)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} & \text{if } 0 < t < t^\diamond \\ -\kappa_r(r_t - \phi_\pi \pi_t - \phi_x x_t - r^*)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t} & \text{if } t \geq t^\diamond \end{cases}$$

and initially, the short rate at $t = 0$ is at the peg: $r_0 = r^\diamond$. Inflation and the output gap still evolve according to eqs. (2) and (3).

Since the policy rate is the only state variable, the affine functional form is the same as in the rigid price model. Using the same approach as before, I now turn to solving for the rational expectations equilibrium dynamics and the initial level of inflation π_0 and the output gap x_0.

Proposition 5 (Forward guidance, sticky prices)**.** *Consider the forward guidance sticky price model. In general equilibrium:*

1. $\frac{\partial \pi_0}{\partial r^\diamond} \leq 0$ *and* $\frac{\partial x_0}{\partial r^\diamond} \leq 0$*. Both are increasing in* a*, and approach* 0 *as* $a \to \infty$*.*
2. $\frac{\partial^2 \pi_0}{\partial r^\diamond \partial t^\diamond} \leq 0$ *and* $\frac{\partial^2 x_0}{\partial r^\diamond \partial t^\diamond} \leq 0$*. Both are increasing in* a*, and approach* 0 *as* $a \to \infty$*.*

Note that inflation and the output gap fall if the central bank increases the level of the peg, and this effect grows with the length of the peg. So the first result of Prop. 5 says

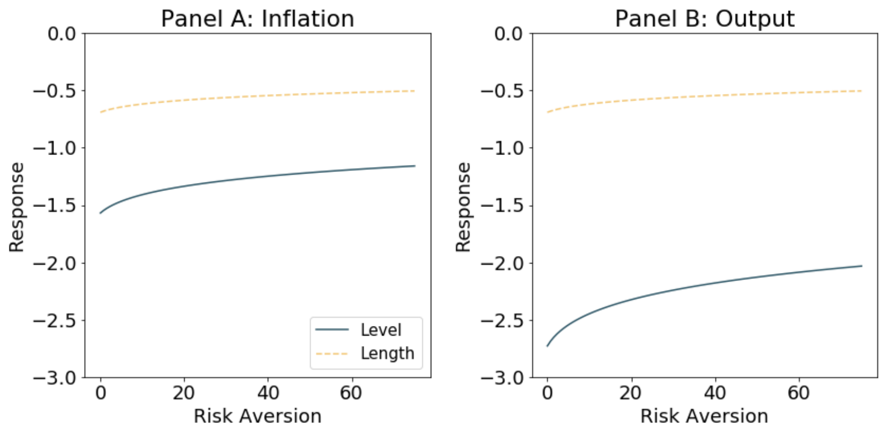

Figure 6: Inflation and Output Responses to Forward Guidance
Notes: panel A plots $\frac{\partial \pi_0}{\partial r^\diamond}$ ("level"; the interaction of the level of the peg and inflation) and $\frac{\partial^2 \pi_0}{\partial r^\diamond \partial t^\diamond}$ ("length"; the interaction of the length of the peg and inflation). Panel B plots the corresponding level and length interaction terms for output x_0. These objects are plotted for various levels of risk aversion (x-axis).

that current inflation and output become less sensitive to the size of the forward guidance shock as the risk-bearing capacity of arbitrageurs falls, and eventually become completely insensitive; while the second result says the same regarding length of the peg.

Figure 6 shows the results regarding inflation and output graphically. The dark "level" line in the first panel corresponds to $\frac{\partial \pi_0}{\partial r^\diamond}$, while the lighter "length" line corresponds to $\frac{\partial^2 \pi_0}{\partial r^\diamond \partial t^\diamond}$. The second panel plots the same objects for the output gap x_0. As risk aversion increases, both of this effects are mitigated.

4.6 Quantitative Easing

I now modify the sticky price model to allow for QE shocks. Suppose that the central bank also purchases long-term bonds, so that the demand shifter in eq. (4) evolves according to eq. (19).

First, I solve the macroeconomic dynamics, taking as given the affine coefficients. Write the model in matrix form according to eq. (8), where

$$\Upsilon = \begin{bmatrix} \kappa_r & 0 & -\kappa_r \phi_\pi & -\kappa_r \phi_x \\ 0 & \kappa_\beta & 0 & 0 \\ 0 & 0 & -\rho & \delta \\ -\varsigma^{-1} \hat{A}_r & -\varsigma^{-1} \hat{A}_\beta & \varsigma^{-1} & 0 \end{bmatrix}.$$

Lemma 7 (Characterizing $\hat{A}_r$ and $\hat{A}_\beta$, sticky prices)**.** *Consider the sticky price QE model.*

1. *Υ has exactly two eigenvalues with positive real part if and only if the condition eq. (24) is satisfied. Further, these stable roots are real. One of these eigenvalues is κ_β, the other*

is $\lambda_1 > 0$.

2. $\hat{A}_r$ *is given by eq.* (25).

3. *The rational expectations equilibrium dynamic matrices are given by eqs.* (A5) *and* (A6).

Given the equilibrium value of λ_1, $\hat{A}_r$ is the same as the sticky price model. The determinacy condition is also equivalent. As in the rigid price model, this is because I treat QE as a zero-probability event. Moreover, conditional on how the state evolves in general equilibrium, the arbitrageurs' optimality conditions and portfolio problem is the same as considered in Lemma 5. Therefore, putting everything together shows how QE shocks impact the economy in general equilibrium.

Proposition 6 (QE, sticky prices)**.** *Consider the QE sticky price model. In general equilibrium, the model is determinate then*

$$\hat{A}_\beta \geq 0 \implies \frac{\partial \pi_t}{\partial \beta_t} \leq 0, \; \frac{\partial x_t}{\partial \beta_t} \leq 0$$

with equality if and only if $a = 0$.

Prop. 6 confirms that in the sticky price model, when the determinacy condition is satisfied QE works in the same way as in the rigid price model. Expansionary QE shocks move both inflation and output in the same direction, but only when arbitrageurs are not risk-neutral.

5 General Numerical Model

The analysis thus far has focused on delivering analytical results, but this comes at the cost of realism. This section generalizes the model in order to move closer to the data and allows me to take a first step towards quantifying the effects of unconventional policies in and out of financial crises.

This section first extends the model to allow for a richer set of shocks and develops the tools to solve the model numerically. Since this requires taking a stance on parameter values, I next turn to estimating the model. I then use the estimated model to quantify the effects of unconventional policies. The extended model allows for the study of not only standard QE policies, but also more complicated LSAP programs such as Operation Twist, where the Federal Reserve bought long-term bonds while selling shorter term securities. I also study a counterfactual LSAP policy whereby the central bank conducts QE endogenously according to a Taylor-type of rule. Finally, I use the model to study optimal policy as a function of financial frictions.

5.1 Macroeconomic Dynamics and Term Structure Determination

I assume prices are sticky but not fully rigid. Besides monetary policy shocks, I add demand shocks and cost-push shocks. I also include shocks that shift demand for bonds coming from the idiosyncratic preferred habitat investors. This model allows me to explore the robustness of the results in the previous section, as well as explore the implications for unconventional monetary policy. Adding multiple demand factors not only makes the model more realistic, but also allows me to explore more complicated LSAP programs such as Operation Twist, where the Fed simultaneously purchased and sold Treasuries of long- and short-term maturities, respectively.

The aggregate dynamics of the extended model are as follows:

$$\mathrm{d}\pi_t = (\rho\pi_t - \delta x_t - z_{\pi,t})\,\mathrm{d}t \tag{26}$$

$$\mathrm{d}x_t = \varsigma^{-1}\,(\tilde{r}_t - \pi_t - \bar{r} - z_{x,t})\,\mathrm{d}t \tag{27}$$

$$\mathrm{d}r_t = -\kappa_r(r_t - \phi_\pi\pi_t - \phi_x x_t - r^*)\,\mathrm{d}t + \sigma_r\,\mathrm{d}B_{r,t}\,. \tag{28}$$

The new aggregate variables are a cost-push shock to the Phillips curve $(z_{\pi,t})$ and a demand shock to the Euler equation $z_{x,t}$.These shocks follow simple Ornstein-Uhlenbeck processes:

$$dz_{i,t} = -\kappa_{z_i} z_{i,t}\,\mathrm{d}t + \sigma_{z_i}\,\mathrm{d}B_{z_i,t}\,.$$

Additionally, I assume there are factors $\beta_{k,t}$ that affect the demand of preferred habitat investors. These shocks also follow simple Ornstein-Uhlenbeck processes:

$$d\beta_{k,t} = -\kappa_{\beta_k}\beta_{k,t}\,\mathrm{d}t + \sigma_{\beta_k}\,\mathrm{d}B_{\beta_k,t}\,.$$

This is the only change to the term structure side of the model: the demand shifter $\beta_{t,\tau}$ from idiosyncratic preferred habitat investors is given by

$$\beta_{t,\tau} = \bar{\beta}(\tau) + \sum_k \beta_{k,t}\theta_k(\tau)$$

where $\theta_k(\tau)$ governs how shifts in the demand factor $\beta_{k,t}$ affects the level of demand for τ bonds. Unlike the QE models explored in previous sections, the habitat demand factors are now stochastic.

As before, start with the conjecture that bonds are affine in the state variables. Now the state consists of the short rate r_t, the demand shock and cost push shocks $z_{x,t}$ and $z_{\pi,t}$, and the preferred habitat demand factors $\beta_{k,t}$. If $\mathbf{Y}_t$ is the vector of all variables and the state is denoted by a vector $\mathbf{y}_t$, then effective rate is therefore also affine in the state variables:

$$\tilde{r}_t = \mathbf{y}_t^T\hat{\mathbf{A}} + \hat{C}, \tag{29}$$

and the preferred habitat demand shifter is

$$\beta_{t,\tau} = \bar{\beta}(\tau) + \mathbf{y}_t^T \Theta(\tau)$$

where $\Theta(\tau)$ is a vector of that collects the $\theta_k(\tau)$ functions corresponding to each $\beta_{k,t}$ demand shocks (and is zero for the other state variables).

Using matrix notation, the dynamics matrix from eq. (8) is a function of the affine coefficients $\hat{\mathbf{A}}$ from eq. (29). Therefore, so is the state dynamics matrix $\Gamma(\hat{\mathbf{A}})$, given by eq. (C10). Prop. 7 characterizes the affine coefficients and general equilibrium solution in this setup.

Proposition 7 (General equilibrium characterization)**.** *Suppose the aggregate economy evolves according to eq. (8), where the dynamics matrix is a function of affine coefficients $\hat{\mathbf{A}}$ in eq. (29). Define the matrix*

$$\mathbf{M} = \Gamma(\hat{\mathbf{A}})^T - a\left[\int_0^T \alpha(\tau)\left(\tau\Theta(\tau) - \mathbf{A}(\tau)\right)\mathbf{A}(\tau)^T \,\mathrm{d}\tau\right]\mathbf{\Sigma} \tag{30}$$

as a function of $\hat{\mathbf{A}}$. Letting $\mathbf{e}_1$ be the first standard basis coordinate vector, if $\mathbf{M}$ is diagonalizable and invertible then $\mathbf{A}(\tau)$ solves

$$\mathbf{A}(\tau) = \mathbf{G}\mathbf{D}^{-1}\left[\mathbf{I} - \exp(-\mathbf{D}\tau)\right]\mathbf{1} \tag{31}$$

$$\implies \hat{\mathbf{A}} = \mathbf{G}\mathbf{D}^{-1}\int_0^T \frac{\eta(\tau)}{\tau}\left[\mathbf{I} - \exp(-\mathbf{D}\tau)\right]\mathrm{d}\tau\,\mathbf{1} \tag{32}$$

where $\mathbf{D}$ is the diagonal matrix of the eigenvalues of $\mathbf{M}$, and $\mathbf{G}$ is the matrix of corresponding eigenvectors, normalized such that $\mathbf{G1} = \mathbf{e}_1$, where $\mathbf{1}$ is a vector of ones and $\mathbf{e}_1$ is the first standard basis coordinates. Then the general equilibrium solution is a fixed point of the matrix function defined in eq. (30).

Note that the matrix $\mathbf{M}$ is nothing more than the multidimensional generalization of the scalar ν from Lemma 3. Thus Prop. 7 is similar to the fixed point problem which defines the parameter ν, but is now complicated by the fact that the problem is no longer scalar-valued. Except in special cases such as when $a = 0$, the problem no longer lends itself to tractable solutions but instead must be solved numerically. An algorithm for solving the model is described in Appendix B.

5.2 Calibration

In order to implement the numerical solution method I need to parameterize the model. I calibrate the model by separating the parameters into two groups: the macroeconomic dynamics parameters and the term structure preferred habitat parameters. The macroeconomic parameters consist of: the weighting function in the effective borrowing rate ($\eta(\tau)$);

the preference parameters (ρ and ς^{-1}); nominal rigidity (δ); the Taylor rule coefficients (ϕ_π and ϕ_x); the mean reversion of monetary shocks, cost-push shocks, and demand shocks (κ_r, κ_{z_π}, and κ_{z_x}); and the volatility of these shocks (σ_r^2, $\sigma_{z_\pi}^2$, and $\sigma_{z_x}^2$). The term structure parameters consist of: arbitrageur risk aversion (a); preferred habitat demand elasticities ($\alpha(\tau)$); the number and location of demand factors ($\theta_k(\tau)$); and the mean reversion and volatility of these demand factors (κ_{β_k} and $\sigma_{\beta_k}^2$).[6]

I estimate the first group of parameters using data from the U.S. from 1985-2007. Since 1985-2007 was largely a period of financial calm in the U.S., I estimate these parameters by assuming $a \approx 0$. Given estimates of the macroeconomic parameters, I estimate the term structure parameters in a second step by focusing on the term structure response to QE during 2009.

5.2.1 Effective Borrowing Rate Weights

A key input is the weighting function in the effective borrowing rate $\eta(\tau)$. This function governs the household allocation of borrowing across the term structure, and therefore the sensitivity of the effective borrowing rate is to short- and long-term rates.

I set $\eta(\tau)$ to match the average maturity structure of outstanding U.S. Treasury Bills, Notes, and Bonds from 1985-2007; data is from the monthly CRSP U.S. Treasury database. While it would be possible to match this distribution non-parametrically, as explained in Appendix B the numerical algorithm to solve the model requires being able to solve closed-form solutions to many integral expressions involving $\eta(\tau)$. To that end, I assume that $\eta(\tau)$ is equal to the probability density function of a (truncated) Gamma distribution with shape parameter 2 and rate parameter η_1 (or scale parameter $1/\eta_1$), so $\eta(\tau) \propto \tau \exp(-\eta_1 \tau)$. I estimate the rate parameter η_1 in order to minimize the distance between the parameterized $\eta(\tau)$ and the distribution of outstanding U.S. Treasuries.

5.2.2 Macroeconomic Parameters

In order to estimate the parameters governing the macroeconomic dynamics, I take a moments-matching approach. I target 9 moments in the data: the variance of the short rate, inflation, and the output gap; the respective covariances; and the respective one-year autocovariances. I use data from the U.S. from 1985-2007. For the short rate I use the 3-month Treasury Bill rate; for inflation I use PCE; and for the output gap I use the CBO's nominal potential GDP. The data series are from FRED.

I choose the period 1985-2007 because this was largely a period of financial calm. Hence, I solve the model and compute the model analogues of the variance-covariances assuming that risk aversion $a = 0$. This also allows me to defer estimating many of the parameters

[6] The remaining parameters $\bar{\beta}(\tau)$, $\bar{r}$, and r^* affect the steady state but play no role governing the dynamics of the model; I focus on equilibrium dynamics linearized around a zero steady state. Formally, I assume that given any parameterization, the central bank sets the target rate r^* to deliver a zero steady state as discussed in Cor. 1.2.

from the finance side of the model. I additionally set $\varsigma^{-1} = 1$ and the discount factor $\rho = 0.04$. This leaves 9 parameters to be estimated: the inertia terms κ_r, κ_{z_π}, and κ_{z_x}; the Taylor rule coefficients ϕ_π and ϕ_x; the nominal rigidity term δ; and the shock variances σ_r^2, $\sigma_{z_\pi}^2$, and $\sigma_{z_x}^2$.

Since I have 9 parameters to estimate, the model is able to match the target 9 moments perfectly. Although each moment is sensitive to each parameter, intuitively it is useful to discuss which moments respond most strongly to which parameters. The auto-covariances are largely dependent on the inertia terms, while the overall volatility of the economy is a function of the volatility of the fundamental shocks. Finally, the covariances between the short rate, inflation, and the output gap depend on the degree of nominal rigidity and how the central bank changes the policy rate in response to deviations in inflation and output.

5.2.3 Term Structure Parameters

The other set of parameters to estimate come from the preferred habitat side of the model. I estimate these parameters in order to match the change in the yield curve following the FOMC announcement regarding QE1 on March 18, 2009. Since the model makes predictions about zero-coupon yields, the response I target is the daily change in zero-coupon yields as taken from Gurkaynak et al. (2007).

Before estimating the model, I make some simplifying assumptions. First, I assume that there are two demand factors $\beta_{s,t}$ and $\beta_{\ell,t}$ that are otherwise identical (same inertia parameter κ_β and variance σ_β^2) but concentrated at short and long maturities (different functions $\theta_s(\tau)$ and $\theta_\ell(\tau)$). I assume that these functions are entirely concentrated at maturities of length 2 and 10 years, respectively.[7] Second, I assume that the preferred habitat demand elasticities $\alpha(\tau)$ are constant across maturities. Since the risk aversion coefficient a always enters multiplicatively with $\alpha(\tau)$, I normalize $\alpha(\tau) = 1$. Hence, the estimated coefficient a should be interpreted as a mix of both the arbitrageur's preferences for risk as well as the preferred habitat investor's sensitivity to price movements. The remaining parameters to be estimated are the inertia parameter κ_β, variance σ_β^2, and the risk aversion term a. In the model, I assume that QE1 was a ten standard deviation shock; this coincides with the findings in Gorodnichenko and Ray (2017) that QE purchases were roughly ten times larger than typical private demand shocks for long-term Treasuries.

5.2.4 Calibration Results

Table 1 summarizes the results of the calibration. In this section I briefly discuss the results in more detail.

During 1985-2007, on average roughly 40% of Treasury debt was less than one year, 35%

[7] Formally, I assume that the functions $\theta_s(\tau) = \delta(\tau - 2)$ and $\theta_\ell(\tau) = \delta(\tau - 10)$, where $\delta(\cdot)$ is the Dirac delta function. The Dirac delta function can be interpreted as the limit of a mean=zero normal distribution as the variance approaches 0.

Table 1: Calibration Results

Parameter	Value	Description	Target
Effective Borrowing Rate			
η_1	1.7069	Weight Scaling Factor	Treasury Maturity Distribution
Macroeconomic Dynamics			
ρ	0.0400	Discount Factor	Long-Run Interest Rate
ς^{-1}	1.0000	Intertemporal Elasticity	Balanced Growth
κ_r	0.9473	Monetary Policy Inertia	$\mathrm{Cov}[r_t, r_{t-1}] = 3.5013$
κ_{z_π}	0.5863	Cost-Push Shock Inertia	$\mathrm{Cov}[\pi_t, \pi_{t-1}] = 0.9141$
κ_{z_x}	0.2554	Demand Shock Inertia	$\mathrm{Cov}[x_t, x_{t-1}] = 2.2908$
ϕ_π	2.0420	Inflation Taylor Coeff.	$\mathrm{Cov}[r_t, \pi_t] = 1.0006$
ϕ_x	0.9709	Output Taylor Coeff.	$\mathrm{Cov}[r_t, x_t] = 0.7722$
δ	0.0459	Nominal Rigidity	$\mathrm{Cov}[\pi_t, x_t] = -0.3015$
σ_r	0.0116	Monetary Shock Vol.	$\mathrm{Var}[r_t] = 2.7066$
σ_{z_π}	0.0068	Cost-Push Shock Vol.	$\mathrm{Var}[\pi_t] = 0.5097$
σ_{z_x}	0.0126	Demand Shock Vol.	$\mathrm{Var}[x_t] = 1.5192$
Term Structure			
$\theta_s(\tau)$	$\delta(\tau - 2)$	Short Factor Location	LSAP Targets
$\theta_\ell(\tau)$	$\delta(\tau - 10)$	Long Factor Location	LSAP Targets
$\alpha(\tau)$	1.0000	Habitat Elasticity	Normalized
κ_β	0.1710	Habitat Factor Inertia	QE1 Yield Curve Response
σ_{z_β}	0.0142	Habitat Factor Vol.	QE1 Yield Curve Response
a	1559.7	Risk Aversion	QE1 Yield Curve Response

Notes: results of the calibration exercise. The effective borrowing weight term η_1 is the rate factor in a (truncated) Gamma distribution: $\eta(\tau) \propto \tau \exp(-\eta_1 \tau)$. For the macroeconomic dynamics coefficients, each parameter is listed alongside the covariance target which is most sensitive to changes in the given parameter; however, the parameters are jointly estimated in order to match all the target moments. Variances and covariances are expressed in percentage points. The short and long demand factor location functions are Dirac delta functions $\delta(\cdot)$.

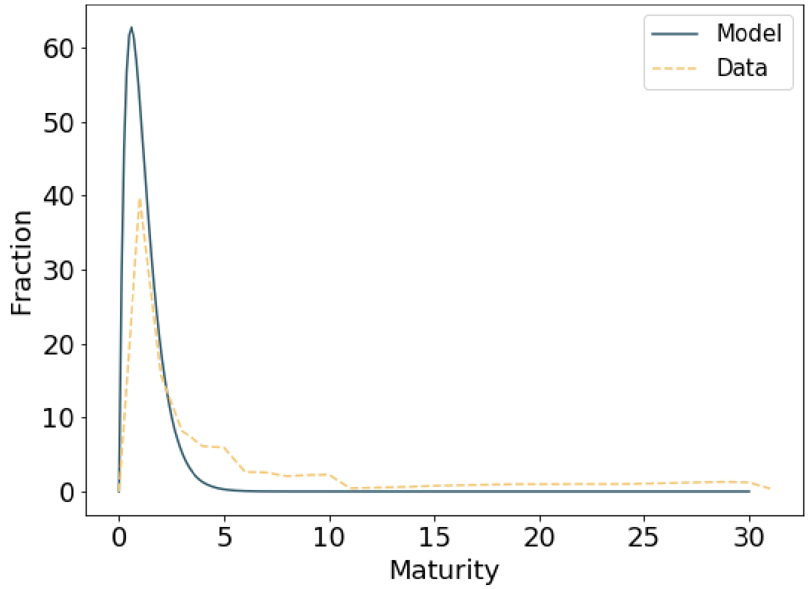

Figure 7: Estimated Borrowing Weights $\eta(\tau)$
Notes: the estimated effective borrowing weights $\eta(\tau)$ and the distribution of outstanding Treasury debt in the data.

was between 1 and 5 years, and the remaining 25% was between 5 and 30 years. Targeting these moments, I estimate that the rate parameter in the (truncated) Gamma distribution is $\eta_1 \approx 1.7$. As Figure 7 shows, the model analogue matches the short end of the distribution but somewhat understates the fraction of long-term debt. Although the concentration is highest for shorter maturities, the weighting implies that over 60% of the distribution of borrowing is weighted towards maturities over 1 year. This will imply substantial deviations from benchmark models where borrowing is entirely concentrated at the short rate.

Figure 8 compares the yield curve response to QE1 in the data vs. the model.

5.3 Responses to Conventional and Unconventional Monetary Shocks

Using the estimated parameters, I now explore the implications for monetary policy in general equilibrium. I study the model for varying degrees of risk aversion; from very low $a \approx 0$ to even higher than the the level estimated during the QE1 period, in order to understand how policy interacts with different degrees of financial crisis.

5.3.1 Expansionary Monetary Shock

I first study the macroeconomic response to a standard monetary shock. The shock is a 50 basis point fall in the policy rate (an expansionary shock).

The first panel of Figure 9 plots the immediate response of the yield curve (in terms of deviations from steady state). Lighter lines plot the response for low levels of risk aversion;

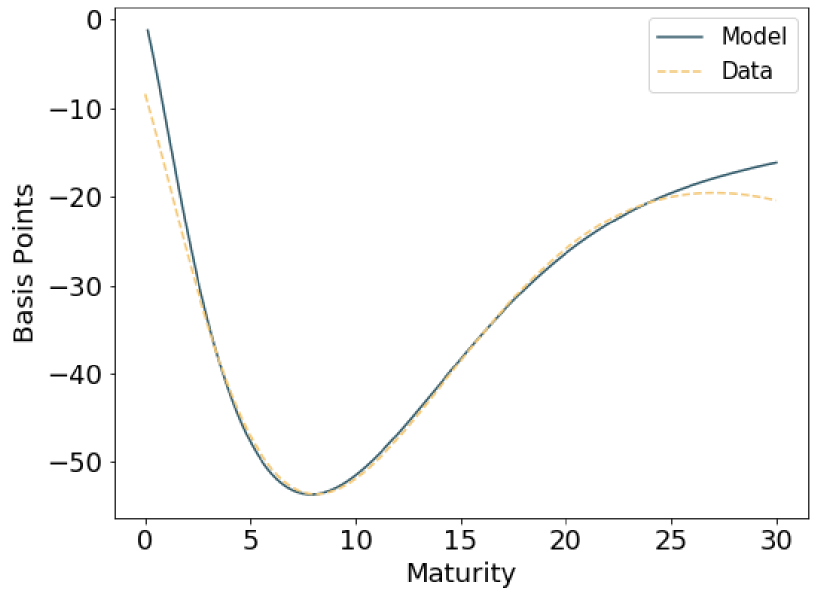

Figure 8: Yield Curve Response to QE1, Model vs. Data
Notes: the estimated yield curve response to a QE shock as compared to the actual response.

the darker lines correspond to high levels of risk aversion. When risk aversion is very low, the entire yield curve shifts down significantly. But as risk aversion increases, the darker lines begin to move closer to zero. While short-term rates still respond strongly, long-term rates become nearly unresponsive. The expected path of the policy rate is not that different, but arbitrageurs do not equalize the expected returns of all bonds and hence long-term rates under-react to the change in the policy rate.

The bottom panels of Figure 9 plot the immediate responses of output and inflation, with the level of risk aversion on the x-axis. These results show that as risk aversion increases, the immediate macroeconomic effects of a monetary shock fall. This follows from the results regarding the yield curve. Since the entire yield curve becomes less responsive to a given monetary shock, the household effective borrowing rate similarly responds less. Hence the consumption response (and output gap response) is smaller. Smaller output gaps into the future imply that inflation also responds less.

The exercise in Figure 9 also allows for a quantification of the relative effectiveness of monetary policy in and out of financial crises. Comparing the output response to a monetary shock during normal times ($a \approx 0$) to periods of high financial distress (the dotted black line, corresponding to the estimated value of risk aversion a), output responds by about 20% less to a monetary shock in a financial crisis than out. As risk aversion increases, the effectiveness falls even further. This suggests that had the zero lower bound on the policy rate had not been binding (or alternatively, had the Fed pursued a policy of negative rates), monetary policy still would have struggled to boost output during the recent financial crisis.

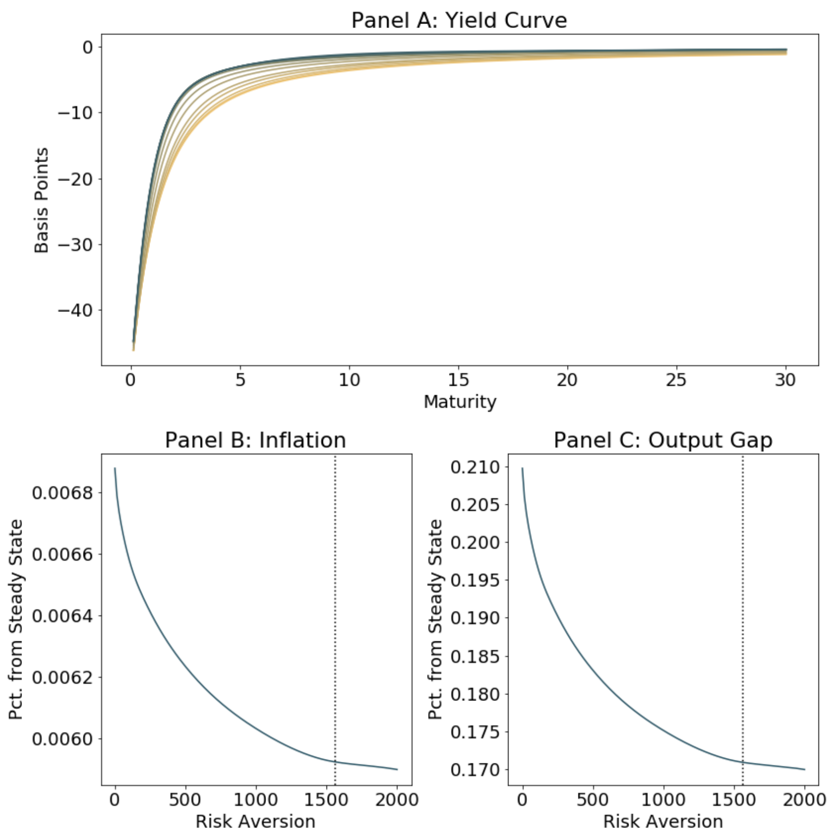

Figure 9: Monetary Policy Shock

Notes: Panel A is the contemporaneous response of the yield curve to a 50 basis point monetary policy shock. Responses are plotted as deviations from steady state, in terms of basis points. The x-axis is maturity. Lighter lines correspond to models where risk aversion is low; darker lines to models with high risk aversion. Panels B and C are the contemporaneous response of inflation and the output gap to the same shock. Responses are plotted as deviations from steady state, in terms of percentage points. The x-axis is level of risk aversion. The dotted black line corresponds to the estimated level of risk aversion.

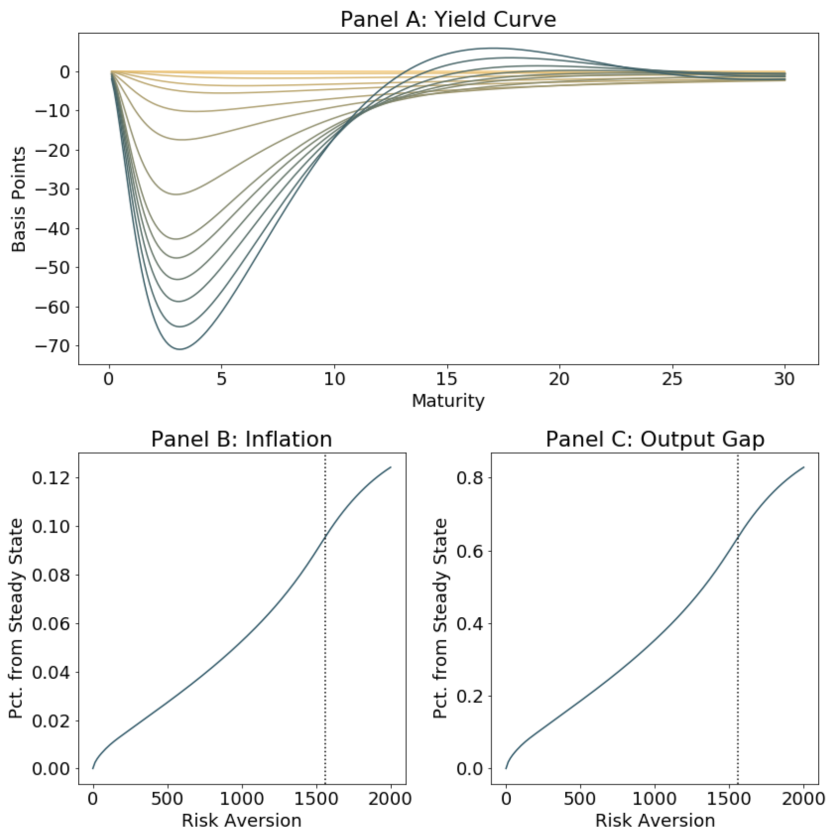

Figure 10: QE Shock (short-term purchases)

Notes: Panel A is the contemporaneous response of the yield curve to a QE shock, where purchases are concentrated at shorter term bonds. Responses are plotted as deviations from steady state, in terms of basis points. The x-axis is maturity. Lighter lines correspond to models where risk aversion is low; darker lines to models with high risk aversion. Panels B and C are the contemporaneous response of inflation and the output gap to the same shock. Responses are plotted as deviations from steady state, in terms of percentage points. The x-axis is level of risk aversion. The dotted black line corresponds to the estimated level of risk aversion.

5.3.2 QE Shocks

I now study the response to two types of QE shocks. In the model, these correspond to shocks to the demand factors which I label as $\beta_{s,t}$ (short-term shock, purchases concentrated at 2-year maturities) and $\beta_{\ell,t}$ (long-term shock, purchases concentrated at 10-year maturities).

Figure 10 plots the response to a shock to the short demand factor $\beta_{s,t}$. The top panel plots how the yield curve changes immediately, where darker lines correspond to increasing levels of risk aversion. The bottom panels plot immediate responses of output and inflation (with the level of risk aversion on the x-axis).

When risk aversion is very low, there is little to no response to a QE shock. This is because arbitrageurs are able to fully absorb the purchases without requiring any changes

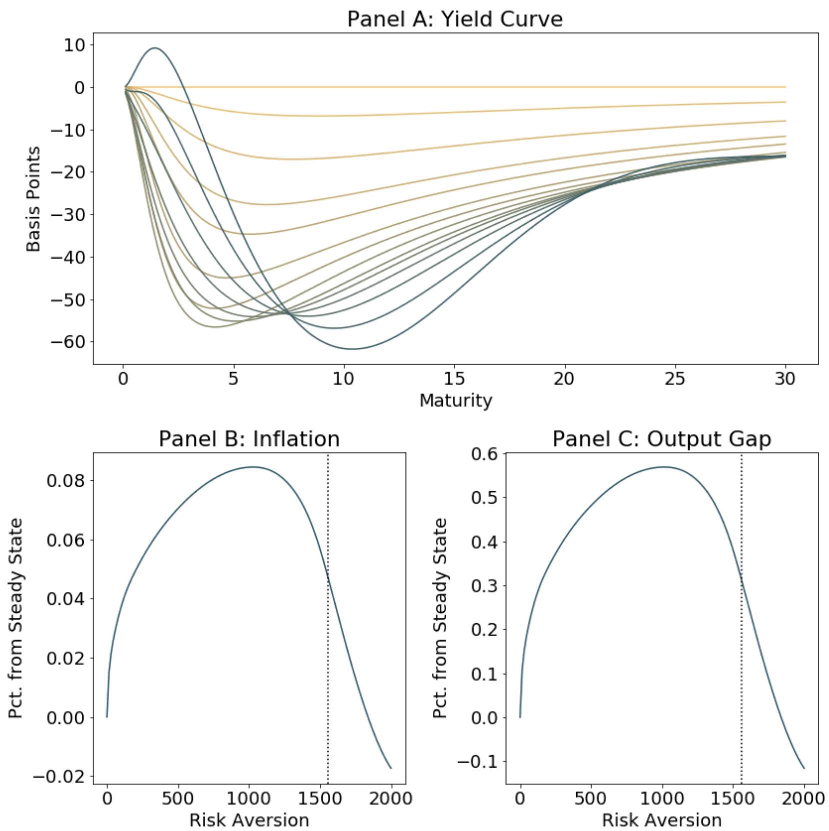

Figure 11: QE Shock (long-term purchases)

Notes: Panel A is the contemporaneous response of the yield curve to a QE shock, where purchases are concentrated at longer term bonds. Responses are plotted as deviations from steady state, in terms of basis points. The x-axis is maturity. Lighter lines correspond to models where risk aversion is low; darker lines to models with high risk aversion. Panels B and C are the contemporaneous response of inflation and the output gap to the same shock. Responses are plotted as deviations from steady state, in terms of percentage points. The x-axis is level of risk aversion. The dotted black line corresponds to the estimated level of risk aversion.

in the returns of their portfolio. This implies that there is very little reaction in the yield curve, hence there is little feedback to household borrowing. But as risk aversion increases, the effects increase. Again this is due to the behavior of arbitrageurs. QE purchases offload some risk from the portfolio of arbitrageurs. Hence, they require smaller excess returns to hold other bonds, which pushes down interest rates. This leads to a decline in the effective borrowing rate of households, which boosts consumption (and hence output) on impact.

Figure 11 repeats the above exercise for shocks to the long demand factor $\beta_{\ell,t}$. Once again, there is nearly no response when arbitrageurs are close to risk neutral; as risk aversion increases, so too does the magnitude of the response of the yield curve and macroeconomic variables.

The intuition is similar for both "short" and "long" QE shocks, but comparing the differential impacts also reveals interesting results. Studying the yield curve responses in Figures 10 and 11 reveals that there are four “regimes” where the yield curve responses are qualitatively different. The first regime corresponds to very low levels of risk aversion, where QE shocks have essentially no effect. In the second regime, when financial markets are somewhat disrupted, both QE shocks put downward pressure on interest rates but the response of the yield curve to both short-term and long-term purchases is very similar in shape. The response is hump-shaped, with the peak response occurring at shorter to intermediate maturities, and pushes both short- and long-term spot rates in the same direction. Only the magnitude differs, with the long-term shock leading to somewhat larger responses.

However, when risk aversion becomes sufficiently high, the effects become more localized. This is the third regime, when financial markets start to become severely disrupted; this regime corresponds to the level of risk aversion estimated from the QE1 data. In this case, both long and short QE shocks push down all interest rates, but now short-term QE purchases have larger effects on short-term yields than long-term QE purchases. Finally, the fourth regime corresponds to extreme levels of financial crisis (higher than what was estimated). The localization of demand shocks becomes so extreme that long-term demand shocks actually lead to increases in short-term rates, while short-term demand shocks also put upward pressure on long-term rates.

The increasing localization of demand shocks is a key partial equilibrium result from Vayanos and Vila (2009); this exercise shows that the finding holds in general equilibrium. What causes this localization? Intuitively, consider the limiting case when arbitrageurs only want to minimize the variance of the change in their wealth. Hence they allocate their entire portfolio to the short (risk-free) rate, taking no positions in any other longer-term bonds. In response to a long-term demand shock from preferred habitat investors, arbitrageurs would be unwilling to make any changes their portfolio allocation to accomodate the shift. The only way for the net supply condition to be satisfied is if the prices of the bonds that are affected by the demand shock respond. Moreover, these are the only bonds that see price changes. In other words, the demand shock has only local effects.

Of course, even when arbitrageurs are very risk-averse, the bond market will not exhibit this type of extreme segmentation. But the qualitative behavior of the term structure response is different for low and high levels of risk aversion. When risk aversion is low, demand shocks have a (small) effect, but the location of the shock does not matter. A the key source of risk that arbitrageurs are concerned with is short-rate risk. Every bond has some sensitivity to this source of risk, hence shifts in demand from preferred habitat investors change the portfolio allocations of arbitrageurs. Thus this changes the market price of short-rate risk, and as this impacts all bonds this has a global effect regardless of where the demand shock originates. For example, in response to positive demand shifts, arbitrageurs sell bonds, reducing their exposure to short-rate risk. Hence they require lower expected returns to hold bonds, pushing down rates. The bonds that respond the most are those most sensitive to short-rate risk, which does not depend on the location of the shock, but only on the stochastic (mean-reversion) properties of the shocks.

But when risk aversion is very high, the location of the shock matters. When risk aversion is very high and there are multiple sources of risk (short-rate risk, multiple demand factors, and other structural shocks), arbitrageurs try to limit their exposure to these sources of risk. This means that demand shocks must have mostly local effects, since otherwise arbitrageurs would be exposing themselves to other sources of risk. In other words, if a demand shock for long-term debt had big effects on short-term debt, this would be because arbitrageurs were making large changes to their holdings of short-term debt. But this changes their exposure to other sources of risk. Hence they choose not to integrate the markets across maturities, and the shocks stay more localized.

5.3.3 Operation Twist Shock

Next, I look at a policy mimicking "Operation Twist," where the Federal Reserve purchased long-term securities and sold short-term securities. To model this, I analyze the effect of a simultaneous increase in long-term purchases ($\beta_{\ell,t}$) and decrease in short-term purchases ($\beta_{s,t}$). Figure 12 plots the yield curve and real responses.

The previous QE exercises showed that when risk aversion is low, the yield curve reacts similarly to the short and long demand factors; the only difference is the magnitude of the response to the long demand factor is larger. When the two shocks are combined but with opposite signs, this implies that the long demand factor dominates, leading to a decline in the term structure. The macroeconomic response is thus a similar but muted version of the QE responses observed previously.

Recall that in the calibration, the effective borrowing rate is weighted mostly towards maturities between 1 and 5 years; there is little weight on interest rates 10 years and above. However, when financial markets are relatively healthy ("regime 2" discussed above), it is still the case that QE purchases concentrated at long-term rates is more effective at push up output than short-term purchases.

But as can be seen in Figure 12, when risk aversion is high enough, the short and long demand factors have differential impacts on the yield curve. The combination of the two shocks leads to declining long-term rates, but at some point leads to increasing short-term rates. For intermediate values of risk aversion, real effective borrowing rates still fall, leading to increases in output and inflation. However, as risk aversion continues to increase, the magnitude of the macroeconomic effects declines. For high values of risk aversion, the sign eventually flips: after the Operation Twist shock, both inflation and the output gap actually decline.

The exercise implies that if Operation Twist had taken place during March 2009 when risk aversion was very high (the dotted black line in the bottom panels of Figure 12),

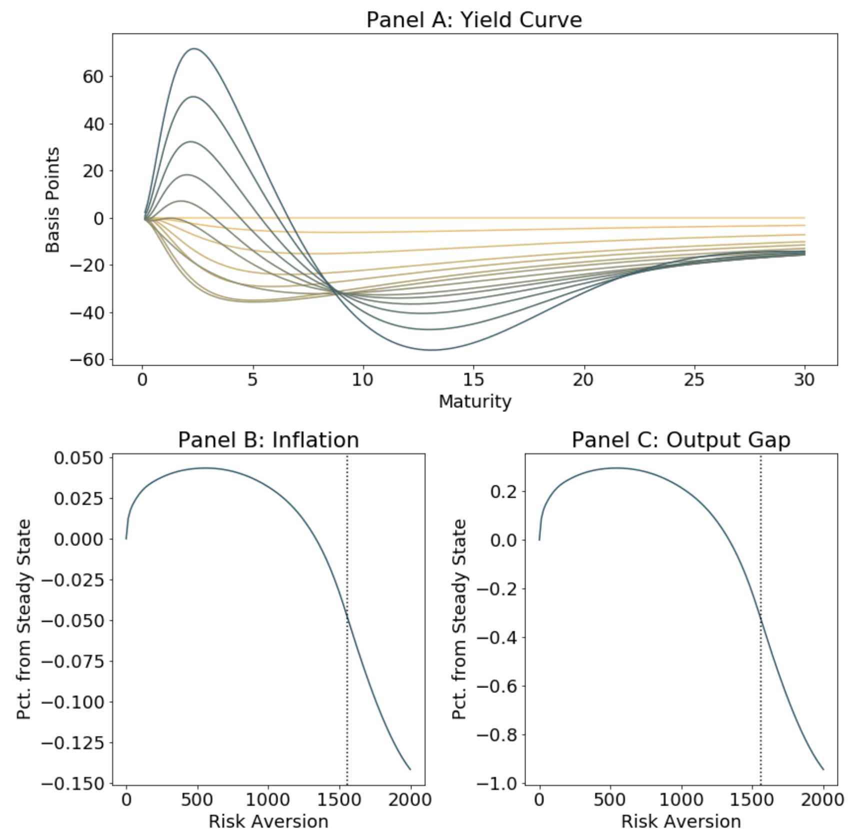

Figure 12: QE Shock (long-term purchases)

Notes: Panel A is the contemporaneous response of the yield curve to an Operation Twist shock, where longer term bonds are purchased and shorter term bonds are sold. Responses are plotted as deviations from steady state, in terms of basis points. The x-axis is maturity. Lighter lines correspond to models where risk aversion is low; darker lines to models with high risk aversion. Panels B and C are the contemporaneous response of inflation and the output gap to the same shock. Responses are plotted as deviations from steady state, in terms of percentage points. The x-axis is level of risk aversion. The dotted black line corresponds to the estimated level of risk aversion.

the aggregate effect would have actually been contractionary. Moreover, the actual yield curve responses around September 21, 2011 (when Operation Twist was announced by the FOMC) saw long-term rates fall while short-term rates increased slightly. While financial market disruptions in 2011 had subsided relative to the peak of the crisis, this suggests that financial frictions were still high. Hence, the aggregate effects of Operation Twist were likely muted at best.

One key result in all of these exercises is that the transmission of these policies depends crucially on the health of financial markets. As with conventional monetary policy explored in previous sections, the effectiveness of QE policies varies with the level of risk aversion of arbitrageurs. But the interaction is the opposite of conventional policy. QE has no effect when arbitrageurs are risk neutral; as risk aversion increases, QE becomes more and more effective at moving the yield curve and hence boosting output.

Policies like Operation Twist may have more ambiguous effects. When financial markets are highly disrupted, bond markets become much more segmented. Targeted buying and selling of securities throughout the term structure will have highly localized effects. Depending on which maturities are the most important for household borrowing, this could end up having the opposite of the intended effect, leading to falling consumption.

5.4 Can LSAPs Be Stabilizing? Endogenous QE Rules and Determinacy

The results above show that, under the right financial conditions, QE can move output and inflation. A separate question is: can the central bank use QE to achieve determinacy and stabilize the economy?

Suppose that the central bank conducts QE operations in a similar manner to how it sets the policy rate. I study this by modifying the idiosyncratic demand factor from eq. (19) so that it endogenously reacts to inflation:

$$\mathrm{d}\beta_t = -\kappa_\beta \left(\beta_t - \phi_\pi^\beta \pi_t\right) \mathrm{d}t \,. \tag{33}$$

To simplify, I assume this is the only demand factor. Further, I assume that the central bank does not react to the output gap ($\phi_x = 0$). This will imply that the model moves towards the region of indeterminacy for intermediate to high levels financial frictions. However, by choosing $\phi_\pi^\beta > 0$, it is possible to re-establish determinacy.

Figure 13 shows how the determinacy condition varies with different levels of risk aversion (a) and QE responsiveness to inflation (ϕ_π^β). When risk aversion is very low, the model is determinate for all values of ϕ_π^β. Moreover, changes in the responsiveness to either inflation or output have no effect on the determinacy condition (the value of the unstable eigenvalues). As risk aversion increases, the baseline model with no endogenous QE responses quickly moves towards the region of indeterminacy. But allowing for a stabilizing QE policy that reacts to inflation ($\phi_\pi^\beta > 0$) moves the model back towards the region of determi-

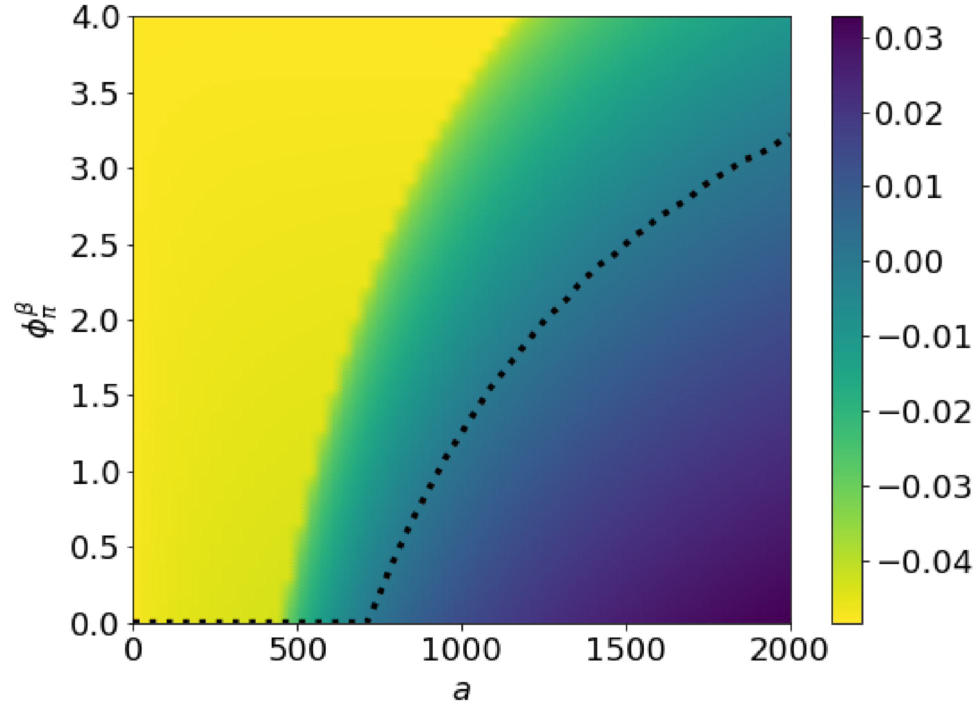

Figure 13: Regions of Determinacy

Notes: the heatmaps show the region of determinacy (lighter regions, upper left) and indeterminacy (darker regions, lower right). The x-axis is the level of risk aversion a, and the y-axis is the level of the parameter ϕ_π^β, which governs how strongly QE reacts to inflation. The values correspond to the smallest unstable (real) eigenvalue; the model is determinate when this is negative. The dotted black line delineates the region of indeterminacy from determinacy.

nacy. For strong enough responses of QE to inflation, the model regains determinacy.

Intuitively, an endogenous QE rule works by picking up the slack left by conventional policy. When financial markets are healthy, a standard Taylor rule is stabilizing. As financial frictions increase, the pass-through of conventional policy deteriorates. If this is the only policy rule, then eventually the model becomes unstable. However, as seen above, QE becomes more effective as financial frictions increase. Hence, just as conventional policy is failing, the endogenous QE rule becomes more and more effective.

5.5 Optimal Policy

A full treatment of a planner's optimal policy problem is beyond the scope of this paper. Allowing for the central bank to choose a path of the policy rate that differs from the Taylor-type rules considered in this paper will in general lead to bond prices that are not affine functions of the state variables. Instead, I consider a simpler optimal policy problem. This section analyzes how the central bank would choose to set the response to inflation ϕ_π and the inertia term κ_r optimally, as a function of financial market health (and further simplifies the problem by assuming no response to output: $\phi_x = 0$). I assume a quadratic loss function for the planner, given by

$$\min_{\phi_\pi, \kappa_r} E_0 \int_0^\infty e^{-\rho t} \left(w_\pi \pi_t^2 + w_x x_t^2 \right) \mathrm{d}t$$

where $w_\pi \geq 0$ and $w_x \geq 0$ are the weights that the planner assigns to inflation and output, respectively. Appendix C.4 derives the general expression for this present discounted value of future second moments of the jump variables in this class of models.

Figure 14 plots the optimal coefficients ϕ_π (Panel A) and κ_r (Panel B) as a function of risk aversion. I suppose that the planner cares more about inflation relative to output by setting the planner weights as $w_\pi = 1, w_x = 0.02$.[8]

The optimal response to inflation is increasing (higher ϕ_π) as arbitrageur risk aversion increases (higher a). Further, optimal inertia is also increasing (lower κ_r). Since monetary policy relies on the pass-through provided by financial markets, as the risk-bearing capacity of arbitrageurs becomes disrupted, monetary policy efficacy is weakened. Moreover, as financial market disruptions increase, the economy becomes less stable and the output and inflation responses to these shocks increase. In order to effectively stabilize the economy (minimize the volatility of inflation and output), the optimal response becomes more agressive and longer-lasting.

[8] Note that in a full optimal policy experiment, the quadratic loss function is an approximation to the welfare loss function, and the planner weights would be derived from this welfare function.

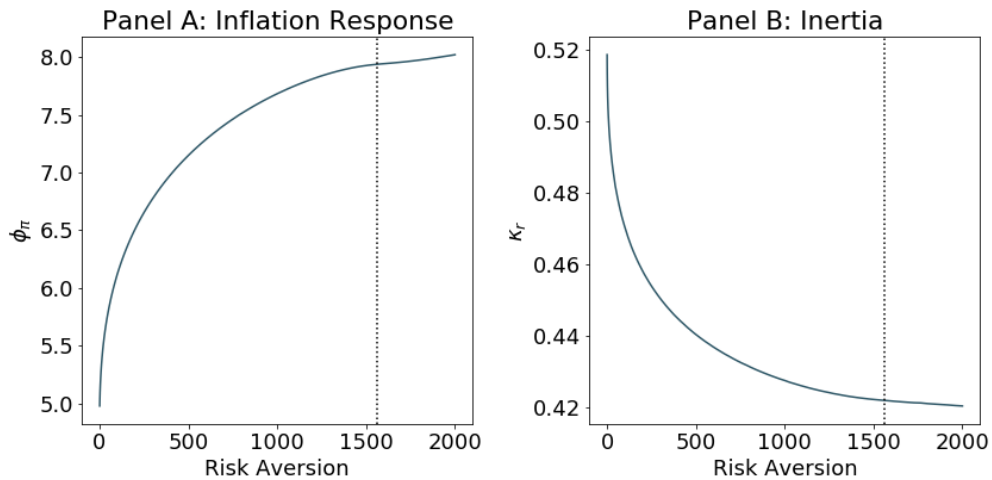

Figure 14: Optimal Response to Shocks

Notes: optimal policy coefficients as risk aversion increases. The planner weights are set to $w_\pi = 1, w_x = 0.02$. Panel A plots the optimal Taylor rule coefficient on inflation (ϕ_π), while Panel B plots the optimal inertia term (κ_r). The response Taylor rule coefficient on output (ϕ_x) is set to 0. The dotted black line corresponds to the estimated value of risk aversion.

6 Concluding Remarks

This paper studies conventional and unconventional monetary policy through the lens of a general equilibrium "preferred habitat" New Keynesian model. When output depends on both short- and long-term borrowing rates, the transmission of monetary policy to the macroeconomy depends on imperfect financial markets. As a result, standard monetary policy becomes less effective when financial markets are disrupted. For the same reason, the efficacy of forward guidance is weakened. However, financial crises open the door to other unconventional policies such as quantitive easing, which can push down long-term rates and stabilize output and inflation.

The framework considered in this paper suggests promising avenues for future work. The results relating financial health and the determinacy of the model are important for understanding how policymakers can achieve macroeconomic stability. The model suggests revisiting the empirical work studying the stability properties of central bank policy rules, as the requirements for stability are state-dependent. Furthermore, the findings provide a possible justification for macro-prudential policies. The model assumes that risk aversion is fixed, but in reality the risk-bearing capacity of financial markets is endogenous. When this capacity is too low, the result is macroeconomic instability. If investors in financial markets do not internalize this, then there is an externality that can be addressed by policy. Extending the model to allow for endogenous risk-bearing capacity is useful for understanding how such macro-prudential policies should be carried out.

The model can also be extended to allow for a more realistic treatment of household borrowing. For example, the model does not allow for default risk, but in reality the key borrowing rates for households will not be default-free. Moreover, a major part of some QE programs was targeting mortgage-backed securities. The localization results suggest that during financial disruptions, LSAP programs will be most effective when they target borrowing markets in which households are most active. But the sensitivity of real activity to different borrowing rates, and the stability of this relationship over time, is an open empirical question.

This paper shows that LSAPs should be a tool in central bankers' arsenal to stabilize the economy during financial crises. However, the predictions of the model are not only dependent on the health of financial markets but also sensitive to interactions with the location of the purchases in maturity space, how long the purchases last, how sensitive the real economy is to long-term rates, and the structure of the other fundamental shocks in the economy. Policymakers are likely to face a great deal of uncertainty about these factors, which raises the possibility that a better approach may be to target specific long-term rates and allow for flexibility in the open-market operations used to achieve these targets. More generally, the model should serve as the basis for more quantitative analysis in order to design optimal policies.

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Appendix A Proofs

Proof of Lemma 1. The results follow from analyzing the characteristic polynomial of Υ:

$$c(\lambda) = \lambda^2 - \kappa_r \lambda - \varsigma^{-1} \kappa_r \phi_x \hat{A}_r$$

1. Note that $c(\lambda)$ is increasing iff $\lambda > \frac{1}{2}\kappa_r$. Additionally, $c(\lambda) \to \infty$ whenever $\lambda \to \pm\infty$. Note

   $$c(0) = c(\kappa_r) = -\varsigma^{-1} \kappa_r \phi_x \hat{A}_r$$

   Hence if $\hat{A}_r > 0$, the expression above is negative which implies $c(\lambda)$ has two roots: $\lambda_1 > \kappa_r$ and $\lambda_2 < 0$.

2. Setting $c(\lambda)$ to zero and solving for $\hat{A}_r$ gives eq. (11). Note that $h(\lambda)$ is a positive quadratic with zeros at $\lambda = 0$ and $\lambda = \kappa_r$; for $\lambda \in (0, \kappa_r)$, $h(\lambda)$ is negative.

3. The eigenvector associated with λ_1 is

   $$q_1 \equiv \begin{bmatrix} -\frac{\lambda_1}{\varsigma^{-1}\hat{A}_r} \\ 1 \end{bmatrix}$$

   In this case Ω, the matrix governing the equilibrium dynamics of the jump variables in eq. (10), is simplify a scalar, given by $\omega_x \equiv \frac{q_{21}}{q_{11}}$. Substituting the solution for $\hat{A}_r$ gives the result.

□

Proof of Lemma 2. This result is the scalar counterpart of the optimality conditions derived in Prop. 7. The expected instantaneous return simplifies to

$$\mu_{t,\tau} = A_r'(\tau) r_t + C'(\tau) + \lambda(r_t - r^{SS}) A_r(\tau) + \frac{1}{2}\sigma_r^2 A_r(\tau)^2 \tag{A1}$$

□

Proof of Lemma 3. The functional forms of $A_r(\tau)$, $C(\tau)$, and ν are again the scalar analogues of the results in Prop. 7. In this case, $\mathbf{M} \equiv \nu$ and simplifies to eq. (16) so $C(\tau)$ simplifies to

$$C(\tau) = n_1(\tau) Z_C - \frac{1}{2}\sigma_r^2 n_2(\tau) \tag{A2}$$

where

$$
\begin{aligned}
Z_C &= \frac{a\sigma_r^2 N_2 + \lambda r^{SS}}{1 + a\sigma_r^2 N_1} \\
n_1(\tau) &= \int_0^{\tau} A_r(u)\,\mathrm{d}u \\
n_2(\tau) &= \int_0^{\tau} A_r(u)^2\,\mathrm{d}u \\
N_1 &= \int_0^{T} \alpha(\tau) A_r(\tau) n_1(\tau)\,\mathrm{d}\tau \\
N_2 &= \int_0^{T} \alpha(\tau) A_r(\tau) \left(\bar{\beta}(\tau)\tau + \frac{1}{2}\sigma_r^2 n_2(\tau)\right) \mathrm{d}\tau
\end{aligned}
$$

If $a = 0$ it immediately follows that $\nu = \lambda$, so throughout assume that $a > 0$. In this case, solving for ν is a fixed point problem. Since $\alpha(\tau) > 0\ \forall \tau$, this implies

$$
a\sigma_r^2 \int_0^{T} \alpha(\tau)\tau^2 f(\nu\tau)^2\,\mathrm{d}\tau > 0
$$

Hence, the right-hand-side of eq. (16) is strictly greater than λ.

Next, note

$$
\frac{\partial f(\nu\tau)^2}{\partial \nu} = 2\tau f(\nu\tau) f'(\nu\tau)
$$

$f(\cdot)$ is a strictly decreasing, convex function, approaching 0 as $x \to \infty$. So the above expression is negative. This implies the right-hand-side of eq. (16) is strictly decreasing in ν. Further, as $\nu \to \infty$, if $\tau > 0$ then $f(\nu\tau) \to 0$. Thus

$$
\int_0^{T} \alpha(\tau)\tau^2 f(\nu\tau)^2\,\mathrm{d}\tau \to 0
$$

which implies the right-hand-side of eq. (16) approaches λ as $\nu \to \infty$.

Hence there exists a unique $\nu \in (\lambda, \infty)$ such that eq. (16) is satisfied. Treating ν as a function of λ, I derive some additional properties. Recall $\nu(\lambda) > \lambda$. Further,

$$
\begin{aligned}
\frac{\partial f(\nu\tau)^2}{\partial \lambda} &= 2\tau f(\nu\tau) f'(\nu\tau) \frac{\partial \nu}{\partial \lambda} \\
\implies \frac{\partial \nu}{\partial \lambda} &= \left(1 - 2a\sigma_r^2 \int_0^{T} \alpha(\tau)\tau^3 f(\nu\tau) f'(\nu\tau)\,\mathrm{d}\tau\right)^{-1}
\end{aligned}
$$

Also note $\alpha(\tau)\tau^3 f(\nu\tau) \geq 0$ and $f'(\nu\tau) \leq 0$, so

$$
0 < \frac{\partial \nu}{\partial \lambda} < 1
$$

Fixing λ, note that

$$\frac{\partial \nu}{\partial a} = \frac{\sigma_r^2 \int_0^T \alpha(\tau)\tau^2 f(\nu\tau)^2 \,\mathrm{d}\tau}{1 - 2a\sigma_r^2 \int_0^T \alpha(\tau)\tau^3 f(\nu\tau) f'(\nu\tau)\,\mathrm{d}\tau}$$

Hence ν is increasing in a.

Finally, $\hat{A}_r$ is given by

$$\hat{A}_r \equiv \int_0^T \frac{\eta(\tau)}{\tau} A_r(\tau)\,\mathrm{d}\tau$$

eq. (17) is obtained by substituting eq. (15). If $\eta(\tau)$ is the Dirac delta function, then $g(\lambda) = 1\ \forall\lambda$. Otherwise,

$$g'(\lambda) = \int_0^T \eta(\tau)\tau f'(\nu\tau)\,\mathrm{d}\tau \,\frac{\partial \nu}{\partial \lambda}$$

hence $g'(\lambda) < 0$ and additionally $g(\lambda) \to 0$ as $\lambda \to \infty$.

□

Proof of Prop. 1. In general equilibrium, λ_1 and $\hat{A}_r$ are determined by the intersection of eqs. (17) and (11). Recall from the proof of Lemma 3, for $\lambda > 0$, $g(\lambda)$ is strictly positive, decreasing, and approaches 0 as $\lambda \to \infty$. From the proof of Lemma 1, $h(\lambda)$ is negative for $0 < \lambda < \kappa_r$, but is strictly increasing and grows without bound when $\lambda \geq \kappa_r$. Hence there exists a unique $\lambda_1 > \kappa_r > 0$ such that $g(\lambda_1) = h(\lambda_1)$. Figure 2 plots examples of the intersection of g and h.

Given any value of r^*, the steady state is

$$r^{SS} = \frac{\bar{r} - \hat{C}}{\hat{A}_r}$$
$$x^{SS} = \frac{r^* - r^{SS}}{\phi_x}$$

□

Proof of Corollary 1.1. Recall from Prop. 1 that $\hat{A}_r$ (and the associated eigenvalue λ_1) are determined by the intersection of a downward sloping curve $g(\lambda)$ and an upward sloping (in the neighborhood of the intersection λ_1) curve $h(\lambda)$. And while these curves depend on the parameters of the model, the parameter dependence is disjoint: $g(\cdot)$ only depends on the parameters governing the term structure side of the model, while $h(\cdot)$ depends on the macro parameters. This greatly simplifies studying comparative statics.

1. The proof of Lemma 3 showed that ν is increasing in a. Then

$$\frac{\partial g(\lambda)}{\partial a} = \int_0^T \eta(\tau) f'(\nu\tau)\,\mathrm{d}\tau\,\frac{\partial \nu}{\partial a}$$

which implies that for any given λ, an increase in a leads to a downward shift in $g(\lambda)$. Since $h(\lambda)$ is unchanged, this implies that the point of intersection shifts downward and to the left.

Further, note that as $a \to \infty$, $\nu \to \infty$. To see why, suppose instead that $\nu \to \nu^* < \infty$. Then $f(\nu\tau) \to f(\nu^*\tau) > 0$. But then

$$a\sigma_r^2 \int_0^T \alpha(\tau)\tau^2 f(\nu\tau)^2\,\mathrm{d}\tau \to \infty$$

and therefore the right-hand side eq. (16) grows without bound, a contradiction. This immediately implies that $f(\nu\tau) \to 0$ and therefore so too does $g(\lambda)$ for $\lambda > 0$. As before, this says that the point of intersection of g and h continues shifting downward and to the left until $\hat{A}_r = 0$ and $\lambda_1 = \kappa_r$.

Note that if $a = 0$, $g(\lambda)$ is independent of changes in σ_r. But when $a \neq 0$, the same arguments above show that

$$\frac{\partial \hat{A}_r}{\partial \sigma_r} < 0, \quad \frac{\partial \lambda_1}{\partial \sigma_r} < 0$$
$$\lim_{\sigma_r \to \infty} \hat{A}_r = 0, \quad \lim_{\sigma_r \to \infty} \lambda_1 = \kappa_r$$

2. Differentiating eq. (11) with respect to κ_r gives

$$-\frac{\lambda^2}{\varsigma^{-1}\kappa_r^2\phi_x} < 0 \;\; \forall \lambda$$

Hence the intersection shifts down and to the right.

3. Differentiating eq. (11) with respect to ϕ_x gives

$$-\frac{\lambda(\lambda - \kappa_r)}{\varsigma^{-1}\kappa_r\phi_x^2}$$

which is negative whenever $\lambda > \kappa_r$. Hence the intersection shifts down and to the right.

4. Note for arbitrary functions f and g, where f and g satisfy the following:

$$f(x) > 0, \quad f'(x) < 0$$
$$g(x) > 0 \iff x < x^*, \quad \int g(x)\,\mathrm{d}x = 0$$

Then $\int f(x)g(x)\,\mathrm{d}x > 0$. So set $f(\nu\tau)$ as f, and $\eta^s(\tau) - \eta^\ell(\tau)$ as g, which gives

$$\int_0^T (\nu\tau)\eta^s(\tau)\,\mathrm{d}\tau > \int_0^T (\nu\tau)\eta^\ell(\tau)\,\mathrm{d}\tau$$

which says that $g^s(\lambda) > g^\ell(\lambda)$ (using the notation from a previous Lemma). In other words, g shifts down moving from the η^s weights to the η^ℓ weights. Since h is unchanged, the equilibrium results follow.

□

Proof of Corollary 1.2. The expression for the optimal target eq. (18) follows from the steady state results derived in the proof of Prop. 1.

Next, consider demand functions $\bar{\beta}^h(\tau) \geq \bar{\beta}^h(\tau)$. Letting superscripts h and ℓ denote the equilibrium outcomes under $\bar{\beta}^h(\tau)$ and $\bar{\beta}^\ell(\tau)$, note that the coefficient function $A_r^h(\tau) = A_r^\ell(\tau)$, so the only difference comes about due to changes in the constant function. Then note that

$$\begin{aligned}
\hat{C}^h - \hat{C}^\ell &= -\hat{A}_r(r^{*h} - r^{*\ell}) \\
\implies r^{*h} - r^{*\ell} &= -\frac{a\sigma_r^2\hat{n}_1(N_2^h - N_2^\ell)}{\lambda\hat{n}_1 + \hat{A}_r(1 + a\sigma_r^2 N_1)}
\end{aligned}$$

where the second line follows from eq. (A2), and the expressions defined in the proof of Lemma 3. Then, since $\bar{\beta}^h(\tau) \geq \bar{\beta}^\ell(\tau)$,

$$\begin{aligned}
N_2^h - N_2^\ell = \int_0^T \alpha(\tau)\tau A_r(\tau)\left(\bar{\beta}^h(\tau) - \bar{\beta}^\ell(\tau)\right)\mathrm{d}\tau > 0 \\
\implies r^{h*} \leq r^{\ell*}
\end{aligned}$$

where the inequalities are strict whenever $a > 0$.

□

Proof of Prop. 2. To solve for x_0, for $t \leq t^\diamond$ the dynamics of the policy rate under the peg imply that

$$\begin{aligned}
E_0 r_t &= r^\diamond + e^{-\kappa_r^\diamond t}(r_0 - r^\diamond) \\
\implies E_0 r_{t^\diamond} &= r^\diamond
\end{aligned}$$

In order to satisfy transversality conditions, once the central bank returns to a Taylor rule, the output gap must satisfy $x_{t^\diamond} = \omega_x(r_{t^\diamond} - r^{SS})$. Thus,

$$E_0 x_{t^\diamond} = \omega_x(r^\diamond - r^{SS})$$

Solving for $E_0 x_{t^\diamond}$ given x_0 gives

$$E_0 x_{t^\diamond} = x_0 + t^\diamond \varsigma^{-1}(\hat{A}_r^\diamond r^\diamond + \hat{C}^\diamond - \bar{r})$$

Hence,

$$\begin{aligned}\frac{\partial x_0}{\partial r^\diamond} &= \omega_x - t^\diamond \varsigma^{-1} \hat{A}_r^\diamond \\ \frac{\partial^2 x_0}{\partial r^\diamond \partial t^\diamond} &= -\varsigma^{-1} \hat{A}_r^\diamond\end{aligned}$$

Then the result follows since

$$\frac{\partial \hat{A}_r^\diamond}{\partial a} < 0, \quad \left|\frac{\partial \omega_x}{\partial a}\right| < 0$$

□

Proof of Lemma 4. Following the same steps as in Lemma 1:

$$\hat{A}_r = \frac{\lambda_1(\lambda_1 - \kappa_r)}{\varsigma^{-1}\kappa_r \phi_x}$$

Substituting this into Υ, the eigenvalue decomposition is

$$\Lambda = \begin{bmatrix}\lambda_1 & 0 & 0 \\ 0 & \kappa_\beta & 0 \\ 0 & 0 & -\lambda_1 + \kappa_r\end{bmatrix}$$

$$Q = \begin{bmatrix}\frac{\kappa_r \phi_x}{-\lambda_1 + \kappa_r} & \frac{\kappa_r \phi_x}{\kappa_r - \kappa_\beta} & \frac{\kappa_r \phi_x}{\lambda_1} \\ 0 & \frac{\kappa_\beta^2 - \kappa_\beta \kappa_r - \lambda_1^2 + \lambda_1 \kappa_r}{(\kappa_r - \kappa_\beta)\varsigma^{-1}\hat{A}_\beta} & 0 \\ 1 & 1 & 1\end{bmatrix}$$

Solving for the rational expectations equilibrium gives

$$\Gamma = \begin{bmatrix}\lambda_1 & \frac{\kappa_r \phi_x \varsigma^{-1}\hat{A}_\beta}{\lambda_1 - \kappa_r + \kappa_\beta} \\ 0 & \kappa_\beta\end{bmatrix}$$

$$\Omega = \begin{bmatrix}\frac{-\lambda_1 + \kappa_r}{\kappa_r \phi_x} & \frac{-\varsigma^{-1}\hat{A}_\beta}{\lambda_1 - \kappa_r + \kappa_\beta}\end{bmatrix}$$

□

Proof of Lemma 5. This follows from the general case in Prop. 7, where

$$\Gamma = \begin{bmatrix}\gamma_1 & \gamma_{12} \\ 0 & \gamma_2\end{bmatrix}, \quad \Sigma = \begin{bmatrix}\sigma_r^2 & 0 \\ 0 & 0\end{bmatrix} \implies \mathbf{M} = \begin{bmatrix}\nu & 0 \\ \nu_{12} & \gamma_2\end{bmatrix}$$

and the elements of $\mathbf{M}$ are given by

$$\begin{aligned}
\nu &= \gamma_1 + a\sigma_r^2 \int_0^T \alpha(\tau) A_r(\tau)^2 \,\mathrm{d}\tau \\
\nu_{12} &= \gamma_{12} + a\sigma_r^2 \int_0^T \alpha(\tau) \left[A_\beta(\tau) - \tau\theta(\tau)\right] A_r(\tau) \,\mathrm{d}\tau
\end{aligned}$$

The eigenvector decomposition gives

$$\mathbf{D} = \begin{bmatrix} \nu & 0 \\ 0 & \gamma_2 \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 1 & 0 \\ \frac{\nu_{12}}{\nu - \gamma_2} & \frac{\nu_{12}}{\gamma_2 - \nu} \end{bmatrix}$$

and thus

$$\begin{aligned}
& \mathbf{A}(\tau) = \mathbf{G}\mathbf{D}^{-1} \left[\mathbf{I} - \exp(-\mathbf{D}\tau)\right] \mathbf{1} \\
\implies & A_r(\tau) = \tau f(\nu\tau) \\
& A_\beta(\tau) = \frac{\nu_{12}}{\nu - \gamma_2} \tau (f(\nu\tau) - f(\gamma_2\tau))
\end{aligned}$$

Hence, rewriting the expression for ν gives

$$\nu = \gamma_1 + a\sigma_r^2 \int_0^T \alpha(\tau) \tau^2 f(\nu\tau)^2 \,\mathrm{d}\tau$$

which is equivalent to eq. (16).

Substituting the solution for $A_\beta(\tau)$ into the expression for ν_{12} and solving gives

$$\nu_{12} = \frac{\gamma_{12} - a\sigma_r^2 \int_0^T \alpha(\tau)\tau^2 f(\nu\tau)\theta(\tau)}{1 - \frac{a\sigma_r^2}{\nu - \gamma_2} \int_0^T \alpha(\tau)\tau^2 (f(\nu\tau) - f(\gamma_2\tau)) \,\mathrm{d}\tau} \tag{A3}$$

Thus, integrating and weighting by $\frac{\eta(\tau)}{\tau}$ gives

$$\begin{aligned}
\hat{A}_r &= \int_0^T \eta(\tau) f(\nu\tau) \,\mathrm{d}\tau \\
\hat{A}_\beta &= \frac{\nu_{12}}{\nu - \gamma_2} \int_0^T \eta(\tau)(f(\nu\tau) - f(\gamma_2\tau)) \,\mathrm{d}\tau
\end{aligned}$$

and hence, since the fixed point problem that solves ν is equivalent to the baseline rigid price model, so is the expression for $\hat{A}_r$.

□

Proof of Prop. 3. 1. Combining the coefficients used in Lemmas 4 and 5:

$$\begin{aligned}
\gamma_1 &= \lambda_1 \\
\gamma_2 &= \kappa_\beta \\
\gamma_{12} &= \frac{\kappa_r \phi_x \varsigma^{-1} \hat{A}_\beta}{\lambda_1 - \kappa_r + \kappa_\beta} \\
&= \frac{\kappa_r \phi_x \varsigma^{-1}}{\lambda_1 - \kappa_r + \kappa_\beta} \left(\frac{\nu_{12}}{\nu - \gamma_2} \int_0^T \eta(\tau)(f(\nu\tau) - f(\gamma_2\tau))\,\mathrm{d}\tau \right)
\end{aligned}$$

Substituting these expressions into eq. (A3) and rearranging gives

$$\nu_{12} = \frac{-a\sigma_r^2 N_1}{1 - \frac{\phi_x \kappa_r \varsigma^{-1}}{(\lambda_1 - \kappa_r + \kappa_\beta)(\nu - \kappa_\beta)} N_2 - \frac{a\sigma_r^2}{\nu - \kappa_\beta} N_3} \tag{A4}$$

where N_1, N_2, N_3 are integral expressions:

$$\begin{aligned}
N_1 &= \int_0^T \alpha(\tau)\tau^2 f(\nu\tau)\theta(\tau)\,\mathrm{d}\tau \\
N_2 &= \int_0^T \eta(\tau)(f(\nu\tau) - f(\kappa_\beta\tau))\,\mathrm{d}\tau \\
N_3 &= \int_0^T \alpha(\tau)\tau^2(f(\nu\tau) - f(\kappa_\beta\tau))\,\mathrm{d}\tau
\end{aligned}$$

First, focusing on the denominator of eq. (A4),

$$\nu > \kappa_\beta \iff f(\nu\tau) < f(\kappa_\beta\tau)$$

hence the denominator is strictly positive. Then, since by assumption $\theta(\tau) \geq 0$, the numerator of eq. (A4) is negative (strictly if $a > 0$). Therefore, $\nu_{12} \leq 0$ and $\hat{A}_\beta \geq 0$ (with strict inequalities when $a > 0$).

2. Taking the limit as $\kappa_\beta \to \infty$, from eq. (A4) the limiting value of ν_{12} is

$$\nu_{12} \to -a\sigma_r^2 \int_0^T \alpha(\tau)\tau^2 f(\nu\tau)\theta(\tau)$$

Since ν is independent of κ_β, this limit is bounded. Further, $f(\kappa_\beta\tau) \to 0$ as well. Hence, taking the limit of eq. (22) gives

$$\frac{\nu_{12}}{\nu - \kappa_\beta} \int_0^T \eta(\tau)(f(\nu\tau) - f(\kappa_\beta\tau))\,\mathrm{d}\tau \to 0$$

$\square$

Proof of Lemma 6. 1. The characteristic polynomial of Υ is

$$c(\lambda) \equiv \lambda^3 - (\kappa_r - \rho)\lambda^2 - \varsigma^{-1}(\hat{A}_r\kappa_r\phi_x + \delta + \kappa_r\rho\varsigma)\lambda - \varsigma^{-1}\kappa_r(\hat{A}_r(\delta\phi_\pi + \rho\phi_x) - \delta)$$

Hence:

$$\begin{aligned}
\lim_{\lambda\to+\infty} c(\lambda) &= +\infty \\
\lim_{\lambda\to-\infty} c(\lambda) &= -\infty \\
c(\kappa_r) &= -\varsigma^{-1}\kappa_r\hat{A}_r(\kappa_r\phi_x + \delta\phi_\pi + \rho\phi_x) < 0
\end{aligned}$$

Hence there is some (real) $\lambda_1 > \kappa_r > 0$ such that $c(\lambda_1) = 0$. For the other two roots, note

$$\begin{aligned}
c'(\lambda) &= 3\lambda^2 - 2(\kappa_r - \rho)\lambda - \varsigma^{-1}(\hat{A}_r\kappa_r\phi_x + \delta + \kappa_r\rho\varsigma) \\
c(0) &= -\varsigma^{-1}\kappa_r(\hat{A}_r(\delta\phi_\pi + \rho\phi_x) - \delta) \\
c'(0) &= -\varsigma^{-1}(\hat{A}_r\kappa_r\phi_x + \delta + \kappa_r\rho\varsigma)
\end{aligned}$$

So $c'(0) < 0$. If $c(0) > 0$, since $c(\kappa_r) < 0$ there will be another value $\lambda_2 \in (0, \kappa_r)$ such that $c(\lambda_2) = 0$. Hence $c(0) < 0$ is necessary for determinacy. This is satisfied iff eq. (24) holds.

To see that this condition is sufficient, since $c'(0) < 0$, $c(\lambda)$ has a local maximum for some $\lambda < 0$, denoted $c(z) = y$. If $y > 0$ then there are two real negative zeros, one less than z and one greater than z. If $y = 0$ then there are two duplicated real negative zeros (equal to z). Finally, if $y < 0$ then there are two conjugate complex zeros. The line that intersects $(\lambda_1, 0)$ and is tangent to $c(\lambda)$ is upward sloping, and is tangent to $c(\lambda)$ at a point (M, H) where $M < z$. The real components of the complex zeros is given by M.

2. Setting $c(\lambda)$ to zero and solving for $\hat{A}_r$ gives eq. (25). The fact that $h(\lambda) \to \infty$ as $\lambda \to \infty$ follows for the sign restrictions on the parameters.

Note that the zeros of $h(\lambda)$ are

$$\left\{\kappa_r,\ \frac{1}{2}\left(-\rho \pm \sqrt{\rho^2 + 4\delta\varsigma^{-1}}\right)\right\}$$

so there are two positive (real) zeros, and one negative (real) zero. Order these $z_1 > z_2 > 0 > z_3$. Then $z_2 + z_3 \leq -\rho < 0$ (equality holds if $z_1 = \kappa_r$). Write

$$h(\lambda) = \frac{(\lambda - z_1)(\lambda - z_2)(\lambda - z_3)}{\varsigma^{-1}\kappa_r\left(\delta\phi_\pi + \rho\phi_x + \lambda\phi_x\right)}$$

Differentiating with respect to λ gives

$$h'(\lambda) = h(\lambda)\left[\frac{1}{\lambda - z_1} + \frac{1}{\lambda - z_2} + \frac{1}{\lambda - z_3} + \frac{-\phi_x}{\delta\phi_\pi + \rho\phi_x + \lambda\phi_x}\right]$$

If $\lambda \in [0, z_2)$, then $h(\lambda) > 0$ while the first and fourth terms in brackets are negative. Then

$$\frac{1}{\lambda - z_2} + \frac{1}{\lambda - z_3} = \frac{2\lambda - (z_2 + z_3)}{(\lambda - z_2)(\lambda - z_3)} < 0$$

since $z_3 < 0 < \lambda < z_2$ by assumption, and $z_2 + z_3 < 0$. Hence for $\lambda \in [0, z_2)$, $h'(\lambda) < 0$ and hence $h(0)$ is the maximum in this range.

Finally, $h(\lambda) > 0$ whenever $\lambda > z_1$. The second and third in brackets are positive, while

$$\frac{1}{\lambda - z_1} - \frac{\phi_x}{\delta\phi_\pi + \rho\phi_x + \lambda\phi_x} = \frac{\delta\phi_\pi + \rho\phi_x + z_1\phi_x}{(\lambda - z_1)(\delta\phi_\pi + \rho\phi_x + \lambda\phi_x)} > 0$$

hence for $\lambda > z_1$, $h(\lambda)$ is a positive strictly increasing function.

Also, note that the condition for determinacy is equivalent to

$$\hat{A}_r > h(0)$$

3. Substituting the expression for $\hat{A}_r$ into Υ and carrying out the eigenvalue decomposition and solving for Ω gives the result

□

Proof of Prop. 4. 1. In general equilibrium, $\hat{A}_r$ is determined by the intersection of eqs. (17) and (25). Recall that for $\lambda \geq z_1$ where z_1 is the largest root of h, h is strictly increasing and grows without bound. Further, g is a positive, strictly decreasing function approaching 0. Hence there exists a unique $\lambda_1 \geq z_1$ such that

$$\hat{A}_r = g(\lambda_1) = h(\lambda_1)$$

2. For uniqueness, since g is defined for positive values only I show that there is no other $\lambda' \in [0, z_1]$ such that $g(\lambda') = h(\lambda')$. From Lemma 6, $h(0) > h(\lambda')$ in this range. Additionally, Lemma 3 says that $g(\lambda)$ is decreasing, so that $g(\lambda_1) > g(\lambda')$. If the model is determinate, then the condition from Lemma 6 gives $g(\lambda_1) > h(0)$. So $g(\lambda') > h(\lambda')$ for all $\lambda' \in [0, z_1]$. Hence $\hat{A}_r$ is unique. Figure 5 plots examples of intersections of g and h.

□

Proof of Corollary 4.1. The proofs are similar to Cor. 1.1.

□

Proof of Corollary 4.2. Since $\delta > 0$, the right hand side of the inequality from eq. (24) is strictly greater than zero. Then since $\hat{A}_r \to 0$ as $a \to \infty$, there is some $\overline{a}$ for which $a > \overline{a}$ implies

$$\hat{A}_r < \frac{\delta}{\delta\phi_\pi + \rho\phi_x}$$

□

Proof of Prop. 5. When the determinacy condition eq. (24) is satisfied, the proof is the same as in Prop. 2. The result follows since

$$\frac{\partial \hat{A}_r^{\diamond}}{\partial a} < 0, \quad \left|\frac{\partial \omega_\pi}{\partial a}\right| < 0, \quad \left|\frac{\partial \omega_x}{\partial a}\right| < 0$$

□

Proof of Lemma 7. The proof is the same as Lemma 4, except that the rational expectations equilibrium matrices are now more complicated. The rational expectations dynamics matrices are given by

$$\Gamma = \begin{bmatrix} \lambda_1 & \gamma_{12} \\ 0 & \kappa_\beta \end{bmatrix} \tag{A5}$$

$$\Omega = \begin{bmatrix} \frac{\delta(-\lambda_1+\kappa_r)}{\kappa_r(\phi_\pi\delta+\phi_x(\lambda_1+\rho)} & \omega_{2,1} \\ \frac{(\lambda_1+\rho)(-\lambda_1+\kappa_r)}{\kappa_r(\phi_\pi\delta+\phi_x(\lambda_1+\rho)} & \omega_{2,2} \end{bmatrix} \tag{A6}$$

where γ_{12}, $\omega_{2,1}$ and $\omega_{2,2}$ are rational functions of the eigenvalue λ_1.

□

Proof of Prop. 6. Given the determinacy condition is satisfied, the the proof is similar to Prop. 3.

□

Proof of Prop. 7. Given the affine functional form assumption, I use Ito's Lemma to compute instantaneous returns. Write $P_{t,\tau}$ explicitly as a function of time t and variables $\mathbf{y}_t$:

$$P(t, \mathbf{y}) = \exp\left\{-\mathbf{y}^T \mathbf{A}(\tau(t)) - C(\tau(t))\right\}$$

Note that the dependence on the first argument t comes through τ in the coefficient functions $\mathbf{A}(\tau)$ and $C(\tau)$. Of course $\frac{\mathrm{d}\tau}{\mathrm{d}t} = -1$, which implies

$$\frac{\partial P}{\partial t} = P_{t,\tau}\left(\mathbf{y}_t^T \mathbf{A}'(\tau) + C'(\tau)\right)$$

The gradient and Hessian with respect to $\mathbf{y}$ are

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{y}} P &= -P_{t,\tau}\mathbf{A}(\tau) \\ \mathrm{H}_{\mathbf{y}} P &= P_{t,\tau}\left(\mathbf{A}(\tau)\mathbf{A}(\tau)^T\right) \end{aligned}$$

Therefore, Ito's Lemma implies the instantaneous return is

$$\frac{\mathrm{d}P_{t,\tau}}{P_{t,\tau}} = \mu_{t,\tau}\,\mathrm{d}t - \mathbf{A}(\tau)^T \mathbf{S}\,\mathrm{d}\mathbf{B}_t \tag{A7}$$

$$\mu_{t,\tau} = \mathbf{y}_t^T \mathbf{A}'(\tau) + C'(\tau) + [\Gamma(\mathbf{y}_t - \overline{\mathbf{y}})]^T \mathbf{A}(\tau) + \frac{1}{2} Tr\left[\mathbf{\Sigma}\mathbf{A}(\tau)\mathbf{A}(\tau)^T\right] \tag{A8}$$

The arbitrageur's optimality conditions are given by:

$$\frac{\partial E_t\,\mathrm{d}W_t}{\partial b_{t,\tau}} = \frac{a}{2}\frac{\partial Var_t\,\mathrm{d}W_t}{\partial b_{t,\tau}}$$

Use eq. (A7) to compute the expectation and variance of the change in arbitrageur wealth:

$$E_t\,\mathrm{d}W_t = \left[\left(W_t - \int_0^T b_{t,\tau}\,\mathrm{d}\tau\right) r_t + \int_0^T b_{t,\tau}\mu_{t,\tau}\,\mathrm{d}\tau\right]\mathrm{d}t$$

$$\implies \frac{\partial E_t\,\mathrm{d}W_t}{\partial b_{t,\tau}} = (\mu_{t,\tau} - r_t)\,\mathrm{d}t$$

and the variance is

$$Var_t\,\mathrm{d}W_t = \left(\int_0^T b_{t,\tau}\mathbf{A}(\tau)^T\,\mathrm{d}\tau\right)\mathbf{\Sigma}\left(\int_0^T b_{t,\tau}\mathbf{A}(\tau)\,\mathrm{d}\tau\right)\mathrm{d}t$$

$$\implies \frac{\partial Var_t\,\mathrm{d}W_t}{\partial b_{t,\tau}} = 2\left(\int_0^T b_{t,\tau}\mathbf{A}(\tau)^T\,\mathrm{d}\tau\right)\mathbf{\Sigma}\mathbf{A}(\tau)\,\mathrm{d}t$$

Note that

$$b_{t,\tau} = -\tilde{b}_{t,\tau} = \alpha(\tau)\left[\mathbf{y}_t^T\left(\Theta(\tau)\tau - \mathbf{A}(\tau)\right) + \bar{\beta}(\tau)\tau - C(\tau)\right] \tag{A9}$$

Substitute eq. (A9) into the optimality conditions derived above. Equating the coefficients on $\mathbf{y}_t$ terms (and assuming that r_t is ordered first) gives:

$$\mathbf{A}'(\tau) + \mathbf{M}\mathbf{A}(\tau) - \mathbf{e}_1 = 0$$

where $\mathbf{M}$ is defined by eq. (30). Note that since $\mathbf{M}$ is itself a function of integral terms involving $\mathbf{A}(\tau)$ this is a fixed point problem with no simple solution. However, treating $\mathbf{M}$ as fixed, with initial conditions $\mathbf{A}(\tau) = 0$ the general solution is given by eq. (31).

Suppose $\mathbf{M}$ is diagonalizable, with $\mathbf{M} = \mathbf{G}\mathbf{D}\mathbf{G}^{-1}$. Normalize $\mathbf{G}$ such that $\mathbf{G}\mathbf{1} = \mathbf{e}_1$ (that is, its first row sum is 1, and all other rows sum to 0). Then

$$\int_0^\tau \exp(-\mathbf{M}s)\,\mathrm{d}s\ \mathbf{e_1} = \mathbf{G}\int_0^\tau \exp(-\mathbf{D}s)\,\mathrm{d}s\ \mathbf{G}^{-1}\mathbf{e_1}$$

$$= \mathbf{G}\int_0^\tau \exp(-\mathbf{D}s)\,\mathrm{d}s\ \mathbf{1}$$

and if $\mathbf{M}$ is invertible then eq. (31) is obtained.

Again substitute eq. (A9) into the optimality conditions. Equating constant coefficients gives:

$$C'(\tau) - (\Gamma\overline{\mathbf{y}})^T\mathbf{A}(\tau) + \frac{1}{2}Tr\left[\mathbf{\Sigma}\mathbf{A}(\tau)\mathbf{A}(\tau)^T\right] \\ = a\mathbf{A}(\tau)^T\mathbf{\Sigma}\int_0^T \alpha(\tau)\left(\tau\bar{\beta}(\tau) - C(\tau)\right)\mathbf{A}(\tau)\,\mathrm{d}\tau$$

Define the vector

$$\mathbf{Z}_C \equiv a\mathbf{\Sigma}\int_0^T \alpha(\tau)\mathbf{A}(\tau)\left[\tau\bar{\beta}(\tau) - C(\tau)\right]\mathrm{d}\tau + \Gamma\overline{\mathbf{y}} \tag{A10}$$

Imposing the initial condition $C(0) = 0$ and integrating, the solution for $C(\tau)$ given by

$$C(\tau) = \mathbf{n}_1(\tau)^T\mathbf{Z}_C - \frac{1}{2}Tr\left[\mathbf{\Sigma}\mathbf{n}_2(\tau)\right] \tag{A11}$$

$$\mathbf{n}_1(\tau) = \int_0^\tau \mathbf{A}(u)\,\mathrm{d}u \tag{A12}$$

$$\mathbf{n}_2(\tau) = \int_0^\tau \mathbf{A}(u)\mathbf{A}(u)^T\,\mathrm{d}u \tag{A13}$$

Substitute $C(\tau)$ from (A11) into (A10) and solve for $\mathbf{Z}_C$:

$$\mathbf{Z}_C = \left[\mathbf{I} + a\mathbf{\Sigma}\mathbf{N}_1\right]^{-1}\left[a\mathbf{\Sigma}\mathbf{N}_2 + \Gamma\overline{\mathbf{y}}\right] \tag{A14}$$

where $\mathbf{I}$ is the identity matrix and

$$\mathbf{N}_1 = \int_0^T \alpha(\tau)\mathbf{A}(\tau)\mathbf{n}_1(\tau)^T\,\mathrm{d}\tau \tag{A15}$$

$$\mathbf{N}_2 = \int_0^T \alpha(\tau)\mathbf{A}(\tau)\left(\bar{\beta}(\tau)\tau + \frac{1}{2}Tr\left[\mathbf{\Sigma}\mathbf{n}_2(\tau)\right]\right)\mathrm{d}\tau \tag{A16}$$

□

Appendix B Numerical Solution Algorithm

This section describes the numerical solution method used to solve the generalized model. This also requires obtaining closed-form solutions to a number of integral expressions.

B.1 Closed-Form Integral Expressions

Define the (scalar) functions:

$$
\begin{aligned}
\phi_0(\nu, \tau) &\equiv \int_0^\tau u f(\nu u)\, \mathrm{d}u \\
&= \frac{\tau}{\nu} \left(1 - f(\nu\tau)\right) \\
\phi_1(\nu, \tau) &\equiv \int_0^\tau u^2 f(\nu u)\, \mathrm{d}u \\
&= \frac{\tau}{\nu^2} \left(\exp(-\nu\tau) - f(\nu\tau) + \frac{1}{2}\nu\tau\right) \\
\phi_2(\nu_i, \nu_j, \tau) &\equiv \int_0^\tau u^2 f(\nu_i u) f(\nu_j u)\, \mathrm{d}u \\
&= \frac{\tau}{\nu_i \nu_j} \left(1 - f(\nu_i\tau) - f(\nu_j\tau) + f((\nu_i + \nu_j)\tau)\right)
\end{aligned}
$$

For any function $F(\tau)$, define

$$
\tilde{F} \equiv \int_0^T \eta(\tau) F(\tau)\, \mathrm{d}\tau\,, \quad \hat{F} \equiv \int_0^T \frac{\eta(\tau)}{\tau} F(\tau)\, \mathrm{d}\tau
$$

In particular, define

$$
\tilde{f}(\nu) \equiv \int_0^T \eta(\tau) f(\nu\tau)\, \mathrm{d}\tau
$$

Recall $\int_0^T \eta(\tau)\, \mathrm{d}\tau \equiv 1$. Therefore,

$$
\begin{aligned}
\hat{\phi}_0(\nu, \tau) &= \frac{1}{\nu}(1 - \tilde{f}(\nu)) \\
\hat{\phi}_2(\nu_i, \nu_j, \tau) &= \frac{1}{\nu_i \nu_j}(1 - \tilde{f}(\nu_i) - \tilde{f}(\nu_j) + \tilde{f}(\nu_i + \nu_j))
\end{aligned}
$$

For the integral terms involving $\alpha(\tau)$, define

$$
\begin{aligned}
\phi_0^\alpha(\nu, \tau) &\equiv \int_0^\tau \alpha(u) u f(\nu u)\, \mathrm{d}u \\
\phi_1^\alpha(\nu, \tau) &\equiv \int_0^\tau \alpha(u) u^2 f(\nu u)\, \mathrm{d}u \\
\phi_2^\alpha(\nu_i, \nu_j, \tau) &\equiv \int_0^\tau \alpha(u) u^2 f(\nu_i u) f(\nu_j u)\, \mathrm{d}u
\end{aligned}
$$

For the integral terms involving $\theta_k(\tau)$, define

$$
\phi_1^{\alpha,\theta_k}(\nu, \tau) \equiv \int_0^\tau \alpha(u) u^2 \theta_k(u) f(\nu u)\, \mathrm{d}u
$$

Since $\eta(\tau)$ is proportional to the pdf of a truncated Gamma distribution, this term can be written as

$$\tilde{f}(\nu) = \left(\int_0^T \tau \exp(-\eta_1 \tau) \, \mathrm{d}\tau \right)^{-1} \frac{(\nu \exp(\eta_1 T) - \eta_1 - \nu + \exp(-\nu T)\eta_1) \exp(-\eta_1 T)}{(\nu \eta_1 (\eta_1 + \nu))}$$

Further, I consider a more general form of $\alpha(\tau)$, given as follows:

$$\alpha(\tau) = \alpha_0 e^{-\alpha_1 \tau}$$

In the estimation, I simplify by setting $\alpha_1 = 0$ and $\alpha_0 = 1$.

Note that this implies

$$\alpha(u) f(\nu u) = \frac{\alpha_0}{\nu} \left[(\alpha_1 + \nu) f((\alpha_1 + \nu) u) - \alpha_1 f(\alpha_1 u) \right]$$

Therefore the integral expressions become

$$\begin{aligned}
\phi_0^\alpha(\nu, \tau) &= \frac{\alpha_0}{\nu} \left[(\alpha_1 + \nu) \phi_0(\alpha_1 + \nu, \tau) - \alpha_1 \phi_0(\alpha_1, \tau) \right] \\
\phi_1^\alpha(\nu, \tau) &= \frac{\alpha_0}{\nu} \left[(\alpha_1 + \nu) \phi_1(\alpha_1 + \nu, \tau) - \alpha_1 \phi_1(\alpha_1, \tau) \right] \\
\phi_2^\alpha(\nu_i, \nu_j, \tau) &= \frac{\alpha_0}{\nu_i} \left[(\alpha_1 + \nu_i) \phi_2(\alpha_1 + \nu_i, \nu_j, \tau) - \alpha_1 \phi_2(\alpha_1, \nu_j, \tau) \right]
\end{aligned}$$

For $\theta_k(\tau)$, I assume

$$\theta_k(\tau) = \delta(T_k - \tau)$$

where $T_k \in [0, T]$, and where δ is the Dirac delta function. Therefore, when $\tau > T_k$,

$$\phi_1^{\alpha, \theta_k}(\nu, \tau) = \alpha(T_k) T_k^2 f(\nu T_k)$$

Also, note that $\tau^2 f(\nu \tau) \to 0$ as $\tau \to 0$. Hence if $T_k = 0$, $\phi_1^{\alpha, \theta_k}(\nu, \tau) = 0$.

Next, extend these scalar functions to their multi-dimensional components.

$$\mathbf{F}(\mathbf{x}) = \mathbf{x}^{-1} \left(\mathbf{I} - e^{-\mathbf{x}} \right) \mathbf{1}$$

This implies the coefficient function can be written

$$\mathbf{A}(\tau) = \mathbf{G} \tau \mathbf{F}(\mathbf{D} \tau)$$

where

$$[\mathbf{F}(\mathbf{D}\tau)]_i = f(\nu_i \tau)$$

since $\mathbf{D} = diag\,[\nu_1, \ldots, \nu_J]$.

$$\mathbf{\Phi}_0(T) = \int_0^T \tau \mathbf{F}(\mathbf{D}\tau)\,\mathrm{d}\tau$$
$$\mathbf{\Phi}_1(T) = \int_0^T \tau^2 \mathbf{F}(\mathbf{D}\tau)\,\mathrm{d}\tau$$
$$\mathbf{\Phi}_2(T) = \int_0^T \tau^2 \mathbf{F}(\mathbf{D}\tau)\mathbf{F}(\mathbf{D}\tau)^T\,\mathrm{d}\tau$$

For integral terms involving the function $\alpha(\tau)$, define

$$\mathbf{\Phi}_1^{\alpha}(T) = \int_0^T \alpha(\tau)\tau^2 \mathbf{F}(\mathbf{D}\tau)\,\mathrm{d}\tau$$
$$\mathbf{\Phi}_2^{\alpha}(T) = \int_0^T \alpha(\tau)\tau^2 \mathbf{F}(\mathbf{D}\tau)\mathbf{F}(\mathbf{D}\tau)^T\,\mathrm{d}\tau$$

For integral terms involving the function $\mathbf{\Theta}(\tau)$, define

$$\mathbf{\Phi}_1^{\alpha,\theta}(T) = \int_0^T \alpha(\tau)\tau^2 \mathbf{\Theta}(\tau)\mathbf{F}(\mathbf{D}\tau)^T\,\mathrm{d}\tau$$

For integral terms involving the function $\eta(\tau)$, define

$$\tilde{\mathbf{F}} = \int_0^T \eta(\tau)\mathbf{F}(\mathbf{D}\tau)\,\mathrm{d}\tau$$
$$\hat{\mathbf{\Phi}}_0 = \int_0^T \frac{\eta(\tau)}{\tau}\mathbf{\Phi}_0(\tau)\,\mathrm{d}\tau$$
$$\hat{\mathbf{\Phi}}_2 = \int_0^T \frac{\eta(\tau)}{\tau}\mathbf{\Phi}_2(\tau)\,\mathrm{d}\tau$$

Therefore, all the vector functions can be written as

$$\mathbf{n}_1(\tau) = \mathbf{G}\mathbf{\Phi}_0(\tau)$$
$$\mathbf{n}_2(\tau) = \mathbf{G}\mathbf{\Phi}_2(\tau)\mathbf{G}^T$$
$$\mathbf{M} = \Gamma^T - a\left[\mathbf{\Phi}_1^{\alpha,\theta} - \mathbf{G}\mathbf{\Phi}_2^{\alpha}\right]\mathbf{G}^T\mathbf{\Sigma}$$
$$\mathbf{N}_1 = \mathbf{G}\left[\mathbf{\Phi}_1^{\alpha}\mathbf{1} - \mathbf{\Phi}_2^{\alpha}\right]\mathbf{D}^{-1}\mathbf{G}^T$$
$$\mathbf{N}_2 = \mathbf{G}\left[\mathbf{\Phi}_1^{\alpha,\beta} + \frac{1}{2}\mathbf{\Phi}_3^{\alpha} vec\left[\mathbf{G}^T\mathbf{\Sigma}\mathbf{G}\right]\right]$$
$$\hat{\mathbf{A}} = \mathbf{G}\tilde{\mathbf{F}}$$
$$\hat{\mathbf{n}}_1 = \mathbf{G}\hat{\mathbf{\Phi}}_0$$
$$\hat{\mathbf{n}}_2 = \mathbf{G}\hat{\mathbf{\Phi}}_2\mathbf{G}^T$$

I compute these integral vectors and matrices element by element, using the scalar functions above.

$$\begin{aligned}
[\mathbf{\Phi}_0]_i &= \phi_0(\nu_i, T) \\
[\mathbf{\Phi}_1]_i &= \phi_1(\nu_i, T) \\
[\mathbf{\Phi}_2]_{i,j} &= \phi_2(\nu_i, \nu_j, T)
\end{aligned}$$

For $\alpha(\tau)$ integrals:

$$\begin{aligned}
[\mathbf{\Phi}_1^\alpha]_i &= \phi_1^\alpha(\nu_i, T) \\
[\mathbf{\Phi}_2^\alpha]_{i,j} &= \phi_2^\alpha(\nu_i, \nu_j, T)
\end{aligned}$$

For $\eta(\tau)$ integrals:

$$\begin{aligned}
\left[\tilde{\mathbf{F}}\right]_i &= \tilde{f}(\nu_i) \\
\left[\hat{\mathbf{\Phi}}_0\right]_i &= \hat{\phi}_0(\nu_i, T)
\end{aligned}$$

For $\theta_k(\tau)$ integrals:

$$\left[\mathbf{\Phi}_1^{\alpha,\theta}\right]_{k,i} = \phi_1^{\alpha,\theta_k}(\nu_i, \tau)$$

B.2 Equilibrium Algorithm

This section describes a solution method for solving the fixed point problem described in Prop. 7. Define the function

$$F(\mathbf{M}) = \Gamma(\hat{\mathbf{A}})^T - a\left[\int_0^T \alpha(\tau)\left(\tau\Theta(\tau) - \mathbf{A}(\tau)\right)\mathbf{A}(\tau)^T \,\mathrm{d}\tau\right]\mathbf{\Sigma} - \mathbf{M}$$

Equilibrium is defined as a root of this function: $F(\mathbf{M}) = 0$.

The following algorithm is used to compute F. Given some initial value of the matrix $\mathbf{M}$:

1. Solve the eigen-decomposition to get $\mathbf{D}$, $\mathbf{G}$, which gives the implied $\mathbf{A}(\tau)$ and $\hat{\mathbf{A}}$.
2. Construct the implied dynamics matrix $\Upsilon(\hat{\mathbf{A}})$ and solve for the rational expectations equilibrium matrix $\Gamma(\hat{\mathbf{A}})$ (assuming stability conditions are met).
3. Using the integral expressions derived above, solve for $F(\mathbf{M})$.

Using the above algorithm, standard root-finding algorithms can be used to solve for the equilibrium (note that while $\mathbf{D}$ and $\mathbf{G}$ may contain complex values, $\mathbf{M}$ and F will always be real-valued).

For the dynamics matrix functions $\Upsilon(\hat{\mathbf{A}})$ considered in Section 5, when $a = 0$ the equilibrium is unique. However, for generic dynamics matrix functions $\Upsilon(\hat{\mathbf{A}})$ and when

$a > 0$, multiple equilibria are possible. In order to rule out pathological equilbiria, I focus on the equilibrium which approaches the $a = 0$ equilibrium as $a \to 0$.

Appendix C Microfoundations

This section describes the model from first principles. Time is continuous. The model consists of households, firms, arbitrageurs, and a government. The government conducts monetary policy (changes in policy rate), passive fiscal policy (changes in taxes, which play no role), and QE (changes in holdings of long-term bonds).

Households face radically incomplete markets: they can only borrow through a passive mutual fund offering an instantaneous return which is a weighted average of all yields. Infinitesimally-lived arbitrageurs are the marginal investors in financial markets.

The short rate is the main policy tool, and as in Woodford (2003) I consider a "cashless limit" economy.

C.1 Households and Firms

An infinitely-lived representative household seeks to maximize expected utility by choosing consumption, labor, and savings. The household problem is standard, except they are restricted to borrowing at an effective rate $\tilde{r}_t$. The household problem is

$$\max_{\{C_t, N_t\}} E_0 \int_0^\infty e^{-\rho t} \left(\frac{C_t^{1-\varsigma}}{1-\varsigma} - \frac{N_t^{1+\xi}}{1+\xi} \right) \mathrm{d}t \tag{C1}$$

$$\text{s.t. } \mathrm{d}W_t^H \leq \left(\tilde{r}_t W_t^H - P_t C_t + w_t N_t - T_t^H \right) \mathrm{d}t \tag{C2}$$

$$\lim_{T\to\infty} E_t[Q_{t,T}^H W_T^H] = 0 \tag{C3}$$

eq. (C2) is the household's flow budget constraint; eq. (C3) is a transversality condition. The parameter ς is the coefficient of relative risk aversion; ξ is the labor supply elasticity; ρ is the discount rate. W_t^H is household nominal wealth; C_t is consumption of composite good; N_t is labor; w_t is nominal wage; T_t^H is the net taxes and transfers to households.

$Q_{t,T}^H$ is the household's discount factor, given by

$$Q_{t,T}^H = \exp\left[- \int_t^T (\tilde{r}_s - \pi_s) \, \mathrm{d}s \right] \tag{C4}$$

Note that markets are not complete (and so eq. (C4) is not the relevant discount factor for pricing securities), but eqs. (C2) and (C3) can be replaced with an equivalent single intertemporal budget constraint

$$W_0^H = E_0 \int_0^\infty Q_{0,t}^H \left(P_t C_t - w_t N_t + T_t^H \right) \mathrm{d}t \tag{C5}$$

where W_0^H is given.

Then the intra-temporal optimality condition determining labor supply is

$$C_t^{\varsigma} N_t^{\xi} = \frac{w_t}{P_t} \tag{C6}$$

and linearized intertemporal optimality conditions give

$$\mathrm{d}c_t = \varsigma^{-1} \left(\tilde{r}_t - \pi_t - \rho\right) \mathrm{d}t \tag{C7}$$

Firms face Rotemberg (1982) pricing adjustment costs, and their problem is unchanged relative to benchmark New Keynesian models.

C.2 Arbitrageurs and Preferred Habitat Investors

Arbitrageurs face a mean-variance trade-off in their wealth:

$$\max_{b_{t,\tau}} E_t \, \mathrm{d}W_t^A - \frac{a}{2} Var_t \, \mathrm{d}W_t^A$$

subject to their flow budget constraint

$$\begin{aligned} \mathrm{d}W_t^A = \left(W_t^A - T_t^A - M_t^A - \int_0^T b_{t,\tau} \, \mathrm{d}\tau\right) r_t \, \mathrm{d}t + \int_0^T b_{t,\tau} \frac{\mathrm{d}P_{t,\tau}}{P_{t,\tau}} \, \mathrm{d}\tau \\ + M_t^A r_t^M \, \mathrm{d}t \end{aligned} \tag{C8}$$

This is the same as the one considered in the main text, with the exception that arbitrageurs can also hold monetary balances M_t^A (with a return r_t^m, which must satisfy $r_t \geq r_t^m$ with equality if money supply is non-zero), and they face taxes/transfers T_t^A.

The "preferred habitat" investors are aggregated into a single passive mutual fund, which offers households the effective borrowing rate, and invests the rest of its wealth in long-term bonds. The flow budget constraint is thus

$$\mathrm{d}W_t^F = -W_t^H \tilde{r}_t \, \mathrm{d}t + \int_0^T \tilde{b}_{t,\tau} \frac{\mathrm{d}P_{t,\tau}}{P_{t,\tau}} \, \mathrm{d}\tau + \left(W_t^F - W_t^H - T_t^G - \int_0^T \tilde{b}_{t,\tau} \, \mathrm{d}\tau\right) r_t \, \mathrm{d}t \tag{C9}$$

and also face taxes/transfers.

The demand for bonds $\tilde{b}_t$ is given by eq. (4). Thus the demand from the mutual fund for long-term bonds is reduced-form. Demand of this form could additionally be derived by adding another set of finite-lived, infinitely risk-averse investors as in the appendix of Vayanos and Vila (2009). When studying QE, I assume that the demand shocks operate through the preferred habitat investors. In this case, it may be useful to view this mutual fund as a type of government sponsored entereprise.

C.3 Rational Expectations Equilibrium

Assuming that all of the profits from the arbitrageurs and mutual funds are transfered lump-sum to the households, the log-linearized solution is the same as a benchmark model but with the effective rate $\tilde{r}_t$ in place of the short rate r_t in the IS curve.

After linearizing around the steady state and imposing affine functional forms to the term structure, the aggregate dynamics can be written in matrix form as in eq. (8). The solution to the rational expectations equilibrium is found by following Buiter (1984), the continuous time analogue of Blanchard and Kahn (1980). In general, let $\mathbf{Y}_t = [\mathbf{y}_t \;\; \mathbf{x}_t]^T$ where $\mathbf{x}_t$ are the "jump" variables and $\mathbf{y}_t$ are the state variables. Assuming that Υ is diagonalizable, partition the eigenvalues and eigenvectors as follows:

$$\Upsilon = Q\Lambda Q^{-1}$$
$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

where the partitions correspond to the state and jump variables. If the number of "stable" eigenvalues (non-negative real parts) equals the number of state variables, then given some transversality conditions the rational expectations equilibrium dynamics are given by eq. (10), with

$$\Gamma = Q_{11}\Lambda_1 Q_{11}^{-1} \tag{C10}$$
$$\Omega = Q_{21} Q_{11}^{-1} \tag{C11}$$

C.4 Conditional and Unconditional Distributions

The conditional distribution of the state variables $\mathbf{y}_t$ given initial state $\mathbf{y}_0$ is normal:

$$\mathbf{y}_t|\mathbf{y}_0 \sim N\left(\boldsymbol{\mu}_t, \boldsymbol{\Sigma}_t\right)$$

The conditional mean is

$$\boldsymbol{\mu}_t = \mathbf{y}^{SS} + e^{-\Gamma t}(\mathbf{y}_0 - \mathbf{y}^{SS})$$

The conditional variance-covariance is

$$\boldsymbol{\Sigma}_t = \int_0^t e^{\Gamma(u-t)} \boldsymbol{\Sigma} e^{\Gamma^T(u-t)} \, \mathrm{d}u$$

where again $\boldsymbol{\Sigma} = \mathbf{S}\mathbf{S}^T$. Then this simplifies to

$$\begin{aligned} \operatorname{vec} \boldsymbol{\Sigma}_t &= \left(\int_0^T e^{(\Gamma \oplus \Gamma)(u-t)} \, \mathrm{d}u \right) \operatorname{vec} \boldsymbol{\Sigma} \\ &= (\Gamma \oplus \Gamma)^{-1} \left(\mathbf{I} - e^{-(\Gamma \oplus \Gamma)t} \right) \operatorname{vec} \boldsymbol{\Sigma} \end{aligned}$$

where $\oplus$ is the Kronecker sum. Taking the limit as $t \to \infty$, the unconditional variance-covariance is given by

$$\operatorname{vec} \mathbf{\Sigma}_\infty = (\Gamma \oplus \Gamma)^{-1} \operatorname{vec} \mathbf{\Sigma}$$

and the present discounted value is computed as

$$\begin{aligned}
\widetilde{\mathbf{\Sigma}}_\infty &\equiv \int_0^\infty e^{-\rho t} \mathbf{\Sigma}_t \, \mathrm{d}t \\
\operatorname{vec} \widetilde{\mathbf{\Sigma}}_\infty &= (\Gamma \oplus \Gamma)^{-1} (\rho \mathbf{I} + \Gamma \oplus \Gamma)^{-1} \operatorname{vec} \mathbf{\Sigma}
\end{aligned}$$

Then from eq. (10), the conditional distribution of the jump variables $\mathbf{x}_t$ given initial state $\mathbf{y}_0$ (recall the initial values $\mathbf{x}_0$ are endogenous) is normal:

$$\mathbf{x}_t | \mathbf{y}_0 \sim N\left(\Omega(\boldsymbol{\mu}_t - \mathbf{y}^{SS}) + \mathbf{x}^{SS}, \mathbf{\Sigma}_t^{\mathbf{x}}\right)$$

and the equivalently defined covariances for the jump variables are easily computed as

$$\begin{aligned}
\mathbf{\Sigma}_t^{\mathbf{x}} &= \Omega \mathbf{\Sigma}_t \Omega^T \\
\mathbf{\Sigma}_\infty^{\mathbf{x}} &= \Omega \mathbf{\Sigma}_\infty \Omega^T \\
\widetilde{\mathbf{\Sigma}}_\infty^{\mathbf{x}} &= \Omega \widetilde{\mathbf{\Sigma}}_\infty \Omega^T
\end{aligned}$$

Hence if $\mathbf{x}^{SS} = 0$, $E_0\left[\mathbf{x}_t \mathbf{x}_t^T\right] = \Omega \mathbf{\Sigma}_t \Omega^T$.