Homework 10

DUE: Wednesday, 17. Nov. 2021

Exercise 1. Recall from class that, given $K \in \mathsf{Set}_\Delta$, we define $E_K \in \mathsf{Cat}_\Delta$, by $\mathrm{Obj}(E_K) = K_0$, and

$$E_K(x,y) := \underset{(\mathsf{Nec}_{/K})_{x,y}}{\mathrm{colim}} \mathfrak{C}[\mathcal{N}](x,y).$$

1. Show that the assignment

$$\begin{aligned} \mathsf{Set}_\Delta &\longrightarrow \mathsf{Cat}_\Delta \\ K &\longmapsto E_K \end{aligned}$$

defines a functor.

2. Show that there are functors

$$\mu_K : E_K \longrightarrow \mathfrak{C}[K]$$

of simplicially enriched categories which piece together to define a natural transformation

$$\mu : E_{(-)} \Longrightarrow \mathfrak{C}.$$

CORRECTION: In the previous exercise, we used the opposite 2-morphism convention. We will use the following convention for the rest of the course.

Definition. Recall the definition of a 2-category from Homework 1. Denote by Cat_2 the category whose objects are (small) 2-categories, and whose morphisms are strict 2-functors. Denote by $\mathbb{\Delta}^n$ the 2-category with objects $0, 1, \ldots, n$, and such that

- For every $i \leq j$, there is a morphism $\phi_{i,j} : i \to j$ such that $\phi_{i,i} = \mathrm{id}_i$.
- For every $S = \{i = i_0 \leq i_1 \leq \cdots \leq i_k = j\}$, and every $T = \{i_{\ell_t}\}_{0 \leq t \leq r} \subset S$ containing i and j, there is a unique 2-morphism

$$\phi_{i_{k-1},j} \circ \cdots \circ \phi_{i_1,i_2} \circ \phi_{i,i_1} \Leftarrow \phi_{i_{t_{r-1}},j} \circ \cdots \circ \phi_{i_{t_1},i_{t_2}} \circ \phi_{i,i_{t_1}}$$

Exercise 2. Recall the *2-nerve* $N_2 : \mathsf{Cat}_2 \to \mathsf{Set}_\Delta$ is given by

$$N_2(\mathbb{C})_n := \mathrm{Fun}_2(\mathbb{\Delta}^n, \mathbb{C}).$$

1. Let $\mathbb{C} \in \mathsf{Cat}_2$. Show that there is a simplicially enriched category $N_M(\mathbb{C})$ with $\mathrm{Obj}(N_M(\mathbb{C})) = \mathrm{Obj}(\mathbb{C})$, and

$$N_M(\mathbb{C})(x,y) := N(\mathbb{C}(x,y)).$$

2. Show that the above definition extends to a functor

$$N_M : \mathsf{Cat}_2 \longrightarrow \mathsf{Cat}_\Delta$$

Exercise 3.

1. Show that there is a commutative diagram

$$\begin{tikzcd} & \mathsf{Cat}_\Delta \arrow[dr, "N_{\mathrm{HC}}"] & \\ \mathsf{Cat}_2 \arrow[ur, "N_M"] \arrow[rr, "N_2"'] & & \mathsf{Set}_\Delta \end{tikzcd}$$

   where N_{HC} denotes the homotopy coherent nerve.

2. Suppose $\mathbb{C} \in \mathsf{Cat}_2$ has non-invertible 2-morphisms. Show that $N_2(\mathbb{C})$ is not a quasi-category.