Physically-based Simulation in Computer Graphics

Exercise 1

Kevin Wallimann, Andri Schmidt, Marc Maetz

Berhard Thomaszewski,
Amit Bermano
Disney Research Zurich

Barbara Solenthaler
CGL ETH Zrich

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Problem 1

Backward Euler update formula derivation

The general update formula is

$$y_{k+1} = y_k + \mathrm{dt} \cdot f(t_{k+1}, yk + 1). \tag{1}$$

In concrete terms for the equations of motion, it is

$$x_{k+1} = x_k + \mathrm{dt} \cdot v_{k+1} \tag{2a}$$

$$v_{k+1} = v_k + \mathrm{dt} \cdot \left(\frac{k(-x_{k+1} - L)}{m} - \frac{d \cdot v_{k+1}}{m} - g \right) \tag{2b}$$

Insert (2a) into (2b) and reorder,

$$v_{k+1} = v_k + \mathrm{dt} \cdot \left(\frac{k(-x_{k+1} - L)}{m} - \frac{d \cdot v_{k+1}}{m} - g \right) \tag{3a}$$

$$= \mathrm{dt} \left(-\frac{k}{m}\mathrm{dt} - \frac{d}{m} \right) v_{k+1} + v_k + \mathrm{dt} \left(\frac{k}{m}(-x_k - L) - g \right) \tag{3b}$$

$$= \frac{v_k - \mathrm{dt} \left(\frac{k}{m}(x_k + L) - g \right)}{1 + \mathrm{dt} \left(\frac{k \cdot \mathrm{dt} + d}{m} \right)}. \tag{3c}$$

Problem 2

2.1. Analytic solution

First lets do a sketch as seen in figure 1 to omit any misunderstanding of the problem. We would like to find the constants c_1, c_2 of the equation

$$y(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t) - L - \frac{mg}{k}, \tag{4}$$

where the constraints are the rest state initial conditions, which means there is no energy in the system except gravitational energy at $t = 0$. This has the same physical meaning as the spring having its rest length L [1] and no kinetic energy at $t = 0$. This can be expressed as

$$y(t = 0) = -L, \tag{5}$$

and

$$\dot{y}(t = 0) = 0. \tag{6}$$

The position of p_2 computed with the given solution Eq. 4 is
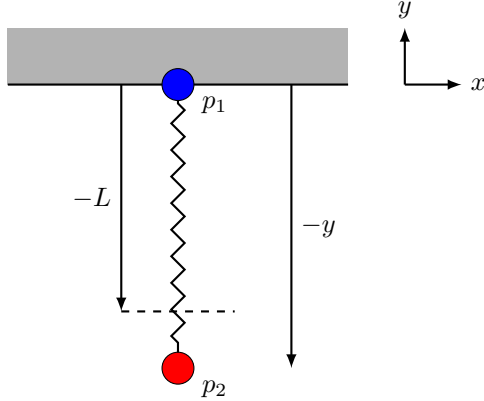
$$y(t = 0) = c_1 - L - \frac{mg}{k}. \tag{7}$$

Figure 1: Sketch of the spring problem, p_2 being not in rest state.

Comparing Eq. 5 and Eq. 7 gives us

$$c_1 = \frac{mg}{k}. \tag{8}$$

To get c_2 we need to compute $\dot{y}(t = 0)$ from Eq. 4

$$\dot{y}(t = 0) = \left(c_1\alpha + c_2\beta - L - \frac{mg}{k} \right), \tag{9}$$

where $c_1 = mg/k$ leads to

$$\dot{y}(t = 0) = \frac{mg}{k}\alpha + c_2\beta - L - \frac{mg}{k}. \tag{10}$$

Using the constraint in Eq. 6 finally gives us

$$c_2 = \frac{mg}{\beta k}(1 - \alpha) + \frac{L}{\beta}. \tag{11}$$

2.2. Error convergence analysis

The error convergence plots in figures 2,3 and 4 are all quite similar with respect to different damping factors. The three Euler methods show the lowest average error convergence, in comparison to the Midpoint method. This is what we expect, since the Midpoint method is known to be accurate of order 2, while Euler methods are only accurate of order 1.

In concrete terms, the following values for the error convergence $\frac{e_{i+1}}{e_i}$ (damping factor 0) were obtained.
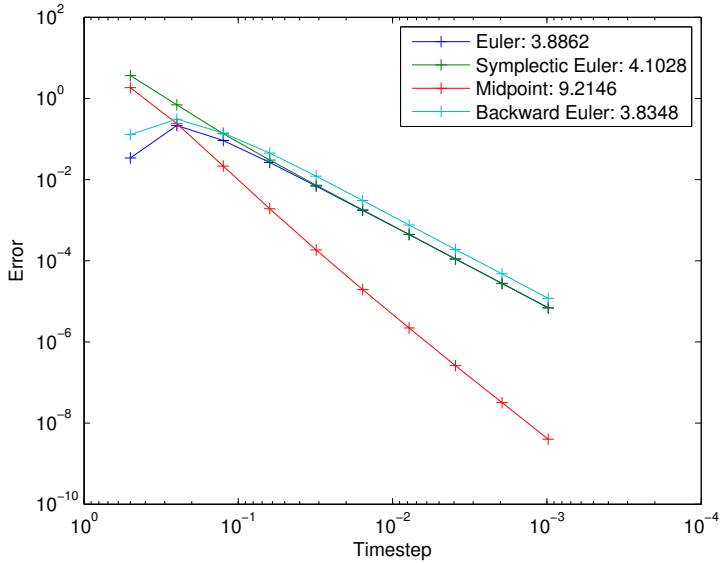
Figure 2: Error convergence analysis with damping factor $d = 0.01$.

Euler	3.89
Symplectic Euler	4.10
Backwards Euler	3.84
Midpoint	9.57

In order to get meaningful results, the first three measurements were ignored when computing the average error convergence.

2.3. Stability analysis

The plots of the stability analysis can be seen in figures 5 and 6. As expected, the backwards Euler method is unconditionally stable. The symplectic Euler method is also stable, which was pointed out during the lecture. Both midpoint and Euler method are not stable, which is also as expected.

However, for a damping coefficient of $d = 1.9$, all methods seem to be stable. But for a general setting, the Midpoint and Euler method remain unstable.

Since an accurate method produces a smaller error per step, the error propagation is smaller, therefore it is more likely to be stable.
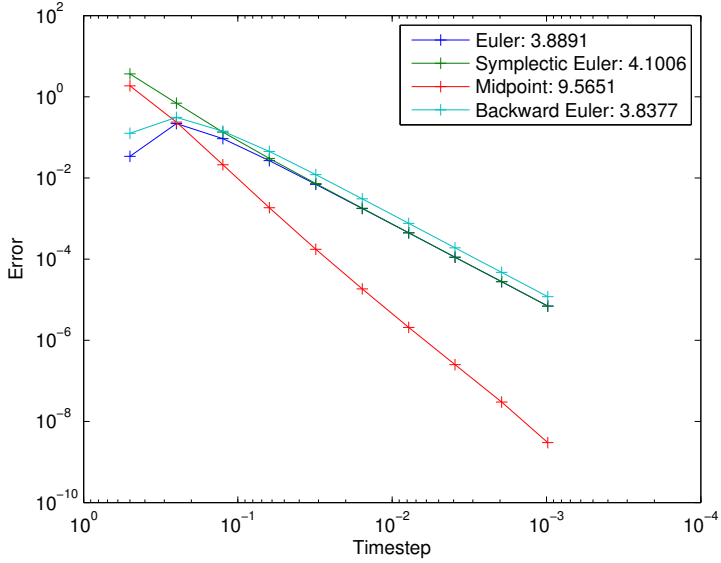
Figure 3: Error convergence analysis with damping factor $d = 0$.
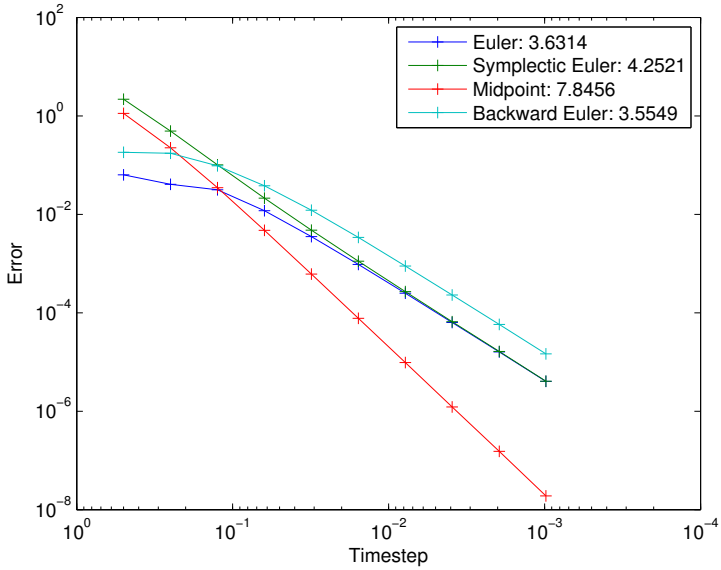

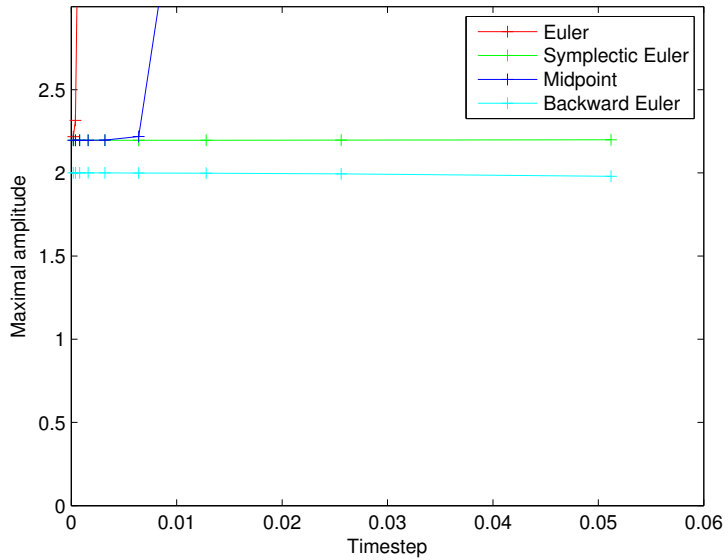Figure 4: Error convergence analysis with damping factor $d = 1$.

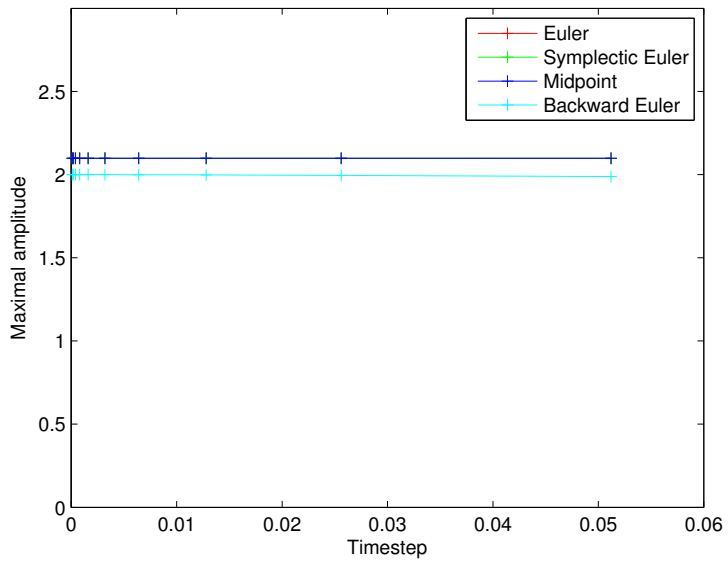Figure 5: Stability analysis with damping factor $d = 0$.

Figure 6: Stability analysis with damping factor $d = 1.9$.