Physically-based Simulation in Computer Graphics

Summary of slides

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Finite elements

1.1 Poisson's equation

$$-\Delta u(x,y) = f(x,y) \tag{1}$$

1.2 Finite Difference

Example 1.1 (1D problem).

$$-u''(x) = f(x), \quad x \in \Omega = (0,1) \tag{2}$$

$$u(0) = u(1) = 0 \tag{3}$$

$$u[i] = u(i \cdot h), \quad i \in (0,\dots,n) \tag{4}$$

$$\begin{aligned} u''(x) &= \frac{u'[i] - u'[i-1]}{h} \\ &= \frac{u[i+1] - 2u[i] + u[i-1]}{h^2} \end{aligned} \tag{5}$$

$$h^2 f[i] = u[i-1] + 2u[i] + u[i+1] \tag{6}$$

1.3 Finite Elements

Example 1.2 (1D problem).

$$-u''(x) = f(x), \quad x \in \Omega = (0,1) \tag{7a}$$
$$u(0) = u(1) = 0 \tag{7b}$$

Assume PDF is satisfied pointwise*, then*

$$-\int_\Omega \mathrm{d}x\, u''(x) = \int_\Omega \mathrm{d}x\, f(x) \tag{7c}$$

and also

$$-\int_\Omega \mathrm{d}x\, u''(x) \cdot v(x) = \int_\Omega \mathrm{d}x\, f(x) \cdot v(x) \tag{7d}$$

for arbitrary functions $v : \Omega \to \mathbb{R}$.

Example 1.3.

- *Assume v sufficiently smooth and*

$$v(0) = v(1) = 0\,. \tag{8}$$

- *Integration by parts*

$$\int_a^b \mathrm{d}x\, f'(x) g(x) = [f(x) \cdot g(x)]_a^b - \int_a^b \mathrm{d}x\, f(x) g'(x)\,. \tag{9}$$

$$-\int_\Omega \mathrm{d}x\, u''(x) \cdot v(x) = \int_\Omega \mathrm{d}x\, f(x) \cdot v(x) \tag{10a}$$

$$\int_\Omega \mathrm{d}x\, u'(x) \cdot v'(x) = \int_\Omega \mathrm{d}x\, f(x) \cdot v(x) + [u'(x) \cdot v(x)]_0^1 \tag{10b}$$

with the imposed boundary conditions of $v(x)$ this leads to

$$\int_\Omega \mathrm{d}x\, u'(x) \cdot v'(x) = \int_\Omega \mathrm{d}x\, f(x) \cdot v(x) \tag{10c}$$

Weak *form because of the* weaker *continuity requirements.*

1.3.1 Ritz-Galerkin Approach

So far, $u(x)$ and $f(x)$ continuous function, but now choose finite-dimensional subspace for $u(x)$ and $f(x)$. Solve problem in *weak* form in subspace (projection). $\to$ Discretize u

- Sample with nodal positions x_i
- Nodal coefficients u_i
- Basis functions $N_i(x)$

$$\rightsquigarrow u(x) = \sum_i u_i N_i(x) \tag{11}$$

Discretize both u and v on n-dimensional space

$$\begin{aligned}
u(x) &= \sum_{i=1}^{N} u_i N_i(x) \\
\rightsquigarrow \frac{\partial}{\partial x} u(x) &= \sum_{i=1}^{n} u_i \frac{\partial}{\partial x} N_i(x)
\end{aligned}$$

insert into *weak* formulation

$$\begin{aligned}
\int_\Omega \mathrm{d}x\, u'(x) v'(x) &= \int_\Omega \mathrm{d}x\, f(x) v(x) \\
\rightsquigarrow \int_\Omega \mathrm{d}x \sum_{i\,=1}^{n} u_i \frac{\partial N_i}{\partial x} \cdot \sum_{j\,=1}^{n} v_j \frac{\partial N_j}{\partial x} &= \int_\Omega \mathrm{d}x\, f \cdot \sum_{j=1}^{n} v_j N_j(x) \\
\rightsquigarrow \sum_{i,j\,=1}^{n} u_i v_j \int_\Omega \mathrm{d}x\, \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} &= \sum_{j=1}^{n} v_j \int_\Omega \mathrm{d}x\, f(x) N_j(x)
\end{aligned}$$

$$\rightsquigarrow \sum_{j\,=1}^{n} v_j \left[\sum_{i=1}^{n} u_i \int_\Omega \mathrm{d}x\, \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} - \int_\Omega \mathrm{d}x\, f(x) N_j(x) \right] = 0\,,$$

with an arbitrary v_j (assumed)

$$\sum_{i\,=1}^{n} u_i \int_\Omega \mathrm{d}x\, \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} - \int_\Omega \mathrm{d}x\, f(x) N_i(x) = 0, \quad \forall i, \ldots, n.$$

Now we have n linear equations for n unknowns, which can be written as

$$K\mathbf{u} = \mathbf{f} \tag{12a}$$

$$\underbrace{\begin{pmatrix} K_{11} & \cdots & K_{1n} \\ \vdots & \ddots & \vdots \\ K_{n1} & \cdots & K_{nn} \end{pmatrix}}_{K} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}}_{\mathbf{u}} = \underbrace{\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}}_{\mathbf{f}} \tag{12b}$$

$$K_{ij} = \int_\Omega \mathrm{d}x\, \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \tag{13a}$$

$$f_i = \int_\Omega \mathrm{d}x\, f(x) N_i(x) \tag{13b}$$

This matrix is

- symmetric (definition of K_{ij})
- positive-definite (eliptic PDE)
- sparse (N_i have compact support)

$\rightsquigarrow$ use sparse solvel, e.g., conjugate gradients

Choice of function space (types of Finite Elements)

$$u(x) = \sum_i u_i N_i(x) \tag{14}$$

- smooth enough $\rightsquigarrow$ once differentiable
- simple $\rightsquigarrow$ polynomial functions
- interpolation $\rightsquigarrow$ $N_i(\mathbf{x}_j) = \delta_{ij}$
- compact support $\rightsquigarrow$ defined piecewise on simple geometry

use piecewise linear basis functions $\rightsquigarrow$ piece wise linear approximation $u(x)$.

Linear simplicial elements Basis functions are uniquely defined through

- Geometry $\mathbf{x}_j$ and
- interpolations requirement $N_i(\mathbf{x}_j) = \delta_{ij}$

$$\begin{aligned} \mathbf{x}_j &= x_j, \quad \text{in 1D,} \\ &= (x_j, y_j)^t, \quad \text{in 2D,} \\ &= (x_j, y_j, z_j)^t, \quad \text{in 3D}\,. \end{aligned} \tag{15a}$$

- Basis functions are linear

$$N_i(x, y) = a_i x + b_i y + c \tag{16}$$

- Due to $N_i(\mathbf{x}_j) = \delta_{ij}$ we have

$$\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{pmatrix} \tag{17a}$$

$$\begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{pmatrix} \tag{17b}$$

What is a finite element A finite element is a triple consisting of

- a closed subset $\Omega_e \subset \mathbb{R}^d$
- a set of n basis function $N_i : \Omega_E \to \mathbb{R}$
- a set of n associated nodal variables $\mathbf{x}_i$

Summary of problem

- 1D Poisson problem given as

$$u''(x) = f(x) \tag{18}$$

- Weak form + galerkin approach gives linear system

$$Ku = f \tag{19a}$$

with

$$K_{ij} = \int_\Omega \mathrm{d}x \, \frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} \,, \tag{19b}$$

and

$$\mathbf{f}_i = \int_\Omega \mathrm{d}x \, f(x) N_i(x) \,. \tag{19c}$$

- Use simple elements with linear basis functions

What is left is

- tesseate domain into elements
- evaluate integrals (basis functions & derivatives)
- assemble the system matrix and right hand side

1.4 Exercise 2

Task: use linear triangle elements

- evaluate integrals
- assemble the global system matrix and right hand side

Goal: evaluate

$$\mathbf{f}_i = \int_\Omega \mathrm{d}x\,\mathrm{d}y\, f(x) N_i(x) \tag{20}$$

- N_i extends ovel all $n_{e,i}$ elements incident to $\mathbf{x}_i$
- N_i is zero outside $\Omega_i \subset \Omega$
- Consider restrictions N_i^e onto elements

It holds

$$\int_\Omega \mathrm{d}x\, f(x) N_i(x) = \sum_{\ell=1}^{n_{\ell,i}} \int_{\Omega_\ell} \mathrm{d}x\, f(x) N_i^\ell(x), \quad \text{with } \Omega_i = \bigcup_\ell \Omega_\ell \tag{21a}$$

evaluate integrals over Ω_ℓ with *quadratic rule*

$$\int_{\Omega_\ell} \mathrm{d}x\mathrm{d}y\, f(x) N_i^e(x) \approx f(x_q, y_q) \cdot N_i^\ell(x_q, y_q) \cdot A_\ell\,, \tag{21b}$$

where A_ℓ area of element ℓ and (x_ℓ, y_ℓ) the single quadrature point at borycenter

Goal: evaluate

$$\mathbf{K}_{ij} = \int_\Omega \mathrm{d}x\mathrm{d}y \int_{\Omega_\ell} \mathrm{d}x\mathrm{d}y\, N_i^\ell N_j^\ell \tag{22a}$$

$$> 0$$

i.e., $\exists$ an element containing vertices $\mathbf{x}_i$ and $\mathbf{x}_j$. Let S_{ij} denote a set of all these elements, then

$$K_{ij} = \sum_{\ell \in S_{ij}} \int_{\Omega_\ell} \mathrm{d}x\mathrm{d}y\, \frac{\partial N_i^\ell}{\partial x}\frac{\partial N_j^\ell}{\partial x} + \frac{\partial N_i^\ell}{\partial y}\frac{\partial N_j^\ell}{\partial y}\,. \tag{22b}$$

This is an element-centered implementation. For each element ℓ, compute basis function derivatives

$$\frac{\partial N_k^\ell}{\partial \mathbf{x}}, \quad k = 1, \ldots, 3 \tag{23a}$$

form products and integrate

$$\frac{\partial N_k^\ell}{\partial \mathbf{x}} \frac{\partial N_m^\ell}{\partial \mathbf{x}}, \quad k = 1, \ldots, 3;\, m = k, \ldots, 3\,, \tag{23b}$$

which is a constant on element and add to global matrix (local vs. global numbering)

$$K_{ij} \mathrel{+}= A_\ell \left(\frac{\partial N_k^\ell}{\partial \mathbf{x}} \frac{\partial N_m^\ell}{\partial \mathbf{x}} + \frac{\partial N_k^\ell}{\partial \mathbf{y}} \frac{\partial N_m^\ell}{\partial \mathbf{y}} \right) \tag{23c}$$

Advantages of FEM (over Finite Differences)

- can easily deal with comple geometries
- solution in nodes is naturally interpolated inside elements using basis functions
- weaker smoothness requirements on solution
- weak form is natural for many applications