1 Introduction to Linear Programming (LP)

1.1 Definition of LP

A LP is a problem to minimize or maximize a linear so called objective function, satisfying a finite set of linear constraints. **All** of the given linear constraints must be fulfilled. The number of variables and the number of linear constraints need not be the same.

Objective function	$c_1x_1 + c_2x_2 + ... + c_nx_n = c^Tx \quad c \in \mathbb{R}^n, \quad x$ are variables
Linear constraints	$a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = b_i$
	$a_{j1}x_1 + a_{j2}x_2 + ... + a_{jn}x_n \geqslant b_j$
	$a_{k1}x_1 + a_{k2}x_2 + ... + a_{kn}x_n \leqslant b_k$
	i, j, k stand for different row indices, $\quad b \in \mathbb{R}^m \quad A \in \mathbb{R}^{m,n}$
	Strict inequalities are not allowed!
Compact LP notation	$\begin{aligned} \{\text{min,max}\} \quad & c^Tx \\ \text{subject to} \quad & Ax = b \\ & A'x \geqslant b' \\ & A''x \leqslant b'' \end{aligned}$
Example	$\begin{array}{llllll} \min & 2x_1 & - & x_2 & - & 3x_3 \\ \text{s.t.} & x_1 & + & 4x_2 & & \leqslant 4 \\ & & & x_2 & + & x_3 \leqslant 4 \\ & x_1 \geqslant 0 & & x_2 \geqslant 0 & & \end{array}$ $\quad c^T = [2, -1, -3] \quad A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$
	$\Leftrightarrow \min c^Tx$ s.t. $Ax \leqslant b \quad b^T = [4, 4, 0, 0] \quad x = [x_1, x_2, x_3]$

1.2 Solution of LP's

Ref. P.13

Theorem 1.3	Every LP is either **infeasible**, **unbounded** or it has an **optimal** solution
Infeasibility	The constraints are such that they cannot be fulfilled at the same time. (Insert graphics for 2D case)
Unboundedness	The constraints form a feasible region which is infinite in the direction of the objective function (unbounded) (Insert graphics for 2D case)

2 LP Basics I

2.1 The Dual

2.1.1 Rationale for the dual

Ref. P.16-17
A strategy to prove the optimality of a LP is to make a linear combination of the constraints such that the summed constraint "lower-equals" the objective function. The right hand side of the summed constraint will be the optimal solution.
Example:

$$
\begin{array}{llllll}
\max & 6x_1 & + & x_2 & \\
\text{s.t.} & & & & \\
\text{E1} & 2x_1 & + & x_2 & \leqslant 4 \\
\text{E2} & x_1 & & & \leqslant 3 \\
\text{E3} & x_1 & \geqslant 0 & x_2 \geqslant 0 &
\end{array}
$$

Summed constraint: $1 \cdot E1 + 4 \cdot E2 = 6x_1 + x_2 \leqslant 16$
That means that our objective value cannot exceed 16, but it still may be smaller. In fact, $6x_1 + x_2 \leqslant 3 \cdot E1 + 0 \cdot E2 = 6x_1 + 3x_2 \leqslant 12$ is the smallest upper bound.

In general, we might not know or see the coefficients for the linear combination easily. Thus we formulate a problem to find them, which turns out to be a LP again. As above, we can't calculate the solution directly, instead we overestimate it, aiming to find the smallest upper bound for the solution.
Consider the LP

$$
\begin{array}{llllll}
\max & c_1x_1 & + & c_2x_2 & \\
\text{s.t.} & & & & \\
\text{E1:} & a_{11}x_1 & + & a_{12}x_2 & \leqslant b_1 \\
\text{E2:} & a_{21}x_1 & + & a_{22}x_2 & \leqslant b_2 \\
\text{E3:} & a_{31}x_1 & + & a_{32}x_2 & \leqslant b_3 \\
\text{E4:} & x_1 & \geqslant 0 & x_2 \geqslant 0 &
\end{array}
$$

In the following y_i denote the coefficients we want to find. Linear combination of the constraints yields:

$$y_1 \cdot E_1 + y_2 \cdot E_2 + y_3 \cdot E_3 = (a_{11}y_1 + a_{21}y_2 + a_{31}y_3)x_1 + (a_{12}y_1 + a_{22}y_2 + a_{32}y_3)x_2 \leqslant y_1b_1 + y_2b_2 + y_3b_3$$

Now we overestimate the solution: (Note that E_4 must hold in order the following be an over- and not an underestimation)

$$
\begin{array}{ll}
(a_{11}y_1 + a_{21}y_2 + a_{31}y_3) & \geqslant c_1 \\
(a_{12}y_1 + a_{22}y_2 + a_{32}y_3) & \geqslant c_2
\end{array}
$$

To make the overestimation smallest possible, we have a minimization problem, therefore the dual problem is:
Dual problem:

$$
\begin{array}{llllllll}
\min & y_1b_1 & + & y_2b_2 & + & y_3b_3 & \\
\text{s.t.} & a_{11}y_1 & + & a_{21}y_2 & + & a_{31}y_3 & \geqslant c_1 \\
 & a_{12}y_1 & + & a_{22}y_2 & + & a_{32}y_3 & \geqslant c_2 \\
 & y_i \geqslant 0 & & & & &
\end{array}
$$

2.1.2 Forms of Dual LP

Ref. P.22
Note two equivalences of linear constraints

1. $a^T x = b \Leftrightarrow a^T x \leqslant b$ AND $-a^T x \leqslant -b$
2. x_j free $\Leftrightarrow x_j = x'_j - x''_j$, where $x'_j, x''_j \geqslant 0$

max	$c^T x$	min	$b^T y$
s.t.	$Ax \leqslant b$	s.t.	$A^T y \geqslant c$
	$x \geqslant 0$		$y \geqslant 0$
max	$c^T x$	min	$b^T y$
s.t.	$Ax = b$	s.t.	$A^T y \geqslant c$
	$x \geqslant 0$		y free
max	$c^T x$	min	$b^T y$
s.t.	$Ax \leqslant b$	s.t.	$A^T y = c$
	x free		$y \geqslant 0$

To prove these alternative forms of Dual LP, use the equivalences 1 and 2, and then try to transform it to the standard primal-dual problem in row 1 of this table (i.e. try to transform equalities into inequalities, and free variables into variables $\geqslant 0$

2.2 Weak and Strong Duality

Ref. P.18-19

Weak duality	For any pair of primal and dual feasible solutions x and y: $\mathbf{c^T x \leqslant b^T y}$
Proof	*Ref. Lecture, Blackboard* Let (x, y) be feasible solutions of a primal LP and its corresponding dual. Therefore 1. $A\bar{x} \leqslant b$ 3. $\bar{y} \geqslant 0$ 2. $\bar{x} \geqslant 0$ 4. $A^T\bar{y} \geqslant c$ Then $b^T y \underbrace{\geqslant}_{1.,3.} (Ax)^T y = x^T A^T y \underbrace{\geqslant}_{4.,2.} x^T c = c^T x$
Remark II	If dual is unbounded / infeasible $\Rightarrow$ primal is infeasible / unbounded, because Weak Duality: $c^T x \leqslant b^T y$; $y \to -\infty$
Strong duality	If an LP has an optimal solution $\bar{x}$, then the dual has an optimal solution $\bar{y}$, and $\mathbf{c^T \bar{x} = b^T \bar{y}}$
Remark I *Proof for "$\Leftarrow$"*	If $\exists \bar{x}, \bar{y}$ s.t. $c^T\bar{x} = b^T\bar{y}$, then $\bar{x}$ and $\bar{y}$ are both optimal as by Weak duality $c^T x \leqslant b^T y$ $\forall y$, thus $\max c^T x \leqslant b^T y \Leftrightarrow c^T x = b^T y$

2.3 Complementary Slackness Conditions

Ref. P.19

The following conditions are equivalent for a dual pair of solutions $\bar{x}$ and $\bar{y}$

(a)	$\bar{x}$ and $\bar{y}$ are optimal solutions
(b)	$c^T\bar{x} = b^T\bar{y}$
(c)	$\bar{y}^T(b - A\bar{x}) = 0 \wedge \bar{x}^T(A^T\bar{y} - c) = 0$
(c')	$\bar{y}_i(b - A\bar{x})_i = 0 \ \forall i \wedge {\bar{x}_j}^T(A^T\bar{y} - c)_j = 0 \ \forall j$
(c")	$\bar{y}_i > 0 \Rightarrow (b - A\bar{x})_i = 0 \forall i \wedge \bar{x}_j > 0 \Rightarrow (A^T\bar{y} - c)_j = 0 \ \forall j$

Proof:

(b) $\Rightarrow$ (a)	by weak duality
(a) $\Rightarrow$ (b)	by strong duality
(b) $\Leftrightarrow$ (c')	Let $(\bar{x}, \bar{y})$ be feasible solutions.

Therefore $\begin{array}{llll} 1. & A\bar{x} \leqslant b & 3. & \bar{y} \geqslant 0 \\ 2. & \bar{x} \geqslant 0 & 4. & A^T\bar{y} \geqslant c \end{array}$

$$
\begin{aligned}
0 \underbrace{=}_{(b)} b^T\bar{y} - c^T\bar{x} &= b^T\bar{y} - \underbrace{\bar{y}^T A\bar{x}}_{\text{Skalar}} + \bar{y}^T A\bar{x} - c^T\bar{x} \\
&= \bar{y}^T(b - A\bar{x}) + \bar{x}^T(A^T\bar{y} - c) \\
&= \sum_{i=1}^{m} \underbrace{\bar{y}_i}_{3.)\geqslant 0} \underbrace{(b - A\bar{x})_i}_{1.)\geqslant 0} + \sum_{i=1}^{n} \underbrace{\bar{x}_j}_{2.)\geqslant 0} \underbrace{(A^T\bar{y} - c)_j}_{4.)\geqslant 0} = 0
\end{aligned}
$$

Therefore, $\bar{y}_i(b - A\bar{x})_i = 0 \wedge \bar{x}_j(A^T\bar{y} - c)_j = 0$ (c')

2.4 Recognition of Infeasibility

Ref. P.19-20

In principle, we just have to find two contradictory constraints or linear combinations of constraints. After Farkas' Lemma, this kind of proof is always possible.

Farkas' Lemma:	A System of inequalities $Ax \leqslant b, x \geqslant \mathbf{0}$ is **infeasible** $\Leftrightarrow \exists y \geqslant \mathbf{0} : A^T y \geqslant \mathbf{0} \wedge b^T y < 0$
Proof for $\Rightarrow$	$Ax \leqslant b \underbrace{\Rightarrow}_{y \geqslant 0} \underbrace{y^T A}_{\geqslant \mathbf{0}^T} x \nleqslant \underbrace{y^T b}_{<0} \ \forall x$

2.5 Recognition of Unboundedness

Ref. P.20-21
If there is a vector z, which can be multiplied by any $\alpha > 0$ and added to a feasible solution x, (i.e. $x' = x+\alpha z$), the LP is unbounded. Add a graphic (P.21)

Theorem 2.5 A LP $\max c^T x$, s.t. $Ax \leqslant b, x \geqslant 0$ is unbounded if and only if it has a feasible solution x and there exists (a direction) z such that $z \geqslant 0, Az \leqslant 0, c^T z > 0$

Proof Let $x' = x + \underbrace{\alpha}_{>0} z$

Because of...		this must hold
$x' \geqslant 0$	$\Rightarrow$	$z \geqslant 0$
$Ax' \leqslant b \Leftrightarrow Ax + A\alpha z \leqslant b$	$\Rightarrow$	$A\alpha z \leqslant 0 \Rightarrow Az \leqslant 0$
$c^T x' > c^T x \Leftrightarrow c^T x + c^T \alpha z) > c^T x$	$\Rightarrow$	$c^T \alpha z > 0 \Rightarrow c^T z > 0$

Important! x must be a **feasible** solution!

2.6 Coexistent Primal and Dual

		Dual		
		optimal	infeasible	unbounded
Primal	optimal	✓, s.D.	×, s.D.	×, w.D.
	infeasible	×, s.D.	✓	✓, w.D.
	unbounded	×, w.D.	✓, w.D.	×, w.D.

s.D. stands for "proven by strong duality", w.D. stands for "proven by weak duality". A LP which is primal infeasible and dual infeasible can be constructed as follows:

Primal:
$$\begin{array}{llrcl} \max & x_1 & & & \\ \text{s.t.} & & x_2 & \leqslant & -1 \\ & -x_1 & & \leqslant & 0 \\ & & x_1, x_2 & \geqslant & 0 \end{array}$$

Infeasible, because $-1 \geqslant x_2 \geqslant 0$ contradictory

Dual:
$$\begin{array}{llrcl} \min & -y_1 & & & \\ \text{s.t.} & & -y_2 & \geqslant & 1 \\ & y_1 & & \geqslant & 0 \\ & & y_1, y_2 & \geqslant & 0 \end{array}$$

Infeasible, because $1 \leqslant -y_2 \leqslant 0$ contradictory

3 LP Basics II

Omitted in this overview.

4 LP Algorithms

Ref. P.31

A system of inequalities can be converted into a system of equalities by introducing so called slack (dt: Schlupf) variables

$$\begin{array}{ccccccccccccc} a_{11}x_1 & + & a_{12}x_2 & \leqslant b_1 & \iff & a_{11}x_1 & + & a_{12}x_2 & + & x_3 & & & = b_1 \\ a_{21}x_1 & + & a_{22}x_2 & \leqslant b_2 & & a_{21}x_1 & + & a_{22}x_2 & & & + & x_4 & = b_2 \\ x_1 & \geqslant 0 & x_2 & \geqslant 0 & & x_1 & \geqslant 0 & x_2 & \geqslant 0 & x_3 & \geqslant 0 & x_4 & \geqslant 0 \end{array}$$

We can also introduce a slack variable for the objective function, simply by

$$c_1x_1 + c_2x_2 = x_f$$

By solving the above system of equalitites for x, we can rewrite the LP problem:

$$\begin{array}{lcccccccc} \max & x_f & = & 0 \cdot x_g & + & c_1x_1 & + & c_2x_2 \\ \text{s.t.} & x_3 & = & b_1 \cdot x_g & - & a_{11}x_1 & - & a_{12}x_2 \\ & x_4 & = & b_2 \cdot x_g & - & a_{21}x_1 & - & a_{22}x_2 \\ & & & x_g & = 1 \\ & x_i \geqslant 0, & i = \{1,2,3,4\} \end{array}$$

This can be written in short as $\mathbf{x_B} = \mathbf{D}\mathbf{x_N}$, where $B = \{f,3,4\} = \{f\} \cup B_0$,

$N = \{g,1,2\} = \{g\} \cup N_0$,

$D = \begin{bmatrix} 0 & c^T \\ b & -A \end{bmatrix}$

4.1 Transformation from canonical into dictionary form

Ref. P.32-33

$$\begin{array}{lllllll} & \multicolumn{2}{l}{\text{Canonical}} & \multicolumn{2}{l}{\text{Dictionary}} \\ \text{Primal} & \max & c^Tx & \max & x_f \\ & \text{s.t.} & Ax \leqslant b & \text{s.t.} & x_B & = & Dx_N \\ & & x \geqslant 0 & & x_g & = & 1 \\ & & & & x_j & \geqslant & 0 \qquad \forall j \setminus \{f,g\} \end{array}$$

The original x is replaced by x_{N_0}, while the new vector x has indices from $B \cup N$

$$\begin{array}{lllllll} \text{Dual} & \min & b^Ty & \max & y_g \\ & \text{s.t.} & A^Ty \geqslant c & \text{s.t.} & y_N & = & -D^Ty_B \\ & & y \geqslant 0 & & y_f & = & 1 \\ & & & & y_j & \geqslant & 0 \qquad \forall j \setminus \{f,g\} \end{array}$$

Proof for the latter is given by replacing y_f with 1 and removing y_{N_0}, i.e. replace the equality by a inequality, and finally by setting $B_0 = \{1,2,...,n\}$

4.2 Basic solutions and basic directions

Ref. P.35
For any primal and dual feasible solutions x and y

$$x^T y = x_B^T y_B + x_N^T y_N = 0$$

Proof: Impose $x_B = Dx_N$ and $y_N = -D^T y_B$
Insert handwritten images on P.35

Basic Vector	The basic vector $x(B,j)$ for $j \in N$ is the unique solution $\bar{x}$ to $x_B = Dx_N$ such that
	$\bar{x}_j = 1$
	$\bar{x}_{N-j} = \mathbf{0}$
	$\bar{x}_B = D._j$
Basic Solution	$x(B,g)$
Basic Directions	$x(B,j), j \in N - g$
Dual Basic Vector	The Dual basic vector $y(B,i)$ for $i \in B$ is the unique solution $\bar{y}$ to $y_N = -D^T y_B$ s.t.
	$\bar{y}_i = 1$
	$\bar{y}_{B-i} = \mathbf{0}$
	$\bar{y}_N = -(D_i)^T$
Dual Basic Solution	$y(B,f)$
Basic Directions	$y(B,i), i \in B - f$

4.3 LP Weak and Strong Duality

Ref. P.35, 36

LP Weak Duality For any LP in dictionary form and for any primal and dual feasible solutions x and y

$x_f + y_g \leqslant 0$

Proof

$$\begin{aligned} x_f + y_g &= x_f y_f + x_g y_g \quad x_g = y_f = 1 \\ &= x^T y - \sum_j \setminus \{f,g\} x_j y_j \quad x^T y = 0 \text{ (see subsection before)} \\ &= -\sum_{j \setminus \{f,g\}} x_j y_j \leqslant 0 \quad x_j, y_j \geqslant 0 \text{ by definition of LP} \end{aligned}$$

Corollary $\exists x, y : x_f + y_g = 0 \Rightarrow x, y$ are optimal

Proof If not, $\exists c > 0 : x_f + y_g = -c < 0$

Then, one can always find a better x_f, namely $x_f + c$

LP Strong Duality For any LP in dictionary form

a. If the primal and dual LPs are both feasible then both have optimal solutions x_f, y_g and $x_f + y_g = 0$

b. If either the primal or dual LP is infeasible, then neither of the LPs has an optimal solution. If the dual (primal) LP is infeasible, the primal (dual) LP is either infeasible or unbounded.

4.4 Types of Dictionaries

Ref. P.37, Figures 4.1 and 4.2
Proposition 4.4
For any LP in dictionary form,

a) If the dictionary is feasible then the associated basic solution is feasible.

b) If the dictionary is dual feasible then the associated dual solution is feasible.

c) If the dictionary is both primal and dual feasible then the associated basic solution $\bar{x}$ and the associated dual basic solution $\bar{y}$ are optimal, and furthermore $\bar{x}_f + \bar{y}_g = 0$

Proof

a) If the dictionary is feasible, the basic solution $x(B, g)$ satisfies all the nonnegativity conditions and thus is feasible.

b) If the dictionary is dual feasible, the dual basic solution $y(B, f)$ satisfies all the nonnegativity conditions and thus is dual feasible.

c) Assume the dictionary is both primal and dual feasible. Since $x(B, g)_f = d_{fg}$ and $y(B, f)_g = -d_{fg}$, they add up to zero. By the LP Weak Duality, they are both optimal solutions.

Proposition 4.5
For any LP in dictionary form,

a) If the dictionary is **inconsistent** then the LP is infeasible and the **dual** LP is **unbounded**.

b) If the dictionary is **dual inconsistent** then the LP is infeasible and the **primal** LP is **unbounded**.

4.5 Pivot Operation

Ref. P.40

$$D = \begin{array}{c|ccccc}
 & & j & & s & \\
\hline
 & & \vdots & & \vdots & \\
i & \cdots & d_{ij} & \cdots & d_{is} & \cdots \\
 & & \vdots & & \vdots & \\
r & \cdots & d_{rj} & \cdots & d_{rs} & \cdots \\
 & & \vdots & & \vdots & \\
\hline
\end{array} \qquad i \in B - r, j \in N - s$$

$$\text{Pivot on (r,s):} \begin{array}{l} \text{Update basis: } B' = B - r + s, \\ \text{Update nonbasis: } N' = N + r - s, \\ \text{Update Dictionary: } \Downarrow \end{array}$$

$$D' = \begin{array}{c|ccccc}
 & & j & & r & \\
\hline
 & & \vdots & & \vdots & \\
i & \cdots & d_{ij} - \dfrac{d_{is} \cdot d_{rj}}{d_{rs}} & \cdots & \dfrac{d_{is}}{d_{rs}} & \cdots \\
 & & \vdots & & \vdots & \\
s & \cdots & -\dfrac{d_{rj}}{d_{rs}} & \cdots & \dfrac{1}{d_{rs}} & \cdots \\
 & & \vdots & & \vdots & \\
\hline
\end{array} \qquad i \in B - s, j \in N - r$$

Proposition 4.6
Suppose D' is obtained by a pivot operation on D on the position (r,s)

a) (Reversibility) The result of a pivot operation of D' on (s, r) is D

b) (Equivalence) The associated equation systems are equivalent $\Rightarrow V(D) = V(D')$, V denotes the set of solutions

c) (Dual equivalence) The associated dual equation systems are equivalent $\Rightarrow V(-D^T) = V(-D'^T)$

4.6 Criss Cross Method

Ref. Figure on P.72 and Pseudocode on P.43
Theorem 4.7, Corollary 4.8, Lemma 4.9 are from pages 43 to 46. There is nothing to summarize, it is already quite compact.
Most important results: Criss Cross Method is finite and ends in optimal, inconsistent or dual inconsistent state. Once a state is achieved, no other state can be achieved. Thus, the strong duality holds.

4.7 Simplex Method

Ref. Figure on P.71 and Pseudocode P.49

Simplex Pivot

A Simplex Pivot is a Criss Cross Pivot with the following additional constraints.

a) $d_{rs} < 0$

b) $-\frac{d_{rg}}{d_{rs}} = \min\left\{-\frac{d_{ig}}{d_{is}} : i \in B - f, d_{is} < 0\right\}$

Additionally, it should be chosen following the smallest index rule, as with the Criss Cross pivot. Choosing the pivot like this preserves feasibility for a basic solution (i.e. d_{ig} stays positive after a pivot operation)

Proof: Let (r,s) the Simplex pivot with the above constraints.

$$\text{Pivot Operation for } \; d_{ig}, i \neq r: \quad d_{ig} - \frac{d_{rg}}{d_{rs}} \cdot d_{is} \geqslant^{?} 0$$

$$\Leftrightarrow -\frac{d_{rg}}{d_{rs}} \cdot d_{is} \geqslant^{?} -d_{ig}$$

$$d_{is} > 0 \swarrow \quad \Big| \quad \searrow d_{is} < 0$$

$$\underbrace{-\frac{d_{rg}}{d_{rs}}}_{\geqslant 0} \geqslant \underbrace{-\frac{d_{ig}}{d_{is}}}_{\leqslant 0} \checkmark \quad \Big| \quad -\frac{d_{rg}}{d_{rs}} \leqslant -\frac{d_{ig}}{d_{is}} \checkmark \text{ b.) LHS is most negative}$$

$$\text{Pivot Operation for } \; d_{rg}: \quad -\frac{d_{rg}}{d_{rs}} \geqslant^{?} 0 \checkmark \text{ a.) } d_{rs} < 0$$

Input handwritten Simplex Method Phase I

4.8 Implementing Pivot Operations

Ref. P.51

Consider a LP $\max c^T x$, s.t. $Ax = b$, $x \geqslant 0$, i.e. with slack variables (See P.31). The large $m \times E$ Matrix A can be seen as follows: A =

B	$N = E \setminus B$
$A_{\cdot B}$	$A_{\cdot N}$

Note that $A_{\cdot B}$ is quadratic and invertible. Essentially, we split everything in terms of basic and nonbasic parts. Therefore,

$$\begin{array}{rrcl} Ax = b & \Leftrightarrow \quad A_{\cdot B}x_B + A_{\cdot N}x_N & = & b \\ & \Leftrightarrow \quad A_{\cdot B}^{-1}(A_{\cdot B}x_B + A_{\cdot N}x_N) & = & A_{\cdot B}^{-1}b \\ & \Leftrightarrow \quad x_B & = & A_{\cdot B}^{-1}b - A_{\cdot B}^{-1}A_{\cdot N}x_N \end{array}$$

Plugging in x_B in the objective function yields:

$$\begin{array}{rcl} c^T x & = & c_B^T x_B + c_N^T x_N \\ & = & c_B^T(A_{\cdot B}^{-1}b - A_{\cdot B}^{-1}A_{\cdot N}x_N) + c_N^T x_N \\ & = & c_B^T A_{\cdot B}^{-1}b + (c_N^T - c_B^T \cdot A_{\cdot B}^{-1}A_{\cdot N})x_N \end{array}$$

As a consequence, a dictionary can be rewritten as follows:

$$D = \begin{array}{c} \\ f \\ B-f \end{array} \begin{array}{c} \begin{array}{cc} \quad g \quad & \qquad N-g \end{array} \\ \begin{pmatrix} c_B^T A_{\cdot B}^{-1}b & c_N^T - c_B^T A_{\cdot B}^{-1}A_{\cdot N} \\ A_{\cdot B}^{-1}b & -A_{\cdot B}^{-1}A_{\cdot N} \end{pmatrix} \end{array}$$

For $B - f$, D can be read as

$$\begin{array}{rrcl} & x_B & = & A_{\cdot B}^{-1}b - A_{\cdot B}^{-1}A_{\cdot N}x_N \\ \Leftrightarrow & x_B + A_{\cdot B}^{-1}A_{\cdot N}x_N & = & A_{\cdot B}^{-1}b \\ \Leftrightarrow & (I \mid A)\begin{pmatrix} x_B \\ x_N \end{pmatrix} & = & A_{\cdot B}^{-1}b \end{array}$$

Recall the pivot operation: It means to swap a certain basic variable (r) with a certain nonbasic variable (s), i.e. basis and nonbasis get updated as follows: $B' = B - r + s, N' = N + r - s$. Here, we don't need to look at the operations defined by the pivot operation for the actual dictionary values, as we have decomposed the dictionary in terms of $b, c, A_{\cdot B}^{-1}$ and $A_{\cdot N}$.
So, the only difficulty is to get the updated basis inverse $A_{\cdot B'}^{-1}$. Note that $A_{\cdot B'} = A_{\cdot B-r+s}$, i.e. this is only a column swap. Luckily, there is a matrix T, which performs the update of the basis inverse, i.e. $T \cdot A_{\cdot B}^{-1} = A_{\cdot B'}^{-1}$

$$T = \begin{pmatrix} 1 & & & -\frac{d_{1s}}{d_{rs}} & & & \\ & \ddots & & \vdots & & & \\ & & 1 & -\frac{d_{(r-1)s}}{d_{rs}} & & & \\ & & & \frac{1}{d_{rs}} & & & \\ & & & -\frac{d_{(r+1)s}}{d_{rs}} & 1 & & \\ & & & \vdots & & \ddots & \\ & & & -\frac{d_{ms}}{d_{rs}} & & & 1 \end{pmatrix}$$

(Caution! This is wrongly stated in the Musterlösung of Ex.6)

4.9 Computing Sensitivity

Ref. P.53
Thanks to the above gained alternative writing of D, feasibility certificates can also be reexpressed.

Primal feasibility	$A_{\cdot B}^{-1} b \geqslant \mathbf{0}$
Dual feasibility	$c_N^T - c_B^T A_{\cdot B}^{-1} A_{\cdot N} \leqslant \mathbf{0}$

Having these certificates, we can easily compute the sensitivity of a certain b_i (supply) or c_i (price)
Insert Example (Ex.6 No.2)

4.10 Geometry of Pivots

Ref. P.57ff.

Combinatorial Diameter	Ω: Maximum of the lengths (in terms of edges) of all shortest pathes between any pair of vertices
Hirsch conjecture	For bounded polyhedra, Ω is at most $n - d$, where $n = \#$ inequalities, $d =$ dimension
Corollary	If you can choose between computation of primal or dual, always take the one with the higher dimension, as pivot algorithms have to check all vertices in worst-case.
Convexity of feasible region	The feasible region Ω is convex. *Proof:* Let $x_1, x_2 \in \Omega$ and let x be a convex combination of x_1, x_2. We want to show, that $Ax \leqslant b$ $Ax = A(\lambda x_1 + (1-\lambda)x_2) = \lambda A x_1 + (1-\lambda) A x_2 \leqslant \lambda b + (1-\lambda) b = b$