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Dynamics of Large-Scale Atmospheric Flows

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Basics

Notation

$$\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\frac{D}{Dt} = \frac{D_h}{Dt} + w\frac{\partial}{\partial z}$$

$$i = (1,0,0)$$
$$j = (0,1,0)$$
$$k = (0,0,1)$$

Coordinate system

Spherical coordinate system

$$x = r\sin\phi\cos\lambda$$
$$y = r\sin\phi\sin\lambda$$
$$z = r\cos\phi$$

Unit vectors on the sphere

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}: i = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$

$$j = \begin{pmatrix} -\sin\theta\cos\lambda \\ -\sin\theta\sin\lambda \\ \cos\theta \end{pmatrix}$$

$$k = \begin{pmatrix} \cos\theta\cos\lambda \\ \cos\theta\sin\lambda \\ \sin\theta \end{pmatrix}$$

Wind vectors on the sphere

$$\frac{Di}{Dt} := u = r\cos\phi\frac{D\lambda}{Dt}$$
$$\frac{Dj}{Dt} := v = r\frac{D\phi}{Dt}$$
$$\frac{Dk}{Dt} := w = \frac{Dr}{Dt}$$

Basic equations

6 equations for 6 unknowns u, v, w, ρ, p, T

Equations of motion (Navier-Stokes)

x, y, z are spherical unit coordinates (eastward, northward, vertical). Equations are simplified through β-plane.

β-plane approximation

$f = f_0 + \beta y, \beta = \frac{2\Omega\cos\phi_0}{a}$

a = Earth radius

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Continuity equation

$$\frac{D\rho}{Dt} + \rho(\nabla\cdot\boldsymbol{u}) = 0$$

Equation of state

$$pV = nRT$$
$$\Leftrightarrow p = \rho RT$$

Thermodynamic equation

$$\frac{Dln\theta}{Dt} = \frac{1}{\theta}\frac{D\theta}{Dt} = \frac{1}{c_pT}\underbrace{\frac{DH}{Dt}}_{\sum diabatic\ processes}$$

Large-scale approximations (Synoptic-Scale Motions)

Large-scale approximations hold for the sea breeze and cumulus scale.

Vertical component decomposition

Due to the strong vertical stratification of the atmosphere, it is useful to decompose a field variable χ as follows.

$$\chi = \chi_0(z) + \chi^*(x, y, z, t)$$

At a certain height,

$$\chi_0 \gg |\chi^*|$$

This holds for $\chi = p, \rho, \theta, T$

Equation of state approx

$$\frac{\rho^*}{\rho_0} = \frac{p^*}{p_0} - \frac{T^*}{T_0}$$

Potential temperature approx

$$\frac{\rho^*}{\rho_0} \approx -\frac{\theta^*}{\theta_0}$$

Vertical momentum equation approx

$$\frac{Dw}{Dt} \approx -\left(\frac{\partial}{\partial z} - \frac{N^2}{g}\right)\frac{p^*}{p_0} + g\frac{\theta^*}{\theta_0}$$

Hydrostatic approximation

Using: Vertical momentum equation

$$\frac{Dw}{Dt} \approx 0 \Rightarrow \frac{\partial p}{\partial z} = -\rho g$$

Geostrophic wind

Using: Horizontal momentum equation

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Equivalently,

$$v_G = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} = k \times \frac{1}{f\rho}\nabla_h p$$

- Geostrophic wind field is non-divergent
$$\nabla_h(\rho_0 v_G) = -\left(\frac{1}{f}\frac{\partial f}{\partial y'}\right)\rho_0 v_G < 0$$
- Geostrophic approximation not valid for large Rossby numbers (Ro >> 1)

Thermal wind

Using: Hydrostatic & geostrophic approximations & equation of state

$$\underbrace{\frac{\partial v_G}{\partial z}}_{v_{th}} = \frac{g}{fT}k \times \nabla_h T$$

- Left turning of geostrophic wind => Cold air advection
- Right turning of geostrophic wind => Warm air advection

Aka thermal wind relationship is the following

$$\frac{g}{\theta_0}\frac{\partial \theta^*}{\partial x} = f_0\frac{\partial v_G}{\partial z}$$

Vorticity

ζ: Vertical component of vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \boldsymbol{k} \times (\nabla \times \boldsymbol{u})$$

$$\zeta = \begin{cases} \text{cyclonic circulation,} & \zeta > 0 \\ \text{anticyclonic circulation,} & \zeta < 0 \end{cases}$$

Vorticity equation

$$\frac{D\zeta}{Dt} + \beta v = \begin{array}{c} \underbrace{-(\zeta + f)(\nabla_h v)}_{divergence\ effect} \\ \underbrace{-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}_{twisting/tilting} \\ \underbrace{+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial p}{\partial y}\frac{\partial p}{\partial x}\right)}_{solenoidal\ effect} \end{array}$$

Synoptic scale approximation

$$\frac{D_h}{Dt}\zeta + \beta v = -f_0(\nabla_h \boldsymbol{v})$$

βv: Meridional excursion

$-f_0(\nabla_h \boldsymbol{v})$: Flow convergence / divergence

Assumptions

- Invalid in frontal regions with $L \approx 10^5, \zeta \approx f, w \approx 10^{-1}\mathrm{ms}^{-1}$
- Caution: $f \leq 10^{-5}\mathrm{s}^{-1}$ in tropical and equatorial regions
- If $\nabla_h v \approx 0$, then the vorticity equation is reduced to the **barotropic vorticity equation**.

Quasi-geostrophic (QG) approximation

Simplifications

$$f \approx f_0$$
$$\zeta \approx \zeta_G$$
$$\frac{D_h}{Dt} \approx \frac{D_G}{Dt}$$
$$\Psi = \frac{p^*}{f_0\rho_0}$$

Subscript G denotes the usage of the geostrophic wind, i.e. $f_G(\boldsymbol{u}) = f(\boldsymbol{u_G})$

QG Equations

Geostrophic approximation	$u_G = -\frac{\partial \Psi}{\partial y}; v_G = \frac{\partial \Psi}{\partial x}$

Hydrostatic approximation	$g\frac{\theta^*}{\theta_0} = f_0\frac{\partial\Psi}{\partial z}$
Vorticity equation	$\frac{D_G}{Dt}\zeta_G + \beta\frac{\partial\Psi}{\partial x} = -f_0(\nabla_h\boldsymbol{v})$
Mass conservation	$\nabla_h\boldsymbol{v} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w) = 0$
Thermodynamic equation	$\frac{D_G}{Dt}\left(f_0\frac{\partial\Psi}{\partial z}\right) + N^2 w = 0$

Implications on vorticity / westward tilt

- $\zeta_G = \nabla_h^2\Psi \propto -p^*$
  Positive / negative values of ζ_G are associated with low / high pressure
- $\frac{\partial}{\partial z}\zeta_G = \nabla_h^2\left(\frac{\partial}{\partial z}\Psi\right) = \frac{g}{f_0\theta_0}\nabla_h^2\theta^* \propto -\theta^*$
  Positive / negative values of $\frac{\partial}{\partial z}\zeta_G$ are associated with low / high temperature
- Both effects together lead to the westward slope of cyclones and anticyclones.

Ageostrophic wind

Split velocity in basic state and perturbation. The geostrophic wind $\boldsymbol{u_G}$ is the basic state while the ageostrophic wind $\boldsymbol{u_a}$ is the perturbation.

$$\boldsymbol{u} = \underbrace{(u_G, v_G, 0)}_{\boldsymbol{u_G}} + \underbrace{(u_a, v_a, w)}_{\boldsymbol{u_a}}$$

Solution of QG inconsistency

Geostrophic approximation ⇒ $\nabla_h\boldsymbol{v_G} = 0$, but
Mass conservation ⇒ $\nabla_h\boldsymbol{v} \neq 0$

Solution: $\boldsymbol{v} = (u_G, v_G) + (u_a, v_a)$ where $\nabla_h(u_a, v_a) \neq 0$, so $\nabla_h v = \underbrace{\nabla_h\boldsymbol{v_G}}_{=0} + \nabla_h\boldsymbol{v_a} \neq 0$

Some properties

- $div\ \boldsymbol{v_G} = 0$
- $div\ \boldsymbol{v_a} \neq 0$
- $|\boldsymbol{u_a}| \ll |\boldsymbol{v_G}|$

Linkage of ageo- and geostrophic wind

Assumptions: $N^2 = const., \beta = 0, \rho_0$=const.

From thermodynamic equation and vorticity equation using hydrostatic approximation.

$$N^2\frac{\partial w}{\partial x} - f_0^2\frac{\partial u_a}{\partial z} = 2Q_1$$

$$N^2\frac{\partial w}{\partial y} - f_0^2\frac{\partial v_a}{\partial z} = 2Q_2$$

$$Q_1 = f_0\left(\frac{\partial u_G}{\partial z}\frac{\partial v_G}{\partial x} + \frac{\partial v_G}{\partial z}\frac{\partial v_G}{\partial y}\right) = -\frac{g}{\theta_0}\left(\frac{\partial u_G}{\partial x}\frac{\partial\theta^*}{\partial x} + \frac{\partial v_G}{\partial x}\frac{\partial\theta^*}{\partial y}\right)$$

$$Q_2 = -f_0\left(\frac{\partial u_G}{\partial z}\frac{\partial u_G}{\partial x} + \frac{\partial v_G}{\partial z}\frac{\partial u_G}{\partial y}\right) = -\frac{g}{\theta_0}\left(\frac{\partial u_G}{\partial y}\frac{\partial\theta^*}{\partial x} + \frac{\partial v_G}{\partial y}\frac{\partial\theta^*}{\partial y}\right)$$

Diagnostic w equation (vertical wind)

$$2\,\nabla\cdot\boldsymbol{Q} = N^2(\nabla_h^2 w) + f_0^2\left(\frac{\partial^2 w}{\partial z^2}\right)$$

With $2\nabla\cdot\boldsymbol{Q} = 2\left(\frac{\partial}{\partial x}Q_1 + \frac{\partial}{\partial y}Q_2\right)$

Ageostrophic wind is completely determined by geostrophic wind!

4-step golden rule

1. $\nabla\cdot Q \approx \nabla^2 w \approx -w$
2. $\frac{D_G}{Dt}\zeta_G \propto \frac{\partial}{\partial z}(\rho_0 w)$
3. $\frac{D_G}{Dt}\zeta_G \begin{cases} > 0 & \text{Cyclogenesis} \\ < 0 & \text{Anticyclogenesis} \end{cases}$

1. W-equation
2. Combination of vorticity equation and mass conservation
3. Approximating ζ with $\frac{D_G}{Dt}\zeta$?

Finding $\nabla\cdot Q$

1. Find largest Q-vector. Arrowhead is zone of convergence ($\nabla\cdot Q < 0$), tail is zone of divergence
2. $Q = -\frac{g}{\theta_0}|\nabla_h\theta^*|\left(\boldsymbol{k}\times\frac{\partial}{\partial\xi}\boldsymbol{v_G}\right)$
3. Find largest temperature gradient and a strong wind change along the isentrope

Thermal steering effect

Neighboring low and high pressure cells tend to move perpendicular to the isentropes. This coincides with the direction of thermal wind.

Development on the left exit of a jet

From a jetstream's point of view, on the left side of its exit, there is a zone of divergence and associated with it, there is upward motion.

QG Potential Vorticity

1. Combination of vorticity equation and mass conservation.
2. Combine this with thermodynamic equation
3. Interchange $\frac{\partial}{\partial z}$ and $\frac{D_G}{Dt}$

$$\frac{D_G}{Dt}\left[\underbrace{\zeta_G + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial \Psi}{\partial z}\right)}_{=:q=QGPV} + \beta y\right] = 0$$

- $q \approx \nabla^2 \Psi$
- q = relative vorticity + static stability
  - Static stability: From hydrostatic equation: $\frac{\partial}{\partial z}\theta^* \propto \frac{\partial^2}{\partial z^2}\Psi$
- Only valid for adiabatic flow.
- Boundary condition: $\frac{D_G}{Dt}\left(f_0 \underbrace{\frac{\partial \Psi}{\partial z}}_{g\theta^*/\theta_0}\right) = 0$

QG Prognostic system

1. $q(t_0)$ known with B.C. $\theta^*(t_0, z = 0)$
2. Main diagnostic step
   a. Solve $q = \nabla^2 \Psi$
   b. Ψ yields u_G, v_G and θ^*
   c. These yield the Q-vectors
   d. The Q-vectors yield w
3. Do prognostic step using passive advection of $\frac{D_G}{Dt}(q + \beta y) = 0$ and go to 2.

QGPV key properties

- QG flow determined by interior distribution of QGPV and surface distribution of θ^*
- Conservation: $\frac{D_G}{Dt}(q + \beta y) = 0$
- Invertibility: $q = \nabla^2 \Psi$
- Partition and attribution: $q = \sum_j q_j$

Prototype vortex for QG flow

Since QGPV can be decomposed into "atomic" vortices, we can analyse prototype vortices.

Given:

- $q = \begin{cases} p, & r < a \\ 0, & r \geq a \end{cases}$

Then:

- $v = \begin{cases} \Omega R, & r < a \\ \frac{k}{r^2}, & r \geq a \end{cases}$
- Here, r is the spherical symmetrical coordinate, and R is the "x" coordinate

Ertel PV

Idea: Analogue derivation to QGPV, but starting from full NSE.

Applicable only for inviscid and adiabatic flow.

$$\frac{D}{Dt}\left[\underbrace{\frac{1}{\rho}\boldsymbol{\eta} \cdot \nabla\theta}_{Ertel\ PV}\right] = 0$$

- $\boldsymbol{\eta} = rot\ \boldsymbol{u} + 2\boldsymbol{\Omega}$, absolute vorticity vector

Isentropic PV

$$\frac{D}{Dt}\left[\underbrace{\frac{1}{\rho}(\zeta + f) \cdot \frac{\partial}{\partial z}\theta}_{IPV}\right] = 0$$

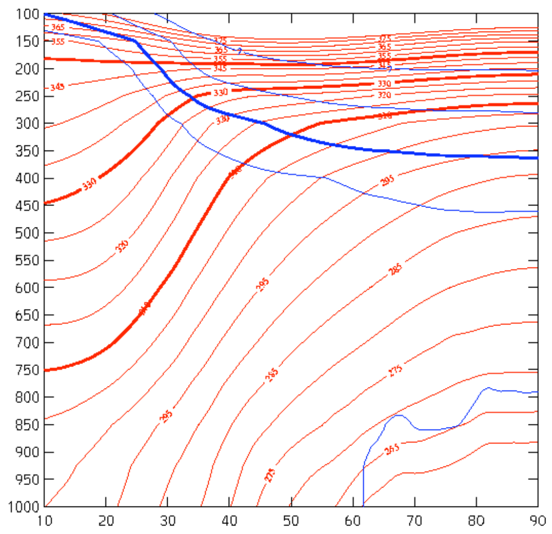
- IPV = z-component of PV
- $\frac{1}{\zeta} \propto \frac{\partial}{\partial z}\theta$

Preliminaries for PV chart analysis

- Unit for PV: 1 pvu. 2 pvu are defined as the tropopause.
- $\frac{\partial \theta}{\partial z} > 0$ & θ increases towards equator.
- Max(PV) at equator due to Coriolis parameter.
- $\frac{\partial PV}{\partial z} > 0$, because $\frac{\partial \theta}{\partial z} > 0$ & $\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right) > 0$
- For adiabatic and inviscid flow
$$\frac{D\theta}{Dt} = 0 \ \& \ \frac{DPV}{Dt} = 0$$

PV anomaly / cyclogenesis

- Adiabatic movement of air parcel towards south => stays on isentrope.
- As it must retain PV, it "drags" its PV value along the isentrope, especially when intersecting the 2 pvu tropopause. A PV anomaly occurs.
- Because of far-field effect of a strong PV gradient (=> strong wind), a cyclone forms at ground.



Idealised PV situations

Effect: Strong PV gradient w.r.t latitude => Strong wind.
Reason: High PV => CCW flow, low PV => CW flow. Strongest wind at strongest gradient. C.f. Jetstream and PV

Effect: Positive / negative upper-level PV anomaly ⇒ CCW / CW circulation and cold / warm below, warm / cold above
Reason: Positive upper-level PV anomaly => higher ζ (=> CCW flow) and lower $\frac{\partial\theta}{\partial z}$

Effect: Positive / negative surface temp. anomaly ⇒ CCW / CW circulation and warm / cold above
Reason: Negative sfc temp: Isentropes squeezed, higher $\frac{\partial\theta}{\partial z}$ => lower ζ => CW flow

Effect: Diabatically produce low-level PV anomalies
Reason: Microphysical processes

Effect: PV tower. If all anomalies are aligned on top of each other, they intensify and form a devastating storm.

Diabatic PV

$$\frac{D}{Dt}PV \approx -g(f+\zeta)\frac{\partial\dot{\theta}}{\partial p}$$

Important: $\frac{D}{Dt}PV$ is only proportional to $\frac{\partial\dot{\theta}}{\partial p}$, not $\dot{\theta}$

Insert: Image with diabatic PV through a cloud

QG Wave theory

Examples

- Large scale flows in mid and upper-troposphere
- Planetary scale quasi-stationary and quasi-steady wave features in stratosphere

Waves on a uniform zonal flow

- Set: $\Psi(y) = -\mathcal{U}y$
- Split quantities in basic state and perturbations
  $\psi = \Psi(y) + \psi'(x,y,z,t), |\psi'| \ll |\Psi|$

From the QG potential vorticity equation, inserting ψ and linearizing (i.e. discarding terms like $a'b'$) yields the perturbation equation:

$$\left(\frac{\partial}{\partial t} + \mathcal{U}\frac{\partial}{\partial x}\right)q' + \frac{\partial\bar{q}}{\partial y}v' = 0$$

- $\Psi(y) = -\mathcal{U}y$, $\nabla_h^2\Psi = 0$ and $\frac{\partial}{\partial z}\Psi = 0$, thus $\bar{q} = 0 + \beta y = \beta y$
- $\frac{\partial\bar{q}}{\partial y} = \beta$
- $q' = \nabla_h^2\psi' + \frac{f_0^2}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2}\frac{\partial\psi'}{\partial z}\right)$
  QGPV definition

2d wave solution (aka Rossby wave)

Solution on a mid-latitude band (all longitudes, latitudes 30-90°N)

$$\psi' = A\sin(kx - \omega t)\sin\big(l(y+d)\big)$$

Dispersion relationship

$$\omega = \mathcal{U}k - \frac{\beta k}{k^2 + l^2}$$

- $k = \frac{2\pi}{\mathcal{L}} = \frac{2m}{a}$
  - East-West wavelength: $m\mathcal{L} = \pi a$
  - m=1 => One full sine period
- $l = \frac{2\pi}{\mathcal{M}} = \frac{3m'}{a}$
  - North-South half-wavelength: $m'\left(\frac{\mathcal{M}}{2}\right) =$
  -

Phase velocities

- $u_p = \frac{\omega}{k} = \mathcal{U} - \frac{\beta}{k^2+l^2}$
- $v_p = \frac{\omega}{l}$

Effects

$m' = 1; m$	1	2	3	4
$-\dfrac{\beta}{k^2+l^2}$	-35.5	-18.5	-6.0	-1.7

- **Short waves (large m) typically move with $\mathcal{U}$**
- **Long waves (small m) move against $\mathcal{U}$**

Group velocities

- $u_g = \frac{\partial \omega}{\partial k} = \mathcal{U} + \frac{\beta(k^2-l^2)}{(k^2+l^2)^2}$
- $v_g = \frac{\partial \omega}{\partial l} = 2\beta \frac{kl}{(k^2+l^2)^2}$

Effects

- $k^2 < l^2 \Rightarrow (u_g - \mathcal{U}) < 0$
- $k^2 > l^2 \Rightarrow (u_g - \mathcal{U}) > 0$
- Longitudinal waves propagate eastwards
- Latitudinal waves propagate westwards

More effects

$$\frac{v_g}{u_g} = \tan\alpha = \frac{2kl}{k^2 - l^2}$$

Westward translation of a sinusoidal displacement of a fluid

3d wave solution

$$\psi'(x,y,z,t) = \chi'(x,y,z,t)e^{\frac{z}{2H_0}}$$

Yields a slightly different perturbation equation with solutions of the form:

$$\chi' = \psi(z)\sin(kx - \omega t)\sin l(y+d)$$

With the following constraints

- $\psi_{zz} + n^2\psi = 0$
- $n^2 = \left(\frac{N}{af_0}\right)^2 \left[\frac{\beta a^2}{\mathcal{U}-c} - (4m^2 + 9m'^2 + \alpha^2 a^2)\right]$
- $(\omega - \mathcal{U}k)\left(\psi_z + \frac{1}{2H_0}\psi\right) = 0$

Effect: Vertical propagability

- $\psi \propto e^{\pm inz}$ for $n^2 > 0$
- $\psi \propto e^{\mp nz}$ for $n^2 < 0$

Discard exponential solution. Vertically propagating waves only occur for $n^2 > 0$

$\boldsymbol{n^2 > 0} \Leftarrow$

- $(\mathcal{U} - c) > 0$: Wave velocity c must be easterly relative to zonal flow $\mathcal{U}$
- $(\mathcal{U} - c) < \frac{\beta}{k^2+l^2+\alpha^2}$: For typical values of β, k, l, α
- The above criteria are only satisfied for low zonal wavenumbers m, since

Zonal wavenumber	$\dfrac{\beta}{k^2+l^2+\alpha^2}$
m=1	$55\ ms^{-1}$
m=2	$38\ ms^{-1}$
m=3	$24\ ms^{-1}$

Real effect: **From the troposphere, only m=1 or m=2 waves propagate to the stratosphere.**

Orographic forcing

Setting: Stationary, incompressible wave ($\frac{D}{Dt} = 0, \rho_0 = const.$), subject to sinusoidal terrain

PV equation

$$\mathcal{U}\frac{\partial}{\partial x}\left[\psi'_{xx} + \psi'_{yy} + \frac{f_0^2}{N^2}\psi'_{zz}\right] + \beta\frac{\partial \psi'}{\partial x} = 0$$

Terrain (η) and Thermodynamic B.C.

$$\eta = \eta_0 \cos(kx)\sin\left[\frac{\pi}{2d}(y+d)\right]"$$

$$\frac{\partial \eta}{\partial x} = \frac{\omega'}{\mathcal{U}}$$

Solution

$$\psi' \propto -\sin(kx + nz)\sin\left[\frac{\pi}{2d}(y+d)\right], n^2 > 0$$

$$\psi' \propto \cos(kx)e^{-\mu z}\sin\left[\frac{\pi}{2d}(y+d)\right], n^2 < 0$$

$$n^2 = f(\mathcal{U}, \dots)\begin{cases} > 0, & m = 1,2,3; m' = 1,2 \\ < 0, & m > 3 \end{cases}$$

There are large Rossby waves induced by (large) mountains such as the Rocky mountains, Himalaya or Greenland.

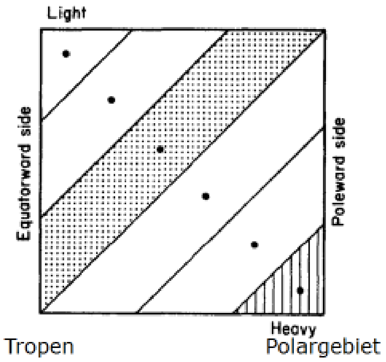
They propagate vertically for low wavenumbers (m <= 3) and decay vertically for high wavenumbers (m > 3)

Diabatic forcing

Large planetary scale diabatic heating distributions generate **vertically propagating waves at high altitudes. At the surface, the low and high pressure centres are displaced**

about ¼ wavelength eastward from centres of the diabatically heated and cooled regions.

Baroclinic instability



Baroclinic atmosphere

Eady problem

Set: $\mathcal{U} = \frac{\Lambda z}{d}$, where d = height of atmosphere

PV equation

$$\left(\frac{\partial}{\partial t} + \mathcal{U}\frac{\partial}{\partial x}\right)\left(\frac{\partial^2 \psi'}{\partial x^2} + \frac{f_0^2}{N^2}\frac{\partial^2 \psi'}{\partial z^2}\right) = 0$$

Thermodynamic boundary condition

$$\left(\frac{\partial}{\partial t} + \mathcal{U}\frac{\partial}{\partial x}\right) f_0 \frac{\partial \psi'}{\partial z} - \left(f_0 \frac{\Lambda}{d}\right)\frac{\partial \psi'}{\partial x} = 0$$

Ansatz

$$\psi' = \Psi(z) e^{i(kx - \omega t)}$$

Solution

$$\Psi(z) = A \sinh(\mu z) + B \cosh(\mu z)$$

Where $\mu^2 = \frac{k^2 N^2}{f_0^2}$

$$\omega = k\sqrt{\gamma}$$

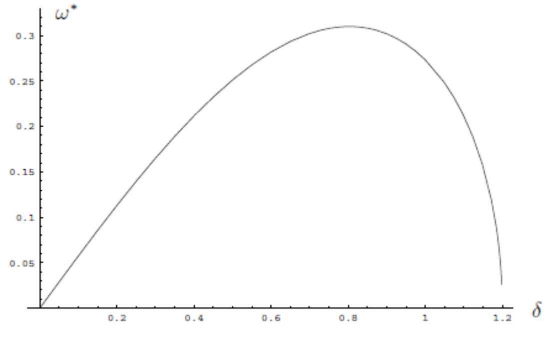
Where $\gamma = -\left(\frac{\Lambda}{\mu d}\right)^2 (\delta - \tanh\delta)(\coth\delta - \delta)$

Where $\delta = \frac{1}{2}\mu d$

Exponential growth solution

If $\sqrt{\gamma} \in \mathbb{C}$ $(\Leftrightarrow \gamma < 0)$, then $\psi' \propto e^{\omega t}$

$\gamma < 0$ for $0 < \delta < 1.1997$



Eady growth rate

Nature of solution

- Wavelength $(2\pi/k)$ for $\delta = 1.1997$
  $$\mathcal{L} \approx 2500\ km$$
- Wavelength for δ_{max}
  $$\mathcal{L} \approx 4000\ km$$
- Growth rate for δ_{max}
  $$T_e = \frac{1}{\omega} \approx 1.1\ days$$
- **All perturbations of wavelength greater than 2500 km are unstable**

Notions

- Cyclonic = Counter clockwise = Left turn
- Anticyclonic = Clockwise = Right turn
- Zonal wavenumber = m
- North-south petal count = m'
- Large waves ⇔ small m
- Short waves ⇔ large m
- $\boldsymbol{u} = (u, v, w)$
- $\boldsymbol{v} = (u, v) = \boldsymbol{u}(0,1)$

Numerical values

Earth radius	a	$6.37 \cdot 10^6 \mathrm{m}$
Gas constant of air	R	$2.87 \cdot 10^2 \mathrm{JK^{-1}kg^{-1}}$
Specific heat at constant volume	c_V	$7.17 \cdot 10^2 \mathrm{JK^{-1}kg^{-1}}$
Specific heat at constant pressure	c_p	$10.04 \cdot 10^2 \mathrm{JK^{-1}kg^{-1}}$
Coriolis parameter on mid-latitude	$f(45°)$	$\approx 10^{-4}$

Quantities

Coriolis parameter

Full form

$f = 2\Omega \sin\varphi$

Ω: Rotation rate of Earth
φ: Latitude

β-plane approximation

$f = f_0 + \beta y$

$$f_0 = 2\Omega \sin \phi_0$$
$$\beta = \frac{2\Omega}{a} \cos \phi_0$$

a = radius of earth

$f \approx 10^{-4}$ for mid-latitudes

$f > 0$ for the Northern hemisphere

$f < 0$ for the Southern hemisphere

Brunt-Vaisala frequency

$$N^2 = \frac{g}{\theta_0} \frac{d\theta_0}{dz}$$

Rossby number

Ratio of horizontal advection to Coriolis term

$$\frac{u \frac{\partial u}{\partial x}}{fv} \approx \frac{U}{fL} \equiv Ro$$

Geopotential height

Geopotential

$$\Phi(h) = \int_0^h g(\phi, z) dz$$

ϕ:latitude ,z: geometric height

The geopotential could also be expressed as a function of pressure

Geopotential height

- As a function of geometric height
  $$Z_g^h(h) = \frac{\Phi(h)}{g_0}$$
  g_0: Standard gravity at mean sea level
- As a function of pressure
  $$Z_g^p(p) = \frac{\Phi(p)}{g_0}$$
- The geostrophic wind v_G is parallel to the Z_g^p contours, and its magnitude is proportional to the distance between the Z_g^p contours.

Potential temperature

Temperature which an air parcel would acquire if adiabatically brought from level 1 to level 0.

$$T_0 = T_1 \left(\frac{p_0}{p_1}\right)^\kappa \equiv \theta(T_1, p_1)$$

$\kappa \equiv {}^R/_{c_p} = 0.286, p_0 = 1000 \text{ hPa}$

- $\frac{\partial \theta}{\partial z} > 0 \Rightarrow$ Stable atmosphere
- $\frac{\partial \theta}{\partial z} < 0 \Rightarrow$ Unstable atmosphere
- $\theta(300K, p_0) = 300$
  $\theta(220K, 200hPa) = 348$

Circulation and vorticity

Vorticity

$$\zeta = k \cdot (\nabla \times u)$$

Mean vorticity

$$\bar{\zeta} = \frac{1}{S} \iint_S \zeta dS$$

Circulation

$$C = \iint_S \zeta dS = \bar{\zeta} S$$

$$C = \oint v_g dr$$

Mathematical tricks

Interchange of $\frac{\partial}{\partial z}$ and $\frac{D}{Dt}$

$$\frac{\partial}{\partial z}\frac{D}{Dt} = \frac{D}{Dt}\frac{\partial}{\partial z} + \frac{\partial u}{\partial z}\frac{\partial}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial}{\partial y} + \frac{\partial w}{\partial z}\frac{\partial}{\partial z}$$

Product rule reversed

The product rule is normally used to expand a derivative of a product to a product of derivatives. This can be done the other way around, too.

Neglecting small terms

$$\Psi = \bar{\Psi} + \Psi^*$$

where $\Psi^* \ll \bar{\Psi}$ and Ψ is an arbitrary quantity. $\bar{\Psi}$ is the basic state and Ψ^* is the perturbation.

$$\Psi \cdot \Phi = (\bar{\Psi} + \Psi^*)(\bar{\Phi} + \Phi^*)$$
$$\approx \bar{\Psi}\bar{\Phi} + \bar{\Psi}\Phi^* + \bar{\Phi}\Psi^*$$

Approximate functions