Dynamics of Large-Scale Atmospheric Flows

Basics

Notation

$$\frac{D_h}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\frac{D}{Dt} = \frac{D_h}{Dt} + w\frac{\partial}{\partial z}$$

$$i = (1,0,0)$$
$$j = (0,1,0)$$
$$k = (0,0,1)$$

Coordinate system

Spherical coordinate system

$$x = r\sin\phi\cos\lambda$$
$$y = r\sin\phi\sin\lambda$$
$$z = r\cos\phi$$

Unit vectors on the sphere

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} : i = \begin{pmatrix} -\sin\lambda \\ \cos\lambda \\ 0 \end{pmatrix}$$
$$j = \begin{pmatrix} -\sin\theta\cos\lambda \\ -\sin\theta\sin\lambda \\ \cos\theta \end{pmatrix}$$
$$k = \begin{pmatrix} \cos\theta\cos\lambda \\ \cos\theta\sin\lambda \\ \sin\theta \end{pmatrix}$$

Wind vectors on the sphere

$$\frac{Di}{Dt} \coloneqq u = r\cos\phi\frac{D\lambda}{Dt}$$
$$\frac{Dj}{Dt} \coloneqq v = r\frac{D\phi}{Dt}$$
$$\frac{Dk}{Dt} \coloneqq w = \frac{Dr}{Dt}$$

Basic equations

6 equations for 6 unknowns u, v, w, ρ, p, T

Equations of motion (Navier-Stokes)

x, y, z are spherical unit coordinates (eastward, northward, vertical). Equations are simplified through β-plane.

β-plane approximation

$$f = f_0 + \beta y, \beta = \frac{2\Omega\cos\phi_0}{a}$$

a = Earth radius

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

Continuity equation

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \boldsymbol{u}) = 0$$

Equation of state

$$pV = nRT$$
$$\Leftrightarrow p = \rho RT$$

Thermodynamic equation

$$\frac{Dln\theta}{Dt} = \frac{1}{\theta}\frac{D\theta}{Dt} = \frac{1}{c_pT}\underbrace{\frac{DH}{Dt}}_{\substack{\sum diabatic \\ processes}}$$

Large-scale approximations (Synoptic-Scale Motions)

Vertical component decomposition

Due to the strong vertical stratification of the atmosphere, it is useful to decompose a field variable χ as follows.

$$\chi = \chi_0(z) + \chi^*(x, y, z, t)$$

At a certain height,

$$\chi_0 \gg |\chi^*|$$

This holds for $\chi = p, \rho, \theta, T$

Density – potential temperature relation

$$\frac{\rho^*}{\rho_0} \approx -\frac{\theta^*}{\theta_0}$$

This holds for the sea breeze and cumulus scale.

Hydrostatic approximation

Using: Vertical momentum equation

$$\frac{Dw}{Dt} \approx 0 \Rightarrow \frac{\partial p}{\partial z} = -\rho g$$

Geostrophic wind

Using: Horizontal momentum equation

$$-fv \approx -\frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$fu \approx -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

Equivalently,

$$v_G = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} = k \times \frac{1}{f\rho}\nabla_h p$$

Thermal wind

Using: Hydrostatic & geostrophic approximations & equation of state

$$\underbrace{\frac{\partial v_G}{\partial z}}_{v_{th}} = \frac{g}{fT} k \times \nabla_h T$$

- Left turning of geostrophic wind => Cold air advection
- Right turning of geostrophic wind => Warm air advection

Vorticity

ζ: Vertical component of vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \boldsymbol{k} \times (\nabla \times \boldsymbol{u})$$

$$\zeta = \begin{cases} \text{cyclonic circulation,} & \zeta > 0 \\ \text{anticyclonic circulation,} & \zeta < 0 \end{cases}$$

Vorticibty equation

$$\frac{D\zeta}{Dt} + \beta v$$
$$= \underbrace{-(\zeta + f)(\nabla_h v)}_{divergence\ effect} \underbrace{-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}_{twisting/tilting} \underbrace{+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)}_{solenoidal\ effect}$$

Synoptic scale approximation

$$\frac{D_h}{Dt}\zeta + \beta v = -f_0(\nabla_h \boldsymbol{v})$$

Quasi-geostrophic (QG) approximation

Simplifications

$$f \approx f_0$$
$$\zeta \approx \zeta_G$$
$$\frac{D_h}{Dt} \approx \frac{D_G}{Dt}$$
$$\Psi = \frac{p^*}{f_0\rho_0}$$

Subscript G denotes the usage of the geostrophic wind, i.e. $f_G(\boldsymbol{u}) = f(\boldsymbol{u_G})$

QG Equations

Geostrophic approximation	$u_G = -\frac{\partial \Psi}{\partial y}; v_G = \frac{\partial \Psi}{\partial x}$
Hydrostatic approximation	$g\frac{\theta^*}{\theta_0} = f_0\frac{\partial \Psi}{\partial z}$
Vorticity equation	$\frac{D_G}{Dt}\zeta_G + \beta\frac{\partial \Psi}{\partial x} = -f_0(\nabla_h \boldsymbol{v})$
Mass conservation	$\nabla_h \boldsymbol{v} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\partial_0 w) = 0$
Thermodynamic equation	$\frac{D_G}{Dt}\left(f_0\frac{\partial \Psi}{\partial z}\right) + N^2 w = 0$

Implications on vorticity / westward tilt

- $\zeta_G = \nabla_h^2 \Psi \propto -p$
  Positive / negative values of ζ_G are associated with low / high pressure
- $\frac{\partial}{\partial z}\zeta_G = \nabla_h^2\left(\frac{\partial}{\partial z}\Psi\right) = \frac{g}{f_0\theta_0}\nabla_h^2\theta^* \propto -T$
  Positive / negative values of $\frac{\partial}{\partial z}\zeta_G$ are associated with low / high temperature
- Both effects together lead to the westward slope of cyclones and anticyclones.

Ageostrophic wind

$$\boldsymbol{u} = \boldsymbol{v_G} + \boldsymbol{u_a}$$
$$\boldsymbol{v_G} = (u_G, v_G, 0)$$
$$\boldsymbol{u_a} = (u_a, v_a, w)$$

Some properties

- $div\ \boldsymbol{v_G} = 0$
- $div\ \boldsymbol{u_a} \neq 0$
- $|\boldsymbol{u_a}| \ll |\boldsymbol{v_G}|$

Linkage of ageo- and geostrophic wind

Assumptions: $N^2 = const., \beta = 0$

From thermodynamic equation and vorticity equation using hydrostatic approximation.

$$N^2\frac{\partial w}{\partial x} - f_0^2\frac{\partial u_a}{\partial z} = 2Q_1$$

$$N^2\frac{\partial w}{\partial y} - f_0^2\frac{\partial v_a}{\partial z} = 2Q_2$$

$$Q_1 = f_0 \left(\frac{\partial u_G}{\partial z} \frac{\partial v_G}{\partial x} + \frac{\partial v_G}{\partial z} \frac{\partial v_G}{\partial y} \right) = -\frac{g}{\theta_0} \left(\frac{\partial u_G}{\partial x} \frac{\partial \theta^*}{\partial x} + \frac{\partial v_G}{\partial x} \frac{\partial \theta^*}{\partial y} \right)$$

$$Q_2 = -f_0 \left(\frac{\partial u_G}{\partial z} \frac{\partial u_G}{\partial x} + \frac{\partial v_G}{\partial z} \frac{\partial u_G}{\partial y} \right) = -\frac{g}{\theta_0} \left(\frac{\partial u_G}{\partial y} \frac{\partial \theta^*}{\partial x} + \frac{\partial v_G}{\partial y} \frac{\partial \theta^*}{\partial y} \right)$$

Correct? Inserted minus and replaced v_G with u_G

Diagnostic w equation (vertical wind)

$$2\,\nabla \cdot \boldsymbol{Q} = N^2 (\nabla_h^2 w) + f_0^2 \left(\frac{\partial^2 w}{\partial z^2} \right)$$

With $2\nabla \cdot \boldsymbol{Q} = 2 \left(\frac{\partial}{\partial x} Q_1 + \frac{\partial}{\partial y} Q_2 \right)$

4-step golden rule

1. $\nabla \cdot Q \approx \nabla^2 w \approx -w$
2. $\frac{D_G}{Dt} \zeta_G \propto \frac{\partial}{\partial z} (\rho_0 w)$
3. $\frac{D_G}{Dt} \zeta_G \begin{cases} > 0 & \text{Cyclogenesis} \\ < 0 & \text{Anticyclogenesis} \end{cases}$

1. W-equation
2. Combination of vorticity equation and mass conservation
3. Approximating ζ with $\frac{D_G}{Dt}\zeta$?

Finding $\nabla \cdot Q$

1. Find largest Q-vector. Arrowhead is zone of convergence ($\nabla \cdot Q < 0$), tail is zone of divergence
2. $Q = -\frac{g}{\theta_0} |\nabla_h \theta^*| \left(\boldsymbol{k} \times \frac{\partial}{\partial \xi} \boldsymbol{v_G} \right)$
3. Find largest temperature gradient and a strong wind change along the isentrope

QG Potential Vorticity

1. Combination of vorticity equation and mass conservation.
2. Combine this with thermodynamic equation
3. Interchange $\frac{\partial}{\partial z}$ and $\frac{D_G}{Dt}$

$$\frac{D_G}{Dt} \left[\underbrace{\zeta_G + \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z} \left(\frac{\rho_0}{N^2} \frac{\partial \Psi}{\partial z} \right)}_{=:q=QGPV} + \beta y \right] = 0$$

- $q \approx \nabla^2 \Psi$
- q = relative vorticity + static stability
    - Static stability: From hydrostatic equation: $\frac{\partial}{\partial z} \theta^* \propto \frac{\partial^2}{\partial z^2} \Psi$
- Only valid for adiabatic flow.

QG Prognostic system

1. $q(t_0)$ known with B.C. $\theta^*(t_0, z = 0)$
2. Main diagnostic step
    a. Solve $q = \nabla^2 \Psi$
    b. Ψ yields u_G, v_G and θ^*
    c. These yield the Q-vectors
    d. The Q-vectors yield w
3. Do prognostic step using passive advection of $\frac{D_G}{Dt}(q + \beta y) = 0$ and go to 2.

QGPV key properties

- Conservation: $\frac{D_G}{Dt}(q + \beta y) = 0$
- Invertibility: $q = \nabla^2 \Psi$
- Partition and attribution: $q = \sum_j q_j$

Prototype vortex for QG flow

Since QGPV can be decomposed into "atomic" vortices, we can analyse prototype vortices.

Given:

- $q = \begin{cases} p, & r < a \\ 0, & r \geq a \end{cases}$

Then:

- $v = \begin{cases} \Omega R, & r < a \\ \frac{k}{r^2}, & r \geq a \end{cases}$
- Here, r is the spherical symmetrical coordinate, and R is the "x" coordinate

Ertel PV

Idea: Analogue derivation to QGPV, but starting from full NSE.

Applicable only for inviscid and adiabatic flow.

$$\frac{D}{Dt} \left[\underbrace{\frac{1}{\rho} \boldsymbol{\eta} \cdot \boldsymbol{\nabla} \theta}_{Ertel\ PV} \right] = 0$$

- $\boldsymbol{\eta} = rot\, \boldsymbol{u} + \boldsymbol{2\Omega}$, absolute vorticity vector

Isentropic PV

$$\frac{D}{Dt}\left[\underbrace{\frac{1}{\rho}(\zeta + f)\cdot\frac{\partial}{\partial z}\theta}_{IPV}\right] = 0$$

- IPV = z-component of PV
- $\frac{1}{\zeta} \propto \frac{\partial}{\partial z}\theta$

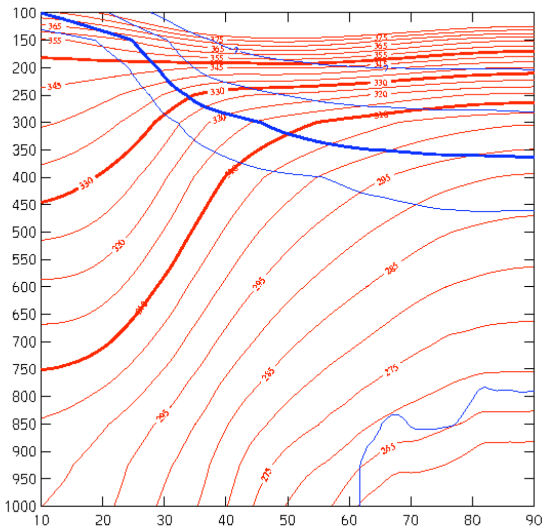
Preliminaries for PV chart analysis

- Unit for PV: 1 pvu. 2 pvu are defined as the tropopause.
- $\frac{\partial\theta}{\partial z} > 0$ & θ increases towards equator.
- Max(PV) at equator due to Coriolis parameter. ?? Coriolis parameter zero at Equator
- $\frac{\partial PV}{\partial z} > 0$, because $\frac{\partial\theta}{\partial z} > 0$ & $\frac{\partial}{\partial z}\left(\frac{1}{\rho}\right) > 0$
- For adiabatic and inviscid flow

$$\frac{D\theta}{Dt} = 0 \,\&\, \frac{DPV}{Dt} = 0$$

PV anomaly / cyclogenesis

- Strong PV gradient (w.r.t what coordinate? T,x?) => Strong wind. Why? Because of ζ?
- Adiabatic movement of air parcel towards south => stays on isentrope.
- As it must retain PV, it “drags” its PV value along the isentrope. A PV anomaly occurs.
- Because of far-field effect => why? of a strong PV gradient (=> strong wind), a cyclone forms at ground.



Idealised PV situations

Positive upper-level PV anomaly.

CCW circulation (Why?)

Negative upper-level PV anomaly.

CW circulation (Why?)

Negative surface temp. anomaly.

Higher $\frac{\partial\theta}{\partial z}$ => lower ζ

Positive surface temp. anomaly

Lower $\frac{\partial\theta}{\partial z}$ => higher ζ Why are windspeeds not only lower but opposite in sign (negative and positive sfc temp. anomaly)?

Diabatic PV

$$\frac{D}{Dt}PV \approx -g(f+\zeta)\frac{\partial\dot{\theta}}{\partial p}$$

Insert: Image with diabatic PV through a cloud

QG Wave theory

Examples

- Large scale flows in mid and upper-troposphere
- Planetary scale quasi-stationary and quasi-steady wave features in stratosphere

Waves on a uniform zonal flow

- Set: $\Psi(y) = -\mathcal{U}y$
- Split quantities in basic state and perturbations
  $\psi = \Psi(y) + \psi'(x,y,z,t), |\psi'| \ll |\Psi|$

From the QG potential vorticity equation, inserting ψ and linearizing (i.e. discarding terms like $a'b'$) yields the perturbation equation:

$$\left(\frac{\partial}{\partial t} + \mathcal{U}\frac{\partial}{\partial x}\right)q' + \frac{\partial\bar{q}}{\partial y}v' = 0$$

- $\Psi(y) = -\mathcal{U}y$, $\nabla_h^2\Psi = 0$ and $\frac{\partial}{\partial z}\Psi = 0$, thus $\bar{q} = 0 + \beta y = \beta y$
- $\frac{\partial\bar{q}}{\partial y} = \beta$

- $q' = \nabla_h^2 \psi' + \frac{f_0^2}{\rho_0} \frac{\partial}{\partial z}\left(\frac{\rho_0}{N^2} \frac{\partial \psi'}{\partial z}\right)$
  QGPV definition

2d wave solution (aka Rossby wave)

Solution on a mid-latitude band (all longitudes, latitudes 30-90°N)

$$\psi' = A \sin(kx - \omega t) \sin\big(l(y+d)\big)$$

Dispersion relationship

$$\omega = \mathcal{U}k - \frac{\beta k}{k^2 + l^2}$$

- $k = \frac{2\pi}{\mathcal{L}} = \frac{2m}{a}$
  - East-West wavelength: $m\mathcal{L} = \pi a$
  - m=1 => One full sine period
- $l = \frac{2\pi}{\mathcal{M}} = \frac{3m'}{a}$
  - North-South half-wavelength: $m'\left(\frac{\mathcal{M}}{2}\right) =$
  -

Phase velocities

- $u_p = \frac{\omega}{k} = \mathcal{U} - \frac{\beta}{k^2+l^2}$
- $v_p = \frac{\omega}{l}$

Effects

$m' = 1; m$	1	2	3	4
$-\frac{\beta}{k^2+l^2}$	-35.5	-18.5	-6.0	-1.7

- **Short waves (large m) typically move with $\mathcal{U}$**
- **Long waves (small m) move against $\mathcal{U}$**

Group velocities

- $u_g = \frac{\partial \omega}{\partial k} = \mathcal{U} + \frac{\beta(k^2-l^2)}{(k^2+l^2)^2}$
- $v_g = \frac{\partial \omega}{\partial l} = 2\beta \frac{kl}{(k^2+l^2)^2}$

Effects

- $k^2 < l^2 \Rightarrow (u_g - \mathcal{U}) < 0$
- $k^2 > l^2 \Rightarrow (u_g - \mathcal{U}) > 0$
- Longitudinal waves propagate eastwards
- Latitudinal waves propagate westwards

More effects

$$\frac{v_g}{u_g} = \tan\alpha = \frac{2kl}{k^2 - l^2}$$

Westward translation of a sinusoidal displacement of a fluid

3d wave solution

$$\psi'(x,y,z,t) = \chi'(x,y,z,t) e^{\frac{z}{2H_0}}$$

Yields a slightly different perturbation equation with solutions of the form:

$$\chi' = \psi(z) \sin(kx - \omega t) \sin l(y+d)$$

With the following constraints

- $\psi_{zz} + n^2 \psi = 0$
- $n^2 = \left(\frac{N}{af_0}\right)^2 \left[\frac{\beta a^2}{\mathcal{U}-c} - (4m^2 + 9m'^2 + \alpha^2 a^2)\right]$
- $(\omega - \mathcal{U}k)\left(\psi_z + \frac{1}{2H_0}\psi\right) = 0$

Effect: Vertical propagability

- $\psi \propto e^{\pm inz}$ for $n^2 > 0$
- $\psi \propto e^{\mp nz}$ for $n^2 < 0$

Discard exponential solution. Vertically propagating waves only occur for $n^2 > 0$

$\boldsymbol{n^2 > 0} \Leftarrow$

- $(\mathcal{U} - c) > 0$: Wave velocity c must be easterly relative to zonal flow $\mathcal{U}$
- $(\mathcal{U} - c) < \frac{\beta}{k^2+l^2+\alpha^2}$: For typical values of β, k, l, α
- The above criteria are only satisfied for low zonal wavenumbers m, since

Zonal wavenumber	$\frac{\beta}{k^2+l^2+\alpha^2}$
m=1	$55\ ms^{-1}$
m=2	$38\ ms^{-1}$
m=3	$24\ ms^{-1}$

Real effect: From the troposphere, only m=1 or m=2 waves propagate to the stratosphere.

Orographic forcing

Setting: Stationary, incompressible wave ($\frac{D}{Dt} = 0, \rho_0 = const.$), subject to sinusoidal terrain

Perturbation equation

$$\mathcal{U}\frac{\partial}{\partial x}\left[\psi'_{xx} + \psi'_{yy} + \frac{f_0^2}{N^2}\psi'_{zz}\right] + \beta\frac{\partial \psi'}{\partial x} = 0$$

Terrain (η) and B.C.

$$\eta = \eta_0 \cos(kx) \sin\left[\frac{\pi}{2d}(y+d)\right]$$

$$\frac{\partial \eta}{\partial x} = \frac{\omega'}{\mathcal{U}}$$

Solution

$$\psi' \propto -\sin(kx+nz)\sin\left[\frac{\pi}{2d}(y+d)\right], n^2 > 0$$

$$\psi' \propto \cos(kx)e^{-\mu z}\sin\left[\frac{\pi}{2d}(y+d)\right], n^2 < 0$$

$$n^2 = f(\mathcal{U}, \dots)\begin{cases} > 0, & m = 1,2,3; m' = 1,2 \\ < 0, & m > 3 \end{cases}$$

There are large Rossby waves induced by (large) mountains such as the Rocky mountains, Himalaya or Greenland.

How is ψ related to our streamfunction from QG system?

Notions

- Cyclonic = Counter clockwise = Left turn
- Anticyclonic = Clockwise = Right turn
- Zonal wavenumber = m
- North-south petal count = m'
- Large waves ⇔ small m
- Short waves ⇔ large m