1 Wave Equation

Form	$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$
Boundary Conditions	$u(0,t) = 0 = u(L,t)$
Initial Conditions	$u(x,0) = f(x)$
	$\frac{\partial u}{\partial t}(x,0) = g(x)$

1.1 Solution

The Ansatz Separation of variables is applicable if PDE and BC are linear homogenuous.

Ansatz (Sep. of vars) $\mathbf{u}(\mathbf{x},\mathbf{t}) = \mathbf{X}(\mathbf{x})\mathbf{T}(\mathbf{t})$

$$\frac{\partial^2 u}{\partial t^2} = X(x)T''(t); \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$\Rightarrow \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \Leftrightarrow X(x)T''(t) = c^2 X''(x)T(t)$$

$$\Leftrightarrow \frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = k$$

ODE-System
$$\begin{array}{l\|l} I & X''(x) - kX(x) = 0 \\ II & T''(t) - kc^2T(t) = 0 \end{array}$$

Imposing BC
$$\left\| \begin{array}{ll} u(0,t) = X(0)T(t) = 0 & \forall t \\ u(L,t) = X(L)T(t) = 0 & \end{array} \right.$$

$$\Rightarrow \left\| \begin{array}{ll} X'' - kX = 0 & k = -p^2 < 0 \\ X(0) = 0 & \\ X(L) = 0 & \end{array} \right.$$

Consider cases
$$\begin{array}{l\|ll}
k = 0 & \lambda_{1,2} = 0,0 & \rightarrow \text{ only trivial solution} \\
 & X(x) = Ax + B & \\
 & X(0) = A = 0 & \\
 & X(L) = 0 + B = 0 & \\
k = \mu^2 > 0 & \lambda_{1,2} = \pm\mu & \rightarrow \text{ only trivial solution} \\
 & X(x) = Ae^{\mu x} + Be^{-\mu x} & \\
 & X(0) = A + B = 0 & \\
 & X(L) = A\left(e^{\mu L} - e^{-\mu L}\right) = 0 & \\
k = -p^2 < 0 & \lambda_{1,2} = \pm ip & \\
 & X(x) = A\cos(px) + B\sin(px) & \\
 & X(0) = A = 0 & \\
 & X(L) = \sin(pL) = 0 \Rightarrow pL = n\pi &
\end{array}$$

$$\Rightarrow \boxed{X_n(x) = \sin\left(\frac{n\pi}{L}x\right)} \quad n \in \mathbb{Z}^{>0} \quad k = -\left(\frac{n\pi}{L}\right)^2$$

$$\Rightarrow \boxed{T_n(t) = C_n\cos(\lambda_n t) + D_n\sin(\lambda_n t)} \quad \lambda = \frac{n\pi c}{L}$$

General solution
$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cdot\left(C_n\cos(\lambda_n t) + D_n\sin(\lambda_n t)\right) \quad n \in \mathbb{Z}^{>0}, \lambda_n = \frac{cn\pi}{L}$$

Superposition
$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\cdot\left(C_n\cos(\lambda_n t) + D_n\sin(\lambda_n t)\right) \quad n \in \mathbb{Z}^{>0}, \lambda_n = \frac{cn\pi}{L}$$

Imposing IC
$$u(x,0) = \sum_{n=1}^{\infty} C_n\sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$u_t(x,0) = \sum_{n=1}^{\infty} D_n\lambda_n\sin\left(\frac{n\pi x}{L}\right) = g(x)$$

Fourier-Coefficients
$$C_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$D_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\frac{n\pi x}{L}dx$$

2 Heat equation

Form	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
Boundary Conditions	$u(0,t) = 0 = u(L,t)$
Initial Conditions	$u(x,0) = f(x)$

2.1 Solution

Ansatz (Sep. of vars)

$$\mathbf{u(x,t) = X(x)T(t)}$$

$$\frac{\partial u}{\partial t} = X(x)T'(t); \quad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)$$

$$\Rightarrow \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \Leftrightarrow X(x)T'(t) = c^2 X''(x)T(t)$$

$$\Leftrightarrow \frac{1}{c^2}\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = k$$

ODE-System

$$\begin{array}{l|l} I & X''(x) - kX(x) = 0 \\ II & T'(t) - kc^2 T(t) = 0 \end{array}$$

Imposing BC

$$\left\| \begin{array}{l} u(0,t) = X(0)T(t) = 0 \qquad \forall t \\ u(L,t) = X(L)T(t) = 0 \end{array} \right.$$

$$\Rightarrow \left\| \begin{array}{l} X'' - kX = 0 \qquad k = -p^2 < 0 \\ X(0) = 0 \\ X(L) = 0 \end{array} \right.$$

$$\Rightarrow \boxed{X_n(x) = \sin\left(\frac{n\pi}{L}x\right)} \quad n \in \mathbb{Z}^{>0} \quad k = -\left(\frac{n\pi}{L}\right)^2$$

$$\Rightarrow \boxed{T_n(t) = B_n e^{-\lambda_n^2 t}} \quad \lambda = \frac{n\pi c}{L}$$

General solution

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right) \cdot B_n e^{-\lambda_n^2 t} \quad n \in \mathbb{Z}^{>0}, \lambda_n = \frac{cn\pi}{L}$$

Superposition

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \cdot B_n e^{-\lambda_n^2 t} \quad n \in \mathbb{Z}^{>0}, \lambda_n = \frac{cn\pi}{L}$$

Imposing IC

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

Fourier-Coefficients

$$B_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$$

3 Other types of boundary conditions

3.1 Nonzero BC

Solution: "Homogen-partikulär"-Ansatz

Particular Solution	1. Find solution $u_P(x)$ of: $\left\| \begin{array}{ll} \frac{d^2u_P}{dx^2} = 0 & \Rightarrow \quad u_P(x) = ax + b \\ u_P(0) = U_0 & \Rightarrow \quad b = U_0 \\ u_P(L) = U_L & \Rightarrow \quad a = \frac{U_L - U_0}{L} \end{array} \right.$
Homogeneous Solution	2. Find solution $u_H(x,t)$ of: $\left\| \begin{array}{l} \frac{\partial u_H}{\partial t} = c^2 \frac{\partial^2 u_H}{\partial x^2} \\ u_H(0,t) = u_H(L,t) = 0 \\ u_H(x,0) = f(x) - u_P(x) \end{array} \right.$
Final solution	$u(x,t) = u_H(x,t) + u_P(x)$ $u(x,t)$ satisfies: $\begin{array}{lll} \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} & \Leftrightarrow & 0 + \frac{\partial u_H}{\partial t} = 0 + c^2 \frac{\partial^2 u_H}{\partial x^2} \\ u(0,t) = U_0, u(L,t) = U_L & \Leftrightarrow & 0 + U_0, 0 + U_L \\ u(x,0) = f(x) & \Leftrightarrow & f(x) - u_P(x) + u_P(x) \end{array}$

3.2 Adiabatic Conditions (Insulated ends)

$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$

$\Rightarrow$ Do separation of variables! This yields:

ODE-System	$\begin{array}{l\|l} I & X''(x) - kX(x) = 0 \\ II & T'(t) - kc^2T(t) = 0 \end{array}$
Imposing BC	$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$ $X'(0)T(t) = X'(L)T(t) = 0 \quad \forall t$ $\Rightarrow \left\| \begin{array}{l} X'' - kX = 0 \qquad k = -p^2 < 0 \\ X'(0) = 0 \\ X'(L) = 0 \end{array} \right.$ $\Rightarrow \boxed{X_n(x) = \cos\left(\frac{n\pi}{L}x\right)} \quad n \in \mathbb{Z}^{\geqslant 0} \quad k = -\left(\frac{n\pi}{L}\right)^2$ $\Rightarrow \boxed{T_n(t) = A_n e^{-\lambda_n^2 t}} \quad \lambda = \frac{n\pi c}{L}$
Superposition	$u(x,t) = \sum_{n=0}^{\infty} \cos\left(\frac{n\pi x}{L}\right) \cdot A_n e^{-\lambda_n^2 t} \quad n \in \mathbb{Z}^{>0}, \lambda_n = \frac{cn\pi}{L}$
Imposing IC	$u(x,0) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$
Fourier-Coefficients	$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx$
$t \to \infty$	$\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{L} \int_0^L f(x)dx$

3.3 No BC (infinite bar)

Form	$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$
Boundary Conditions	$None$
Initial Conditions	$u(x,0) = f(x)$

3.4 Solution

ODE-System
$$\begin{array}{c} I \\ II \end{array}\left\| \begin{array}{l} X''(x) - kX(x) = 0 \\ T'(t) - kc^2T(t) = 0 \end{array}\right.$$

Discussing k

$k > 0 : T(t) = De^{kc^2T} \Rightarrow$ Exponentially increasing temperature. No physical interest

$k = -p^2 \leqslant 0$

$$\left\| \begin{array}{l} T(t) = De^{-p^2c^2T} \\ X(x) = A\cos(px) + B\sin(px) \quad X(x) = Ax + B \text{ is physically uninteresting } (k=0) \end{array}\right.$$

General solution $u_p(x,t) = (A_p\cos(px) + B_p\sin(px)) \cdot e^{-p^2c^2T} \quad \forall p \geqslant 0$

Integration
$$u(x,t) = \int u_p(x,t)dp$$
$$u(x,t) = \int_0^\infty (A_p\cos(px) + B_p\sin(px)) \cdot e^{-p^2c^2T}dp$$

Imposing IC
$$u(x,0) = \int_0^\infty (A_p\cos(px) + B_p\sin(px))\, dp = f(x)$$

Fourier-Coefficients
$$A_p = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(pv)dv$$
$$B_p = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(pv)dv$$

Imposing FC
$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\int_0^\infty \cos(px - pv)e^{-p^2c^2T}dpdv$$
$$\int_0^\infty \cos(px-pv)e^{-p^2c^2T}dp = \frac{1}{c\sqrt{t}}\frac{\sqrt{\pi}}{2}e^{-\frac{(x-v)^2}{4c^2T}}$$
$$\boxed{u(x,t) = \frac{1}{2c\sqrt{t\pi}}\int_{-\infty}^{\infty} f(v)\exp\left(-\frac{(x-v)^2}{4c^2T}\right)dv}$$

4 Heat equation on a solid ball

General form
$$\frac{\partial u}{\partial t} = c^2\Delta u$$

Symmetry
$$u(x,y,z,t) = u(r,t) \quad r = \sqrt{x^2+y^2+z^2}$$

Laplacian
$$\Delta u(x,y,z) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$
$$\Delta u(r) = \frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\cdot\frac{2}{r}$$

Concrete form
$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\cdot\frac{2}{r}\right)$$

Boundary Conditions $u(L,t) = 0, t > 0, x^2+y^2+z^2 = r^2 = L^2$

Initial Conditions $u(r,0) = U_0, 0 \leqslant r \leqslant L$

4.1 Solution

Ansatz (Sep. of vars) $\mathbf{u(r,t) = R(r)T(t)}$

$$R(r)T'(t) = c^2\left(R''(r) + R'(r)\frac{2}{r}\right)T(t)$$

$$\Leftrightarrow \frac{T'}{c^2T}\frac{R'' + \frac{2}{r}R'}{R} = k$$

ODE-System

$$\begin{array}{l\|l} I & R''(x) + \frac{2}{r}R' - kR = 0 \quad \text{(Bessel equation)} \\ II & T'(t) - kc^2T(t) = 0 \end{array}$$

Impose BC $u(L,t) = R(L)T(t) = 0 \quad \forall t \quad \Rightarrow R(L) = 0$

Solve ODE I Substitution:

$$G(r) := rR(r) \quad \Rightarrow \quad R(r) = \frac{G(r)}{r}$$

$$G'(r) = R(r) + rR'(r) \quad \Rightarrow \quad R'(r) = \frac{1}{r}G'(r) - \frac{1}{r^2}G(r)$$

$$G''(r) = 2R'(r) + rR''(r) \quad \Rightarrow \quad R''(r) = \frac{1}{r}G''(r) - \frac{2}{r^2}G'(r) + \frac{2}{r^3}G(r)$$

Rewrite ODE I:

$$\begin{array}{lcccccl} & R'' & + & \frac{2}{r}R' & - & kR & = 0 \\ \Leftrightarrow & \frac{1}{r}G''(r) - \frac{2}{r^2}G'(r) + \frac{2}{r^3}G(r) & + & \frac{2}{r^2}G'(r) - \frac{2}{r^3}G(r) & - & \frac{k}{r}G & = 0 \\ \Leftrightarrow & G'' - kG = 0 & & & & & \end{array}$$

Solve rewritten ODE and get well-defined solution for R:

$$k = -p^2 < 0 \quad \Rightarrow \quad G(r) = A\cos pr + B\sin pr$$

$$R(L) = 0 = \frac{G(L)}{L} \quad \Rightarrow \quad \frac{A\cos pL + B\sin pL}{L} = 0$$

$$\exists \lim_{r\to 0} R(r)\forall r \quad \Rightarrow \quad A = 0$$

$$\Rightarrow \boxed{R_n(r) = \frac{1}{r}\sin\left(\frac{n\pi r}{L}\right)} \quad k = -\left(\frac{n\pi}{L}\right)^2$$

Solve ODE II

$$\Rightarrow \boxed{T_n(t) = B_n e^{-\lambda_n^2 t}} \quad \lambda = \frac{n\pi c}{L}$$

General solution

$$u_n(r,t) = \frac{1}{r}\sin\left(\frac{n\pi r}{L}\right)\cdot B_n e^{-\lambda_n^2 t} \quad n \in \mathbb{Z}^{>0}, \lambda_n = \frac{n\pi c}{L}$$

Superposition

$$u(r,t) = \sum_{n=1}^{\infty} u_n(x,t)$$

$$u(r,t) = \sum_{n=1}^{\infty}\frac{1}{r}\sin\left(\frac{n\pi r}{L}\right)\cdot B_n e^{-\lambda_n^2 t} \quad n \in \mathbb{Z}^{>0}, \lambda_n = \frac{cn\pi}{L}$$

Imposing IC

$$u(r,0) = \sum_{n=1}^{\infty} B_n\frac{1}{r}\sin\left(\frac{n\pi r}{L}\right) = U_0$$

Fourier-Coefficients

$$B_n = \frac{2}{L}\int_0^L \frac{U_0}{r}\sin\frac{n\pi r}{L}dr$$

5 Solve PDE by Fouriertransformation

Trick: By Fouriertransformation, turn the PDE into a easily solvable ODE in Fourier-Space.
Here, the Fouriertransform of $u(x,t)$ is $\hat{u}(k,t)$,i.e. FT in x

PDE		**O**DE in Fourier-Space
$u_{tt} + u_{xxxx} = 0$	$\xrightarrow{FT}$	$\hat{u}_{tt} + k^4 \hat{u} = 0$
$u(x,0) = e^{-x^2}$	$\xrightarrow{FT}$	$\hat{u}(k,0) = \sqrt{\pi} e^{\frac{-k^2}{4}}$
$u_t(x,0) = 0$	$\xrightarrow{FT}$	$\hat{u}_t(k,0) = 0$

Solve ODE	$\hat{u}_{tt} + k^4 \hat{u} = 0$
Ansatz	$\hat{u}(k,t) = e^{\lambda t}$
General solution	$\hat{u}(k,t) = Ae^{ik^2 t} + Be^{-ik^2 t}$
Impose conditions	$... \Rightarrow A = B = \frac{\sqrt{\pi}}{2} e^{-\frac{k^2}{4}}$
Solution	$\hat{u}(k,t) = \frac{\sqrt{\pi}}{2} e^{-\frac{k^2}{4}} \left(e^{ik^2 t} + e^{-ik^2 t} \right)$

Getting Inverse Fourier Transform

$\hat{u}_1 := \frac{\sqrt{\pi}}{2} e^{-\frac{k^2}{4}} \cdot (e^{ik^2 t} = \frac{\sqrt{\pi}}{2} \exp\left(-k^2(\frac{1}{4} - it)\right)$

We seek to rearrange $\hat{u}_1$ such, that we can apply a known inverse Fouriertransform on it.

$$\frac{\sqrt{\pi}}{2} \exp\left(-k^2(\tfrac{1}{4} - it)\right) = \frac{\sqrt{\pi}}{2} \exp\left(-\left(k\sqrt{\tfrac{1}{4} - it}\right)^{\mathbf{2}}\right) = \frac{\sqrt{\pi}}{2} \exp\left(-\frac{\left(\mathbf{4} \cdot k\sqrt{\frac{1}{4} - it}\right)^2}{\mathbf{16}}\right) =$$

$$= \frac{1}{\mathbf{4\sqrt{\frac{1}{4} - it}}} \cdot \underbrace{\mathbf{4\sqrt{\tfrac{1}{4} - it}}}_{b} \cdot \underbrace{\frac{\sqrt{\pi}}{2} \exp\left(-\frac{\left(4\sqrt{\frac{1}{4} - it} \cdot k\right)^2}{16}\right)}_{\hat{f}(b \cdot k)}$$

Now we can apply our knowledge about Fourierbacktransformations.

$\hat{f}(k) = \sqrt{\frac{2\pi}{a}} \exp\left(-\frac{k^2}{2a}\right)$	$\xrightarrow{InvFT}$	$f(x) = \exp\left(\frac{-ax^2}{2}\right)$	
$\hat{g}(k) = b \cdot \hat{f}(b \cdot k)$	$\xrightarrow{InvFT}$	$g(x) = f\left(\frac{x}{b}\right)$	Skalierungssatz
$c \cdot \hat{h}(k)$	$\xrightarrow{InvFT}$	$c \cdot h(x)$	Konstante

In our case, we have

$a = 8$	$\Rightarrow$	$f(x) = \exp\left(-4x^2\right)$
$b = 4\sqrt{\frac{1}{4} - it}$	$\Rightarrow$	$g(x) = \exp\left(-4\left(\frac{x}{4\sqrt{\frac{1}{4} - it}}\right)^2\right) = \exp\left(-\frac{x^2}{1 - 4it}\right)$
$c = \frac{1}{4\sqrt{\frac{1}{4} - it}}$	$\Rightarrow$	$h(x) = u_1(x,t) = \frac{1}{2\sqrt{1 - 4it}} \cdot g(x)$

6 Solve PDE by Double Fourier-Series

PDE	$u_{tt} = u_{xx} + u_{yy}$
BC	$u(x,y,t) = 0$ for boundaries $x = 0, x = L, y = 0, y = L$
IC	$u(x,y,0) = f(x)$
	$u_t(x,y,0) = g(x)$
Ansatz	$u(x,y,t) = S(x,y) \cdot T(t)$
	$u_{tt} = S(x,y)T''(t) \quad u_{xx} + u_{yy} = \Delta S(x,y)T(t) \quad = (X''Y + XY'')T$
	PDE: $S(x,y)T''(t) = \Delta S(x,y)T(t) = 0$
	$\Leftrightarrow \dfrac{T''(t)}{T(t)} = \dfrac{\Delta S(x,y)}{S(x,y)} = -p^2$
ODE-PDE System	$I. \quad T''(t) + p^2 T(t) = 0$
	$II. \quad \Delta S(x,y) + p^2 S(x,y) = 0$
PDE II	$S(x,y) = X(x)Y(y)$
	PDE II: $\dfrac{X''(x)}{X(x)} = -\dfrac{Y''(y)}{Y(y)} - p^2 = -k^2 \quad p^2 > k^2$
ODE System	$I. \quad T''(t) + p^2 T(t) = 0$
	$II._1 \quad X''(x) + k^2 X(x) = 0$
	$II._2 \quad Y''(y) + (p^2 - k^2)Y(y) = 0$
Impose BC	Get basic solutions: $X_m(x), Y_n(y), T_{mn}(t)$
General solution	$u_{mn}(x,y,t) = X_m(x)Y_n(y)T_{mn}(t)$
Superposition	$u(x,y,t) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} u_{mn}(x,y,t)$
Impose IC	Die Fourierkoeffizienten der inneren FR berechnen, diesen Ausdruck einsetzen und als

7 Rounded domains

Sturm-Liouville-Operator	$(p(x) \cdot X')' + q(x) \cdot X = \lambda w(x) \cdot X$
Dirichlet problem (BC of 1st kind)	$u(0,t) = ... \& u(L,t) = ...$
Neumann problem (BC of 2nd kind)	$\dfrac{\partial u}{\partial x}(0,t) = ... \& \dfrac{\partial u}{\partial x}(L,t) = ...$
Robin problem (BC of 3rd kind)	$\alpha u(0,t) + \beta \dfrac{\partial u}{\partial x}(0,t) = 0 \& \gamma u(L,t) + \delta \dfrac{\partial u}{\partial x}(L,t) = 0$
Laplace Equation	$\boxed{\Delta u = 0}$
Δu	Cartesian (3D) $\quad \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$
	Polar (2D) $\quad \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \varphi^2}$
	Cylindrical (3D) $\quad \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \varphi^2} + \dfrac{\partial^2 u}{\partial z^2}$
	Spherical (3D) $\quad \Delta u = \dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{2}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \theta^2} + \dfrac{\cot\theta}{\rho^2}\dfrac{\partial u}{\partial \theta} + \dfrac{1}{\rho^2 \sin^2\theta}\dfrac{\partial^2 u}{\partial \varphi^2}$

7.1 Polar coordinates

Laplace equation	$\Delta u = 0 \Leftrightarrow$ $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} = 0$
Ansatz	$u(r,\varphi) = R(r)P(\varphi)$
ODE-System	$\left\Vert \begin{array}{ll} I & P'' + kP = 0 \\ II & r^2R'' + rR' - kR = 0 \end{array}\right.$
Solution to ODE I	Hidden BC: $P(\varphi) = P(\varphi + 2\pi)$ $P_n(\varphi) = a_n \cos(n\varphi) + b_n \sin(n\varphi), \quad k = n^2$
Solution to ODE II	2nd order Euler equation $\Rightarrow$ Use Ansatz $R(r) = r^q$ $\begin{cases} R_n = ar^n + br^{-n} & n = 1,2,3,... \\ R_0 = a + b\ln(r) & n = 0 \end{cases}$

Additional constraints

If domain...	Take (to get bounded solutions)	
includes origin, excludes infinity	$R_n(r) = a_n r^n$	$n \geqslant 0$
excludes origin, includes infinity	$R_n(r) = b_n r^{-n}$	$n \geqslant 0$
excludes both origin and infinity	$R_n(r) = a_n r^n + b_n r^{-n}$	$n > 0$
	$R_0(r) = a + b\ln(r)$	$n = 0$

7.2 Bessel equations

Bessel eqn of order α	$x^2X'' + xX' + (x^2 - \alpha^2)X = 0$
Solution	$A \cdot J_\alpha(x) + B \cdot Y_\alpha(x)$
Modified Bessel eqn of order α	$x^2X'' + xX' + (-x^2 - \alpha^2)X = 0$
Solution	$A \cdot I_\alpha(x) + B \cdot K_\alpha(x)$

Properties of Bessel fns

	$x \to 0$	$x \to \infty$	Behaviour	Roots
$I_\alpha(x)$	bounded	unbounded	like e^x	maybe at $x = 0$
$K_\alpha(x)$	unbounded	bounded	like $e^{1/x}$	none
$Y_\alpha(x)$	singular		oscillating	countable many
$J_\alpha(x)$			oscillating	countable many

7.3 Example for Bessel equation

Almost Bessel equation	$r^2R'' + rR' + (p^2r^2 - n^2)R = 0$ Substitute: $\varrho = p \cdot r$ $\bar{R}(\varrho) = R\left(\frac{\varrho}{p}\right)$ $\bar{R}'(\varrho) = \left(R\left(\frac{\varrho}{p}\right)\right)' = \frac{1}{p}R'\left(\frac{\varrho}{p}\right)$ $\bar{R}''(\varrho) = \left(R\left(\frac{\varrho}{p}\right)\right)'' = \frac{1}{p^2}R''\left(\frac{\varrho}{p}\right)$
Impose substitutions	$r^2R''(r) + rR'(r) + (p^2r^2 - n^2)R(r) = 0$ $\frac{\varrho^2}{\cancel{p^2}}\cancel{p^2}\bar{R}''(\varrho) + \frac{\varrho}{\cancel{p}}\cancel{p}\bar{R}'(\varrho) + \left(\varrho^2 - n^2\right)\bar{R}(\varrho) = 0$
Bessel function	The above is now a Bessel equation of order n. Its solution is given by: $\bar{R}(\varrho) = R(r) = AJ_n + BY_n$ If we want $\|R(0)\| < \infty$, then $B = 0$

7.4 Cylindrical coordinates

Laplace equation

$$\Delta u = 0 \Leftrightarrow$$
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

Ansatz

$$u(r,\varphi,z) = R(r)P(\varphi)Z(z)$$

ODE-System

$$\left\| \begin{array}{ll} I & P'' + k_2 P = 0 \\ II & r^2 R'' + rR' + (k_1 r^2 - k_2)R = 0 \\ III & Z'' - k_1 Z = 0 \end{array} \right.$$

Solution to ODE's

Hope that the solution is independent from $\varphi \Rightarrow u(r,\varphi,z) = R(r)Z(z), k_2 = 0$

Trick 77

$u = u_1 + u_2$

$$\begin{array}{l|l} \Delta u_1 = 0 & \Delta u_2 = 0 \\ u_1(r,0) = 0 & u_2(r,0) = f(r) \\ u_1(r,L) = 0 & u_2(r,L) = g(r) \\ u_1(\mu,z) = h(z) & u_2(\mu,z) = 0 \end{array}$$

Solve u_1 Problem

$Z'' - k_1 Z = 0, Z(0) = Z(L) = 0$

$$Z_{n1}(z) = \sin\frac{n\pi z}{L}, \quad n = 1,2,... \quad k_1 = -\left(\frac{n\pi}{L}\right) = -p_n^2$$

$r^2 R'' + rR' - p^2 r^2 R = 0$

By substituting $\varrho = p \cdot r$, we get $R_{n1}(r) = A_n I_0(p_n \cdot r) + B_n K_0(p_n \cdot r)$

Depending on domain, A_n or B_n might be set to 0

$$u_1(r,z) = \sum_{n\geqslant 1} R_{n1}(r)Z_{n1}(z)$$

Solve u_2 Problem

1. Try to determine the sign which k_1 must have with BC $u_2(\mu,z) = 0 \Rightarrow R(\mu) = 0$
2. In order to have $R(\mu) = 0$, R must not be the modified Bessel function.
3. Here: Choose $k_1 = p^2 > 0$

Case $k_1 = 0$

$$\begin{array}{lll} & rR'' + R' = 0 & \Leftrightarrow \quad (rR')' = 0 \\ \Rightarrow & rR' = b & \Leftrightarrow \quad R' = \dfrac{b}{r} \\ \Rightarrow & R(r) = a + b\ln(r) & \end{array}$$

It depends on the domain if this is a nontrivial result

7.5 Spherical coordinates

Spherical coordinates $\Delta u = 0 \Leftrightarrow$

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{\rho^2}\frac{\partial u}{\partial \theta} + \frac{1}{\rho^2 \sin^2\theta}\frac{\partial^2 u}{\partial \varphi^2} = 0$$

Ansatz $u(\rho, \varphi, \theta) = R(\rho)S(\varphi)T(\theta)$

Sep. of vars.

$$\rho^2 R''ST + 2\rho R'ST + RST'' + \cot\theta RST' + \frac{1}{\sin^2\theta}RS''T = 0 \Leftrightarrow$$

$$\frac{\rho^2 R'' + 2\rho R'}{R} = -\frac{T'' + \cot\theta T'}{T} - \frac{1}{\sin^2\theta}\frac{S''}{S} = k_1$$

$$\frac{\sin^2\theta T'' + \cos\theta\sin\theta T'}{T} + k_1\sin^2\theta = -\frac{S''}{S} = k_2$$

ODE-System

$$\left\| \begin{array}{ll} I & \rho^2 R'' + 2\rho R' - k_1 R = 0 \\ II & S'' + k_2 S = 0 \\ III & \sin^2\theta T'' + \sin\theta\cos\theta T' + (k_1\sin^2\theta - k_2)T = 0 \end{array} \right.$$

Solve ODE I

2nd order Euler eqn $\Rightarrow R(\rho) = \rho^m \quad k_1 = l(l+1), l \in \mathbb{Z}^{\geqslant 0}$ (because of ODE III)

... $m = l$ or $m = -(l+1)$

$R_l(\rho) = A_l\rho^l + B_l\rho^{-(l+1)}, l \in \mathbb{Z}^{\geqslant 0}$

Solve ODE II

$$\left\| \begin{array}{l} S'' + k_2 S = 0, S(\varphi) = S(\varphi + 2\pi) \\ \text{Nontrivial if } k_2 = m^2 \quad m \in \mathbb{Z}^{\geqslant 0} \\ S_m(\varphi) = a_m\cos(m\varphi) + b_m\sin(m\varphi) \end{array} \right.$$

Solve ODE III

Substitution: $w = \cos\theta \Rightarrow T'(w(\theta)) = w'(\theta) \cdot T'(w(\theta)) = -\sin\theta \cdot T'(\cos\theta)$

$(1-w^2)\frac{d^2T(w)}{dw^2} - 2w\frac{dT}{dw} + \left(k_1 - \frac{k_2}{1-w^2}\right)T = 0 \quad -1 \leqslant w \leqslant 1$General Legendre eq

$k_1 = l(l+1), k_2 = m^2$

$T_l^m(\theta) = P_l^m(\cos\theta) \quad 0 \leqslant m \leqslant l \quad P =$ associated Legendre Polynomials

General solution

$$u(\rho, \varphi, \theta) = \sum_{l=0}^{\infty}\sum_{m=0}^{l}(A_l\rho^l + B_l\rho^{-(l+1)}) \cdot (a_m\cos(m\varphi) + b_m\sin(m\varphi)) \cdot P_l^m(\cos\theta)$$

Spherical harmonics

$$Y_{lm}(\theta, \varphi) = \begin{cases} N_{lm}P_l^m(\cos\theta)\cos(m\varphi) & \text{if } m > 0 \\ N_{lm}P_l^0(\cos\theta) & \text{if } m = 0 \\ N_{lm}P_l^{-m}(\cos\theta)\sin(m\varphi) & \text{if } m < 0 \end{cases} \quad N_{lm} = \text{Normalization constant}$$

$S(\varphi) \cdot T(\theta) \equiv Y_{lm}$ form a complete set of orthonormal functions (like Fourier-Series)

Legendre Polynomials

$P_0(x) = 1 \qquad P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2 - 1) \qquad P_3(x) = \frac{1}{2}(5x^3 - 3x)$

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \qquad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$

$$P_n(x) = \frac{1}{2^n\, n!} \cdot \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[(x^2-1)^n\right]$$

Associated Legendre Polynomials

$$P_\ell^m(x) = (-1)^m\ (1-x^2)^{m/2}\ \frac{d^m}{dx^m}\left(P_\ell(x)\right), \quad m \geqslant 0$$

$$P_l^m(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x)$$

8 2D Laplace-Equation: Complex Analysis

Idea: We might have a solution for a problem in an easy domain, like a stripe. What if the boundaries of that stripe are not straight, but bent? We can try a conformal mapping. If we apply this mapping to our solution in the easy domain, we will get a solution for the complicated domain.

u harmonic $\qquad \Delta u = 0$

f analytic $\qquad f(x+iy) = u(x,y) + iv(x,y) \quad x, y \in \mathbb{R}$

f analytic $\Leftrightarrow u_x = v_y \wedge u_y = -v_x$

v is *conjugate harmonic* of u

Orthogonality $\qquad \langle \text{grad } u, \text{grad } v \rangle = 0$

Equipotential lines $\qquad$ Example: Eq.Lines of $u(x) = x^2 - y^2$ are the lines $x^2 - y^2 = c$ for constant c

Conformal Mapping

$$
\begin{tikzcd}
\boxed{D \subset \mathbb{C}} \arrow[r, bend left, "g(z) = w"] & \boxed{D^* \subset g(D)} \arrow[l, bend left, "z = g^{-1}(w)"]
\end{tikzcd}
$$

$z = x + iy \qquad\qquad w = s + it$

$$
\begin{tikzcd}
f(z) \arrow[d] & & f^*(g(z)) \arrow[ll, "="'] \\
u(x,y) = \mathrm{Re}(f(z)) \arrow[d, mapsto] & & u^*(s,t) = \mathrm{Re}(f^*(w)) \arrow[u, mapsto] \\
f\left(g^{-1}(w)\right) \arrow[rr, "="] & & f^*(w) \arrow[u]
\end{tikzcd}
$$

$g(z) = s(x,y) + it(x,y) \quad$ analytic (thus invertible) in D

Ellipse $\qquad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad a =$ Abstand Ursprung Brennpunkt, $b =$ Abstand Ursprung y-Achsenschr

Hyperbola $\qquad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \quad a =$ Abstand Ursprung x-Achsenschnittpunkt

Example

$$
\begin{aligned}
g(z) &= \sin z = w \\
&= \underbrace{\sin x \cosh y}_{=s(x,y)} + i \underbrace{\cos x \sinh y}_{=t(x,y)}
\end{aligned}
$$

A straight line $z = x_0 + iy$ will be transformed as follows:

$$
g\begin{pmatrix} x = x_0 \\ y = y \end{pmatrix} = \begin{pmatrix} s = \sin x_0 \cosh y \\ t = \cos x_0 \sinh y \end{pmatrix}
$$

$$
\Leftrightarrow \frac{s^2}{\sin^2 x_0} - \frac{t^2}{\cos^2 x_0} = \cosh^2 y - \sinh^2 y = 1 \quad \text{(Hyperbola)}
$$

Maximum Principle $\qquad$ Let $u(x,y)$ be a potential in a bounded region with domain D and boundary ∂D

1. If $u(x,y)$ is not constant, then it cannot have any max or min point inside D
2. Max & Min of $u(x,y)$ must be attained in ∂D
3. If $u(x,y)$ constant on ∂D then it must be constant in D

8.1 Poisson Integral Formula

Usable for Laplace equations on disks. In the following,

R = Radius of domain $\quad r, \varphi$ = Coordinates of evaluation point, θ = Integration variable

Dirichlet problem	$\Delta u = 0$ on B_R $\boxed{u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} u(R,\theta)\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\varphi - \theta)}d\theta}$
Neumann problem	$\Delta u = 0$ on $B_R, \frac{\partial u}{\partial n} = h$ on ∂B_R $u(r,\varphi) = A_0 + \frac{R}{\pi}\int_0^{2\pi} -\frac{1}{2}\ln\left(1 - 2\frac{r}{R}\cos(\varphi - \theta) + \left(\frac{r}{R}\right)^2\right)h(\theta)d\theta$

9 Distributions

A distribution T is defined by its value on test functions φ.

φ must be infinitely differentiable and must have compact support, i.e. $\varphi(\infty) = \varphi(-\infty) = 0$

Mathematical definition	$T(\varphi) := \int_{\mathrm{R}^n} f(x)\varphi(x) \equiv \langle f, \varphi\rangle$
Example for test function	$\varphi(x) = \begin{cases} \exp\left(\frac{1}{x^2 - 1}\right) & \lvert x\rvert < 1 \\ 0 & \lvert x\rvert \geqslant 1 \end{cases}$
Dirac Delta distribution	$\delta(x - \xi) = \begin{cases} \infty & x = \xi \\ 0 & x \neq \xi \end{cases}$ $\int_{-\infty}^{\infty} \delta(x - \xi)\varphi(x)dx = \varphi(\xi)$ Notation: $\delta_\xi(x) \equiv \delta(x - \xi)$
Heaviside distribution	$h(x - \xi) = \begin{cases} 1 & x > \xi \\ 0 & x \leqslant \xi \end{cases}$ $H(x - \xi) := \int_{-\infty}^{\infty} h(x - \xi)\varphi(x)dx = \int_0^{\infty} \varphi(x)dx$ Notation: $H_\xi(x) \equiv H(x - \xi)$ $\boxed{H'_\xi = \delta_\xi}$
Derivatives	$\int_{-\infty}^{\infty} T'(x)\varphi(x)dx = \underbrace{\int_{-\infty}^{\infty}(T\varphi)'(x)dx}_{=0} - \int_{-\infty}^{\infty} T(x)\varphi'(x)dx$
n = 1	$\langle T', \varphi\rangle = -\langle T, \varphi'\rangle$
n = 2	$\left\langle \frac{\partial^{k+l}T}{\partial x^k \partial y^l}, \varphi\right\rangle = (-1)^{k+l}\left\langle T, \frac{\partial^{k+l}}{\partial x^k \partial y^l}\varphi\right\rangle$
Laplace	$\langle \Delta T, \varphi\rangle = \langle T, \Delta\varphi\rangle$

10 Green's functions

A Green's function for the laplacian Δ on a region D is a family of functions $G_D(x, y; \xi, \eta)$ defined for $(x, y) \neq (\xi, \eta)$ and satisfying

1. $\Delta G(\vec{x}; \vec{\xi}) = \delta(\vec{x} - \vec{\xi})$
2. $G(\vec{x}; \vec{\xi}) = 0 \quad$ for $\vec{x} \in \partial D$

Green's 2nd identity	$\int\int_D v\Delta u - u\Delta v = \int_{\partial D} v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}$
Fundamental solution of Δ on $\mathbb{R}^2$	$\Gamma(\underbrace{x, y}_{\text{Vars}}; \underbrace{\xi, \eta}_{\text{Params}}) = \frac{1}{4\pi}\ln\big(\underbrace{(x-\xi)^2 + (y-\eta)^2}_{=r^2}\big)$
Green's functions	
Trick 1	$G_D(x, y; \xi, \eta) = \Gamma(x, y; \xi, \eta) + g(x, y; \xi, \eta)$ where $\Delta\Gamma = \delta(\vec{x} - \vec{\xi})$ $\Delta g = 0$ on D $g = -\Gamma$ on ∂D
Trick 2 (*Reflection Principle*)	Take $g(x, y; \xi, \eta) = -\Gamma(x, y, \underbrace{\xi^*, \eta^*}_{\notin D}) + c$
D_1 = upper half plane	$G_{D_1}(x, y; \xi, \eta) = \Gamma(x, y; \xi, \eta) - \Gamma(x, y; \xi, -\eta) = \frac{1}{4\pi}\ln\left(\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2}\right)$
D_2 = disk B_R	$G_{D_2}(x, y; \xi, \eta) = \Gamma(x, y; \xi, \eta) - \Gamma\left(x, y; \frac{R^2}{r}\xi, \frac{R^2}{r}\eta\right) + \frac{1}{4\pi}\ln\left(\frac{R^2}{r^2}\right) = \frac{1}{4\pi}\ln\left(\frac{R^2}{r^2}\cdot\frac{(x-\xi)^2 + (y-\eta)^2}{(x-\frac{R^2}{r}\xi)^2 + (y-\frac{R^2}{r}\eta)^2}\right)$ $G_{D_3}(\rho, \theta; r, \varphi) = \frac{1}{4\pi}\ln\left(\frac{\rho^2}{r^2}\cdot\frac{\rho^2 + r^2 - 2\rho r\cos(\theta-\varphi)}{\rho^2 + \frac{R^4}{r^2} - 2\rho\frac{R^2}{r}\cos(\theta-\varphi)}\right)$ where $x = \rho\cos\theta, y = \rho\sin\theta; \xi = r\cos\varphi, \eta = r\sin\varphi$
Poisson equation	$\boxed{\begin{array}{ll} \Delta u = f & \text{on } D \\ u = g & \text{on } \partial D \end{array}}$
Solution for Poisson eqn	$u(\vec{\xi}) = \int\int_D G(\vec{x}, \vec{\xi})f(\vec{x})d\vec{x} + \int_{\partial D} g(\vec{x})\frac{\partial G}{\partial n}(\vec{x}; \vec{\xi})$ Tips: For a circle, the normal derivative is just the derivative w.r.t r. Don't forget the Jacobian determinant when using the polar form of a Green's fn.
Maxwell's reciprocity	$G_D(\vec{x}; \vec{\xi}) = G_D(\vec{\xi}; \vec{x})$