Minimum Mean Cycle (Karp '78)

1 Definitions

1.1 General definitions

Given a directed graph $G = (V, E)$ and a cost function on the edges, $c : E \to \mathbb{R}$, the following definitions will be used:

1. A *path* P from u to v is a sequence of vertices $P = \langle v_0, v_1, \dots, v_k \rangle$ s.t. $\forall_{1 \le i \le k}, (v_{i-1}, v_i) \in E$ and also $v_0 = u$ and $v_k = v$. The *length* of the path is defined as k, and is denoted by $|P|$.
2. The *cost* of the path is defined as $c(P) = \sum_{i=1}^{k} c(e)$.
3. A path of length 0 is a path from a vertex to itself with a cost of 0.
4. A path that begins and ends in the same vertex, with a length greater than 0, is called a *cycle*.
5. The *mean cost* of a path is defined as $\mu(P) = \frac{c(P)}{|P|}$
6. Denote by A the set of all cycles in G. The *minimum mean cost* of the cycles in A is:

$$\mu^* = \min_{C \in A} \mu(C)$$

7. A cycle C is a *critical cycle* if $\mu(C) = \mu^*$ (note that in different contexts the same term might stand for the maximum mean cycle).

1.2 Problem definition

The **Minimum Mean Cycle Problem** is defined as follows:
Input: A directed and strongly connected graph, $G = (V, E)$, and a cost function on the edges, $c : E \to \mathbb{R}$.
Output: A critical cycle in G.

2 The main theorem

Denote an arbitrary vertex by r and treat it as the root. Let $F_k(v)$ denote the set of paths of length k from r to v. Define:

$$d_k(v) = \min_{P \in F_k(v)} c(P)$$

(if $F_k(v) = \emptyset$, let $d_k(v) = \infty$). In what follows it should be assumed that $\infty - \infty = \infty$.

Theorem:

$$\mu^* = \min_{v \in V} \max_{0 \leq k \leq n-1} \left\{ \frac{d_n(v) - d_k(v)}{n-k} \right\}$$

Proof. Distinguish between two cases: $\mu^* = 0$ and $\mu^* \neq 0$.

First, notice that the following holds:

$$\min_{v \in V} \max_{0 \leq k \leq n-1} \left\{ \frac{d_n(v) - d_k(v)}{n-k} \right\} = 0 \Leftrightarrow \min_{v \in V} \max_{0 \leq k \leq n-1} \{d_n(v) - d_k(v)\} = 0$$

Thus, for the case of $\mu^* = 0$, the following holds:

1. It is sufficient to prove that:

$$\min_{v \in V} \max_{0 \leq k \leq n-1} \{d_n(v) - d_k(v)\} = 0$$

2. There is a cycle in the graph whose cost is 0.
3. There are no negative cycles in the graph.
4. For every vertex v, there exists a **simple** path from r to v whose cost is the minimum cost for a path from r to v.

Let $d(v)$ denote the cost of the minimum cost path from r to v. Since the appropriate path is simple, its length is smaller than n and thus:

$$\max_{0 \leq k \leq n-1} \{d_n(v) - d_k(v)\} = d_n(v) - d(v)$$

for every vertex v. Thus, it is left to prove that:

$$\min_{v \in V} \{d_n(v) - d(v)\} = 0.$$

- Since $\forall_{v \in V}, d_n(v) - d(v) \geq 0$,

$$\min_{v \in V} \{d_n(v) - d(v)\} \geq 0.$$

- Let C be a cycle whose cost is 0, and let u be a vertex in C. Let P be a simple path from r to u with the minimum cost. Concatenate to the path P the cycle C (starting from the vertex u) as many times as needed in order to get a path whose length is at least n. The result is a path P' whose cost is $c(P)$, and thus it is one of the paths with the minimum cost from r to u. Let v be the $(n+1)$-th vertex of P'. Since any prefix of P' is also a path with a minimum cost from r to its ending vertex, the prefix of P' of length n is a path from r to v with the minimum cost, and thus $d_n(v) = d(v)$. Hence,

$$\min_{v \in V} \{d_n(v) - d(v)\} = 0.$$

It remains to prove the theorem for the case of $\mu^* \neq 0$.
Let Δ denote a constant number and define a new cost function to the edges: $c^{\Delta}(e) = c(e) - \Delta$. Let $c^{\Delta}(P), \mu^{\Delta}, {\mu^*}^{\Delta}, d_k^{\Delta}(v)$ denote the respectively previously defined terms but this time with respect to the new cost function. For every path P, $c^{\Delta}(P) = c(P) - |P| \cdot \Delta$ and thus for any cycle C, $\mu^{\Delta}(C) = \mu(C) - \Delta$, which implies ${\mu^*}^{\Delta} = \mu^* - \Delta$. Thus, a cycle which was a critical cycle under the cost function c, is also a critical cycle under the new cost function c^{Δ}. Moreover, if Δ is chosen to be μ^*, it follows that ${\mu^*}^{\Delta} = 0$ and thus, according to the first case in which $\mu^* = 0$:

$$\begin{aligned}
0 = {\mu^*}^{\Delta} &= \min_{v \in V} \max_{0 \le k \le n-1} \left\{ \frac{d_n^{\Delta}(v) - d_k^{\Delta}(v)}{n-k} \right\} \\
&= \min_{v \in V} \max_{0 \le k \le n-1} \left\{ \frac{d_n(v) - n \cdot \Delta - (d_k(v) - k \cdot \Delta)}{n-k} \right\} \\
&= \min_{v \in V} \max_{0 \le k \le n-1} \left\{ \frac{d_n(v) - d_k(v)}{n-k} - \Delta \cdot \frac{n-k}{n-k} \right\} \\
&= \min_{v \in V} \max_{0 \le k \le n-1} \left\{ \frac{d_n(v) - d_k(v)}{n-k} \right\} - \Delta \\
\Rightarrow \mu^* = \Delta &= \min_{v \in V} \max_{0 \le k \le n-1} \left\{ \frac{d_n(v) - d_k(v)}{n-k} \right\}
\end{aligned}$$

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3 Finding the minimum mean cycle

The following algorithm solves the minimum mean cycle problem. It implicitly uses a data structure to hold the values it calculates so it can easily access them when needed.

1. Initialize: $d_0(r) = 0$ and $\forall_{v \in V \setminus \{r\}}, d_0(v) = \infty$.
2. For $i = 1$ to n:

   For every $v \in V$ calculate:

$$d_i(v) = \min_{u:(u,v) \in E} \{d_{i-1}(u) + c((u,v))\},$$

$$f_i(v) = \arg \min_{u:(u,v) \in E} \{d_{i-1}(u) + c((u,v))\}.$$

3. Calculate μ^* using the main theorem. Let u denote the vertex for which the value of μ^* was achieved.
4. Let P be the directed path of length n from r to u (constructed by backtracking the values $f_i(v)$). Output any directed cycle $C \subseteq P$ (there exists such a cycle since P is not a simple path).

Note that the complexity of the above algorithm is $O(nm)$.

Claim: *The above algorithm outputs a critical cycle.*

Proof. If $\mu^* = 0$, then:

- There exists $0 \leq k < n$ s.t. $d_k(u) = d_n(u) = d(u)$, and $\forall_i d_i(u) \geq d_k(u)$.
- There are no negative cycles in the graph.

Assume, for the purpose of contradiction, that there exists a cycle $C \subseteq P$ such that $c(C) > 0$. Then, $P \setminus C$ is a path from r to u with a cost of $c(P) - c(C) < c(P) = d_n(u) = d(u)$ which is a contradiction. Thus, any cycle $C \subseteq P$ has a cost of 0.
The proof for the case of $\mu^* \neq 0$ is analogous to the proof for that case for the main theorem. ■