Advanced Analysis of Algorithms - Homework IV (Solutions)

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1 Problems

1. Problem $26.2(9)$ on Page 664 of [CLRS01].

   Solution: Let $G = \langle V, E\rangle$ denote the undirected graph and let $s \in V$ denote an arbitrary vertex. Transform G into a flow network G' as follows:

   (i) Replace each undirected edge $(u, v) \in E$ with two directed edges (u, v) and (v, u), each with capacity 1.

   (ii) Let s be the source vertex of the flow network.

   Run the max-flow algorithm treating each vertex $u \in V - \{s\}$ as the sink t, recording the value of the maximum flow (mf_u). The edge-connectivity of G is then $\min_{u\in V-\{s\}} mf_u$. The crucial observation is that any cut of G, the vertex s is always on one side and hence it suffices to consider only the cases in which s is the source vertex in G'. It is straightforward to see that a directed cut in G' corresponds to a cut in G with the same capacity and vice versa. □

2. Problem $26.3(3)$ on Page 668 of [CLRS01].

   Solution: Without loss of generality, assume that $|L| \leq |R|$. Any augmenting path has the following structure:

   $$s \rightarrow L \rightarrow R \rightarrow L \ldots \rightarrow R \rightarrow t$$

   In other words, the path enters L from s, alternates between L and R and then exits to t. In the worst case, the path could have an incoming edge into and an outgoing edge from each vertex of L. In this case, the total path length is $2 \cdot |L| + 1$.

   In the case that $|R| \leq L$, we can similarly conclude that the length of the augmenting path cannot exceed $2 \cdot |R| + 1$.

   We thus conclude that an upper bound of the length of the augmenting path is $2 \cdot \min(|L|, |R|) + 1$. □

3. Problem $29.4(3)$ on Page 810 of [CLRS01].

   Solution: Let $G = \langle V, E, s, t, \mathbf{c}\rangle$ denote the flow network with s and t denoting the source and sink respectively and c_{ij} denoting the capacity of arc $(v_i, v_j) \in E$. Let x_{ij} denote the flow on arc (v_i, v_j) of the network.

   To recapitulate, the linear programming formulation of the max flow problem is:

   $$\begin{aligned} \max \sum_{(s,v)\in E} x_{(s,v)} & & \\ s.t. \sum_{(v_i,v_k\in E)} x_{ik} - \sum_{(v_k,v_r)\in E} x_{kr} &= 0 \ \forall v_k \in V - \{s, t\} \\ x_{ij} &\leq c_{ij},\ \forall (v_i, v_j) \in E \\ x_{ij} \geq 0,\ \forall (v_i, v_j) \in E & & \end{aligned}$$

This formulation is a more efficient formulation than the one discussed in class and corresponds to Exercise 29.2-(5) in [CLRS01].

To obtain the dual, we associate the variable y_v for the flow-conservation constraints and the variable z_ij for the capacity constraints. Accordingly, the dual can be written as:

$$
\begin{array}{rll}
\min \displaystyle\sum_{(v_i,v_j)\in E} c_{ij}\cdot z_{ij} & & \\
s.t.\ \forall (v_i,v_j)\in E & y_j - y_i + z_{ij} \geq 0, & if\ v_i \neq s, v_j \neq t \\
 & y_j + z_{ij} \geq 1, & if\ v_i = s, v_j \neq t \\
 & -y_i + z_{ij} \geq 0, & if\ v_i \neq s, v_j = t \\
 & z_{ij} \geq 1, & if\ v_i = s, v_j = t \\
\forall (v_i,v_j)\in E & z_{ij} \geq 0 &
\end{array}
$$

Writing the dual takes a lot of careful thought.

The fact that the optimal solution to the dual program (OPT) is at most the min-cut of the network can be established as follows: Given any (S, T) cut of the network, we set z_{ij} to 1, if (v_i, v_j) crosses the cut and to 0 otherwise. Likewise, y_i is set to 1 if $v_i \in S$ and 0 otherwise. It is easy to see that this assignment satisfies all the constraints. Thus, $OPT \leq MIN - CUT$. However, by strong duality, we know that OPT is equal to the max-flow, which by the max-flow min-cut theorem is equal to the min-cut of the network.

□

4. Problem $29.5(7)$ on Page 817 of [CLRS01].

   Solution: The dual of the given linear program is:

$$
\begin{array}{ccc}
 & \min\ s\cdot y & \\
r\cdot y & \geq & t \\
y & \geq & 0
\end{array}
$$

   We consider the four cases:

   (i) Both primal and dual have optimal solutions with finite objective value - In this case, both primal and dual must be feasible. This is possible only in one of these ways:
      (a) $r = 0$, $s \geq 0$ and $t \leq 0$,
      (b) $r > 0$, $s \geq 0$ and t arbitrary.
      (c) $r < 0$, $s \leq 0$ and $t \leq 0$.

   (ii) The primal is feasible, but the dual is not - There are two ways in which this is possible:
      (a) $r \leq 0$, $s \leq 0$ and $t > 0$.
      (b) $r > 0$, $s \leq 0$ and $t > 0$.

   (iii) The dual is feasible, but the primal is not - This is possible, only if $r \geq 0$ and $s < 0$.

   (iv) Both primal and dual are infeasible - This is possible, only if $r = 0, s < 0$ and $t > 0$.

□

5. Problem 29-$1(b)$ on Page 818 of [CLRS01].

   Solution: Without loss of generality, assume that the linear program to be solved is:

$$
\begin{array}{ccc}
 & \max\ \vec{\mathbf{c}}\cdot\vec{\mathbf{x}} & \\
\mathbf{A}\cdot\vec{\mathbf{x}} & \leq & \vec{\mathbf{b}} \\
\vec{\mathbf{x}} & \geq & \vec{\mathbf{0}}
\end{array}
\tag{1}
$$

It follows that the dual is:

$$
\begin{array}{rcl}
 & \min\ \vec{\mathbf{b}} \cdot \vec{\mathbf{y}} & \\
\vec{\mathbf{y}} \cdot \mathbf{A} & \geq & \vec{\mathbf{c}} \\
\vec{\mathbf{y}} & \geq & \vec{\mathbf{0}}
\end{array}
$$

Consider the following linear inequality system:

$$
\begin{array}{rcl}
\mathbf{A} \cdot \vec{\mathbf{x}} & \leq & \vec{\mathbf{b}} \\
\vec{\mathbf{y}} \cdot \mathbf{A} & \geq & \vec{\mathbf{c}} \\
\vec{\mathbf{x}} & \geq & \vec{\mathbf{0}} \\
\vec{\mathbf{c}} \cdot \vec{\mathbf{x}} & \geq & \vec{\mathbf{b}} \cdot \vec{\mathbf{y}} \\
\vec{\mathbf{y}} & \geq & \vec{\mathbf{0}}
\end{array} \tag{2}
$$

From the principle of Strong Duality, any solution to System (2) is the optimal solution of System (1). □

References

[CLRS01] T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to Algorithms.* MIT Press, 2001.