Solution: In this case, we can no longer replace each edge with a directed edge that assures us that no cycles exist. Therefore our best method in this general case is to use Dijkstra's algorithm after transforming the graphs as above replacing each level road with two directed arrows.

Problem 9-2. Karp's minimum mean-weight cycle algorithm

Let $G = (V, E)$ be a directed graph with weight function $w : E \to \mathbb{R}$, and let $n = |V|$. We define the ***mean weight*** of a cycle $c = \langle e_1, e_2, \ldots, e_k \rangle$ of edges in E to be

$$\mu(c) = \frac{1}{k} \sum_{i=1}^{k} w(e_i) .$$

Let $\mu^* = \min_c \mu(c)$, where c ranges over all directed cycles in G. A cycle c for which $\mu(c) = \mu^*$ is called a ***minimum mean-weight cycle***. This problem investigates an efficient algorithm for computing μ^*.

Assume without loss of generality that every vertex $v \in V$ is reachable from a source vertex $s \in V$. Let $\delta(s, v)$ be the weight of a shortest path from s to v, and let $\delta_k(s, v)$ be the weight of a shortest path from s to v consisting of *exactly* k edges. If there is no path from s to v with exactly k edges, then $\delta_k(s, v) = \infty$.

(a) Show that if $\mu^* = 0$, then G contains no negative-weight cycles and $\delta(s, v) = \min_{0 \le k \le n-1} \delta_k(s, v)$ for all vertices $v \in V$.

Solution: If there were a negative-weight cycle, then $\mu^* < 0$ because the minimum would have to be negative, therefore there are no negative weight cycles. Given that there are no negative weight cycles, then the shortest path will not take any cycles and can only be at most $n - 1$ edges long.

(b) Show that if $\mu^* = 0$, then

$$\max_{0 \le k \le n-1} \frac{\delta_n(s, v) - \delta_k(s, v)}{n - k} \ge 0$$

for all vertices $v \in V$. (*Hint:* Use both properties from part (a).)

Solution: First, we know $n - k$ is strictly positive because $k \le n - 1$. Then $\delta_n(s, v) - \delta(s, v) \ge 0$ because the shortest path when no negative weight cycles exist is going to cost more with n nodes than the shortest path with fewer nodes that is the actual shortest path.

(c) Let c be a 0-weight cycle, and let u and v be any two vertices on c. Suppose that the weight of the path from u to v along the cycle is x. Prove that $\delta(s, v) = \delta(s, u) + x$. (*Hint:* The weight of the path from v to u along the cycle is $-x$.)

Solution: We know $\delta(s, u) \le \delta(s, v) + x$ because the shortest path from s to u might use v. Alternatively we also know that $\delta(s, v) \le \delta(s, u) - x$ because the shortest path to v from s might go through u and around the zero weight cycle for a cost of $-x$. Therefore with these two inequalities we know $\delta(s, v) = \delta(s, u) + x$.

(d) Show that if $\mu^* = 0$, then there exists a vertex v on the minimum mean-weight cycle such that

$$\max_{0 \le k \le n-1} \frac{\delta_n(s,v) - \delta_k(s,v)}{n-k} = 0 .$$

(*Hint:* Show that a shortest path to any vertex on the minimum mean-weight cycle can be extended along the cycle to make a shortest path to the next vertex on the cycle.)

Solution: Get to the cycle along some shortest path and then extend the path along the cycle to make a shortest path of length n. If v is the vertex we end up at, then $\delta_n(s,v) = \delta(s,v)$. Then since for we took the shortest possible path to the cycle, there cannot exist any shorter path to the node with fewer steps, only equal path lengths.

(e) Show that if $\mu^* = 0$, then

$$\min_{v \in V} \max_{0 \le k \le n-1} \frac{\delta_n(s,v) - \delta_k(s,v)}{n-k} = 0 .$$

Solution: We know that there exists some vertex with a maximum difference of 0, and all differences are greater than 0, so the minimum must be 0.

(f) Show that if we add a constant t to the weight of each edge of G, then μ^* is increased by t. Use this to show that

$$\mu^* = \min_{v \in V} \max_{0 \le k \le n-1} \frac{\delta_n(s,v) - \delta_k(s,v)}{n-k} .$$

Solution: Adding t to each edge increases μ^* by t. It also increases $\delta_n(s,v)$ by nt and decreases $-\delta_k(s,v)$ by kt. Manipulate, and both sides increase by t and the equation is maintained. Thus, by picking $t = -\mu^*$, we can use the previous part.

(g) Give an $O(VE)$-time algorithm to compute μ^*.

Solution: Compute $\delta_k(s,v)$ for $k = 0, 1, \ldots, n$ in $O(VE)$ time by evaluating the recurrence $\delta_{k+1}(s,v) = \min_u \delta_k(s,u) + w(u,v)$. In $O(V^2)$ time, determine the minimum of the maximum of the fraction.