

# PI MU EPSILON JOURNAL

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**PI MU EPSILON JOURNAL**  
**THE OFFICIAL PUBLICATION OF THE**  
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**ON MUTUAL AND PAIRWISE INDEPENDENCE:  
SOME COUNTEREXAMPLES**

*Anwer Khurshid and Haredo Sahai  
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In elementary probability theory the concepts of pairwise and mutual or stochastic independence play a useful role. Three events A, B, and C, defined on the same sample space, are said to be **pairwise independent** if

$$P(A \cap B) = P(A)P(B),$$

$$P(A \cap C) = P(A)P(C),$$

$$P(B \cap C) = P(B)P(C).$$

The events are said to be **completely independent** if

$$P(A \cap B \cap C) = P(A)P(B)P(C).$$

The events are said to be **mutually independent** if both conditions hold. One might think that the first relations imply the second, i. e., that pairwise independence implies complete independence. Generally, it is almost always true but there are instances where the events are pairwise independent and yet the second condition does not hold. However, such occurrences are not very common and it takes some effort to construct a nontrivial natural example. In fact, Feller (1957, p. 117) has remarked that "... practical examples of pairwise independent events that are not mutually independent apparently do not exist." It was the famous Russian mathematician S. N. Bernstein who first gave an artificial example to illustrate such a possibility. Similarly, the second condition does not imply the first, i. e., events which are completely independent are not necessarily pairwise independent.

The fact that pairwise independence is a strictly weaker condition than mutual independence is noted with some surprise by most students in a probability course. The purpose of this note is to assemble some examples when the events are pairwise independent but not completely independent and *vice versa*. The examples are readily constructed and can be easily presented in an elementary probability course.

**Pairwise independence does not imply complete independence.**

**Example 1.** Suppose that a regular tetrahedron has one red face, one green face, one blue face, and its fourth face colored with red, green, and blue stripes. Toss the tetrahedron and observe the face that appears (on the bottom). Let A be the event that red appears, B the event that green appears, and C the event that blue appears. We see that A, B, and C each have probability 1/2. Furthermore, each of  $A \cap B$ ,  $A \cap C$ , and  $B \cap C$  has probability 1/4, since the only

way that two colors can appear is for the striped face to appear. So, A, B, and C are **pairwise** independent. But  $A \cap B \cap C$  also has probability  $1/4$ , so that A, B, and C are not completely independent.

This example appears in [Gnedenko, 1963, p. 62], where credit is given to S. N. Bernstein. It also appears in many other texts. For example, [Freund, 1973, p. 151] gives a Venn diagram with those probabilities. But most of the time, "Bernstein's tetrahedron" wears some kind of disguise. Here are some other examples. We leave it to the reader to figure out the disguises.

1a: [Cramér, 1946, p. 162; Lindgren, 1976, p. 46]. Let the sample space be  $S = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$  with all points equally likely. Let A (respectively, B, C) be the event that the first (respectively, second, third) coordinate is a 1.

1b: [Goldberg, 1960, p. 111]. In order to maintain quality control in a manufacturing process, each item undergoes three inspections. Of four units in a sample, unit 1 passed only the first inspection, unit 2 passed only the second inspection, unit 3 passed only the third, and unit 4 passed all three. Select one of the four units at random. Let A (respectively, B, C) be the event that the unit passed the first (respectively, second, third) inspection.

1c: [Eisen, 1969, p. 49]. The following four combinations of symbols for apples, pears, and lemons appear on the face of a slot machine: (a, a, a), (p, p, p), (l, l, l), and (a, p, l), each with probability  $1/4$ . Let A (respectively, B, C) be the event that an apple (respectively, a pear, a lemon) shows up.

1d: [Parzen, 1960, p. 90; Hogg and Tanis, 1988, p. 42]. An urn contains balls numbered 1 to 4. Draw a ball at random. Let A be the event that ball number 1 or 2 is selected, B the event that ball number 1 or 3 is selected, and C the event that ball number 1 or 4 is selected.

1e: [Ash, 1972, p. 204; Subrahmaniam, 1979, p. 109; Berman, 1968, p. 69; Larsen and Marx, 1986, p. 61; Mendenhall, Schaeffer, and Wackerly, 1986, p. 1091]. Suppose that a fair coin is tossed twice. Let A be the event that the first toss is a head, B the event that the second toss is a head, and C the event that both tosses yield the same outcome.

#### Example 2.

Roll two fair dice. Let A be the event of an odd number showing up on the first die, B the event of an odd number showing up on the second die, and C the event of an odd total from the two faces. Each of A, B and C has probability  $1/2$  and each of  $A \cap B$ ,  $A \cap C$ , and  $B \cap C$  has probability  $1/4$ , so A, B, and C are **pairwise** independent. But  $A \cap B \cap C$  has probability 0, since two odd numbers have an even sum, so the events are not completely independent.

This example appears in [Mood, Graybill, and Boes, 1974, p. 42; Feller,

1968, p. 143; Larsen and Marx, 1986, p. 63]. Similar examples based on rolling **two** dice appear in several textbooks. We leave to the reader the calculation of the probabilities.

2a: [Goldberg, 1960, p. 111]. Let A be the event that 6 shows up on the first die, B the event that 6 shows up on the second die, and C the event-Of, an odd total.

2b: [Blake, 1979, p. 122]. Let A be the event that 1 shows up on the first die, B the event that 1 shows up on the second die, and C the event that the same number shows up on both dice.

2c: [Ash, 1970, p. 28]. Let A be the event that the first die shows a 1, 2, or 3, B the event that the second die shows a 4, 5, or 6, and C the event that the sum is 7.

Example 3 [Feller, 1968, p. 127]. Consider the sample space consisting of nine triplets: the six permutations of a, b, and c together with (a, a, a), (b, b, b), and (c, c, c), each with probability  $1/9$ . Let A (respectively, B, C) be the event that the **first** coordinate is a (respectively, the second coordinate is b, the third coordinate is c). Each of A, B, and C has probability  $1/3$  and each of  $A \cap B$ ,  $A \cap C$ , and  $B \cap C$  has probability  $1/9$ , so A, B, and C are **pairwise** independent. But  $A \cap B \cap C$  has probability  $1/9$  also, so the events are not completely independent.

Example 4 [Larsen and Marx, 1986, p. 62]. A roulette wheel has 36 numbered slots colored red or black as follows:

red: (1, 2, 3, 4, 5, 10, 11, 12, 13, 24, 25, 26, 27, 32, 33, 34, 35, 36)

black: (6, 7, 8, 9, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 28, 29, 30, 31).

Spin the wheel once and observe the **number** and color of the slot in which the ball lands. Let A be the event that the slot is red, B the event that the slot has an even number, and C the event that its number is less than or equal to 18. Then each of A, B, and C has probability  $1/2$ , and each of  $A \cap B$ ,  $A \cap C$ , and  $B \cap C$  has probability  $1/4$ . But  $A \cap B \cap C$  has probability  $1/9$ .

Example 5 [Geisser and Mantel, 1962, p. 290]. Consider a sphere and select, randomly and independently, three great circle segments on it. Let A be the event that segments 1 and 2 intersect, B the event that segments 1 and 3 intersect, and C the event that segments 2 and 3 intersect. The specific probabilities involved will depend on the lengths of the segments, but whatever they are, **pairwise** independence is inherent in the experiment. If any two of **the** three events occur, then one of the segments intersects both of the **other** two; intuitively, this increases **the** chance that the other two segments will intersect each other. So, complete independence fails.

Example 6 [Geisser and Mantel, 1962, p. 290]. Suppose that three halls

are distributed independently and at random into two or more urns. Let A be the event that balls 1 and 2 are placed in the same urn, B the event that balls 1 and 3 are placed in the same urn, and C the event that balls 2 and 3 are placed in the same urn. If there are exactly two urns, then we have yet another disguised example of the Bernstein tetrahedron. If there are exactly three urns, then we have the same probabilities as in Example 3. If there are more than three urns, then the probabilities are new.

Geisser and Mantel actually gave a generalized Example 6, where the number of balls is  $n \geq 3$ . Then we get  $n(n - 1)/2$  events  $A_{ij}$ , where  $A_{ij}$  occurs if balls  $i$  and  $j$  are placed in the same urn. Similarly, Bernstein's example has been generalized to any number of dimensions in [Lancaster, 1965].

Other examples, not significantly different from those above can be found in [Neuts, 1973, p. 76; Kreysig, 1970, p. 57; Giri, 1974, p. 33; DeGroot, 1975, p. 42].

### Complete independence does not imply pairwise independence

#### Example 1 [Ash, 1970, p. 27].

Suppose that two dice are tossed and let  $S$  be the sample space comprising all ordered pairs  $\{(i, j)\}$ ,  $i, j = 1, 2, 3, 4, 5, 6$ , with a probability of  $1/36$  assigned to each point. Let A be the event that the face of the first die is 1, 2, or 3; B the event that the face of the first die is 3, 4, or 5; and C the event that the sum of the two faces is 9. It is easily verified that  $P(A) = P(B) = 1/2$ ,  $P(C) = 1/9$ ,  $P(A \cap B) = 1/6$ ,  $P(A \cap C) = 1/36$ ,  $P(B \cap C) = 1/12$ , and  $P(A \cap B \cap C) = 1/36$ . Thus, it follows that A, B, and C are completely independent but not pairwise independent.

#### Example 2 [Larsen and Marx, pp. 61-62].

In Example 1, let A be the event that the face of the first die is 1 or 2; B the event that the face of the second die is 3, 4, or 5; and C the event that the sum of the two faces is 4, 11, or 12. It is readily shown that A, B, and C are completely independent but not pairwise independent.

#### Example 3 [Crow, 1957, pp. 716-717].

An urn contains one red, one blue, one white, two yellow, and three black balls. A ball is drawn randomly from the urn. Let A be the event that a red, yellow, or white ball is drawn; B the event that a blue, yellow, or white ball is drawn; and C the event that a black or white ball is drawn. It is easily shown that  $P(A) = P(B) = P(C) = 1/2$ ,  $P(A \cap B) = 3/8$ ,  $P(A \cap C) = P(B \cap C) = 1/8$ , and  $P(A \cap B \cap C) = 1/8$ . Thus, it follows that A, B, and C are completely independent but not pairwise independent. Essentially the same example is given in [Lindgren (1976, p. 48)].

#### Example 4 [Subrahmaniam, 1979, p. 110].

In a certain town there are three editions of a daily newspaper: morning ( $M$ ), evening ( $E$ ), and weekend ( $W$ ). Suppose the probability of a randomly chosen household subscribing to any one of the editions is illustrated by the Venn diagram in Figure 1. From the figure,  $P(M) = .50$ ,  $P(E) = .80$ ,  $P(W) = .60$ ,  $P(M \cap E) = .42$ ,  $P(M \cap W) = .24$ ,  $P(E \cap W) = .42$  and  $P(M \cap E \cap W) = .24$ . Thus, M, E, and W are completely independent but not pairwise independent.

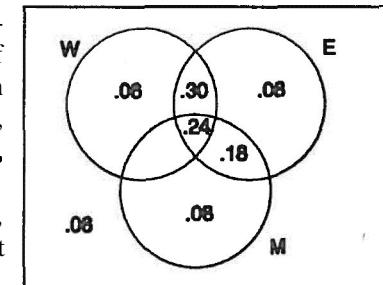


Figure 1

#### Example 5 [Freund, 1962, p. 50].

Suppose the events A, B and C and their respective probabilities are illustrated by the Venn diagram in Figure 2, where the sample space contains 100 equally likely points and the numbers on the diagram indicate the number of distinct outcomes contained in the respective events. From the figure,  $P(A) = 1/2$ ,  $P(B) = 1/4$ ,  $P(C) = 8/100$ ,  $P(A \cap B) = 1/4$ ,  $P(A \cap C) = P(B \cap C) = 11/100$ , and  $P(A \cap B \cap C) = 1/100$ . Thus, it follows that A, B, and C are completely independent but not pairwise independent.

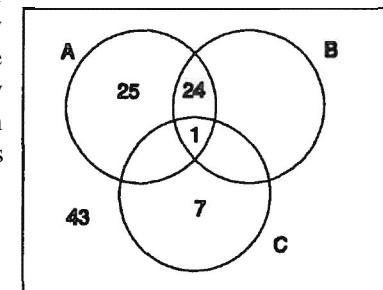


Figure 2

#### Example 6 [Goldberg, 1960, p. 112].

A card is selected at random from a standard deck of 52 cards. Let A be the event that the selected card is a spade or club, B the event that it is a spade, and C the event that it is the ace of spades or the ace, king, ..., 8 of diamonds. It is readily verified that A, B, and C are completely independent but not pairwise independent.

#### Example 7 [Mood, Graybill, and Boes, 1974, p. 43].

Consider two events A and B that are not independent and another event C of probability zero. Then it immediately follows that A, B, and C are completely independent but not pairwise independent.

As suggested by the examples given in this paper, some authors give only one type of counterexample while others include both types. Furthermore, most examples can be criticized as "trivial" in nature, or at least artificial. We

conclude with two examples which definitely are not frivolous but have important practical applications.

**Example 1** [Geisser and Mantel, 1962, p. 2901].

Let  $r_{12}$ ,  $r_{13}$ , and  $r_{23}$  be the pairwise sample correlation coefficients based on a random sample of  $n$  observations from a trivariate nonsingular normal distribution having a diagonal variance-covariance matrix. Using the methods of mathematical statistics, it can be proven that the joint density of the sample correlation coefficients,  $r_{12}$ ,  $r_{13}$ , and  $r_{23}$  is given by

$$f(r_{12}, r_{13}, r_{23}) = C(n)(1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23})^{(n-5)/2}$$

when  $1 - r_{12}^2 - r_{13}^2 - r_{23}^2 + 2r_{12}r_{13}r_{23} > 0$  and zero elsewhere. Now, using the considerations of continuity and positive definiteness of the correlation matrix, it can be shown that the three random variables are not mutually independent. However, it can be shown directly that the variables  $r_{12}$ ,  $r_{13}$ , and  $r_{23}$  are pairwise independent.

The above result can be extended to the general case of  $p(p - 1)/2$  (with  $p > 3$ ) jointly distributed correlation coefficients when a random sample of  $n$  observations is drawn from a  $p$ -variate nonsingular normal distribution having a diagonal variance-covariance matrix. The result has an important statistical application to the effect that it simplifies the evaluation of the variance, in the null case (when the corresponding population correlations are zero), i.e.,

$$\text{Var}\left(\sum_{i=1}^p \sum_{j=i+1}^p a_{ij} r_{ij}\right) = \sum_{i=1}^p \sum_{j=i+1}^p a_{ij}^2 \text{Var}(r_{ij}).$$

**Example 2** [Driscoll, 1978, p. 432].

Consider two independent random variables  $X$  and  $Y$  each having the rectangular distribution on the unit interval. Further, define a random variable  $Z = (X + Y) \pmod{1}$ , i.e.

$$Z = \begin{cases} X + Y & \text{if } 0 \leq X + Y \leq 1 \\ X + Y - 1 & \text{if } 1 < X + Y \leq 2. \end{cases}$$

Then, using the methods of calculus and analytical probability, it can be shown that  $X$ ,  $Y$ , and  $Z$  are identically distributed and pairwise independent but not mutually independent.

The above result has an important application in the characterization of the rectangular distribution. For example, using the methods of advanced probability, it can be proven that among the absolutely continuous distributions having the closed unit interval for their support, the rectangular distribution is the only one satisfying the above properties [Driscoll, 1978]. The result also extends to intervals other than  $[0, 1]$ , as well as to discrete rectangular distributions, and thus provides characterizations for all rectangular distributions.

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### Chapter Reports

Professors Rudy Gideon and Mary Jean **Brod**, faculty advisors of the MONTANA ALPHA chapter (University of Montana), report that there are thirty new members and plans for biweekly meetings. Four students presented talks to the chapter last year:

Dean **Risinger**—Finiteness and rings of continuous functions

Doug Holstein—Exact sequences of topological spaces

Patricia **Olsen**—Optimal jury selection

Scott McRae—Analysis of a heuristic algorithm for optimally scheduling assignments with constraints.

The NEW YORK OMEGA chapter (St. Bonaventure University) held five meetings and, in cooperation with the MAA student chapter, sponsored the Mathematics Forum, a series of ten lectures, including one by this year's Frame Lecturer, George Andrews.

### ON THE SOLUTIONS OF $a^a = b^b$

Jeffrey D. Bomberger  
University of *Nebraska—Lincoln*

The equation  $x^y = y^x$  was solved by **Euler** and has been considered many times since. However, the equation  $x^x = y^y$  is not as well known. In this paper we will find all pairs  $(a, b)$  which satisfy  $a^a = b^b$ .

Let us consider the function  $f(x) = x^x$ ,  $x > 0$ . Calculation shows that  $f$  attains its minimum at  $x = 1/e$  and that  $\lim_{x \rightarrow 0^+} x^x = 1$ . Thus the graph of  $f$  is as shown in Figure 1. From the graph we can see that if  $e^{-1/e} < y < 1$ , then there is a unique pair of real numbers  $(a, b)$  that satisfies  $a^a = b^b$ . Therefore, by allowing  $y$  to vary in the interval  $(e^{-1/e}, 1)$ , it follows that there are infinitely many pairs  $(a, b)$  that satisfy  $a^a = b^b$ .

Let  $S$  denote the set of all pairs  $(a, b)$  with  $a < b$  which satisfy  $a^a = b^b$ . Then it is clear that if  $(a, b) \in S$  then  $0 < a < 1/e < b < 1$ . Also, since  $f$  is one-to-one on each of  $(0, 1/e)$  and  $(1/e, 1)$ , if  $(a, b)$  and  $(a, c)$  are in  $S$ , then  $b = c$ , and if  $(a, b)$  and  $(c, b)$  are in  $S$ , then  $a = c$ .

It is not hard to see that  $(1/4, 1/2) \in S$ . Thus, there is  $y$  so that  $(1/2y, 1/y) \in S$ . Is there  $y$  so that  $(1/3y, 1/y) \in S$ ? If so, then  $(1/3y)^{1/3y} = (1/y)^{1/y}$ ,  $(1/3y) = (1/y)^3$ , and  $y^3 - 3y = 0$ . Solving, we get  $y = 3^{1/2}$ . Replacing 3 with 4 gives  $y^4 - 4y = 0$  and  $y = 4^{1/3}$ .

In general, if  $(1/xy, 1/y) \in S$ , we have  $y^x - xy = 0$ , from which we get

**LEMMA 1:** For any  $x > 1$ ,  $(x^{-x/(x-1)}, x^{-1/(x-1)}) \in S$ .

**Proof.** Let  $a = x^{-x/(x-1)}$  and  $b = x^{-1/(x-1)}$ . Then  $0 < a < b$  and since  $xa = b$ ,

$$a^a = (x^{-x/(x-1)})^a = (x^{-1/(x-1)})xa = b^b.$$

**LEMMA 2:** For  $x > 1$ , let  $A(x) = x^{-x/(x-1)}$  and  $B(x) = x^{-1/(x-1)}$ . Then

- (i)  $A$  and  $B$  are continuous on  $(1, \infty)$ .
- (ii)  $A$  is strictly decreasing and  $B$  is strictly increasing on  $(1, \infty)$ .

$$\begin{array}{ll} \text{(iii)} & \lim_{x \rightarrow 1^+} A(x) = e^{-1}, \text{ and } \lim_{x \rightarrow \infty} A(x) = 0. \\ \text{(iv)} & \lim_{x \rightarrow 1^+} B(x) = e^{-1}, \text{ and } \lim_{x \rightarrow \infty} B(x) = 1. \end{array}$$

Proof: (i): Since  $A(x) = \exp(-x \ln x/(x-1))$  and  $x \ln x/(x-1)$  is continuous on  $(1, \infty)$ , then so is A. Similarly, B is continuous on  $(1, \infty)$ . In fact, A and B are infinitely differentiable on  $(1, \infty)$ . (ii): Let  $h(x) = x - 1 - \ln x$ . Then  $h'(x) = 1 - 1/x > 0$  for  $x > 1$ . So, h is strictly increasing on  $(1, \infty)$ , and  $x - 1 - \ln x = h(x) > h(1) = 0$  for all  $x > 1$ . Now, since  $A'(x) = -h(x)A(x)/(x-1)^2$  and  $A(x) > 0$ , it follows that  $A'(x) < 0$  for  $x \in (1, \infty)$ . The proof that  $B'(x) > 0$  for  $x \in (1, \infty)$  is similar. Here, we need only show that  $x \ln x - (x-1) > 0$  for all  $x > 1$ , which is not difficult. Finally, (iii) and (iv) follow from L'Hôpital's rule.

**THEOREM:**  $S = \{(x^{-x/(x-1)}, x^{-1/(x-1)}) \mid 1 < x < \infty\}$ .

Proof: By Lemma 1, we need only show that

$$S \subset \{(x^{-x/(x-1)}, x^{-1/(x-1)}) \mid 1 < x < \infty\}.$$

Let  $(a, b) \in S$  and  $A(x), B(x)$  as in Lemma 2. Then, from Lemma 2,  $A(x)$  and  $B(x)$  are one-to-one continuous functions on  $(1, \infty)$ ,  $A(1, \infty) = (0, e^{-1})$ , and  $B(1, \infty) = (e^{-1}, 1)$ . Now, since  $a \in (0, e^{-1})$  and  $b \in (e^{-1}, 1)$  then by the intermediate value theorem, there exist unique numbers  $x$  and  $z \in (1, \infty)$  such that  $A(x) = a$  and  $B(z) = b$ . So,  $(A(x), B(z)) \in S$ , but  $(A(x), B(x)) \in S$  by Lemma 1. So,  $B(z) = B(x)$ . Since  $B(x)$  is one-to-one on  $(1, \infty)$ , then  $z = x$ . Hence,  $(a, b) = (A(x), B(x))$  for a unique  $x \in (1, \infty)$ , completing the proof.

By letting  $x = n/(n-1)$  for integer  $n > 1$ , we get an infinite sequence of rational solutions  $(a, b)$ :  $\{((n-1)/n)^n, (n-1)/n^{n-1}\}$ ,  $n = 2, 3, 4, \dots$ . The first four terms of this sequence are  $(1/4, 1/2)$ ,  $(8/27, 4/9)$ ,  $(811256, 27/64)$ ,  $(1024/3125, 256/625)$ .

Jeffrey Bomberger is an actuarial science major at the University of **Nebraska—Lincoln**. This paper was written when he was a first-year student in calculus, under the direction of Professor Mohammad **Rammaha**.

### Chapter Report

Professor Paul Eloë reports from the OHIO ZETA chapter (University of Dayton) that Kristine Fromm and Kristen Toft participated in summer research programs in 1992, that nine students participated in the Putnam Examination, and that a Dayton team was entered in the annual **mathematical** modeling contest.

### ELLIPSES AS PROJECTIONS OF CIRCLES

Ali R. Amir-Moéz  
Texas Tech University

One can study some properties of an ellipse through the orthogonal projection of circles. Two properties which are carried through the projection are quite interesting.

(i) Tangency is transformed into tangency.

(ii) Areas are all multiplied by the cosine of the angle between the plane of the circle and the plane of its projection.

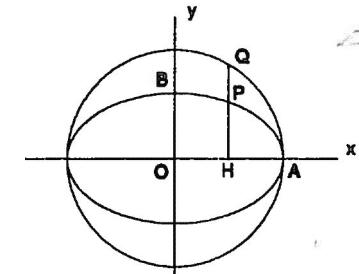


Figure 1

1. The Projection of a Circle: Each ellipse can be considered as the orthogonal projection of its principle circle. Let Q be a point on the principle circle of an ellipse (Fig. 1). Then the perpendicular to OA through Q intersects the ellipse at P. Let the foot of the perpendicular be H. We may say that HP is the projection of HQ. In order to make this idea clear, we rotate the principle circle about the x-axis through an angle  $\theta$  such that  $\cos \theta = b/a$ ; then the projection of this circle on the xy-plane is the ellipse. A cross section, with OA being the edge of the plane of the circle, is shown in Figure 2.

Let the equation of the principle circle be

$$(1) \quad X^2 + Y^2 = a^2.$$

Then Q has coordinates  $(X, Y)$  which satisfy (1). Since

$$\frac{PH}{QH} = \cos \theta = \frac{b}{a},$$

if the coordinates of P are  $(x, y)$  then we observe that

$$x = X \quad \text{and} \quad y = \frac{b}{a} Y.$$

Substituting in (1) for X and Y, we obtain

$$x^2 + \frac{a^2}{b^2}y^2 = a^2$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

2. Tangency: We observe that a tangent line to an ellipse at a point  $P$  and the tangent line to the corresponding point  $Q$  of the principle circle intersect at a point  $K$  which is on the  $x$ -axis (Fig. 3). The proof is quite simple. Now suppose we would like to draw a tangent line to the ellipse from any point  $L$  outside the ellipse (Fig. 3). We draw the perpendicular line to the  $x$ -axis through  $L$ . Then we obtain the point  $M$  on this line such that

$$\frac{LN}{MN} = \cos \theta = \frac{b}{a}.$$

Then the tangent line through  $M$  to the circle intersects the  $x$ -axis at  $K$ . The line  $KL$  is tangent to the ellipse.

3. Areas: It is well-known that a projection of an area (as in Figure 2) is equal to the area of the original surface multiplied by the cosine of the angle between the two planes. So the area of the ellipse with semi-major axis  $a$  and semi-minor axis  $b$  will be

$$A = na^2 \cos \theta = \pi a^2 \left( \frac{b}{a} \right) = n ab.$$

#### 4. Hippocrates' Theorem:

Let  $ABCD$  be a square inscribed in a circle. Draw four half circles with diameters  $AB$ ,  $BD$ ,  $DC$ , and  $CA$ . We obtain four crescent-shaped configurations over the arcs (Fig. 4). The sum of the areas of these crescent shapes is the same as the area of the square  $ABCD$ . The proof is quite simple and can be found in [1].

Now consider the orthogonal projection of this configuration on a plane through the line  $CB$ . We shall get an ellipse and four crescent-shaped areas that

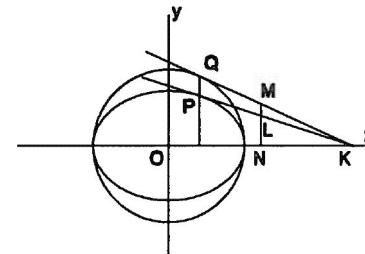


Figure 3

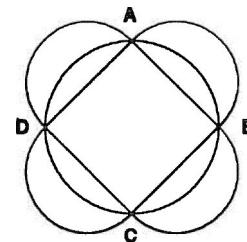


Figure 4

we would like to describe. Each of the areas is situated between two ellipses; one of them is the projection of the circumscribed circle of the square. We shall look at the outer ellipse in the first quadrant. The center  $M$  of the circle with diameter  $AB$  projects into  $M'$ , the midpoint of  $A'B'$  (Fig. 5). The half circle of diameter  $AB$  projects into half of an ellipse. Let the projection of the circumscribed circle be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then half of the major axis of the other ellipse will be  $a\sqrt{2}/2$  and half of its minor axis will be

$$\frac{a\sqrt{2}}{2} \cos \theta = \frac{a\sqrt{2}}{2} \cdot \frac{b}{a} = \frac{b\sqrt{2}}{2}.$$

The set of coordinates of  $M$  is  $(a/2, a/2)$ . So the set of coordinates of  $M'$  will be

$$\left( \frac{a}{2}, \frac{a}{2} \cos \theta \right) = \left( \frac{a}{2}, \frac{b}{2} \right).$$

Consequently, the equation of this ellipse is

$$\frac{(x - a/2)^2}{a^2/2} + \frac{(y - b/2)^2}{b^2/2} = 1.$$

This way the four crescent-shaped areas become four areas the sum of which is the same as the area of the rectangle, that is, the projection of the square  $ABCD$ .

#### Reference

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Ali R. Amir-Moéz, Professor Emeritus since 1988, is the author of, among many other things, *Elements of Multilinear Algebra* and *Three Persian Tales*.

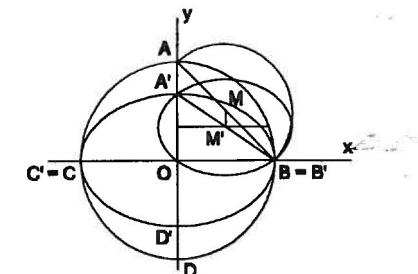


Figure 5

## ***PSL(2, 7) IS SIMPLE, BY COUNTING***

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Proving that  $PSL(2, 7)$  is a simple group can be done by the method of conjugation. This paper will use the more efficient method of counting. The paper will cover some definitions that are referred to in the proof and then address the proof. This method of proof also applies to  $A_5$ . I will begin by defining the group.

$SL(2, 7)$  is the group of 2-by-2 matrices with determinant one with entries from  $F_7$ , the integers modulo 7. To find its order, we look at the following elements, with  $a, b, c, d$  nonzero entries. There are 6 of the form

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \text{ 6 of the form } \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \text{ and 36 of each of the four forms } \begin{bmatrix} 0 & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & 0 \\ c & d \end{bmatrix},$$

$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ , and  $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ . The entries  $a$  and  $b$  in  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  can be chosen in 36 ways and

for each choice there are 5 ways of choosing  $c$  and  $d$  (since both are nonzero). So, the order of  $SL(2, 7)$  is  $2 \cdot 6 + 4 \cdot 36 + 5 \cdot 36 = 336$ .

The center of a group is

$$Z(G) = \{s \in G \mid sg = gs \text{ for all } g \in G\}.$$

$PSL(2, 7)$  is the group  $SL(2, 7)/Z(SL(2, 7))$ . To find the order of  $PSL(2, 7)$ , we find  $|Z(SL(2, 7))|$ . We are looking for the matrices that commute with all the elements of  $SL(2, 7)$ . Since such a matrix commutes with all the elements, in

particular it commutes with matrices of the form  $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ , with  $a$  and  $b$  not equal to 0. Thus

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & d \\ e & f \end{bmatrix} = \begin{bmatrix} c & d \\ e & f \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

or

$$\begin{bmatrix} ac & ad \\ be & bf \end{bmatrix} = \begin{bmatrix} ca & db \\ ea & fb \end{bmatrix}.$$

So we know that  $be = ea$  for all  $b, a$  in  $F_7$ , so  $(b - a)e = 0$  from which it follows that  $e = 0$ . We also know that  $ad = db$  for all  $a, b$ , so we have that  $d$

= 0. Since  $d = e = 0$ , any matrix in the center has the form  $\begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}$ . Now,  $\begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix}$  also commutes with  $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$  so

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix} = \begin{bmatrix} c & 0 \\ 0 & f \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & f \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & c \\ f & 0 \end{bmatrix},$$

from which it follows that  $f = c$ . Since  $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$  must have determinant 1,  $c^2 = 1$  and so  $c = 1$  or  $c = -1 = 6$  (in  $F_7$ ). Also, it is easy to see that  $\begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$  does

indeed commute with all the elements of  $SL(2, 7)$ . Hence the order of  $PSL(2, 7)$  is  $33612 = 168$ .

Now we state and prove a theorem.

**THEOREM:** Let  $H$  be a normal subgroup of a finite group  $G$  and let  $x$  be in  $G$ . If  $\gcd(|x|, |G/H|) = 1$ , then  $x$  is in  $H$ . ([1], p. 156)

**Proof:** Let  $|x| = r$ ,  $|G/H| = s$ . Then  $x^r = e$ , so  $|xH| \mid r$  because  $(xH)^r = (x^r)H = eH$ . Now, by Lagrange's Theorem,  $|xH| \mid |G/H|$ , and so  $|xH| \mid s$ . Since  $|xH| \mid r$  and  $|xH| \mid s$ , we have that  $|xH| \mid \gcd(r, s)$ . Since  $\gcd(r, s) = 1$ , we have  $|xH| = 1$ , and  $x$  is in  $H$ .

A straightforward but long calculation shows that the elements of  $PSL(2, 7)$  have the following orders:

Order	1	2	3	4	7
Number of elements	1	21	56	42	48

Now we can prove that  $PSL(2, 7)$  is simple. If  $PSL(2, 7)$  has a nontrivial proper subgroup  $H$ , then

$$|H| = 2, 3, 4, 6, 7, 8, 12, 14, 21, 24, 28, 42, 56, \text{ or } 84.$$

We consider different cases.

i.) If  $|H| = 7, 14, 21, 28$ , or  $42$ , then  $|PSL(2, 7)/H|$  is relatively prime to 7, and so by the theorem,  $H$  would have to contain all 48 elements of order 7. Since  $|H| \leq 42$ , that is impossible.

ii.) If  $|H| = 3, 6, 12$ , or  $24$ , then  $|PSL(2, 7)/H|$  is relatively prime to 3, and so by the theorem,  $H$  would have to contain all 56 elements of

order 3, but  $|H| \leq 24$ .

iii.) If  $|H| = 8$ , then  $|\mathbf{PSL}(2,7)/H|$  is relatively prime to 4, and so by the theorem H would have to contain all 42 elements of order 4, but  $|H| = 8$ .

iv.) If  $|H| = 56$ , then  $|\mathbf{PSL}(2,7)/H|$  is relatively prime to both 4 and 7, and so by the theorem, H would have to contain all 90 elements of orders 4 and 7, but  $|H| = 56$ .

v.) If  $|H| = 84$ , then  $|\mathbf{PSL}(2,7)/H|$  is relatively prime to both 3 and 7, and so by the theorem, H would have to contain all 104 elements of orders 3 and 7, but  $|H| = 84$ .

vi.) If  $|H| = 2$  or  $4$ , then  $|\mathbf{PSL}(2,7)/H| = 84$  and  $42$ . We know from the Sylow theorems that any group of order 84 and any group of order 42 has only one Sylow-7 subgroup, which is therefore normal. If H is normal in  $\mathbf{PSL}(2,7)$ , then  $\mathbf{PSL}(2,7)/H$  has a normal subgroup  $N_H$  of order 7. So now we consider the projection homomorphism

$$\psi: \mathbf{PSL}(2,7) \rightarrow \mathbf{PSL}(2,7)/H.$$

$\psi^{-1}(N_H)$  is a normal subgroup of G and is of order  $2 \cdot 7$  or  $4 \cdot 7$ . However, we have already shown that there does not exist a normal subgroup of order 14 or 28, so this case also is impossible.

We have examined all possible orders of nontrivial proper normal subgroups H of  $\mathbf{PSL}(2,7)$  and shown that all lead to contradictions. Hence  $\mathbf{PSL}(2,7)$  has no nontrivial proper normal subgroups and is a simple group of order 168.

#### Reference

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Sandra Lepsi, now a graduate of Illinois Benedictine, wrote this *paper* under the direction of Dr. Lisa Townsley Kulich. She plans to continue studying mathematics.

3507! - 1 is a big prime. How many digits?

## EQUIVALENCE CLASSES IN THE REAL WORLD

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Ziff-Davis Publishing Company

### 1. Introduction — what does a mathematician do in business?

Perhaps the last two words in the title of this section are **superfluous**—many people simply wonder what **mathematicians** do in general without restricting the question to the "business world". But while I was teaching I was often asked precisely that question by my students and, at the time, I really couldn't answer them with any kind of authority. I knew that there were many jobs in which a knowledge of elementary statistics was useful. And I even suspected that one might be able to use some linear algebra or calculus, but aside from jobs that are viewed as "technical", I didn't really have much of a clue as to the real value of a mathematical education to the business community.

Over the last twenty years in business I have come to realize that the value of **mathematical** studies is not specifically in the mastery of certain tools. Rather, it is that the study of mathematics leads one into a "mathematical" approach to problem solving that places great emphasis upon precise definitions. The fact of the matter is that most people are not skilled in, nor do they appreciate the importance of, precise definition-making. Yet a mathematician, almost by instinct, will turn a question of problem-solving into one of problem-identification.

The matching problem outlined below gives a good example of a problem that is of great strategic and tactical interest to the publishing industry and which has been "solved" before many times. However, as you will see, a mathematician's approach will give a new twist to this old problem.

### 2. The matching problem.

Recognizing whether or not two things are the same is a **theme** common to many parts of mathematics, as well as of human existence. Even the statement that  $1 + 1 = 2$  is a simple illustration of this theme, but it has great consequences in the development of the Peano Postulates for Arithmetic.

The matching I am interested in is that of names and addresses. In the magazine industry the inadvertent entry onto the subscription files of two orders for the same person is usually a minor annoyance to the subscriber, but it is a costly mistake for the company. Not only will the company waste copies by sending unwanted copies to the subscriber, but the failure to detect such duplicates has the undesirable side effect of reducing the total "paid circulation" of the magazine. Paid circulation is a technically defined term that is used by advertisers to indicate the number of people who have paid to receive copies of the magazine. It is the basis for advertising rates in the magazine and is there-

fore of great interest to the advertiser and publisher alike.

Correctly matching names and addresses has many other ramifications. For example, credit reporting bureaus link together records of transactions such as credit card payments or loan payments and prepare credit reports requested by **companies** looking to extend credit to borrowers. An example that is very important to direct mail advertisers (magazines themselves are often sold through direct mail promotions) has to do with name suppression. If a list of prospects is rented from an outside source, before the list is mailed one would want to identify all the prospects on the list who are already subscribers to the magazine and "suppress" their names from the **mailing**.

Making an error in the suppression of names is not nearly as serious as erroneously reporting credit information. In the former case it simply **means** than an existing subscriber gets a wasted promotion; in the latter case an erroneous credit report could result in a costly lawsuit.

But consider the cost of mailing promotions to already subscribing prospects. The cost of a **promotion** **may** be as great as 50 cents per name. It is not unusual to **mail** out as **many** as 1 million pieces in a single campaign. If as few as 5% of these are already on file, a not uncommon duplication percentage, some \$25,000 will be wasted trying to promote people who already subscribe!

### 3. The traditional solution.

The common approach to this problem is quite straightforward. For each name and address record a "**matchcode**" is defined by extracting portions of the record in a specific way. Two records having the same **matchcode** are, by definition, declared to be duplicates.

For example, a **matchcode** might be defined by concatenating the zip code with the subscriber's last name. Such a simple **matchcode** would have a couple of obvious failings. In the first place, any two people having the same last name actually residing in the same zip code area would be declared duplicates. A glance in any phone book would indicate that it is, in fact, relatively common for families to have **members** with the same surname living near each other. Another **problem** comes from the **fact** that **many** lists are constructed in such a way that the last name is not easily extracted. Sometimes the last **name** **might** be first, sometimes it **might be** last. There might be a suffix (**Ph.D.** or Jr.) or a prefix (Mr. or Ms.) that one has to avoid. Several techniques can help minimize these shortcomings. For example, instead of using the entire last name, a common trick is to drop all non-initial vowels from the last **name** instead. This partially compensates for two of the **most** common errors in transcribing names: transposition of characters and the interchange of vowels within a name. Another idea is to append the first initial of the first name—this helps distinguish different family members. Sometimes a portion of the address

is added too. Clearly any **matchcoding** scheme can fall prey to two errors—two different records **might** give rise to the same **matchcode** (a type II error, in statistical terms) or really duplicate records give rise to different **matchcodes** (a type I error). It is well-known that it is impossible to minimize both types of errors simultaneously. So, the question usually boils down to which type of **error** is the more serious in the context. The publisher can decide to use a "tight" **matchcode** (one that extracts many characters **from** each record and requires matching on all characters) or a "loose" **matchcode**. A tight **matchcode** minimizes making an error in saying that two records are duplicates (when they are not) but will overlook duplicate records that are not nearly perfect duplicates; a loose **matchcode** minimizes the chance of overlooking two duplicate records by asserting **some** are equal even when they are not. Aside from the observations above, **matchcodes** have an extremely vulnerable shortcoming—they are hierarchical. That is, in comparing two **matchcodes**, even a very simple error early in the hierarchy will cause two **match codes** to be significantly different. Two records could be nearly identical, but a **mistake** in the first letter of the last names would be fatal.

### 4. A High Tech Approach.

I like to think of the process of matching to a "search and rescue" operation. I think of each record as broadcasting a signal that indicates its presence while a matching procedure picks up these signals and determines (by a scoring mechanism) whether two records' signals are sufficiently strong to determine them as duplicates. The signals broadcast are called "tokens" and are extracts of the subscriber record chosen in a way that they "represent" portions of the record. For example, a name field might be represented by a first name token, a middle **name** token, a last name token, and a set of initials. Thomas J. Watson might become four tokens: THMS, J, WTSN, TJW. The same concept could apply to companies: International Business Machines would become INTRN, BSNSS, MCHNS, IBM. In a high tech approach each record is first broken into tokens and the tokens are put into an indexed file. Then every record is again read sequentially and, for each token, every record having a matching token is selected and put into a pool of matching "candidates". A scoring mechanism can be set up that is used to determine whether a potential match has a sufficiently high score to be declared an actual match. This method has the advantage of minimizing the chance of missing records which are, in fact, duplicates (providing the scoring mechanism is good). But there is a tremendous cost. Clearly there will be **many, many** instances where one, two, or even more tokens might match but the records are simple not matches. Several techniques could be used to reduce the **number** of potential matching records, but still the number of data fetches could overwhelm even a very

powerful computer.

### 5. A mathematician's approach.

We have seen two approaches to the matching problem. The **matchcode** technique is fast but is very sensitive to variations in the name and address. A token approach decreases the chance of missing duplicate records but requires tremendous computer capacity since many records are read hundreds of times. What would a mathematician do? A mathematician is trained to look for ways in which a problem can be broken down into **smaller** problems—hopefully ones which are simpler and perhaps have even already been solved. Also **reasoning by analogy** is a common approach and so one tries to find analogies to **prior experiences**. First note that it is possible to think of matching records as a **metric** problem; or more generally, as a problem in defining an **equivalence relation** whereby two records are "related" if they match. Secondly, it is important to realize that matching has two important **components**—(1) the search for candidates for matching and (2) a scoring mechanism to declare actual matches. Considering the two aspects of matching—searching and **scoring**—gives a key idea. We want a search procedure that leverages prior search activities. That is, suppose we had five candidates for matching a given record. Because we want matching to be an equivalence relation, each of these records should be a candidate for matching each other. Therefore, when we create a candidate pool for the "first" of these records we want to use this pool for **all** these records simultaneously. This leads to the idea of a "window" of candidates. We sort all the records by some criterion that maximizes the chances that candidates are "near" each other and then open a sliding window that considers all of the records visible in the window as potential candidates for matching. The advantage of this view is that records are read only once and **are** candidates for matching with all other records in the window. From a **processing** standpoint, since each record is read only once the process is linear with **the number** of records.

Now that we have a way to locate candidates we still need a **scoring** mechanism to declare matches. The scoring mechanism should be **symmetric** since the order of comparing two records should be irrelevant. This suggests some kind of additivity, the **simplest** being just to add points for matching tokens from both records. We will declare that two records match if the score exceeds some predetermined threshold. The easiest way of insuring transitivity is simply to take the transitive closure of this matching relation—i.e. if a record matches any member of a family it is defined to **be** in that family.

This idea also has an extremely valuable consequence that considerably widens its usefulness. A "helper file" is a file which contains some certified linking field that exists between records within the file. For **example**, it may be

possible to obtain a file that contains "certified" variations of a company name all linked by a single company number. Sometimes these variations are non-trivial, as in the case of the Scripps Oceanographic Institute—a part of the University of California at San Diego! If the helper file links these two records and is merged with the file to be matched, the transitivity of the matching relation insures that otherwise unmatchable records are, in fact, matched.

Without going into further detail here I just want to recall the point that an equivalence relation gives rise to equivalence classes so that two records are "related" (or **matched**) if they are in the same class. This point is fundamental to a person who has been trained in mathematics but it would **be completely** overlooked by a non-mathematician.

### 6. Epilogue.

There are many further applications of mathematical thought in the matching problem. Just to name a couple, consider the problem of measuring the accuracy of a matching run. To assert that a given **match** is either correct or incorrect requires that you have some underlying scoring **method** (other than the matching program itself). This would be the case if, for example, one has a certified linkage between records (perhaps you secretly have social security numbers). The problem, then, is how to define a **metric** that reflects the accuracy of the matching run.

Another **problem** would be how to best define a match "threshold", a score that minimizes the chances of erroneously declaring matches and which **simultaneous** maximizes the likelihood of correctly declaring matches (it is mathematically impossible to actually do both, but still one needs to define a threshold for use).

So, how can one recognize uses of mathematics in the "real world"? The key is to be able to recognize general structures and the procedures used to build and analyze such structures. And while the importance of precise definitions is well understood by a **mathematician**, it is rarely understood just how important precision is in understanding **the exact nature of a problem before trying to solve it**.

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By the Cayley-Hamilton Theorem, every square matrix satisfies its characteristic equation. Let  $A$  be a  $2 \times 2$  invertible matrix with real entries. When will  $A^{-1}$  satisfy the characteristic equation of  $A$ ? This note answers the question.

Let  $p(\lambda)$  denote the characteristic polynomial of  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ . Then  $p(\lambda) = |A - \lambda I| = \lambda^2 - (\text{tr } A)\lambda + |A|$ , where  $|A| = ad - bc$  and  $\text{tr } A = a + d$  (which is called the trace of  $A$ ). Since  $A$  is invertible,  $|A| \neq 0$ .

**THEOREM.**  $p(A^{-1}) = O$  if and only if  $|A| = 1$  or  $(\text{tr } A)A = (1 + |A|)I$ .

*Proof.* Since  $p(A) = 0$ ,

$$\begin{aligned} A^2 p(A^{-1}) &= A^2 p(A^{-1}) - |A| p(A) \\ &= A^2[(A^{-1})^2 - (\text{tr } A)A^{-1} + |A|I] - |A|[A^2 - (\text{tr } A)A + |A|I]. \end{aligned}$$

This simplifies to give

$$\begin{aligned} A^2 p(A^{-1}) &= I - (\text{tr } A)A + (|A|)A - |A|^2 I, \\ &= (1 - |A|^2)I + (|A| - 1)(\text{tr } A)A, \\ &= (1 - |A|)((1 + |A|)I - (\text{tr } A)A). \end{aligned}$$

If  $p(A^{-1}) = O$ , then either  $|A| = 1$  or  $(\text{tr } A)A = (1 + |A|)I$ .

Conversely, if  $|A| = 1$  or  $(\text{tr } A)A = (1 + |A|)I$ , then  $A^2 p(A^{-1}) = O$  and so  $p(A^{-1}) = O$ .

**COROLLARY.** If  $|A| = -1$  and  $\text{tr } A = 0$ , then  $p(A^{-1}) = O$ .

*Proof.* Since  $|A| = -1$  and  $\text{tr } A = 0$ ,  $(\text{tr } A)A = (1 + |A|)I = O$  and so  $p(A^{-1}) = O$  from the theorem.

The restriction  $\text{tr } A = 0$  in the corollary is necessary: if  $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ , then  $|A| = -1$ ,  $\text{tr } A = 2$ , and  $p(\lambda) = \lambda^2 - 2\lambda - 1$ . Also,  $A^{-1} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$  and

$$p(A^{-1}) = \begin{bmatrix} 4 & -8 \\ -4 & 4 \end{bmatrix} \neq O.$$

The question of when a  $3 \times 3$  matrix  $A$  has an inverse that satisfies the characteristic equation of  $A$  remains open.

Russell Euler, the author of many analytical and pedagogical papers, received his Ph. D. degree from the University of *Missouri-Kansas* City. He is a professor in the department of mathematics and statistics at Northwest Missouri State University.

### Are Proofs Hard?

Here is an amazing theorem, giving a necessary and sufficient condition for two numbers to be equal, connecting the operations of addition and multiplication:

**THEOREM:**  $x = y$  if and only if  $xy = (x + y)^2/4$ .

Prove half of the theorem, your choice of which half.

There, was that hard? Now, if you feel like it, prove the other half. Then see how the theorem would be changed if the last equation had a 2 instead of a 4 in the denominator. Then, if you still feel like it, see what a  $2n$  instead of a 4 would do, thus proving infinitely many theorems all at once. Then let  $n \rightarrow \infty$ . Then what? I don't know—mathematics is endless.

### Chapter Report

The Historian of the OHIO XI chapter (Youngstown State University), Lori Kaminski produced an eight-page chapter Newsletter, not all of which can be reproduced here. The winner of the annual calculus competition, and \$50—who says that mathematics does not pay?—was a major in mechanical engineering, but second place was taken by Patrick DiRusso, a mathematics major. The Pi Mu Epsilon T-shirt Sale resulted in twenty-seven purchases. Among the advantages of preparing a student paper are

A student paper presentation at a national meeting will often draw a positive reaction from interviewers when discussing a resume.

The chapter held a mathematics careers panel discussion at which six alumni served as panel members. Their occupations are: Vice President of a software company, Senior Actuarial Analyst at a pension consulting firm, Computer Analyst for the National Security Agency, Business Relations Specialist for Electronic Data Systems, Mathematics Instructor at a high school, and graduate, student in mathematics.

Don Bloomquist, Jr.  
Albertson College

## HOW TO FIND SINES WITHOUT KNOWING ANY

Andrew Cusumano  
Great Neck, New York

This note describes an easily programmable procedure for approximating the sine of a given angle, using the double-angle formula for the sine.

The approach is to divide the angle in half enough times until we obtain an angle small enough to approximate its sine as the angle itself. We can then work backwards, repeatedly using the double-angle formula, until we have the sine of the original angle.

Since  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ , the error in approximating  $\sin x$  by  $x$  is  $< x^3/6$ . If we want 10-place accuracy, then we can use the approximation if  $x^3/6 < 5 \cdot 10^{-11}$ , or  $x < \sqrt[3]{3 \cdot 10^{-10}} \approx .0006694$ . For an angle between  $0^\circ$  and  $90^\circ$ , we will need no more than  $k$  divisions by 2, where  $\frac{\pi/2}{2^k} < .0006694$ , so  $k = 12$  will do. Then, since  $\sin 2x = 2\sin x \cos x = 2\sin x \sqrt{1 - \sin^2 x}$ , we apply

$$(1) \quad \sin a_{n+1} = 2 \sin a_n \sqrt{1 - \sin^2 a_n}$$

$k$  times, with  $\sin a_0 = a_0$ , where  $a_0$  is the original angle bisected  $k$  times.

The procedure can easily be carried out on non-programmable calculators without any data entry other than the original angle as long as one memory location is available to store the current value of  $\sin a$ . It takes only a few minutes to go from  $\frac{\pi/6}{2^{10}} = .0005113$  back to  $.5000000$  by applying (1) 10 times.

Andrew Cusumano was a mathematics major at C. W. Post College, graduating in 1976. He is now a software engineer and, besides that, is interested in sequences, series, and geometry.

That ancient calculating device, the abacus, can be used to do calculations in base 16, the base in which the contents of computer files are usually displayed.

The Chinese version of the abacus, the *suan pan*, is divided into two sections, as indicated in Figure 1. In the lower section there are five beads, the one-point beads, on each reed. Each one-point bead on the rightmost reed—the unit's reed—represents one unit. In the upper section there are two beads, the five-point beads, on each reed. When the suan pan is used to represent decimal numbers, each one-point bead represents ten beads on the reed to its right, and each five-point bead represents five one-point beads on the same reed. Figure 2 shows the decimal number 1993 represented on the suan pan.

The suan pan, with its five one-point beads and two five-point beads, has more beads than it needs for calculating in decimal. (In contrast, the modern Japanese version of the abacus, the *soroban*, has only four one-point and one five-point bead on each reed, and thus has no beads to spare.) Some numbers can be displayed in more than one way. For example, 10 can be shown in three different ways: one one-point bead on the second reed from the right, both five-point beads on the rightmost reed, or one five-point bead and the five one-point beads on the rightmost reed. If they were used with maximum efficiency, the beads on the units reed could be used to represent all of the integers from 0 to 15 ( $2 \cdot 5 + 5$ ).

Thus, one reed of a suan pan can represent all of the digits (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F) used in hexadecimal (base-16) arithmetic. In order to represent all numbers in hexadecimal on a suan pan, each one-point bead on the second reed from the right (the sixteen's reed) must represent sixteen one-point beads on the unit's reed. Each one-point bead on the third reed from

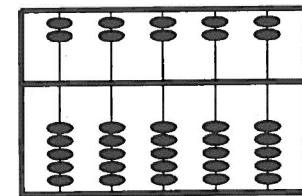


Figure 1: Suan pan, displaying 0

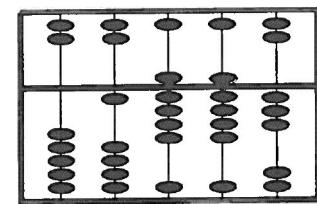


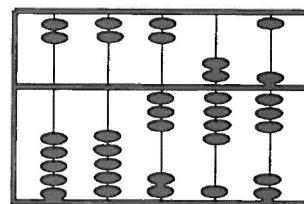
Figure 2: 1993 (decimal)

the right (the 256's reed) represents sixteen one-point beads on the sixteen's reed. Similarly, each one-point bead on the fourth reed from the right (the 4096's reed) represents sixteen one-point beads on the 256's reed, and so on. Starting with the rightmost reed, the value that each one-point bead represents increases by powers of sixteen.

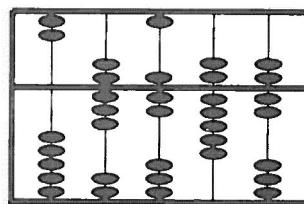
Zero is displayed on the suan pan by pushing all beads away from the middle bar. To display other numbers, push the appropriate beads toward the middle bar. For example, to display 1000 (**3E8** in hexadecimal), push three one-point beads and one five-point bead to the middle bar on the unit's reed. Then push four one-point and both **five-point beads** to the middle bar on the sixteen's reed to display the E; finally, push three one-point beads on the 256's reed to the middle bar. See Figure 3 for this representation, and Figure 4 for the display of **D7FC** (854012 in decimal).

At times there are two ways of representing hexadecimal numbers on the suan pan. The digit 5 can be represented by pushing one five-point bead or five one-point beads to the middle bar. The digit A can be represented by pushing the two five-point beads to the center bar or by moving one five-point and all the one-point beads to the middle bar.

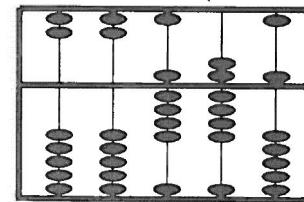
To add two numbers, for example **7CE** + **217**, first display **7CE** on the suan pan. Display E by pushing two five-point beads and four one-point beads to the middle bar on the unit's reed. Next, display C on the sixteen's reed by pushing two five-point and two one-point beads to the **middle** bar. Lastly, display the 7 on the 256's reed. To add **217**, first add 7 to the unit's reed. This manipulation cannot be done immediately because there are not enough one- and five-point beads to move to the **middle** bar. Carrying is necessary: since  $7 = 16 - 9$ , add 16 by moving a one-point bead on the six-



**Figure 3:** **3E8** (hexadecimal)



**Figure 4:** **D7FC** (hexadecimal)

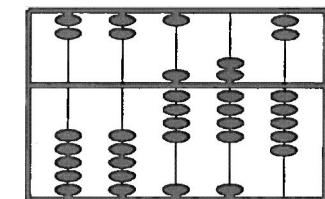


**Figure 5:** **9E5** (hexadecimal)

teen's reed to the middle bar. Then subtract 9 from the unit's reed by moving one five-point and four one-point beads away from the **middle** bar. Another way to subtract 9 is to slide both five-point beads on the unit's reed away from the middle bar (subtracting 10) and then to move one unit bead to the middle bar. No carrying is needed to add 1 on the sixteen's reed and 2 on the 256's reed. The correct sum, **9E5**, is now displayed. See Figures 5 and 6 for the two ways the result could appear, depending on which of the two methods described was used to subtract 9.

Sometimes, as in adding **57** to **4EF**, it will be necessary to carry more than one hexadecimal place to the left, as it is necessary to carry more than one decimal place in the addition of **57** to **489** in base 10.

Subtraction may be done as easily as addition. With some practice, a person with a suan pan will be able to do hexadecimal arithmetic much more quickly than almost anyone who uses pencil and paper; however, computers will still be faster.



**Figure 6:** **Also 9E5** (hexadecimal)

*Don Bloomquist, Jr., a mathematics major, is in his junior year at the Albertson College of Idaho. He wrote this paper during his freshman year under the direction of Dr. L. R. Tanner.*

### Was Dirichlet Smart?

Even the great mathematicians can make mistakes. As J. W. Dauben tells us on page 7 of **Georg Cantor** (Princeton University Press, 1979), Cauchy once made an assertion equivalent to saying that if  $2a_n$  converges and if  $\lim_{n \rightarrow \infty} a_n/b_n = 1$ , then  $2b_n$  converges also. Dirichlet, who read critically, found an example showing that Cauchy was wrong. Can you do as well? We have all been taught, as Cauchy and Dirichlet were not, that Cauchy's assertion is true if the series have positive terms, so we know to look first at alternating series. Even with that hint, it is not all that likely that an average, or even above-average, student of mathematics would be likely to duplicate Dirichlet's accomplishment, which is why Dirichlet's example is on page 598.

## SQUARE-FREE LUCAS PSEUDOPRIMES

Paul S. Bruckman  
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The purpose of this paper is to give a condition necessary and sufficient for all Lucas pseudoprimes to be square-free.

We will begin with some preliminaries. The Fibonacci numbers are defined by

$$F_{n+2} = F_{n+1} + F_n, \quad n = 0, 1, 2, \dots; \quad F_0 = 0, \quad F_1 = 1.$$

The sequence  $\{F_n\}_0^\infty$  is called the Fibonacci sequence, after the 13th-century mathematician Leonardo of Pisa, also known as Fibonacci. The sequence has non-negative terms and, for  $n \geq 2$ , is strictly increasing.

The Lucas numbers are defined similarly, but with different initial values:

$$L_{n+2} = L_{n+1} + L_n, \quad n = 0, 1, 2, \dots; \quad L_0 = 2, \quad L_1 = 1.$$

The Lucas sequence  $\{L_n\}_0^\infty$  is named after the 19th-century French mathematician Edouard Lucas, whose seminal work [5] generated much of the subsequent research into the sequences and their generalizations. The sequence has positive terms and, for  $n \geq 1$ , is strictly increasing.

If  $\alpha = (1 + \sqrt{5})/2$  and  $\beta = (1 - \sqrt{5})/2$ , then

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n = 1, 2, \dots.$$

Broadly speaking, there are two categories of properties of (and relationships between) the Fibonacci and Lucas numbers: additive and divisibility. The first includes representations of integers as sums (or differences) of Fibonacci (or Lucas) numbers. The second, as its name implies, includes representations of integers as products (or ratios) of Fibonacci (or Lucas) numbers. In this paper, we shall be most concerned with the divisibility properties.

We list below (without derivation) some of the divisibility properties that we will make frequent use of in this paper. For derivations, see, for example, [1, 6].

$$(1) \quad F_{2n} = F_n L_n.$$

$$(2) \quad (F_m, F_n) = F_{(m,n)}, \text{ where } (a, b) \text{ denotes the greatest common divisor of } a \text{ and } b.$$

divisor of a and b.

$$(3) \quad (F_n, F_{n+1}) = (L_n, L_{n+1}) = 1.$$

$$(4) \quad F_m | F_n \text{ if and only if } m | n.$$

$$(5) \quad L_{2n+1} - 1 = \begin{cases} 5F_n F_{n+1}, & n \text{ even} \\ L_n L_{n+1}, & n \text{ odd.} \end{cases}$$

We next define the Fibonacci entry-point. It may be shown that any integer  $m$  is a divisor of some element of the Fibonacci sequence. This is by no means a foregone conclusion. Not all integers  $m$  divide some element of the Lucas sequence since 5 divides no Lucas number. (The reason for this is that the Lucas numbers, modulo 5, are 2, 1, 3, 4, 2, 1, 3, 4, 2, ....) The smallest positive index  $n$  such that  $m | F_n$  (where  $m > 1$ ) is called the Fibonacci entry-point of  $m$  in the Fibonacci sequence, and is denoted by  $Z(m)$ . Other authors have referred to  $Z(m)$  as the "rank of apparition" of  $m$ , an odious appellation brought about by a mistranslation of the French word apparition, which means "appearance", not "apparition", in English. Another more acceptable and frequently used alternative is "rank of appearance". To illustrate, since  $F_1 = F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, F_7 = 13, \dots$ , we find  $Z(2) = 3, Z(3) = 4, Z(5) = 5, Z(8) = 6, Z(13) = 7$ . We may also verify that  $Z(4) = 6, Z(6) = 12, Z(7) = 8, Z(9) = 12, Z(10) = 15, Z(11) = 10, Z(12) = 12$ , and so on.

For all  $m > 1$  we have the following properties:

$$(6) \quad Z(m) = n \text{ if and only if } m | F_n \text{ and } m \nmid F_r \text{ for all } r \text{ with } 1 \leq r \leq n-1 \text{ (this is actually the definition of } Z(m) \text{);}$$

$$(7) \quad Z(m) | Z(n) \text{ if and only if } m | n \text{ (if and only if } F_m | F_n, \text{ by (4));}$$

$$(8) \quad m | F_n \text{ if and only if } Z(m) | n;$$

$$(9) \quad \text{if } m = \prod_{i=1}^n p_i^{e_i} \text{ is the prime-power decomposition of } m, \text{ then}$$

$$(10) \quad Z(m) = \text{LCM}\{Z(p_i^{e_i})\};$$

$$(11) \quad \text{if } p \text{ is any prime and } e \geq 1 \text{ any integer, then } Z(p^e) = p^f Z(p), \text{ for some integer } f \text{ with } 0 \leq f < e;$$

$$(12) \quad \text{if } p \text{ is any prime with } p^e \neq 2 \text{ for some integer } e \geq 1, \text{ and if } Z(p^{e+1}) \neq Z(p^e), \text{ then } Z(p^r) = p^{r-e} Z(p^e) \text{ for all } r \geq e+1.$$

See, for example, [2, 6].

We need one more property which involves  $(x/p)$ , the Legendre symbol, defined for odd primes  $p$  and integers  $x$  relatively prime to  $p$  as follows. If an

integer  $y$  exists such that  $y^2 \equiv x \pmod{p}$ , we call  $x$  a quadratic residue  $(\pmod p)$  and write  $(x/p) = 1$ ; otherwise we write  $(x/p) = -1$ . The **Legendre symbol** is also known as the quadratic character of  $x \pmod p$ . The final property is

$$(13) \quad \text{if } p \neq 2, 5 \text{ is prime, then } Z(p) \mid (p - (p/5)).$$

Since for odd primes  $\neq 5$

$$(p/5) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{10} \\ -1 & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$$

we may restate (13) as

$$\text{if } p \neq 2, 5 \text{ is prime, then } \begin{cases} Z(p) \mid (p - 1) & \text{if } p \equiv \pm 1 \pmod{10}, \\ Z(p) \mid (p + 1) & \text{if } p \equiv \pm 3 \pmod{10}. \end{cases}$$

In a 1982 paper [8], H. C. Williams reported that he had verified that, for  $p \neq 2, 5$ ,

$$(14) \quad p^2 \nmid F_{p-(5/p)}$$

for all primes  $p < 10^9$ . Although he did not assert that (14) holds for *all* primes  $\neq 2, 5$ , we will nevertheless call (14) the "Williams conjecture". In light of (8), we may restate it as

$$(15) \quad Z(p^2) \nmid (p - (5/p)) \text{ for all primes } p \neq 2, 5.$$

However, (11) implies that  $Z(p^2) = Z(p)$  or  $pZ(p)$ . From (12), we see that (15) implies that

$$(16) \quad Z(p^2) = pZ(p), \text{ for all primes } p \neq 2, 5.$$

On the other hand, if we assume that (16) holds and if it were true that  $p^2 \mid F_{p-(5/p)}$ , then, using (8),  $Z(p^2) = pZ(p) \mid (p - (5/p))$ . This is impossible since  $p - (5/p) = p \pm 1$ , and cannot be divisible by  $p$ . Therefore (16) implies (14) and we have shown that (14) and (16) are equivalent. Moreover,  $Z(2^2) = Z(4) = 6 = 2 \cdot 3 = 2Z(2)$ . We can also verify that  $Z(25) = Z(5^2) = 25 = 5 \cdot 5 = 5Z(5)$ . Therefore we may restate the Williams conjecture in the slightly stronger form

$$Z(p^2) = pZ(p) \text{ for all primes } p.$$

In a 1984 paper [3], J. J. Heed, evidently unaware of Williams' prior work, verified the conjecture for all  $p < 10^6$ .

The Williams conjecture is related to the study of Wieferich primes and their generalizations. A Wieferich prime is a prime  $p$  that satisfies  $2^{p-1} \equiv 1 \pmod{p^2}$ . There are only two Wieferich primes  $< 6 \cdot 10^9$ , namely 1093 and 3511. More generally, prime solutions of  $a^{p-1} \equiv 1 \pmod{p^2}$ , where  $a > 1$  is not a multiple of  $p$ , are exceedingly rare. Accordingly, we should expect solutions to the "counter-conjecture"  $Z(p^2) = Z(p)$  to be also very rare. Indeed, the Williams conjecture states that such solutions are non-existent.

Note that (13) implies  $p \mid F_{p-(5/p)}$ , or, equivalently,

$$F_{p-(5/p)} \equiv 0 \pmod{p}, \text{ for all primes } p \neq 2, 5.$$

We can ask if the congruence

$$(17) \quad F_{n-(5/n)} \equiv 0 \pmod{n}, \text{ with } (n, 10) = 1,$$

might hold for composite values of  $n$ , with  $(5/n)$  being the Jacobi symbol, the generalization of the Legendre symbol to composite  $n$  defined by  $(x/n) = \prod_{p \mid n} (x/p)$ . The answer is affirmative. Any composite integer which satisfies (17) is called a Fibonacci *pseudoprime*, or FPP. The first two FPP's are  $323 = 17 \cdot 19$  and  $377 = 13 \cdot 29$  and it is known that there are infinitely many [4].

It is known that  $L_p \equiv 1 \pmod{p}$ , holds for all primes  $p$  [7]. We can also ask if there are composite  $n$  such that  $L_n \equiv 1 \pmod{n}$ . Again, the answer is affirmative, and such  $n$  are called Lucas *pseudoprimes* (or LPP's). The first three LPP's are  $705 = 3 \cdot 5 \cdot 47$ ,  $2465 = 5 \cdot 17 \cdot 29$ , and  $2737 = 7 \cdot 17 \cdot 23$ , and it is known that there are infinitely many [7]. It is also known that all LPP's are odd [7].

All known FPP's and LPP's are products of distinct primes, and so are square-free. It is not known if this is true in general, in spite of efforts made to prove it. Though it will not be proved here either, we will establish that a slightly weaker version of the Williams conjecture is equivalent to the conjecture that all LPP's are square-free. (A comparable version may be shown to be equivalent to the conjecture that all FPP's are square-free, but we will not prove that here.) We will attempt to show that

$$(*) \quad Z(p^2) = pZ(p) \text{ for all primes } p$$

and

$$(**) \quad \text{All LPP's are square-free}$$

Suppose (\*) is true and that  $n$  is a LPP. Also, assume that  $p^2 \mid n$  for some prime  $p$ . Since all LPP's are odd,  $p$  is odd. Since, by definition,  $n \mid (L_n - 1)$ , we have  $p^2 \mid (L_n - 1)$ . Let  $m = (n - 1)/2$ . We consider three cases.

CASE 1:  $n \equiv 1 \pmod{4}$ ,  $5 \nmid n$ .

By (5),  $L_n - 1 = 5F_m F_{m+1}$ . Since  $p \neq 5$ ,  $p^2 \mid F_m F_{m+1}$ . By (3),  $F_m$  and  $F_{m+1}$  are relatively prime, so either  $p^2 \mid F_m$  or  $p^2 \mid F_{m+1}$ . That is,  $p^2 \mid F_{m+0}$ , where  $0 = 0$  or  $1$ . By (1),  $F_{2m+20} = F_{m+0} L_{m+0}$ . Thus  $p^2 \mid F_{2m+20}$ , i.e.,  $p^2 \mid F_{n+1}$ . By (8),  $Z(p^2) \mid (n \pm 1)$ , which implies  $p \mid (n \pm 1)$ . However, since  $p^2 \mid n$ ,  $p \mid n$  also. This is impossible.

CASE 2:  $n \equiv 3 \pmod{4}$ .

By (5),  $L_n - 1 = L_m L_{m+1}$ , which implies  $p^2 | L_m L_{m+1}$ . By (3),  $(L_m, L_{m+1}) = 1$ , which implies either  $p^2 | L_m$  or  $p^2 | L_{m+1}$ . As in Case 1,  $p^2 | L_{m+1}$ , which again implies  $p^2 | F_{2m+20}$ . The remaining steps are identical to those in Case 1.

CASE 3:  $n \equiv 5 \pmod{20}$ .

If  $p \neq 5$ , we proceed as in Case 1. If  $p = 5$ , then  $5^2 | 5F_m F_{m+1}$ , so  $5 | F_m F_{m+1}$ . Proceeding as above, we conclude that  $5 | (n \pm 1)$ . However,  $5 | n$  also, since  $5^2 | n$ . Once again, we are led to a contradiction.

Since the three cases are exhaustive, we conclude that (\*) implies (\*\*).

Conversely, suppose that (\*\*) is true. Let  $n$  be any LPP and suppose that  $p | n$ , where  $p$  is an odd prime. By hypothesis,  $p^2 \nmid n$ . By definition,  $n | (L_n - 1)$ . Following the steps used to show that (\*) implies (\*\*), we find that  $p | F_{n \pm 1}$  and  $p^2 \nmid F_{n \pm 1}$ . Then  $Z(p) | (n \pm 1)$ , but  $Z(p^2) \nmid F_{n \pm 1}$ . Hence,  $Z(p^2) \neq Z(p)$  for all primes  $p$  which divide some LPP. Since  $Z(2^2) = 2Z(2)$ , we see that all that is needed to complete the proof that (\*\*) implies (\*) is the assertion that all odd primes  $p$  divide some LPP. We will forego the proof of this assertion here, since it involves concepts that are somewhat more complicated than intended in the scope of this paper.

So, all that we have proven is the weaker equivalence

$$(\#) \quad Z(p^2) = pZ(p) \text{ for all prime } p \text{ dividing some LPP.}$$

All LPP's are square-free.

However, as stated above, (#) may be replaced by the stronger statement (\*).

Based on the empirical evidence, it appears highly likely that (\*) and (\*\*) are true, but the proof of either conjecture is equally likely to meet with considerable difficulties.

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Paul Bruckman received an M. S. degree in mathematics from the University of Illinois in 1971. He is a frequent contributor to *The Fibonacci Quarterly*, has been a member of The Fibonacci Society for more than twenty years, and subscribed to the *Pi Mu Epsilon Journal* in 1992.

## Chapter Report

The FLORIDA EPSILON chapter (University of South Florida) held thirteen meetings during the 1992-93 academic year, devoted to talks by students, faculty members, and visitors. A selection of titles is

Off into space

An explanation of Strang's strange figures

Applied mathematics in engineering

The Euler summation formula

Computer image techniques in medical imaging

Perfect and perfectly useless numbers

Dynamics for the college student

Careers for mathematics majors

Paradoxes in mathematics

Laser sensing of the atmosphere.

In addition, there was a mathematical game party before the Christmas vacation and the chapter, in collaboration with the student chapter of the MAA, sponsored the twice-yearly Hillsborough County Math Bowl competitions, with more than 200 participants in each. Professor Fredric Zerla also noted that Suzanne Josephs, the chapter's Outstanding Scholar, completed her university career with a perfect 4.0 grade-point average.

## COUNTING ICE-CREAM FLAVORS TO PROVE AN IDENTITY

James Chew  
North Carolina A & T State University

The identity in question is

$$1 + 2 + \dots + n = \frac{n(n+1)}{2}$$

How many kinds of double-dipped cones can a customer order at an ice-cream parlor that sells  $n$  flavors of ice cream? We will allow two scoops of the same flavor but will consider chocolate the same as vanilla. Let  $N$  be the number of different cones.

One line of reasoning is to think of  $N$  as being the sum of the number of double-flavored cones and the number of single-flavored cones, so

$$N = \binom{n}{2} + n = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}.$$

The same result comes from thinking that  $N$  = total number of possible pairs - number of duplicated pairs, so

$$N = n^2 - \binom{n}{2} = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}.$$

Let us take a different approach, with  $n = 4$  for definiteness. Let A, B, C, and D be the flavors. There are 4 single-flavored cones: AA, BB, CC, and DD. The rest are double-flavored. There are 3 cones in which A is picked first: AB, AC, and AD. Next come the 2 cones in which B is picked first: BC and BD. Finally, we have the 1 cone in which C is picked first: CD. Hence  $N = 4 + 3 + 2 + 1$ .

The argument generalizes to  $n$  flavors, so that

$$N = n + (n-1) + \dots + 2 + 1.$$

James Chew has lived in Indonesia, Australia, and Ethiopia, though his Ph. D. degree is from the Virginia Polytechnic Institute and he has been at NC A&T State for the past fifteen years.

## A REGULAR POLYGON EQUATION

See Chin Woon  
Imperial College, London

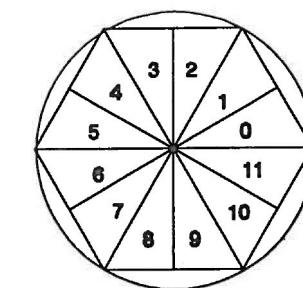


Figure 1

In analytic geometry, geometrical objects are defined algebraically by equations. In this note, we give a polar coordinate equation for a regular polygon with any number of sides.

Let  $R$  be the distance from the center of the polygon to a vertex. Place the polygon so that its center is at the origin and a vertex is at  $(R, 0)$ .

For  $q \in [0, 2\pi]$ , let  $\theta' \equiv q \pmod{\pi/n}$ ,

so that

$$\theta = \theta' + q\pi/n, \text{ with } 0 \leq \theta' < \pi/n \text{ and } 0 \leq q < 2n.$$

The polygon can be divided into triangles with different values of  $q$ , as illustrated in Fig. 1 for a hexagon.

Let

$$\phi(q) = \begin{cases} 0 & \text{if } q \text{ is odd} \\ 1 & \text{if } q \text{ is even.} \end{cases}$$

The function may also be defined in a single equation as  $\phi(q) = 2[q/2] - q + 1$  or as  $\phi(q) = ((-1)^q + 1)/2$ .

In Fig. 2, if  $q$  is odd, then

$$\frac{OQ}{OP} = \cos \theta \quad \text{so} \\ OP = \frac{R \cos \pi/n}{\cos \theta'}$$

If  $q$  is even,  $\frac{OP'}{OQ} = \cos(\pi/n - \theta')$ ,  
so

$$OP' = \frac{R \cos \pi/n}{\cos(\pi/n - \theta')}$$

The two equations for  $OP$  and  $OP'$  may be combined into

$$r = \frac{R \cos \pi/n}{\cos(\phi(q)\pi/n + (-1)^{\phi(q)}\theta')}$$

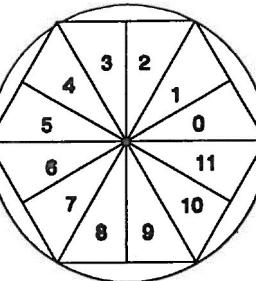


Figure 2

This gives the following single polar coordinate equation for the regular polygon:

$$r = \frac{R \cos \pi/n}{\cos(\phi(q)\pi/n + (-1)^{\phi(q)}(\theta - q\pi/n))}.$$

As  $n \rightarrow \infty$  this equation becomes  $r = R$ , as it should.

See Chin Woon a member of the Class of 1994 at Imperial College. He says that he has sometimes felt that some of the concepts in mathematics seem really ahead of their time.

### Yes, He Was

Dirichlet said, let

$$a_n = \frac{(-1)^n}{\sqrt{n}} \quad \text{and} \quad b_n = \frac{(-1)^n}{\sqrt{n}} \left(1 + \frac{(-1)^n}{\sqrt{n}}\right).$$

Then it is clear that  $a_n/b_n$  approaches 1 as  $n \rightarrow \infty$ , but  $a_n - b_n = -1 - 1/2 - 1/3 - \dots$ , which diverges. Thus, not both  $2a_n$  and  $2b_n$  can converge.

### Chapter Reports

Professor Joan Wyzkoski Weiss reports that CONNECTICUT GAMMA (Fairfield University) had twenty-three new initiates in the spring. Members of the chapter assisted in coordinating the activities for Math Counts, a mathematics contest for junior high school students.

MASSACHUSETTS GAMMA (Bridgewater State College) sponsored a colloquium, "Aspects of real-time object-oriented systems" by John McNulty of the MITRE Corporation. Vice-president Keith Desrosiers also reports that the chapter's advisor, Professor Thomas E. Moore, was awarded the MAA's Northeastern Section award for distinguished teaching.

### SOME PARTITIONS OF THE INTEGERS

Joseph M Moser and Genoveno Lopez  
San Diego State University

Let  $m$  be an arbitrary positive integer. The congruence relation,  $\equiv$  modulo  $m$ , on the set  $\mathbf{Z}$  of all integers is defined by

$$x \equiv y \pmod{m} \quad \text{if and only if} \quad x - y = km \quad \text{for some integer } m.$$

The congruence relation is an equivalence relation with equivalence classes  $\{x + km \mid k \in \mathbf{Z}\}$ ,  $x = 0, 1, \dots, m - 1$ . It is likely that other equivalence relations on the set  $\mathbf{Z}$  are not well known. We would like to present a few more equivalence classes on  $\mathbf{Z}$  which may be useful as exercises or examples.

First, for a fixed integer  $k$ , define  $R_k$  by

$$x R_k y \quad \text{if and only if} \quad x^2 + kx = y^2 + ky.$$

It is easy to show that  $R_k$  is an equivalence relation which partitions the integers into an infinite number of classes,  $\{x, -(x+k)\}$ .

Next, let  $i = \sqrt{-1}$ , and define  $R_i$  by

$$x R_i y \quad \text{if and only if} \quad i^x = i^y.$$

It again is easy to show that  $R_i$  is an equivalence relation which partitions the integers into four equivalence classes. Also,  $R_i = R_m$  when  $m = 4$ .

Let us define  $R_e$  by

$$x R_e y \quad \text{if and only if} \quad e^{2\pi ix/m} = e^{2\pi iy/m}.$$

It is not difficult to see that  $R_e = R_m$ , where  $R_m$  is as defined above.

Now, let us define  $R_{\sin}$  by

$$x R_{\sin} y \quad \text{if and only if} \quad \left| \sin \frac{\pi y}{m} \right| = \left| \sin \frac{\pi x}{m} \right|.$$

It is easy to show that  $R_{\sin}$  is an equivalence relation which partitions the integers into  $k$  equivalence classes, where  $m = 2k$  or  $m = 2k - 1$ . For example, when  $m = 5$ , the equivalence classes are

$$\{0, \pm 5, \pm 10, \pm 15, \dots\}, \quad \{\pm 1, \pm 4, 26, \pm 9, \dots\}, \quad \{\pm 2, \pm 3, \pm 7, \pm 8, \dots\}.$$

Finally, let us define  $R^*$  by  $x R^* y$  if and only if

$$\begin{aligned} & (-1)^{[y/m]} \left( \frac{y}{m} - \left[ \frac{y}{m} \right] \right) + \frac{1 + (-1)^{[y/m] + 1}}{2} \\ &= (-1)^{[x/m]} \left( \frac{x}{m} - \left[ \frac{x}{m} \right] \right) + \frac{1 + (-1)^{[x/m] + 1}}{2} \end{aligned}$$

where  $[z]$  is the greatest integer in  $z$ . It is easy to show that  $R^*$  is an equivalence relation. The partition of the integers is

$$\begin{aligned} & \{x = mk \mid k \in \mathbb{Z}\}, \\ & \{x = 1 + 2mk \mid k \in \mathbb{Z}\} \cup \{x = (2m - 1) + 2mk \mid k \in \mathbb{Z}\}, \\ & \{x = 2 + 2mk \mid k \in \mathbb{Z}\} \cup \{x = (2m - 2) + 2mk \mid k \in \mathbb{Z}\}, \dots \\ & \{x = m - 1 + 2mk \mid k \in \mathbb{Z}\} \cup \{x = (2m - m + 1) + 2mk \mid k \in \mathbb{Z}\}. \end{aligned}$$

Professors Moser and Lopez earned Ph. D. degrees *from*, respectively, St. Louis ***U.*** and ***UCLA*** before joining ***SDSU***.

### Chapter Reports

At the OHIO NU chapter (University of Akron), **nineteen** new members were inducted in April and **thirty-five** awards were made, including seventeen memberships in various mathematical **organizations** and thirteen scholarships.

One of the new **initiates** of the MARYLAND DELTA chapter (**Hood** College) has an interdisciplinary major in the political economy of the third world. The **annual** Pi Mu Epsilon Lecture was given by Dr. Lida K. Barrett, on "Emmy Noether and Grace Chisholm Young: **two** women mathematicians **of this** century".

### A Steeply Puzzling Question

Does anyone, anywhere, know why ***m*** is always and invariably used to denote the slope of a line? If so, many readers of the Journal would like to know as well,

## NON-EXISTENCE OF CERTAIN UNITARY PERFECT NUMBERS

*Jennifer DeBoer*

Michigan Technological University

Let ***N*** be a positive integer. The **unitary divisors** of ***N*** are all the integers ***d*** such that ***d* | *N*** and **(d, N/d) = 1**. A positive integer ***N*** is unitary perfect when the sum of its unitary divisors is **2*N***.

We use  **$\sigma^*(N)$**  to denote the sum of the unitary divisors of ***N***. It can be easily shown that if ***N* =  $p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$** , where  **$p_1, \dots, p_k$**  are distinct primes, then

$$\sigma^*(N) = (p_1^{a_1} + 1)(p_2^{a_2} + 1) \dots (p_k^{a_k} + 1).$$

Therefore, ***N*** is unitary perfect if and only if

$$\frac{\sigma^*(N)}{N} = \frac{p_1^{a_1} + 1}{p_1^{a_1}} \dots \frac{p_k^{a_k} + 1}{p_k^{a_k}} = 2.$$

As Guy [2] pointed out, any **unitary** perfect number must be even. Subbarao and Warren [4] proved that the first four unitary perfect numbers are

$$(A) \quad 6 = 2 \cdot 3, \quad 60 = 2^2 \cdot 3 \cdot 5, \quad 90 = 2 \cdot 3^2 \cdot 5, \quad \text{and} \quad 87360 = 2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$$

Wall [5] has shown that

$$2^{18} \cdot 3 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313$$

is the fifth unitary perfect number. Furthermore, Graham [1] has shown that the only three unitary perfect numbers of the form  **$2^m s$** , where ***s*** is odd and squarefree, are the first, second, and fourth numbers in (A).

In this paper, we will look at the third **unitary** perfect number and show that it is the only one of the form  **$2^m 3^2 s$** .

For a given ***m***, there is a simple procedure to find out if there is a unitary perfect number of the form  **$2^m 3^2 s$** . For example, suppose ***m* = 1**. Then we want

$$\frac{2^1 + 1}{2^1} \frac{3^2 + 1}{3^2} \frac{\sigma^*(s)}{s} = 2.$$

or

$$\frac{5}{3} \frac{\sigma^*(s)}{s} = 2.$$

This cannot equal 2 unless there is some factor ***p<sub>i</sub>* ≠ 5** in the denominator to

cancel the 5 in the numerator. Therefore,  $5 \mid s$ . On writing  $s = 5 \cdot s_1$ , we get

$$\frac{5}{3} \frac{6}{5} \frac{\sigma^*(s_1)}{s_1} = 2.$$

This is true when  $s_1 = 1$ , so we see that  $2 \cdot 3^2 \cdot 5$  is unitary perfect.

However, this does not work for every  $m$ . For example,  $m = 2$  does not yield an unitary perfect number of the form  $2^m \cdot 3^2 \cdot s$ , for in the product

$$\frac{5}{2^2} \frac{2 \cdot 5}{3^2} \frac{2 \cdot 3}{5} \dots$$

there is no way to cancel out both factors of 5 in the numerator with a squarefree  $s$ .

As we see, a number  $N$  cannot be unitary perfect if  $5 \mid (2^m + 1)$  or if there exists  $p_i \mid s$  such that  $5 \mid (p_i + 1)$ . In addition, since  $s$  is squarefree,  $s$  must not be divisible by the squares of any odd primes.

For another example,

$$\frac{3^2}{2^3} \frac{2 \cdot 5}{3^2} \frac{2 \cdot 3}{5} \dots$$

cannot equal 2 because there are three factors of 3 in the numerator, and the highest power of 3 in the denominator is 2. Therefore, if  $3^2 \mid (2^m + 1)$ , then  $N$  is not unitary perfect. Finally, suppose  $m = 5$ . Then

$$\frac{3 \cdot 11}{2^5} \frac{2 \cdot 5}{3^2} \frac{2 \cdot 3}{5} \frac{2^3 \cdot 3}{11} \dots$$

cannot equal 2, again because there are too many factors of 3 in the numerator.

So, if one of the following cases holds, then  $2^m \cdot 3^2 \cdot s$  cannot be unitary perfect:

- (1)  $3^2 \mid (2^m + 1)$ .
- (2)  $3 \mid (2^m + 1)$  and there exists  $p_i \mid s$  such that  $3 \mid (p_i + 1)$ .
- (3)  $5 \mid (2^m + 1)$  or there exists  $p_i \mid s$  such that  $5 \mid (p_i + 1)$ .
- (4) there exists  $p_i \mid s$  such that  $p_i + 1$  is divisible by the square of an odd prime.
- (5) there exists  $p_i \mid s$  and  $p_j \mid s$  such that  $3 \mid (p_i + 1)$  and  $3 \mid (p_j + 1)$ .

Now we always have  $5 \mid s$  since  $5 \mid (3^2 + 1)$ . For the remainder of this paper, we will write  $N = 2^m \cdot 3^2 \cdot 5 \cdot s$ , where  $s$  is such that  $(s, 3) = (s, 5) = 1$ , and we will let  $s = p_1 p_2 \dots p_k$ , where  $p_1, \dots, p_k$  are distinct primes. If  $N$  is unitary perfect, then

$$\frac{2^m + 1}{2^m} \frac{4}{3} \frac{p_1 + 1}{p_1} \dots \frac{p_k + 1}{p_k} = 2.$$

Furthermore, if any of criteria (1) through (5) hold, then  $N$  cannot be unitary perfect.

We will use criteria (1) through (5) to establish the result **in-all-cases** except  $m = 0 \pmod{8}$ .

**LEMMA 1.** If  $m \equiv 3 \pmod{6}$ , then  $2^m \cdot 3^2 \cdot 5 \cdot s$  cannot be unitary perfect.

**Proof.** If  $m \equiv 3 \pmod{6}$ , then  $9 \mid (2^m + 1)$ . Thus we may apply case (1).

**LEMMA 2.** If  $m \equiv 5 \pmod{6}$ , then  $2^m \cdot 3^2 \cdot 5 \cdot s$  cannot be unitary perfect.

**Proof.** Suppose  $m \equiv 5 \pmod{6}$ . Then  $2^m \equiv 5 \pmod{9}$ , so  $2^m + 1 \equiv 6 \pmod{9}$ , and  $3 \mid (2^m + 1)$ . Also,  $a = (2^m + 1)/3 \equiv 2 \equiv -1 \pmod{3}$ . Now  $a$  has at least one prime divisor  $p \equiv -1 \pmod{3}$ , for if all prime divisors of  $a$  are  $\equiv 1 \pmod{3}$ , then  $a \equiv 1 \pmod{3}$ . Thus,  $3 \mid (p + 1)$ . This meets case (2).

**LEMMA 3.** If  $m \equiv 1 \pmod{6}$  and  $m > 1$ , then  $2^m \cdot 3^2 \cdot 5 \cdot s$  cannot be unitary perfect.

**Proof.** Let  $m \equiv 1 \pmod{6}$ . Then  $2^m \equiv 2 \pmod{9}$ , so  $3 \mid (2^m + 1)$ , and  $(2^m + 1)/3 \equiv 1 \pmod{3}$ . Furthermore,  $2^m + 1 \equiv 1 \pmod{8}$ , and  $(2^m + 1)/3 \equiv 3 \pmod{8}$ .

We may assume that every prime divisor  $p$  of  $(2^m + 1)/3$  has  $p \equiv 1 \pmod{3}$ ; otherwise, criterion (2) is satisfied. If  $p \mid (2^m + 1)/3$  then  $2^m \equiv -1 \pmod{p}$ ; consequently,  $2^{6a+2} \equiv -2 \pmod{p}$ , where  $m = 6a + 1$ .

We now see that  $x = 2^{3a+1}$  is a solution of  $x^2 \equiv -2 \pmod{p}$ . A result from quadratic residue theory tells us that this **congruence** is solvable if and **only** if  $p \equiv 1$  or  $3 \pmod{8}$ . Thus every prime divisor of  $(2^m + 1)/3$  satisfies  $p \equiv 1$  or  $3 \pmod{8}$ .

Now  $(2^m + 1)/3 = p_1 \dots p_r \equiv 3 \pmod{8}$ ,  $p_i \equiv 1$  or  $3 \pmod{8}$ , and  $p_i \equiv 1 \pmod{3}$  for every  $i$ . There must be at least one  $i$  such that  $p_i \equiv 3 \pmod{8}$ . Then  $p_i + 1 \equiv 4 \pmod{8}$ , and  $(p_i + 1)/4 \equiv 2 \pmod{3}$ . Therefore, there exists  $j$  such that  $p_j \mid (p_i + 1)$  and  $p_j \equiv 2 \equiv -1 \pmod{3}$ . But  $3 \mid (p_j + 1)$  and we noted previously that  $3 \mid (2^m + 1)$ . This again meets case (2), so  $2^m \cdot 3^2 \cdot 5 \cdot s$  cannot be unitary perfect.

**LEMMA 4.** If  $m \equiv 2 \pmod{4}$ , then  $2^m \cdot 3^2 \cdot 5 \cdot s$  cannot be **unitary perfect**.

**Proof.** Suppose  $m \equiv 2 \pmod{4}$ . Then  $2^m \equiv -1 \pmod{5}$  and

$5 \mid (2^m + 1)$ . This is case (3).

LEMMA 5. If  $m \equiv 4 \pmod{8}$ , then  $2^m \cdot 3^2 \cdot 5 \cdot s$  cannot be unitary perfect.

Proof. Let  $m \equiv 4 \pmod{8}$ . Then  $2^m \equiv -1 \pmod{17}$ ,  $17 \mid (2^m + 1)$ , and so  $17 \mid s$ . However,  $17 + 1 = 18$ , which is case (4).

From these lemmas, we see that if  $2^m \cdot 3^2 \cdot 5 \cdot s$  is unitary perfect, then  $m \equiv 0 \pmod{8}$ . The proof for this case requires a different technique. To begin with, we will show that there exists at least one Mersenne prime  $p_i$  such that  $p_i \mid s$ . Note that  $2^m + 1 \equiv 2 \pmod{3}$ , and there exists a unique  $p_1 \mid (2^m + 1)$  such that  $p_1 \equiv -1 \pmod{3}$ . At least one such term is needed to cancel out one of the factors of 3, but two such numbers would meet 5 and violate our conditions. The other primes  $p_2, \dots, p_k$  are  $\equiv 1 \pmod{3}$ . Let  $p_i$  be the smallest prime in  $p_2, \dots, p_k$ . Then  $p_i + 1$  is not divisible by any  $p_j$  in  $\{p_2, \dots, p_k\}$ , so  $p_i + 1 = 2^n$  for some integer  $n$ . Therefore  $p_i$  is a Mersenne prime.

Now, we will look at "chains" of prime divisors of  $s$ . To explain this concept, we examine the unitary perfect number  $2^6 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ . (This is not germane to our main theorem, but it is a helpful example.) The Mersenne prime divisors of this number are 3 and 7. First, we look for a prime  $p \nmid s$  such that  $3 \mid (p+1)$ . By inspection, we see that this number is 5. Furthermore,  $5 \mid (2^6 + 1)$ , completing the chain. If 5 did not divide  $2^6 + 1$ , then we would look for another prime  $p_2 \mid s$  such that  $5 \mid (p_2 + 1)$ . The "chain" would continue in this way until we reached a prime that did divide  $2^6 + 1$ . Thus, the elements in this chain are 3 and 5. Next, we consider the chain beginning with the same prime 7. Because  $7 \mid (13 + 1)$  and  $13 \mid (2^6 + 1)$ , the elements in this chain are 7 and 13. (Note also that if either of our Mersenne primes had divided  $2^6 + 1$ , that chain would have contained only the single prime.)

More formally, there is a set of Mersenne primes  $q_1, \dots, q_l$  that divide  $s$ . Each  $q_i$  begins a chain of primes  $R_1(q_i), \dots, R_j(q_i)$ , where  $q_i \mid (R_1(q_i) + 1)$ ,  $R_a(q_i) \mid (R_{a+1}(q_i) + 1)$  for every  $a$  such that  $1 \leq a \leq j-1$ , and, finally,  $R_j(q_i) \mid (2^m + 1)$ . The primes found in these chains, in addition to  $p_1$ , account for every prime that divides  $s$ . Some factors may be repeated; for example, if  $433 \mid s$ , the Mersenne primes 7 and 31 both lead up to 433, since  $7 \mid (433 + 1)$  and  $31 \mid (433 + 1)$ . From this point, we will let  $Q_i = q_i R_1(q_i) R_2(q_i) \dots R_j(q_i)$ , or the product of all the primes in the chain beginning with  $q_i$ . If  $N$  is unitary perfect, then

$$(B) \quad \frac{2^m + 1}{2^m} \cdot \frac{4}{3} \cdot \frac{p_1 + 1}{p_1} \cdot \frac{\sigma^*(Q_1)}{Q_1} \cdots \frac{\sigma^*(Q_l)}{Q_l} \geq 2 = \frac{\sigma^*(N)}{N}.$$

To complete the proof, we will show that (B) does not happen when  $m \equiv 0 \pmod{8}$ .

First, we may assume that  $m \geq 8$ , so Second, we will look at  $p_1$ , which is  $\equiv 2 \pmod{3}$ . We may write  $m = 8b$ . Then, as  $p_1 \mid (2^m + 1)$ ,  $2^{8b} \equiv -1 \pmod{p_1}$ . Thus  $2^{16b} \equiv 1 \pmod{p_1}$ . This

$$\frac{2^m + 1}{2^m} \leq \frac{257}{256}.$$

shows that the order of  $2^b \pmod{p_1}$  is 16. Therefore,  $16 \mid (p_1 - 1)$ . (Note that  $p - 1$  is the order of  $Z_p$  and that the order of  $2^b$  must divide  $p - 1$ .) Now, with a little arithmetic, one can see that the smallest prime  $p$  such that  $p \equiv 2 \pmod{3}$ ,  $p \equiv 1 \pmod{16}$ , and  $p + 1$  has odd squarefree part is 113. Therefore,

$$\frac{p_1 + 1}{p_1} \leq \frac{114}{113}.$$

Finally, we will look at the terms contributed by each  $Q^i$ . This is a product of the form

$$\frac{q+1}{q} \frac{p+1}{p} \cdots,$$

where each numerator is divisible by the previous denominator. The product is largest if  $p_{i+1} + 1 = 2p_i$  for every  $i$ . Then the product is at most

$$(C) \quad \frac{q+1}{q} \frac{2q}{2q-1} \frac{4q-2}{4q-3} \cdots.$$

Now an easy induction shows that the  $n$ th partial product of (C) is

$$\frac{2^n q + 2^n}{2^n q - (2^n - 1)}$$

Thus

$$\frac{\sigma^*(Q)}{Q} \leq \lim_{n \rightarrow \infty} \frac{2^n q + 2^n}{2^n q - (2^n - 1)} = \frac{q+1}{q-1},$$

where  $Q$  is any  $Q_i$ .

Remember that  $q \neq 3$ . The next two Mersenne primes are 7 and 31, so

$$(D) \quad \frac{\sigma^*(Q_1)}{Q_1} \frac{\sigma^*(Q_2)}{Q_2} \leq \frac{8}{6} \frac{32}{30} = \frac{4}{3} \frac{16}{15}.$$

To find the upper bound for the contribution of the Mersenne primes  $> 31$ , we will suppose that every odd power of 2 yields a Mersenne prime. Then the next Mersenne prime is 127, and

$$(D) \quad \frac{\sigma^*(Q_3)}{Q_3} \frac{\sigma^*(Q_4)}{Q_4} \cdots \leq \frac{64}{63} \frac{156}{255} \frac{1024}{1023} \cdots.$$

This is bounded by the product obtained by replacing each numerator with 4 times the previous denominator, so is

$$(E) \quad \leq \frac{64}{63} \frac{252}{251} \frac{1004}{1003} \cdots.$$

It is easy to show that the  $n$ th partial product of (E) is

$$\frac{4^{n-1} \cdot 64}{4^{n-1} \cdot 64 - (4^{n-2} + 4^{n-2} + \dots + 1)} = \frac{64}{64 - (1 + 4^{-1} + \dots + 4^{-n+1})}$$

On letting  $n$  tend to infinity, we get

$$\frac{64}{64 - 4/3} = \frac{48}{47}.$$

Now we are ready to **finish** the proof by noting that

$$\frac{\sigma(N)}{N} \leq \frac{257}{256} \cdot \frac{4}{3} \cdot \frac{114}{113} \cdot \frac{4}{3} \cdot \frac{16}{15} \cdot \frac{48}{84} = 1.96\dots < 2.$$

This contradicts (B). Therefore, there are no unitary perfect numbers of the form  $2^m \cdot 3^2 \cdot 5 \cdot s$ , when  $m > 1$ .

**Acknowledgements.** This work was supported by a grant through the Michigan Technological University Undergraduate Research Project in Mathematics. I also wish to thank S. W. Graham for his guidance throughout this project.

## References

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Jennifer DeBoer will graduate from Michigan Technological University in November 1993 with a B. S. degree in mathematics and teaching as well as a physics minor. She plans to continue mathematical study in graduate school.

## SURPRISE! AN IMPROPER INTEGRAL

Howard B Lambert  
East Texas State University

Computer mathematics packages, useful as they are, must be treated in the same way as pocket calculators: whenever something appears on the display screen, the user should ask, "Is the answer reasonable?" (For that **matter**, the same question could profitably be asked after many paper-and-pencil calculations as well.)

Some packages can, for some functions  $f$ , find in closed form the antiderivative  $F$ . When asked for the exact value of  $\int_a^b f(x) dx$ , they will return  $F(b) - F(a)$ . They can also approximate the value using formulas, such as Simpson's Rule, that do not depend on the antiderivative.

As an example, Derive (**version 1.62**), asked for the exact value of  $\int_0^\pi f(x) dx$  with  $f(x) = 1/(3\cos^2 x + 1)$ , gave 0. This is obviously wrong since  $f(x) > 0$  on  $[0, \pi]$ . When asked for an approximation, the program gave a value of 1.57079.

When asked for the antiderivative of  $f$ , the answer was

$$F(x) = (1/2)\arctan((1/2)\tan x).$$

Sure enough,  $F(\pi) - F(0) = 0$ , and a few minutes' work with pencil and paper shows that  $F'(x) = f(x)$ .

So what went **wrong**? Let us look at one way of stating the Fundamental Theorem of Calculus:

If  $f(x)$  is continuous on  $[a, b]$  then  $f$  has an antiderivative  $G$  defined by

$$G(x) = \int_a^x f(t) dt$$

and if  $H$  is any antiderivative of  $f$  on  $[a, b]$  then  $G - H$  is a constant on  $[a, b]$  and

$$\int_a^b f(x) dx = H(b) - H(a).$$

For  $G$  and  $H$  to be antiderivatives, they must have antiderivatives on  $[a, b]$  and are thus continuous there. In the example,  $f$  is continuous on  $[0, \pi]$  but  $F$  is not. Thus  $F$  is not an antiderivative on the entire interval  $[a, b]$  and cannot be used as the function  $H$  to evaluate  $\int_0^\pi f(x) dx$ .

If we begin with  $F(x) = (1/2)\arctan((1/2)\tan x)$ , since  $F$  is undefined at  $\pi/2$  we have that

$$F'(x) = f(x) = \frac{1}{3\cos^2 x + 1}$$

is not defined at  $\pi/2$  and thus is discontinuous. (The discontinuity, of course, is removable.)

Surprise! We have an improper integral. To evaluate it, we have

$$\begin{aligned} \int_0^\pi f(x) dx &= \lim_{t \rightarrow \pi/2^-} \int_0^t f(x) dx + \lim_{t \rightarrow \pi/2^+} \int_t^\pi f(x) dx \\ &= \lim_{t \rightarrow \pi/2^-} F(x) \Big|_0^t + \lim_{t \rightarrow \pi/2^+} F(x) \Big|_t^\pi \\ &= F(\pi) - F(0) - (\lim_{t \rightarrow \pi/2^+} F(t) - \lim_{t \rightarrow \pi/2^-} F(t)) \\ &= F(\pi) - F(0) - (-\pi/4 - \pi/4) = \pi/2. \end{aligned}$$

Note that  $\lim_{t \rightarrow \pi/2^+} F(t) - \lim_{t \rightarrow \pi/2^-} F(t)$  is the jump of  $F(x)$  at  $x = \pi/2$ .

If we ask the computer to draw the graphs of  $F(x)$  and the continuous antiderivative  $G(x) = \int_0^x f(t) dt$  we get Figures 1 and 2. On any closed interval where both are continuous their values differ at most by a constant and they have the same shape. If we define a new function  $F^*$  by

$$F^*(x) = \begin{cases} F(x) & \text{if } 0 \leq x < \pi/2 \\ \lim_{x \rightarrow \pi/2^-} F(x) & \text{if } x = \pi/2 \\ F(x) - J & \text{if } \pi/2 < x \leq \pi \end{cases}$$

where  $J$  is the jump in  $F$  at  $\pi/2$ , then  $F^*$  is a continuous antiderivative off and in the example is equal to  $G$ . It follows that  $\int_0^\pi f(x) dx = F^*(\pi) - F^*(0) = F(\pi) - J - F(0) = \pi/2$ . An updated version of Derive (2.06) gives the anti-derivative off as

$$\frac{1}{2} \arctan\left(\frac{\tan x}{2}\right) + \frac{x}{2} - \frac{1}{2} \arctan(\tan x),$$

whose only discontinuities are removable.

The discontinuities of  $(1/2) \arctan((1/2)\tan x)$  are smoothed out by  $x/2 - (1/2) \arctan(\tan x)$ , which has a jump at odd multiples of  $\pi/2$ .

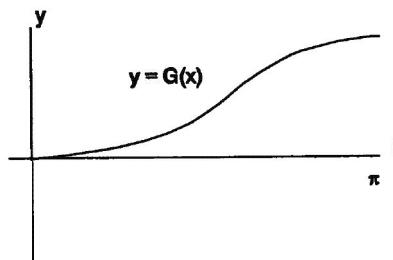


Figure 1

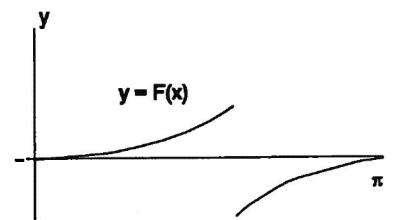


Figure 2

Other programs give different antiderivatives. A version of Maple gives

$$\frac{1}{2} \arctan\left(2 \tan \frac{x}{2} - \sqrt{3}\right) + \frac{1}{2} \arctan\left(2 \tan \frac{x}{2} + \sqrt{3}\right),$$

discontinuous at  $(2k + 1)\pi$ ; a version of Mathematica

$$-\frac{1}{2} \arctan(2 \cot x),$$

discontinuous at  $k\pi$ ; and another version of Mathematica

$$\frac{i}{4} \ln(1 + 3(\cos 2x - i \sin 2x)) - \frac{i}{4} \ln(3 + \cos 2x - i \sin 2x),$$

discontinuous at  $(2k + 1)\pi/2$ .

The proper rephrasing of the Fundamental Theorem of Calculus for a function  $f$  with a piecewise antiderivative, continuous except for jumps, is left for the reader.

So, when using the computer to find a definite integral, we should ask for both the exact answer and the approximate answer. The program uses different methods to find them, so if they agree we should feel confident with the result. If they do not agree, then the antiderivative should be examined closely.

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Howard Lambert is a graduate of Texas Tech University. In 1992, he received a Teaching Excellence Award **from** the East Texas State University chapter of the Texas **Association** of College Teachers.

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### Pin Prices Going Up

The cost of a Pi Mu Epsilon Pin will increase from \$8 to \$12 at **midnight**, June 30, 1994. Those of you who have lost yours and need a replacement or who are buying quantities of pins for their investment value are thus advised to act soon. Pins are available **from** the Secretary-Treasurer, Professor Robert M. Woodside, Department of Mathematics\*East Carolina University, **Greenville**, North Carolina 27858.

## USING POWER SERIES TO COMPLETE THE BASIC INTEGRATION FORMULA

Margaret Webb  
Penn State University, New Kensington

Recently [1], Schaumberger used the mean-value theorem to show that

$$\lim_{n \rightarrow -1} \int_a^b x^n dx = \int_a^b \frac{dx}{x}.$$

In this note we will use power series to obtain the same result. We have

$$\begin{aligned} \lim_{n \rightarrow -1} \int_a^b x^n dx &= \lim_{n \rightarrow -1} \frac{b^{n+1} - a^{n+1}}{n+1} \\ &= \lim_{n \rightarrow -1} \frac{e^{(n+1)\ln b} - e^{(n+1)\ln a}}{n+1} \\ &= \lim_{n \rightarrow -1} \frac{1}{n+1} \sum_{k=0}^{\infty} \left[ \frac{((n+1)\ln b)^k}{k!} - \frac{((n+1)\ln a)^k}{k!} \right] \\ &= \lim_{n \rightarrow -1} \left[ \ln b - \ln a + \sum_{k=2}^{\infty} \frac{(n+1)^{k+1}}{k!} \cdot ((\ln b)^k - (\ln a)^k) \right] \\ &= \ln b - \ln a = \int_a^b \frac{dx}{x}. \end{aligned}$$

The interchange of limit and summation is justified by the uniform convergence of the series.

## Reference

1. N. Schaumberger, Using the mean value theorem to complete the basic integration formula, this Journal 9 (1991), 226-227

The author prepared *this* paper under the supervision of Professor Javier Gomez-Calderon while she was a freshman at Penn State University—New Kensington campus.

## EVALUATING A DEFINITE INTEGRAL USING PROBABILITY

M. A. Khan  
Research Design and Standards Organisation  
Lucknow, India

We shall show that

$$\int_0^1 x^p (1-x)^q dx = \frac{p! q!}{(p+q+1)!}.$$

The integral is known as Euler's Integral (see, for example [1], [2]). It is interesting that it can be evaluated by using a probabilistic argument.

Let us select  $r+q$  random numbers from the interval  $[0, 1]$  and call them  $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_q$ . What is the probability that  $a_i < b_j$  for all  $i$  and  $j$ ? We will answer the question in two ways.

First, consider the  $(r+q)!$  ways of arranging the  $r+q$  numbers. If we are to have  $a_i < b_j$ , then the as must come first. Thus, the first number may be chosen in  $r$  ways, the second in  $r-1$ , and so on until the last of the as is selected. Then the first of the bs may be chosen in  $q$  ways, the second in  $q-1$  ways, and so on. There are then  $q!$  ways to have the numbers appear in the desired fashion, and so the probability  $a_i < b_j$  for all  $i$  and  $j$  is  $\frac{r! q!}{(r+q)!}$ .

Second, consider the location of the largest of the as on  $[0, 1]$ . Suppose it is between  $x$  and  $dx$ , which has probability  $dx$ . (See Figure 1.) Then, if we are to have  $a < b$ , all of the bs must be greater than  $x$ , and the probability of this is  $(1-x)^q$ . Also, the other  $r-1$  as must be less than  $x$ , and the probability of this is  $x^{r-1}$ . Finally, the largest of the as may be selected in  $r$  ways, so the probability of a suitable arrangement with the largest of the as between  $x$  and  $dx$  is  $x^{r-1} r dx (1-x)^q$ . The total probability is obtained by integrating over all possible values of  $x$ , and thus is

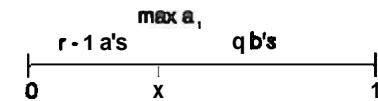


Figure 1

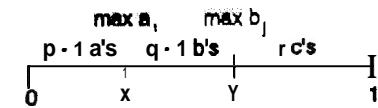


Figure 2

$$\int_0^1 rx^{r-1}(1-x)^q dx.$$

Since the probability is the same no matter which method we use to find it, we have

$$\int_0^1 rx^{r-1}(1-x)^q dx = \frac{r! q!}{(r+q)!}.$$

Putting  $r = p + 1$ , we have the result.

The argument may be generalized to any number of sets. For example, with three sets of numbers,  $\{a_1, \dots, a_p\}$ ,  $\{b_1, \dots, b_q\}$ ,  $\{c_1, \dots, c_r\}$ , the probability that  $a_i < b_j < c_k$  for all  $i, j, k$  is, by considering arrangements,  $\frac{p! q! r!}{(p+q+r)!}$

On the other hand (see Figure 2) it is

$$\iint_{0 \leq x \leq y \leq 1} px^{p-1} q(y-x)^{q-1} (1-y)^r dx dy.$$

Figure 3

The integral may be put in various other forms by change of variables. One of these forms, known as Dirichlet's Integral, is

$$\iiint_{\substack{x, y, z \geq 0 \\ x+y+z \leq 1}} x^{p-1} y^{q-1} z^{r-1} dx dy dz = \frac{(p-1)! (q-1)! (r-1)!}{(p+q+r)!}$$

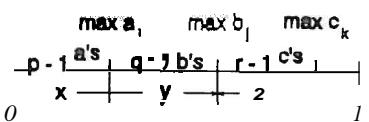
This result can be obtained on similar lines by a slight modification of the number line (see Figure 3). The details are left to the reader.

#### References

- Spiegel, Murray R., *Theory and Problems of Advanced Calculus*, McGraw-Hill, Singapore, 1981.
- Srivastava, R. S. L., *Engineering Mathematics*, vol. 1, Tata McGraw-Hill, New Delhi, 1980.

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M. A. Khan has a degree in electrical engineering and is presently Deputy Director of RDSO, Lucknow. Besides mathematics, his interests include contract bridge.



#### LETTER TO THE EDITOR

Dear Professor Poss,

I am sending here a brief followup to my article "Fractorial!" which appeared in the Fall 1992 issue of the *Journal*. Even when I had finished it I was troubled by what is a very cumbersome development to get from Definition 3, the definition of factorial, to Theorems 1 and 2. I have simplified the whole thing greatly by slightly reformulating the definition of **fractorial** so that now the results of Theorems 1 and 2 follow in a very straightforward manner.

**DEFINITION 3.**  $a!_b = a(a-b)(a-2b)\dots(a-kb)$ , where  $a$  and  $b$  are positive real numbers,  $a \geq b$ , and  $k$  is a natural number such that  $k < a/b \leq k+1$ , so that  $0 < a - kb \leq b$ .

**THEOREM.** For all natural numbers  $a, b, c$ , and  $d$ , the number,  $n$ , of factors in the factorial expression  $(a/c)!_{(b/d)}$  is given by  $n = \left[ \frac{ad-1}{bc} \right] + 1$ .

*Proof.* From  $k < a/b \leq k+1$  we get  $0 < a - kb \leq b$ , or  $l \leq a - kb < b+1$ . Replace  $a$  with  $ad$  and  $b$  with  $bc$  to get  $l \leq ad - kbc < bc+1$ . Dividing by  $bc$  gives  $1/bc \leq ad/bc - k < 1/bc$ , so  $k \leq ad/bc - 1/bc < k+1$ .

Since  $n = k + 1$ , the result follows.

Sincerely yours,

Nataniel Greene  
Yeshiva University

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#### New Giant Twin Primes

$$4650828 \cdot 10^{3429} \pm 1.$$

Questions for the reader: 1. Do they add to our store of mathematical knowledge? 2. Can we trust the computer that calculated them? If your answers are, respectively, "Not much" and "Not necessarily" then 3. Why were they included here and why did you read about them?

## Mathacrostics

Solution to Mathacrostic 36, by Charlotte Maines, (Spring, 1993).

Words:

- |                                |                           |
|--------------------------------|---------------------------|
| A. Monte <b>Carlo</b> method   | P. <b>Hooton</b>          |
| B. abstract space              | Q. interface              |
| C. rheostat                    | R. <b>Noether</b>         |
| D. venetian white              | S. gentes                 |
| E. inappetent                  | T. servomechanism         |
| F. Nathan                      | U. folium of Descartes    |
| G. Sylvester's dialytic method | V. anthotaxy              |
| H. hones                       | W. <b>lenten</b>          |
| I. imaginary circle            | X. lethe                  |
| J. nephews                     | Y. annotate               |
| K. batch                       | Z. point of osculation    |
| L. <b>requiescat</b>           | a. <b>athanor</b>         |
| M. objects                     | b. rule of false position |
| N. trident of Newton           | c. two-throw              |
| O. trajectories                |                           |

Author and title: Marvin **Shinbrot**, Things Fall Apart,

Quotation: From the seventeenth century to the nineteenth, the heart of all physics and much of mathematics as well was Newton's three laws of motion: a body not subject to external force remains at rest or moves with constant speed in a straight line, the acceleration of an object is proportional to the forces acting on it, and to every action there is an equal and opposite reaction.

Solvers: THOMAS BANCHOFF, Brown University, JEANETTE BICKLEY, St. Louis Community College—Meramec, CHARLES R. DIMINNIE, St. Bonaventure University, VICTOR G. FESER, University of Mary, ROBERT FORSBERG, Lexington, Massachusetts, META HARRSEN, Durham, North Carolina, THEODOR KAUFMAN, Brooklyn, New York, HENRY S. LIEBERMAN, Waban, Massachusetts, DON PFAFF, University of Nevada, Reno, STEPHANIE SLOYAN, Georgian Court College

Mathacrostic 37, proposed by PATTI VAHEDI, follows. To be listed as a solver, send your solution to Underwood Dudley, Pi Mu Epsilon Journal, Math. Dept., DePauw Univ., Greencastle, Indiana 46135.

## MATHACROSTICS

- A. Vector cross-product convention.  
[Hyph + wd]  
B. He secured election of first  
German Pope. [2 wds]  
C. Enlightened one. [Sanskrit]

- D. A solution to the Sturm-Liouville  
problem.  
E. FDR's director of the U. S. Mint.  
[Surname]  
F. Belgian art group of *la Belle  
Epoque*. [2 wds]  
G. It salivated when the bell rang.  
[2 wds]  
H. Its limit, by definition. =  
integral of function. [2 wds]  
I. German gelato or sorbet.

- J. Fictive beast, clarifies  
uncertainty principle. [2 wds]  
K. Simple problem in astronomy.  
[Hyph]  
L. Incidental remark; if from a  
judge, it has no bearing on the  
case and is not binding. [2 wds]  
M. It brought attention to  
fundamental law in classical  
mechanics. [2 wds]  
N. Descartes' parabola = — —  
Newton. [2 wds]  
O. Narrow-minded; prejudiced.

- P. He estimated earth radius with  
trig, circa 230 BC  
Q.  $0.4 \cos(\pi/4)$ .

- R. Indian mathematician b. 476 BC.  
S. Polynomial expression of a  
function. [2 wds]  
T. Fool's dullness = "Whetstone  
— ." [3 wds, from As You Like  
It]  
U. Realm of  $a + bi$ . [2 wds]  
V. Designer, intensity interferometer  
[Init, surname]  
W. Chinese moon guitar [Hyph]

5	127	158	111	119	183	21	239	36	216	58	170	
234	114	39	79	184	136	65						207
139	47	199	226	95	71	2	147	218	178	162		
194	241	186	152	118	230	37	92	15	75			
121	150	84	38									
25	64	100	8	52	224	163	81	122				
78	68	99	55	10	173	211	204	229	138			
180	4	134	16	157	23	109	69	197	228			
203	238	96										
149	20	53	42	72	94	106	80	176	117	131	164	
130	145	220	233	24	103	181			206	217	246	
88	26	9	155	179	62	237	171	124	104	212	74	
57	129	50	76	210	11	107	143	86	153	198	102	
191	19	227	208	35	22	49	142	73				
222	146	240	28	168	51	120	3	91				
161	101	12	221	46	245	209	126	90	188	172	33	
1	159	133	41	223	113							
215	202	140	7	60	156	77	110	132				
67	98	125	201	193	34	177	6	219	169	242		
196	48	144	166	32	225	174	44	89				
97	167	83	66	13	231	39	87	135	190	185	56	
232	214	141	31	54	165	189						
154	85	18	40	213	115	187	175					

X. Its examples include 5, 23, 14009.

[2 wds]

Y. \_\_ Atlas; \_\_ telescope; \_\_ time;  
\_\_ expansion.

Z. 19th century pioneer in theory  
and analysis of Word U. [2 wds]

30 160 45 59 112 137 244 151 123 27 236

105 192 200 17 128 61

43 82 243 205 63 182 108 14 116 235 70 195

148 93

11 M	12 P	13 U	14 Z	15 D	16 H	17 Y	18 W	19 N	20 J	21 A	22 N	23 H
24 K	25 F		26 L	27 X	28 D	29 U	30 X	31 V	32 T	33 P	34 S	35 N
		37 D	38 E	39 B	40 W	41 Q	42 J	43 Z	44 T	45 X	46 P	47 C
48 T		49 N	50 M	51 O	52 F	53 J	54 V	55 G	56 U	57 M	58 A	59 X
60 R	61 Y	62 L	63 Z	64 F	65 B	66 U	67 S	68 G	69 H	70 Z	71 C	72 J
73 N		74 L	75 D	76 M	77 R	78 G	79 B	80 J	81 F	82 Z	83 U	84 E
86 M	87 U	88 L	89 T	90 P	91 O	92 D	93 Z	94 J	95 C	96 I	97 U	98 S
100 F	101 P	102 M	103 K		104 L	105 Y	106 J	107 M	108 Z	109 H		110 R
112 X		113 Q	114 B	115 W		116 Z	117 J	118 D	119 A	120 O	121 E	122 F
124 L		125 S	126 P	127 A	128 Y	129 M		130 K	131 J	132 R	133 Q	134 H
137 X	138 G		139 C	140 R		141 V	142 N	143 M	144 T		145 K	146 O
	149 J	150 E	151 X	152 D		153 M	154 W	155 L	156 R	157 H	158 A	159 Q
162 C	163 F	164 J		165 V	166 T	167 U		168 O	169 S	170 A	171 L	172 P
175 W	176 J		177 S	178 C	179 L	180 H	181 K	182 Z	183 A	184 B	185 U	186 D
188 P		189 V	190 U	191 N	192 Y	193 S	194 D		195 Z	196 T	197 H	198 M
200 Y	201 S		202 R	203 I	204 G	205 Z	206 J	207 A	208 N	209 P	210 M	211 G
212 L	213 W	214 V		215 R		216 A	217 J	218 C	219 S	220 K		221 P
224 F	225 T		226 C	227 N	228 H		229 G	230 D	231 U	232 V	233 K	234 B
237 L		238 I	239 A		240 O	241 D	242 S	243 Z	244 X	245 P	246 J	236 X

## PROBLEM DEPARTMENT

Edited by Clayton W. Dodge

University of Maine

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (\*) preceding a problem number indicates that the proposer did not submit a solution.

All communications should be addressed to C. W. Dodge, 5752 Neville/Math, University of Maine, Orono, ME 04469-5752. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name and address. Solutions to problems in this issue should be mailed by July 1, 1994.

This department seeks to present a wide variety of problems in each issue, preferably not more than two problems from any one category. Hence appropriate proposals are sought for all categories, but especially for those that are empty or nearly so. To aid (and entice) you in your submissions, we list each category along with the number of problem proposals in its file: algebra 10, alphametics 1, analysis 10, geometry 3, logic and combinatorics 0, number theory 3, probability and statistics 0, trigonometry 0, and miscellaneous 0.

### Corrections

Several times in the Spring 1993 Problem Department the name of proposer and solver David Iny was inadvertently printed as David Ivy. Our sincere apologies.

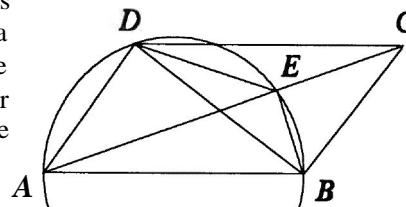
Thanks to William Peirce for pointing out that in the solution to problem 765 on page 475 of the Fall 1992 issue, the last equation should read  $410^2 = (12122_4)^2 = 0221002210_4$ .

**780. [Spring 1992, Fall 1992] Corrected again. Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.**

Let ABCD be a parallelogram with  $\angle A = 60^\circ$ . Let the circle through A, B, and D intersect AC at E. See the figure. Prove that  $BD^2 + AB \cdot AD = AE \cdot AC$ .

*Editor's comment: The statement of the problem was correct. The*

figure was perhaps misleading: ABCD is a parallelogram, not necessarily a rhombus. The accompanying figure shows this situation more clearly. Our apologies for any inconvenience the original figure may have caused.



### Problems for Solution

**810.** Proposed by Alan Wayne, Holiday, Florida.

In the following base eight multiplication, the digits of the two multipliers have been replaced in a one to one manner by letters:

$$(I)(CLUED) = 437152.$$

Restore the digits. Similarly replace 437152 to find out who might have said "I clued."

**811.** Proposed by Tom Moore, Bridgewater State College, Bridgewater, Massachusetts.

If  $a < b < c$  are positive integers with  $\gcd(a, b) = 1$  and  $a^2 + b^2 = c^2$ , then  $(a, b, c)$  is called a primitive Pythagorean triple (PPT). If both  $a$  and  $c$  are primes, then we shall call it a prime PPT ( $P^3T$ ).

- a) If  $(a, b, c)$  is a  $P^3T$ , deduce that  $b = c - 1$ .
- b) Find all  $P^3Ts$  in which  $a$  and  $c$  are
  - i) twin primes.
  - ii) both Mersenne primes.
  - iii) both Fermat primes.
  - iv) one a Mersenne, the other a Fermat prime.

**812.** Proposed by George P. Evanovich, Saint Peter's College, Jersey City, New Jersey.

If  $n \geq 2$  is a positive integer, prove that

$$\sum_{j=1}^n \cos\left(\frac{2j\pi}{n}\right) = \sum_{j=1}^n \sin\left(\frac{2j\pi}{n}\right) = 0.$$

**813.** Proposed by the late Jack Garfunkel, Flushing, New York.

Given a triangle  $ABC$  with sides  $a, b, c$  and a triangle  $A'B'C'$  with sides

$(b+c)/2, (c+a)/2, (a+b)/2$ . Prove that  $r' \geq r$ , where  $r$  and  $r'$  are the inradii of triangles  $ABC$  and  $A'B'C'$  respectively.

**814.** Proposed by Nathan Jasper, Stevens Institute of Technology, Hoboken, New Jersey.

For any decimal integer  $n$ , prove that  $n^5$  and  $n$  end in the same digit, that  $n^6$  and  $n^2$  end in the same digit, that  $n^7$  and  $n^3$  end in the same digit, and so forth.

**815.** Proposed by Bill Correll, Jr., Cincinnati, Ohio.

Let  $[x]$  denote the greatest integer not exceeding  $x$ . Solve for  $x$ :

$$\left[ \frac{x+1}{2} \right] \left[ \frac{x+2}{3} \right] \left[ \frac{x+3}{4} \right] = 819.$$

**816.** Proposed by Robert C. Gebhardt, Hopatcong, New Jersey.

a) From the integers  $1, 2, 3, \dots, n$ , a state lottery selects at random  $k$  numbers ( $k < n$ ). A person who had previously chosen at random  $m$  of those  $k$  numbers ( $m \leq k$ ) is a winner. Find the probability of being a winner.

\*b) The Tri-State Megabucks (Maine, New Hampshire, and Vermont) tickets cost \$1 each. A participant selects  $m = 6$  numbers out of  $n = 40$  and is a winner if all six numbers match the  $k = 6$  numbers the game selects. The winnings are paid in 20 equal annual installments. How large does the pot have to be before a ticket is worth \$1?

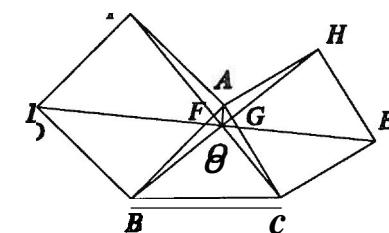
**817.** Proposed by Andrew Cusumano, Great Neck, New York.

In the accompanying figure squares  $CEHA$  and  $AIDE$  are erected externally on sides  $CA$  and  $AB$  of triangle  $ABC$ . Let  $BH$  meet  $IC$  at  $O$  and  $AC$  at  $G$ , and let  $BE$  meet  $IC$  at  $F$ ,  $AE$  at  $H$ , and

$E$  are collinear.

b) Prove that angles  $HOE$ ,  $EOC$ ,  $AOH$ , and  $AOI$  are each  $45^\circ$ .

c) If  $ACB$  is a right angle, then prove that  $E, F$ , and  $G$  are collinear.



Find an "elegant" proof for parts (a) and (b), both of which are known to be true whether the squares are erected both externally or both internally (see The American Mathematical Monthly, problem E831, vol. 56, 1949, 406-407). Part (c) is a delightful result that also should be known, but appears to be more difficult to prove.

**\*818.** Proposed by **Dmitry P.** Mavlo, Moscow, Russia.

From the SYMP-86 Entrance Examination, solve the inequality

$$\frac{1}{x^3 - x} \leq \frac{1}{|x|}.$$

**819.** Proposed by Morris **Katz**, Macwahoc, Maine.

Evaluate the integral

$$\int \ln x \sin^{-1} x dx.$$

**820.** Proposed by William **Moser, McGill** University, Montreal, Quebec, Canada.

Let  $a_{n,k}$  ( $0 \leq k < n$ ) denote the number of  $n$ -bit strings (sequences of **0's** and **1's** of length  $n$ ) with exactly  $k$  occurrences of two consecutive **0's**. Show that

$$a_{n,k} = \sum_{r=0}^{n-1} \binom{r-k}{k} \binom{n-r-1}{r-k},$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!}$  if  $0 \leq k \leq n$  and  $\binom{n}{k} = 0$  otherwise.

**821.** Proposed by Zeev **Barel, Hendrix** College, **Conway**, Arkansas.

Problem B-2 at the fifty-second annual William Lowell Putnam Mathematical Competition (1991) stated: Suppose  $f$  and  $g$  are non-constant, differentiable, real-valued functions defined on  $(-\infty, \infty)$ . Furthermore, suppose that for each pair of real numbers  $x$  and  $y$ ,

$$f(x+y) = f(x)f(y) - g(x)g(y) \quad \text{and} \quad g(x+y) = f(x)g(y) + g(x)f(y).$$

If  $f'(0) = 0$ , prove that  $(f(x))^2 + (g(x))^2 = 1$  for all  $x$ .

In fact, one can do a little more under the same hypothesis. Prove that there exists a real number  $k$  such that  $f(x) = \cos kx$  and  $g(x) = \sin kx$  for all  $x$ .

**822.** Proposed by Stanley **Rabinowitz**, **MathPro** Press, **Westford**, Massachusetts.

If  $a$  is a root of the equation  $x^5 + x - 1 = 0$ , then find an equation that has  $a^4 + 1$  as a root.

## Solutions

**777.** [Spring 1992, Fall 1992] Corrected. Proposed by **Seung-Jin Bang**, Seoul, Korea.

It is well known that, for  $n \geq 2$ ,  $\ln(n+1) < S_n < 1 + \ln n$ , where

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}.$$

It is also known (Crux Mathematicorum 11 (1985), 109) that, for  $n \geq 2$ ,

$$n(n+1)^{1/n} - n < S_n \leq n - (n-1)n^{-1/(n-1)}.$$

Prove that

$$\ln(n+1) < n(n+1)^{1/n} - n \quad \text{and} \quad n - (n-1)^{-1/(n-1)} < 1 + \ln n$$

for all  $n \geq 2$ .

Solution by Alma College Problem Solving Group, Alma College, Alma, Michigan.

Recall that  $e^u > 1 + u$  for all  $u \neq 0$ , so that  $\ln x < x - 1$  for all  $x > 0$ ,  $x \neq 1$ , by setting  $x = u + 1$ .

Because  $(n+1)^{1/n} > 0$  (and never equal to 1) when  $n \geq 1$ , then

$$\ln(n+1)^{1/n} < (n+1)^{1/n} - 1$$

for all  $n \geq 1$ . Finally, multiply both sides by  $n$  to get that

$$\ln(n+1) = n \ln(n+1)^{1/n} < n(n+1)^{1/n} - n.$$

Again using  $x > 1 + \ln x$ , we have, for  $n \geq 2$ ,

$$n^{-1/(n-1)} > 1 + \ln n^{-1/(n-1)}, \quad (n-1)n^{-1/(n-1)} > (n-1) - \ln n,$$

and

$$n - (n-1)n^{-1/(n-1)} < 1 + \ln n.$$

Also solved by PAUL S. BRUCKMAN, Everett, WA, RICHARD I. HESS, Rancho Palos Verdes, CA, DAVID INY, Westinghouse Electric Corporation, Baltimore, MD, MURRAY S. KLAMKIN, University of Alberta, Canada, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, YOSHINOBU MURAYOSHI, Eugene, OR, PAUL D.

SHOCKLEE, Memphis, TN, JORGE-NUNO SILVA, Albany, CA, REX H. WU, Brooklyn, NY, and the PROPOSER.

784. [Fall 1992] Proposed by Alan Wayne, Holiday, Florida.  
Restore the enciphered digits in the decimal computation:

$$(TWO)(TWO + TWO) = EIGHT.$$

Solution by Kenneth M. Wilke, *Topeka*, Kansas.

Since  $98765 \geq EIGHT = 2(TWO)^2$ , then  $TWO < 223$ , T is even so  $T = 2$ , and O = 1, 4, 6, or 9. Therefore  $TWO = 201, 204, 206, 209, 214, 216$ , or 219. Testing these possibilities, we find that only  $TWO = 209$  yields a solution, and then  $EIGHT = 87362$ .

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, CHARLES ASHBACHER, Cedar Rapids, IA, SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, SCOTT H. BROWN, Auburn University, AL, PAUL S. BRUCKMAN, Everett, WA, MARK EVANS, Louisville, KY, VICTOR G. FESER, University of Mary, Bismarck, ND, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho *Palos Verdes*, CA, PETE JOHNSON, Hebron, CT, JOHN D. MOORES, Westbrook, ME, YOSHINOBU MURAYOSHI, Eugene, OR, PAUL D. SHOCKLEE, Memphis, TN, LAURA SILVA, Albany, CA, SONNY VU, University of Illinois at *Urbana-Champaign*, REX H. WU, Brooklyn, NY, and the PROPOSER.

785. [Fall 1992] Proposed by Charles *Ashbacher*, Cedar Rapids, Iowa, and dedicated to the memory of Joseph *Konhauser*. Student solutions are especially solicited.

A tiling of the plane by non-overlapping, non-congruent rectangles  $P_1, P_2, \dots$  is defined in the following way:  $P_1$  is an arbitrary  $x$  by  $y$  rectangle;  $P_2, P_3, \dots$  are all squares such that the side of each square  $P_{k+2}$  is equal to the sum of the sides of the two previous squares  $P_k$  and  $P_{k+1}$  for all  $k > 1$ . Show this tiling.

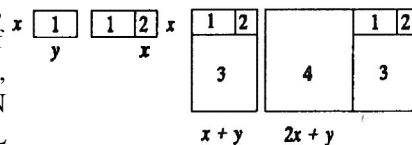
Solution by Matthew Amoroso, St. *Bonaventure* University, St. *Bonaventure*, New York.

To begin, let rectangle 1 be an  $x$  by  $y$  rectangle. See the figure on the next page. Let rectangle 2 be a square of side  $x$  and place it so that it is resting on a side of rectangle 1 with length  $x$ . Again see the figure. Next, rectangle 3 is a square of side  $x+y$  which we place to rest against both rectangles 1 and 2. Now rectangle 4 is a square of side  $2x+y$  that rests against the two immediately preceding rectangles 2 and 3. Continue in this manner to tile the plane in the

prescribed way. Since each square must have its side equal to the sum of the sides of the two previous squares, the coefficients of  $x$  and of  $y$  in these side lengths are successive Fibonacci numbers:  $f_0 = 0, f_1 = 1$  and  $f_{n+2} = f_n + f_{n+1}$  for  $n \geq 1$ . That is, the side of square  $n$  is equal to  $f_{n-1}x + f_{n-2}y$  for  $n \geq 2$ .

Also solved by PAUL S. BRUCKMAN, Everett, WA, RICHARD

I. HESS, Rancho *Palos Verdes*, CA, MURRAY S. KLAMKIN, University of Alberta, Canada, TOM MOORE, Bridgewater State College, MA, JOHN D. MOORES, Westbrook, ME, PAUL D. SHOCKLEE, Memphis, TN, REX H. WU, Brooklyn, NY, and the PROPOSER.



786. [Fall 1992] Proposed by Dmitry P. Mavlo, Moscow, Russia.

From two towns A and B, 48 km apart, two groups of hikers march toward each other starting at the same time. The group leaving A marches at 4 km/hr by marches of not more than 6 hr at one time. The group from B hikes at 6 km/hr for not more than 2 hr at a time. After marching  $t$  hr, the first group must rest for at least  $t$  hr. The second group has to rest not less than  $2t$  hr after  $t$  hr of hiking. Find the least time until the two groups meet and describe the hiking patterns necessary for that solution.



Solution by William H. Peirce, Delray *Beach*, Florida,

The time until the two groups meet will be least if each group hikes as long as possible without resting. Therefore, we let them meet as A is completing a 6-hr hike at 4 km/hr and B is completing a 2-hr hike at 6 km/hr, assuming the time needed to meet is at least 6 hours. These final hikes span  $24 + 12 = 36$  km.

If the total time until they meet is  $T > 6$  hours, then group A must hike and rest equal times for the first  $T - 6$  hours, averaging 2 km/hr during that time. Group B has  $T - 2$  hours during which they average 2 km/hr. Hence

$$48 = 36 + 2(T - 6) + 2(T - 2)$$

and  $T = 7$  hours. Group A must hike 30 minutes and rest 30 minutes during the first hour, and then hike the last 6 hours. Group B hikes 1 hour 40 minutes and rests 3 hours 20 minutes during the first 5 hours, and then hikes the last 2 hours. Except for the fact that the initial hike-and-rest period can be broken down into smaller segments, this solution is unique.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, CHARLES ASHBACHER, Cedar Rapids, IA, PAUL S. BRUCKMAN, Everett, WA, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, JOHN D. MOORES, Westbrook, ME, WILLIAM H. PEIRCE (second solution), Delray Beach, FL, PAUL D. SHOCKLEE, Memphis, TN, MANUEL SILVA, Albany, CA, and REX H. WU, Brooklyn, NY. One incorrect solution was received.

787. [Fall 1992] Proposed by R. S. Luthar, University of Wisconsin Center, Janesville, Wisconsin.

If  $a, b, c, d$  are the roots of

$$x^4 + px^3 + qx^2 + rx + s = 0,$$

then evaluate the expression

$$(a + b + c - 2d)(b + c + d - 2a)(c + d + a - 2b)(d + a + b - 2c)$$

in terms of  $p, q, r$ , and  $s$ .

Solution by Harry Sedinger, St. Bonaventure University, St. Bonaventure, New York.

It is well known (and easily seen) that if

$$x^4 + px^3 + qx^2 + rx + s = (x - a)(x - b)(x - c)(x - d),$$

then

$$a + b + c + d = -p, \quad ab + ac + ad + bc + bd + cd = q,$$

$$abc + acd + abd + bcd = -r, \quad \text{and} \quad abcd = s.$$

Then the given expression is equal to

$$\begin{aligned} & (-p - 3d)(-p - 3a)(-p - 3b)(-p - 3c) \\ &= p^4 + 3(a + b + c + d)p^3 + 9(ab + ac + ad + bc + bd + cd)p^2 \end{aligned}$$

## PROBLEMS AND SOLUTIONS

$$\begin{aligned} & + 27(abc + acd + abd + bcd)p + 81abcd \\ &= p^4 - 3p^4 + 9qp^2 - 27rp + 81s \\ &= -2p^4 + 9qp^2 - 27rp + 81s. \end{aligned}$$



Also solved by SEUNG-JIN BANG, Seoul, Korea, SCOTT H. BROWN, Auburn University, AL, PAUL S. BRUCKMAN, Everett, WA, JILL CARNAHAN, Eastern Kentucky University, Richmond, BILL CORRELL, JR., Cincinnati, OH, DAVID DELSESTO, North Scituate, RI, RUSSELL EULER, Northwest Missouri State University, Maryville, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, MURRAY S. KLAMKIN, University of Alberta, Canada, DAYONG LI, Eastern Kentucky University, Richmond, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, YOSHINOBU MURAYOSHI, Eugene, OR, WILLIAM H. PEIRCE, Delray Beach, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, PAUL D. SHOCKLEE, Memphis, TN, LAWRENCE SKAGGS, Eastern Kentucky University, Richmond, KENNETH M. WILKE, Topeka, KS, J. ERNEST WILKINS, JR., Clark Atlanta University, GA, REX H. WU, Brooklyn, NY, and the PROPOSER.

788. [Fall 1992] Proposed by the late Jack Garfunkel, Flushing, New York.

Given positive numbers  $x, y, z$  such that  $x + y + z = 1$ , prove that

$$xy + yz + zx \geq x^2y^2 + y^2z^2 + z^2x^2 + 8xyz.$$

I. Solution by Sammy Yu and Jimmy Yu, students, Vermillion Middle School, Vermillion, South Dakota.

Since  $x + y + z = 1$ , the desired inequality is equivalent to

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} + 8,$$

and hence to

$$(1) \quad \frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z} \geq \frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} + 5.$$

Now consider that

$$\frac{(y+z)^2}{x} + \frac{(z+x)^2}{y} + \frac{(x+y)^2}{z} - \frac{yz}{x} - \frac{zx}{y} - \frac{xy}{z}$$

$$\begin{aligned}
 &= \frac{y^2 + yz + z^2}{x} + \frac{z^2 + zx + x^2}{y} + \frac{x^2 + xy + y^2}{z} \\
 &= \left( \frac{y^2}{x} + \frac{zx}{y} + \frac{y^2}{z} \right) + \left( \frac{yz}{x} + \frac{x^2}{y} + \frac{x^2}{z} \right) + \left( \frac{z^2}{x} + \frac{z^2}{y} + \frac{xy}{z} \right) \\
 &\geq 3y + 3x + 3z = 3
 \end{aligned}$$

by applying the arithmetic mean-geometric mean inequality to each parentheses separately. Equality holds if and only if  $x = y = z$ . Thus

$$(2) \quad \frac{(y+z)^2}{x} + \frac{(z+x)^2}{y} + \frac{(x+y)^2}{z} \geq \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} + 3.$$

The left side of (2) can be rewritten as

$$\begin{aligned}
 &(y+z)\left(\frac{x+y+z}{x}-1\right) + (z+x)\left(\frac{x+y+z}{y}-1\right) + (x+y)\left(\frac{x+y+z}{z}-1\right) \\
 &= (x+y+z)\left(\frac{y+z}{x} + \frac{z+x}{y} + \frac{x+y}{z}\right) - 2.
 \end{aligned}$$

Hence equation (2) implies (1). The desired result follows with equality if and only if  $x = y = z$ .

**II.** Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.

We rewrite the inequality in the following homogeneous form:

$$(yz + zx + xy)(x + y + z) \geq y^2z^2 + z^2x^2 + x^2y^2 + 8xyz(x + y + z)$$

or

$$(1) \quad T_1^2T_2 \geq T_2^2 - 2T_1T_3 + 8T_1T_3 = T_2^2 + 6T_1T_3$$

where  $T_1 = x + y + z$ ,  $T_2 = yz + zx + xy$ , and  $T_3 = xyz$ .

We give a stronger result by proving that the "best inequality" of the form

$$(2) \quad T_1^2T_2 \geq aT_2^2 + bT_1T_3$$

is

$$(3) \quad T_1^2T_2 \geq 4T_2^2 - 3T_1T_3$$

or equivalently,

$$(3') \quad yz(y-z)^2 + zx(z-x)^2 + xy(x-y)^2 \geq 0,$$

so (3) is clearly true. For (2) to be valid we have, on setting  $x = y = z = 1/3$ , that  $9 \geq 3a + b$ ; on setting  $x = y = 1/2$  and  $z = 0$ , that  $4 \geq a$ . That (3) is "best" will now follow if

$$(4) \quad 4T_2^2 - 3T_1T_3 \geq aT_2^2 + bT_1T_3$$

for all  $a$  and  $b$  satisfying the latter two inequality conditions. Since  $T_2^2 \geq 3T_1T_3$  is a known equality equivalent to  $\Sigma x^2(y-z)^2 \geq 0$ , then (4) follows if

$$\frac{b-3}{4-a} \leq 3,$$

which is equivalent to the condition  $9 \geq 3a + b$ .

Working backwards from (3), we find that the "best inequality" for the original problem is

$$xy + yz + zx \geq 4(x^2y^2 + y^2z^2 + z^2x^2) + 5xyz.$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, DAVID E. MANES, SUNY at Oneonta, YOSHINOBU MURAYOSHI, Eugene, OR, WILLIAM H. PEIRCE, Delray Beach, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, J. ERNEST WILKINS, JR., Clark Atlanta University, GA, and the PROPOSER.

789. [Fall 1992] Proposed by David Iny, Baltimore, Maryland.  
Evaluate the integral

$$\int_0^1 \frac{y-1}{\sqrt{y} \ln y} dy.$$

I. Solution composed from those submitted by Paul S. Bruckman, Everett, Washington, and George P. Evanovich, St. Peter's College, Jersey City, New Jersey.

Let the integrand be denoted by  $f(y)$ . Using L'Hôpital's rule, we see that

$$\lim_{y \rightarrow 1^-} \frac{y-1}{\ln y} = \lim_{y \rightarrow 1^-} \frac{1}{1/y} = 1,$$

so  $\lim_{y \rightarrow 1^-} f(y) = 1$  and  $f$  is continuous at  $y = 1$ . Accordingly, we make the definition

$$(1) \quad I_n = \int_{e^{-2n}}^1 f(y) dy, \quad n = 1, 2, 3, \dots$$

Now if  $\lim_{n \rightarrow \infty} I_n$  exists, then this limit must be equal to the given integral, which we denote as  $I$ .

We make the substitution  $y = e^{-2x}$  in (1). Then  $dy = -2e^{-2x} dx$ , and we obtain

$$I_n = \int_0^n \frac{e^{-x} - e^{-3x}}{x} dx, \text{ so } I = \int_0^\infty \frac{e^{-x} - e^{-3x}}{x} dx,$$

if it exists.

Now, more generally, we let

$$F(a, b) = \lim_{n \rightarrow \infty} \int_0^n \frac{e^{-bx} - e^{-ax}}{x} dx.$$

Then

$$\frac{\partial F(a, b)}{\partial b} = \lim_{n \rightarrow \infty} \int_0^n (-e^{-bx}) dx = \lim_{n \rightarrow \infty} \frac{1}{b} e^{-bx} \Big|_0^n = -\frac{1}{b}.$$

Therefore

$$F(a, b) = -\int \frac{1}{b} db = -\ln b + g(a) + C$$

for some function  $g$  and constant  $C$ . Since  $F(b, a) = -F(a, b)$ , it follows that  $g(a) = \ln a$ . Also  $F(a, a) = 0 = \ln a - \ln a + C$ , so  $C = 0$ . That is,

$$F(a, b) = \ln a - \ln b = \ln \frac{a}{b}, \text{ whence } I = \ln 3.$$

**II. Solution by Murray S. Klamkin, University of Alberta, Edmonton, Alberta, Canada.**

More generally, we evaluate

$$I(m, n) = \int_0^1 \frac{y^m - 1}{y^n \ln y} dy.$$

Let  $y = e^{-x}$ , so that

$$I(m, n) = \int_0^w \frac{e^{-(1-n)x} - e^{-(m+1-n)x}}{x} dx.$$

This is a Frellani integral and it is known that

$$\int_0^\infty \frac{F(ax) - F(bx)}{x} dx = F(0) \cdot \ln \frac{b}{a},$$

provided the integral exists and that

$$\lim_{n \rightarrow \infty} \int_{n/b}^{n/a} \frac{F(bx)}{x} dx = 0.$$

Hence  $I(m, n) = \ln [(m+1-n)/(1-n)]$ . For the given problem,  $I(1, 1/2) = \ln 3$ .

For the case when  $\lim_{x \rightarrow \infty} F(x) = F(\infty)$ , then Elliot has shown that

$$\int_0^\infty \frac{F(ax) - F(bx)}{x} dx = [F(\infty) - F(0)] \ln \frac{a}{b}.$$

As an example,

$$\int_0^\infty \frac{\tan^{-1} ax - \tan^{-1} bx}{x} dx = \frac{\pi}{2} \ln \frac{a}{b}.$$

Also solved by BORIS BAEUMER, Louisiana State University, Baton Rouge, SEUNG-JIN BANG, Seoul, Korea, FRANK P. BATTLES, Massachusetts Maritime Academy, Buzzards Bay, SCOTT H. BROWN, Auburn University, AL, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, HARRY SEDINGER, St. Bonaventure University, NY, STAN WAGON, Macalester College, St. Paul, MN, J. ERNEST WILKINS, JR., Clark Atlanta University, GA, and the PROPOSER.

Several solvers found references to the integrals  $F(a, b)$  of Solution I and  $I(m, n)$  of Solution II: Sokolnikoff, Advanced Calculus, p. 364, Gradshteyn & Ryzhik, Table of Integrals, Series, and Products, 4th ed, Academic Press, 1965, Formula 8, p. 543, Borden, A Course in Advanced Calculus, North-Holland, 1983, Problem 37, p. 364. A common method of solution for integral  $F(a, b)$  was to rewrite it as a double integral and then apply Fubini's theorem to reverse the order of integration.

**790. [Fall 1992]** Proposed by Florentin Smarandache, Phoenix, Arizona.

In base 6 how many digits does the  $n$ th prime contain?

Solution by Paul T. Bateman, University of Illinois, Urbana, Illinois.

More generally, the number of digits of the number  $N$  to the base  $b$  is the integral part of  $1 + (\ln N)/(\ln b)$ . The known formula

$$n \ln n < p_n < n \ln n (1 + \delta(n)),$$

where  $\delta(n)$  is a positive quantity that approaches zero when  $n$  gets large, makes it possible to approximate the number of digits in the  $n$ th prime  $p_n$  within one unit for large  $n$ . That is, the number of digits in  $p_n$  to base  $b$  is given by the integral part of

$$1 + \frac{\ln(n \ln n)}{\ln b}$$

for large  $n$ .

Also solved by PAUL S. BRUCKMAN, Everett, WA, and DAVID E. MANES, SUNY at Oneonta.

**791.** [Fall 1992] Proposed by Seung-Jin Bang, Seoul, Republic of Korea.

Prove that  $2^n + 1$ , where n is a nonnegative integer, is never a multiple of 143.

I. Solution by Kenneth M. Wilke, Topeka, Kansas.

Suppose that  $2^n + 1 \equiv 0 \pmod{143}$  for some nonnegative integer n. Then n simultaneously satisfies both congruences  $2^n + 1 \equiv 0 \pmod{11}$  and  $2^n + 1 \equiv 0 \pmod{13}$ . By trial one finds that  $2^5 + 1 = 33 \equiv 0 \pmod{11}$ . That is,

$$2^5 \equiv -1 \pmod{11}, \text{ so } 2^{10} \equiv (-1)^2 \equiv 1 \pmod{11}.$$

Thus the congruence  $2^n + 1 \equiv 0 \pmod{11}$  is satisfied when  $n = 5 + 10r$  for any nonnegative integer r. By similar trial one finds that  $2^n + 1 \equiv 0 \pmod{13}$  is satisfied when  $n = 6 + 12s$  for any nonnegative integers. Hence we must have that  $5 + 10r = 6 + 12s$  for integral r and s. Since the left side of the equation is always odd while the right side is always even, there is no solution.

II. Solution by David E. Manes, State University of New York College, Oneonta, New York.

We disregard 0 since  $2^0 + 1 = 2$  is not a multiple of 143. The order of 2 modulo 143 is 60; i.e., 60 is the smallest positive integer t such that  $2^t \equiv 1 \pmod{143}$ . Also  $2^m \equiv 1 \pmod{143}$  if and only if m is a multiple of 60. Now assume there is a positive integer n such that  $2^n + 1 \equiv 0 \pmod{143}$ . Then  $2^n \equiv -1 \pmod{143}$ , which implies  $2^{2n} \equiv 1 \pmod{143}$ . Then  $2n = 60k$  for some positive integer k, or  $n = 30k$ . We have, however, that  $2^{30} \equiv -12 \pmod{143}$ . Accordingly,

$$2^2 = 2^{30k} = \begin{cases} 1 \pmod{143}, & \text{if } k \text{ is even} \\ -12 \pmod{143}, & \text{if } k \text{ is odd;} \end{cases}$$

that is,  $2^n \not\equiv -1 \pmod{143}$  for any positive integer n.

Also solved by CHARLES ASHBACHER, Cedar Rapids, IA, who calculated that  $2^{60} = 1152921504606846976$ , SCOTT H. BROWN, Auburn University, AL, PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., Cincinnati, OH, CHARLES R. DIMINNIE, St. Bonaventure University, NY, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, MURRAY S. KRAMKIN, University of Alberta, Canada, HENRY S. LIEBERMAN, Waban, MA, JOHN D. MOORES, Westbrook, ME, KANDASAMY MUTHUVEL, University of Wisconsin-Oshkosh, WILLIAM H. PEIRCE, Delray Beach, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, HARRY SEDINGER, St.

Bonaventure University, NY, REX H. WU, Brooklyn, NY, and the PROPOSER.

**792.** [Fall 1992] Proposed by Seung-Jin Bang, Seoul, Republic of Korea.

Given any thirteen distinct real numbers, prove that there exists at least one subset  $\{x, y, z\}$  of three of them such that

$$0 < \frac{(x-y)(y-z)(x-z)}{(1+xy)(1+yz)(1+xz)} < \frac{1}{3\sqrt{3}}.$$

Solution by Oxford Running Club, University of Mississippi, University, Mississippi.

The arctangents of the thirteen numbers are thirteen distinct numbers in the interval  $(-\pi/2, \pi/2)$ . Then some three of these must lie in one of the subintervals  $(-\pi/2, -\pi/3)$ ,  $(-\pi/3, -\pi/6)$ ,  $(-\pi/6, 0)$ ,  $(0, \pi/6)$ ,  $(\pi/6, \pi/3)$ ,  $(\pi/3, \pi/2)$ . Say that  $x > y > z$  are three of the original thirteen and that  $\arctan x > \arctan y > \arctan z$  are in the same subinterval. Then each of the differences

$$\arctan x - \arctan y, \quad \arctan y - \arctan z, \quad \arctan x - \arctan z$$

is positive and strictly less than  $\pi/6$ . Hence, by the increasing nature of the tangent function, each difference has a tangent between 0 and  $1/\sqrt{3}$ , so the product of their three tangents is less than  $1/(3\sqrt{3})$ . Since

$$\tan(\arctan x - \arctan y) = \frac{x-y}{1+xy}$$

and two similar relationships, the result follows.

Also solved by PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., Cincinnati, OH, and the PROPOSER.

**793.** [Fall 1992] Proposed by Dieter Bennewitz, Koblenz, Germany.

Given any trapezoid, its diagonals divide its interior area into four triangular areas: A and B adjacent to the parallel bases, and C and D adjacent to the nonparallel sides, as shown in the figure.

a) Prove that the areas C and D are equal and that  $A \cdot B = C \cdot D$ .

b) Find area C in terms of the lengths of the altitude and the bases of the trapezoid.

Solution by Richard I. Hess, Rancho Palos Verdes, California.

Let the upper and lower bases of the trapezoid have lengths a and b and let its altitude be h. Let the segment of the diagonal common to triangles A and D have length x and that common to B and C have length y. See the figure.

a) Since the triangle formed by  $B + C$  and that formed by  $B + D$  have the same base  $b$  and altitude  $h$ ,

$$B + C = B + D = bh/2$$

and hence  $C = D$ . Similarly,  $A + C = A + D = ah/2$ . Thinking of the diagonal  $x + y$  as base, then triangles  $B$  and  $D$  have a common altitude, so  $D/B = x/y$ , whence  $AB = CD$ . Similarly  $A/C = x/y$ , whence  $AB = CD$ .

b) Since  $AB = CD = C^2/A$ , then  $B = C^2/A$  and we have

$$B + C = \frac{C^2}{A} + C = \frac{bh}{2} \quad \text{and} \quad A + C = \frac{ah}{2}.$$

We solve these equations simultaneously to get that

$$C = D = \frac{ab}{2(a+b)}, \quad A = \frac{a^2h}{2(a+b)}, \quad \text{and} \quad B = \frac{b^2h}{2(a+b)}.$$

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., Cincinnati, OH, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, HENRY S. LIEBERMAN, Waban, MA, BARBARA J. LEHMAN, Brigantine, NJ, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, YOSHINOBU MURAYOSHI, Eugene, OR, HARRY SEDINGER, St. Bonaventure University, NY, REX H. WU, Brooklyn, NY, SAMMY YU and JIMMY YU (jointly), Vermillion, SD, and the PROPOSER.

**794.** [Fall 1992] Proposed by Peter A. Lindstrom, North Lake College, Irving, Texas.

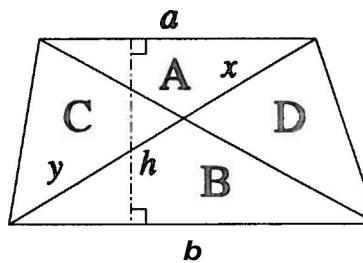
For  $-3 \leq x \leq 6$ , show that  $2\pi$  is equal to the sum of the zeros of

$$f(x) = \sin(x + \cos x).$$

Solution by George P. Evanovich, St. Peter's College, Jersey City, New Jersey.

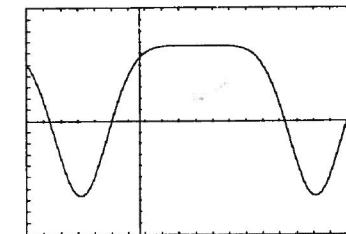
Because  $\cos(-1) \approx .54$  and  $\cos 0 = 1$  and the cosine function is continuous, there exists a number  $a$  such that  $0 < a < 1$  and  $\cos(-a) = a$ . (Actually,  $a \approx 0.739085$ .) Therefore  $-a$  is a zero of  $\sin(x + \cos x)$ . Also note that

$$\cos(a + \pi) = -a, \cos(-a + 2\pi) = a, \text{ and } \cos(a - \pi) = -a$$



## PROBLEMS AND SOLUTIONS

Thus  $-a$ ,  $a + \pi$ ,  $-a + 2\pi$ , and  $a - \pi$  are the zeros of  $\sin(x + \cos x)$  in the interval  $(-3, 6)$ . Their sum is  $2\pi$ . See the figure for the graph of  $y = \sin(x + \cos x)$  over this interval.



$$y = \sin(x + \cos x).$$

Also solved by SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, MARK EVANS, Louisville, KY, MURRAY S. Klamkin, University of Alberta, Canada, HENRY S. LIEBERMAN, Waban, MA, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, OXFORD RUNNING CLUB, University of Mississippi, University, BOB PRIELIPP (who supplied the figure), University of Wisconsin-Oshkosh, REX H. WU, Brooklyn, NY, and the PROPOSER.

**795.** [Fall 1992] Proposed by Russell Euler, Northwest Missouri State University, Maryville, Missouri.

Find all solutions on the interval  $[0, 2\pi]$  to

$$2\cos^3 x - 2\cos x + 1 = 0.$$

I. Solution by Henry S. Lieberman, Waban, Massachusetts.

The equation has no solutions. We have that

$$1 = 2\cos x - 2\cos^3 x = 2\cos x(1 - \cos^2 x) = 2\sin^2 x \cos x = \sin x \sin 2x.$$

Then  $\sin x = \sin 2x = \pm 1$ . Now  $\sin x = \pm 1$  for  $x = k\pi/2$  where  $k$  is an odd integer. Then  $\sin 2x = \sin 2k\pi/2 = 0$ . Hence, there are no solutions to  $\sin x \sin 2x = 1$ , and hence to the original equation also.

II. Solution by Rex H. Wu, Brooklyn, New York.

If  $t = \cos x$ , then we must solve

$$(1) \quad 2t^3 - 2t + 1 = 0, \text{ where } -1 \leq t \leq 1.$$

That is,  $t^3 = t - 112$ . However,  $t^3 > t - 112$  in the interval  $(-1, 1)$ . Hence the original equation has no real solution. In fact, the only real solution to equation (1) is

$$t = \left( -\frac{1}{4} + \sqrt{\frac{11}{432}} \right)^{1/3} + \left( -\frac{1}{4} - \sqrt{\frac{11}{432}} \right)^{1/3} \approx -1.1915 < -1.$$

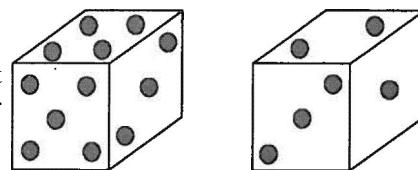
Also solved by CHARLES ASHBACHER, Cedar Rapids, IA, SEUNG-JIN BANG, Seoul, Korea, PAUL S. BRUCKMAN, Everett, WA, BILL CORRELL, JR., Cincinnati, OH, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, MURRAY S. KLAMKIN, University of Alberta, Canada, PETER A. LINDSTROM, North Lake College, Irving, TX, DAVID E. MANES, SUNY at Oneonta, JOHN D. MOORES, Westbrook, ME, OXFORD RUNNING CLUB, University of Mississippi, University, WILLIAM H. PEIRCE, Delray Beach, FL, BOB PRIELIPP, University of Wisconsin-Oshkosh, HARRY SEDINGER, St. Bonaventure University, NY, KENNETH M. WILKE, Topeka, KS, J. ERNEST WILKINS, JR., Clark Atlanta University, GA, and the PROPOSER.

796. [Fall 1992] Proposed by Michael W. Ecker, Clarks Summit, Pennsylvania.

a) A die is thrown until a prescribed face (e.g. say 3) shows. What is the mathematically expected number of throws required for this to occur?

b) Same question, but suppose a throw now consists of rolling 2 dice. In particular, should we expect this expectation to be half that of part (a)?

c) What is the smallest whole number of dice needed to constitute one throw, if we wish to have the mathematically expected number of throws required to roll our prescribed number not exceed 2?



*Solution by Charles Ashbacher, Cedar Rapids, Iowa.*

a) If an event has probability  $p$  and all trials are equally likely, then the expected number  $E$  of trials until the event occurs is given by

$$E = \sum_{k=1}^{\infty} kq^{k-1}p = \frac{1}{p},$$

where  $q = 1 - p$ . Hence the expected number of trials for a six-sided die to

show a 3 is  $1/(1/6) = 6$ .

b) With two dice the probability of throwing at least one 3 is  $1 - (5/6)^2 = 11/36$ , so  $E = 1/(11/36) = 36/11 \approx 3.3$  trials.

c) For three dice the probability is  $1 - (5/6)^3 = 91/216$ , so  $E = 216/91 \approx 2.37$ .

For four dice,  $p = 1 - (5/6)^4 = 67111296$ , so  $E = 12961671 \approx 1.93 < 2$ , so the answer is that four dice are necessary.

Note that the die you have in the sketch is wrong! You have 3 and 4 both showing on one die, but 3 and 4 are *always* on opposite sides of a die.

Also solved by ALMA COLLEGE PROBLEM SOLVING GROUP, MI, PAUL S. BRUCKMAN, Everett, WA, GEORGE P. EVANOVICH, Saint Peter's College, Jersey City, NJ, MARK EVANS, Louisville, KY, STEPHEN I. GENDLER, Clarion University of Pennsylvania, RICHARD I. HESS, Rancho Palos Verdes, CA, HENRY S. LIEBERMAN, Waban, MA, JOHN D. MOORES, Westbrook, ME, OXFORD RUNNING CLUB, University of Mississippi, University, HARRY SEDINGER, St. Bonaventure University, NY, and the PROPOSER. Only Ashbacher noticed the error in the die on the left: the top face should be a six, not a four. The right die is correct.

### Anecdotes Wanted

Mathematics is full of peoples' names: Fermat's theorem, Newton's method, the Euclidean algorithm, Bernoulli numbers, Euler's  $\phi$ -function, L'Hôpital's rule, Pell's equation, Gauss's lemma, ... .

However, the people behind the names are hardly known at all. Writers of mathematical biography tend to concentrate on their subjects' mathematical life and ignore the rest. Pick up a book of mathematical history at random and see if that is not true. Here is an excerpt from the middle of page 154 of *Hilbert*, by Constance Reid (Springer-Verlag, New York, 1970):

The next summer Hilbert lectured on relativity theory as part of a University series for all of the faculties.

In the middle of page 154 of *Niels Henrik Abel*, by Oystein Ore (U. of Minnesota Press, 1957) we find

With very few acquaintances and low ebb in his purse Abel could do little else than write mathematics, and the last months in Paris turned out to be extremely fruitful. A few days after his great memoir had been submitted to the Institute, he completed a lesser paper on

equations, which he presented to **Gergonne's Annals**.

The middle of page 154 of Carl Friedrich Gauss, by G. Waldo Dunnington (Hafner, New York, 1955):

By January, 1832, he had thrown himself with all force into the investigation of magnetism, and by February of that year had succeeded in reducing the intensity of terrestrial magnetism to absolute units.

Page 154 (middle) of Joseph Fourier, by I. Grattan-Guinness (MIT Press, Cambridge, 1972):

On calculera de même la valeur de  $d$  pour le cas de quatre inconnues et on multipliera cette valeur par  
 $9^2/(9^2 - 7^2)$ ,  $11^2/(11^2 - 7^2)$ ,  $13^2/(13^2 - 7^2)$ , ...

It is hard to make mathematics human. Mathematicians mostly live dull lives, so colorful anecdotes bringing them alive as people are rare. It is a shame that mathematicians as people have been so neglected. (If you doubt that they have been neglected, can you tell which of Lagrange, Laplace, Legendre, L'Hôpital, and Lhulier was the tallest, or had the most children? Can you even tell them apart?) The names of mathematicians are not the names of people, they are the names of gods who produce theorems.

This is too bad because mathematicians were and are people, and mathematics is a human activity. When we cannot make a connection between our subject and the **humans** who were and are responsible for it, mathematics can be viewed as inhuman, artificial, sterile, and boring. In fact, it is so viewed by a rather large number of people.

To fix this, we need a supply of anecdotes about mathematicians. The state of mathematical anecdotes is now so bad that even the false anecdotes are no good. For **example**, you can find in print in more than one place the story of how, when he was old, De Moivre each night slept for fifteen minutes **more** than the previous night until he slept the clock around; then he died. It was clearly made up by a non-mathematician, and not by a clever **non-mathematician**. It is obviously false and pointless even if true. The often-printed anecdote about Euler's algebraic proof of the existence of God is another example. Why would Euler deliver such nonsense as " $(a^n + b)/c = n$ ; hence God exists."? Why would Diderot who according to the anecdote knew no **mathematics** (not true) have consented to listen? Ridiculous! That such feeble stories should gain acceptance and be constantly repeated shows how easy it is for counterfeit anecdotes to get into circulation and stay there.

We need good anecdotes about **mathematicians**, illustrating their human qualities. We have George Washington chopping down the cherry tree, but we have nothing similar for Euler. Benjamin Franklin flew a kite: what did Gauss

## ANECDOTES WANTED

do?

Since the historians of mathematics are not going to supply us with what we need, we must turn elsewhere. Let us turn to the readers of the *Pi Mu Epsilon Journal*. This is an appeal for you to make up anecdotes about mathematicians. Length and subject are immaterial as long as the anecdote is **good**. It should be memorable, it should illustrate something, it can be funny, touching, sentimental, stirring: anything as long as it is **good**. It does not have to be true. Many of the best anecdotes, for example G. W. and the cherry tree, are not. Nevertheless, they serve valuable purposes.

The best anecdotes received will be printed in a future issue of the *Journal*, either anonymously or with attribution, as their authors choose.

## The 1993 National Pi Mu Epsilon Meeting

The meeting took place at the summer meeting of the American Mathematical Society, the Mathematical Association of America, and the Canadian Mathematical Society, in Vancouver, British Columbia, from August 15 to 19, 1993.

There were **thirty-three** student papers delivered in five sessions:

Upper chromatic numbers, by Aaron Abrams (University of California)  
Optimal material layout in a problem of heat transfer, by Ray V. Adams (Worcester Polytechnic Institute)

Math anxiety, by Dawn Boyung (St. Norbert College)  
The analytic hierarchy process with Bayes' Theorem, by Frank Castro (Youngstown State University)

An introduction to the theory of  $K_2(R)$ , by John Davenport (Miami University)

Functional integrals in a theory of absolute integration, by Anthony F. De Lia (University of Central Florida)

Riemann Zeta Function on the distribution of prime numbers, by Rondel DeLong (Marshall University)

A marble drop method for solving linear programs, by Vladimir Dimitrijevic (Youngstown State University)

Math methods down under, by Sandra S. Gestl (St. Norbert College)

Sailing down the river of  $3x^2 + 6xy - 5y^2$ , by Francis Fung (Kansas State University)

Bayesian probability and credibility theory in insurance **ratemaking**, by

Jennifer Garrett (Miami University)

Differential hyperbolic geometry, by Lauren D. **Hartman** (Washington and Lee University)

Data structures in the implementation of the **Huffman** algorithm, by Jon Hester (Hendrix College)

Allocations for matching games on weighted graphs, by Jennifer Howes (Drew University)

The economics of exhaustible resources, by Benjamin Keen (Miami University)

Stokes<sup>7</sup> Theorem and its application, by Deborah Kellogg (East Carolina University)

A matrix-balancing problem, by Julie Labbiento (Youngstown State University)

**Hilbert's** seventeenth problem, by Pasquale **Lapomarda III** (College of the Holy Cross)

Some proofs without words, by Cheryl **McClellan** (Youngstown State University)

One sample study of variance focusing on type I error, by Julie Mullett (Miami University)

The life table approach in determining actuarial mortality, by Umagasan C. Naidoo (Miami University)

Automatic differentiation, by Mai Nguyen (Miami University)

Dead horses in the desert, by Kathryn Nyman (Carthage College)

A combinatorial queuing model related to the ballot problem, by David C. Ogden (Wichita State University)

Three proofs of **Kaplansky's** Theorem, by Thomas **Peppard** (John Carroll University)

**Jentzsch's** Theorem in two complex variables, by **Xiaoling** Qian (University of Illinois)

The Shapley value and partially defined games, by Jennifer **Rich** (Drew University)

A generalization of triangular numbers, by Bonnie A. Sadler (East Carolina University)

Implementation of diva procedure calls on a ring of processors, by Scott Shauf (University of Richmond)

Continued fractions, by **Traca** Slusher (Youngstown State University)

Values of games in partition function form, by Maria Theoharidas (Drew University)

Matrices and AIDS, by Jeffrey A. Wallace (University of West Florida)

A study of the representations of even numbers as the sum of two primes, by Joel M. Wisdom (University of Tennessee, Chattanooga).

Some of these papers, the editor hopes, will appear in forthcoming issues of the *Journal*.

Five prizes, for papers of unusual merit, were awarded to Vladimir Dimitrijevic, Jennifer Garrett, Lauren D. **Hartman**, Jon Hester, and Joel M. Wisdom.

At the meeting of the Pi Mu Epsilon Council, it was announced that the National Security Agency had again granted Pi Mu Epsilon \$5000 for the support and encouragement of student speakers and that the American Mathematical Society had contributed \$1000 towards prize awards. In addition, a donor who wished to remain anonymous has made a contribution that will more than double the fund for the Richard V. **Andree** Awards, given to the best student papers that appear in the *Journal*. The Council expressed gratification and thanks for the support. In other business, the Council decided to increase the cost of Pi Mu Epsilon pins from \$8 to \$12, but to leave all other fees unchanged.

After the annual Pi Mu Epsilon banquet, an inexpensive and well-attended event, the J. Sutherland Frame lecture was delivered by Professor George E. Andrews of Pennsylvania State University. His topic was "Ramanujan for students" and with energy and clarity he went from how Fibonacci numbers could aid travelers in Canada to properties of continued fractions, giving the impression, as skilled lecturers can, that mathematics is really a simple subject.

The next meeting of Pi Mu Epsilon will take place in conjunction with the summer meeting of the MAA and **AMS** in Minneapolis, August 15-17, 1994.

### Editorial Statement

The *Journal* is always seeking manuscripts from student members of Pi Mu Epsilon.

The main purpose of the *Journal* is to interest and inform its readers, who are mostly undergraduates or recent graduates. Thus, the results of specialized research, of interest to experts in a field, are not in general appropriate for the readers of the *Journal* who, along with its editor, are experts in **no** Geld. Papers should give background information and place the results of research in context. The audience that authors should keep in mind is a group of bright young mathematicians who know next to nothing about your area of expertise, but who are able and willing to learn.

## Miscellany

Those "problems" of the form "Find the next number in the following sequence" are not problems in the usual mathematical sense at all. You may think that the next number in the sequence 2, 4, 6, 8, ... is 10, but you are wrong. It is  $\pi$ , because the formula that I had in mind when writing the terms for  $n = 1, 2, 3, 4$  was

$$2n + \frac{\pi - 10}{24} (n - 1)(n - 2)(n - 3)(n - 4).$$

The problems would be better stated as, "What was in the mind of the author as the following sequence was being written?" This of course is a problem of psychology and not of mathematics.

Nevertheless, such problems can be entertaining, especially if you can guess the next number. The British periodical *Eureka*, a publication of the undergraduate mathematical society at Cambridge University, had a tradition of posing such problems. Here is a selection of 40-year-old sequences, which incidentally illustrates the timeless and eternal nature of mathematics. They range from the fairly obvious to the completely impossible, which is why solutions will be given in the next issue of the Journal.

1, 3, 6, 10, 15, ...	(#17, 1954)
1, 2, 4, 8, 1, 6, 3, 2, 6, ...	(#11, 1949)
0, 1, 3, 7, 15, 31, ...	(#17, 1954)
3, 5, 11, 13, 17, 19, 29, ...	(#18, 1955)
3, 4, 6, 8, 12, 14, 18, ...	(#19, 1957)
4, 6, 9, 10, 14, 15, 21, 22, 25, 26, 33, ...	(#17, 1954)
3, 2, 1, 7, 4, 1, 1, 8, 5, 2, 9, 8, ...	(#13, 1950)
1, 15, 29, 12, 26, 12, 26, 9, 23, ...	(#13, 1952)
5, 11, 15, 16, 17, 18, 23, 25, ...	(#18, 1955)

While looking through old issues of *Eureka*, I discovered another reason why students should publish papers in the Journal. In #17 (1954) there was a problem by Roger Penrose, then (I think) a student, now eminent in the extreme (deviser of Penrose tiles, author of the recent best-seller *The Emperor's New Mind*, Rouse Ball Professor of Mathematics at Oxford, etc.). Here it is (it is not trivial): in a semigroup, where multiplication is associative but there are no inverses, you are given that  $aba = a$ ,  $bab = b$ ,  $ab = ba$ , and  $ac = ca$ . Show that  $bc = cb$ . Though the problem may be difficult, the conclusion to be drawn from its existence is easy: if Roger Penrose published while a student, and Roger Penrose has gone on to impressive achievements, then if you are a student, it follows that ... .

## Eighth Annual

# MORAVIAN COLLEGE STUDENT MATHEMATICS CONFERENCE

Bethlehem, Pennsylvania

Saturday, February 26, 1994

We invite you to join us, whether to present a talk or just to listen and socialize. The invited speaker will be Diane Souvaine, Acting Director of DIMACS (The Center for Discrete Mathematics and Theoretical Computer Science), and associate professor of Computer Science at Rutgers University. Her topic will be "Geometric Computation and Applications." The conference will start at 9:00 a.m. and continue into late afternoon. After the morning invited address, the rest of the day will be devoted to student talks. Talks may be fifteen or thirty minutes long. They may be on any topic related to mathematics, operations research, statistics or computing. We encourage students doing research or honors work to present their work here. We also welcome expository talks, talks about interesting problems or applications and talks about internships, field studies and summer employment. We need your title, time of presentation (15 or 30 minutes) and a 50 word (approximate) abstract by February 18, 1994.

Sponsored by the Moravian College Chapter of Pi Mu Epsilon and the Lehigh Valley Association of Independent Colleges.

Please contact:

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