Data mining and statistical learning approaches for the Auto dataset, including visualization of decision boundary for predictive models

Bradley Wallace

1 ABSTRACT

Analysis of the well-known *auto-mpg* dataset in machine learning and data mining literature was undertaken using a range of classification machine learning architectures. This dataset contains fuel efficiency, mechanical and structural characteristics for a range of American, European (German and French), and Japanese automobiles from 1970-1982. In general, the machine learning models were used to predict the fuel efficiency (mpg: miles per gallon) of different automobiles. The performance of several classical machine learning models was compared to two renowned ensemble methods (Random Forest and AdaBoost classifiers). Further, Monte Carlo Cross Validation (CV) was performed to both: tune hyperparameters of model architectures; and undertake a sensitivity analysis of model error when models are exposed to resamples of the dataset. Stepwise Regression methods, PCA and application of domain expertise were utilized to select a 'core' set of predictor variables and reduce model complexity. Results of study concluded that the ensemble methods yielded better performance than other classical 'standalone' machine learning models, using 1000-CV Monte Carlo resamples. In particular, it was found that the ensemble methods yield more dynamic and accurate decision boundaries when the data is visualized in two dimensions. It was noted that ensemble methods have significantly longer runtimes due to their inherent architecture and thus, the trade-off between performance of ensemble methods versus computational runtime and resource usage was identified as an opportunity for further research.

2 INTRODUCTION

The topic of fuel efficiency of internal combustion engine (ICE) automobiles has been a primary concern for both consumers and industry since the widespread adoption of automobiles in modern society. On one hand, end consumers pursue a trade-off between power and performance with fuel efficiency (range), and the automobile manufacturing industry endeavors to both satisfy and influence these demands. On the other hand, the oil production industry has vested interests in keeping fuel consumption high and may have lobbied and interfered with technological progress and regulations relating to ICE automobile design and distribution. In saying that, as detailed by Sivak (2019), thankfully widespread technological progress has prevailed and we find that fuel economy of cars and light trucks has generally trended upward over time. Refer *Figure 1* below.

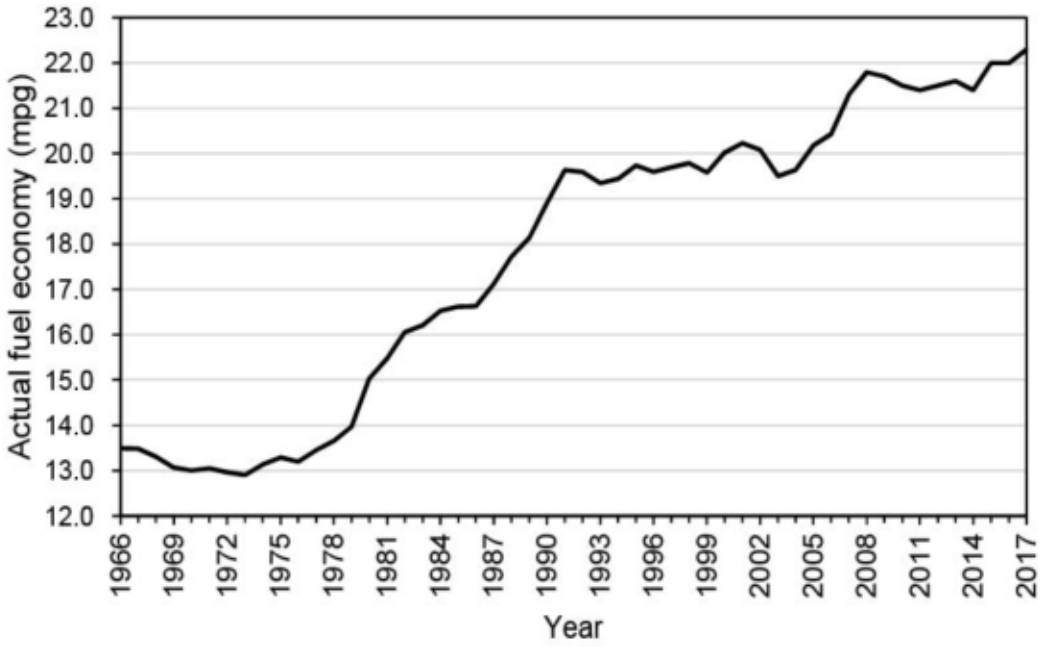


Figure 1 — Automobile Fuel Economy (Efficiency) vs. Year, 1966-2017 (Sivak M., 2019).

This study aimed to explore this topic further using the famous *auto-mpg* dataset which contains fuel efficiency, mechanical and structural characteristics for a range of American, European and Japanese automobiles from 1970-1982. The study endeavored to both: discover the mechanical and structural characteristics which are most significant and useful in inferring fuel efficiency; and develop

machine learnings that can accurately predict fuel efficiency, using binary labels of low and high, by utilizing these significant characteristics as inputs.

A range of statistical and machine learning methods were used in this study. The methods used includes (in this order): Correlation analysis and Stepwise search using AIC (Akaike Information Criterion) for Linear Regression (for variable selection only), Linear Discriminant Analysis (LDA), Logistic Regression (Logit), K-Nearest Neighbors (KNN), Classification And Regression Tree (CART), Random Forest Classification, and AdaBoost Classification, and Principal Component Analysis (PCA, for dimensionality reduction). Monte Carlo CV was used to tune classification model hyperparameters and evaluate the robustness (or sensitivity) of the supervised learning architectures.

Data mining challenges were encountered at various stages of the analysis, as detailed below.

In the exploratory data analysis and model formulation stages, the following issues were encountered: the high covariance shared between some predictor variables, the insignificance of some predictor variables, the relationship of automobiles being split across three manufacturing regions (USA, Europe and Japan) and how the models can deal with this (i.e. Can the relationship be represented as a single ordinal integer variable, or does each region need to be treated as a dummy variable, as per a categorical variable?). These issues were addressed using a combination of statistical methods, balanced by domain expertise and analytical interpretation of results. This led to a 'core' set of predictors being used as inputs in the later machine learning model architectures investigated.

In the coding of the machine learning models, there were great syntactical challenges as to how to best run a successful Grid Search using Monte Carlo CV across a range of hyperparameters for each machine learning model architecture investigated. Robust problem solving and logic were applied, and a successful process to perform Monte Carlo CV Grid Search was formulated. It was decided that for each Monte Carlo resample, the 'optimal' hyperparameter, i.e. lowest test error and lowest complexity, for each model architecture was to be recorded in a database. Then, the hyperparameters selection was counted

Finally, there were data mining challenges with how to best compare models and visualize results. Comparing model parameters, coefficients and weights are not

particularly insightful to the average reader, so it was decided to explore opportunities of visualize model decision boundaries in lower dimensions. In the case of this study, it was demonstrated that the 'core' set of predictor variables could be projected into two PCA directions successfully and logically. This allowed the results to be approximated into two PCA directions to further explore the performance and provide insight as to how the models work when classifying an automobile as having high or low fuel efficiency.

Results of study concluded that the ensemble methods (Random Forest and AdaBoost classifiers) yielded better performance than the other classical machine learning models (LDA, Logistic Regression, KNN and CART), using 1000-CV Monte Carlo resamples. The ensemble methods yielded lower average testing misclassification error rates and lower variance for these errors, across all 1000-CV Monte Carlo resamples. Further to this, it was also found that the ensemble methods yield more dynamic and accurate decision boundaries when the data is visualized in two dimensions. In particular, AdaBoost is able to segment isolate clusters of classification (e.g. some low fuel efficiencies automobiles surrounded by a larger cluster of high fuel efficiency vehicles).

During the computation of these machine learning models, it was noted that ensemble methods have significantly longer runtimes due to their inherent architecture of being comprised of 50 or more standalone models. For example, the final Random Forest Classifier is composed of 75 un-pruned classification tree model voting in majority, whereas the final CART model is one pruned classification tree, and thus, it makes logical sense that the computational time of the Random Forest Classifier is an order of magnitude larger than the single classification tree. Long runtimes were also experienced for the final AdaBoost classifier comprised of 200 decision stumps (or base functions). Therefore, the trade-off between performance of ensemble methods versus computational runtime and resource usage was identified as an opportunity for further research.

3 PROBLEM STATEMENT AND DATA SOURCE

The purpose of this homework is to analyze the well-known *auto-mpg* dataset in machine learning and data mining literature from Quinlan (1993) and Carnegie Mellon University (available from: *https://archive.ics.uci.edu/ml/datasets/auto+mpg*), using a range of classification machine learning architectures,

as well as using Monte Carlo CV to tune model hyperparameters and perform a sensitivity analysis of model error when models are exposed to resamples of the dataset.

The *auto-mpg* dataset contains fuel efficiency, mechanical and structural characteristics for a range of American, European (German and French), and Japanese automobiles from 1970-1982. The dataset is comprised of 398 observations (rows) and 9 variables (1 response and 8 predictor variables), as described below. The document-provided descriptions of variables were fairly limited, and so, further investigation and domain knowledge was required to adequately describe these variables.

Variable descriptions:

1. **mpg** *(response)*: continuous (miles per gallon i.e. fuel efficiency)
2. **cylinders**: multi-valued discrete (number of cylinders in IC engine)
3. **displacement**: continuous (engine size, as volume)
4. **horsepower**: continuous (a common measure of engine power)
5. **weight**: continuous (lbs)
6. **acceleration**: continuous (assumed to be seconds)
   (This is presumably the time taken to go from 0-100 mph ~or something like this i.e. the higher this value is, the slower the car is)
7. **model year:** multi-valued discrete (years in format 19YY <70 to 82>, the dataset is from 1993, so YYYY notation is shorted to 19YY)
8. **origin:** categorical, 3 levels
   (From investigation of 'car name', this is actually a categorical variable, where 1 = US, 2 = Europe, 3 = Japan)
9. **car name:** string (unique for each instance)

After preliminary investigation into the data set, the *auto-mpg* dataset essentially aims to describe miles-per-gallon (i.e. fuel efficiency), using four measures of car engine power and acceleration, car weight, year of manufacture and country of origin. The years of manufacture range from 1970 to 1982. The original study was published in *1983 American Statistical Association Exposition*, but then donated to open research 10 years later in 1993. It has been featured in a range of noteworthy conference papers since this time. These papers are referenced on the source page.

In addition, the provided data set is a cleaned data set that has had 8 observations removed due to unknown response variable values ('mpg') i.e. 416 observations reduced to 398 observations. No imputation of the missing values is undertaken in this investigation, as these observations (rows of data) are already removed.

4 METHODOLOGY

The methods used include (in this order): Correlation analysis and Stepwise search using AIC (Akaike Information Criterion) for Linear Regression (for variable selection only), Linear Discriminant Analysis (LDA), Quadratic Discriminant Analysis (QDA), Gaussian Naïve Bayes (GNB), Logistic Regression (Logit), K-Nearest Neighbors (KNN), Principal Component Analysis (PCA, for dimensionality reduction), and finally, Classification And Regression Tree (CART). As stated, the robustness (or sensitivity) of these supervised learning architectures are then further explored using Monte Carlo CV.

The methodology was implemented as per Python code in the *Appendix*. In particular, the various classification models are all implemented using the respected machine learning library in Python, *scikit-learn*. Variable selection via stepwise regression is implemented from scratch using *statsmodels.api* as a core to output statistical measures. Furthermore, other Python libraries are utilized: *pandas* and *numpy* are used for matrix, array and table operations, and *matplotlib.pyplot* and *seaborn* are used for plots and visualizations.

Steps in the methodology can be summarized as follows:

1. Based on the correlation analysis among variables, and their potential redundancy if all used as input predictor variables in our models, undertake predictor variable selection using:
    - The thorough correlation analyses and application of domain knowledge undertaken in *Exploratory Data Analysis*; and
    - Stepwise search using AIC (Akaike Information Criterion) for Linear Regression on the entire data set, and investigating the selected predictor variables.
2. Split the dataset into an 80% training set and a 20% testing set for all Monte Carlo Cross Validation and final model re-training. Justification for this split are as follows: most of the data (80%) is used for training the models and a reasonable remaining portion (20%) is reserved for evaluating the models'

performances; random split reduces potential order bias that may be present in the data set; the 80/20 split allows conveniently for 5-fold Cross Validation, if required; and the 80/20 split is pedagogical standard that is repeated many times in our course work. It is also commonly seen across studies and is considered an industry standard.

3. Perform Monte Carlo 200-CV Grid Search on a range of hyperparameters for six different classification models on the training set and evaluate their performance on the testing set. The test error is the misclassification rate of the predictions versus the actual label (response). This test error is used to select the 'optimal' (lowest error and lowest complexity) hyperparameter for each model, for each Monte Carlo resample. The six classification model architectures, and their associated tuning hyperparameters are as follows:
    1. **Linear Discriminant Analysis (LDA)**, shrinkage rate;
    2. **Logistic Regression (Logit)**, **l-1** ratio (**l-1**: LASSO, **l-2**: Ridge);
    3. **K-Nearest Neighbors (KNN)**, k number of neighbors;
    4. **Classification And Regression Tree (CART)**, tree depth;
    5. **Random Forest Classification**, number of trees and tree depth;
    6. **AdaBoost Classification**, number of estimators and learning rate.
4. Based on the Monte Carlo 200-CV Grid Search results for finding the 'optimal' hyperparameters for each of the six classification models, tabulate results and retrain the final model models using those 'optimal' hyperparameters values which yield the lowest test error the most often.
5. In order to evaluate the robustness (specifically sensitivity to splits of training and test sets) of these six models, undertake Monte Carlo 1000-CV and train 100 models i.e. (B=100) for each classification model architecture detailed in Step #3. Store the testing error rate for each model (there will be 6000 models, 6 model architectures * 1000-CV data splits). Then, calculate the average testing error (misclassification rate) across each of the six model architectures (an average of 1000 models each). Finally, calculate the testing error variance by summing the squared differences of testing errors and scaling it by 1/B-1. Of particular note, each of the six model architectures is trained and tested on the same Monte Carlo resample at each step of the outer loop in the Python code.
6. Visualize the decision boundaries of each model to help explain their predictive performance for classification. It is not particularly insightful to compare the coefficients, hyperparameters and weights of the different model

architectures that have substantially different underlying architectures. In order to do achieve this, each of the model architectures is projected into 2 principal components for visualization, and a decision boundary is created and visualized by calculating model predictions across a x-y grid (PC1 and PC2).

4.1 Particular methodology comments

- *scikit-learn* can be used to implement all but one of model architectures, the stepwise search using AIC. This is because the scikit-learn tends to shy away from statistical inference methods. Stepwise search using AIC was written mostly from-scratch utiizing Ordinary Least Squares (OLS) regression in *statsmodel.api* to output statistical measures.
- The 2-dimensional projections of each classification model's decision boundary is not strictly the same as the multi-dimensional models for which testing error (misclassification rate) is calculated. They are as close-as-possible approximations for the purpose of visualizations.

5 DATA CLEANING

Before performing exploratory data analysis, it is necessary to create binary variable of the *'mpg'* (miles per gallon) variable to use in our range of classification models. This binary variable *'mpg01'* shall be the response variable for the models used in this investigation. *'mpg01'* equals 1 if the corresponding *'mpg'* value for a given car (row or observation) is above the median value of *'mpg'* (which is 22.75), otherwise it is 0. The logic can be summarized as follows:

- *'mpg01'* = 1 -> the car is average/above average (me in fuel efficiency
- (in regards of the scope of cars in the data set)
- *'mpg01'* = 0 -> the car is below average in fuel efficiency (in regards of the scope of cars in the data set)

Finally, *'car name'* is dropped from the data set because this text/string data is not useful to the numerical analysis in this investigation. However, it is worth noting that 'car name' was originally useful to determine the categorical variable *'origin'* = 1, 2, 3 descriptions of US, Europe, Japan, respectively.

Please note that *'mpg'* is not used to train or make predictions with any classification models as it is continuous. Instead, the binary variable *'mpg01'* is used.

6 EXPLORATORY DATA ANALYSIS

6.1 Value ranges of continuous and discrete predictor variables

Value ranges, mean, standard deviation of the continuous and discrete predictor variables can be seen below (*Table 1*). *'origin'* and *'mpg01'* are omitted from this analysis. Please note that *'origin'* is a 3-level categorical variable, where 1 = USA, 2 = Europe and 3 = Japan, and *'mpg01'* is as described above.

Table 1 — Value ranges of continuous and discrete predictors

	cylinders	displacement	horsepower	weight	acceleration	year
count	392.000000	392.000000	392.000000	392.000000	392.000000	392.000000
mean	5.471939	194.411990	104.469388	2977.584184	15.541327	75.979592
std	1.705783	104.644004	38.491160	849.402560	2.758864	3.683737
min	3.000000	68.000000	46.000000	1613.000000	8.000000	70.000000
25%	4.000000	105.000000	75.000000	2225.250000	13.775000	73.000000
50%	4.000000	151.000000	93.500000	2803.500000	15.500000	76.000000
75%	8.000000	275.750000	126.000000	3614.750000	17.025000	79.000000
max	8.000000	455.000000	230.000000	5140.000000	24.800000	82.000000

6.2 Correlation analysis of the training data

Correlation coefficients are indicators of the strength of the linear relationship between variables. Therefore, a correlation analysis using a correlation matrix is a highly useful tool for undertaking preliminary data analysis prior to developing the multiple linear regression models. *Figure 2* depicts the strength of correlation coefficients between the 7 predictor variables and single binary response variable, where a light is a strong positive correlation and dark is a strong negative (or inverse) correlation.

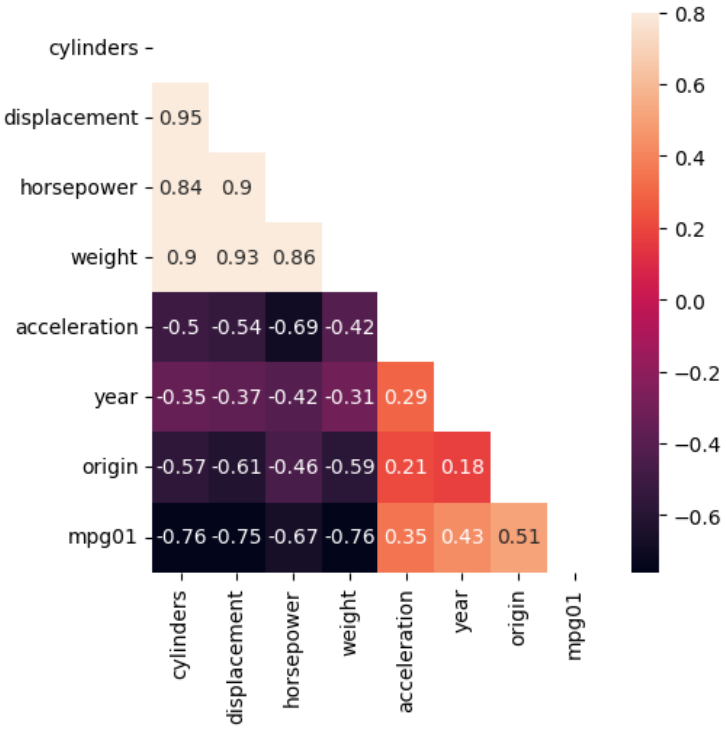


Figure 2 — Correlation matrix of the 7 predictor and 1 response ('mpg01') variables

It is worth noting now that there is particular convenience of defining *'origin'* with the following values, where 1 = USA, 2 = Europe and 3 = Japan. If the order had 2 = Europe not in-between 1 = USA and 3 = Japan, the relationship that we later see is no longer linear.

The correlation analysis can be summarized as follows:

- 'cylinders', 'displacement', 'horsepower' and 'weight' are inversely correlated with 'mpg01'.
- In descending order, 'origin', 'year' and 'acceleration' are positively-correlated with 'mpg01'.
- 'cylinders', 'displacement', 'horsepower' and 'weight' are all highly-correlated with one another. This group is inversely-correlated with 'acceleration', 'year' and 'origin'.
- 'acceleration', 'year' and 'origin' are poorly-correlated with one another.

Applying domain knowledge to make sense of these linear relationships, we find very logical explanations. *'mpg01'* represents fuel efficiency, and this logically makes sense to be inversely-related to the amount of 'cylinders' an engine has,

how large the engine is ('displacement'), how powerful the engine is ('horsepower') and the overall 'weight' of the automobile. It takes more fuel to fuel a larger engine, produce more power and move more weight. Similarly, the newer an automobile is ('year') and the slower a car accelerates ('acceleration'), the better fuel efficiency it is likely to have. The categorical predictor variable 'origin' requires some further investigation to explain, but is can be quickly summarized with the stereotype that US cars ('origin' = 1) are more powerful, larger/heavier and have lower fuel efficiency compared to Japanese cars ('origin' = 3), and, conveniently, European cars ('origin' = 2) lie somewhere in the middle of car engine power, size/weight vs. fuel efficiency. This leads to a convenient ordered (increasing or decreasing magnitude of other characteristics) relationship along the number line of this categorical variable. The in-depth analysis of 'origin' can be viewed in the *Appendix*.

6.3 Stepwise Regression and Final Predictor Variable Selection

It is useful to reduce the redundancy and intra-correlation amongst predictor variables in order to reduce our classification model complexity and reduce any risks that models might have to highly-correlated data (for example, LASSO regression has weaknesses with highly-correlated data). In particular, it is useful to utilize a machine learning algorithm to select which variables are statistically significant and balance this against domain expertise and judgement.

Stepwise search using AIC (Akaike Information Criterion) for Linear Regression was used on the entire data set to investigate the statistical significance of all predictor variables. The full analysis can be viewed in *Appendix*. After 5-steps the algorithm converged and found that 5 of the 7 predictor variables are significant for predicting 'mpg01'. They are 'cylinders', 'horsepower', 'weight', 'year' and 'origin'.

Therefore, for classification model implementation I will retain and drop the following predictor variables:

I will retain both 'cylinders' and 'horsepower' because: 'cylinders' has a piecewise or non-linear relationship underlying it and I want to capture this (this can be viewed in the box plot of 'cylinders' in *Appendix*); 'horsepower' is standard measure of power of an engine. Presumably, we should see more powerful

engines have lower fuel efficiency, and vice versa; In addition, these two variables were also selected as significant variables by the stepwise regression model.

I will drop 'displacement' (engine size) because 'cylinders' and 'horse power' are sufficient proxy predictor variables; it was not selected as a significant variable by the stepwise regression model. The high correlation between these predictor variables and the design philosophy that 'the size of an engine is directly proportional to its power' supports this logic.

I will retain 'weight' because this is the only proxy measurement of chassis and the weight of an automobile is not solely-determined by its engine size or power. Amenities, seating capacity, and so on, will all effect weight.

I will drop 'acceleration' because: it has the lowest correlation with 'mpg01' and 'mpg'; it was not selected as a significant variable by the stepwise regression model; and a check of the box plot of 'acceleration' in *Appendix* shows it has higher variability and less descriptive power than 'year'.

I will retain 'year' because: it has reasonable positive linear correlation with 'mpg01' and 'mpg'; it was selected as a significant variable by the stepwise regression model; and it can be explained with domain knowledge and advances in technology, that 'the newer the car is, the higher the fuel efficiency'.

I will retain 'origin' because: despite being categorical, holds value as the highest positively-correlated predictor variable; it was selected as a significant variable by the stepwise regression model; it is a positively linearly-correlated predictor variable solely because of the orientation of the categories, therefore, if in a different order where 'origin': 2 != Europe, it would capture non-linear relationships; and its richness as a categorical variable can be captured by some classification model architectures, such as a classification tree (CART) and Random Forest Classification.

7 ANALYSIS AND RESULTS

7.1 Hyperparameter tuning using Monte Carlo 200-CV Grid Search

The 'optimal' hyperparameters for each model, for each Monte Carlo resample was recorded. The most frequent 'optimal' hyperparameters were chosen as the final hyperparameters for each model architecture. Full results can be viewed in

Appendix. However, the top 3 frequency hyperparameter values for each model can be viewed in *Table 2* below.

As a result of this Grid Search, the final optimal hyperparameters selected are as follows: LDA, shrinkage = 0.0; Logit: l1-ratio = 1.0 i.e. Ridge; KNN, k: 1.0; CART, tree depth = 5; Random Forest, n trees = 5, tree depth = 6; and AdaBoost, n estimators = 200, learning rate = 0.35. These are highlight in blue in *Table 2* below.

Table 2 — Optimal hyperparameter results 6 classification models, for Monte Carlo 200-CV

***Hyperparameter values by top 3- frequency, more frequent is most 'optimal' value**

1. LDA Shrinkage	Frequency	**2. Logit** l1-ratio	Frequency	**3. KNN** k neighbors	Frequency
0	188	1.0	197	1.0	59
0.1	11	0.9	2	3.0	48
1	1	0.2	1	5.0	21
4. **CART** Tree depth	Frequency	**5. Random Forest** N trees	Frequency	Tree depth	Frequency
5	52	5	75	6	62
4	51	10	54	5	54
2	40	25	32	4	39
6. AdaBoost N estimators	Frequency	Learning rate	Frequency		
200	200	0.35	38		
		0.8	37		
		0.5	26		

7.2 Performance Sensitivity Analysis, Monte Carlo 1000-CV

In order to assess the performance of the six classification models, primarily with the view to compare the classical 'standalone' models (LDA, Logistic Regression, KNN and CART) with the two ensemble methods (Random Forest and AdaBoost classifiers), the models were trained and validated using 1000 Monte Carlo CV resamples of dataset, split into training and testing data. This yielded the following **mean testing error rates** and **testing error rate variance.** This helps us to understand the robustness of each model. Here, we take an average with variance for the 1000 models created for each classification model architecture. Refer *Table 3* below.

Table 3 — 1000 Monte Carlo CV Models results: 7 Classification models

	model num	mean test error rate	var test error rate
0	1: LDA	0.092772	0.000938
1	2: Logistic Reg	0.098671	0.000885
2	3: KNN	0.136241	0.001138
3	4: Classf Tree	0.093165	0.001013
4	5: Randm Frst Classf	0.086380	0.000876
5	6: AdaBst Classf	0.086646	0.000885

8 FINDINGS AND CONCLUSIONS

8.1 Performance of classification models

Reviewing the performance of the classification model results (refer *Table 3* above), we find that the mean test error and variance of the test error are very low for all classification models, <14% mean misclassification rate and <0.012 variance for misclassification rate. All models have test error and test error variance in the same order of magnitude. This suggests that all classification models selected are fairly robust models for this dataset. We do find, however, that KNN (k=1) has the worst performance (by about 50% more than all other models) with a mean misclassification rate of 13.6%, versus the 8.6-9.9% mean misclassification rate of the other models. KNN (k=1) also has the highest test error variance. Moreover, we find that the ensemble methods of Random Forest and AdaBoost classifiers yield the lowest mean test error rates and the smallest variances at

8.64% and 8.66% respectively. We manage to confirm an initial hypothesis that the ensemble methods should outperform the singular standalone models and we find that they are outperforming LDA, Logit and CART models by approximately 7% and KNN (k=1) by approximately 57%, in terms of relative mean test error rates.

The best performance being achieved by the ensemble methods should not come as too surprising given the fact that these methods average (majority vote) across a larger number of models, compared to the singular standalone models. The collective classification ability of an ensemble method is more complex and thus should be able to discern any classification boundaries (i.e. the true relationship) more accurately. The majority vote performed by the final Random Forest classifier composed of five unpruned classification trees is fairly straight-forward and it appears that the a majority vote by five classification trees out performs a single pruned classification tree. On the other hand, the AdaBoost classifier composed of 200 base functions (or stumps) is actually able to capture isolated clusters of certain classifications in a much more accurate way than KNN (k=1).

It was noted that ensemble methods have significantly longer runtimes due to their inherent architecture and thus, the trade-off between performance of ensemble methods versus computational runtime and resource usage could be an opportunity for further research.

8.2 'Optimal' Hyperparameter selection

The LDA model shrinkage parameter was 'optimal' at 0, which means the final LDA model just works as per its 'normal mode'. It assumes that the class covariance matrices are all equal and are estimated using the global sample covariance i.e. across the entire dataset. This means the potential complexity of the LDA model was reduced.

The Logit model l1-ratio was 'optimal' at 1, which means that the final Logit model found Ridge regression solutions which are sparse. This means the potential complexity of the Logit model was also fairly reduced.

Next, the KNN model k-neighbors parameter was 'optimal' at 1, which means the final KNN model simply selected the classification of the most 'similar' automobile in the training set. Logically, it likely isn't worthwhile running a KNN (k=1) model, as there is likely a simpler algorithm that achieves the same

outcome; however, intuitively, it makes sense to make estimations about an unknown automobile's fuel efficiency (i.e. a new data point) using the most 'similar' vehicle you already know about.

The CART model tree depth parameter was 'optimal' at 5 (and almost 4), and this was most likely the balancing point between having a tree that was too simple in its branching and structures that are too fine in resolution. I note that a tree depth of 2 was fairly successful, so it appears it was necessary for the CART model to branch by several more levels to achieve better performance.

Regarding the Random Forest classifier, it was interesting to find that the ensemble method performed most 'optimally' with a low number of decision trees, at 5 trees. It was also interesting to find that the 'optimal' tree depth value was 6, one level deeper than the CART model. As covered, a Random Forest's individual trees are unpruned, so it would appear the 5-tree Random Forest model needed more branching and depth in order to outperform the single CART decision tree model using a majority vote.

Finally, the AdaBoost classifier performed 'optimally' for all Monte Carlo CV resamples using 200 estimators (base functions or stumps), and with a learning rate of 0.35 (with 0.8 being a close second optimal). The learning rate of an AdaBoost classifier controls how much the previous iteration of boosting contributes to the current iteration, where vanilla AdaBoost simply sets the learning rate to 1. It is interesting that the 'optimal' values for the learning rate are 0.35 and 0.8. Further to this, the literature suggests that, for large amounts of estimators, a good rule-of-thumb is that the learning rate for boosting algorithms should be <0.1. Therefore, further exploration of AdaBoost classifiers with higher numbers of estimators and more tuning of the learning rate could be an opportunity for further research.

8.3 Visualization of Decision Boundaries

As the six classification models have predictive performance in the same order of magnitude, it is useful to visualize their decision boundaries using two principal directions in an attempt to understand what is going on under the hood. A thorough analysis of the two principal directions of the dataset using only 'cylinders', 'horsepower', 'weight', 'origin' and 'year' as predictor variables can be seen in *Appendix (end)*, and shows that the 1st principal direction corresponds to

'cylinders', 'horsepower', 'weight' and 'origin', and the 2nd principal direction corresponds to 'year'. Refer to the six figures below.

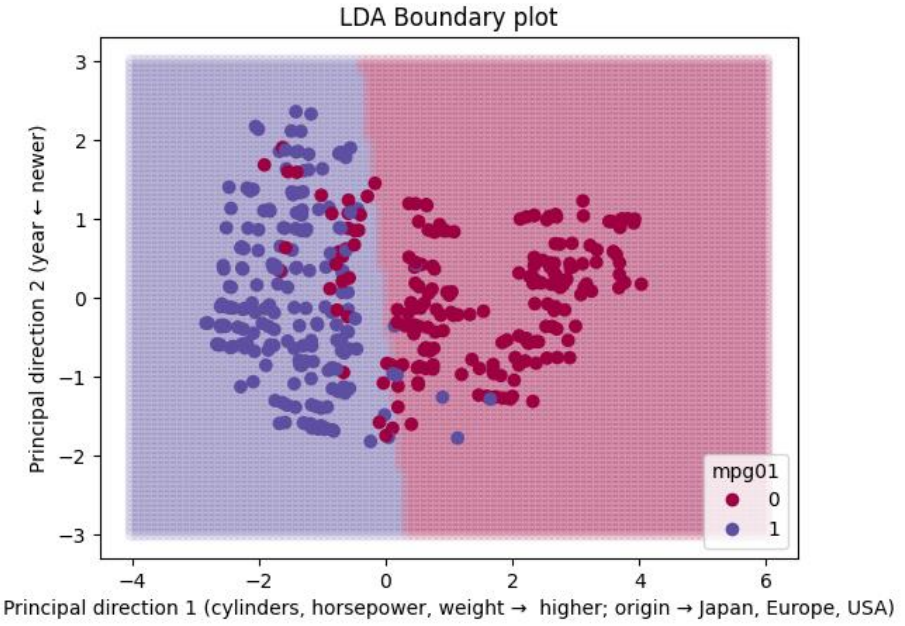
In the figures, the transparent areas are the decision boundaries and the points are the true labels for 'mpg01'. In general, we find that 'mpg01'=0 and 'mpg01'=1 (less fuel efficient and more fuel efficient) cars are not perfectly linearly-separable in PC 2-dimensional space. In fact, the transitionary regions between the two classification groups are fairly mixed.

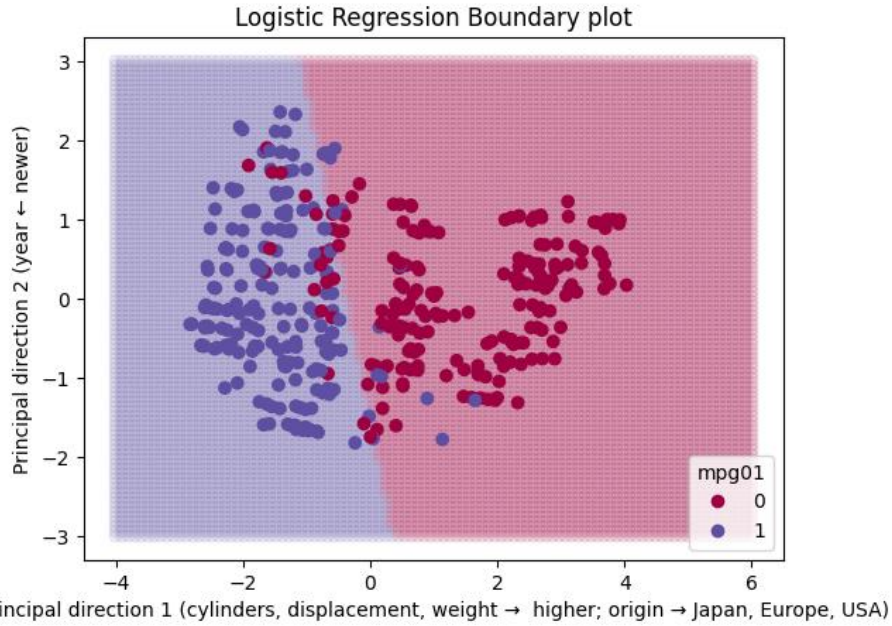
Interestingly, LDA manages to create a simple linear boundary that is the best singular model in terms of performance, and outperforms the linear boundary of Logistic Regression, with 9.28% versus 9.87% average misclassification rate, respectively.

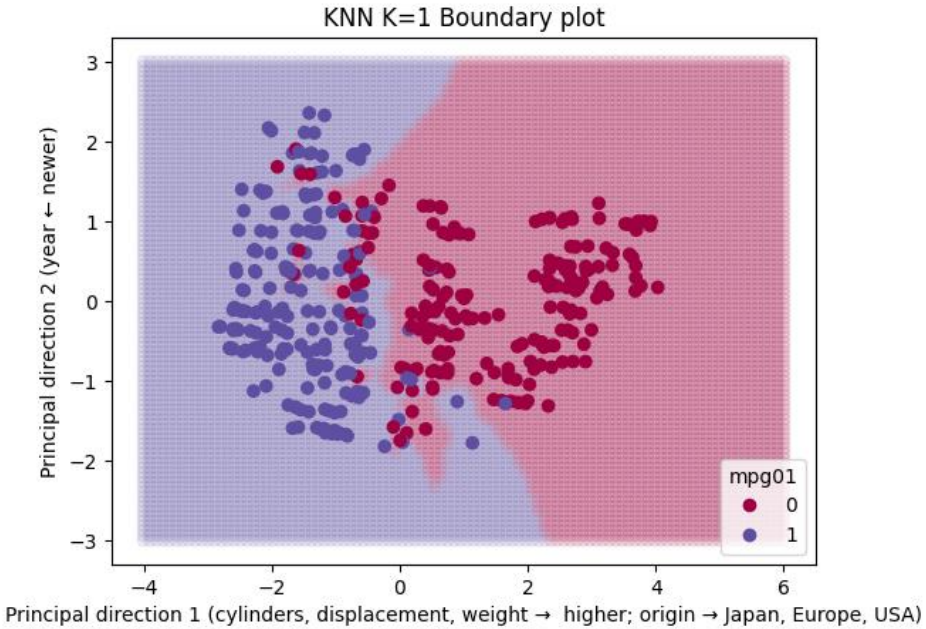
We also find some very dynamic boundaries created by KNN (K=1) classifier, as expected. It essentially blobs a classification region around each data point present in the training set, where the most 'similar' car 'mpg01' value will be applied to any new data points.

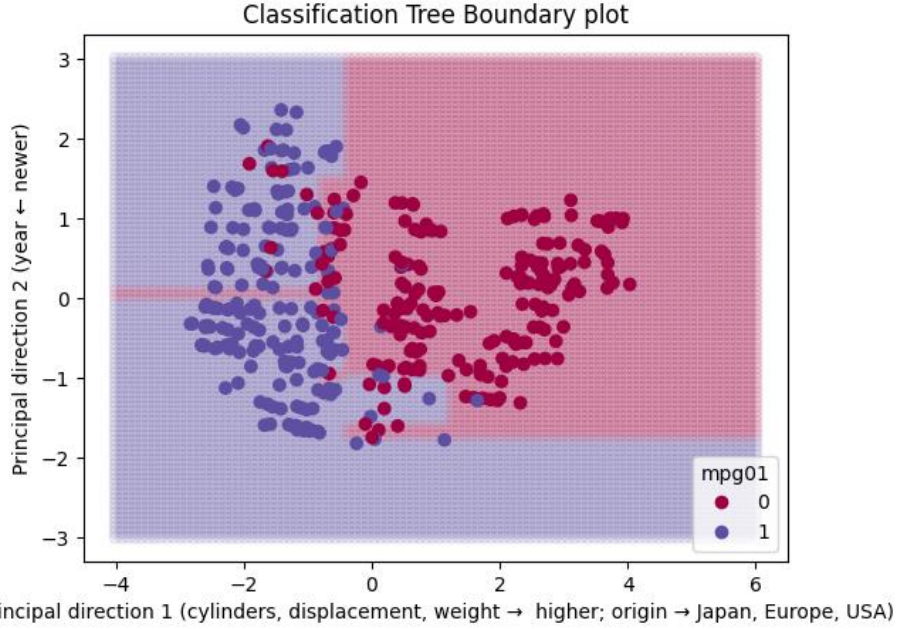
The CART decision tree model and Random Forest models exhibit somewhat similar decision boundaries, with average misclassification rates of 9.32% and 8.64%, respectively. The non-sensical thin rectangular sections of classification that appear to 'shoot-off' to the axes highlight the how decision tree models are splitting in a binary manner using one variable at a time. For example, the CART model is implying that any automobile manufactured in the middle of the 1970-1983 period is immediately low fuel efficiency, seemingly regardless of other predictor values. The Random Forest classifier also exhibits similar behavior at PC2 = 1.2.

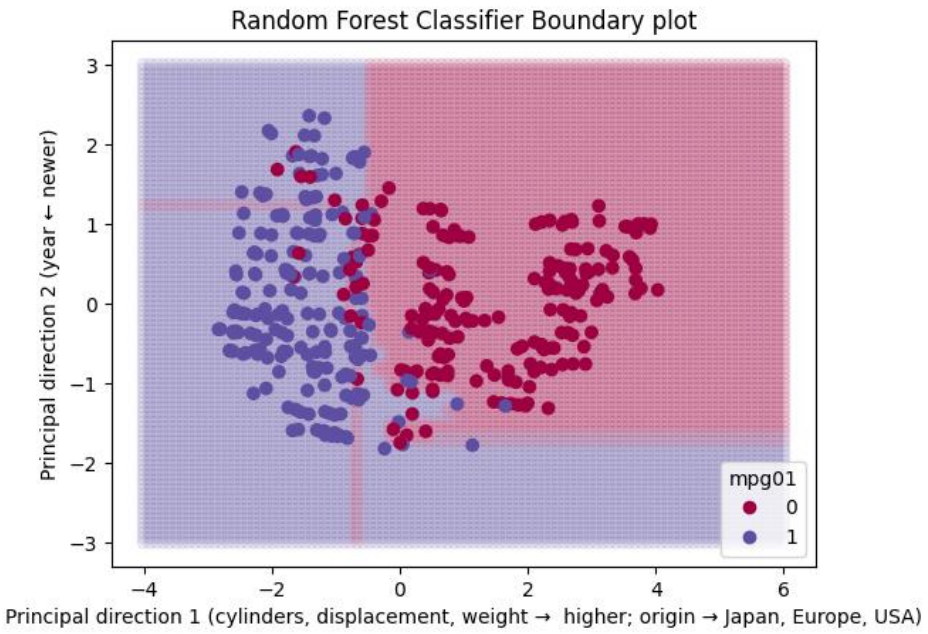
Finally, we find that the AdaBoost classifier model creates a decision boundary with a 'step-like' pattern which is typical of the ensemble of lower level base function that are very basic and form very simple 'step-like' or simply straight decision boundaries across solution space. Most interestingly, the AdaBoost classifier model is able to localize a small cluster of 'mpg01'=0 at PC1 = -1.8, PC2 = 1.5. This demonstrates how dynamic the boosting ensemble methods can be. Similar ensemble decision boundaries can be demonstrated in simpler examples done by hand.

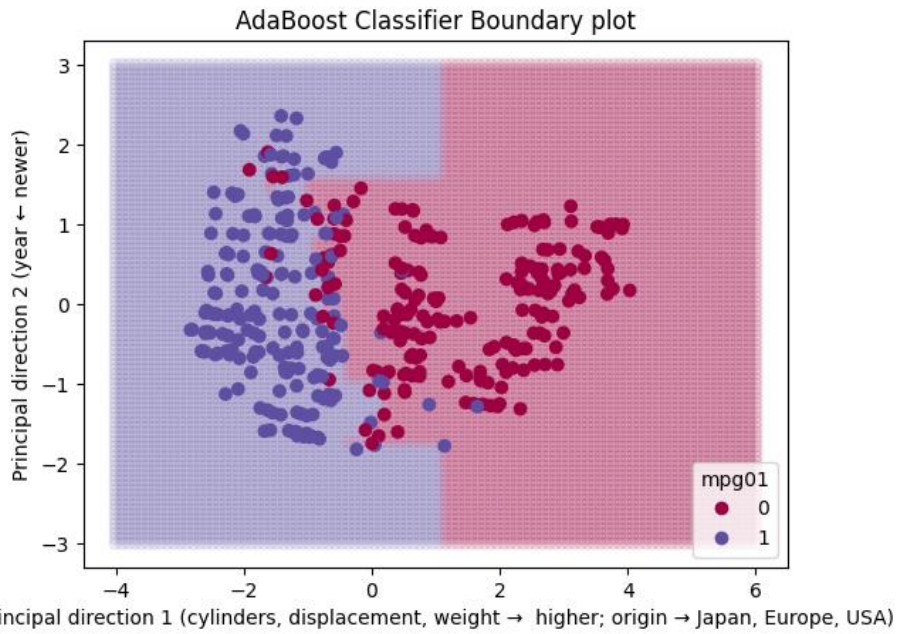

LDA Boundary plot
Principal direction 2 (year ← newer)
Principal direction 1 (cylinders, horsepower, weight → higher; origin → Japan, Europe, USA)
mpg01
0
1


Logistic Regression Boundary plot
Principal direction 2 (year ← newer)
Principal direction 1 (cylinders, displacement, weight → higher; origin → Japan, Europe, USA)
mpg01
0
1


KNN K=1 Boundary plot
Principal direction 2 (year ← newer)
Principal direction 1 (cylinders, displacement, weight → higher; origin → Japan, Europe, USA)
mpg01
0
1


Classification Tree Boundary plot
Principal direction 2 (year ← newer)
Principal direction 1 (cylinders, displacement, weight → higher; origin → Japan, Europe, USA)
mpg01
0
1


Random Forest Classifier Boundary plot
Principal direction 2 (year ← newer)
Principal direction 1 (cylinders, displacement, weight → higher; origin → Japan, Europe, USA)
mpg01
0
1


AdaBoost Classifier Boundary plot
Principal direction 2 (year ← newer)
Principal direction 1 (cylinders, displacement, weight → higher; origin → Japan, Europe, USA)
mpg01
0
1

9 REFERENCE

1. Quinlan,R. (1993). Combining Instance-Based and Model-Based Learning. In Proceedings on the Tenth International Conference of Machine Learning, 236-243, University of Massachusetts, Amherst. Morgan Kaufmann. Available at: https://archive.ics.uci.edu/ml/datasets/auto+mpg (Accessed: 11 March 2023)
2. Sivak, S. (2019). Actual fuel economy of cars and light trucks: 1966-2017. Available at: https://www.greencarcongress.com/2019/09/20190930-sivak.html (Accessed: 25 March 2023)

10 APPENDIX

See next page

In [1]:
```
# Load dependencies/packages
import numpy as np # for array operations
import pandas as pd # for neat tables, database-like grouping/summary tables & csv_read
import matplotlib.pyplot as plt # for the occasional plot
import seaborn as sns # for the occasional plot

from sklearn.model_selection import train_test_split # train / test split method

# for variable selection investigation, use correlation from pandas and stepwise linear regression
import statsmodels.api as sm # to help create Stepwise AIC Linear Regression
import random # for random selection of starting variables/features/predictors

### Classification Model Architectures ###
# 'previous' baseline methods
from sklearn.discriminant_analysis import LinearDiscriminantAnalysis
# (1) LDA with 'eigenvalue decomposition' solver, parameter to tune: 'shrinkage'
from sklearn.linear_model import LogisticRegression
# (2) Logistic Regression with ElasticNet penalty, parameter to tune: 'L1 ratio (Lasso) vs. L2 ratio (Ridge)'
from sklearn.neighbors import KNeighborsClassifier # (3) KNN, parameter to tune: 'k neighbors'
from sklearn import tree
# (4) Classification Tree (CART) split by 'gini', parameter to tune: ...
# ... 'max depth' i.e. extent of how much the tree is grown before pruning

# 'new' ensemble methods
from sklearn.ensemble import RandomForestClassifier
# (5) Random Forest (classification) split by 'gini', parameter to tune: ...
# ... 'max depth' i.e. extent of how much the are individual trees grown AND 'n_estimators'
from sklearn.ensemble import AdaBoostClassifier
# (6) Boosting (classification), parameter to tune: 'learning rate' AND 'n_estimators'

# PCA scaling, transformation
from sklearn.pipeline import make_pipeline # the create a pipeline of transforms necessary to undertake PCA Regression
from sklearn.preprocessing import StandardScaler # first, scale all data as required for PCA
from sklearn.decomposition import PCA # second, perform PCA and take k principal directions of the new linear transformation of X
# then finally use KNN

# Ignore convergence warnings
from warnings import simplefilter
from sklearn.exceptions import ConvergenceWarning

simplefilter("ignore", category=ConvergenceWarning)
```

In [2]:
```
# Method to append to dict
def add_to_dict(dict_, key, value):
```

```
    if key not in dict_:
        dict_[key] = []
    dict_[key].append(value)

# alternatively, I could have used *from collections import defaultdict*
```

In [3]:

```
# Load Data, Create Binary Var 'mpg01'

### load Cleaned Data ###
# Originally, 398 rows/data points and 9 car attributes/features.
# Attribute/feature description:
# 1. mpg: continuous (miles per gallon i.e. fuel efficiency)
# 2. cylinders: multi-valued discrete
# 3. displacement: continuous (engine size, as volume)
# (hard to determine, the ranges are too small to be car length, so it likely some chassis track/wheelbase measurement)
# 4. horsepower: continuous
# 5. weight: continuous (lbs)
# 6. acceleration: continuous (presumably 0-100 mph or something like this, the higher it is, the slower the car is)
# 7. model year: multi-valued discrete (years are YY, the dataset is from 1993, so YYYY notation is 19YY)
# 8. origin: multi-valued discrete
# from investigation of car names, this is actually categorical 1 = US, 2 = Europe, 3 = Japan (this is before SK cars)
# 9. car name: string (unique for each instance)

# But, 6 rows/data pointshave values missing...
# ... NO IMPUTATION (estimation of missing attribute/feature values using the rest of the data set) as directed.
# ... Instead, these 6 rows/data points are DELETED.

# Furthermore, 1 attribute/feature is 'car name' and is REMOVED as it is non-numerical / cannot be categorized in a useful way.

auto_data = pd.read_csv('data/auto.csv') # Load cleaned data

### Create Binary Variable 'mpg01' ###
print(auto_data['mpg'].describe()) # check summary of 'mpg'
mpg_median = auto_data['mpg'].median() # mpg_median = 22.75
# if mpg >= mpg_median, then 1, otherwise 0 . int conversion required from true / false to 1 / 0.
auto_data['mpg01'] = (auto_data['mpg'] >= mpg_median).astype(int)

# 'mpg01' (basically) represents:
# if 1 -> the car is average/above average (in terms of the scope of cars in the data set) in fuel efficiency,
# if 0 -> the car is below average (in terms of the scope of cars in the data set) in fuel efficiency
```

```
count    392.000000
mean      23.445918
std        7.805007
min        9.000000
25%       17.000000
50%       22.750000
75%       29.000000
max       46.600000
Name: mpg, dtype: float64
```

In [4]:
```
# Data Exploration
auto_data # Now 392 rows x 9 cols. 'mpg01' is the response/label, cols 1-7 are the predictors 'cylinders'~'origin'
```

Out[4]:

	mpg	cylinders	displacement	horsepower	weight	acceleration	year	origin	mpg01
0	18.0	8	307.0	130	3504	12.0	70	1	0
1	15.0	8	350.0	165	3693	11.5	70	1	0
2	18.0	8	318.0	150	3436	11.0	70	1	0
3	16.0	8	304.0	150	3433	12.0	70	1	0
4	17.0	8	302.0	140	3449	10.5	70	1	0
...	...	...	...	...	...	...	...	...	...
387	27.0	4	140.0	86	2790	15.6	82	1	1
388	44.0	4	97.0	52	2130	24.6	82	2	1
389	32.0	4	135.0	84	2295	11.6	82	1	1
390	28.0	4	120.0	79	2625	18.6	82	1	1
391	31.0	4	119.0	82	2720	19.4	82	1	1

392 rows × 9 columns

In [5]:
```
auto_data.iloc[:,1:7].describe() # don't numerically summarize 'mpg' or the categorical or binary values, there's no value
# of note, the cars are from 1970 to 1982
```

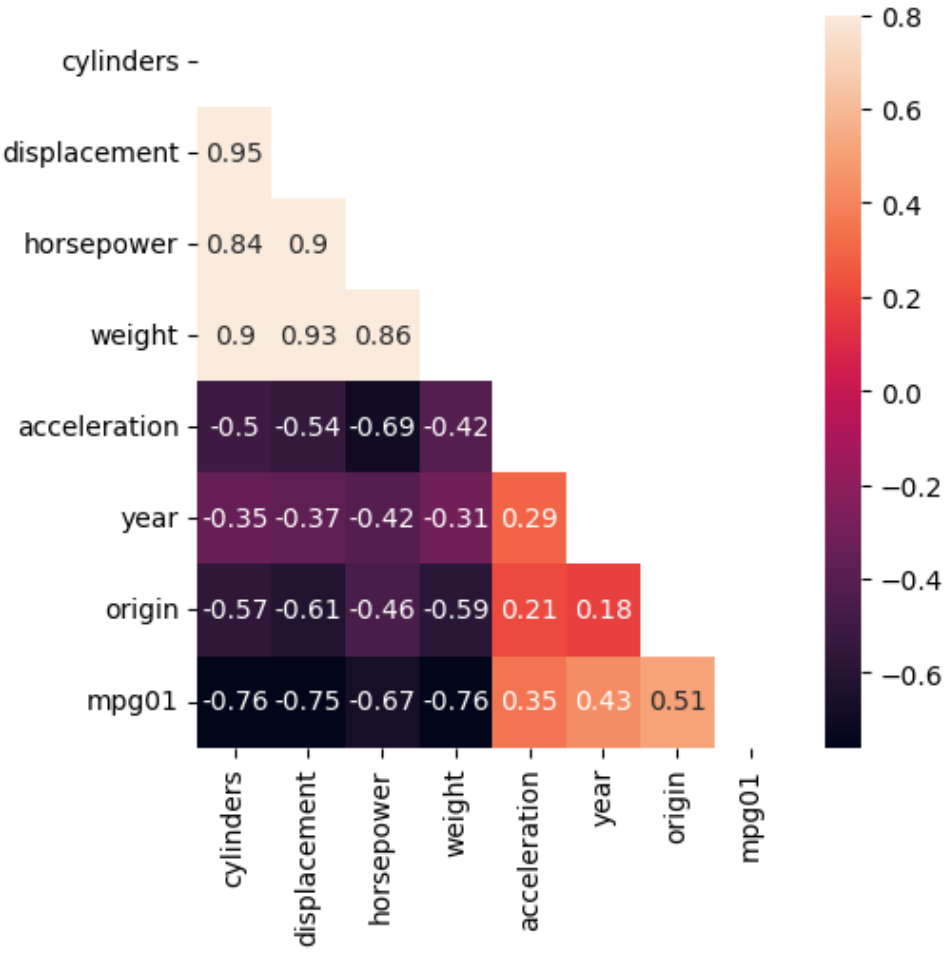
Out[5]:

	cylinders	displacement	horsepower	weight	acceleration	year
count	392.000000	392.000000	392.000000	392.000000	392.000000	392.000000
mean	5.471939	194.411990	104.469388	2977.584184	15.541327	75.979592
std	1.705783	104.644004	38.491160	849.402560	2.758864	3.683737
min	3.000000	68.000000	46.000000	1613.000000	8.000000	70.000000
25%	4.000000	105.000000	75.000000	2225.250000	13.775000	73.000000
50%	4.000000	151.000000	93.500000	2803.500000	15.500000	76.000000
75%	8.000000	275.750000	126.000000	3614.750000	17.025000	79.000000
max	8.000000	455.000000	230.000000	5140.000000	24.800000	82.000000

In [6]:
```
auto_data_corr = auto_data.iloc[:,1:].corr() # ignore 'mpg'
auto_data_corr['mpg01']
```

Out[6]:
```
cylinders      -0.759194
displacement   -0.753477
horsepower     -0.667053
weight         -0.757757
acceleration    0.346822
year            0.429904
origin          0.513698
mpg01           1.000000
Name: mpg01, dtype: float64
```

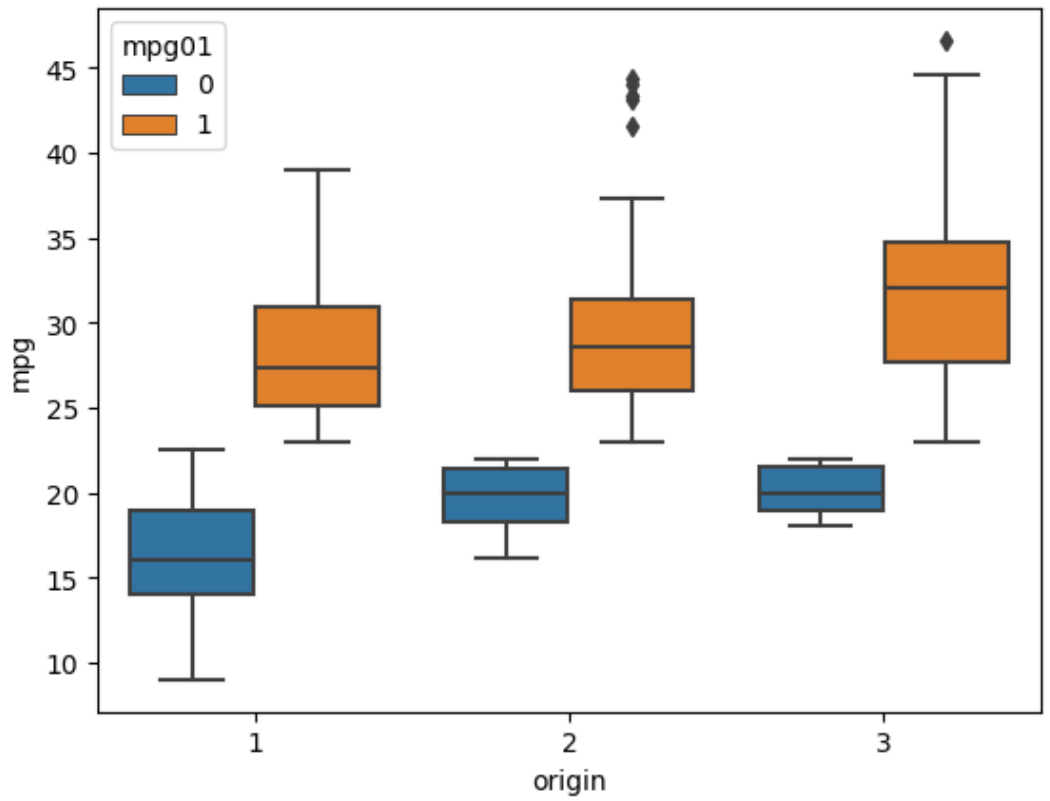
In [7]:
```
mask = np.triu(np.ones_like(auto_data_corr)) # add a mask
plt.subplots(figsize=(5, 5))
sns.heatmap(auto_data_corr, vmax=0.8, annot=True, mask=mask)
plt.show()
```



In [8]:
```
### Boxplots split by origin 1 US, 2 Europe, 3 Japan ###
sns.boxplot(data=auto_data, x="origin", y="mpg", hue="mpg01")
# origin, despite being categorical, holds value as the highest positively-correlated predictor variable.
# It demonstrates that US vehicles have lower fuel efficiency, then European cars, ...
# ... then Japanese cars have higher fuel efficiency.
# This is somewhat of a stereotype, but the data reflects this.
# This occurrence is typically explained economic and resource-availability.
# Firstly, the US is a oil-producing nation, therefore the economic pressures for fuel efficiency are much lower than in Japan.
# Japan historically was regarded as natural resource and energy poor, so there are economic pressures to reduce imports of oil
# and push for fuel efficiency. Whereas, in the US, with the availability of domestic oil, there are less pressures ...
# ... to develop higher levels of fuel efficiency. When also throwing in urban sprawl (population density),
# ... longer driving distances (intra and intercity) and lobbying by the domestic oil companies in the US, ...
# it can become clear why the automotive industries in both nations evolved as they did.
```

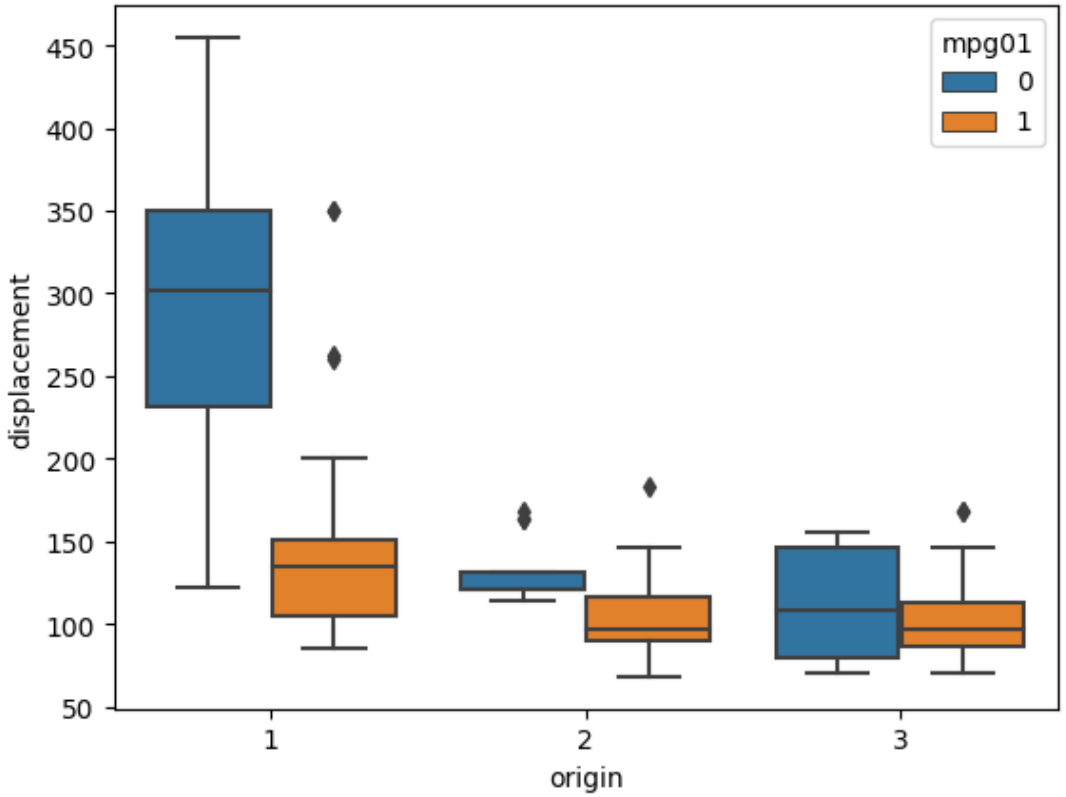
```
# There are plenty of retrospective papers on US vs Japan car manufacturing, should you wish to learn more.
# Without dragging on, we find that, stereotypically, European cars fall in-between the US and Japan cars.
```

Out[8]: <AxesSubplot:xlabel='origin', ylabel='mpg'>



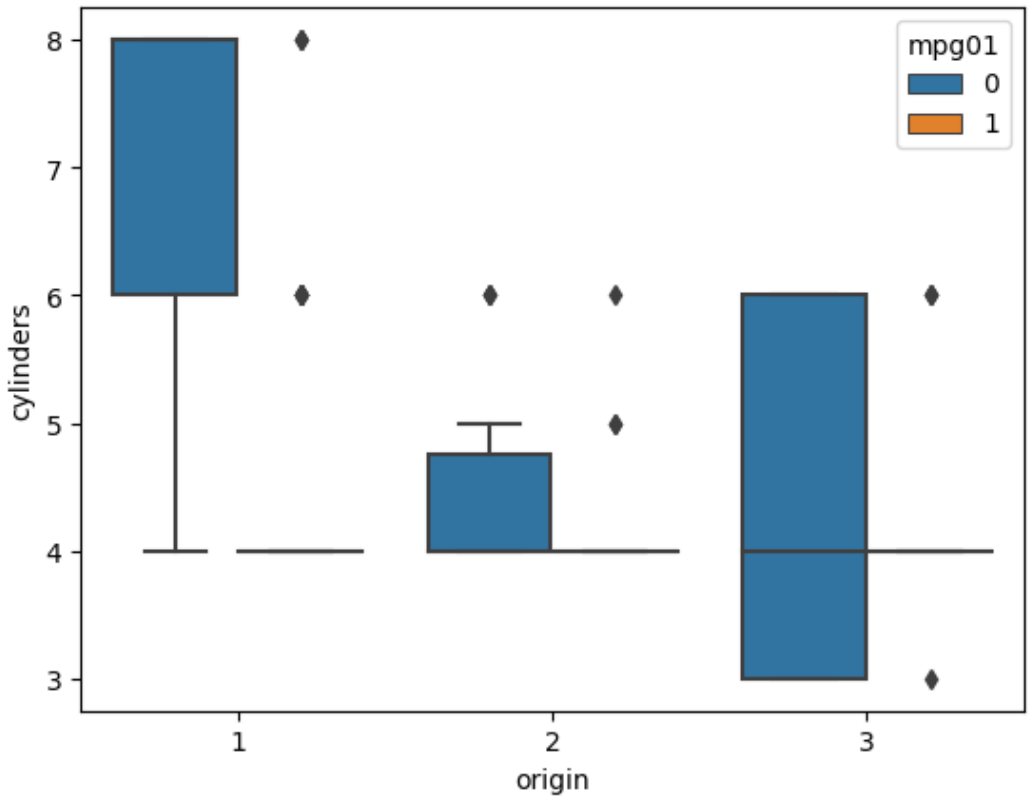
In [9]:
```
sns.boxplot(data=auto_data, x="origin", y="displacement", hue="mpg01")
# smaller engines, have higher fuel efficiency
# Again we find that European and Japanese cars have smaller engines than US cars.
# This is likely due to the great socio-economic influences mentioned above.
# We also find bigger variability in engine size in the US.
# In saying that, we find that the vehicles with above average fuel efficiency tend to have a narrower range of engine size,
# compared to those with below average fuel efficiency. This likely due to the fact that a vast majority of the ...
# ... fuel efficient cars are all 4 cylinder vehicles and thus have a fairly narrow range of engine sizes.
# We are working on the idea that 'engines aren't bigger or heavier than they need to be'.
```

Out[9]: <AxesSubplot:xlabel='origin', ylabel='displacement'>



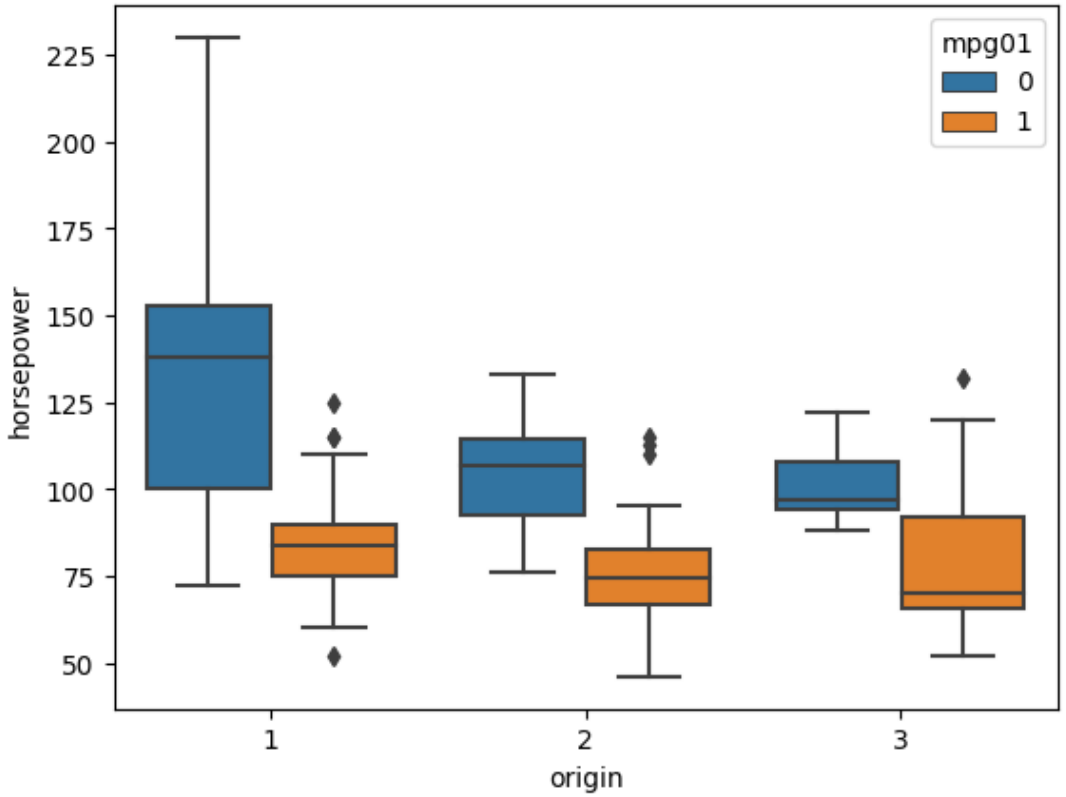
In [10]:
```
sns.boxplot(data=auto_data, x="origin", y="cylinders", hue="mpg01")
# non-linear relationship because of below 4 cylinders generally being the floor value
# Generally-speaking, 4 cylinder or less vehicles have good fuel efficiency, there are some outliers in each
# origin region (some 5, 6 or 8 cylinder cars). Also, stereotypically, we find that V8's are usually only US cars.
# We also find that European cars are mostly 4 to 5 cylinders only. Whereas Japanese cars will be 3 to 6 (V6) cylinders.
```

Out[10]:
```
<AxesSubplot:xlabel='origin', ylabel='cylinders'>
```



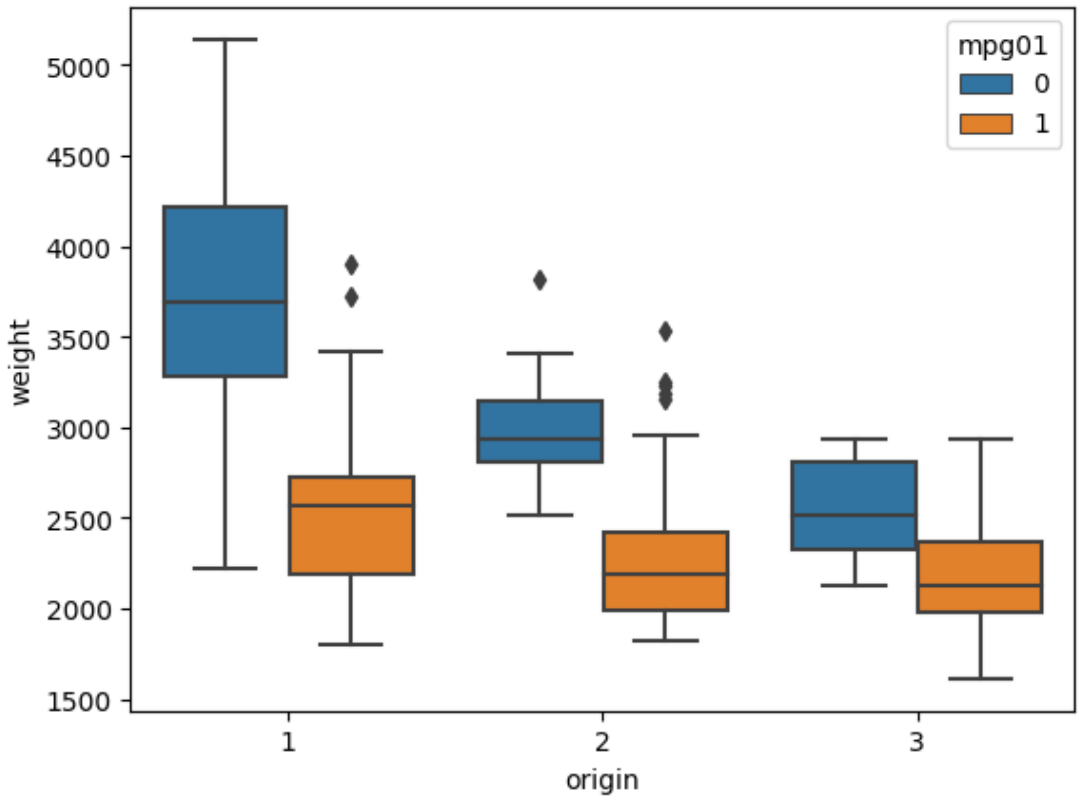
In [11]:
```
sns.boxplot(data=auto_data, x="origin", y="horsepower", hue="mpg01") # the higher the horsepower, the lower the fuel efficiency
```

Out[11]:
```
<AxesSubplot:xlabel='origin', ylabel='horsepower'>
```



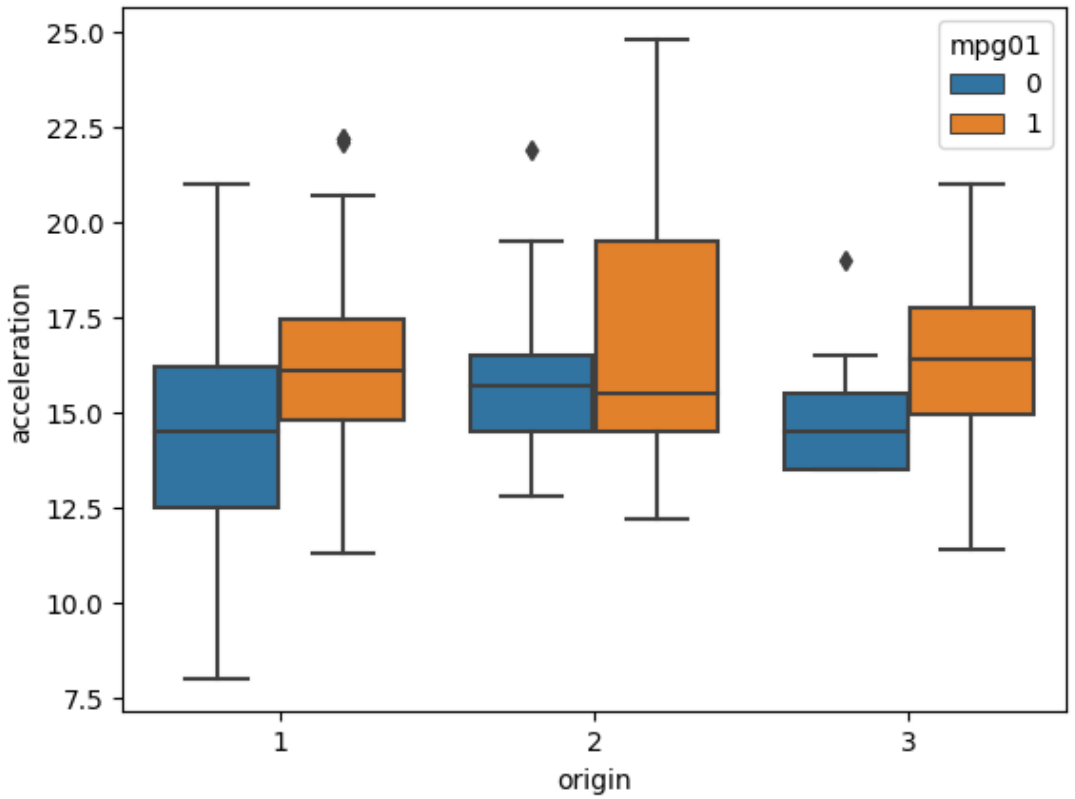
In [12]:
```
sns.boxplot(data=auto_data, x="origin", y="weight", hue="mpg01") # the higher the weight (lbs), the lower the fuel efficiency
```

Out[12]:
```
<AxesSubplot:xlabel='origin', ylabel='weight'>
```



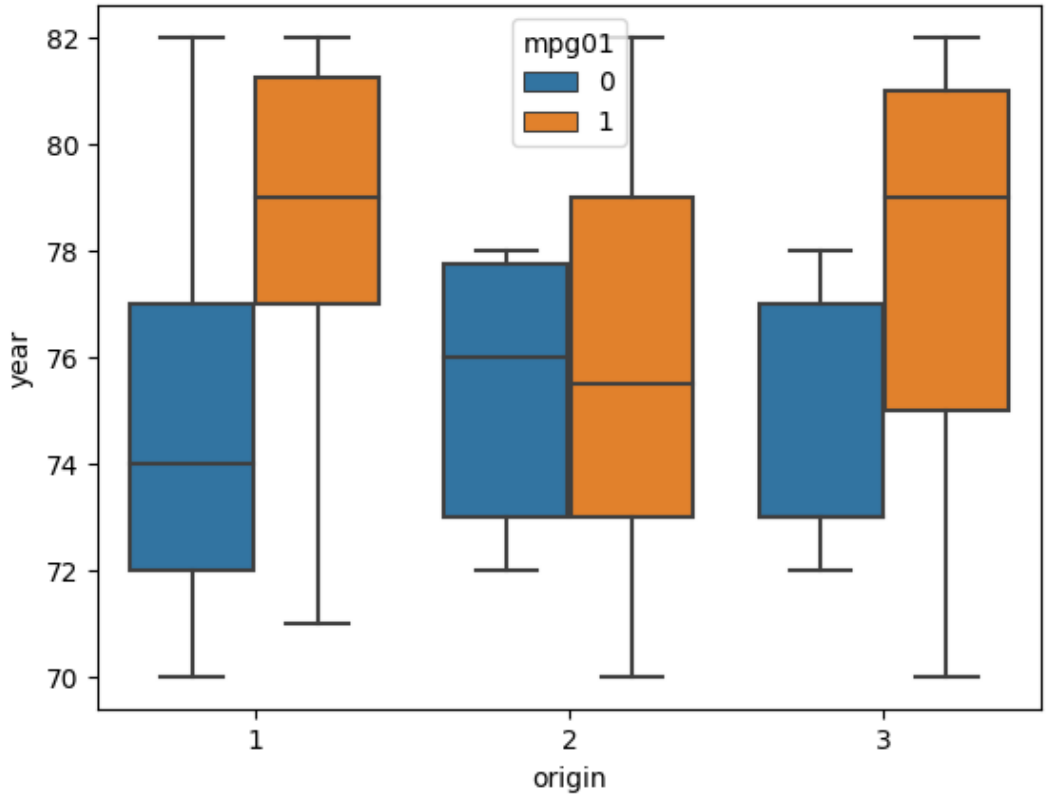
In [13]:
```
sns.boxplot(data=auto_data, x="origin", y="acceleration", hue="mpg01")
# the longer the vehicle takes to accelerate (0-100 mph) i.e. the slower it accelerates, the higher the fuel efficiency
```

Out[13]:
```
<AxesSubplot:xlabel='origin', ylabel='acceleration'>
```



In [14]:
```
sns.boxplot(data=auto_data, x="origin", y="year", hue="mpg01") # the newer the car is, the higher the fuel efficiency
```

Out[14]:
```
<AxesSubplot:xlabel='origin', ylabel='year'>
```



In [15]:
```
pred_var_names = list(auto_data.columns)
pred_var_names = pred_var_names[1:8]
```

In [16]:
```
# Linear regression with variables (stepwise) selected using AIC
def stepw_reg_mod(n_test, train_data, test_data, k_names, label):

    k_start_vars = 0;
    current_vars = random.sample(k_names, k_start_vars) # this works with k=0

    # Initialize
    # OLS(Y,X + C'), Y-label is first, X-data is last. Need to assign a column of 1's to allow for intercept
    current_stepw_reg = sm.OLS(train_data[label], train_data[current_vars].assign(intercept=1)).fit()
    current_aic = current_stepw_reg.aic

    step = 0
    max_steps = 20

    #best_forward_vars = current_vars[:]
```

```
while step < max_steps:
    print('current vars:', current_vars)
    print('current AIC:', current_aic)

    ### Forward Step, try adding vars one at a time that improve AIC ###
    print('--- forward step', step, '---')
    best_forward_aic = 10000000 # initialize a dummy var for inner loop of forward step

    # 1. Get the list of potential vars to add
    potential_vars = k_names[:]
    for i in current_vars:
        potential_vars.remove(i) # get the full list and remove those in the model

    # 2. For each potential var, create a model and measure its AIC
    for var in potential_vars:
        forward_vars = current_vars[:]
        forward_vars.append(var)
        forward_stepw_reg = sm.OLS(train_data[label], train_data[forward_vars].assign(intercept=1)).fit()
        forward_aic = forward_stepw_reg.aic

        # 3. Store the potential var with the best AIC performance, and assoc. model, AIC and all model vars
        if forward_aic <= best_forward_aic:
            best_forward_var_to_add = var
            best_forward_vars = forward_vars[:]
            best_forward_aic = forward_aic
            best_forward_stepw_reg = forward_stepw_reg

    print('new var to add:', best_forward_var_to_add)
    print('new vars (forward):', best_forward_vars)
    print('new AIC (forward):', best_forward_aic)
    print('----------------------')

    ### Backwards Step ###
    # 1. Delete the var with the highest p-value > 0.05 signif level, from the current model
    # 2. Rebuild the model with the var deleted

    print('--- backward step', step, '---')
    backward_aic = 10000000 # initialize a dummy var for inner loop of backward step

    current_pvalues = current_stepw_reg.pvalues[:-1] # remove intercept from p signif test
    print('current p values:')
    print(current_pvalues)

    if np.max(current_pvalues) > 0.05:
```

```
            current_highest_p_var = current_pvalues.index[np.argmax(current_pvalues)]

            print('current var to remove:', current_highest_p_var)
            backward_vars = current_vars[:] # copy current_vars list because remove affects the original list due to pointers
            backward_vars.remove(current_highest_p_var)
            print('new vars (backward):', backward_vars)
            backward_stepw_reg = sm.OLS(train_data[label], train_data[backward_vars].assign(intercept=1)).fit()
            print(backward_stepw_reg.pvalues[:-1])
            backward_aic = backward_stepw_reg.aic
            print('new AIC (backward):', backward_aic)
            print("-----------------------")

        ### Model Selection for Step ###
        if best_forward_aic < current_aic and best_forward_aic < backward_aic:
            print('****chose forward step****')
            print('')
            current_vars = best_forward_vars[:] # copy current_var list because remove affects the original list due to pointers
            current_aic = best_forward_aic
            current_stepw_reg = best_forward_stepw_reg # save the entire model for prediction and testing analysis

        elif backward_aic < current_aic and backward_aic < best_forward_aic:
            print('****chose backward step****')
            print('')
            current_vars = backward_vars[:] # copy backward_vars list because remove affects the original list due to pointers
            current_aic = backward_aic
            current_stepw_reg = backward_stepw_reg # save the entire model for prediction and testing analysis

        else:
            print('****current model has best performance****')
            break

        step += 1

    final_stepw_reg = current_stepw_reg

    stepw_reg_pred_test = final_stepw_reg.predict(test_data[current_vars].assign(intercept=1))
    stepw_reg_test_error = (1/n_test)*np.sum(test_data[label] - stepw_reg_pred_test)**2
    print('')
    print('test error:', stepw_reg_test_error)
    print(final_stepw_reg.params)

    return final_stepw_reg
```

In [17]:
```
stepw_reg_var_check = stepw_reg_mod(len(auto_data), auto_data, auto_data, pred_var_names, 'mpg01')
```

```
current vars: []
current AIC: 571.0204204734663
--- forward step 0 ---
new var to add: cylinders
new vars (forward): ['cylinders']
new AIC (forward): 236.3286833590464
----------------------
--- backward step 0 ---
current p values:
Series([], dtype: float64)
****chose forward step****

current vars: ['cylinders']
current AIC: 236.3286833590464
--- forward step 1 ---
new var to add: year
new vars (forward): ['cylinders', 'year']
new AIC (forward): 207.6800734976772
----------------------
--- backward step 1 ---
current p values:
cylinders    9.689540e-75
dtype: float64
****chose forward step****

current vars: ['cylinders', 'year']
current AIC: 207.6800734976772
--- forward step 2 ---
new var to add: weight
new vars (forward): ['cylinders', 'year', 'weight']
new AIC (forward): 178.2850544766037
----------------------
--- backward step 2 ---
current p values:
cylinders    7.118247e-64
year         3.560436e-08
dtype: float64
****chose forward step****

current vars: ['cylinders', 'year', 'weight']
current AIC: 178.2850544766037
--- forward step 3 ---
new var to add: horsepower
new vars (forward): ['cylinders', 'year', 'weight', 'horsepower']
new AIC (forward): 172.299306655927
----------------------
```

```
--- backward step 3 ---
current p values:
cylinders    1.848348e-06
year         9.787597e-09
weight       2.534289e-08
dtype: float64
****chose forward step****

current vars: ['cylinders', 'year', 'weight', 'horsepower']
current AIC: 172.299306655927
--- forward step 4 ---
new var to add: origin
new vars (forward): ['cylinders', 'year', 'weight', 'horsepower', 'origin']
new AIC (forward): 170.94578333710763
----------------------
--- backward step 4 ---
current p values:
cylinders     8.308923e-08
year          3.013200e-10
weight        4.720938e-10
horsepower    5.015817e-03
dtype: float64
****chose forward step****

current vars: ['cylinders', 'year', 'weight', 'horsepower', 'origin']
current AIC: 170.94578333710763
--- forward step 5 ---
new var to add: displacement
new vars (forward): ['cylinders', 'year', 'weight', 'horsepower', 'origin', 'displacement']
new AIC (forward): 172.71244519894935
----------------------
--- backward step 5 ---
current p values:
cylinders     5.319309e-07
year          3.782985e-10
weight        2.192915e-08
horsepower    1.373326e-02
origin        6.936757e-02
dtype: float64
current var to remove: origin
new vars (backward): ['cylinders', 'year', 'weight', 'horsepower']
cylinders     8.308923e-08
year          3.013200e-10
weight        4.720938e-10
horsepower    5.015817e-03
dtype: float64
```

```
new AIC (backward): 172.299306655927
-----------------------
****current model has best performance****

test error: 1.87030952025939e-25
cylinders    -0.109289
year          0.029271
weight       -0.000270
horsepower    0.002144
origin        0.043070
intercept    -0.612454
dtype: float64
```

In [18]:

```
# We saw that stepwise regression performed on the entire dataset selected 'cylinders', 'horsepower', ' weight', ...
# ... 'year' and 'origin' as its significant variables.
# Further to this, and as with the golf putting example in the material, we can use domain knowledge as ...
# ... a balance to address gaps/holes in the numerical analysis.

# For Part E, will retain and drop the following predictor variables:

# 'cylinders', 'displacement', 'horsepower' and 'weight' are all highly negatively-correlated predictor variables with 'mpg01'
# ... i.e. fuel efficiency. This makes sense, given that fuel efficiency is likely to decrease the bigger and more powerful ...
# ... the engine is and the heavier the car is.

# I will retain both 'cylinders' and 'horsepower' because 'cylinders' likely has a piece-wise or non-linear relationship ...
# ... underlying it and I want to capture this. We saw the golf putting example how physical thresholds/caps are ...
# ... non-linear relationships and we need to use domain knowledge to address model shortcomings/pitfalls.
# 'horsepower' is standard measure of power of an engine. Presumably, we should see more powerful engines have lower fuel ...
# ... efficiency, and vice versa.
# In addition, these two variables were also selected as significant variables by the stepwise regression model.

# I will drop 'displacement' (engine size) because 'cylinders' and 'horse power' are sufficient proxy predictor variables,
# and also because it was NOT selected as a significant variable by the stepwise regression model.
# The high correlation between these predictor variables and the design philosophy and logic that 'the size of ...
# ... an engine is directly proportional to its power' supports these ideas.

# I will retain 'weight' because this is the only proxy measurement of chassis and the weight of a vehicle is not
# ... solely-determined by its engine size or power. Amenities, seating capacity, and so on, will all effect.

# I will drop 'acceleration' due to it having the lowest correlation with 'mpg01' and 'mpg', and also because it was ...
# ... NOT selected as a significant variable by the stepwise regression model. Furthermore, a check of the box plots shows ...
# ... it has higher variability and less descriptive power than 'year'.

# I will retain 'year' as it has reasonable positive linear correlation with 'mpg01' and 'mpg', and also because it was ...
# ... selected as a significant variable by the stepwise regression model.
# Furthermore, it follows the general idea that 'the newer the car is, the higher the fuel efficiency'.
```

```
# I will retain 'origin', despite being categorical, holds value as the highest positively-correlated predictor variable, ...
# ... , and also because it was selected as a significant variable by the stepwise regression model.
# It is a positively-correlated predictor variable solely because of the orientation of the categories.
# 1 = US, 2 = Europe, 3 = Japan. If these categories were arbitrarily reversed (which is completely fine to do), ...
# ... this predictor variable would become negatively-correlated to the same magnitude.
# As stated earlier, 'origin' falls into this stereotype of how US cars have lower fuel efficiency than Japanese cars, and ...
# ... European cars fall somewhere in the middle in terms of fuel efficiency.
# It will be used much like a continuous variable in many of the model architectures, but its richness as categorical variable
# ... will be able to be captured by the KNN and CART decision tree models.
```

In [19]:
```
# Train/Test Split
new_auto_data = auto_data.drop(columns=['mpg', 'displacement', 'acceleration'])
# DELETE the original 'mpg' column it's unncessary
# DELETE 'displacement' due to redundancy vs 'horsepower' and 'cylinders'
# DELETE 'acceleration' due to low descriptive power

X_train, X_test, y_train, y_test = train_test_split(
    new_auto_data.iloc[:,:5], new_auto_data['mpg01'], test_size=0.20, random_state=55) # 80% train, 20% test

n_test = len(y_test)
```

In [20]:
```
# Model Applications ~> One pass runs tests

one_pass_results = [] # create an empty list to store one-pass (i.e. non-cv) model results

# (1) Linear Discriminant Analysis (LDA)
lda_0 = LinearDiscriminantAnalysis()
lda_0 = lda_0.fit(X_train, y_train)
lda_0_y_pred = lda_0.predict(X_test)
lda_0_test_error = np.array([lda_0_y_pred != y_test]).sum()/n_test
one_pass_results.append(['lda', lda_0_test_error]) # store test error
print('1. lda test error:', lda_0_test_error)
print('lda coefficients: ', lda_0.coef_[0])
print('')

# (2) Logistic Regression
logit_0 = LogisticRegression(max_iter = 250)
logit_0 = logit_0.fit(X_train, y_train)
# When provided a binary response/label, sk-learn logistic regression fits a probability for each label i.e. '0' or '1'
# Thus, we avoid needing to tune 'c' (the threshold between a 0 or 1 label) as per the lecture material because ...
# The logistic regression model calculates two probabilities for a given point, prob(x|Y='0') or prob(x|Y='1').
# It then predicts this given point as the label with the highest probability value.
logit_0_y_pred = logit_0.predict(X_test)
logit_0_test_error = np.array([logit_0_y_pred != y_test]).sum()/n_test
```

```
one_pass_results.append(['logit', logit_0_test_error]) # store test error
print('2. logistic regression test error:', logit_0_test_error)
print('logit variable coefficients: ', logit_0.coef_[0])
print('logit intercept: ', logit_0.intercept_)
print('')

# (3) K-Nearest Neighbors (KNN)

# KNN Models
results = [] # list to store results

k_values = [1, 3, 5, 7, 9, 11, 13, 15, 17]

for k in k_values:

    knn_0 = KNeighborsClassifier(n_neighbors=k)
    knn_0.fit(X_train, y_train)

    knn_0_y_pred = knn_0.predict(X_test)
    knn_0_test_error = np.array([knn_0_y_pred != y_test]).sum()/n_test

    results.append([k, knn_0_test_error])

knn_results_col_names = ['k-neighbors', 'test error']
knn_results_df = pd.DataFrame(results, columns = knn_results_col_names)
# plot visualization, test error vs K-neighbors
#plt.plot(knn_results_df['k-neighbors'], knn_results_df['test error'], 'g', linewidth=2)
#plt.title('Test Error vs K-neighbors')
#plt.xticks(k_values)
#plt.xlabel('K neighbors used (model architecture)')
#plt.ylabel('Test Error (misclass_amount/n_test_size)')

# choose K = 7
k_final = 7
knn_0 = KNeighborsClassifier(n_neighbors=k_final)
knn_0.fit(X_train, y_train)
knn_0_y_pred = knn_0.predict(X_test)
knn_0_test_error = np.array([knn_0_y_pred != y_test]).sum()/n_test
one_pass_results.append(['knn', knn_0_test_error]) # store test error
print('3. knn test error:', knn_0_test_error)
print('knn: 7 neighbors')
print('')

# (4) Classification Tree
cart_0 = tree.DecisionTreeClassifier(max_depth=4)
cart_0.fit(X_train, y_train)
```

```
cart_0_y_pred = cart_0.predict(X_test)
cart_0_test_error = np.array([cart_0_y_pred != y_test]).sum()/n_test
one_pass_results.append(['classf tree', cart_0_test_error]) # store test error
print('4. classification tree test error:', cart_0_test_error)
print('max depth: 4 levels')
print('')

# (5) Random Forest Classifier
rand_for_0 = RandomForestClassifier(n_estimators=100, max_depth=4, random_state=0)
rand_for_0.fit(X_train, y_train)
rand_for_0_y_pred = rand_for_0.predict(X_test)
rand_for_0_test_error = np.array([rand_for_0_y_pred != y_test]).sum()/n_test
one_pass_results.append(['classf tree', rand_for_0_test_error]) # store test error
print('5. random forest classifier test error:', rand_for_0_test_error)
print('n trees: 100 estimators')
print('max depth: 4 levels')
print('')

# (6) AdaBoost Classifier
adaboost_0 = AdaBoostClassifier(n_estimators=100, learning_rate = 0.25, random_state=0)
adaboost_0.fit(X_train, y_train)
adaboost_0_y_pred = adaboost_0.predict(X_test)
adaboost_0_test_error = np.array([adaboost_0_y_pred != y_test]).sum()/n_test
one_pass_results.append(['classf tree', adaboost_0_test_error]) # store test error
print('6. adaboost classifier test error:', adaboost_0_test_error)
print('n stumps (base) functions: 100 estimators')
print('learning rate: 0.25')
```

```
1. lda test error: 0.10126582278481013
lda coefficients:  [-1.36805246  0.02561226 -0.00306702  0.35504921  0.76991011]

2. logistic regression test error: 0.11392405063291139
logit variable coefficients:  [-0.15437936 -0.04374057 -0.00453577  0.47469815  0.63945316]
logit intercept:  [-19.42920671]

3. knn test error: 0.10126582278481013
knn: 7 neighbors

4. classification tree test error: 0.16455696202531644
max depth: 4 levels

5. random forest classifier test error: 0.0759493670886076
n trees: 100 estimators
max depth: 4 levels

6. adaboost classifier test error: 0.08860759493670886
n stumps (base) functions: 100 estimators
learning rate: 0.25
```

In [21]:

```python
# Functions of the 6 model architectures for Cross Validation and Tuning of parameters

# (1) Linear Discriminant Analysis (LDA)
def lda_mod(n_test, shrinkage_, X_train, X_test, y_train, y_test):
    lda_m = LinearDiscriminantAnalysis(solver='eigen', shrinkage=shrinkage_)
    lda_m = lda_m.fit(X_train, y_train)
    lda_m_y_pred = lda_m.predict(X_test)
    lda_m_test_error = np.array([lda_m_y_pred != y_test]).sum()/n_test
    return lda_m_test_error

# (2) Logistic Regression
def logit_mod(n_test, l1_ratio_, X_train, X_test, y_train, y_test):
    logit_m = LogisticRegression(penalty='elasticnet', solver='saga',
                                 max_iter=500, l1_ratio=l1_ratio_)
    logit_m = logit_m.fit(X_train, y_train)
    logit_m_y_pred = logit_m.predict(X_test)
    logit_m_test_error = np.array([logit_m_y_pred != y_test]).sum()/n_test
    return logit_m_test_error

# (3) K-Nearest Neighbors (KNN)
def knn_mod(n_test, k_, X_train, X_test, y_train, y_test):
    knn_m = KNeighborsClassifier(n_neighbors=k_)
    knn_m.fit(X_train, y_train)
    knn_m_y_pred = knn_m.predict(X_test)
    knn_m_test_error = np.array([knn_m_y_pred != y_test]).sum()/n_test
```

```
    return knn_m_test_error

# (4) Classification Tree
def cart_mod(n_test, depth_, X_train, X_test, y_train, y_test):
    cart_m = tree.DecisionTreeClassifier(max_depth=depth_)
    cart_m.fit(X_train, y_train)
    cart_m_y_pred = cart_m.predict(X_test)
    cart_m_test_error = np.array([cart_m_y_pred != y_test]).sum()/n_test
    return cart_m_test_error

# (5) Random Forest Classifier
def rforest_mod(n_test, n_trees_, depth_, X_train, X_test, y_train, y_test):
    rforest_m = RandomForestClassifier(n_estimators=n_trees_, max_depth=depth_, random_state=0)
    # fix random state = 0, rforest varies by n_trees, depth and Monte Carlo CV resamples
    rforest_m.fit(X_train, y_train)
    rforest_m_y_pred = rforest_m.predict(X_test)
    rforest_m_test_error = np.array([rforest_m_y_pred != y_test]).sum()/n_test
    return rforest_m_test_error

# (6) AdaBoost Classifier
def adaboost_mod(n_test, n_estimators_, learn_rate_, X_train, X_test, y_train, y_test):
    adaboost_m = AdaBoostClassifier(n_estimators=n_estimators_, learning_rate = learn_rate_, random_state=0)
    # fix random state = 0, rforest varies by n_trees, depth and Monte Carlo CV resamples
    adaboost_m.fit(X_train, y_train)
    adaboost_m_y_pred = adaboost_m.predict(X_test)
    adaboost_m_test_error = np.array([adaboost_m_y_pred != y_test]).sum()/n_test
    return adaboost_m_test_error
```

In [22]:

```
### Cross validation - parameter tuning ###
# Monte Carlo 200 Resamples Cross Validation for parameter tuning
# Ideally, I would have chosen more resamples, but 200 seems adequate for such an extensive Grid Search on parameters
# Run time was 20+ minutes.

test_ratio = 0.2 # 80/20 Train/Test as before

b_cv = 200 # B = 200
#n_test = len(y_test) = 79

cv_results = []

# parameters for different models
shrinkage_list = [0, 0.1, 0.2, 0.4, 0.6, 0.8, 1] # 1. LDA
l1_ratio_list = [0, 0.1, 0.2, 0.35, 0.5, 0.65, 0.8, 0.9, 1] # 2. Logistic Regression
# priortize l1_ratio = 1 i.e. LASSO yields sparse solutions
k_list = [1, 3, 5, 7, 9, 11, 13, 15, 17] # 3. KNN
depth_list = [2, 3, 4, 5, 6] # 4. CART,  & 5. Random Forest Classifier
```

```
# 5. Random Forest Classifier
n_estimators_list = [5, 10, 25, 50, 75, 100, 150, 200] # parameter 1
#depth_list = [2, 3, 4, 5, 6] # parameter 2
# 6. AdaBoost Classifier
#n_estimators_list = [5, 10, 25, 50, 75, 100, 150, 200] # parameter 1
learn_rate_list = [0.01, 0.05, 0.1, 0.2, 0.35, 0.5, 0.65, 0.8, 0.9, 1] # parameter 2

# Perform Grid Search on Monte Carlo CV to find optimal parameters
# i.e. search a range of parameter values using inner & outer loops and ...
# ... record the parameter values that achieve the lowest test error for each model for each CV resample.
# The parameters are ordered in way that will prioritize reduced model complexity in the event of tie for lowest test error

for b in range(b_cv):
    cv_X_train, cv_X_test, cv_y_train, cv_y_test = train_test_split(
    new_auto_data.iloc[:,:5], new_auto_data['mpg01'], test_size=0.20, random_state=b) # 80% train, 20% test

    # for each monte carlo CV run, save the best (lowest test misclassification) parameter value

    # (1) Linear Discriminant Analysis (LDA)
    results = []
    for shrinkage in shrinkage_list:
        results.append([shrinkage, lda_mod(n_test, shrinkage, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])
    results_df = pd.DataFrame(results, columns=['shrinkage', 'test misclass'])

    cv_results.append([1, b+1, results_df['shrinkage'][results_df['test misclass'].idxmin()], 'na'])

    # (2) Logistic Regression
    results = []
    for l1_ratio in reversed(l1_ratio_list):
        results.append([l1_ratio, logit_mod(n_test, l1_ratio, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])
    results_df = pd.DataFrame(results, columns=['l1 ratio', 'test misclass'])

    cv_results.append([2, b+1, results_df['l1 ratio'][results_df['test misclass'].idxmin()], 'na'])

    # (3) K-Nearest Neighbors (KNN)
    results = []
    for k in k_list:
        results.append([k, knn_mod(n_test, k, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])
    results_df = pd.DataFrame(results, columns=['k', 'test misclass'])

    cv_results.append([3, b+1, results_df['k'][results_df['test misclass'].idxmin()], 'na'])

    # (4) Classification Tree
    results = []
    for depth in depth_list:
        results.append([depth, cart_mod(n_test, depth, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])
```

```
    results_df = pd.DataFrame(results, columns=['depth', 'test misclass'])

    cv_results.append([4, b+1, results_df['depth'][results_df['test misclass'].idxmin()], 'na'])

    # (5) Random Forest Classifier
    results = []
    for n_trees in n_estimators_list:
        for depth in depth_list:
            results.append([n_trees, depth,
                            rforest_mod(n_test, n_trees, depth, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])
    results_df = pd.DataFrame(results, columns=['n trees', 'depth', 'test misclass'])

    cv_results.append([5, b+1, results_df['n trees'][results_df['test misclass'].idxmin()],
                       results_df['depth'][results_df['test misclass'].idxmin()]])

    # (6) AdaBoost Classifier
    results = []
    for n_estimators in n_estimators_list:
        for learn_rate in learn_rate_list:
            results.append([n_trees, learn_rate,
                            adaboost_mod(n_test, n_estimators, learn_rate, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])
    results_df = pd.DataFrame(results, columns=['n estimators', 'learn rate', 'test misclass'])

    cv_results.append([6, b+1, results_df['n estimators'][results_df['test misclass'].idxmin()],
                       results_df['learn rate'][results_df['test misclass'].idxmin()]])
```

In [23]:

```
cv_results_df = pd.DataFrame(cv_results, columns=['model num', 'cv run', 'param 1', 'param 2'])
#cv_results_df
model_num_list = [1, 2, 3, 4, 5, 6]

best_param = []
for i in model_num_list:
    i_step = cv_results_df.loc[cv_results_df['model num'] == i].drop(columns='model num')['param 1'].value_counts()
    best_param.append([i, i_step])
for i in [5, 6]:
    i_step = cv_results_df.loc[cv_results_df['model num'] == i].drop(columns='model num')['param 2'].value_counts()
    best_param.append([i, i_step])

#best_param
```

In [25]:

```
best_param
```

Out[25]:
```
[[1,
  0.0    188
  0.1     11
  1.0      1
  Name: param 1, dtype: int64],
 [2,
  1.0    200
  Name: param 1, dtype: int64],
 [3,
  1.0     59
  3.0     48
  5.0     21
  15.0    18
  7.0     16
  13.0    12
  11.0    10
  17.0    10
  9.0      6
  Name: param 1, dtype: int64],
 [4,
  4.0    58
  5.0    48
  2.0    41
  3.0    38
  6.0    15
  Name: param 1, dtype: int64],
 [5,
  5.0      75
  10.0     54
  25.0     32
  50.0     27
  100.0     6
  75.0      3
  150.0     2
  200.0     1
  Name: param 1, dtype: int64],
 [6,
  200.0    200
  Name: param 1, dtype: int64],
 [5,
  6    62
  5    54
  4    39
  3    28
  2    17
  Name: param 2, dtype: int64],
```

```
[6,
 0.35    38
 0.80    37
 0.50    26
 0.20    23
 0.65    20
 0.90    18
 1.00    14
 0.01    11
 0.10     7
 0.05     6
 Name: param 2, dtype: int64]]
```

In [26]:

```
### Cross validation - test classification performance comparison ###

# Monte Carlo 1000 Resamples Cross Validation
# I acknowledge that 1000 resamples Monte Carlo CV is great for sensitivity analysis of Test Error and TE Variance

test_ratio = 0.2 # 80/20 Train/Test as before

b_cv = 1000 # B = 1000
#n_test = len(y_test) = 79

# best parameters
best_shrinkage = 0
best_l1_ratio = 1
best_k = 1
best_depth_cart = 5
best_n_trees = 5
best_depth_rf = 6
best_n_estimators = 200
best_learn_rate = 0.35

cv_results = []

# Perform Grid Search on Monte Carlo CV to find optimal parameters
# i.e. search a range of parameter values using inner & outer loops and ...
# ... record the parameter values that achieve the lowest test error for each model for each CV resample.
# The parameters are ordered in way that will prioritize reduced model complexity in the event of tie for lowest test error

for b in range(b_cv):
    cv_X_train, cv_X_test, cv_y_train, cv_y_test = train_test_split(
    new_auto_data.iloc[:,:5], new_auto_data['mpg01'], test_size=0.20, random_state=b) # 80% train, 20% test

    # for each monte carlo CV run, record misclass rate for each model
```

```
    # (1) Linear Discriminant Analysis (LDA)
    cv_results.append([1, b+1, lda_mod(n_test, best_shrinkage, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])

    # (2) Logistic Regression
    cv_results.append([2, b+1, logit_mod(n_test, best_l1_ratio, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])

    # (3) K-Nearest Neighbors (KNN)
    cv_results.append([3, b+1, knn_mod(n_test, best_k, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])

    # (4) Classification Tree
    cv_results.append([4, b+1, cart_mod(n_test, best_depth_cart, cv_X_train, cv_X_test, cv_y_train, cv_y_test)])

    # (5) Random Forest Classifier
    cv_results.append([5, b+1, rforest_mod(n_test, best_n_trees, best_depth_rf,
                                           cv_X_train, cv_X_test, cv_y_train, cv_y_test)])

    # (6) AdaBoost Classifier
    cv_results.append([6, b+1, adaboost_mod(n_test, best_n_estimators, best_learn_rate,
                                            cv_X_train, cv_X_test, cv_y_train, cv_y_test)])
```

In [27]:
```
cv_results_col_names = ['model num', 'b', 'test error']
cv_results_df = pd.DataFrame(cv_results, columns = cv_results_col_names)

final_results = []
model_num_list = [1, 2, 3, 4, 5, 6]
# Analyze model results for model_num = 1, 2, ..., 6
for i in model_num_list:
    model_i_results_df = cv_results_df[cv_results_df['model num'] == i]

    model_i_mean_test_err = model_i_results_df['test error'].mean()

    model_i_results_df.insert(loc=3,
                              column='squared error',
                              value=(model_i_results_df['test error']-model_i_mean_test_err)**2)
    model_i_var_test_err = model_i_results_df['squared error'].sum()/(b_cv-1)

    final_results.append([i, model_i_mean_test_err, model_i_var_test_err])

final_results_col_names = ['model num', 'mean test error rate', 'var test error rate']
final_results_df = pd.DataFrame(final_results, columns = final_results_col_names)
```

In [28]:
```
final_results_df['model num'] = ['1: LDA', '2: Logistic Reg', '3: KNN', '4: Classf Tree',
                                 '5: Randm Frst Classf', '6: AdaBst Classf']
final_results_df
```

Out[28]:

	model num	mean test error rate	var test error rate
0	1: LDA	0.092772	0.000938
1	2: Logistic Reg	0.098671	0.000885
2	3: KNN	0.136241	0.001138
3	4: Classf Tree	0.093165	0.001013
4	5: Randm Frst Classf	0.086380	0.000876
5	6: AdaBst Classf	0.086646	0.000885

In [29]:
```
### Visualization of Decision Boundaries in PC-2 Space ###
pc2_pipeline = make_pipeline(StandardScaler(), PCA(n_components=2))
pc2_pipeline.fit(auto_data[['cylinders', 'horsepower', 'weight', 'year', 'origin']]) # fit PCA n=2 to whole data set
# project into 2 PCs for data visualization
pc2_X_train = pd.DataFrame(pc2_pipeline.transform(X_train), columns = ['pc1', 'pc2'])
```

In [30]:
```
pc12_auto_data = pd.DataFrame(pc2_pipeline.transform(auto_data[['cylinders', 'horsepower', 'weight', 'year', 'origin']]),
                              columns = ['pc1', 'pc2'])
pc12_auto_data = pc12_auto_data.join(auto_data)
```

In [31]:
```
pc12_auto_data
```

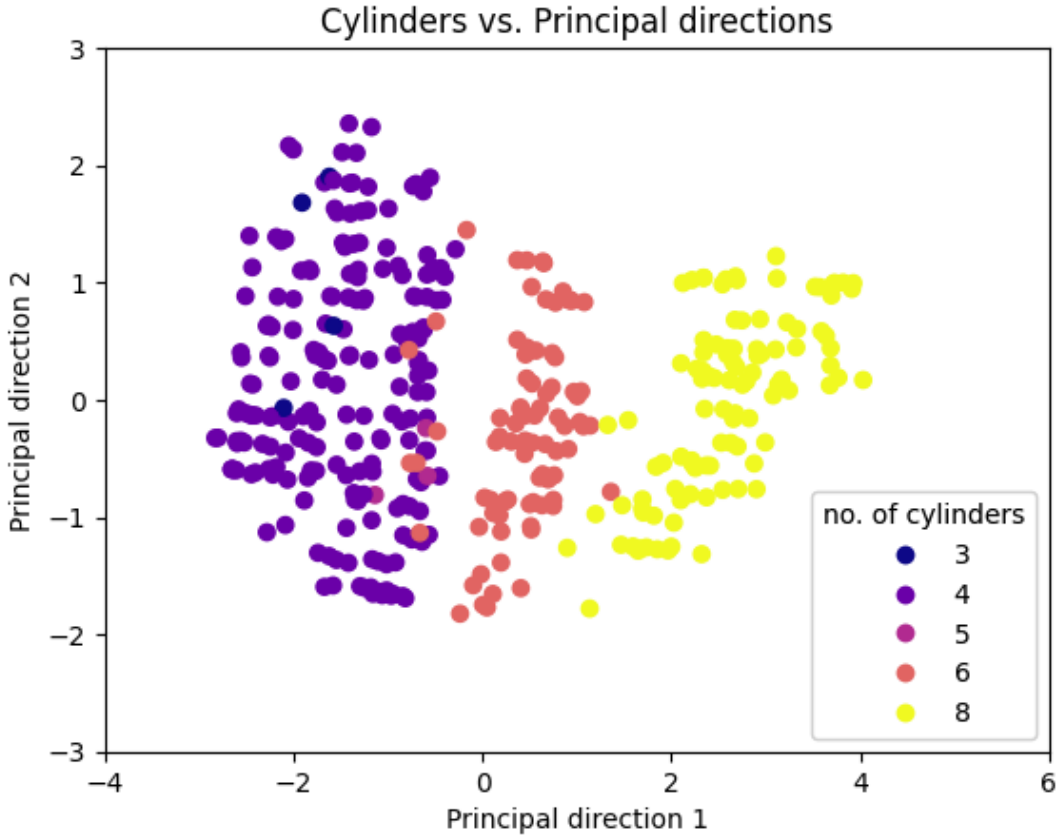
Out[31]:

	pc1	pc2	mpg	cylinders	displacement	horsepower	weight	acceleration	year	origin	mpg01
0	2.123137	0.996304	18.0	8	307.0	130	3504	12.0	70	1	0
1	2.696291	1.021809	15.0	8	350.0	165	3693	11.5	70	1	0
2	2.343180	1.041101	18.0	8	318.0	150	3436	11.0	70	1	0
3	2.341348	1.041616	16.0	8	304.0	150	3433	12.0	70	1	0
4	2.220335	1.022308	17.0	8	302.0	140	3449	10.5	70	1	0
...	...	...	...	...	...	...	...	...	...	...	...
387	-0.966711	-1.670783	27.0	4	140.0	86	2790	15.6	82	1	1
388	-2.283038	-1.129417	44.0	4	97.0	52	2130	24.6	82	2	1
389	-1.295137	-1.589094	32.0	4	135.0	84	2295	11.6	82	1	1
390	-1.159016	-1.654041	28.0	4	120.0	79	2625	18.6	82	1	1
391	-1.061770	-1.665386	31.0	4	119.0	82	2720	19.4	82	1	1

392 rows × 11 columns

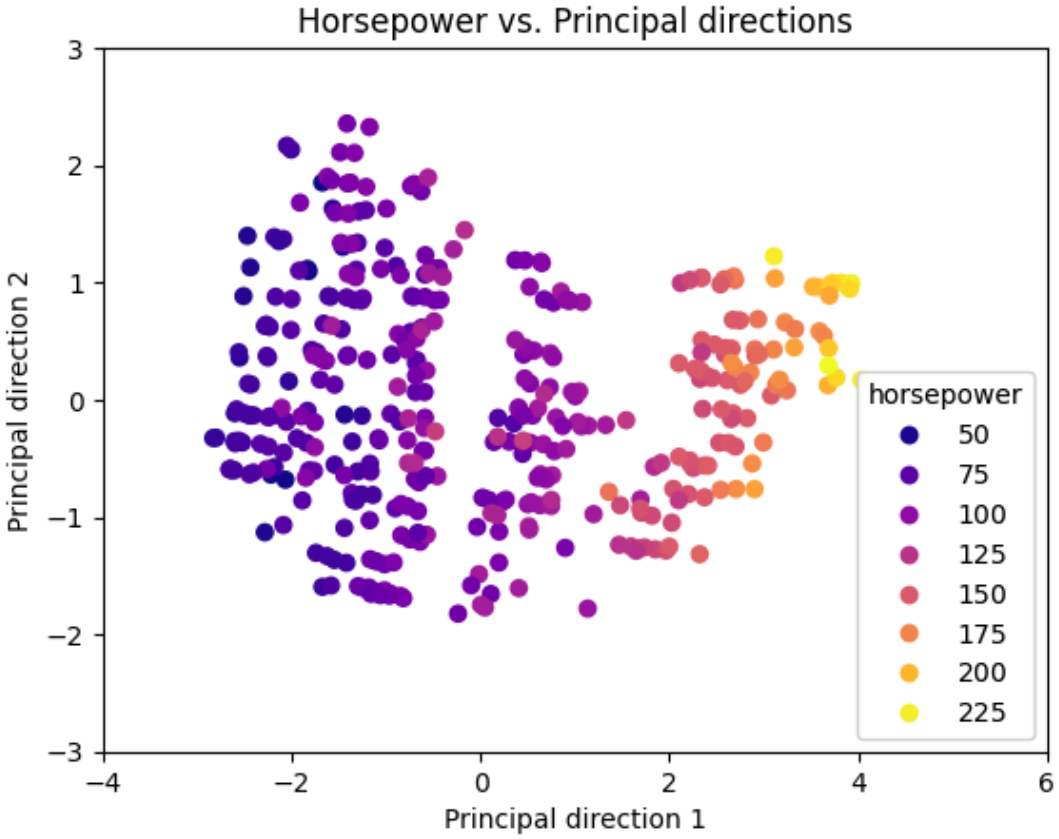
In [32]:
```
# PC1, cylinders, displacement, weight, origin
# PC2, year
```

In [33]:
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
ax1.set(xlim=(-4, 6), ylim=(-3, 3))
scatter = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['cylinders'], cmap='plasma')
legend1 = ax1.legend(*scatter.legend_elements(), loc="lower right", title="no. of cylinders")
ax1.add_artist(legend1)
plt.title('Cylinders vs. Principal directions')
plt.xlabel('Principal direction 1')
plt.ylabel('Principal direction 2')
plt.show()
```

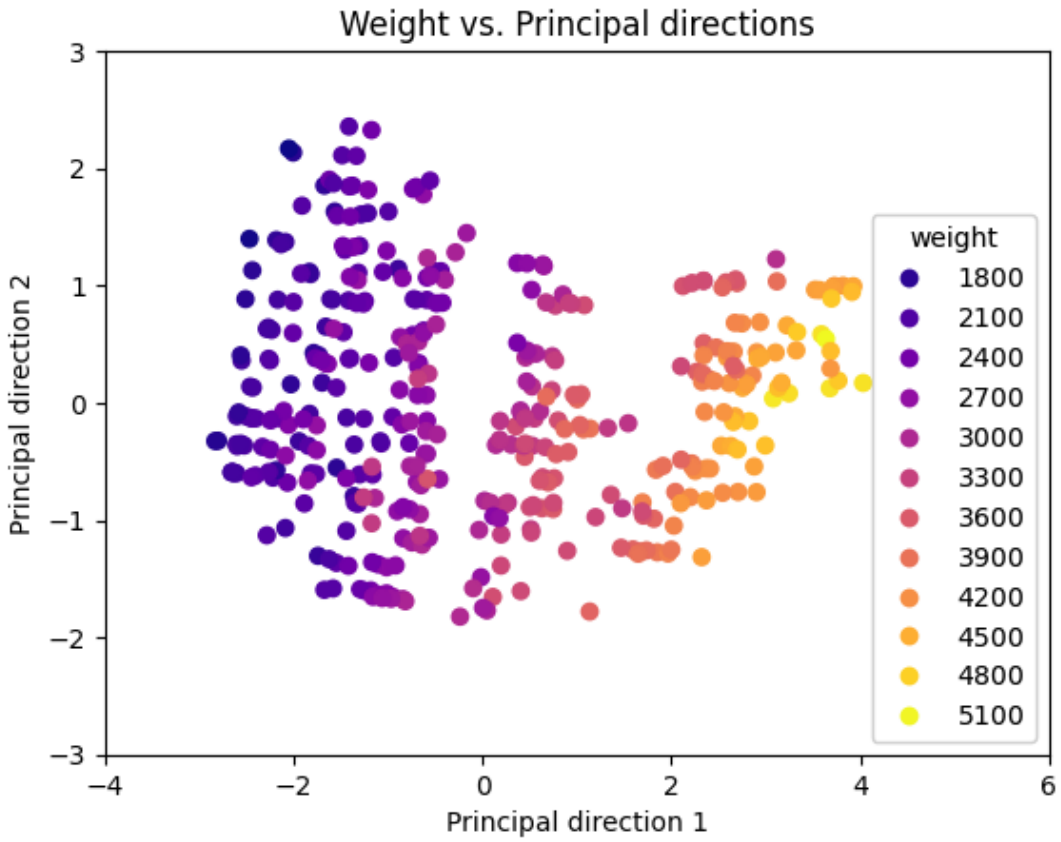


In [34]:

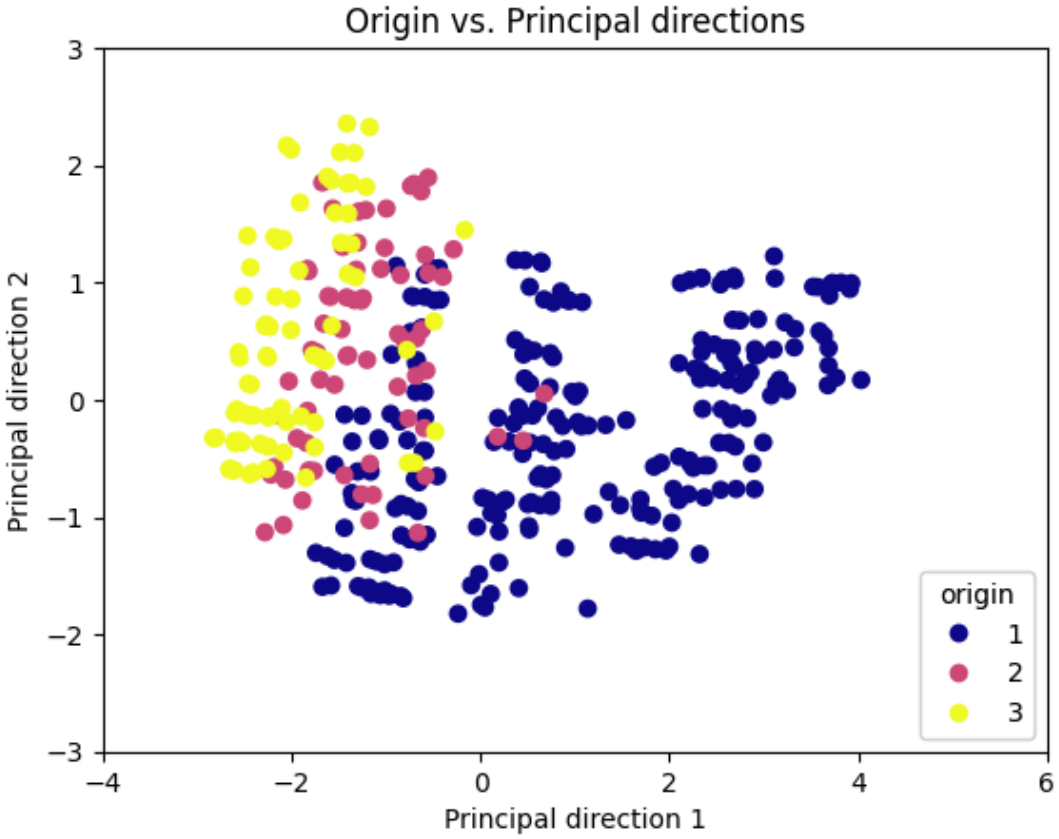
```
fig, ax1 = plt.subplots()
ax1.set(xlim=(-4, 6), ylim=(-3, 3))
scatter = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['horsepower'], cmap='plasma')
legend1 = ax1.legend(*scatter.legend_elements(), loc="lower right", title="horsepower")
ax1.add_artist(legend1)
plt.title('Horsepower vs. Principal directions')
plt.xlabel('Principal direction 1')
plt.ylabel('Principal direction 2')
plt.show()
```



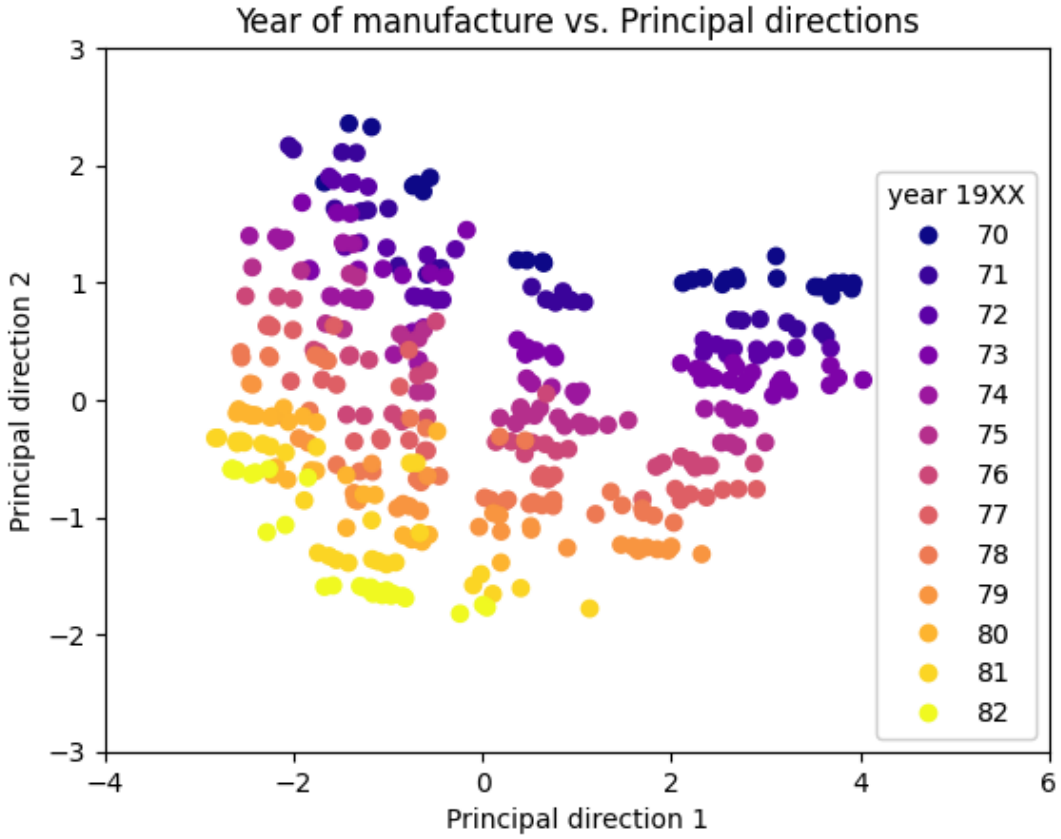
In [35]:
```
fig, ax1 = plt.subplots()
ax1.set(xlim=(-4, 6), ylim=(-3, 3))
scatter = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['weight'], cmap='plasma')
legend1 = ax1.legend(*scatter.legend_elements(), loc="lower right", title="weight")
ax1.add_artist(legend1)
plt.title('Weight vs. Principal directions')
plt.xlabel('Principal direction 1')
plt.ylabel('Principal direction 2')
plt.show()
```



In [36]:
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
ax1.set(xlim=(-4, 6), ylim=(-3, 3))
scatter = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['origin'], cmap='plasma')
legend1 = ax1.legend(*scatter.legend_elements(), loc="lower right", title="origin")
ax1.add_artist(legend1)
plt.title('Origin vs. Principal directions')
plt.xlabel('Principal direction 1')
plt.ylabel('Principal direction 2')
plt.show()
```



In [37]:
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
ax1.set(xlim=(-4, 6), ylim=(-3, 3))
scatter = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['year'], cmap='plasma')
legend1 = ax1.legend(*scatter.legend_elements(), loc="lower right", title="year 19XX")
ax1.add_artist(legend1)
plt.title('Year of manufacture vs. Principal directions')
plt.xlabel('Principal direction 1')
plt.ylabel('Principal direction 2')
plt.show()
```



In [38]:
```python
# Create a grid over the 2D plot axes domain
grid1 = np.linspace(-4,6,num=100)
grid2 = np.linspace(-3,3,num=100)
grid_list = []
for x in grid1:
    for y in grid2:
        grid_list.append([x, y])

grid = pd.DataFrame(grid_list, columns=['pc1', 'pc2'])
```

In [39]:
```python
# Re-specify model architectures

# best parameters
#best_shrinkage = 0
#best_l1_ratio = 1
#best_k = 1
```

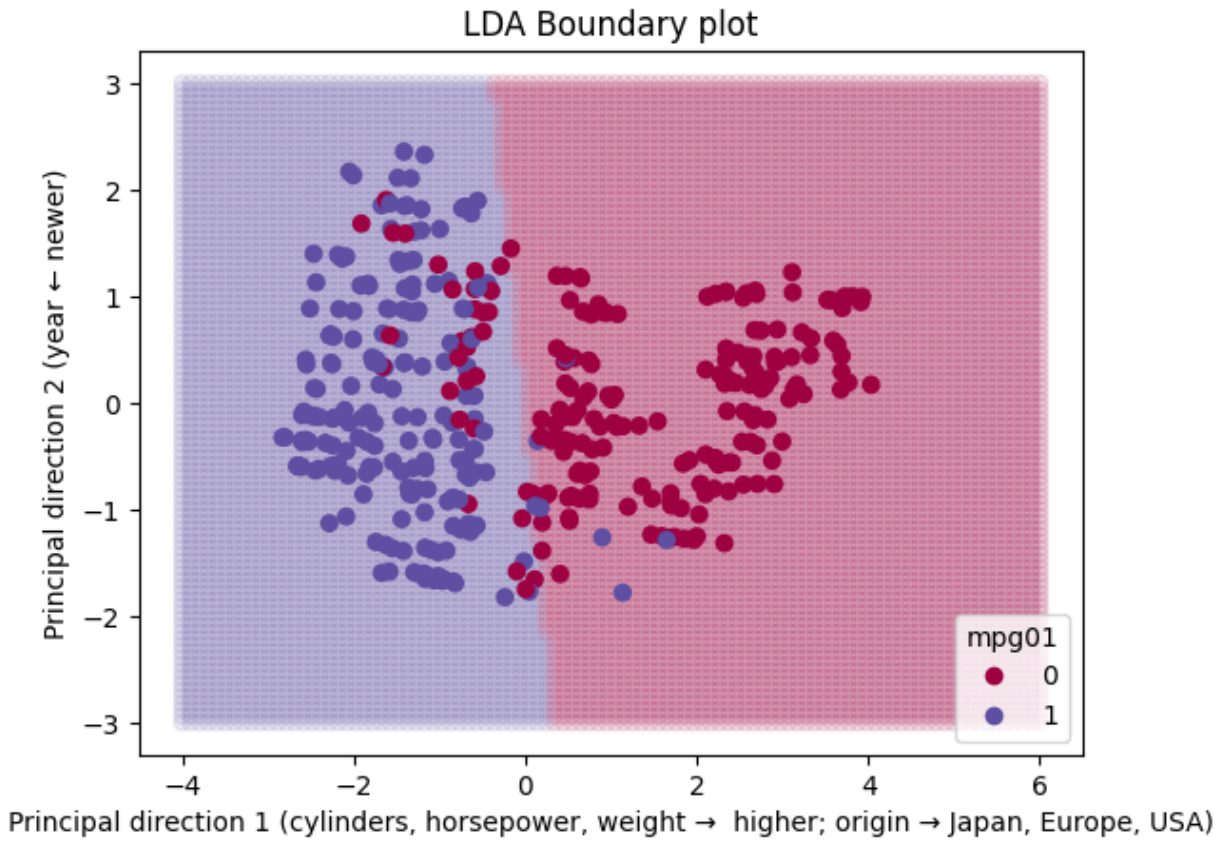
```
#best_depth_cart = 5
#best_n_trees = 5
#best_depth_rf = 6
#best_n_estimators = 200
#best_learn_rate = 0.35

lda_0 = LinearDiscriminantAnalysis(solver='eigen', shrinkage=best_shrinkage)
logit_0 = LogisticRegression(penalty='elasticnet', solver='saga',
                             max_iter=500, l1_ratio=best_l1_ratio)
knn_0 = KNeighborsClassifier(n_neighbors=best_k)
cart_0 = tree.DecisionTreeClassifier(max_depth=best_depth_cart)
rforest_0 = RandomForestClassifier(n_estimators=best_n_trees, max_depth=best_depth_rf, random_state=0)
adaboost_0 = AdaBoostClassifier(n_estimators=best_n_estimators, learning_rate = best_learn_rate, random_state=0)

# Calculate/predict grid 'mpg01' labels for various models re-trained on PC2 data
grid['lda'] = lda_0.fit(pc2_X_train, y_train).predict(grid.iloc[:,:2])
grid['logit'] = logit_0.fit(pc2_X_train, y_train).predict(grid.iloc[:,:2])
grid['knn'] = knn_0.fit(pc2_X_train, y_train).predict(grid.iloc[:,:2])
grid['cart'] = cart_0.fit(pc2_X_train, y_train).predict(grid.iloc[:,:2])
grid['rforest'] = rforest_0.fit(pc2_X_train, y_train).predict(grid.iloc[:,:2])
grid['adaboost'] = adaboost_0.fit(pc2_X_train, y_train).predict(grid.iloc[:,:2])
```

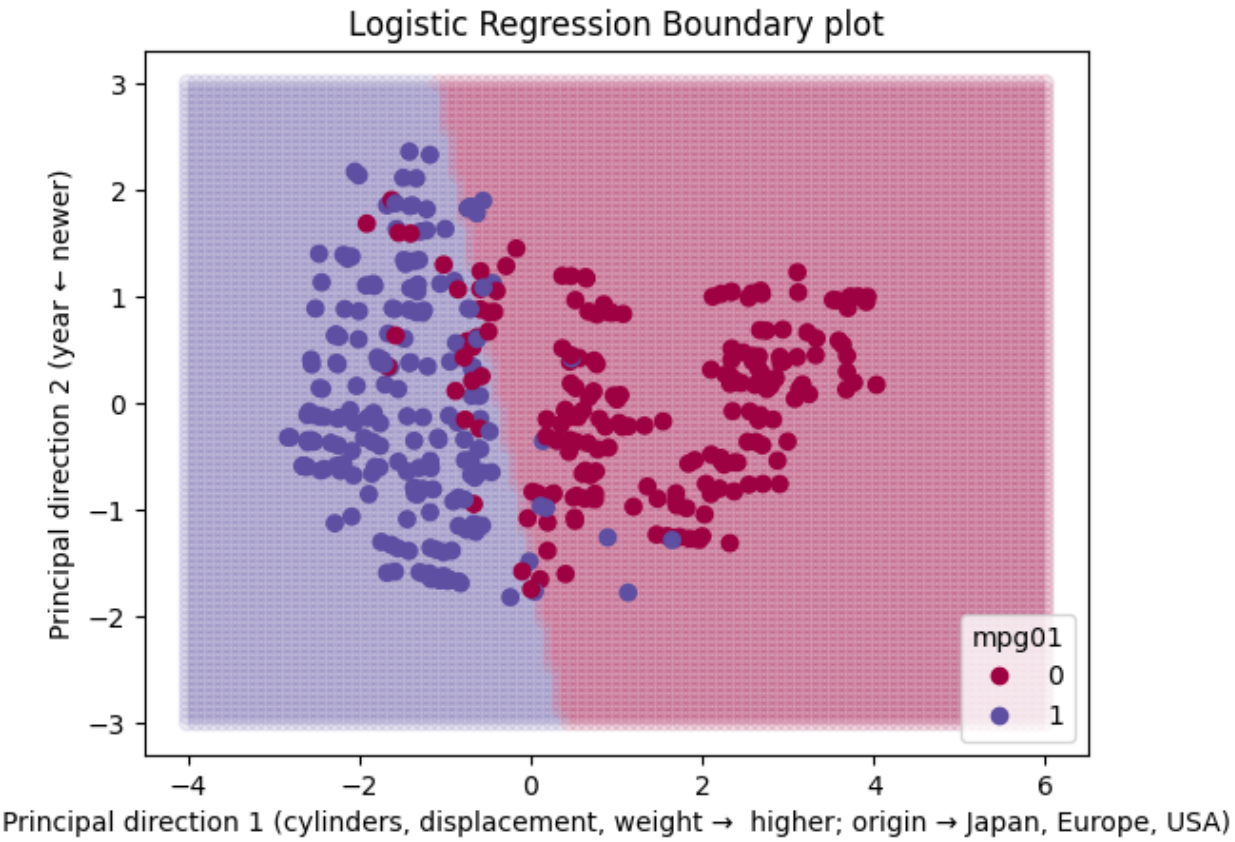
In [40]:

```
fig = plt.figure()
ax1 = fig.add_subplot(111)
scatter1 = ax1.scatter(grid['pc1'], grid['pc2'], c=grid['lda'], cmap='Spectral', alpha=0.1)
scatter2 = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['mpg01'], cmap='Spectral')
legend2 = ax1.legend(*scatter2.legend_elements(), loc="lower right", title="mpg01")
plt.title('LDA Boundary plot')
plt.xlabel('Principal direction 1 (cylinders, horsepower, weight →  higher; origin → Japan, Europe, USA)')
plt.ylabel('Principal direction 2 (year ← newer)')
plt.show()
```

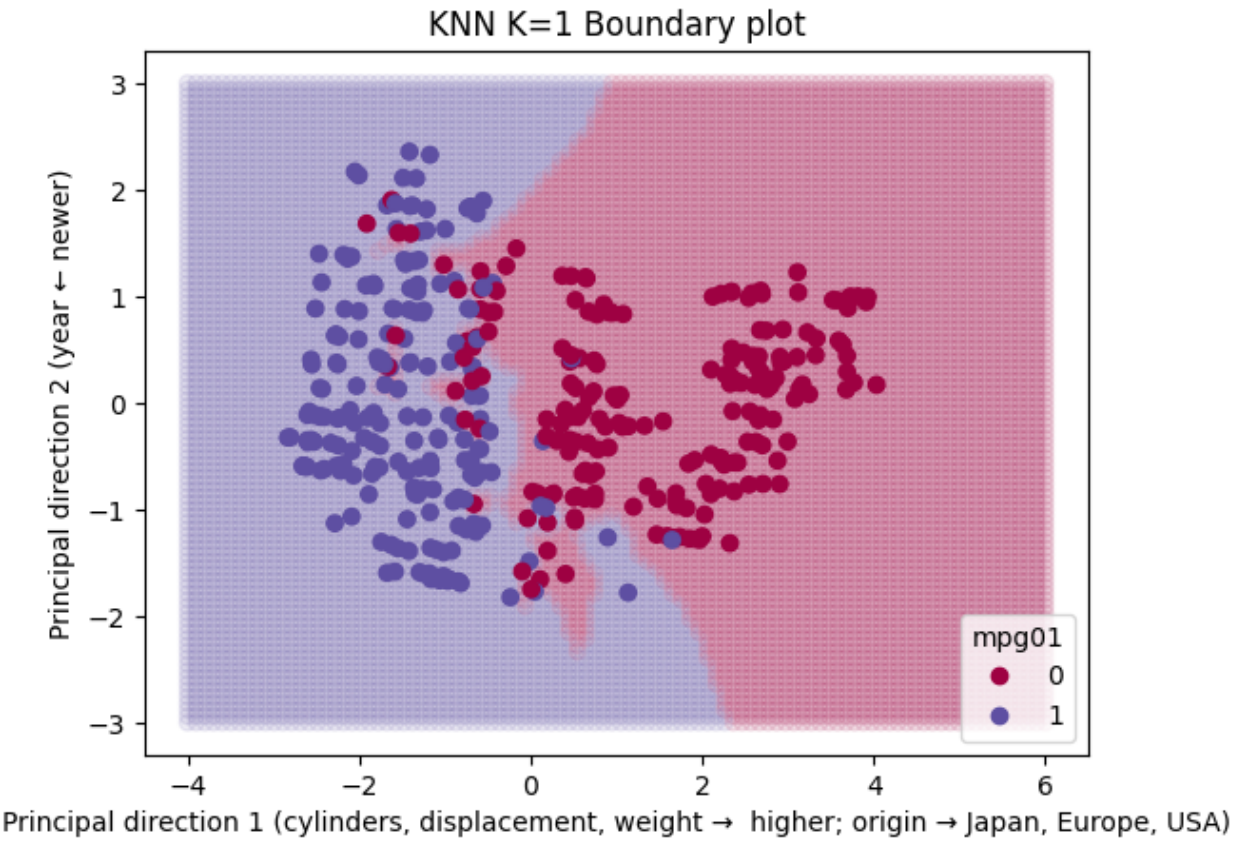


In [41]:

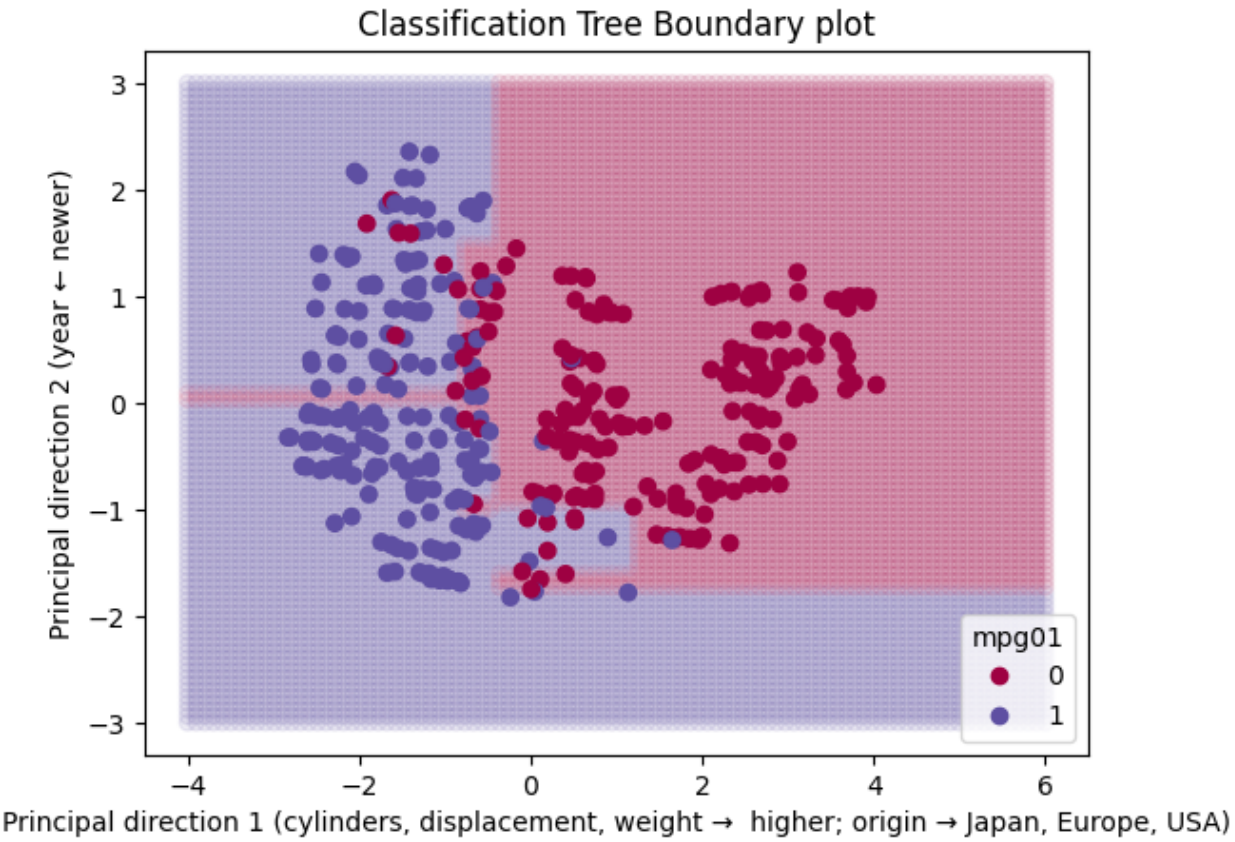
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
scatter1 = ax1.scatter(grid['pc1'], grid['pc2'], c=grid['logit'], cmap='Spectral', alpha=0.1)
scatter2 = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['mpg01'], cmap='Spectral')
legend2 = ax1.legend(*scatter2.legend_elements(), loc="lower right", title="mpg01")
plt.title('Logistic Regression Boundary plot')
plt.xlabel('Principal direction 1 (cylinders, displacement, weight →  higher; origin → Japan, Europe, USA)')
plt.ylabel('Principal direction 2 (year ← newer)')
plt.show()
```



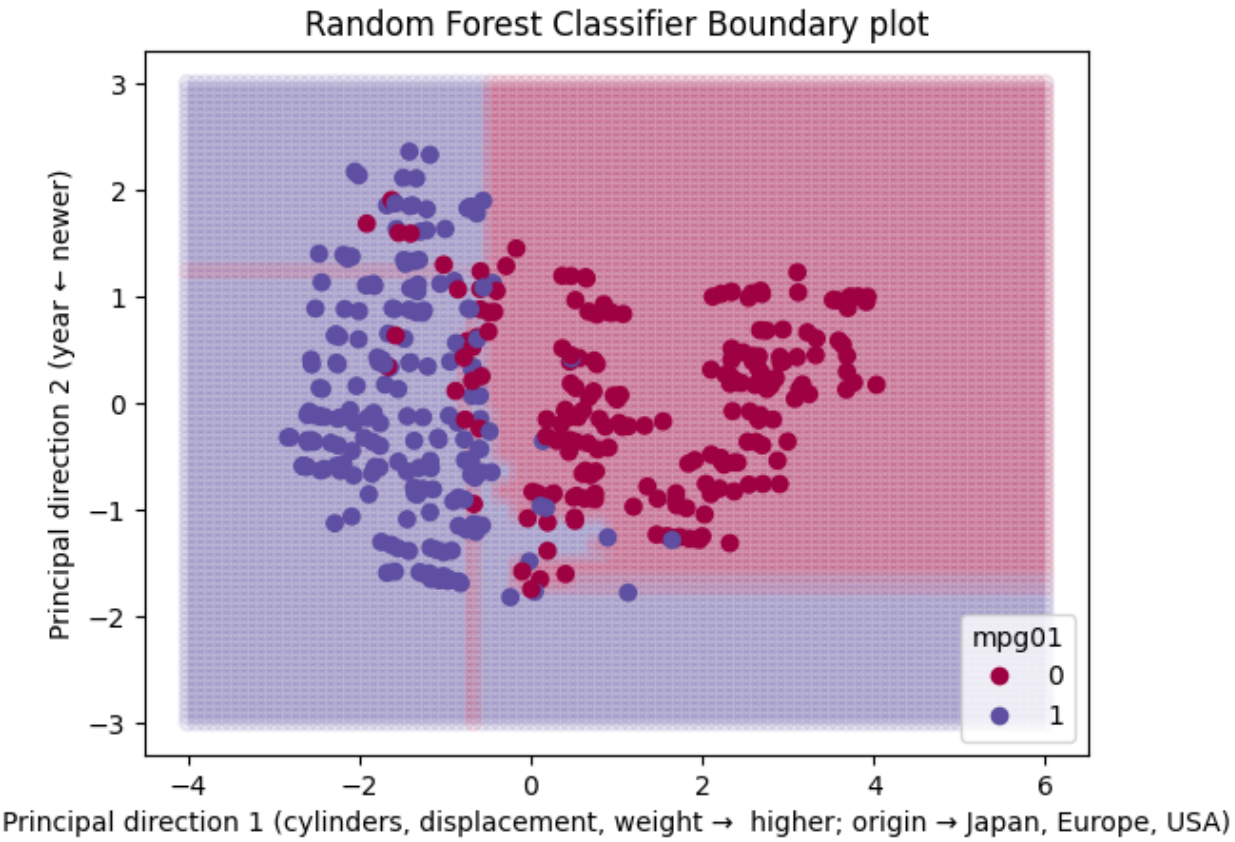
In [42]:
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
scatter1 = ax1.scatter(grid['pc1'], grid['pc2'], c=grid['knn'], cmap='Spectral', alpha=0.1)
scatter2 = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['mpg01'], cmap='Spectral')
legend2 = ax1.legend(*scatter2.legend_elements(), loc="lower right", title="mpg01")
plt.title('KNN K=1 Boundary plot')
plt.xlabel('Principal direction 1 (cylinders, displacement, weight →  higher; origin → Japan, Europe, USA)')
plt.ylabel('Principal direction 2 (year ← newer)')
plt.show()
```



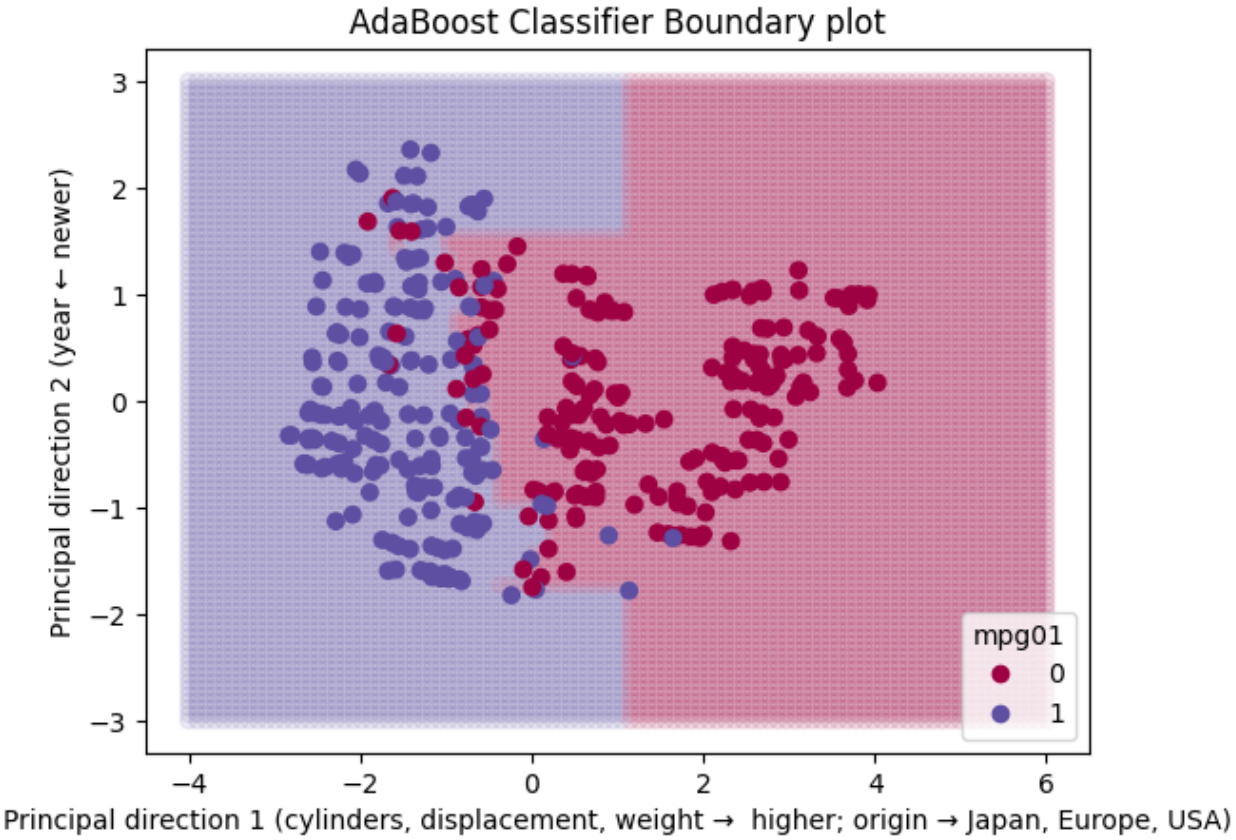
In [43]:
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
scatter1 = ax1.scatter(grid['pc1'], grid['pc2'], c=grid['cart'], cmap='Spectral', alpha=0.1)
scatter2 = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['mpg01'], cmap='Spectral')
legend2 = ax1.legend(*scatter2.legend_elements(), loc="lower right", title="mpg01")
plt.title('Classification Tree Boundary plot')
plt.xlabel('Principal direction 1 (cylinders, displacement, weight →  higher; origin → Japan, Europe, USA)')
plt.ylabel('Principal direction 2 (year ← newer)')
plt.show()
```



In [44]:
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
scatter1 = ax1.scatter(grid['pc1'], grid['pc2'], c=grid['rforest'], cmap='Spectral', alpha=0.1)
scatter2 = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['mpg01'], cmap='Spectral')
legend2 = ax1.legend(*scatter2.legend_elements(), loc="lower right", title="mpg01")
plt.title('Random Forest Classifier Boundary plot')
plt.xlabel('Principal direction 1 (cylinders, displacement, weight →  higher; origin → Japan, Europe, USA)')
plt.ylabel('Principal direction 2 (year ← newer)')
plt.show()
```



In [45]:
```
fig = plt.figure()
ax1 = fig.add_subplot(111)
scatter1 = ax1.scatter(grid['pc1'], grid['pc2'], c=grid['adaboost'], cmap='Spectral', alpha=0.1)
scatter2 = ax1.scatter(pc12_auto_data['pc1'], pc12_auto_data['pc2'], c=pc12_auto_data['mpg01'], cmap='Spectral')
legend2 = ax1.legend(*scatter2.legend_elements(), loc="lower right", title="mpg01")
plt.title('AdaBoost Classifier Boundary plot')
plt.xlabel('Principal direction 1 (cylinders, displacement, weight →  higher; origin → Japan, Europe, USA)')
plt.ylabel('Principal direction 2 (year ← newer)')
plt.show()
```


AdaBoost Classifier Boundary plot
Principal direction 2 (year ← newer)
Principal direction 1 (cylinders, displacement, weight → higher; origin → Japan, Europe, USA)
mpg01
0
1
3
2
1
0
−1
−2
−3
−4
−2
0
2
4
6

In [ ]: