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Contest (1)

template.cpp33 lines

```
#include <bits/stdc++.h>
using namespace std;
#define sws cin.tie(0)->sync_with_stdio(0)

#define endl '\n'
#define ll long long
#define ld long double
#define pb push_back
#define ff first
#define ss second
#define pll pair<ll, ll>
#define vll vector<ll>

#define teto(a, b) ((a+b-1)/(b))
#define LSB(i) ((i) & -(i))
#define MSB(i) (32 - __builtin_clz(i)) //64 - clzll
#define BITS(i) __builtin_popcountll(i) //count set bits

mt19937 rng(chrono::steady_clock::now().time_since_epoch()).count();

#define debug(a...) cerr<<#a<<" ";for(auto b:a)cerr<<b<<" ";
cerr<<endl;
template<typename... A> void dbg(A const&... a){((cerr<<"{"<<a
<<"} ", ...);cerr<<endl;};

const int MAX = 3e5+10;
const int INF = INT32_MAX;
const long long MOD = 1e9+7;
const long long LLINF = INT64_MAX;
const long double EPS = 1e-7;
const long double PI = acos(-1);
```

1

int32_t main(){ sws;

1

}

3

.bashrc1 lines

4

alias comp='g++ -std=c++17 -g3 -ggdb3 -O3 -Wall -Wextra -fsanitize=address,undefined -Wshadow -Wconversion -D_GLIBCXX_ASSERTIONS -o test'

4

hash.sh3 lines

6

Hashes a file, ignoring all whitespace and comments. Use for # verifying that code was correctly typed. CTRL+D to send EOF

6

c++ -dD -P -fpreprocessed | tr -d '[:space:]' | md5sum | cut -c -6

6

troubleshoot.txt52 lines

6

Pre-submit: Write a few simple test cases if sample is not enough. Are time limits close? If so, generate max cases. Is the memory usage fine? Could anything overflow? Make sure to submit the right file.

11

Wrong answer: Print your solution! Print debug output, as well. Are you clearing all data structures between test cases? Can your algorithm handle the whole range of input? Read the full problem statement again. Do you handle all corner cases correctly? Have you understood the problem correctly? Any uninitialized variables? Any overflows? Confusing N and M, i and j, etc.? Are you sure your algorithm works? What special cases have you not thought of? Are you sure the STL functions you use work as you think? Add some assertions, maybe resubmit. Create some testcases to run your algorithm on. Go through the algorithm for a simple case. Go through this list again. Explain your algorithm to a teammate. Ask the teammate to look at your code. Go for a small walk, e.g. to the toilet. Is your output format correct? (including whitespace) Rewrite your solution from the start or let a teammate do it.

11

Runtime error: Have you tested all corner cases locally? Any uninitialized variables? Are you reading or writing outside the range of any vector? Any assertions that might fail? Any possible division by 0? (mod 0 for example) Any possible infinite recursion? Invalidated pointers or iterators? Are you using too much memory? Debug with resubmits (e.g. remapped signals, see Various).

13

Time limit exceeded: Do you have any possible infinite loops? What is the complexity of your algorithm? Are you copying a lot of unnecessary data? (References) How big is the input and output? (consider scanf) Avoid vector, map. (use arrays/unordered_map) What do your teammates think about your algorithm?

Memory limit exceeded:
What is the max amount of memory your algorithm should need?
Are you clearing all data structures between test cases?

Mathematics (2)

2.1 Equations

$$ax^2 + bx + c = 0 \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The extremum is given by $x = -b/2a$.

$$\begin{aligned} ax + by &= e & x &= \frac{ed - bf}{ad - bc} \\ cx + dy &= f & y &= \frac{af - ec}{ad - bc} \end{aligned}$$

In general, given an equation $Ax = b$, the solution to a variable x_i is given by

$$x_i = \frac{\det A'_i}{\det A}$$

where A'_i is A with the i 'th column replaced by b .

2.2 Recurrences

If $a_n = c_1a_{n-1} + \dots + c_ka_{n-k}$, and r_1, \dots, r_k are distinct roots of $x^k - c_1x^{k-1} - \dots - c_k$, there are d_1, \dots, d_k s.t.

$$a_n = d_1r_1^n + \dots + d_kr_k^n.$$

Non-distinct roots r become polynomial factors, e.g. $a_n = (d_1n + d_2)r^n$.

2.3 Trigonometry

$$\begin{aligned} \sin(v + w) &= \sin v \cos w + \cos v \sin w \\ \cos(v + w) &= \cos v \cos w - \sin v \sin w \end{aligned}$$

$$\tan(v + w) = \frac{\tan v + \tan w}{1 - \tan v \tan w}$$

$$\sin v + \sin w = 2 \sin \frac{v + w}{2} \cos \frac{v - w}{2}$$

$$\cos v + \cos w = 2 \cos \frac{v + w}{2} \cos \frac{v - w}{2}$$

$$(V + W) \tan(v - w)/2 = (V - W) \tan(v + w)/2$$

where V, W are lengths of sides opposite angles v, w .

$$a \cos x + b \sin x = r \cos(x - \phi)$$

$$a \sin x + b \cos x = r \sin(x + \phi)$$

where $r = \sqrt{a^2 + b^2}, \phi = \operatorname{atan2}(b, a)$.

2.4 Geometry

2.4.1 Triangles

Side lengths: a, b, c

Semiperimeter: $p = \frac{a + b + c}{2}$

Area: $A = \sqrt{p(p - a)(p - b)(p - c)}$

Circumradius: $R = \frac{abc}{4A}$

Inradius: $r = \frac{A}{p}$

Length of median (divides triangle into two equal-area triangles):
 $m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}$

Length of bisector (divides angles in two):

$$s_a = \sqrt{bc \left[1 - \left(\frac{a}{b + c} \right)^2 \right]}$$

Law of sines: $\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c} = \frac{1}{2R}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc \cos \alpha$

Law of tangents: $\frac{a + b}{a - b} = \frac{\tan \frac{\alpha + \beta}{2}}{\tan \frac{\alpha - \beta}{2}}$

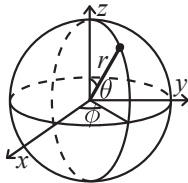
2.4.2 Quadrilaterals

With side lengths a, b, c, d , diagonals e, f , diagonals angle θ , area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin \theta = F \tan \theta = \sqrt{4e^2 f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180° , $ef = ac + bd$, and $A = \sqrt{(p - a)(p - b)(p - c)(p - d)}$.

2.4.3 Spherical coordinates



$$\begin{aligned} x &= r \sin \theta \cos \phi & r &= \sqrt{x^2 + y^2 + z^2} \\ y &= r \sin \theta \sin \phi & \theta &= \arccos(z / \sqrt{x^2 + y^2 + z^2}) \\ z &= r \cos \theta & \phi &= \operatorname{atan2}(y, x) \end{aligned}$$

2.5 Derivatives/Integrals

$$\begin{aligned} \frac{d}{dx} \arcsin x &= \frac{1}{\sqrt{1 - x^2}} & \frac{d}{dx} \arccos x &= -\frac{1}{\sqrt{1 - x^2}} \\ \frac{d}{dx} \tan x &= 1 + \tan^2 x & \frac{d}{dx} \arctan x &= \frac{1}{1 + x^2} \\ \int \tan ax &= -\frac{\ln |\cos ax|}{a} & \int x \sin ax &= \frac{\sin ax - ax \cos ax}{a^2} \\ \int e^{-x^2} &= \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) & \int x e^{ax} dx &= \frac{e^{ax}}{a^2} (ax - 1) \end{aligned}$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

2.6 Sums

$$c^a + c^{a+1} + \dots + c^b = \frac{c^{b+1} - c^a}{c - 1}, c \neq 1$$

$$\begin{aligned} 1 + 2 + 3 + \dots + n &= \frac{n(n + 1)}{2} \\ 1^2 + 2^2 + 3^2 + \dots + n^2 &= \frac{n(2n + 1)(n + 1)}{6} \\ 1^3 + 2^3 + 3^3 + \dots + n^3 &= \frac{n^2(n + 1)^2}{4} \\ 1^4 + 2^4 + 3^4 + \dots + n^4 &= \frac{n(n + 1)(2n + 1)(3n^2 + 3n - 1)}{30} \end{aligned}$$

2.7 Series

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, (-\infty < x < \infty) \\ \ln(1 + x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, (-1 < x \leq 1) \\ \sqrt{1 + x} &= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \dots, (-1 \leq x \leq 1) \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, (-\infty < x < \infty) \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, (-\infty < x < \infty) \end{aligned}$$

2.8 Probability theory

Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x . It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x x p_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation. If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y ,

$$V(aX + bY) = a^2 V(X) + b^2 V(Y).$$

2.8.1 Discrete distributions

Binomial distribution

The number of successes in n independent yes/no experiments, each which yields success with probability p is $\operatorname{Bin}(n, p)$, $n = 1, 2, \dots$, $0 \leq p \leq 1$.

$$p(k) = \binom{n}{k} p^k (1 - p)^{n - k}$$

$$\mu = np, \sigma^2 = np(1 - p)$$

$\operatorname{Bin}(n, p)$ is approximately $\operatorname{Po}(np)$ for small p .

First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each which yields success with probability p is $\operatorname{Fs}(p)$, $0 \leq p \leq 1$.

$$p(k) = p(1 - p)^{k - 1}, k = 1, 2, \dots$$

$$\mu = \frac{1}{p}, \sigma^2 = \frac{1 - p}{p^2}$$

Poisson distribution

The number of events occurring in a fixed period of time t if these events occur with a known average rate κ and independently of the time since the last event is $\operatorname{Po}(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, 2, \dots$$

$$\mu = \lambda, \sigma^2 = \lambda$$

2.8.2 Continuous distributions

Uniform distribution

If the probability density function is constant between a and b and 0 elsewhere it is $U(a, b)$, $a < b$.

$$f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$$
$$\mu = \frac{a+b}{2}, \sigma^2 = \frac{(b-a)^2}{12}$$

Exponential distribution

The time between events in a Poisson process is $\text{Exp}(\lambda)$, $\lambda > 0$.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
$$\mu = \frac{1}{\lambda}, \sigma^2 = \frac{1}{\lambda^2}$$

Normal distribution

Most real random values with mean μ and variance σ^2 are well described by $\mathcal{N}(\mu, \sigma^2)$, $\sigma > 0$.

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ then

$$aX_1 + bX_2 + c \sim \mathcal{N}(\mu_1 + \mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$$

2.9 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let X_1, X_2, \dots be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathbf{P} = (p_{ij})$, with $p_{ij} = \Pr(X_n = i | X_{n-1} = j)$, and $\mathbf{p}^{(n)} = \mathbf{P}^n \mathbf{p}^{(0)}$ is the probability distribution for X_n (i.e., $p_i^{(n)} = \Pr(X_n = i)$), where $\mathbf{p}^{(0)}$ is the initial distribution.

π is a stationary distribution if $\pi = \pi \mathbf{P}$. If the Markov chain is *irreducible* (it is possible to get to any state from any state), then $\pi_i = \frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state i . π_j / π_i is the expected number of visits in state j between two visits in state i .

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, π_i is proportional to node i 's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1). $\lim_{k \rightarrow \infty} \mathbf{P}^k = \mathbf{1}\pi$.

A Markov chain is an A-chain if the states can be partitioned into two sets \mathbf{A} and \mathbf{G} , such that all states in \mathbf{A} are absorbing ($p_{ii} = 1$), and all states in \mathbf{G} leads to an absorbing state in \mathbf{A} . The probability for absorption in state $i \in \mathbf{A}$, when the initial state is j , is $a_{ij} = p_{ij} + \sum_{k \in \mathbf{G}} a_{ik} p_{kj}$. The expected time until absorption, when the initial state is i , is $t_i = 1 + \sum_{k \in \mathbf{G}} p_{ki} t_k$.

Data structures (3)

3.1 Ordered Set

Policy Based Data Structures (PBDS) from gcc compiler

Ordered Multiset can be created using `ordered_set<pll>val, idx`

`order_of_key()` can search for non-existent keys!

`find_by_order()` requires existent key and return the 0-idx position of the given value. Therefore, it returns the numbers of elements that are smaller than the given value;

```
ordered-set.cpp
Description: Set with index operators, implemented by gnu pbds. Remember to compile with gcc!!
Time:  $\mathcal{O}(\log(N))$  but with slow constant
<bits/extc++.h>, <bits/extc++.h> 8578e5, 11 lines

// 0-idx
// find_by_order(i) -> iterator to elem with index i
// order_of_key(val) -> index of key

// Ordered Set
using namespace __gnu_pbds;
template <class T> using ordered_set = tree<T, null_type, less<T>, rb_tree_tag, tree_order_statistics_node_update>;

// Ordered Map
using namespace __gnu_pbds;
template <class K, class V> using ordered_map = tree<K, V, less<K>, rb_tree_tag, tree_order_statistics_node_update>;
```

3.2 Disjoint Set Union

There are two optional improvements:

-Tree Balancing

-Path Compression

If one improvement is used, the time complexity will become $\mathcal{O}(\log N)$

If both are used, $\mathcal{O}(\alpha) \approx \mathcal{O}(5)$

```
dsu.cpp
Description: Disjoint Set Union with path compression and tree balancing
Time:  $\mathcal{O}(\alpha)$ 
0479c4, 22 lines

struct DSU{
    vll group, card;
    DSU (ll n){
        n += 1; // 0-idx -> 1-idx
        group = vll(n);
```

```
        iota(group.begin(), group.end(), 0);
        card = vll(n, 1);
    }
    ll find(ll i){
        return (i == group[i]) ? i : (group[i] = find(group[i]));
    }
    // returns false if a and b are already in the same component
    bool join(ll a ,ll b){
        a = find(a);
        b = find(b);
        if (a == b) return false;
        if (card[a] < card[b]) swap(a, b);
        card[a] += card[b];
        group[b] = a;
        return true;
    }
};
```

3.3 Segment Tree

Each node of the segment tree represents the cumulative value of a range.

Observation: For some problems, such as range distinct values query, considerer offline approach, ordering the queries by L for example.

```
segRecursive.cpp
Description: Basic Recursive Segment Tree for points increase and range sum query. When initializing the segmente tree, remeber to choose the range limits (L, R)
Time:  $\mathcal{O}(N \log N)$  to build,  $\mathcal{O}(\log N)$  to increase or query 156cd2, 70 lines

// [0, n] segtree for range sum query, point increase
ll L=0, R;
struct Segtree {

    struct Node {
        // null element:
        ll ps = 0;
    };

    vector<Node> tree;
    vector<ll> v;

    Segtree(ll n) {
        R = n;
        v.assign(n+1, 0);
        tree.assign(4*(n+1), Node{});
    }

    Node merge(Node a, Node b) {
        return Node {
            // merge operaton:
            a.ps + b.ps
        };
    }

    void build(ll l=L, ll r=R, ll i=1 ) {
        if (l == r) {
            tree[i] = Node {
                // leaf element:
                v[l]
            };
        }
        else {
            ll mid = (l+r)/2;
```

```
        build(l, mid, 2*i);
        build(mid+1, r, 2*i+1);
        tree[i] = merge(tree[2*i], tree[2*i+1]);
    }
}

void increase(ll idx=1, ll val=0, ll l=L, ll r=R, ll i=1 )
{
    if (l == r) {
        // increase operation:
        tree[i].ps += val;
    }
    else {
        ll mid = (l+r)/2;
        if (idx <= mid) increase(idx, val, l, mid, 2*i);
        else increase(idx, val, mid+1, r, 2*i+1);
        tree[i] = merge(tree[2*i], tree[2*i+1]);
    }
}

Node query(ll left=L, ll right=R, ll l=L, ll r=R, ll i=1) {
    // left/right are the range limits for the query
    // l / r are the internal variables of the tree
    if (right < l or r < left){
        // null element:
        return Node{};
    }
    else if (left <= l and r <= right) return tree[i];
    else{
        ll mid = (l+r)/2;
        return merge(
            query(left, right, l, mid, 2*i),
            query(left, right, mid+1, r, 2*i+1)
        );
    }
}

};
```

3.4 Convex Hull Trick

If multiple transitions of the DP can be seen as first degree polynomials (lines). CHT can be used to optimized it

Some valid functions:

$ax + b$

$cx^2 + ax + b$ (ignore cx^2 if c is independent)

```
cht-dynamic.cpp
Description: Dynamic version of CHT, thefore, one can insert lines in any
order. There is no line removal operator
Time: O(log N) per query and per insertion
707da4, 51 lines

// Convex Hull Trick Dinamico
//
// Para float, use LLINF = 1/.0, div(a, b) = a/b
//
// update(x) atualiza o ponto de intersecao da reta x
// overlap(x) verifica se a reta x sobrepoa a proxima
// add(a, b) adiciona reta da forma ax + b
// query(x) computa maximo de ax + b para entre as retas
// se quiser computar o minimo, eh soh fazer (-a)x + (-b)
//
// O(log(n)) amortizado por insercao
// O(log(n)) por query

struct Line {
    mutable ll a, b, p;
```

```
    bool operator<(const Line& o) const { return a < o.a; }
    bool operator<(ll x) const { return p < x; }
};

struct DynamicCHT : multiset<Line, less<>> {
    ll div(ll a, ll b) {
        return a / b - ((a ^ b) < 0 and a % b);
    }

    void update(iterator x) {
        if (next(x) == end()) x->p = LLINF;
        else if (x->a == next(x)->a) x->p = x->b >= next(x)->b ?
            LLINF : -LLINF;
        else x->p = div(next(x)->b - x->b, x->a - next(x)->a);
    }

    bool overlap(iterator x) {
        update(x);
        if (next(x) == end()) return 0;
        if (x->a == next(x)->a) return x->b >= next(x)->b;
        return x->p >= next(x)->p;
    }

    void add(ll a, ll b) {
        auto x = insert({a, b, 0});
        while (overlap(x)) erase(next(x)), update(x);
        if (x != begin() and !overlap(prev(x))) x = prev(x), update
            (x);
        while (x != begin() and overlap(prev(x)))
            x = prev(x), erase(next(x)), update(x);
    }

    ll query(ll x) {
        assert(!empty());
        auto l = *lower_bound(x);
        return l.a * x + l.b;
    }
};
```

3.5 Li-chao Tree

Works for any type of function that has the **transcending property**:

Given two functions f(x),g(x) of that type, if f(t) is greater than/smaller than g(t) for some x=t, then f(x) will be greater than/smaller than g(x) for x≠t. In other words, once f(x) “win/lose” g(x), f(x) will continue to “win/lose” g(x).

The most common one is the line function: $ax + b$

Dynamic Programming (4)

Game theory (5)

5.1 Classic Game

- There are n piles (heaps), each one with x_i stones.
- Each turn, a players must remove t stones (non-zero) from a pile, turning x_i into y_i .
- The game ends when it’s impossible to make any more moves and the player without moves left lose.

5.2 Bouton’s Theorem

Let s be the xor-sum value of all the piles sizes, a state $s = 0$ is a losing position and a state $s! = 0$ is a winnig position

5.2.1 Proof

All wining positions will have at least one valid move to turn the game into a losing position.

All losing positions will only have moves that turns the game into winning positions (except the base case when there are no piles left and the player already lost)

5.3 DAG Representation

Consider all game positions or states of the game as **Vertices** of a graph

Valid moves are the transition between states, therefore, the directed **Edges** of the graph

If a state has no outgoing edges, it’s a dead end and a losing state (degenerated state).

If a state has only edges to winning states, therefore it is a losing state.

if a state has at least one edge that is a losing state, it is a winning state.

5.4 Sprague-Grundy Theorem

Let’s consider a state u of a two-player impartial game and let v_i be the states reachable from it.

To this state, we can assign a fully equivalent game of Nim with one pile of size x . The number x is called the **Grundy value or nim-value or number** of the state u .

If **all transitions** lead to a *winning state*, the current state must be a *losing state* with number 0.

If **at least one transition** lead to a *losing state*, the current state must be a *winning state* with number $i > 0$.

The **MEX** operator satisfies both condition above and can be used to calculate the nim-value of a state:

$nimber_u = \text{MEX of all } nimber_{v_i}$

Viewing the game as a DAG, we can gradually calculate the Grundy values starting from vertices without outgoing edges (number=0).

Note that the MEX operator **guarantees** that all nim-values smaller than the considered number can be reached, which is essentially the nim game with a single heap with pile size = number.

There are only two operations that are used when considering a Sprague-Grundy game:

5.4.1 Composition

XOR operator to compose sub-games into a single composite game

When a game is played with multiple sub-games (as nim is played with multiple piles), you are actually choosing one sub-game and making a valid move there (choosing a pile and subtracting a value from it).

The final result/winner will depend on all the sub-games played. Because you need to play all games.

To compute the final result, one can simply consider the XOR of the numbers of all sub-games.

5.4.2 Decomposition

MEX operator to compute the number of a state that has multiple transitions to other states

A state with number x can be transitioned (decomposed) into all states with number $y < x$

Nevertheless a state may reach several states, only a single one will be used during the game. This shows the difference between **states** and **sub-games**: All sub-games must be played by the players, but the states of a sub-game may be ignored.

To compute the mex of a set efficiently:

```
mex.cpp
Description: Compute MEX efficiently by keeping track of the frequency
of all existent elements and also the missing ones
Time:  $\mathcal{O}(\log N)$  per addition/removal,  $\mathcal{O}(1)$  to get mex value,  $\mathcal{O}(N \log(N))$ 
to initialize
d6f2b9, 27 lines

struct MEX {
    map<ll, ll> freq;
    set<ll> missing;

    // initialize set with values up to {max_valid_value}
    // inclusive
    MEX(ll max_valid_value) { //  $\mathcal{O}(n \log(n))$ 
        for(ll i=0; i<=max_valid_value; i++)
            missing.insert(i);
    }

    ll get() { //  $\mathcal{O}(1)$ 
        if (missing.empty()) return 0;
        return *missing.begin();
    }

    void remove(ll val) { //  $\mathcal{O}(\log(n))$ 
        freq[val]--;
        if (freq[val] == 0)
            missing.insert(val);
    }

    void add(ll val) { //  $\mathcal{O}(\log(n))$ 
        freq[val]++;
        if (missing.count(val))
            missing.erase(val);
    }
};
```

5.5 Variations and Extensions

5.5.1 Nim with Increases

Consider a modification of the classical nim game: a player can now add stones to a chosen pile instead of removing.

Note that this extra rule needs to have a restriction to keep the game acyclic (finite game).

Lemma: This move is not used in a winning strategy and can be ignored.

Proof: If a player adds t stones in a pile, the next player just needs to remove t stones from this pile.

Considering that the game is finite and this ends sooner or later.

Example: If the set of possible outcomes for a state is 0, 1, 2, 7, 8, 9. The number is 3, because the MEX is 3, which is the smallest nim-value you can't transition into and also you can transition to all smaller nim-values.

Note that 7, 8, 9 transitions can be ignored, because you can simply revert the play by subtracting the same amount.

5.6 Misère Game

In this version, the player who takes the last object loses. To consider this version, simply swap the winning and losing player of the normal version.

5.7 Staircase Nim

5.7.1 Description

In Staircase Nim, there is a staircase with n steps, indexed from 0 to $n-1$. In each step, there are zero or more coins. Two players play in turns. In his/her move, a player can choose a step ($i > 0$) and move one or more coins to step below it ($i-1$). The player who is unable to make a move loses the game. That means the game ends when all the coins are in step 0.

5.7.2 Strategy

We can divide the steps into two types, odd steps, and even steps.

Now let's think what will happen if a player A move x coins from an even step(non-zero) to an odd step. Player B can always move these same x coins to another even position and **the state of odd positions aren't affected**

But if player A moves a coin from an odd step to an even step, similar logic won't work. Due to the degenerated case, there is a situation when x coins are moved from stair 1 to 0, and player B can't move these coins from stair 0 to -1 (not a valid move).

From this argument, we can agree that coins in even steps are useless, they don't interfere to decide if a game state is winning or losing.

Therefore, the staircase nim can be visualized as a simple nim game with only the odd steps.

When stones are sent from an odd step to an even step, it is the same as removing stones from a pile in a classic nim game.

And when stones are sent from even steps to odd ones, it is the same as the increasing variation described before.

5.8 Grundy's Game

Initially there is only one pile with x stones. Each turn, a player must divide a pile into two non-zero piles with different sizes. The player who can't do any more moves loses.

5.8.1 Degenerate (Base) States

$x = 1$ (nim-val = 0) (losing)

$x = 2$ (nim-val = 0) (losing)

5.8.2 Other States

nim-val = MEX (all transitions)

Examples

```
x = 3:
{2, 1} -> (0) xor (0) -> 0

nim-val = MEX({0}) = 1

x = 4:
{3, 1} -> (1) xor (0) -> 1

nim-val = MEX({1}) = 0

x = 5:
{4, 1} -> (0) xor (0) -> 0
{3, 2} -> (1) xor (0) -> 1

nim-val = MEX({0, 1}) = 2

x = 6:
{5, 1} -> (2) xor (0) -> 2
{4, 2} -> (0) xor (0) -> 0

nim-val = MEX({0, 2}) = 1
```

Important observation: All numbers for ($n \geq 2000$) are non-zero. (missing proof here and testing for values above $1e6$).

5.9 Insta-Winning States

Classic nim game: if **all** piles become 0, you lose. (no more moves)

Modified nim game: if **any** pile becomes 0, you lose.

To adapt to this version of nim game, we create insta-winning states, which represents states that have a transition to any empty pile (will instantly win). Insta-winning states must have an specific nimber so they don't conflict with other nimbers when computing. A possible solution is number=INF, because no other number will be high enough to cause conflict.

Because of this adaptation, we can now ignore states with empty piles, and consider them with (null value = -1). And the (nimber = 0) now represents the states that only have transitions to insta-winning states.

After this, beside winning states and losing states, we have added two new categories of states (insta-winning and empty-pile). Notice that:

```
empty-pile <- insta-winning <- nimber(0)
```

Therefore, we have returned to the classical nim game and can proceed normally.

OBS: *Empty piles* (wasn't empty before) (nimber = -1) is different from *Non-existent piles* (never existed) (nimber = 0)

Usage Example:
<https://codeforces.com/gym/101908/problem/B>

5.10 References

https://cp-algorithms.com/game_theory/sprague-grundy-nim.html

<https://codeforces.com/blog/entry/66040>

<https://brilliant.org/wiki/nim/>

Numerical (6)

Number theory (7)

Combinatorial (8)

8.1 Permutations

8.1.1 Factorial

<i>n</i>	1	2	3	4	5	6	7	8	9	10
<i>n</i> !	1	2	6	24	120	720	5040	40320	362880	3628800
<i>n</i>	11	12	13	14	15	16	17			
<i>n</i> !	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
<i>n</i>	20	25	30	40	50	100	150	171		
<i>n</i> !	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

Graph (9)

9.1 Network flow

In optimization theory, maximum flow problems involve finding a feasible flow through a flow network that obtains the maximum possible flow rate.

```

dinic.cpp
Description: Run several bfs to compute the residual graph until a max
flow configuration is discovered
Time: General Case,  $\mathcal{O}(V^2E)$ ; Unit Capacity,  $\mathcal{O}\left((V+E)\sqrt{E}\right)$ ; Bipartite
and unit capacity,  $\mathcal{O}\left((V+E)\sqrt{V}\right)$ 
dealb7, 86 lines

// remember to duplicate vertices for the bipartite graph
// N = number of nodes, including sink and source
const ll N = 700;

struct Dinic {
    struct Edge {
        ll from, to, flow, cap;
    };
    vector<Edge> edges;

    vector<ll> g[N];
    ll ne = 0, lvl[N], vis[N], pass;
    ll qu[N], px[N], qt;

    ll run(ll s, ll sink, ll minE) {
        if (s == sink) return minE;
        ll ans = 0;
        for(; px[s] < (int)g[s].size(); px[s]++){
            ll e = g[s][ px[s] ];
            auto &v = edges[e], &rev = edges[e^1];
            if( lvl[v.to] != lvl[s]+1 || v.flow >= v.cap)
                continue;
            ll tmp = run(v.to, sink, min(minE, v.cap - v.flow));
            ;
            v.flow += tmp, rev.flow -= tmp;
            ans += tmp, minE -= tmp;
            if (minE == 0) break;
        }
        return ans;
    }

    bool bfs(ll source, ll sink) {
        qt = 0;
        qu[qt++] = source;
        lvl[source] = 1;
        vis[source] = ++pass;
        for(ll i=0; i<qt; i++) {
            ll u = qu[i];
            px[u] = 0;
            if (u == sink) return 1;
            for(auto& ed :g[u]) {
                auto v = edges[ed];
                if (v.flow >= v.cap || vis[v.to] == pass)
                    continue;
                vis[v.to] = pass;
                lvl[v.to] = lvl[u]+1;
                qu[qt++] = v.to;
            }
        }
        return false;
    }

    ll flow(ll source, ll sink) { // max_flow

```

```

        reset_flow();
        ll ans = 0;
        while(bfs(source, sink))
            ans += run(source, sink, LLINF);
        return ans;
    }

    void addEdge(ll u, ll v, ll c, ll rc = 0) { // c = capacity
        , rc = retro-capacity;
        Edge e = {u, v, 0, c};
        edges.pb(e);
        g[u].pb(ne++);
        e = {v, u, 0, rc};
        edges.pb(e);
        g[v].pb(ne++);
    }

    void reset_flow() {
        for (ll i=0; i<ne; i++) edges[i].flow = 0;
        memset(lvl, 0, sizeof(lvl));
        memset(vis, 0, sizeof(vis));
        memset(qu, 0, sizeof(qu));
        memset(px, 0, sizeof(px));
        qt = 0; pass = 0;
    }

    // cut set cost = minimum cost = max flow
    // cut set is the set of edges that, if removed,
    // will disrupt flow from source to sink and make it 0.
    vector<pll> cut() {
        vector<pll> cuts;
        for (auto [from, to, flow, cap]: edges)
            if (flow == cap and vis[from] == pass and vis[to] <
                pass and cap > 0)
                cuts.pb({from, to});
        return cuts;
    }
};

```

9.1.1 Minimum Cut (S-component, T-component) as a parameter for science and optimization theory, the max-flow min-cut theorem states that in a flow network the maximum amount of flow passing from the source to the sink is equal to the total weight of the edges in a minimum cut. The smallest total weight of the edges in a minimum cut is the smallest total weight of the edges which if removed would disconnect the source from the sink. The cost of an s-t cut C is defined by the sum of the capacities of the edges in the cut-set of C.

The by-product of computing Max Flow is Min Cut! After Max Flow algorithm stops, we run graph traversal (DFS/BFS) from source s again. All reachable vertices from source s using positive weighted edges in the residual graph belong to the S-component. All other unreachable vertices belong to the T-component. All edges connecting the S-component to the T-component belong to the cut-set of C . The Min Cut value is equal to the Max Flow value. This is the minimum over all possible s-t cuts values.

9.1.2 Matching with Flow

By modeling a bipartite graph, with some Vertices (that will choose a match) to be on the L graph and some Vertices (that will be chosen) on the R. Set the correct capacities for these edges and also for the edges that connects the sink and source. After this modeling and running the dinic max flow algorithm, one will generate a possible matching (if it is possible).

A special case of matching is the perfect matching, which includes all vertices from the bipartite graph L and R.

A maximum matching has the maximum cardinality. A perfect matching is a maximum matching. But the opposite is not necessarily true.

It's possible to access `dinic.edges`, which is a vector that contains all edges and also its respective attributes, like the *flow* passing through each edge. Remember to consider that negative flow exist for reverse edges.

9.2 Matching

9.3 Coloring

9.4 Shortest Paths

For weighted directed graphs

9.4.1 Dijkstra

Single Source and there **cannot** be any negative weighted edges.

dijkstra.cpp

Description: By keeping track of the distances sorted using an priority queue, transverse only using the smallest distances.
Time: $\mathcal{O}((V + E) \log V)$

ef1d0b, 19 lines

```
priority_queue<pll, vector<pll>, greater<pll>> pq; // min pq
vector<vector<pll>> g(MAX);
vector<ll> d(MAX, INF);

void dijkstra(ll start){
    pq.push({0, start});
    d[start] = 0;

    while( !pq.empty() ){
        auto [p1, u] = pq.top(); pq.pop();
        if (p1 > d[u]) continue;
        for(auto [v, p2] : g[u]){
            if (d[u] + p2 < d[v]){
                d[v] = d[u] + p2;
                pq.push({d[v], v});
            }
        }
    }
}
```

By inverting the sorting order, Dijkstra can be modified for the opposite operation: *longest paths*.

Furthermore, Dijkstra be extended to keep track of more information, such as:

- how many minimum-price routes are there? (modulo $10^9 + 7$)
- what is the minimum number of flights in a minimum-price route?
- what is the maximum number of flights in a minimum-price route?

extendedDijkstra.cpp

Description: Also counts the numbers of shortest paths, the minimum and maximum number of edges transversed in any shortest path.
Time: $\mathcal{O}((V + E) \log V)$

f93094, 32 lines

```
priority_queue<pll, vector<pll>, greater<pll>> pq; // min pq
vector<vector<pll>> g(MAX);
vector<ll> d(MAX, INF), ways(MAX, 0), mx(MAX, -INF), mn(MAX, INF);
// INF = INT64_MAX

void dijkstra(ll start){
    pq.push({0, start});
    ways[start] = 1;
    d[start] = mn[start] = mx[start] = 0;

    while( !pq.empty() ){
        auto [p1, u] = pq.top(); pq.pop();
        if (p1 > d[u]) continue;
        for(auto [v, p2] : g[u]){
            // reset info, shorter path found, previous ones are discarded
            if (d[u] + p2 < d[v]){
                d[v] = d[u] + p2;
                ways[v] = ways[u];
                mx[v] = mx[u]+1;
                mn[v] = mn[u]+1;

                pq.push({d[v], v});
            }
            // same distance, different path, update info
            else if (d[u] + p2 == d[v]) {
                ways[v] = (ways[v] + ways[u]) % MOD;
                mn[v] = min(mn[v], mn[u]+1);
                mx[v] = max(mx[v], mx[u]+1);
            }
        }
    }
}
```

9.4.2 Bellman-Ford

Single Source and it **supports** negative edges

Conjecture: After at most n-1 (Vertices-1) iterations, all shortest paths will be found.

bellman-ford.cpp

Description: n-1 iterations is sufficient to find all shortest paths
Time: $\mathcal{O}(V * E) \rightarrow \mathcal{O}(N^2)$

d749f1, 15 lines

```
using T = array<ll, 3>;
vector<T> edges;
vector<ll> d(MAX, INF);
// INF = 0x3f3f3f3f3f3f3f3f, to avoid overflow

void BellmanFord(ll src, ll n) {
    d[src] = 0;
    for(ll i=0; i<n-1; i++) { // n-1 iterations
        for(auto [u, v, w] : edges) {
            if (d[u] + w < d[v]) {
                d[v] = d[u] + w;
            }
        }
    }
}
```

By iterating once more, one can check if the last iteration reduced once again any distance. If so, it means that there must be a negative cycle, because the shortest distance should have been found before elseway.

To retrieve the negative cycle itself, one can keep track of the last vertice that reaches a considered vertice

bellman-ford-cycle.cpp

Description: By using the property that n-1 iterations is sufficient to find all shortest paths in a graph that doesn't have negative cycles. Iterate n times and retrieve the path using a vector of parents
Time: $\mathcal{O}(V * E) \rightarrow \mathcal{O}(N^2)$

0506b5, 35 lines

```
using T = array<ll, 3>;
vector<T> edges;
vector<ll> d(MAX, INF), p(MAX, -1);
vector<ll> cycle;
// INF = 0x3f3f3f3f3f3f3f3f, to avoid overflow

void BellmanFordCycle(ll src, ll n) {
    d[src] = 0;
    ll x = -1; // possible node inside a negative cycle
    for(ll i=0; i<n; i++) { // n iterations
        x = -1;
        for(auto [u, v, w] : edges) {
            if (d[u] + w < d[v]) {
                d[v] = d[u] + w;
                p[v] = u;
                x = v;
            }
        }
    }

    if (x != -1) {
        // set x to a node, contained in a cycle in p[]
        for(ll i=0; i<n; i++) x = p[x];

        ll tmp = x;
        do {
            cycle.pb(tmp);
            tmp = p[tmp];
        } while (tmp != x);
        cycle.pb(x);

        reverse(cycle.begin(), cycle.end());
    }
}
```



```
}
```

9.5 Undirected Graph

Bridges and Articulation Points are concepts for undirected graphs!

9.5.1 Bridges (Cut Edges)

Also called **isthmus** or **cut arc**.

A back-edge is never a bridge!

A **lowlink** for a vertice U is the closest vertice to the root reachable using only span edges and a *single* back-edge, starting in the subtree of U .

After constructing a DFS Tree, an edge (u, v) is a bridge \iff there is no back-edge from v (or a descendent of v) to u (or an ancestor of u)

To do this efficiently, it's used $tin[i]$ (entry time of node i) and $low[i]$ (minimum entry time considering all nodes that can be reached from node i).

In another words, a edge (u, v) is a bridge \iff the $low[v] \geq tin[u]$.

bridges.cpp

Description: Using the concepts of entry time (tin) and lowlink (low), an edge is a bridge if, and only if, $low[v] > tin[u]$
Time: $\mathcal{O}(V + E)$

```
vector<vll> g(MAX);
ll timer = 1;
ll tin[MAX], low[MAX];
vector<pll> bridges;

void dfs(ll u, ll p = -1){
    tin[u] = low[u] = timer++;
    for(auto v : g[u]) if (v != p) {
        if (tin[v]) // v was visited ({u,v} is a back-edge)
            // considering a single back-edge:
            low[u] = min(low[u], tin[v]);
        else { // v wasn't visited ({u, v} is a span-edge)
            dfs(v, u);
            // after low[v] was computed by dfs(v, u):
            low[u] = min(low[u], low[v]);
            if (low[v] > tin[u])
                bridges.pb({u, v});
        }
    }
}

void findBridges(ll n) {
    for(ll i=1; i<=n; i++) if (!tin[i])
        dfs(i);
}
```

9.5.2 Bridge Tree

After merging *vertices* of a **2-edge connected component** into single vertices, and leaving only bridges, one can generate a Bridge Tree.

Every **2-edge connected component** has following properties:

- For each pair of vertices A, B inside the same component, there are at least 2 distinct paths from A to B (which may repeat vertices).

9.5.3 Articulation Points

One Vertice in a graph is considered a Articulation Points or Cut Vertice if its removal in the graph will generate more disconnected components

articulation.cpp

Description: if $low[v] \geq tin[u]$, u is an articulation points The root is a corner case
Time: $\mathcal{O}(V + E)$

```
vector<vll> g(MAX);
ll timer = 1;
ll low[MAX], tin[MAX], isAP[MAX];
// when vertex i is removed from graph
// isAP[i] is the quantity of new disjoint components created
// isAP[i] >= 1 {i is a Articulation Point}
void dfs(ll u, ll p = -1) {
    low[u] = tin[u] = timer++;

    for(auto v : g[u]) if (v != p) {
        if (tin[v]) // visited
            low[u] = min(low[u], tin[v]);
        else { // not visited
            dfs(v, u);
            low[u] = min(low[u], low[v]);

            if (low[v] >= tin[u])
                isAP[u]++;
        }
    }

    // corner case: root
    if (p == -1 and isAP[u]) isAP[u]--;
}

void findAP(ll n) {
    for(ll i=1; i<=n; i++) if (!tin[i])
        dfs(i);
}
```

9.5.4 Block Cut Tree

After merging *edges* of a **2-vertex connected component** into single vertices, one can obtain a block cut tree.

2-vertex connected components are also called as biconnected component

Every bridge by itself is a biconnected component

Each edge in the block-cut tree connects exactly an Articulation Point and a biconnected component (bipartite graph)

Each biconnected component has the following properties:

- For each pair of edges, there is a cycle that contains both edges
- For each pair of vertices A, B inside the same connected component, there are at least 2 distinct paths from A to B (which do not repeat vertices).

blockCutTree.cpp

Description: After Merging 2-Vertex Connected Components, one can generate a block cut tree
Time: $\mathcal{O}(V + E)$

```
// Block-Cut Tree (bruno monteiro)
//
// Cria a block-cut tree, uma arvore com os blocos
// e os pontos de articulacao
// Blocos sao as componentes 2-vertice-conexas maximais
// Uma 2-coloracao da arvore eh tal que uma cor sao
```

```
// os componentes, e a outra cor sao os pontos de articulacao
//
// Funciona para grafo nao conexo
//
// isAP[i] responde o numero de novas componentes conexas
// criadas apos a remocao de i do grafo g
// Se isAP[i] >= 1, i eh ponto de articulacao
//
// Para todo i < blocks.size()
// blocks[i] eh uma componente 2-vertice-conexa maximal
// blockEdges[i] sao as arestas do bloco i
//
// tree eh a arvore block-cut-tree
// tree[i] eh um vertice da arvore que corresponde ao bloco i
//
// comp[i] responde a qual vertice da arvore vertice i pertence
//
// Arvore tem no maximo 2n vertices
//
// O(n+m)

// 0-idx graph!!!
vector<vll> g(MAX), tree, blocks;
vector<vector<pll>> blockEdges;
stack<ll> st; // st for vertices,
stack<pll> st2; // st2 for edges
vector<ll> low, tin, comp, isAP;
ll timer = 1;

void dfs(ll u, ll p = -1) {
    low[u] = tin[u] = timer++;

    st.push(u);

    // add only back-edges to stack
    if (p != -1) st2.push({u, p});
    for(auto v : g[u]) if (v != p) {
        if (tin[v] != -1) // visited
            st2.push({u, v});
    }

    for(auto v : g[u]) if (v != p) {
        if (tin[v] != -1) // visited
            low[u] = min(low[u], tin[v]);
        else { // not visited
            dfs(v, u);
            low[u] = min(low[u], low[v]);

            if (low[v] >= tin[u]) {
                isAP[u] += 1;

                blocks.pb(vll(1, u));
                while(blocks.back().back() != v)
                    blocks.back().pb(st.top()), st.pop();

                blockEdges.pb(vector<pll>(1, st2.top())), st2.
                    pop();
                while(blockEdges.back().back() != pair<ll, ll>(
                    v, u))
                    blockEdges.back().pb(st2.top()), st2.pop();
            }
        }
    }

    // corner case: root
    if (p == -1 and isAP[u]) isAP[u]--;
}

void blockCutTree(ll n) {
```

```
// initialize vectors and reset
tree.clear(), blocks.clear(), blockEdges.clear();
st = stack<ll>(), st2 = stack<pll>();
tin.assign(n, -1);
low.assign(n, 0), comp.assign(n, 0), isAP.assign(n, 0);
timer = 1;

// find Articulation Points
for(ll i=0; i<n; i++) if (tin[i] == -1)
    dfs(i);

// set component id for APs
tree.assign(blocks.size(), vll());
for(ll i=0; i<n; i++) if (isAP[i])
    comp[i] = tree.size(), tree.pb(vll());

// set component id for non-APs and construct tree
for(ll u=0; u<(ll)blocks.size(); u++) {
    for(auto v : blocks[u]) {
        if (!isAP[v])
            comp[v] = u;
        else
            tree[u].pb(comp[v]), tree[comp[v]].pb(u);
    }
}
```

9.5.5 Strong Orientation

A **strong orientation** of an undirected graph is an assignment of a direction to each edge that makes it a strongly connected graph. That is, after the orientation we should be able to visit any vertex from any vertex by following the directed edges.

Of course, this cannot be done to every graph. Consider a **bridge** in a graph. We have to assign a direction to it and by doing so we make this bridge "crossable" in only one direction. That means we can't go from one of the bridge's ends to the other, so we can't make the graph strongly connected.

Now consider a DFS through a bridgeless connected graph. Clearly, we will visit each vertex. And since there are no bridges, we can remove any DFS tree edge and still be able to go from below the edge to above the edge by using a path that contains at least one back edge. From this follows that from any vertex we can go to the root of the DFS tree. Also, from the root of the DFS tree we can visit any vertex we choose. We found a strong orientation!

In other words, to strongly orient a bridgeless connected graph, run a DFS on it and let the DFS tree edges point away from the DFS root and all other edges from the descendant to the ancestor in the DFS tree.

The result that bridgeless connected graphs are exactly the graphs that have strong orientations is called **Robbins' theorem**.

9.5.6 Minimum Spanning Tree

A minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight. That is, it is a spanning tree whose sum of edge weights is as small as possible.

kruskal.cpp

Description: Sort all edges in crescent order by weight, include all edges which joins two disconnected trees. In case of tie, choose whichever. Dont include edges that will join a already connected part of the tree.

Time: $\mathcal{O}(E \log E\alpha)$

206ba3, 21 lines

```
// use DSU struct
struct DSU{};

set<array<ll, 3>> edges;

int32_t main(){ sws;
    ll n, m; cin >> n >> m;
    DSU dsu(n+1);
    for(ll i=0; i<m; i++) {
        ll u, v, w; cin >> u >> v >> w;
        edges.insert({w, u, v});
    }
    ll minCost = 0;
    for(auto [w, u, v] : edges) {
        if (dsu.find(u) != dsu.find(v)) {
            dsu.join(u, v);
            minCost += w;
        }
    }
    cout << minCost << endl;
}
```

9.6 Directed Graph

9.6.1 Topological Sort

Sort a directed graph with no cycles (DAG) in an order which each source of an edge is visited before the sink of this edge.

Cannot have cycles, because it would create a contradiction of which vertices whould come before.

It can be done with a DFS, appending in the reverse order of transversal. Also a stack can be used to reverse order

toposort.cpp

Description: Using DFS pos order transversal and inverting the order, one can obtain the topological order

Time: $\mathcal{O}(V + E)$

75f781, 17 lines

```
vector<vll> g(MAX, vll());
vector<bool> vis;
vll topological;

void dfs(ll u) {
    vis[u] = 1;
    for(auto v : g[u]) if (!vis[v]) dfs(v);
    topological.pb(u);
}

// 1 - indexed
```

```
void topological_sort(ll n) {
    vis.assign(n+1, 0);
    topological.clear();
    for(ll i=1; i<=n; i++) if (!vis[i]) dfs(i);
    reverse(topological.begin(), topological.end());
}
```

kosaraju.cpp

Description: By using the fact that the inverted graph has the same SCCs, just do a DFS twice to find all SCCs. A condensated graph can be created if wished. The condensated graph is a DAG!!

Time: $\mathcal{O}(V + E)$

381904, 45 lines

```
struct Kosaraju {
    ll n;
    vector<vll> g, gi, gc;
    vector<bool> vis;
    vector<ll> comp;
    stack<ll, vll> st;

    void dfs(ll u) { // g
        vis[u] = 1;
        for(auto v : g[u]) if (!vis[v]) dfs(v);
        st.push(u);
    }

    void dfs2(ll u, ll c) { // gi
        comp[u] = c;
        for(auto v : gi[u]) if (comp[v] == -1) dfs2(v, c);
    }

    Kosaraju(vector<vll> &g_)
        : g(g_), n(g_.size()-1) { // 1-idx

        gi.assign(n+1, vll());
        for(ll i=1; i<=n; i++) {
            for(auto j : g[i])
                gi[j].pb(i);
        }

        gc.assign(n+1, vll());
        vis.assign(n+1, 0);
        comp.assign(n+1, -1);
        st = stack<ll, vll>();

        for(ll i=1; i<=n; i++) if (!vis[i]) dfs(i);

        while(!st.empty()) {
            auto u = st.top(); st.pop();
            if (comp[u] == -1) dfs2(u, u);
        }

        for(ll u=1; u<=n; u++)
            for(auto v : g[u])
                if (comp[u] != comp[v])
                    gc[comp[u]].pb(comp[v]);
    }
};
```

9.6.3 2-SAT

SAT (Boolean satisfiability problem) is NP-Complete.

2-SAT is a restriction of the SAT problem, in 2-SAT every clause has exactly two variables: $(X_1 \vee X_2) \wedge (X_2 \vee X_3)$

Every restriction or implication are represented in the graph as directed edges.

The algorithm uses kosaraju to check if any $(X$ and $\neg X)$ are in the same Strongly Connected Component (which implies that the problem is impossible).

If it doesn't, there is at least one solution, which can be generated using the topological sort of the same kosaraju (opting for the variables that appers latter in the sorted order)

2sat.cpp

Description: Kosaraju to find if there are SCCs. If there are not cycles, use toposort to choose states

Time: $\mathcal{O}(V + E)$

87417c, 83 lines

```
// 0-idx graph !!!
struct TwoSat {
    ll N; // needs to be the twice of the number of variables
    // node with idx 2x => variable x
    // node with idx 2x+1 => variable !x

    vector<vll> g, gi;
    // g = graph; gi = transposed graph (all edges are inverted)

    TwoSat(ll n) { // number of variables (add +1 faor 1-idx)
        N = 2*n;
        g.assign(N, vll());
        gi.assign(N, vll());
    }

    ll idx; // component idx
    vector<ll> comp, order; // topological order (reversed)
    vector<bool> vis, chosen;
    // chosen[x] == 0 -> x was assigned
    // chosen[x] == 1 -> !x was assigned

    // dfs and dfs2 are part of kosaraju algorithm
    void dfs(ll u) {
        vis[u] = 1;
        for (ll v : g[u]) if (!vis[v]) dfs(v);
        order.pb(u);
    }

    void dfs2(ll u, ll c) {
        comp[u] = c;
        for (ll v : gi[u]) if (comp[v] == -1) dfs2(v, c);
    }

    bool solve() {
        vis.assign(N, 0);
        order = vector<ll>();
        for (ll i = 0; i < N; i++) if (!vis[i]) dfs(i);

        comp.assign(N, -1); // comp = 0 can exist
        idx = 1;
        for(ll i=(ll)order.size()-1; i>=0; i--) {
            ll u = order[i];
            if (comp[u] == -1) dfs2(u, idx++);
        }
    }
};
```

```
chosen.assign(N/2, 0);
for (ll i = 0; i < N; i += 2) {
    // x and !x in the same component => contradiction
    if (comp[i] == comp[i+1]) return false;
    chosen[i/2] = comp[i] < comp[i+1]; // choose latter
    node
}
return true;
}

// a (with flagA) implies => b (with flagB)
void add(ll a, bool fa, ll b, bool fb) {
    // {fa == 0} => a
    // {fa == 1} => !a
    a = 2*a + fa;
    b = 2*b + fb;
    g[a].pb(b);
    gi[b].pb(a);
}

// force a state for a certain variable (must be true)
void force(ll a, bool fa) {
    add(a, fa^1, a, fa);
}

// xor operation: one must exist, and only one can exist
void exclusive(ll a, bool fa, ll b, bool fb) {
    add(a, fa^0, b, fb^1);
    add(a, fa^1, b, fb^0);
    add(b, fb^0, a, fa^1);
    add(b, fb^1, a, fa^0);
}

// nand operation: no more than one can exist
void nand(ll a, bool fa, ll b, bool fb) {
    add(a, fa^0, b, fb^1);
    add(b, fb^0, a, fa^1);
}

};
```

9.7 Trees

lca.cpp

Description: Solves LCA for trees

Time: $\mathcal{O}(N \log(N))$ to build, $\mathcal{O}(\log(N))$ per query

7afc1a, 54 lines

```
struct BinaryLifting {
    ll n, logN = 20; // ~1e6
    vector<vll> g;
    vector<ll> depth;
    vector<vll> up;

    BinaryLifting(vector<vll> &g_)
        : g(g_), n(g_.size() + 1) { // 1-idx
        depth.assign(n, 0);

        while((1 << logN) < n) logN++;
        up.assign(n, vll(logN, 0));
        build();
    }

    void build(ll u = 1, ll p = -1) {
        for(ll i=1; i<logN; i++) {
            up[u][i] = up[ up[u][i-1] ][i-1];
        }

        for(auto v : g[u]) if (v != p) {
            up[v][0] = u;
            depth[v] = depth[u] + 1;
        }
    }
};
```

```

        build(v, u);
    }
}

ll go(ll u, ll dist) { // O(log(n))
    for(ll i=logN-1; i>=0; i--) { // bigger jumps first
        if (dist & (1LL << i)) {
            u = up[u][i];
        }
    }
    return u;
}

ll lca(ll a, ll b) { // O(log(n))
    if (depth[a] < depth[b]) swap(a, b);
    a = go(a, depth[a] - depth[b]);
    if (a == b) return a;

    for(ll i=logN-1; i>=0; i--) {
        if (up[a][i] != up[b][i]) {
            a = up[a][i];
            b = up[b][i];
        }
    }
    return up[a][0];
}

ll lca(ll a, ll b, ll root) { // lca(a, b) when tree is
    rooted at 'root'
    return lca(a, b)^lca(b, root)^lca(a, root); //magic
}
};

```

queryTree.cpp

Description: Binary Lifting for min, max weight present in a simple path
Time: $\mathcal{O}(N \log(N))$ to build; $\mathcal{O}(\log(N))$ per query

75ba37, 67 lines

```

struct BinaryLifting {
    ll n, logN = 20; // ~1e6
    vector<vpll> g;
    vector<ll> depth;
    vector<vll> up, mx, mn;

    BinaryLifting(vector<vpll> &g_)
        : g(g_), n(g_.size() + 1) { // 1-idx
        depth.assign(n, 0);

        while((1 << logN) < n) logN++;
        up.assign(n, vll(logN, 0));
        mx.assign(n, vll(logN, -INF));
        mn.assign(n, vll(logN, INF));
        build();
    }

    void build(ll u = 1, ll p = -1) {

        for(ll i=1; i<logN; i++) {
            mx[u][i] = max(mx[u][i-1], mx[ up[u][i-1] ][i-1]);
            mn[u][i] = min(mn[u][i-1], mn[ up[u][i-1] ][i-1]);
            up[u][i] = up[ up[u][i-1] ][i-1];
        }

        for(auto [v, w] : g[u]) if (v != p) {
            mx[v][0] = mn[v][0] = w;
            up[v][0] = u;
            depth[v] = depth[u] + 1;
            build(v, u);
        }
    }
};

```

```

array<ll, 3> go(ll u, ll dist) { // O(log(n))
    ll mxval = -INF, mnval = INF;
    for(ll i=logN-1; i>=0; i--) { // bigger jumps first
        if (dist & (1LL << i)) {
            mxval = max(mxval, mx[u][i]);
            mnval = min(mnval, mn[u][i]);
            u = up[u][i];
        }
    }
    return {u, mxval, mnval};
}

array<ll, 3> query(ll u, ll v) { // O(log(n))
    if (depth[u] < depth[v]) swap(u, v);

    auto [a, mxval, mnval] = go(u, depth[u] - depth[v]);
    ll b = v;

    if (a == b) return {a, mxval, mnval};

    for(ll i=logN-1; i>=0; i--) {
        if (up[a][i] != up[b][i]) {
            mxval = max({mxval, mx[a][i], mx[b][i]});
            mnval = min({mnval, mn[a][i], mn[b][i]});
            a = up[a][i];
            b = up[b][i];
        }
    }

    mxval = max({mxval, mx[a][0], mx[b][0]});
    mnval = min({mnval, mn[a][0], mn[b][0]});
    return {up[a][0], mxval, mnval};
}
};

```

9.8 Math

Geometry (10)

Strings (11)

11.1 Hashing

Hashing consists in generating a Polynomial for the string, therefore, assigning each distinct string to a specific numeric value. In practice, there will always be some collisions:

$$\text{Probability of collision} = \frac{n^2}{2m}$$

n = Comparissons, m = mod size

when using multiple mods, they multiply: $m = m_1 * m_2$

hashing.cpp

Description: Create a numerical value for a string by using polynomial hashing

Time: $\mathcal{O}(n)$ to build, $\mathcal{O}(1)$ per query

c3a650, 43 lines

```

// s[0]*P^n + s[1]*P^(n-1) + ... + s[n]*P^0
// 0-idx
struct Hashing {
    ll n, mod;
    string s;
    vector<ll> p, h; // p = P^i, h = accumulated hash sum

    const ll P = 31; // can be 53
};

```

```

Hashing(string &s_, ll m)
    : n(s_.size()), s(s_), mod(m), p(n), h(n) {

    for(ll i=0; i<n; i++)
        p[i] = (i ? P*p[i-1] : 1) % mod;

    for(ll i=0; i<n; i++)
        h[i] = (s[i] + P*(i ? h[i-1] : 0)) % mod;
}

ll query(ll l, ll r) { // [l, r] inclusive (0-idx)
    ll hash = h[r] - (l ? (p[r-l+1]*h[l-1]) % mod : 0);
    return hash < 0 ? hash + mod : hash;
}

};

// for codeforces:
mt19937 rng(chrono::steady_clock::now().time_since_epoch().count());

int32_t main() { sws;
    vector<ll> mods = {
        1000000009, 10000000021, 10000000033,
        10000000087, 10000000093, 10000000097,
        1000000103, 1000000123, 1000000181,
        1000000207, 1000000223, 1000000241,
        1000000271, 1000000289, 1000000297
    };

    shuffle(mods.begin(), mods.end(), rng);

    string s; cin >> s;

    Hashing hash(s, mods[0]);
}

```

11.2 Z-Function

Suppose we are given a string s of length n . The Z-function for this string is an array of length n where the i -th element is equal to the greatest number of characters starting from the position i that coincide with the first characters of s (the prefix of s)

The first element of the Z-function, $z[0]$, is generally not well defined. This implementation assumes it as $z[0] = 0$. But it can also be interpreted as $z[0] = n$ (all characters coincide).

Can be used to solve the following simples problems:

- Find all occurrences of a pattern p in another string s . ($p + '$' + s$) ($z[i] == p.size()$)
- Find all borders. A border of a string is a prefix that is also a suffix of the string but not the whole string. For example, the borders of $abcbabcbab$ are ab and $abcb$. ($z[8] = 2$, $z[5] = 5$) ($z[i] = n-i$)
- Find all period lengths of a string. A period of a string is a prefix that can be used to generate the whole string by repeating the prefix. The last repetition may be partial. For example, the periods of $abcabca$ are **abc**, **abcabc** and **abcabca**.

It works because $(z[i] + i \leq n)$ is the condition when the common characters of $z[i]$ in addition to the elements already passed, exceeds or is equal to the end of the string. For example:

abaababab $z[8] = 2$

abaababa is the period; the remaining $(z[i]$ characters) are a prefix of the period; and when all these characters are combined, it can form the string (which has n characters).

zfunction.cpp

Description: For each substring starting at position i , compute the maximum match with the original prefix. $z[0] = 0$

Time: $\mathcal{O}(n)$ 14b37c, 12 lines

```
vector<ll> z_function(string &s) { //  $\mathcal{O}(n)$ 
    ll n = (ll) s.length();
    vector<ll> z(n);
    for (ll i=1, l=0, r=0; i<n; i++) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);

        while (i + z[i] < n and s[z[i]] == s[i + z[i]]) z[i]++;

        if (r < i + z[i] - 1) l = i, r = i + z[i] - 1;
    }
    return z;
}
```

11.3 KMP

KMP stands for Knuth-Morris-Pratt and computes the prefix function.

You are given a string s of length n . The prefix function for this string is defined as an array π of length n , where $\pi[i]$ is the length of the longest proper prefix of the substring $s[0 \dots i]$ which is also a suffix of this substring. A proper prefix of a string is a prefix that is not equal to the string itself. By definition, $\pi[0] = 0$.

For example, prefix function of string *"abcabcd"* is $[0, 0, 0, 1, 2, 3, 0]$, and prefix function of string *"aabaab"* is $[0, 1, 0, 1, 2, 2, 3]$.

kmp.cpp

Description: Computes the prefix function

Time: $\mathcal{O}(n)$ 48408b, 13 lines

```
vector<ll> kmp(string &s) { //  $\mathcal{O}(n)$ 
    ll n = (ll) s.length();
    vector<ll> pi(n);
    for (ll i = 1; i < n; i++) {
        ll j = pi[i-1];
        while (j > 0 && s[i] != s[j])
            j = pi[j-1];
        if (s[i] == s[j])
            j++;
        pi[i] = j;
    }
    return pi;
}
```

11.4 Suffix Array

The suffix array is the array with size n , whose values are the indexes from the longest substring (0) to the smallest substring (n) after ordering it lexicographically. Example:

Let the given string be "banana".

0 banana		5 a
1 anana	Sort the Suffixes	3 ana
2 nana	----->	1 anana
3 ana	alphabetically	0 banana
4 na		4 na
5 a		2 nana

So the suffix array for "banana" is $\{5, 3, 1, 0, 4, 2\}$

Note that the length of the string i is: $(s.size()-sa[i])$

suffix-array.cpp

Description: Creates the Suffix Array

Time: $\mathcal{O}(N \log N)$ 49608b, 20 lines

```
vector<ll> suffixArray(string s) {
    s += "!";
    ll n = s.size(), N = max(n, 260LL);
    vector<ll> sa(n), ra(n);
    for (ll i = 0; i < n; i++) sa[i] = i, ra[i] = s[i];

    for (ll k = 0; k < n; k ? k *= 2 : k++) {
        vector<ll> nsa(sa), nra(n), cnt(N);

        for (ll i = 0; i < n; i++) nsa[i] = (nsa[i]-k+n)%n, cnt[ra[i]]++;
        for (ll i = 1; i < N; i++) cnt[i] += cnt[i-1];
        for (ll i = n-1; i+1; i--) sa[--cnt[ra[nsa[i]]]] = nsa[i];

        for (ll i = 1, r = 0; i < n; i++) nra[sa[i]] = r += ra[sa[i]] != ra[sa[i-1]] or ra[(sa[i]+k)%n] != ra[(sa[i-1]+k)%n];
        ra = nra;
        if (ra[sa[n-1]] == n-1) break;
    }
    return vector<ll>(sa.begin()+1, sa.end());
}
```

Kasai generates an array of size n (like the suffix array), whose values indicates the length of the longest common prefix between $(sa[i]$ and $sa[i+1])$

kasai.cpp

Description: Creates the Longest Common Prefix array (LCP)

Time: $\mathcal{O}(N \log N)$ 913195, 13 lines

```
vector<ll> kasai(string s, vector<ll> sa) {
    ll n = s.size(), k = 0;
    vector<ll> ra(n), lcp(n);
    for (ll i = 0; i < n; i++) ra[sa[i]] = i;

    for (ll i = 0; i < n; i++, k -= !k) {
        if (ra[i] == n-1) { k = 0; continue; }
        ll j = sa[ra[i]+1];
        while (i+k < n and j+k < n and s[i+k] == s[j+k]) k++;
        lcp[ra[i]] = k;
    }
    return lcp;
}
```

}

Problems that can be solved:

Numbers of Distinct Substrings:

- $\frac{n(n+1)}{2} - \text{lcp}[i]$ (for all i)

Longest Repeated Substring:

- biggest $\text{lcp}[i]$. The position can be found in $sa[i]$

Find how many distinct substrings there are for each len in $[1:n]$:

- Use delta encoding and the fact that $\text{lcp}[i]$ counts the repeated substring between $s.\text{substr}(sa[i])$ and $s.\text{substr}(sa[i+1])$, which are the substrings corresponding to the common prefix.

Find the k -th distinct substring:

```
string s; cin >> s;
ll n = s.size();

auto sa = suffix_array(s);
auto lcp = kasai(s, sa);

ll k; cin >> k;

for (ll i=0; i<n; i++) {
    ll len = n-sa[i];
    if (k <= len) {
        cout << s.substr(sa[i], k) << endl;
        break;
    }
    k -= lcp[i] - len;
}
```

11.5 Manacher

Manacher's Algorithm is used to find all palindromes in a string.

For each substring, centered at i , find the longest palindrome that can be formed.

Works best for odd size string, so we convert all string to odd ones by adding and extra characters between the original ones

Therefore, the value stored in the vector cnt is actually $\text{palindrome-len} + 1$.

manacher.cpp

Description: Convert String to odd length to use manacher, which computes all the maximum lengths of all palindromes in the given string

Time: $\mathcal{O}(2n)$ 0c2a2b, 46 lines

```
struct Manacher {
    string s, t;
    vector<ll> cnt;

    // t is the transformed string of s, with odd size
    Manacher(string &s_) : s(s_) {
        t = "#";
    }
}
```

```
for(auto c : s) {
    t += c, t += "#";
}
count();
}

// perform manacher on the odd string
// cnt will give all the palindromes centered in i
// for the odd string t
void count() {
    ll n = t.size();
    string aux = "$" + t + "^";
    vector<ll> p(n + 2);
    ll l = 1, r = 1;
    for(ll i = 1; i <= n; i++) {
        p[i] = max(0LL, min(r - i, p[l + (r - i)]));
        while(aux[i - p[i]] == aux[i + p[i]]) {
            p[i]++;
        }
        if(i + p[i] > r) {
            l = i - p[i], r = i + p[i];
        }
    }
    cnt = vector<ll>(p.begin() + 1, p.end() - 1);
}

// compute a longest palindrome present in s
string getLongest() {
    ll len = 0, pos = 0;
    for(ll i=0; i<(ll)t.size(); i++) {
        ll sz = cnt[i]-1;
        if (sz > len) {
            len = sz;
            pos = i;
        }
    }
    return s.substr(pos/2 - len/2, len);
}
};
```

11.6 Booth

An efficient algorithm which uses a modified version of KMP to compute the least amount of rotation needed to reach the **lexicographically minimal string rotation**.

A rotation of a string can be generated by moving characters one after another from beginning to end. For example, the rotations of *acab* are *acab*, *caba*, *abac*, and *baca*.

booth.cpp

Description: Use a modified version of KMP to find the lexicographically minimal string rotation

Time: $O(n)$

64184b, 30 lines

```
// Booth Algorithm
ll least_rotation(string &s) { // O(n)
    ll n = s.length();
    vector<ll> f(2*n, -1);
    ll k = 0;
    for(ll j=1; j<2*n; j++) {
        ll i = f[j-k-1];
        while(i != -1 and s[j % n] != s[(k+i+1) % n] ) {
            if (s[j % n] < s[(k+i+1) % n])
                k = j - i - 1;
            i = f[i];
        }
        if (i == -1 and s[j % n] != s[(k+i+1) % n] ) {
```

```
        if (s[j % n] < s[(k+i+1) % n])
            k = j;
        f[j - k] = -1;
    }
    else
        f[j - k] = i + 1;
    }
    return k;
}

int32_t main(){ sws;
    string s; cin >> s;
    ll n = s.length();
    ll ans_idx = least_rotation(s);
    string tmp = s + s;
    cout << tmp.substr(ans_idx, n) << endl;
}
```

Miscellaneous (12)

Techniques (A)

techniques.txt

159 lines

Recursion

Divide and conquer

Finding interesting points in N log N

Algorithm analysis

Master theorem

Amortized time complexity

Greedy algorithm

Scheduling

Max contiguous subvector sum

Invariants

Huffman encoding

Graph theory

Dynamic graphs (extra book-keeping)

Breadth first search

Depth first search

* Normal trees / DFS trees

Dijkstra’s algorithm

MST: Prim’s algorithm

Bellman-Ford

Konig’s theorem and vertex cover

Min-cost max flow

Lovasz toggle

Matrix tree theorem

Maximal matching, general graphs

Hopcroft-Karp

Hall’s marriage theorem

Graphical sequences

Floyd-Warshall

Euler cycles

Flow networks

* Augmenting paths

* Edmonds-Karp

Bipartite matching

Min. path cover

Topological sorting

Strongly connected components

2-SAT

Cut vertices, cut-edges and biconnected components

Edge coloring

* Trees

Vertex coloring

* Bipartite graphs (=> trees)

* 3^n (special case of set cover)

Diameter and centroid

K’t h shortest path

Shortest cycle

Dynamic programming

Knapsack

Coin change

Longest common subsequence

Longest increasing subsequence

Number of paths in a dag

Shortest path in a dag

Dynprog over intervals

Dynprog over subsets

Dynprog over probabilities

Dynprog over trees

3^n set cover

Divide and conquer

Knuth optimization

Convex hull optimizations

RMQ (sparse table a.k.a 2^k-jumps)

Bitonic cycle

Log partitioning (loop over most restricted)

Combinatorics

Computation of binomial coefficients

Pigeon-hole principle

Inclusion/exclusion

Catalan number

Pick’s theorem

Number theory

Integer parts

Divisibility

Euclidean algorithm

Modular arithmetic

* Modular multiplication

* Modular inverses

* Modular exponentiation by squaring

Chinese remainder theorem

Fermat’s little theorem

Euler’s theorem

Phi function

Frobenius number

Quadratic reciprocity

Pollard-Rho

Miller-Rabin

Hensel lifting

Vieta root jumping

Game theory

Combinatorial games

Game trees

Mini-max

Nim

Games on graphs

Games on graphs with loops

Grundy numbers

Bipartite games without repetition

General games without repetition

Alpha-beta pruning

Probability theory

Optimization

Binary search

Ternary search

Unimodality and convex functions

Binary search on derivative

Numerical methods

Numeric integration

Newton’s method

Root-finding with binary/ternary search

Golden section search

Matrices

Gaussian elimination

Exponentiation by squaring

Sorting

Radix sort

Geometry

Coordinates and vectors

* Cross product

* Scalar product

Convex hull

Polygon cut

Closest pair

Coordinate-compression

Quadtrees

KD-trees

All segment-segment intersection

Sweeping

Discretization (convert to events and sweep)

Angle sweeping

Line sweeping

Discrete second derivatives

Strings

Longest common substring

Palindrome subsequences

Knuth-Morris-Pratt

Tries

Rolling polynomial hashes

Suffix array

Suffix tree

Aho-Corasick

Manacher’s algorithm

Letter position lists

Combinatorial search

Meet in the middle

Brute-force with pruning

Best-first (A*)

Bidirectional search

Iterative deepening DFS / A*

Data structures

LCA (2^k-jumps in trees in general)

Pull/push-technique on trees

Heavy-light decomposition

Centroid decomposition

Lazy propagation

Self-balancing trees

Convex hull trick (wcipeg.com/wiki/Convex_hull_trick)

Monotone queues / monotone stacks / sliding queues

Sliding queue using 2 stacks

Persistent segment tree