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Contest (1)

template.cpp

33 lines

```
#include <bits/stdc++.h>
using namespace std;
#define sws cin.tie(0)->sync_with_stdio(0)

#define endl '\n'
#define ll long long
#define ld long double
#define pb push_back
#define ff first
#define ss second
#define pll pair<ll, ll>
#define vll vector<ll>

#define teto(a, b) ((a+b-1)/(b))
#define LSB(i) ((i) & -(i))
#define MSB(i) (32 - __builtin_clz(i)) //64 - clzll
#define BITS(i) __builtin_popcountll(i) //count set bits

mt19937 rng(chrono::steady_clock::now().time_since_epoch().
    count());

#define debug(a...) cerr<<#a<<": ";for(auto b:a)cerr<<b<<" ";
    cerr<<endl;
template<typename... A> void dbg(A const&... a){((cerr<<"{"<<a
    <<"} "), ...);cerr<<endl;}

const int MAX = 3e5+10;
const int INF = INT32_MAX;
const long long MOD = 1e9+7;
const long long LLINF = INT64_MAX;
const long double EPS = 1e-7;
const long double PI = acos(-1);

int32_t main(){ sws;

}
```

.bashrc

1 lines

```
alias comp='g++ -std=c++17 -g3 -ggdb3 -O3 -Wall -Wextra -
    fsanitize=address,undefined -Wshadow -Wconversion -
    D_GLIBCXX_ASSERTIONS -o test'
```

hash.sh

3 lines

```
# Hashes a file, ignoring all whitespace and comments. Use for
# verifying that code was correctly typed. CTRL+D to send EOF
cpp -dD -P -fpreprocessed | tr -d '[:space:]'| md5sum | cut -c
    -6
```

troubleshoot.txt

52 lines

```
Pre-submit:
Write a few simple test cases if sample is not enough.
Are time limits close? If so, generate max cases.
Is the memory usage fine?
Could anything overflow?
Make sure to submit the right file.

Wrong answer:
Print your solution! Print debug output, as well.
Are you clearing all data structures between test cases?
Can your algorithm handle the whole range of input?
Read the full problem statement again.
Do you handle all corner cases correctly?
Have you understood the problem correctly?
Any uninitialized variables?
Any overflows?
Confusing N and M, i and j, etc.?
Are you sure your algorithm works?
What special cases have you not thought of?
Are you sure the STL functions you use work as you think?
Add some assertions, maybe resubmit.
Create some testcases to run your algorithm on.
Go through the algorithm for a simple case.
Go through this list again.
Explain your algorithm to a teammate.
Ask the teammate to look at your code.
Go for a small walk, e.g. to the toilet.
Is your output format correct? (including whitespace)
Rewrite your solution from the start or let a teammate do it.

Runtime error:
Have you tested all corner cases locally?
Any uninitialized variables?
Are you reading or writing outside the range of any vector?
Any assertions that might fail?
Any possible division by 0? (mod 0 for example)
Any possible infinite recursion?
Invalidated pointers or iterators?
Are you using too much memory?
Debug with resubmits (e.g. remapped signals, see Various).

Time limit exceeded:
Do you have any possible infinite loops?
What is the complexity of your algorithm?
Are you copying a lot of unnecessary data? (References)
How big is the input and output? (consider scanf)
Avoid vector, map. (use arrays/unordered_map)
What do your teammates think about your algorithm?

Memory limit exceeded:
What is the max amount of memory your algorithm should need?
Are you clearing all data structures between test cases?
```

Mathematics (2)

2.1 Equations

$$ax^2+bx+c=0 \Rightarrow x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

The extremum is given by $x=-b/2a$.

$$\begin{aligned} ax+by=e \\ cx+dy=f \end{aligned} \Rightarrow \begin{aligned} x=\frac{ed-bf}{ad-bc} \\ y=\frac{af-ec}{ad-bc} \end{aligned}$$

In general, given an equation $Ax=b$, the solution to a variable x_i is given by

$$x_i=\frac{\det A_i'}{\det A}$$

where A_i' is A with the i'th column replaced by b.

2.2 Recurrences

If $a_n=c_1a_{n-1}+\cdots+c_ka_{n-k}$, and $r_1,\ldots,r_k$ are distinct roots of $x^k-c_1x^{k-1}-\cdots-c_k$, there are $d_1,\ldots,d_k$ s.t.

$$a_n=d_1r_1^n+\cdots+d_kr_k^n.$$

Non-distinct roots r become polynomial factors, e.g. $a_n=(d_1n+d_2)r^n$.

2.3 Trigonometry

$$\sin(v+w)=\sin v\cos w+\cos v\sin w$$
$$\cos(v+w)=\cos v\cos w-\sin v\sin w$$

$$\tan(v+w)=\frac{\tan v+\tan w}{1-\tan v\tan w}$$
$$\sin v+\sin w=2\sin\frac{v+w}{2}\cos\frac{v-w}{2}$$
$$\cos v+\cos w=2\cos\frac{v+w}{2}\cos\frac{v-w}{2}$$

$$(V+W)\tan(v-w)/2=(V-W)\tan(v+w)/2$$

where V,W are lengths of sides opposite angles v,w.

$$a\cos x+b\sin x=r\cos(x-\phi)$$
$$a\sin x+b\cos x=r\sin(x+\phi)$$

where $r=\sqrt{a^2+b^2}, \phi=\operatorname{atan2}(b,a)$.

2.4 Geometry

2.4.1 Triangles

Side lengths: a, b, c

Semiperimeter: $p = \dfrac{a+b+c}{2}$

Area: $A = \sqrt{p(p-a)(p-b)(p-c)}$

Circumradius: $R = \dfrac{abc}{4A}$

Inradius: $r = \dfrac{A}{p}$

Length of median (divides triangle into two equal-area triangles): $m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}$

Length of bisector (divides angles in two):

$$s_a = \sqrt{bc\left[1-\left(\frac{a}{b+c}\right)^2\right]}$$

Law of sines: $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} = \dfrac{1}{2R}$

Law of cosines: $a^2 = b^2 + c^2 - 2bc\cos\alpha$

Law of tangents: $\dfrac{a+b}{a-b} = \dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}}$

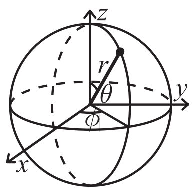
2.4.2 Quadrilaterals

With side lengths a, b, c, d, diagonals e, f, diagonals angle θ, area A and magic flux $F = b^2 + d^2 - a^2 - c^2$:

$$4A = 2ef \cdot \sin\theta = F\tan\theta = \sqrt{4e^2f^2 - F^2}$$

For cyclic quadrilaterals the sum of opposite angles is 180°, $ef = ac + bd$, and $A = \sqrt{(p-a)(p-b)(p-c)(p-d)}$.

2.4.3 Spherical coordinates



$$\begin{aligned} x &= r\sin\theta\cos\phi & r &= \sqrt{x^2+y^2+z^2} \\ y &= r\sin\theta\sin\phi & \theta &= \mathrm{acos}(z/\sqrt{x^2+y^2+z^2}) \\ z &= r\cos\theta & \phi &= \mathrm{atan2}(y,x) \end{aligned}$$

2.5 Derivatives/Integrals

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan x = 1 + \tan^2 x \qquad \frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$
$$\int \tan ax = -\frac{\ln|\cos ax|}{a} \qquad \int x\sin ax = \frac{\sin ax - ax\cos ax}{a^2}$$
$$\int e^{-x^2} = \frac{\sqrt{\pi}}{2}\mathrm{erf}(x) \qquad \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

Integration by parts:

$$\int_a^b f(x)g(x)dx = [F(x)g(x)]_a^b - \int_a^b F(x)g'(x)dx$$

2.6 Sums

$$c^a + c^{a+1} + \cdots + c^b = \frac{c^{b+1}-c^a}{c-1}, c \neq 1$$

$$\begin{aligned} 1+2+3+\cdots+n &= \frac{n(n+1)}{2} \\ 1^2+2^2+3^2+\cdots+n^2 &= \frac{n(2n+1)(n+1)}{6} \\ 1^3+2^3+3^3+\cdots+n^3 &= \frac{n^2(n+1)^2}{4} \\ 1^4+2^4+3^4+\cdots+n^4 &= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} \end{aligned}$$

2.7 Series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, (-\infty < x < \infty)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, (-1 < x \leq 1)$$

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{2x^3}{32} - \frac{5x^4}{128} + \ldots, (-1 \leq x \leq 1)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, (-\infty < x < \infty)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, (-\infty < x < \infty)$$

2.8 Probability theory

Let X be a discrete random variable with probability $p_X(x)$ of assuming the value x. It will then have an expected value (mean) $\mu = \mathbb{E}(X) = \sum_x xp_X(x)$ and variance $\sigma^2 = V(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \sum_x (x - \mathbb{E}(X))^2 p_X(x)$ where σ is the standard deviation. If X is instead continuous it will have a probability density function $f_X(x)$ and the sums above will instead be integrals with $p_X(x)$ replaced by $f_X(x)$.

Expectation is linear:

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$$

For independent X and Y,

$$V(aX + bY) = a^2V(X) + b^2V(Y).$$

2.8.1 Discrete distributions

Binomial distribution

The number of successes in n independent yes/no experiments, each which yields success with probability p is $\mathrm{Bin}(n,p)$, $n = 1, 2, \ldots$, $0 \leq p \leq 1$.

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\mu = np,\ \sigma^2 = np(1-p)$$

$\mathrm{Bin}(n,p)$ is approximately $\mathrm{Po}(np)$ for small p.

First success distribution

The number of trials needed to get the first success in independent yes/no experiments, each which yields success with probability p is $\mathrm{Fs}(p)$, $0 \leq p \leq 1$.

$$p(k) = p(1-p)^{k-1},\ k = 1, 2, \ldots$$

$$\mu = \frac{1}{p},\ \sigma^2 = \frac{1-p}{p^2}$$

Poisson distribution

The number of events occurring in a fixed period of time t if these events occur with a known average rate κ and independently of the time since the last event is $\mathrm{Po}(\lambda)$, $\lambda = t\kappa$.

$$p(k) = e^{-\lambda}\frac{\lambda^k}{k!}, k = 0, 1, 2, \ldots$$

$$\mu = \lambda,\ \sigma^2 = \lambda$$

2.8.2 Continuous distributions

Uniform distribution

If the probability density function is constant between a and b and 0 elsewhere it is $\mathrm{U}(a,b)$, $a<b$.

$$f(x)=\begin{cases}\frac{1}{b-a} & a<x<b\\ 0 & \text{otherwise}\end{cases}$$

$$\mu=\frac{a+b}{2},\ \sigma^2=\frac{(b-a)^2}{12}$$

Exponential distribution

The time between events in a Poisson process is $\mathrm{Exp}(\lambda)$, $\lambda>0$.

$$f(x)=\begin{cases}\lambda e^{-\lambda x} & x\geq 0\\ 0 & x<0\end{cases}$$

$$\mu=\frac{1}{\lambda},\ \sigma^2=\frac{1}{\lambda^2}$$

Normal distribution

Most real random values with mean μ and variance σ^2 are well described by $\mathcal{N}(\mu,\sigma^2)$, $\sigma>0$.

$$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

If $X_1\sim\mathcal{N}(\mu_1,\sigma_1^2)$ and $X_2\sim\mathcal{N}(\mu_2,\sigma_2^2)$ then

$$aX_1+bX_2+c\sim\mathcal{N}(\mu_1+\mu_2+c,a^2\sigma_1^2+b^2\sigma_2^2)$$

2.9 Markov chains

A *Markov chain* is a discrete random process with the property that the next state depends only on the current state. Let $X_1,X_2,\ldots$ be a sequence of random variables generated by the Markov process. Then there is a transition matrix $\mathbf{P}=(p_{ij})$, with $p_{ij}=\Pr(X_n=i|X_{n-1}=j)$, and $\mathbf{p}^{(n)}=\mathbf{P}^n\mathbf{p}^{(0)}$ is the probability distribution for X_n (i.e., $p_i^{(n)}=\Pr(X_n=i)$), where $\mathbf{p}^{(0)}$ is the initial distribution.

π is a stationary distribution if $\pi=\pi\mathbf{P}$. If the Markov chain is *irreducible* (it is possible to get to any state from any state), then $\pi_i=\frac{1}{\mathbb{E}(T_i)}$ where $\mathbb{E}(T_i)$ is the expected time between two visits in state i. π_j/π_i is the expected number of visits in state j between two visits in state i.

For a connected, undirected and non-bipartite graph, where the transition probability is uniform among all neighbors, π_i is proportional to node i's degree.

A Markov chain is *ergodic* if the asymptotic distribution is independent of the initial distribution. A finite Markov chain is ergodic iff it is irreducible and *aperiodic* (i.e., the gcd of cycle lengths is 1). $\lim_{k\to\infty}\mathbf{P}^k=\mathbf{1}\pi$.

A Markov chain is an A-chain if the states can be partitioned into two sets $\mathbf{A}$ and $\mathbf{G}$, such that all states in $\mathbf{A}$ are absorbing ($p_{ii}=1$), and all states in $\mathbf{G}$ leads to an absorbing state in $\mathbf{A}$. The probability for absorption in state $i\in\mathbf{A}$, when the initial state is j, is $a_{ij}=p_{ij}+\sum_{k\in\mathbf{G}}a_{ik}p_{kj}$. The expected time until absorption, when the initial state is i, is $t_i=1+\sum_{k\in\mathbf{G}}p_{ki}t_k$.

Data structures (3)

3.1 Ordered Set

Policy Based Data Structures (PBDS) from gcc compiler

Ordered Multiset can be created using ordered_set<pll>val, idx

order_of_key() can search for non-existent keys!

find_by_order() requires existent key and return the 0-idx position of the given value. Therefore, it returns the numbers of elements that are smaller than the given value;

ordered-set.cpp

Description: Set with index operators, implemented by gnu pbds. Remember to compile with gcc!!

Time: $\mathcal{O}(log(N))$ but with slow constant

<bits/extc++.h>, <bits/extc++.h> 8578e5, 11 lines

```
// 0-idx
// find_by_order(i) -> iterator to elem with index i
// order_of_key(val) -> index of key

// Ordered Set
using namespace __gnu_pbds;
template <class T> using ordered_set = tree<T, null_type, less<
    T>, rb_tree_tag, tree_order_statistics_node_update>;

// Ordered Map
using namespace __gnu_pbds;
template <class K, class V> using ordered_map = tree<K, V, less
    <K>, rb_tree_tag, tree_order_statistics_node_update>;
```

3.2 Disjoint Set Union

There are two optional improvements:

-Tree Balancing

-Path Compression

If one improvement is used, the time complexity will become $O(\log N)$

If both are used, $O(\alpha)\approx O(5)$

dsu.cpp

Description: Disjoint Set Union with path compression and tree balancing

Time: $\mathcal{O}(\alpha)$

0479c4, 22 lines

```
struct DSU{
    vll group, card;
    DSU (ll n){
        n += 1; // 0-idx -> 1-idx
        group = vll(n);
        iota(group.begin(), group.end(), 0);
        card = vll(n, 1);
    }
    ll find(ll i){
        return (i == group[i]) ? i : (group[i] = find(group[i])
            );
    }
    // returns false if a and b are already in the same
        component
    bool join(ll a ,ll b){
        a = find(a);
        b = find(b);
        if (a == b) return false;
        if (card[a] < card[b]) swap(a, b);
        card[a] += card[b];
        group[b] = a;
        return true;
    }
};
```

3.3 Segment Tree

Each node of the segment tree represents the cumulative value of a range.

Observation: For some problems, such as range distinct values query, considerer offiline approach, ordering the queries by L for example.

segRecursive.cpp

Description: Basic Recursive Segment Tree for points increase and range sum query. When initializing the segmente tree, remeber to choose the range limits (L, R)

Time: $\mathcal{O}(N\log N)$ to build, $\mathcal{O}(\log N)$ to increase or query

156cd2, 70 lines

```
// [0, n] segtree for range sum query, point increase
ll L=0, R;
struct Segtree {

    struct Node {
        // null element:
        ll ps = 0;
    };

    vector<Node> tree;
    vector<ll> v;

    Segtree(ll n) {
    R = n;
        v.assign(n+1, 0);
        tree.assign(4*(n+1), Node{});
    }

    Node merge(Node a, Node b) {
        return Node {
            // merge operaton:
            a.ps + b.ps
        };
    }

    void build(ll l=L, ll r=R, ll i=1 ) {
        if (l == r) {
            tree[i] = Node {
                // leaf element:
                v[l]
            };
        }
        else {
            ll mid = (l+r)/2;
```

```
        build(l, mid, 2*i);
        build(mid+1, r, 2*i+1);
        tree[i] = merge(tree[2*i], tree[2*i+1]);
    }
}

void increase(ll idx=1, ll val=0, ll l=L, ll r=R, ll i=1 )
 {
    if (l == r) {
        // increase operation:
        tree[i].ps += val;
    }
    else {
        ll mid = (l+r)/2;
        if (idx <= mid) increase(idx, val, l, mid, 2*i);
        else increase(idx, val, mid+1, r, 2*i+1);
        tree[i] = merge(tree[2*i], tree[2*i+1]);
    }
}

Node query(ll left=L, ll right=R, ll l=L, ll r=R, ll i=1) {
    // left/right are the range limits for the query
    // l / r are the internal variables of the tree
    if (right < l or r < left){
        // null element:
        return Node{};
    }
    else if (left <= l and r <= right) return tree[i];
    else{
        ll mid = (l+r)/2;
        return merge(
            query(left, right, l, mid, 2*i),
            query(left, right, mid+1, r, 2*i+1)
        );
    }
}
};
```

3.4 Convex Hull Trick

If multiple transitions of the DP can be seen as first degree polynomials (lines). CHT can be used to optimized it

Some valid functions:

$ax + b$

$cx^2 + ax + b$ (ignore cx^2 if c is independent)

cht-dynamic.cpp

Description: Dynamic version of CHT, thefore, one can insert lines in any order. There is no line removal operator

Time: $\mathcal{O}(\log N)$ per query and per insertion

707da4, 51 lines

```
// Convex Hull Trick Dinamico
//
// Para float, use LLINF = 1/.0, div(a, b) = a/b
//
// update(x) atualiza o ponto de intersecao da reta x
// overlap(x) verifica se a reta x sobrepoe a proxima
// add(a, b) adiciona reta da forma ax + b
// query(x) computa maximo de ax + b para entre as retas
// se quiser computar o minimo, eh soh fazer (-a)x + (-b)
//
// O(log(n)) amortizado por insercao
// O(log(n)) por query

struct Line {
  mutable ll a, b, p;
  bool operator<(const Line& o) const { return a < o.a; }
  bool operator<(ll x) const { return p < x; }
};

struct DynamicCHT : multiset<Line, less<>> {
  ll div(ll a, ll b) {
    return a / b - ((a ^ b) < 0 and a % b);
  }

  void update(iterator x) {
    if (next(x) == end()) x->p = LLINF;
    else if (x->a == next(x)->a) x->p = x->b >= next(x)->b ?
        LLINF : -LLINF;
    else x->p = div(next(x)->b - x->b, x->a - next(x)->a);
  }

  bool overlap(iterator x) {
    update(x);
    if (next(x) == end()) return 0;
    if (x->a == next(x)->a) return x->b >= next(x)->b;
    return x->p >= next(x)->p;
  }

  void add(ll a, ll b) {
    auto x = insert({a, b, 0});
    while (overlap(x)) erase(next(x)), update(x);
    if (x != begin() and !overlap(prev(x))) x = prev(x), update
        (x);
    while (x != begin() and overlap(prev(x)))
      x = prev(x), erase(next(x)), update(x);
  }

  ll query(ll x) {
    assert(!empty());
    auto l = *lower_bound(x);
    return l.a * x + l.b;
  }
};
```

3.5 Li-chao Tree

Works for any type of function that has the **transcending property**:

Given two functions f(x),g(x) of that type, if f(t) is greater than/smaller than g(t) for some x=t, then f(x) will be greater than/smaller than g(x) for x¿t. In other words, once f(x) "win/lose" g(x), f(x) will continue to "win/lose" g(x).

The most common one is the line function: $ax + b$

Dynamic Programming (4)

Game theory (5)

5.1 Classic Game

- There are n piles (heaps), each one with x_i stones.
- Each turn, a players must remove t stones (non-zero) from a pile, turning x_i into y_i.
- The game ends when it's impossible to make any more moves and the player without moves left lose.

5.2 Bouton's Theorem

Let s be the xor-sum value of all the piles sizes, a state $s = 0$ is a losing position and a state $s! = 0$ is a winnig position

5.2.1 Proof

All wining positions will have at least one valid move to turn the game into a losing position.

All losing positions will only have moves that turns the game into winning positions (except the base case when there are no piles left and the player already lost)

5.3 DAG Representation

Consider all game positions or states of the game as **Vertices** of a graph

Valid moves are the transition between states, therefore, the directed **Edges** of the graph

If a state has no outgoing edges, it's a dead end and a losing state (degenerated state).

If a state has only edges to winning states, therefore it is a losing state.

if a state has at least one edge that is a losing state, it is a winning state.

5.4 Sprague-Grundy Theorem

Let's consider a state u of a two-player impartial game and let v_i be the states reachable from it.

To this state, we can assign a fully equivalent game of Nim with one pile of size x. The number x is called the **Grundy value or nim-value or nimber** of the state u.

If **all transitions** lead to a *winning state*, the current state must be a *losing state* with nimber 0.

If **at least one transition** lead to a *losing state*, the current state must be a *winning state* with nimber ¿ 0.

The **MEX** operator satisfies both condition above and can be used to calculate the nim-value of a state:

$nimber_u$ = MEX of all $nimber_{v_i}$

Viewing the game as a DAG, we can gradually calculate the Grundy values starting from vertices without outgoing edges (nimber=0).

Note that the MEX operator **garantees** that all nim-values smaller than the considered nimber can be reached, which is essentialy the nim game with a single heap with pile size = nimber.

There are only two operations that are used when considering a Sprague-Grundy game:

5.4.1 Composition

XOR operator to compose sub-games into a single composite game

When a game is played with multiple sub-games (as nim is played with multiple piles), you are actually choosing one sub-game and making a valid move there (choosing a pile and subtracting a value from it).

The final result/winner will depend on all the sub-games played. Because you need to play all games.

To compute the final result, one can simply consider the XOR of the nimbers of all sub-games.

5.4.2 Decomposition

MEX operator to compute the nimber of a state that has multiple transitions to other states

A state with nimber x can be transitioned (decomposed) into all states with nimber $y < x$

Nevertheless a state may reach several states, only a single one will be used during the game. This shows the difference between **states** and **sub-games**: All sub-games must be played by the players, but the states of a sub-game may be ignored.

To compute the mex of a set efficiently:

mex.cpp

Description: Compute MEX efficiently by keeping track of the frequency of all existent elements and also the missing ones

Time: $\mathcal{O}(\log N)$ per addition/removal, $\mathcal{O}(1)$ to get mex value, $\mathcal{O}(N \log(N))$ to initialize

d6f2b9, 27 lines

```
struct MEX {
    map<ll, ll> freq;
    set<ll> missing;

    // initialize set with values up to {max_valid_value}
        inclusive
    MEX(ll max_valid_value) { // O(n log(n))
        for(ll i=0; i<=max_valid_value; i++)
            missing.insert(i);
    }

    ll get() { // O(1)
        if (missing.empty()) return 0;
        return *missing.begin();
    }

    void remove(ll val) { // O(log(n))
        freq[val]--;
        if (freq[val] == 0)
            missing.insert(val);
    }

    void add(ll val) { // O(log(n))
        freq[val]++;
        if (missing.count(val))
            missing.erase(val);
    }
};
```

5.5 Variations and Extensions

5.5.1 Nim with Increases

Consider a modification of the classical nim game: a player can now add stones to a chosen pile instead of removing.

Note that this extra rule needs to have a restriction to keep the game acyclic (finite game).

Lemma: This move is not used in a winnig strategy and can be ignored.

Proof: If a player adds t stones in a pile, the next player just needs to remove t stones from this pile.

Considering that the game is finite and this ends sooner or later.

Example: If the set of possible outcomes for a state is 0, 1, 2, 7, 8, 9. The nimber is 3, because the MEX is 3, which is the smallest nim-value you can't transition into and also you can transition to all smaller nim-values.

Note that 7, 8, 9 transitions can be ignored, because you can simply revert the play by subtracting the same amount.

5.6 Misère Game

In this version, the player who takes the last object loses. To consider this version, simply swap the winning and losing player of the normal version.

5.7 Staircase Nim

5.7.1 Description

In Staircase Nim, there is a staircase with n steps, indexed from 0 to n-1. In each step, there are zero or more coins. Two players play in turns. In his/her move, a player can choose a step $(i > 0)$ and move one or more coins to step below it (i-1). The player who is unable to make a move lose the game. That means the game ends when all the coins are in step 0.

5.7.2 Strategy

We can divide the steps into two types, odd steps, and even steps.

Now let's think what will happen if a player A move x coins from an even step(non-zero) to an odd step. Player B can always move these same x coins to another even position and **the state of odd positions aren't affected**

But if player A moves a coin from an odd step to an even step, similar logic won't work. Due to the degenerated case, there is a situation when x coins are moved from stair 1 to 0, and player B can't move these coins from stair 0 to -1 (not a valid move).

From this argument, we can agree that coins in even steps are useless, they don't interfere to decide if a game state is winning or losing.

Therefore, the staircase nim can be visualized as a simple nim game with only the odd steps.

When stones are sent from an odd step to an even step, it is the same as removing stones from a pile in a classic nim game.

And when stones are sent from even steps to odd ones, it is the same as the increasing variation described before.

5.8 Grundy's Game

Initially there is only one pile with x stones. Each turn, a player must divide a pile into two non-zero piles with different sizes. The player who can't do any more moves loses.

5.8.1 Degenerate (Base) States

x = 1 (nim-val = 0) (losing)

x = 2 (nim-val = 0) (losing)

5.8.2 Other States

nim-val = MEX (all transitions)

Examples

x = 3:

```
{2, 1} -> (0) xor (0) -> 0

nim-val = MEX({0}) = 1
```

x = 4:

```
{3, 1} -> (1) xor (0) -> 1

nim-val = MEX({1}) = 0
```

x = 5:

```
{4, 1} -> (0) xor (0) -> 0
{3, 2} -> (1) xor (0) -> 1

nim-val = MEX({0, 1}) = 2
```

x = 6:

```
{5, 1} -> (2) xor (0) -> 2
{4, 2} -> (0) xor (0) -> 0

nim-val = MEX({0, 2}) = 1
```

Important observation: All nimbers for $(n \geq 2000)$ are non-zero. (missing proof here and testing for values above 1e6).

5.9 Insta-Winning States

Classic nim game: if **all** piles become 0, you lose. (no more moves)

Modified nim game: if **any** pile becomes 0, you lose.

To adapt to this version of nim game, we create insta-winning states, which represents states that have a transition to any empty pile (will instantly win). Insta-winning states must have an specific nimber so they don't conflict with other nimbers when computing. A possible solution is nimber=INF, because no other nimber will be high enough to cause conflict.

Because of this adaptation, we can now ignore states with empty piles, and consider them with (null value = -1). And the (nimber = 0) now represents the states that only have transitions to insta-winning states.

After this, beside winning states and losing states, we have added two new categories of states (insta-winning and empty-pile). Notice that:

```
empty-pile <- insta-winning <- nimber(0)
```

Therefore, we have returned to the classical nim game and can proceed normally.

OBS: *Empty piles* (wasn't empty before) (nimber = -1) is different from *Non-existent piles* (never existed) (nimber = 0)

Usage Example:
https://codeforces.com/gym/101908/problem/B

5.10 References

https://cp-algorithms.com/game_theory/sprague-grundy-nim.html

https://codeforces.com/blog/entry/66040

https://brilliant.org/wiki/nim/

Numerical (6)

Number theory (7)

Combinatorial (8)

8.1 Permutations

8.1.1 Factorial

n	1	2	3	4	5	6	7	8	9	10
$n!$	1	2	6	24	120	720	5040	40320	362880	3628800
n	11	12	13	14	15	16	17			
$n!$	4.0e7	4.8e8	6.2e9	8.7e10	1.3e12	2.1e13	3.6e14			
n	20	25	30	40	50	100	150	171		
$n!$	2e18	2e25	3e32	8e47	3e64	9e157	6e262	>DBL_MAX		

Graph (9)

9.1 Fundamentals

9.2 Network flow

In optimization theory, maximum flow problems involve finding a feasible flow through a flow network that obtains the maximum possible flow rate.

dinic.cpp

Description: Run several bfs to compute the residual graph until a max flow configuration is discovered

Time: General Case, $\mathcal{O}\left(V^2E\right)$; Unit Capacity, $\mathcal{O}\left((V+E)\sqrt{E}\right)$; Bipartite and unit capacity, $\mathcal{O}\left((V+E)\sqrt{V}\right)$

dea1b7, 86 lines

```
// remember to duplicate vertices for the bipartite graph
// N = number of nodes, including sink and source
const ll N = 700;

struct Dinic {
    struct Edge {
        ll from, to, flow, cap;
    };
    vector<Edge> edges;

    vector<ll> g[N];
    ll ne = 0, lvl[N], vis[N], pass;
    ll qu[N], px[N], qt;

    ll run(ll s, ll sink, ll minE) {
        if (s == sink) return minE;
        ll ans = 0;
        for(; px[s] < (int)g[s].size(); px[s]++){
            ll e = g[s][ px[s] ];
            auto &v = edges[e], &rev = edges[e^1];
            if( lvl[v.to] != lvl[s]+1 || v.flow >= v.cap)
                continue;
            ll tmp = run(v.to, sink, min(minE, v.cap - v.flow))
                ;
            v.flow += tmp, rev.flow -= tmp;
            ans += tmp, minE -= tmp;
            if (minE == 0) break;
        }
        return ans;
    }

    bool bfs(ll source, ll sink) {
        qt = 0;
        qu[qt++] = source;
        lvl[source] = 1;
        vis[source] = ++pass;
        for(ll i=0; i<qt; i++) {
            ll u = qu[i];
            px[u] = 0;
            if (u == sink) return 1;
            for(auto& ed :g[u]) {
                auto v = edges[ed];
                if (v.flow >= v.cap || vis[v.to] == pass)
                    continue;
                vis[v.to] = pass;
                lvl[v.to] = lvl[u]+1;
                qu[qt++] = v.to;
            }
        }
        return false;
    }

    ll flow(ll source, ll sink) { // max_flow
        reset_flow();
        ll ans = 0;
        while(bfs(source, sink))
            ans += run(source, sink, LLINF);
        return ans;
    }

    void addEdge(ll u, ll v, ll c, ll rc = 0) { // c = capacity
        , rc = retro-capacity;
        Edge e = {u, v, 0, c};
        edges.pb(e);
        g[u].pb(ne++);
        e = {v, u, 0, rc};
        edges.pb(e);
        g[v].pb(ne++);
    }

    void reset_flow() {
        for (ll i=0; i<ne; i++) edges[i].flow = 0;
        memset(lvl, 0, sizeof(lvl));
        memset(vis, 0, sizeof(vis));
        memset(qu, 0, sizeof(qu));
        memset(px, 0, sizeof(px));
        qt = 0; pass = 0;
    }

    // cut set cost = minimum cost = max flow
    // cut set is the set of edges that, if removed,
    // will disrupt flow from source to sink and make it 0.
    vector<pll> cut() {
        vector<pll> cuts;
        for (auto [from, to, flow, cap]: edges)
            if (flow == cap and vis[from] == pass and vis[to] <
                pass and cap > 0)
                cuts.pb({from, to});
        return cuts;
    }
};
```

9.2.1 Minimum Cut

In computer science and optimization theory, the max-flow min-cut theorem states that, in a flow network, the maximum amount of flow passing from the source to the sink is equal to the total weight of the edges in a minimum cut, i.e., the smallest total weight of the edges which if removed would disconnect the source from the sink.

Let's define an s-t cut C = (S-component, T-component) as a partition of V $\in$ G such that source s $\in$ S-component and sink t $\in$ T-component. Let's also define a cut-set of C to be the set (u, v) $\in$ E $\mid$ u $\in$ S-component, v $\in$ T-component such that if all edges in the cut-set of C are removed, the Max Flow from s to t is 0 (i.e., s and t are disconnected). The cost of an s-t cut C is defined by the sum of the capacities of the edges in the cut-set of C.

The by-product of computing Max Flow is Min Cut! After Max Flow algorithm stops, we run graph traversal (DFS/BFS) from source s again. All reachable vertices from source s using positive weighted edges in the residual graph belong to the S-component. All other unreachable vertices belong to the T-component. All edges connecting the S-component to the T-component belong to the cut-set of C. The Min Cut value is equal to the Max Flow value. This is the minimum over all possible s-t cuts values.

9.2.2 Matching with Flow

By modeling a bipartite graph, with some Vertices (that will choose a match) to be on the L graph and some Vertices (that will be chosen) on the R. Set the correct capacities for these edges and also for the edges that connects the sink and source. After this modeling and running the dinic max flow algorithm, one will generate a possible matching (if it is possible).

A special case of matching is the perfect matching, which includes all vertices from the bipartite graph L and R.

A maximum matching has the maximum cadinality. A perfect matching is a maximum matching. But the opposite is not necessarity true.

It's possible to access dinic.edges, which is a vector that contains all edges and also its respective attributes, like the *flow* passing through each edge. Remember to consider that negative flow exist for reverse edges.

9.3 Matching

9.4 Coloring

9.5 Undirected Graph

Bridges and Articulation Points are concepts for undirected graphs!

9.5.1 Bridges (Cut Edges)

Also called **isthmus** or **cut arc**.

A back-edge is never a bridge!

A **lowlink** for a vertice U is the closest vertice to the root reachable using only span edges and a *single* back-edge, starting in the subtree of U.

After constructing a DFS Tree, an edge (u, v) is a bridge $\iff$ there is no back-edge from v (or a descendent of v) to u (or an ancestor of u)

To do this efficiently, it's used $tin[i]$ (entry time of node i) and $low[i]$ (minimum entry time considering all nodes that can be reached from node i).

In another words, a edge (u, v) is a bridge $\iff$ the low[v] ¿ tin[u].

bridges.cpp

Description: Using the concepts of entry time (tin) and lowlink (low), an edge is a bridge if, and only if, low[v] > tin[u]

Time: $\mathcal{O}(V+E)$

87e0d3, 25 lines

```
vector<vll> g(MAX);
ll timer = 1;
ll tin[MAX], low[MAX];
vector<pll> bridges;

void dfs(ll u, ll p = -1){
    tin[u] = low[u] = timer++;
    for(auto v : g[u]) if (v != p) {
        if (tin[v]) // v was visited ({u,v} is a back-edge)
            // considering a single back-edge:
            low[u] = min(low[u], tin[v]);
        else { // v wasn't visited ({u, v} is a span-edge)
            dfs(v, u);
            // after low[v] was computed by dfs(v, u):
            low[u] = min(low[u], low[v]);
            if (low[v] > tin[u])
                bridges.pb({u, v});
        }
    }
}

void findBridges(ll n) {
    for(ll i=1; i<=n; i++) if (!tin[i])
        dfs(i);
}
```

9.5.2 Bridge Tree

After merging *vertices* of a **2-edge connected component** into single vertices, and leaving only bridges, one can generate a Bridge Tree.

Every **2-edge connected component** has following properties:

- For each pair of vertices A, B inside the same component, there are at least 2 distinct paths from A to B (which may repeat vertices).

bridgeTree.cpp

Description: After finding bridges, set an component id for each vertice, then merge vertices that are in the same 2-edge connected component

Time: $\mathcal{O}(V+E)$

6d00bd, 47 lines

```
// g: u -> {v, edge id}
vector<vector<pll>> g(MAX);
vector<vll> gc(MAX);
ll timer = 1;
ll tin[MAX], low[MAX], comp[MAX];
bool isBridge[MAX];

void dfs(ll u, ll p = -1) {
    tin[u] = low[u] = timer++;
    for(auto [v, id] : g[u]) if (v != p) {
        if (tin[v])
            low[u] = min(low[u], tin[v]);
        else {
            dfs(v, u);
            low[u] = min(low[u], low[v]);
            if (low[v] > tin[u])
                isBridge[id] = 1;
        }
    }
}

void dfs2(ll u, ll c, ll p = -1) {
    comp[u] = c;
    for(auto [v, id] : g[u]) if (v != p) {
        if (isBridge[id]) continue;
        if (!comp[v]) dfs2(v, c, u);
    }
}

void bridgeTree(ll n) {
    // find bridges
    for(ll i=1; i<=n; i++) if (!tin[i])
        dfs(i);

    // find components
    for(ll i=1; i<=n; i++) if (!comp[i])
        dfs2(i, i);

    // condensate into a TREE (or TREES if disconnected)
    for(ll u=1; u<=n; u++) {
        for(auto [v, id] : g[u]) {
            if (comp[u] != comp[v]) {
                gc[comp[u]].pb(comp[v]);
            }
        }
    }
}
```

9.5.3 Articulation Points

One Vertice in a graph is considered a Articulation Points or Cut Vertice if its removal in the graph will generate more disconnected components

articulation.cpp

Description: if low[v] >= tin[u], u is an articulation points The root is a corner case

Time: $\mathcal{O}(V+E)$

8707a0, 29 lines

```
vector<vll> g(MAX);
ll timer = 1;
ll low[MAX], tin[MAX], isAP[MAX];
// when vertex i is removed from graph
// isAP[i] is the quantity of new disjoint components created
// isAP[i] >= 1 {i is a Articulation Point}
void dfs(ll u, ll p = -1) {
    low[u] = tin[u] = timer++;

    for(auto v : g[u]) if (v != p) {
        if (tin[v]) // visited
            low[u] = min(low[u], tin[v]);
        else { // not visited
            dfs(v, u);
            low[u] = min(low[u], low[v]);

            if (low[v] >= tin[u])
                isAP[u]++;
        }
    }

    // corner case: root
    if (p == -1 and isAP[u]) isAP[u]--;
}

void findAP(ll n) {
    for(ll i=1; i<=n; i++) if (!tin[i])
        dfs(i);
}
```

9.5.4 Block Cut Tree

After merging *edges* of a **2-vertex connected component** into single vertices, one can obtain a block cut tree.

2-vertex connected components are also called as biconnected component

Every bridge by itself is a biconnected component

Each edge in the block-cut tree connects exactly an Articulation Point and a biconnected component (bipartite graph)

Each biconnected component has the following properties:

- For each pair of edges, there is a cycle that contains both edges
- For each pair of vertices A, B inside the same connected component, there are at least 2 distinct paths from A to B (which do not repeat vertices).

blockCutTree.cpp

Description: After Merging 2-Vertex Connected Components, one can generate a block cut tree

Time: $\mathcal{O}(V+E)$

f752d5, 100 lines

```cpp
// Block-Cut Tree (bruno monteiro)
//
// Cria a block-cut tree, uma arvore com os blocos
// e os pontos de articulacao
// Blocos sao as componentes 2-vertice-conexos maximais
// Uma 2-coloracao da arvore eh tal que uma cor sao
// os componentes, e a outra cor sao os pontos de articulacao
//
// Funciona para grafo nao conexo
//
// isAP[i] responde o numero de novas componentes conexas
// criadas apos a remocao de i do grafo g
// Se isAP[i] >= 1, i eh ponto de articulacao
//
// Para todo i < blocks.size()
// blocks[i] eh uma componente 2-vertce-conexa maximal
// blockEdges[i] sao as arestas do bloco i
//
// tree eh a arvore block-cut-tree
// tree[i] eh um vertice da arvore que corresponde ao bloco i
//
// comp[i] responde a qual vertice da arvore vertice i pertence
//
// Arvore tem no maximo 2n vertices
//
// O(n+m)

// 0-idx graph!!!
vector<vll> g(MAX), tree, blocks;
vector<vector<pll>> blockEdges;
stack<ll> st; // st for vertices,
stack<pll> st2; // st2 for edges
vector<ll> low, tin, comp, isAP;
ll timer = 1;

void dfs(ll u, ll p = -1) {
    low[u] = tin[u] = timer++;

    st.push(u);

    // add only back-edges to stack
    if (p != -1) st2.push({u, p});
    for(auto v : g[u]) if (v != p) {
        if (tin[v] != -1) // visited
            st2.push({u, v});
    }

    for(auto v : g[u]) if (v != p) {
        if (tin[v] != -1) // visited
            low[u] = min(low[u], tin[v]);
        else { // not visited
            dfs(v, u);
            low[u] = min(low[u], low[v]);

            if (low[v] >= tin[u]) {
                isAP[u] += 1;

                blocks.pb(vll(1, u));
                while(blocks.back().back() != v)
                    blocks.back().pb(st.top()), st.pop();

                blockEdges.pb(vector<pll>(1, st2.top())), st2.
                    pop();
                while(blockEdges.back().back() != pair<ll, ll>(
                    v, u))
                    blockEdges.back().pb(st2.top()), st2.pop();
            }
        }
    }

    // corner case: root
    if (p == -1 and isAP[u]) isAP[u]--;
}

void blockCutTree(ll n) {

    // initialize vectors and reset
    tree.clear(), blocks.clear(), blockEdges.clear();
    st = stack<ll>(), st2 = stack<pll>();
    tin.assign(n, -1);
    low.assign(n, 0), comp.assign(n, 0), isAP.assign(n, 0);
    timer = 1;

    // find Articulation Points
    for(ll i=0; i<n; i++) if (tin[i] == -1)
        dfs(i);

    // set component id for APs
    tree.assign(blocks.size(), vll());
    for(ll i=0; i<n; i++) if (isAP[i])
        comp[i] = tree.size(), tree.pb(vll());

    // set component id for non-APs and construct tree
    for(ll u=0; u<(ll)blocks.size(); u++) {
        for(auto v : blocks[u]) {
            if (!isAP[v])
                comp[v] = u;
            else
                tree[u].pb(comp[v]), tree[comp[v]].pb(u);
        }
    }
}
```

9.5.5 Minimum Spanning Tree

A minimum spanning tree (MST) or minimum weight spanning tree is a subset of the edges of a connected, edge-weighted undirected graph that connects all the vertices together, without any cycles and with the minimum possible total edge weight. That is, it is a spanning tree whose sum of edge weights is as small as possible.

kruskal.cpp

Description: Sort all edges in crescent order by weight, include all edges which joins two disconnected trees. In case of tie, choose whichever. Dont include edges that will join a already connected part of the tree.

Time: $\mathcal{O}(E \log E\alpha)$

206ba3, 21 lines

```cpp
// use DSU struct
struct DSU{};

set<array<ll, 3>> edges;

int32_t main(){ sws;
    ll n, m; cin >> n >> m;
    DSU dsu(n+1);
    for(ll i=0; i<m; i++) {
        ll u, v, w; cin >> u >> v >> w;
        edges.insert({w, u, v});
    }
    ll minCost = 0;
    for(auto [w, u, v] : edges) {
        if (dsu.find(u) != dsu.find(v)) {
            dsu.join(u, v);
            minCost += w;
        }
    }
    cout << minCost << endl;
}
```

9.6 Directed Graph

9.6.1 Topological Sort

Sort a directed graph with no cycles (DAG) in an order which each source of an edge is visited before the sink of this edge.

Cannot have cycles, because it would create a contradition of which vertices whould come before.

It can be done with a DFS, appending in the reverse order of transversal. Also a stack can be used to reverse order

toposort.cpp

Description: Using DFS pos order transversal and inverting the order, one can obtain the topological order

Time: $\mathcal{O}(V+E)$

75f781, 17 lines

```cpp
vector<vll> g(MAX, vll());
vector<bool> vis;
vll topological;

void dfs(ll u) {
    vis[u] = 1;
    for(auto v : g[u]) if (!vis[v]) dfs(v);
    topological.pb(u);
}

// 1 - indexed
```

```
void topological_sort(ll n) {
    vis.assign(n+1, 0);
    topological.clear();
    for(ll i=1; i<=n; i++) if (!vis[i]) dfs(i);
    reverse(topological.begin(), topological.end());
}
```

kosaraju.cpp

Description: By using the fact that the inverted graph has the same SCCs, just do a DFS twice to find all SCCs. A condensated graph can be created if wished. The condensated graph is a DAG!!

Time: $\mathcal{O}(V + E)$

381904, 45 lines

```
struct Kosaraju {
    ll n;
    vector<vll> g, gi, gc;
    vector<bool> vis;
    vector<ll> comp;
    stack<ll, vll> st;

    void dfs(ll u) { // g
        vis[u] = 1;
        for(auto v : g[u]) if (!vis[v]) dfs(v);
        st.push(u);
    }

    void dfs2(ll u, ll c) { // gi
        comp[u] = c;
        for(auto v : gi[u]) if (comp[v] == -1) dfs2(v, c);
    }

    Kosaraju(vector<vll> &g_)
      : g(g_), n(g_.size()-1) { // 1-idx

        gi.assign(n+1, vll());
        for(ll i=1; i<=n; i++) {
            for(auto j : g[i])
                gi[j].pb(i);
        }

        gc.assign(n+1, vll());
        vis.assign(n+1, 0);
        comp.assign(n+1, -1);
        st = stack<ll, vll>();

        for(ll i=1; i<=n; i++) if (!vis[i]) dfs(i);

        while(!st.empty()) {
            auto u = st.top(); st.pop();
            if (comp[u] == -1) dfs2(u, u);
        }

        for(ll u=1; u<=n; u++)
            for(auto v : g[u])
                if (comp[u] != comp[v])
                    gc[comp[u]].pb(comp[v]);
    }
};
```

9.6.3 2-SAT

SAT (Boolean satisfiability problem) is NP-Complete.

2-SAT is a restriction of the SAT problem, in 2-SAT every clause has exactly two variables: $(X_1 \vee X_2) \wedge (X_2 \vee X_3)$

Every restriction or implication are represented in the graph as directed edges.

The algorithm uses kosaraju to check if any (X and $\neg X$) are in the same Strongly Connected Component (which implies that the problem is impossible).

If it doesn't, there is at least one solution, which can be generated using the topological sort of the same kosaraju (opting for the variables that appers latter in the sorted order)

2sat.cpp

Description: Kosaraju to find if there are SCCs. If there are not cycles, use toposort to choose states

Time: $\mathcal{O}(V + E)$

87417c, 83 lines

```
// 0-idx graph !!!!
struct TwoSat {
    ll N; // needs to be the twice of the number of variables
    // node with idx 2x => variable x
    // node with idx 2x+1 => variable !x

    vector<vll> g, gi;
    // g = graph; gi = transposed graph (all edges are inverted
        )

    TwoSat(ll n) { // number of variables (add +1 faor 1-idx)
        N = 2*n;
        g.assign(N, vll());
        gi.assign(N, vll());
    }

    ll idx; // component idx
    vector<ll> comp, order; // topological order (reversed)
    vector<bool> vis, chosen;
    //  chosen[x] == 0 -> x was assigned
    //  chosen[x] == 1 -> !x was assigned

    // dfs and dfs2 are part of kosaraju algorithm
    void dfs(ll u) {
        vis[u] = 1;
        for (ll v : g[u]) if (!vis[v]) dfs(v);
        order.pb(u);
    }

    void dfs2(ll u, ll c) {
        comp[u] = c;
        for (ll v : gi[u]) if (comp[v] == -1) dfs2(v, c);
    }

    bool solve() {
        vis.assign(N, 0);
        order = vector<ll>();
        for (ll i = 0; i < N; i++) if (!vis[i]) dfs(i);

        comp.assign(N, -1); // comp = 0 can exist
        idx = 1;
        for(ll i=(ll)order.size()-1; i>=0; i--) {
            ll u = order[i];
            if (comp[u] == -1) dfs2(u, idx++);
        }

        chosen.assign(N/2, 0);
        for (ll i = 0; i < N; i += 2) {
            // x and !x in the same component => contradiction
            if (comp[i] == comp[i+1]) return false;
            chosen[i/2] = comp[i] < comp[i+1]; // choose latter
                node
        }
        return true;
    }

    // a (with flagA) implies => b (with flagB)
    void add(ll a, bool fa, ll b, bool fb) {
        // {fa == 0} => a
        // {fa == 1} => !a
        a = 2*a + fa;
        b = 2*b + fb;
        g[a].pb(b);
        gi[b].pb(a);
    }

    // force a state for a certain variable (must be true)
    void force(ll a, bool fa) {
        add(a, fa^1, a, fa);
    }

    // xor operation: one must exist, and only one can exist
    void exclusive(ll a, bool fa, ll b, bool fb) {
        add(a, fa^0, b, fb^1);
        add(a, fa^1, b, fb^0);
        add(b, fb^0, a, fa^1);
        add(b, fb^1, a, fa^0);
    }

    // nand operation: no more than one can exist
    void nand(ll a, bool fa, ll b, bool fb) {
        add(a, fa^0, b, fb^1);
        add(b, fb^0, a, fa^1);
    }
};
```

9.7 Trees

lca.cpp

Description: Solves LCA for trees

Time: $\mathcal{O}(N \log(N))$ to build, $\mathcal{O}(\log(N))$ per query

7afc1a, 54 lines

```
struct BinaryLifting {
    ll n, logN = 20; // ~1e6
    vector<vll> g;
    vector<ll> depth;
    vector<vll> up;

    BinaryLifting(vector<vll> &g_)
     : g(g_), n(g_.size() + 1) { // 1-idx
        depth.assign(n, 0);

        while((1 << logN) < n) logN++;
        up.assign(n, vll(logN, 0));
        build();
    }

    void build(ll u = 1, ll p = -1) {
        for(ll i=1; i<logN; i++) {
            up[u][i] = up[ up[u][i-1] ][i-1];
        }

        for(auto v : g[u]) if (v != p) {
            up[v][0] = u;
            depth[v] = depth[u] + 1;
```

```cpp
            build(v, u);
        }
    }

    ll go(ll u, ll dist) { // O(log(n))
        for(ll i=logN-1; i>=0; i--) { // bigger jumps first
            if (dist & (1LL << i)) {
                u = up[u][i];
            }
        }
        return u;
    }

    ll lca(ll a, ll b) { // O(log(n))
        if (depth[a] < depth[b]) swap(a, b);
        a = go(a, depth[a] - depth[b]);
        if (a == b) return a;

        for(ll i=logN-1; i>=0; i--) {
            if (up[a][i] != up[b][i]) {
                a = up[a][i];
                b = up[b][i];
            }
        }
        return up[a][0];
    }

    ll lca(ll a, ll b, ll root) { // lca(a, b) when tree is
         rooted at 'root'
        return lca(a, b)^lca(b, root)^lca(a, root); //magic
    }
};
```

queryTree.cpp
Description: Binary Lifting for min, max weight present in a simple path
Time: $\mathcal{O}(N\log(N))$ to build; $\mathcal{O}(\log(N))$ per query 75ba37, 67 lines

```cpp
struct BinaryLifting {
    ll n, logN = 20; // ~1e6
    vector<vpll> g;
    vector<ll> depth;
    vector<vll> up, mx, mn;

    BinaryLifting(vector<vpll> &g_)
     : g(g_), n(g_.size() + 1) { // 1-idx
        depth.assign(n, 0);

        while((1 << logN) < n) logN++;
        up.assign(n, vll(logN, 0));
        mx.assign(n, vll(logN, -INF));
        mn.assign(n, vll(logN, INF));
        build();
    }

    void build(ll u = 1, ll p = -1) {

        for(ll i=1; i<logN; i++) {
            mx[u][i] = max(mx[u][i-1], mx[ up[u][i-1] ][i-1]);
            mn[u][i] = min(mn[u][i-1], mn[ up[u][i-1] ][i-1]);
            up[u][i] = up[ up[u][i-1] ][i-1];
        }

        for(auto [v, w] : g[u]) if (v != p) {
            mx[v][0] = mn[v][0] = w;
            up[v][0] = u;
            depth[v] = depth[u] + 1;
            build(v, u);
        }
    }

    array<ll, 3> go(ll u, ll dist) { // O(log(n))
        ll mxval = -INF, mnval = INF;
        for(ll i=logN-1; i>=0; i--) { // bigger jumps first
            if (dist & (1LL << i)) {
                mxval = max(mxval, mx[u][i]);
                mnval = min(mnval, mn[u][i]);
                u = up[u][i];
            }
        }
        return {u, mxval, mnval};
    }

    array<ll, 3> query(ll u, ll v) { // O(log(n))
        if (depth[u] < depth[v]) swap(u, v);

        auto [a, mxval, mnval] = go(u, depth[u] - depth[v]);
        ll b = v;

        if (a == b) return {a, mxval, mnval};

        for(ll i=logN-1; i>=0; i--) {
            if (up[a][i] != up[b][i]) {
                mxval = max({mxval, mx[a][i], mx[b][i]});
                mnval = min({mnval, mn[a][i], mn[b][i]});
                a = up[a][i];
                b = up[b][i];
            }
        }

        mxval = max({mxval, mx[a][0], mx[b][0]});
        mnval = min({mnval, mn[a][0], mn[b][0]});
        return {up[a][0], mxval, mnval};
    }
};
```

9.8 Math

Geometry (10)

Strings (11)

11.1 Hashing

Hashing consists in generating a Polynomial for the string, therefore, assigning each distint string to a specific numeric value In practice, there will always be some collisions:

Probability of colision: $= \frac{n^2}{2m}$

n = Comparissons, m = mod size

when using multiple mods, they multiply: m = m1 * m2

hashing.cpp
Description: Create a numerical value for a string by using polynomial hashing
Time: $\mathcal{O}(n)$ to build, $\mathcal{O}(1)$ per query c3a650, 43 lines

```cpp
// s[0]*P^n + s[1]*P^(n-1) + ... + s[n]*P^0
// 0-idx
struct Hashing {
    ll n, mod;
    string s;
    vector<ll> p, h; // p = P^i, h = accumulated hash sum

    const ll P = 31; // can be 53

    Hashing(string &s_, ll m)
     : n(s_.size()), s(s_), mod(m), p(n), h(n) {

        for(ll i=0; i<n; i++)
            p[i] = (i ? P*p[i-1] : 1) % mod;

        for(ll i=0; i<n; i++)
            h[i] = (s[i] + P*(i ? h[i-1] : 0)) % mod;
    }

    ll query(ll l, ll r) { // [l, r] inclusive (0-idx)
        ll hash = h[r] - (l ? (p[r-l+1]*h[l-1]) % mod : 0);
        return hash < 0 ? hash + mod : hash;
    }
};

// for codeforces:
mt19937 rng(chrono::steady_clock::now().time_since_epoch().
     count());

int32_t main() { sws;
    vector<ll> mods = {
        1000000009,1000000021,1000000033,
        1000000087,1000000093,1000000097,
        1000000103,1000000123,1000000181,
        1000000207,1000000223,1000000241,
        1000000271,1000000289,1000000297
    };

    shuffle(mods.begin(), mods.end(), rng);

    string s; cin >> s;

    Hashing hash(s, mods[0]);
}
```

11.2 Z-Function

Suppose we are given a string s of length n. The Z-function for this string is an array of length n where the i-th element is equal to the greatest number of characters starting from the position i that coincide with the first characters of s (the prefix of s)

The first element of the Z-function, z[0], is generally not well defined. This implementation assumes it as z[0] = 0. But it can also be interpreted as z[0] = n (all characters coincide).

Can be used to solve the following simples problems:

- Find all ocurrences of a pattern p in another string s. (p + '$' + s) (z[i] == p.size())
- Find all borders. A border of a string is a prefix that is also a suffix of the string but not the whole string. For example, the borders of abcababcab are ab and abcab. (z[8] = 2, z[5] = 5) (z[i] = n-i)
- Find all period lengths of a string. A period of a string is a prefix that can be used to generate the whole string by repeating the prefix. The last repetition may be partial. For example, the periods of *abcabca* are **abc**, **abcabc** and **abcabca**.

It works because (z[i] + i ¿= n) is the condition when the common characters of z[i] in addition to the elements already passed, exceeds or is equal to the end of the string. For example:

abaababaab z[8] = 2

abaababa is the period; the remaining (z[i] characters) are a prefix of the period; and when all these characters are combined, it can form the string (which has n characters).

zfunction.cpp
Description: For each substring starting at position i, compute the maximum match with the original prefix. z[0] = 0
Time: $\mathcal{O}(n)$ 14b37c, 12 lines

```
vector<ll> z_function(string &s) { // O(n)
    ll n = (ll) s.length();
    vector<ll> z(n);
    for (ll i=1, l=0, r=0; i<n; i++) {
        if (i <= r) z[i] = min(r - i + 1, z[i - l]);

        while (i + z[i] < n and s[z[i]] == s[i + z[i]]) z[i]++;

        if (r < i + z[i] - 1) l = i, r = i + z[i] - 1;
    }
    return z;
}
```

11.3 KMP

KMP stands for Knuth-Morris-Pratt and computes the prefix function.

You are given a string s of length n. The prefix function for this string is defined as an array π of length n, where $\pi[i]$ is the length of the longest proper prefix of the substring $s[0 \dots i]$ which is also a suffix of this substring. A proper prefix of a string is a prefix that is not equal to the string itself. By definition, $\pi[0] = 0$.

For example, prefix function of string *"abcabcd"* is $[0, 0, 0, 1, 2, 3, 0]$, and prefix function of string *"aabaaab"* is $[0, 1, 0, 1, 2, 2, 3]$.

kmp.cpp
Description: Computes the prefix function
Time: $\mathcal{O}(n)$ 48408b, 13 lines

```
vector<ll> kmp(string &s) { // O(n)
    ll n = (ll) s.length();
    vector<ll> pi(n);
    for (ll i = 1; i < n; i++) {
        ll j = pi[i-1];
        while (j > 0 && s[i] != s[j])
            j = pi[j-1];
        if (s[i] == s[j])
            j++;
        pi[i] = j;
    }
    return pi;
}
```

11.4 Suffix Array

The suffix array is the array with size n, whose values are the indexes from the longest substring (0) to the smallest substring (n) after ordering it lexicographically. Example:

```
Let the given string be "banana".

0 banana                          5 a
1 anana     Sort the Suffixes     3 ana
2 nana      ---------------->     1 anana
3 ana        alphabetically       0 banana
4 na                              4 na
5 a                               2 nana

So the suffix array for "banana" is {5, 3, 1, 0, 4, 2}
```

Note that the length of the string i is: (s.size()-sa[i])

suffix-array.cpp
Description: Creates the Suffix Array
Time: $\mathcal{O}(N \log N)$ 49608b, 20 lines

```
vector<ll> suffixArray(string s) {
    s += "!";
    ll n = s.size(), N = max(n, 260LL);
    vector<ll> sa(n), ra(n);
    for (ll i = 0; i < n; i++) sa[i] = i, ra[i] = s[i];

    for (ll k = 0; k < n; k ? k *= 2 : k++) {
        vector<ll> nsa(sa), nra(n), cnt(N);

        for (ll i = 0; i < n; i++) nsa[i] = (nsa[i]-k+n)%n, cnt
            [ra[i]]++;
        for (ll i = 1; i < N; i++) cnt[i] += cnt[i-1];
        for (ll i = n-1; i+1; i--) sa[--cnt[ra[nsa[i]]]] = nsa[
            i];

        for (ll i = 1, r = 0; i < n; i++) nra[sa[i]] = r += ra[
            sa[i]] !=
           ra[sa[i-1]] or ra[(sa[i]+k)%n] != ra[(sa[i-1]+k)%n
              ];
        ra = nra;
        if (ra[sa[n-1]] == n-1) break;
    }
    return vector<ll>(sa.begin()+1, sa.end());
}
```

Kasai generates an array of size n (like the suffix array), whose values indicates the lenght of the longest common prefix beetwen (sa[i] and sa[i+1])

kasai.cpp
Description: Creates the Longest Common Prefix array (LCP)
Time: $\mathcal{O}(N \log N)$ 913195, 13 lines

```
vector<ll> kasai(string s, vector<ll> sa) {
    ll n = s.size(), k = 0;
    vector<ll> ra(n), lcp(n);
    for (ll i = 0; i < n; i++) ra[sa[i]] = i;

    for (ll i = 0; i < n; i++, k -= !!k) {
        if (ra[i] == n-1) { k = 0; continue; }
        ll j = sa[ra[i]+1];
        while (i+k < n and j+k < n and s[i+k] == s[j+k]) k++;
        lcp[ra[i]] = k;
    }
    return lcp;
}
```

Problems that can be solved:

Numbers of Distinct Substrings:

- $\frac{n(n+1)}{2} - lcp[i]$ (for all i)

Longest Repeated Substring:

- biggest lcp[i]. The position can be found in sa[i]

Find how many distinct substrings there are for each len in [1:n]:

- Use delta encoding and the fact that lcp[i] counts the repeated substring between s.substr(sa[i]) and s.substr(sa[i+1]), which are the substrings corresponding to the commom prefix.

Find the k-th distinct substring:

```
    string s; cin >> s;
    ll n = s.size();

    auto sa = suffix_array(s);
    auto lcp = kasai(s, sa);

    ll k; cin >> k;

    for(ll i=0; i<n; i++) {
        ll len = n-sa[i];
        if (k <= len) {
            cout << s.substr(sa[i], k) << endl;
            break;
        }
        k += lcp[i] - len;
    }
```

11.5 Manacher

Manacher's Algorithm is used to find all palindromes in a string.

For each substring, centered at i, find the longest palindrome that can be formed.

Works best for odd size string, so we convert all string to odd ones by adding and extra characters between the original ones

Therefore, the value stored in the vector cnt is actually palindrome-len + 1.

manacher.cpp
Description: Covert String to odd length to use manacher, which computes all the maximum lengths of all palindromes in the given string
Time: $\mathcal{O}(2n)$ 0c2a2b, 46 lines

```
struct Manacher {
    string s, t;
    vector<ll> cnt;

    // t is the transformed string of s, with odd size
    Manacher(string &s_) : s(s_) {
        t = "#";
```

```
        for(auto c : s) {
            t += c, t += "#";
        }
        count();
    }

    // perform manacher on the odd string
    // cnt will give all the palindromes centered in i
    // for the odd string t
    void count() {
        ll n = t.size();
        string aux = "$" + t + "^";
        vector<ll> p(n + 2);
        ll l = 1, r = 1;
        for(ll i = 1; i <= n; i++) {
            p[i] = max(0LL, min(r - i, p[l + (r - i)]));
            while(aux[i - p[i]] == aux[i + p[i]]) {
                p[i]++;
            }
            if(i + p[i] > r) {
                l = i - p[i], r = i + p[i];
            }
        }
        cnt = vector<ll>(p.begin() + 1, p.end() - 1);
    }

    // compute a longest palindrome present in s
    string getLongest() {
        ll len = 0, pos = 0;
        for(ll i=0; i<(ll)t.size(); i++) {
            ll sz = cnt[i]-1;
            if (sz > len) {
                len = sz;
                pos = i;
            }
        }
        return s.substr(pos/2 - len/2, len);
    }
};
```

11.6 Booth

An efficient algorithm which uses a modified version of KMP to compute the least amount of rotation needed to reach the **lexicographically minimal string rotation**.

A rotation of a string can be generated by moving characters one after another from beginning to end. For example, the rotations of *acab* are *acab*, *caba*, *abac*, and *baca*.

booth.cpp
Description: Use a modified version of KMP to find the lexicographically minimal string rotation
Time: $\mathcal{O}(n)$ 64184b, 30 lines

```
// Booth Algorithm
ll least_rotation(string &s) { // O(n)
    ll n = s.length();
    vector<ll> f(2*n, -1);
    ll k = 0;
    for(ll j=1; j<2*n; j++) {
        ll i = f[j-k-1];
        while(i != -1 and s[j % n] != s[(k+i+1) % n] ) {
            if (s[j % n] < s[(k+i+1) % n])
                k = j - i - 1;
            i = f[i];
        }
        if (i == -1 and s[j % n] != s[(k+i+1) % n] ) {
            if (s[j % n] < s[(k+i+1) % n])
                k = j;
            f[j - k] = -1;
        }
        else
            f[j - k] = i + 1;
    }
    return k;
}

int32_t main(){ sws;
    string s; cin >> s;
    ll n = s.length();
    ll ans_idx = least_rotation(s);
    string tmp = s + s;
    cout << tmp.substr(ans_idx, n) << endl;
}
```

Miscellaneous (12)

Techniques (A)

techniques.txt

159 lines

```
Recursion
Divide and conquer
  Finding interesting points in N log N
Algorithm analysis
  Master theorem
  Amortized time complexity
Greedy algorithm
  Scheduling
  Max contiguous subvector sum
  Invariants
  Huffman encoding
Graph theory
  Dynamic graphs (extra book-keeping)
  Breadth first search
  Depth first search
  * Normal trees / DFS trees
  Dijkstra's algorithm
  MST: Prim's algorithm
  Bellman-Ford
  Konig's theorem and vertex cover
  Min-cost max flow
  Lovasz toggle
  Matrix tree theorem
  Maximal matching, general graphs
  Hopcroft-Karp
  Hall's marriage theorem
  Graphical sequences
  Floyd-Warshall
  Euler cycles
  Flow networks
  * Augmenting paths
  * Edmonds-Karp
  Bipartite matching
  Min. path cover
  Topological sorting
  Strongly connected components
  2-SAT
  Cut vertices, cut-edges and biconnected components
  Edge coloring
  * Trees
  Vertex coloring
  * Bipartite graphs (=> trees)
  * 3^n (special case of set cover)
  Diameter and centroid
  K'th shortest path
  Shortest cycle
Dynamic programming
  Knapsack
  Coin change
  Longest common subsequence
  Longest increasing subsequence
  Number of paths in a dag
  Shortest path in a dag
  Dynprog over intervals
  Dynprog over subsets
  Dynprog over probabilities
  Dynprog over trees
  3^n set cover
  Divide and conquer
  Knuth optimization
  Convex hull optimizations
  RMQ (sparse table a.k.a 2^k-jumps)
  Bitonic cycle
  Log partitioning (loop over most restricted)
Combinatorics
  Computation of binomial coefficients
  Pigeon-hole principle
  Inclusion/exclusion
  Catalan number
  Pick's theorem
Number theory
  Integer parts
  Divisibility
  Euclidean algorithm
  Modular arithmetic
  * Modular multiplication
  * Modular inverses
  * Modular exponentiation by squaring
  Chinese remainder theorem
  Fermat's little theorem
  Euler's theorem
  Phi function
  Frobenius number
  Quadratic reciprocity
  Pollard-Rho
  Miller-Rabin
  Hensel lifting
  Vieta root jumping
Game theory
  Combinatorial games
  Game trees
  Mini-max
  Nim
  Games on graphs
  Games on graphs with loops
  Grundy numbers
  Bipartite games without repetition
  General games without repetition
  Alpha-beta pruning
Probability theory
Optimization
  Binary search
  Ternary search
  Unimodality and convex functions
  Binary search on derivative
Numerical methods
  Numeric integration
  Newton's method
  Root-finding with binary/ternary search
  Golden section search
Matrices
  Gaussian elimination
  Exponentiation by squaring
Sorting
  Radix sort
Geometry
  Coordinates and vectors
  * Cross product
  * Scalar product
  Convex hull
  Polygon cut
  Closest pair
  Coordinate-compression
  Quadtrees
  KD-trees
  All segment-segment intersection
Sweeping
  Discretization (convert to events and sweep)
  Angle sweeping
  Line sweeping
  Discrete second derivatives
Strings
  Longest common substring
  Palindrome subsequences
  Knuth-Morris-Pratt
  Tries
  Rolling polynomial hashes
  Suffix array
  Suffix tree
  Aho-Corasick
  Manacher's algorithm
  Letter position lists
Combinatorial search
  Meet in the middle
  Brute-force with pruning
  Best-first (A*)
  Bidirectional search
  Iterative deepening DFS / A*
Data structures
  LCA (2^k-jumps in trees in general)
  Pull/push-technique on trees
  Heavy-light decomposition
  Centroid decomposition
  Lazy propagation
  Self-balancing trees
  Convex hull trick (wcipeg.com/wiki/Convex_hull_trick)
  Monotone queues / monotone stacks / sliding queues
  Sliding queue using 2 stacks
  Persistent segment tree
```