EPILOGUE: PART TWO

PT2.4 TRADE-OFFS

Table PT2.3 provides a summary of the trade-offs involved in solving for roots of algebraic and transcendental equations. Although graphical methods are time-consuming, they provide insight into the behavior of the function and are useful in identifying initial guesses and potential problems such as multiple roots. Therefore, if time permits, a quick sketch (or better yet, a computerized graph) can yield valuable information regarding the behavior of the function.

The numerical methods themselves are divided into two general categories: bracketing and open methods. The former requires two initial guesses that are on either side of a root.

TABLE PT2.3 Comparison of the characteristics of alternative methods for finding roots of algebraic and transcendental equations. The comparisons are based on general experience and do not account for the behavior of specific functions.

Method	Initial Guesses	Convergence Rate	Stability	Accuracy	Breadth of Application	Programming Effort	Comments
Direct	—	—	—	—	Limited		
Graphical	—	—	—	Poor	Real roots	—	May take more time than the numerical method
Bisection	2	Slow	Always	Good	Real roots	Easy	
False-position	2	Slow/medium	Always	Good	Real roots	Easy	
Modified FP	2	Medium	Always	Good	Real roots	Easy	
Fixed-point iteration	1	Slow	Possibly divergent	Good	General	Easy	
Newton-Raphson	1	Fast	Possibly divergent	Good	General	Easy	Requires evaluation of $f'(x)$
Modified Newton-Raphson	1	Fast for multiple roots; medium for single	Possibly divergent	Good	General	Easy	Requires evaluation of $f''(x)$ and $f'(x)$
Secant	2	Medium to fast	Possibly divergent	Good	General	Easy	Initial guesses do not have to bracket the root
Modified secant	1	Medium to fast	Possibly divergent	Good	General	Easy	
Müller	2	Medium to fast	Possibly divergent	Good	Polynomials	Moderate	
Bairstow	2	Fast	Possibly divergent	Good	Polynomials	Moderate	

This "bracketing" is maintained as the solution proceeds, and thus, these techniques are always convergent. However, a price is paid for this property in that the rate of convergence is relatively slow.

Open techniques differ from bracketing methods in that they use information at a single point (or two values that need not bracket the root to extrapolate to a new root estimate). This property is a double-edged sword. Although it leads to quicker convergence, it also allows the possibility that the solution may diverge. In general, the convergence of open techniques is partially dependent on the quality of the initial guess and the nature of the function. The closer the guess is to the true root, the more likely the methods will converge.

Of the open techniques, the standard Newton-Raphson method is often used because of its property of quadratic convergence. However, its major shortcoming is that it requires the derivative of the function be obtained analytically. For some functions this is impractical. In these cases, the secant method, which employs a finite-difference representation of the derivative, provides a viable alternative. Because of the finite-difference approximation, the rate of convergence of the secant method is initially slower than for the Newton-Raphson method. However, as the root estimate is refined, the difference approximation becomes a better representation of the true derivative, and convergence accelerates rapidly. The modified Newton-Raphson technique can be used to attain rapid convergence for multiple roots. However, this technique requires an analytical expression for both the first and second derivative.

All the numerical methods are easy to program on computers and require minimal time to determine a single root. On this basis, you might conclude that simple methods such as bisection would be good enough for practical purposes. This would be true if you were exclusively interested in determining the root of an equation once. However, there are many cases in engineering where numerous root locations are required and where speed becomes important. For these cases, slow methods are very time-consuming and, hence, costly. On the other hand, the fast open methods may diverge, and the accompanying delays can also be costly. Some computer algorithms attempt to capitalize on the strong points of both classes of techniques by initially employing a bracketing method to approach the root, then switching to an open method to rapidly refine the estimate. Whether a single approach or a combination is used, the trade-offs between convergence and speed are at the heart of the choice of a root-location technique.

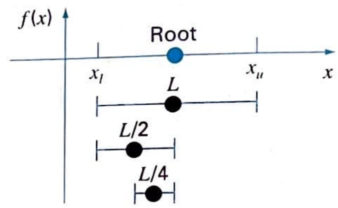
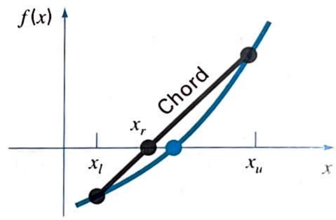
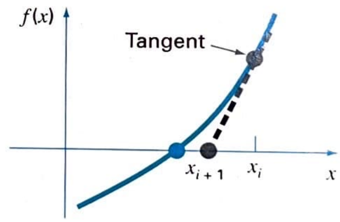
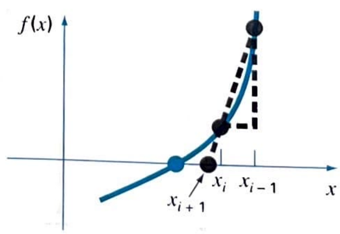
PT2.5 IMPORTANT RELATIONSHIPS AND FORMULAS

Table PT2.4 summarizes important information that was presented in Part Two. This table can be consulted to quickly access important relationships and formulas.

PT2.6 ADVANCED METHODS AND ADDITIONAL REFERENCES

The methods in this text have focused on determining a single real root of an algebraic or transcendental equation based on foreknowledge of its approximate location. In addition, we have also described methods expressly designed to determine both the real and complex

TABLE PT2.4 Summary of important information presented in Part Two.

<table>
<tr><th>Method</th><th>Formulation</th><th>Graphical Interpretation</th><th>Errors and Stopping Criteria</th></tr>
<tr><td></td><td></td><td>Bracketing methods:</td><td></td></tr>
<tr><td>Bisection</td><td>$x_r = \dfrac{x_l + x_u}{2}$
If $f(x_l)f(x_r) < 0,\ x_u = x_r$
$f(x_l)f(x_r) > 0,\ x_l = x_r$</td><td>
</td><td>Stopping criterion:
$\left\lvert \dfrac{x_r^{new} - x_r^{old}}{x_r^{new}} \right\rvert 100\% \le \epsilon_s$</td></tr>
<tr><td>False position</td><td>$x_r = x_u - \dfrac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$
If $f(x_l)f(x_r) < 0,\ x_u = x_r$
$f(x_l)f(x_r) > 0,\ x_l = x_r$</td><td>
</td><td>Stopping criterion:
$\left\lvert \dfrac{x_r^{new} - x_r^{old}}{x_r^{new}} \right\rvert 100\% \le \epsilon_s$</td></tr>
<tr><td>Newton-Raphson</td><td>$x_{i+1} = x_i - \dfrac{f(x_i)}{f'(x_i)}$</td><td>
</td><td>Stopping criterion:
$\left\lvert \dfrac{x_{i+1} - x_i}{x_{i+1}} \right\rvert 100\% \le \epsilon_s$
Error: $E_{i+1} = O(E_i^2)$</td></tr>
<tr><td>Secant</td><td>$x_{i+1} = x_i - \dfrac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$</td><td>
</td><td>Stopping criterion:
$\left\lvert \dfrac{x_{i+1} - x_i}{x_{i+1}} \right\rvert 100\% \le \epsilon_s$</td></tr>
</table>

roots of polynomials. Additional references on the subject are Ralston and Rabinowitz (1978) and Carnahan, Luther, and Wilkes (1969).

In addition to Müller's and Bairstow's methods, several techniques are available to determine all the roots of polynomials. In particular, the *quotient difference (QD) algorithm* (Henrici, 1964, and Gerald and Wheatley, 1989) determines all roots without initial guesses. Ralston and Rabinowitz (1978) and Carnahan, Luther, and Wilkes (1969) contain discussions of this method as well as of other techniques for locating roots of polynomials. As discussed in the text, the *Jenkins-Traub* and *Laguerre's* methods are widely employed.

In summary, the foregoing is intended to provide you with avenues for deeper exploration of the subject. Additionally, all the above references provide descriptions of the basic techniques covered in Part Two. We urge you to consult these alternative sources to broaden your understanding of numerical methods for root location.[1]

[1]Books are referenced only by author here, a complete bibliography will be found at the back of this text.