PART TWO

ROOTS OF EQUATIONS

PT2.1 MOTIVATION

Years ago, you learned to use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{PT2.1}$$

to solve

$$f(x) = ax^2 + bx + c = 0 \tag{PT2.2}$$

The values calculated with Eq. (PT2.1) are called the "roots" of Eq. (PT2.2). They represent the values of x that make Eq. (PT2.2) equal to zero. Thus, we can define the root of an equation as the value of x that makes $f(x) = 0$. For this reason, roots are sometimes called the *zeros* of the equation.

Although the quadratic formula is handy for solving Eq. (PT2.2), there are many other functions for which the root cannot be determined so easily. For these cases, the numerical methods described in Chaps. 5, 6, and 7 provide efficient means to obtain the answer.

PT2.1.1 Noncomputer Methods for Determining Roots

Before the advent of digital computers, there were several ways to solve for roots of algebraic and transcendental equations. For some cases, the roots could be obtained by direct methods, as was done with Eq. (PT2.1). Although there were equations like this that could be solved directly, there were many more that could not. For example, even an apparently simple function such as $f(x) = e^{-x} - x$ cannot be solved analytically. In such instances, the only alternative is an approximate solution technique.

One method to obtain an approximate solution is to plot the function and determine where it crosses the x axis. This point, which represents the x value for which $f(x) = 0$, is the root. Graphical techniques are discussed at the beginning of Chaps. 5 and 6.

Although graphical methods are useful for obtaining rough estimates of roots, they are limited because of their lack of precision. An alternative approach is to use trial and error. This "technique" consists of guessing a value of x and evaluating whether $f(x)$ is zero. If not (as is almost always the case), another guess is made, and $f(x)$ is again evaluated to determine whether the new value provides a better estimate of the root. The process is repeated until a guess is obtained that results in an $f(x)$ that is close to zero.

Such haphazard methods are obviously inefficient and inadequate for the requirements of engineering practice. The techniques described in Part Two represent alternatives that are also approximate but employ systematic strategies to home in on the true root. As elaborated on in the following pages, the combination of these systematic methods and computers makes the solution of most applied roots-of-equations problems a simple and efficient task.

PT2.1.2 Roots of Equations and Engineering Practice

Although they arise in other problem contexts, roots of equations frequently occur in the area of engineering design. Table PT2.1 lists several fundamental principles that are routinely used in design work. As introduced in Chap. 1, mathematical equations or models derived from these principles are employed to predict dependent variables as a function of independent variables, forcing functions, and parameters. Note that in each case, the dependent variables reflect the state or performance of the system, whereas the parameters represent its properties or composition.

An example of such a model is the equation, derived from Newton's second law, used in Chap. 1 for the parachutist's velocity:

$$v = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right) \tag{PT2.3}$$

TABLE PT2.1 Fundamental principles used in engineering design problems.

Fundamental Principle	Dependent Variable	Independent Variable	Parameters
Heat balance	Temperature	Time and position	Thermal properties of material and geometry of system
Mass balance	Concentration or quantity of mass	Time and position	Chemical behavior of material, mass transfer coefficients, and geometry of system
Force balance	Magnitude and direction of forces	Time and position	Strength of material, structural properties, and geometry of system
Energy balance	Changes in the kinetic- and potential-energy states of the system	Time and position	Thermal properties, mass of material, and system geometry
Newton's laws of motion	Acceleration, velocity, or location	Time and position	Mass of material, system geometry, and dissipative parameters such as friction or drag
Kirchhoff's laws	Currents and voltages in electric circuits	Time	Electrical properties of systems such as resistance, capacitance, and inductance

where velocity v = the dependent variable, time t = the independent variable, the gravitational constant g = the forcing function, and the drag coefficient c and mass m = parameters. If the parameters are known, Eq. (PT2.3) can be used to predict the parachutist's velocity as a function of time. Such computations can be performed directly because v is expressed *explicitly* as a function of time. That is, it is isolated on one side of the equal sign.

However, suppose we had to determine the drag coefficient for a parachutist of a given mass to attain a prescribed velocity in a set time period. Although Eq. (PT2.3) provides a mathematical representation of the interrelationship among the model variables and parameters, it cannot be solved explicitly for the drag coefficient. Try it. There is no way to rearrange the equation so that c is isolated on one side of the equal sign. In such cases, c is said to be *implicit*.

This represents a real dilemma, because many engineering design problems involve specifying the properties or composition of a system (as represented by its parameters) to ensure that it performs in a desired manner (as represented by its variables). Thus, these problems often require the determination of implicit parameters.

The solution to the dilemma is provided by numerical methods for roots of equations. To solve the problem using numerical methods, it is conventional to reexpress Eq. (PT2.3). This is done by subtracting the dependent variable v from both sides of the equation to give

$$f(c) = \frac{gm}{c}\left(1 - e^{-(c/m)t}\right) - v \tag{PT2.4}$$

The value of c that makes $f(c) = 0$ is, therefore, the root of the equation. This value also represents the drag coefficient that solves the design problem.

Part Two of this book deals with a variety of numerical and graphical methods for determining roots of relationships such as Eq. (PT2.4). These techniques can be applied to engineering design problems that are based on the fundamental principles outlined in Table PT2.1 as well as to many other problems confronted routinely in engineering practice.

PT2.2 MATHEMATICAL BACKGROUND

For most of the subject areas in this book, there is usually some prerequisite mathematical background needed to successfully master the topic. For example, the concepts of error estimation and the Taylor series expansion discussed in Chaps. 3 and 4 have direct relevance to our discussion of roots of equations. Additionally, prior to this point we have mentioned the terms "algebraic" and "transcendental" equations. It might be helpful to formally define these terms and discuss how they relate to the scope of this part of the book.

By definition, a function given by $y = f(x)$ is algebraic if it can be expressed in the form

$$f_n y^n + f_{n-1} y^{n-1} + \cdots + f_1 y + f_0 = 0 \tag{PT2.5}$$

where f_i = an ith-order polynomial in x. *Polynomials* are a simple class of algebraic functions that are represented generally by

$$f_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \tag{PT2.6}$$

where $n =$ the *order* of the polynomial and the a's = constants. Some specific examples are

$$f_2(x) = 1 - 2.37x + 7.5x^2 \tag{PT2.7}$$

and

$$f_6(x) = 5x^2 - x^3 + 7x^6 \tag{PT2.8}$$

A *transcendental* function is one that is nonalgebraic. These include trigonometric, exponential, logarithmic, and other, less familiar, functions. Examples are

$$f(x) = \ln x^2 - 1 \tag{PT2.9}$$

and

$$f(x) = e^{-0.2x} \sin(3x - 0.5) \tag{PT2.10}$$

Roots of equations may be either real or complex. Although there are cases where complex roots of nonpolynomials are of interest, such situations are less common than for polynomials. As a consequence, the standard methods for locating roots typically fall into two somewhat related but primarily distinct problem areas:

1. *The determination of the real roots of algebraic and transcendental equations.* These techniques are usually designed to determine the value of a single real root on the basis of foreknowledge of its approximate location.
2. *The determination of all real and complex roots of polynomials.* These methods are specifically designed for polynomials. They systematically determine all the roots of the polynomial rather than determining a single real root given an approximate location.

In this book we discuss both. Chapters 5 and 6 are devoted to the first category. Chapter 7 deals with polynomials.

PT2.3 ORIENTATION

Some orientation is helpful before proceeding to the numerical methods for determining roots of equations. The following is intended to give you an overview of the material in Part Two. In addition, some objectives have been included to help you focus your efforts when studying the material.

PT2.3.1 Scope and Preview

Figure PT2.1 is a schematic representation of the organization of Part Two. Examine this figure carefully, starting at the top and working clockwise.

After the present introduction, *Chap. 5* is devoted to *bracketing methods* for finding roots. These methods start with guesses that bracket, or contain, the root and then systematically reduce the width of the bracket. Two specific methods are covered: *bisection* and *false position.* Graphical methods are used to provide visual insight into the techniques. Error formulations are developed to help you determine how much computational effort is required to estimate the root to a prespecified level of precision.

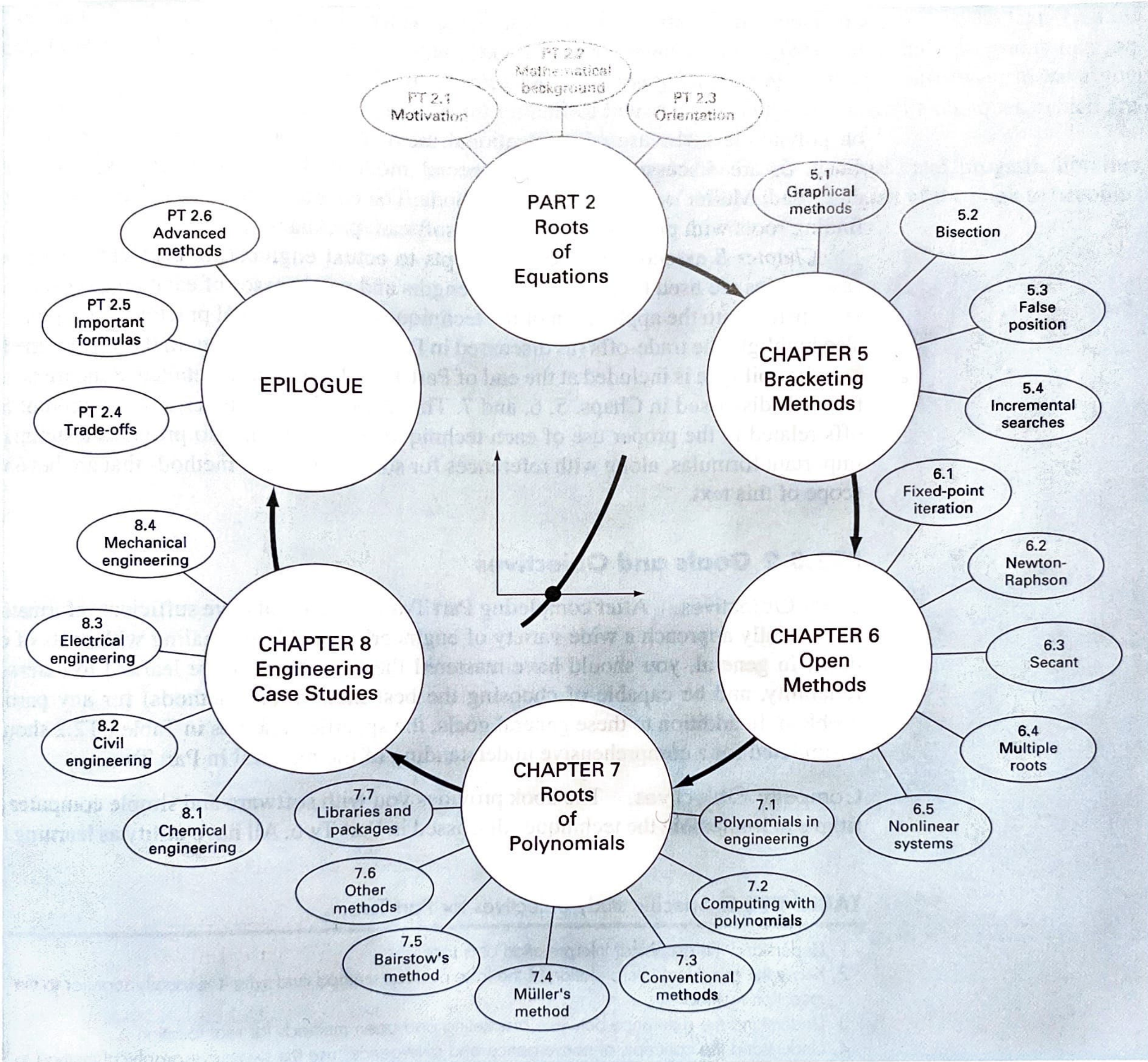

FIGURE PT2.1
Schematic of the organization of the material in Part Two: Roots of Equations.

Chapter 6 covers *open methods.* These methods also involve systematic trial-and-error iterations but do not require that the initial guesses bracket the root. We will discover that these methods are usually more computationally efficient than bracketing methods but that they do not always work. *One-point iteration, Newton-Raphson,* and *secant* methods are described. Graphical methods are used to provide geometric insight into cases where the open methods do not work. Formulas are developed that provide an idea of how fast open methods home in on the root. In addition, an approach to extend the Newton-Raphson method to *systems of nonlinear equations* is explained.

Chapter 7 is devoted to finding the *roots of polynomials.* After background sections on polynomials, the use of conventional methods (in particular the open methods from Chap. 6) are discussed. Then two special methods for locating polynomial roots are described: Müller's and Bairstow's methods. The chapter ends with information related to finding roots with program libraries and software packages.

Chapter 8 extends the above concepts to actual engineering problems. Engineering case studies are used to illustrate the strengths and weaknesses of each method and to provide insight into the application of the techniques in professional practice. The applications also highlight the trade-offs (as discussed in Part One) associated with the various methods.

An epilogue is included at the end of Part Two. It contains a detailed comparison of the methods discussed in Chaps. 5, 6, and 7. This comparison includes a description of trade-offs related to the proper use of each technique. This section also provides a summary of important formulas, along with references for some numerical methods that are beyond the scope of this text.

PT2.3.2 Goals and Objectives

Study Objectives. After completing Part Two, you should have sufficient information to successfully approach a wide variety of engineering problems dealing with roots of equations. In general, you should have mastered the techniques, have learned to assess their reliability, and be capable of choosing the best method (or methods) for any particular problem. In addition to these general goals, the specific concepts in Table PT2.2 should be assimilated for a comprehensive understanding of the material in Part Two.

Computer Objectives. The book provides you with software and simple computer algorithms to implement the techniques discussed in Part Two. All have utility as learning tools.

TABLE PT2.2 Specific study objectives for Part Two.

1. Understand the graphical interpretation of a root
2. Know the graphical interpretation of the false-position method and why it is usually superior to the bisection method
3. Understand the difference between bracketing and open methods for root location
4. Understand the concepts of convergence and divergence; use the two-curve graphical method to provide a visual manifestation of the concepts
5. Know why bracketing methods always converge, whereas open methods may sometimes diverge
6. Realize that convergence of open methods is more likely if the initial guess is close to the true root
7. Understand the concepts of linear and quadratic convergence and their implications for the efficiencies of the fixed-point-iteration and Newton-Raphson methods
8. Know the fundamental difference between the false-position and secant methods and how it relates to convergence
9. Understand the problems posed by multiple roots and the modifications available to mitigate them
10. Know how to extend the single-equation Newton-Raphson approach to solve systems of nonlinear equations

Pseudocodes for several methods are also supplied directly in the text. This information will allow you to expand your software library to include programs that are more efficient than the bisection method. For example, you may also want to have your own software for the false-position, Newton-Raphson, and secant techniques, which are often more efficient than the bisection method.

Finally, packages such as Excel, MATLAB software, and program libraries have powerful capabilities for locating roots. You can use this part of the book to become familiar with these capabilities.