Project Module

by
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Harvesting of Renewable Natural Resources

There are many renewable natural resources that humans desire to use. Examples are salmon and halibut from the Pacific Ocean, lake trout from the Great Lakes, trees from our forests, and the deer and waterfowl that are hunted annually. Nonrenewable resources, such as coal, oil, gas, or minerals, are not considered here. It is desirable that a policy be developed that will allow a maximal harvest of a renewable natural resource and yet not deplete that resource below a sustainable level. The simple mathematical model developed here provides some insights into the planning process. We follow the development given by Clark.*

Without human intervention, we assume that the population would behave logistically—that is, follow the solution given by equation (3) and Figure 3.13 in Section 3.2 of this text. Recall that if $P(t)$ represents the population (expressed in terms of either biomass or number of individuals) at time t (measured in years), then the logistic equation is

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right) = F(P), \tag{1}$$

where $r > 0$ is the **intrinsic growth rate** and K is the **environmental carrying capacity** (also known as the **saturation level** or the **limiting population,** since the population approaches this value over time). The values of these constants are determined experimentally.

In this module we will further assume that humans will be harvesting individuals from an animal population.

Constant Harvesting Rate As a first example, we assume that the **harvest rate** is a constant h. The modification of the differential equation (1) is then

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right) - h = F(P) - h = G(P). \tag{2}$$

Note that the function $G(P) = (-r/K)P^2 + rP - h$ is a quadratic polynomial in P whose graph is concave down. In the normal situation where

*Colin W. Clark, *Mathematical Bioeconomics: The Optimal Management of Renewable Resources,* 2nd ed. (New York: John Wiley & Sons, 1990).

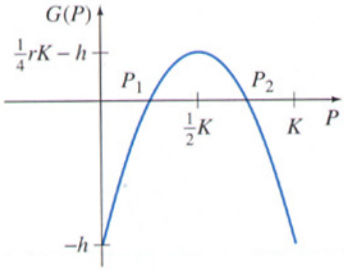

Figure 1

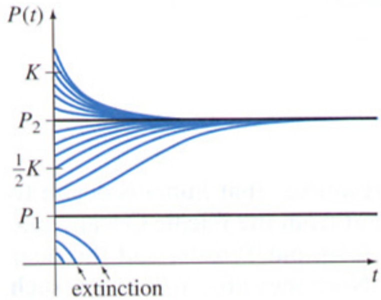

Figure 2

the harvest rate is not too high (that is, $h < \max F(P) = F(\frac{1}{2}K) = \frac{1}{4}rK$), the function G has two real zeros on the interval $[0, K]$. The values of the two zeros P_1 and P_2, shown in Figure 1, are found from the quadratic formula:

$$P_{1,2} = \frac{K \pm \sqrt{K^2 - 4Kh/r}}{2}.$$

We observe that $P(t) = P_1$ and $P(t) = P_2$ are constant solutions, called equilibrium solutions, of (2). Now from Figure 1 we see that the derivative dP/dt is positive on the interval $P_1 < P < P_2$, and so P will increase on that interval, while dP/dt is negative for all other values of P. In addition, by differentiating (2) we get

$$\frac{d^2P}{dt^2} = r\left(1 - \frac{2}{K}P\right)\frac{dP}{dt} = r\left(1 - \frac{2}{K}P\right)G(P),$$

which implies that the graph of $P(t)$ is concave downward for $P < P_1$ and for $\frac{1}{2}K < P < P_2$ and concave upward otherwise. Without solving the differential equation, therefore, we can still deduce the qualitative behavior of the solutions. See Figure 2. Note that if the initial population P_0 is less than P_1, the population $P(t)$ decreases to zero (extinction) in finite time, while otherwise the population $P(t)$ approaches P_2, a value less than K (the limiting population without harvesting). Mathematicians refer to the number P_2 as an **asymptotically stable equilibrium** or an **attractor,** since other solutions that start close to P_2 approach the horizontal line $P = P_2$; the number P_1 is called an **unstable equilibrium** or a **repeller.** Thus we can conclude that the harvest cannot be too large without depleting the resource.

We now ask the next obvious question: How large can the harvest be and still allow a sustainable (that is, long-term) harvest? We observed earlier that there are two real solutions to $G(P) = 0$ if $h < \frac{1}{4}rK$. On the other hand, if $h > \frac{1}{4}rK$, or $\frac{1}{4}rK - h < 0$, it can be seen from Figure 1 that $dP/dt = G(P) < 0$ and so $P(t)$ will decrease to zero. Finally, for $h = \frac{1}{4}rK$ the equation $G(P) = 0$ has the single root $P_1 = \frac{1}{2}K$. This value of P is also a constant solution of the differential equation. The value $h = \frac{1}{4}rK$ is called the **maximum sustainable yield** (MSY). It allows for a constant population of $P_1 = \frac{1}{2}K$ and a constant harvest equal to the MSY. The MSY, in other words, is equal to the population added annually due to reproduction minus death.

A word of caution is in order for anyone using this model to calculate the MSY in actual management practice. The values of r and K may be known only to within an accuracy of 10%. The value $h = \frac{1}{4}rK$ calculated for the MSY might, in fact, be too large for the given population, resulting in a decline to extinction.

So far, we have deduced much qualitative information about solutions of (2) without actually solving the differential equation. A solution of (2) subject to the initial condition $P(0) = P_0$ is easy to find by separation of variables. Since P_1 and P_2 are zeros, $P - P_1$ and $P - P_2$ must be factors of $G(P)$. Writing

$$\frac{dP}{dt} = -\frac{r}{K}(P - P_1)(P - P_2) \quad \text{as} \quad \frac{dP}{(P - P_1)(P - P_2)} = -\frac{r}{K}\,dt,$$

using partial fractions, and then integrating yields

$$\frac{1}{P_2 - P_1} \ln\left|\frac{P - P_2}{P - P_1}\right| = -\frac{r}{K}t + c. \tag{3}$$

By applying $P(0) = P_0$ to (3) and solving for $P(t)$, we arrive at

$$P(t) = \frac{P_2(P_0 - P_1) - P_1(P_0 - P_2)e^{-\alpha t}}{P_0 - P_1 - (P_0 - P_2)e^{-\alpha t}}, \tag{4}$$

where $\alpha = r(P_2 - P_1)/K = r\sqrt{1 - 4h/Kr}$. It should be clear from the explicit solution (4) that $P(t)$ approaches P_2 as time t increases.

Harvesting Proportional to Population In our next model, we assume that the harvest is proportional to the size of the population. The modification of (1) is then

$$\frac{dP}{dt} = F(P) - EP = P\left(r - \frac{r}{K}P\right) - EP = G(P), \tag{5}$$

where $E > 0$ is a constant referred to as the **effort,** since it is a measure of the effort that goes into harvesting the resource. As before, we consider the equilibrium solutions obtained from the equation $G(P) = P(r - E - rP/K) = 0$. We find one positive solution $P_1 = K(1 - E/r)$, as long as the effort E does not exceed the growth rate r. Since (5) can be written $dP/dt = -(r/K)\,P(P - P_1)$, we see that if $0 < P < P_1$ then $dP/dt > 0$, while if $P > P_1$ then $dP/dt < 0$. This indicates that a solution will always approach the equilibrium solution $P(t) = P_1$, making P_1 an asymptotically stable equilibrium, or an attractor. The equilibrium harvest, or **sustainable yield,** in this case is $EP_1 = KE(1 - E/r)$. Note that the left-hand side of this last expression is quadratically dependent upon E (the right-hand side) and has a maximum when $E = \frac{1}{2}r$ and $P_1 = \frac{1}{2}K$. For these latter values the number EP_1 is the maximum sustainable yield, or MSY.

To conclude this model, we note that we can solve (5) analytically. In fact, equation (5) in the form

$$\frac{dP}{dt} = P\left(r - E - \frac{r}{K}P\right) \tag{6}$$

is recognized as the logistic equation (4) of Section 3.2 with $a = r - E$ and $b = r/K$. Thus from (5) of Section 3.2 we obtain the solution

$$P(t) = \frac{(r - E)P_0}{rP_0/K + (r - E - rP_0/K)e^{-(r-E)t}}. \tag{7}$$

It is seen from (7) that the limiting population as $t \to \infty$ is $K(1 - E/r)$.

Antarctic fin whale

EXAMPLE 1 **Antarctic Fin Whale**

As an example, Clark estimates the values $r = 0.08$ and $K = 400{,}000$ for the Antarctic fin whale, with 1976 corresponding to $t = 0$ and $P_0 = P(0) = 70{,}000$.

(a) In the *constant harvesting model,* we find that the MSY is given by $h = \frac{1}{4}rK = 8000$, with a fixed population of $P_1 = \frac{1}{2}K = 200{,}000$. However, since the initial population is $P_0 = 70{,}000 < P_1$, the population will decline to zero, since reproduction the first year (population times growth rate = $70{,}000 \times 0.08 = 5600$) is less than the harvest. In order to guarantee that $P_0 > P_1$, we need to choose h less than 4620 (see part (b) of Problem 3 in the Related Exercises). In this case, the population $P(t)$ would slowly approach $P_2 = 330{,}000$ over time. One management technique for increasing the number of whales faster is to severely limit the harvest initially, with the possibility of a larger annual harvest later.

(b) For the *constant effort model,* let us assume that the effort is one-half the growth rate in part (a)—that is, $E = \frac{1}{2}r = 0.04$. Then $P_1 = \frac{1}{2}K = 200{,}000$, and the first-year harvest is $EP(0) = 0.04 \times 70{,}000 = 2800$. The MSY is $EP_1 = 0.04 \times 200{,}000 = 8000$. ■

RELATED EXERCISES

1. (a) Consider the constant harvesting model

$$\frac{dP}{dt} = P(5 - P) - 4, \quad P(0) = P_0.$$

Since the differential equation is autonomous, use the phase portrait concept discussed in Section 2.1 to sketch representative solution curves corresponding to the cases $P_0 > 4$, $1 < P_0 < 4$, and $0 < P_0 < 1$. Determine the long-term behavior of the population in each case.

(b) Solve the IVP in part (a). Use a graphing utility to verify your results in part (a).

(c) Use the information in parts (a) and (b) to determine whether the population becomes extinct in finite time. If so, find that time.

2. As in Problem 1, investigate the given constant harvesting models both qualitatively and analytically. Determine whether the population becomes extinct in finite time. If so, find that time.

(a) $\dfrac{dP}{dt} = P(5 - P) - \frac{25}{4}$ **(b)** $\dfrac{dP}{dt} = P(5 - P) - 7$

3. This problem concerns the Antarctic fin whale discussed in Example 1.

(a) If no harvesting were allowed, how long would it take for the population to pass $\frac{1}{2}K = 200{,}000$? Plot the solution for $0 < t < 100$, using $P_0 = 70{,}000$.

(b) In the constant harvesting case, calculate the value of $h = h_0$ that allows $P_0 = P_1$. Plot the solution for $0 < t < 100$, using $P_0 = 70{,}000$ and $h = \frac{1}{2}h_0$.

(c) In the constant effort case, what effort E_0 yields the MSY? What is the yield? What is the limiting population? Plot the solution for $0 < t < 100$, using $P_0 = 70{,}000$ and $E = \frac{1}{2}E_0$.

4. (a) The data in Table 3.2 (taken from the 1976 edition of the book by Clark) give a portion of the statistical history of the Peruvian anchovy (*Engraulis ringens*) fishery. We can consider the product of the number of boats B times the number of fishing days D to be a measure of the fishing effort. We let $E = cBD$, where c is a constant of proportionality. If the catch C (harvest) is proportional to both the effort and the size of the population, then

TABLE 3.2

Year	Number of boats	Number of fishing days	Catch (millions of tons)
1959	414	294	1.91
1960	667	279	2.93
1961	756	298	4.58
1962	1069	294	6.27
1963	1655	269	6.42
1964	1744	297	8.86
1965	1623	265	7.23
1966	1650	190	8.53
1967	1569	170	9.82
1968	1490	167	10.26
1969	1455	162	8.96
1970	1499	180	12.27
1971	1473	89	10.28
1972	1399	89	4.45
1973	1256	27	1.78

$C = EP(t) = cBDP(t)$, where $P(t)$ is the population (measured in millions of tons) at time t, and $P(t) = C/(cBD)$. Use a CAS to plot the data (P vs. years) for 1959–1973 with $c = 2 \times 10^{-7}$.

(b) Taking into account the natural scatter in the data and ignoring anomalous points, we could consider the graph produced in (a) as representing a logistic function of the form

$$P(t) = \frac{P_0K}{P_0 + (K - P_0)e^{-rt}},$$

where P_0, K, and r are the initial population, the limiting population, and the growth rate, respectively. Find values of P_0, K, and r for which the graph of $P(t)$ approximates your plot of the data points in part (a) reasonably well. Superimpose the graph of $P(t)$ on the graph of the data.

(c) For the values of P_0, K, and r found in part (b) and for a harvest $h = \frac{1}{8}rK$, plot the solution (4) of the differential equation in the constant harvest case on the interval $0 \leq t \leq 100$.

(d) For the values of P_0, K, and r found in part (b) and for an effort $E = \frac{1}{2}r$, plot the solution (7) of the differential equation in the constant effort case on the interval $0 \leq t \leq 100$.