Recall that an **operator** is just a (continuous) linear transformation between two vectors spaces. A **Banach space** is a complete normed vector space X; being complete means every Cauchy sequence in X converges to some vector in X. There are several notions of expansivity that have been recently studied:

Definition 0.1. Let X be a Banach space, let $S_X = \{x \in X : \|x\| = 1\}$ (meaning S_X is the unit sphere of X), and let $T : X \to X$ be an operator.

- We say T is **positively expansive** (PE) if for all $x \in S_X$, $\sup_{n\in\mathbb{N}} \|T^n x\| = +\infty$.
- We say T is **average positively expansive** (APE) if for all non-zero $x \in X$, $\sup_{n\in\mathbb{N}} \left(\frac{1}{n}\sum_{j=1}^{n} \|T^j x\|\right) = +\infty$.
- We say T is **uniformly positively expansive** (UPE) if $\lim_{n\to\infty} \|T^n x\| = +\infty$ uniformly on S_X, meaning for all $M > 0$, there exists $N \in \mathbb{N}$ such that for all $x \in S_X$ and for all $n \geq N$, $\|T^n x\| \geq M$.

There are reasons why operator theorists care about these definitions, but we'll ignore those reasons for now; what we are interested in is characterizing which forward and backward shifts are PE, APE, and UPE. It may look odd that APE doesn't restrict the x's to the unit sphere S_X, but we leave it as an exercise to prove the following:

$$T \text{ is APE} \iff \forall x \in S_X,\ \sup_n \left(\frac{1}{n}\sum_{j=1}^{n} \|T^j x\|\right) = +\infty.$$

The exercise follows from the linearity of T, and at least one of these useful basic facts about suprema:

- If $c > 0$ and $A \subseteq \mathbb{R}$, then $\sup(cA) = c\sup(A)$.
- $\sup\{a_n : n \in \mathbb{N}\} = +\infty \iff$ for each $k \in \mathbb{N}$, $\sup\{a_n : n \geq k\} = +\infty$.

There are some equivalent characterizations of PE and UPE that were proved in other papers:

- T is PE $\iff \forall x \in S_X, \exists n \in \mathbb{N}$ s.t. $\|T^n x\| \geq 2$.
- T is UPE $\iff \exists n \in \mathbb{N}$ s.t. $\forall x \in S_X$, $\|T^n x\| \geq 2$.

Those characterizations highlight the difference between PE and UPE. We are not sure if there is an analogous characterization of APE, something like:

- T is APE $\iff \forall x \in S_X, \exists n \in \mathbb{N}$ s.t. $\sup_{n\in\mathbb{N}} \left(\frac{1}{n}\sum_{j=1}^{n} \|T^j x\|\right) \geq 2$.

Such a characterization could be worth exploring...

The last thing we'll mention is that it is straightforward to prove

$$\text{UPE} \implies \text{APE} \implies \text{PE},$$

and the reverse implications do not hold in general.

1 Positive Expansivity

From now on and for the rest of this document, let X be $c_0(\mathbb{N})$ or one of the spaces $\ell^p(\mathbb{N})$, $1 \leq p < \infty$. A backward shift on X defined by $B_w(x_1, x_2, x_3, \ldots) = (w_2x_2, w_3x_3, w_4x_4, \ldots)$ is never PE. In fact, any operator T with a non-trivial kernel cannot be PE. If $Tx = 0$ for some non-zero vector x, then $\frac{x}{\|x\|}$ is a unit vector, i.e. $\frac{x}{\|x\|} \in S_X$, and $\|T^n(x/\|x\|)\| = 0$ for all n.

Let's see how forward shifts on X, defined by $F_w(x_1, x_2, x_3, \ldots) = (0, w_1x_1, w_2x_2, \ldots)$, can be PE. For a given basis vector e_v, note that $F^n e_v = w_v w_{v+1} \cdots w_{v+n-1} e_{v+n}$.

Proposition 1.1. F_w is PE on X if and only if $\sup_n |w_1w_2\cdots w_n| = +\infty$.

Proof. ($\implies$) Suppose F_w is PE. Then $e_1 \in S_X$, and since $F^n e_1 = w_1w_2\cdots w_n e_{n+1}$, by positive expansivity we have $\sup_n \|F^n e_1\| = \sup_n |w_1w_2\cdots w_n| = +\infty$.

($\impliedby$) Suppose $\sup_n |w_1w_2\cdots w_n| = +\infty$, from which we can use basic facts about suprema to also conclude that for any $v \in \mathbb{N}$, $\sup_{n\geq v} |w_v \cdots w_{v+n-1}| = +\infty$. Let $x \in S_X$. Since $x \neq 0$, there exists some non-zero term x_v in the sequence x. Then

$$\|F^n x\| \geq \|F^n(x_v e_v)\| = |x_v||w_v w_{v+1} \cdots w_{v+n-1}| = \frac{|x_v|}{|w_1 \cdots w_{v-1}|}|w_1 \cdots w_{v+n-1}|.$$

Whence $\sup_n \|F^n x\| = +\infty$. $\square$

2 Average Positive Expansivity

Proposition 2.1. *F_w is APE on X if and only if $\sup_n \left(\frac{1}{n}\sum_{j=1}^n |w_1 \cdots w_j|\right) = +\infty$.*

Proof. ($\Longrightarrow$) Suppose F is APE. Then $e_1 \in S_X$, and $F^j(e_1) = w_1 \cdots w_j$, so we must have

$$+\infty = \sup_n \left(\frac{1}{n}\sum_{j=1}^n \|F^j e_1\|\right) = \sup_n \left(\frac{1}{n}\sum_{j=1}^n |w_1 \cdots w_j|\right).$$

($\Longleftarrow$) Suppose $\sup_n \left(\frac{1}{n}\sum_{j=1}^n |w_1 \cdots w_j|\right) = +\infty$. Then by basic suprema facts, for any $k \in \mathbb{N}$, we know $\sup_{n\geq k} \left(\frac{1}{n+k}\sum_{j=1}^{n+k} |w_1 \cdots w_j|\right) = +\infty$. Let $x \in S_X$. Since $x \neq 0$, there exists some non-zero term x_v in the sequence x. Since

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^n \|F^j x\| &\geq \frac{1}{n}\sum_{j=1}^n \|F^j(x_v e_v)\| \\
&= \frac{|x_v|}{n}\sum_{j=1}^n |w_v \cdots w_{v+j-1}| \\
&= \frac{1}{n}\frac{|x_v|}{|w_1 \cdots w_{v-1}|}\left(\frac{n+v-1}{n+v-1}\sum_{j=1}^{n+v-1} |w_1 \cdots w_j| - \sum_{j=1}^{v-1} |w_1 \cdots w_j|\right),
\end{aligned}$$

and since $\lim_{n\to\infty} \frac{n+v-1}{n} = 1$ and $\lim_{n\to+\infty} \frac{1}{n}\sum_{j=1}^{v-1} |w_1 \cdots w_j| = 0$, we conclude that $\sup_n \left(\frac{1}{n}\sum_{j=1}^n \|F^j x\|\right) = +\infty$, which had to be shown. $\square$

3 Uniform Positive Expansivity

Proposition 3.1. *F_w is UPE on X if and only if $\lim_{n\to\infty} (\inf_{v\in\mathbb{N}} |w_v \cdots w_{v+n-1}|) = +\infty$.*

Proof. ($\Longrightarrow$) Suppose F_w is UPE. Let $M > 0$ be given. By UPE, there exists some $N \in \mathbb{N}$ such that for all $x \in S_X$ and all $n \geq N$, $\|F^n x\| > M$. In particular, for each unit vector e_v, we have $\|F^n e_v\| = |w_v \cdots w_{v+n-1}| > M$ whenever $n \geq N$. Hence for $n \geq N$, we can conclude

$$\inf_{v \in \mathbb{N}} |w_v \cdots w_{v+n-1}| \geq M,$$

which had to be shown.

($\Longleftarrow$) Suppose $\lim_{n\to\infty} (\inf_{v\in\mathbb{N}} |w_v \cdots w_{v+n-1}|) = +\infty$. Then there exists some $N \in \mathbb{N}$ such that for all $n > N$, $\inf_{v\in\mathbb{N}} |w_v \cdots w_{v+n-1}| \geq 2$. Thus for any $x \in S_X$, we have

$$\|F^n x\| = \sum_{v=1}^{\infty} |w_v \cdots w_{v+n-1}| \cdot |x_v| \geq 2 \sum_{v=1}^{\infty} |x_v| = 2\|x\|,$$

which shows F_w is UPE. $\square$

4 Expansivity on Trees

We already have a characterization of which forward shifts are APE on directed trees. The next steps are to answer the following questions.

- Can we characterize the UPE forward shifts on trees? Said differently, we want to fill in the blank:

  F_λ is UPE on $(V, E) \iff$ ______________________________

- Can we characterize the PE forward shifts on trees?

- Can we characterize the UPE backward shifts? APE? PE? You can look up the definition of a backward shift in the directed trees paper. It might help to know that for $f \in \ell^1(V)$ (or any ℓ^p space or c_0), the nth iterate of the backward shift acting on f satisfies

$$(B_\lambda^n f)(v) = \sum_{u \in \mathrm{Chi}^n(v)} \lambda(v \to u) f(u).$$