1 Erdös-Renyi Model

Definition: $G(n, p)$ is a random graph with n vertices where each possible edge has probability p of existing.

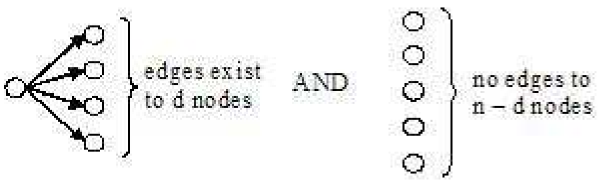
The number of edges in a $G(n, p)$ graph is a random variable with expected value $\binom{n}{2}p$.

A closely related model is of the $G_E(n, e)$ form. Of all possible graphs with n vertices and exactly e edges, one is randomly selected. The two models have very similar properties, but often one will be easier to use in a particular proof. We will be mostly focusing on the Erdoös-Renyi model.

1.1 Properties of vertex degree

1.1.1 Degree Distribution

One question we might ask is what is the probability that a vertex will have a certain degree d. In other words, for some vertex v, what is $Pr[deg(v) = d]$?



Since there are $\binom{n}{d}$ ways to choose d vertices from among n total, and p^d probability that they will have edges, $\binom{n}{d}p^d$ is the probability that a node has edges to d nodes. However, there must be no edges to the rest of the $n - d$ nodes, which occurs with probability $(1 - p)^{n-d}$.

$$Prob[d] = \binom{n}{d} p^d (1 - p)^{n-d}$$

1.1.2 Expected Mean Degree

Next we might ask what is the expected mean degree of a vertex in this graph model? To derive this, recall that for a random variable X, the expectation is $E[X] = \sum_{x \in dom(X)} xPr[X = x]$. In our case X is an edge and for each edge we multiply the probability that it will occur by the number of the edge, and take the sum for all edges. Thus the expected mean degree is:

$$\sum_{d=0}^{n} d \binom{n}{d} p^d (1-p)^{n-d} = np$$

How do we get the answer np? We can use the binomial formula to get the expected mean, recall that:

$$(p+q)^n = \sum_{d=0}^{n} \binom{n}{d} p^d q^{n-d}$$

By differentiating both sides with respect to p, we get:

$$\begin{aligned} n(p+q)^{n-1} &= \sum_{d=0}^{n} d \binom{n}{d} p^{d-1} q^{n-d} \\ &= \frac{1}{p} \sum_{d=0}^{n} d \binom{n}{d} p^d q^{n-d} \end{aligned}$$

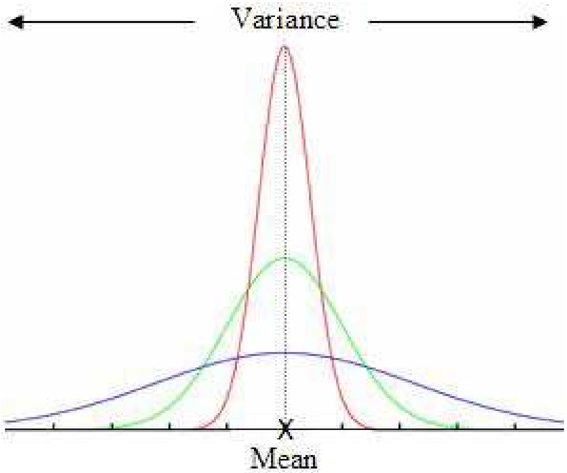
Substituting $q = 1 - p$

$$\begin{aligned} n(p+(1-p))^{n-1} &= \frac{1}{p} \sum_{d=0}^{n} d \binom{n}{d} p^d (1-p)^{n-d} \\ np &= \sum_{d=0}^{n} d \binom{n}{d} p^d (1-p)^{n-d} \end{aligned}$$

The right hand side is recognized as an expression for the mean, np

1.1.3 Variance of Degree Distribution

Beyond the probability of a vertex having a certain degree and the expected value of vertex degree, we are also interested in the variance of the degree distribution.



Since the variance corresponds to the second moment of a probability distribution, we take the 2nd derivative to find the variance. (Ref: check out the sections on `variance` and `moment generating functions` in a probability text if this is not familiar material)

$$Var(d(v)) = n(n-1)(p+q)^{n-2} = \frac{1}{p^2}\sum_{d=0}^{n} d^2 \binom{n}{d} p^d q^{n-d}$$

Substituting $q = 1 - p$

$$n(n-1) = \frac{1}{p^2}\sum_{d=0}^{n} d^2 \binom{n}{d} p^d (1-p)^{n-d}$$

$$n(n-1)p^2 = \sum_{d=0}^{n} d^2 \binom{n}{d} p^d (1-p)^{n-d}$$