Equilibrium Existence, Uniqueness and Stability

230333 Microeconomics 3 (CentER) – Part II
Tilburg University

Introduction

In this section we will discuss three topics:

- **Existence:** Under which assumptions we can be guaranteed that an equilibrium will exist?
  - We will do two proofs, where one deals with the complication of strongly monotonic preferences with zero prices.
- **Uniqueness:** Under which conditions can we be guaranteed that an equilibrium will be unique?
- **Stability:** Under which conditions is an equilibrium stable?
  - If the economy is pushed away from equilibrium (e.g. from a shock), will it adjust back?

Equilibrium in Pure Exchange Economies

- A pure exchange economy is a special case of the general case with $J = 1$ and $Y_1 = -\mathbb{R}^L_+$ (*free disposal*).
- If $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$ and each consumer i has continuous, strictly convex and locally nonsatiated preferences, the equilibrium definition can be restated as:

Definition

$\left(\boldsymbol{x}^\star, \boldsymbol{y}_1^\star\right)$ and $\boldsymbol{p} \in \mathbb{R}^L$ constitute a Walrasian equilibrium in a pure exchange economy iff:

(i) $\boldsymbol{y}_1^\star \leq \mathbf{0}$, $\boldsymbol{p} \cdot \boldsymbol{y}_1^\star = 0$ and $\boldsymbol{p} \geq \mathbf{0}$ (profit maximization).

(ii) $\boldsymbol{x}_i^\star = \boldsymbol{x}_i\left(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_i\right)$ for all i (utility maximization).

(iii) $\sum_{i=1}^I \boldsymbol{x}_i^\star = \sum_{i=1}^I \boldsymbol{\omega}_i + \boldsymbol{y}_1^\star$ (market clearing).

Excess Demand

- The *excess demand function of consumer i* is:

$$\boldsymbol{z}_i(\boldsymbol{p}) = \boldsymbol{x}_i(\boldsymbol{p}, \boldsymbol{p} \cdot \boldsymbol{\omega}_i) - \boldsymbol{\omega}_i$$

- The *aggregate excess demand function* of the economy is:

$$\boldsymbol{z}(\boldsymbol{p}) = \sum_{i=1}^{I} \boldsymbol{z}_i(\boldsymbol{p})$$

- In a pure exchange economy in which preferences are continuous, strictly convex and locally nonsatiated, $\boldsymbol{p} \geq \boldsymbol{0}$ is a Walrasian equilibrium price vector iff $\boldsymbol{z}(\boldsymbol{p}) \leq \boldsymbol{0}$.
  - $\boldsymbol{y}_1^\star = \boldsymbol{z}(\boldsymbol{p})$ is profit-maximizing, because $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$.
    - $\boldsymbol{p} \cdot \boldsymbol{z}_i(\boldsymbol{p}) = 0 \ \forall i$ by Walras' law (LNS), so $\sum_{i=1}^{I} \boldsymbol{p} \cdot \boldsymbol{z}_i(\boldsymbol{p}) = 0$.

Proof of Existence

Proposition

Suppose that $\boldsymbol{z}(\boldsymbol{p})$ is a function defined for all nonzero, nonnegative price vectors $\boldsymbol{p} \in \mathbb{R}^L_+$ and satisfies continuity, homogeneity of degree zero and Walras' law. Then there is a price vector $\boldsymbol{p}^\star$ such that $\boldsymbol{z}\left(\boldsymbol{p}^\star\right) \leq \mathbf{0}$.

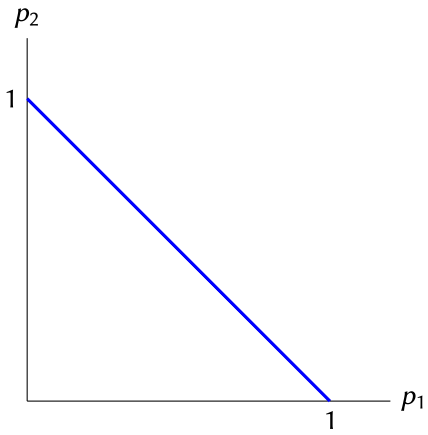
- Because of homogeneity of degree zero, we can normalize prices to the unit simplex:

$$\Delta = \left\{ \boldsymbol{p} \in \mathbb{R}^L_+ : \sum_{\ell=1}^{L} p_\ell = 1 \right\}$$

- Δ is compact (closed and bounded) and convex.

Unit Simplex with $L = 2$

With $L = 2$, the unit simplex is given by the line $p_2 = 1 - p_1$, for $p_1 \in [0, 1]$.



Proof of Existence

- Define the function $f : \Delta \to \Delta$:

$$\{f_\ell(\boldsymbol{p})\}_{\ell=1}^L = \left\{ \frac{p_\ell + \max\{z_\ell(\boldsymbol{p}), 0\}}{1 + \sum_{k=1}^L \max\{z_k(\boldsymbol{p}), 0\}} \right\}_{\ell=1}^L$$

- Because $z_\ell(\boldsymbol{p})$ is continuous $\forall \ell$ and the denominator is bounded away from zero, f is continuous. See notes on Canvas for a formal ε-δ proof of continuity.
- f is a continuous function mapping a compact convex set to itself: Brouwer can be applied.
- By Brouwer's fixed-point theorem, $\exists \boldsymbol{p}^\star \in \Delta$ s.t. $\boldsymbol{p}^\star = f(\boldsymbol{p}^\star)$.

$$\underbrace{0 = \boldsymbol{p}^\star \cdot \boldsymbol{z}(\boldsymbol{p}^\star)}_{\text{Walras' law}} = f(\boldsymbol{p}^\star) \cdot \boldsymbol{z}(\boldsymbol{p}^\star) = \frac{\sum_{\ell=1}^L \left(p_\ell + \max\left\{z_\ell(\boldsymbol{p}^\star), 0\right\}\right) z_\ell(\boldsymbol{p}^\star)}{1 + \sum_{k=1}^L \max\{z_k(\boldsymbol{p}^\star), 0\}}$$

- Therefore $\sum_{\ell=1}^L \max\left\{z_\ell(\boldsymbol{p}^\star), 0\right\} z_\ell(\boldsymbol{p}^\star) = 0$, so $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \boldsymbol{0}$.

Strongly Monotone Preferences

- The previous proof works when demand is continuous over all nonzero, nonnegative prices.
- However, if preferences are strongly monotone, demand is infinite at zero prices
  - This occurs at the boundary of the simplex.
- We will now adapt the proof to handle this case.

Properties of the Aggregate Excess Demand Function

Suppose that, for every consumer i, $X_i = \mathbb{R}^L_+$ and $\succeq_i$ is continuous, strictly convex, and strongly monotone. Suppose also that $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$. Then the aggregate excess demand function, defined for all price vectors $\boldsymbol{p} \gg \mathbf{0}$ satisfies:

(i) $\boldsymbol{z}(\cdot)$ is continuous

(ii) $\boldsymbol{z}(\cdot)$ is homogenous of degree zero.

(iii) $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$ for all $\boldsymbol{p}$ (Walras' law)

(iv) There is an $s > 0$ such that $z_\ell(\boldsymbol{p}) > -s$ for every commodity ℓ and all $\boldsymbol{p}$.

(v) If $\boldsymbol{p}^n$ is a sequence of price vectors converging to $\boldsymbol{p} \neq \mathbf{0}$ and $p_\ell = 0$ for some ℓ, then $z_\ell(\boldsymbol{p}^n) \to \infty$.
   - There is at least one consumer with positive wealth at the limit who demands an infinite amount of the free good.

Existence of Equilibria With Strongly Monotone Preferences

In a pure exchange economy in which consumer preferences are continuous, strictly convex, and strongly monotone, $\boldsymbol{p} \gg \mathbf{0}$ is a Walrasian equilibrium price vector if and only if:

$$\boldsymbol{z}(\boldsymbol{p}) = \mathbf{0}$$

Proposition

Suppose that $\boldsymbol{z}(\boldsymbol{p})$ is a function defined for all $\boldsymbol{p} \in \mathbb{R}^L_{++}$ satisfying conditions (i)-(v) on the previous slide. Then the system of equations $\boldsymbol{z}(\boldsymbol{p}) = \mathbf{0}$ has a solution. Hence, a Walrasian equilibrium exists in any pure exchange economy in which $\bar{\boldsymbol{\omega}} \gg \mathbf{0}$ and every consumer has continuous, strictly convex and strongly monotone preferences.

Unit Simplex

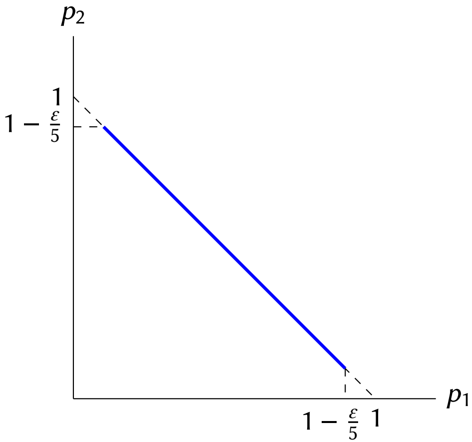
We define a variation on the unit simplex from the last proof.
For a fixed $\varepsilon \in (0, 1)$:

$$\Delta_\varepsilon = \left\{ \boldsymbol{p} : \sum_{\ell=1}^{L} p_\ell = 1 \text{ and } p_\ell \geq \frac{\varepsilon}{1+2L} \forall \ell \right\}$$

- Δ_ε is compact (closed and bounded).
- Δ_ε is convex.
- Δ_ε non-empty:
  - $p_\ell = \frac{1}{L}, \forall \ell$ is an element for any $\varepsilon \in (0, 1)$, because $\sum_{\ell=1}^{L} p_\ell = 1$ and $\frac{1+2L}{L} > \varepsilon$ for $\varepsilon \in (0, 1)$.
- Later we will let $\varepsilon \to 0$.

Δ_ε with $L = 2$

$$\Delta_\varepsilon = \left\{ \boldsymbol{p} : \sum_{\ell=1}^{L} p_\ell = 1 \text{ and } p_\ell \geq \frac{\varepsilon}{1+2L} \forall \ell \right\}$$



Fixed Point Function

Define for each $\boldsymbol{p} \in \Delta_\varepsilon$ a function $\boldsymbol{f}(\boldsymbol{p}) = \{f_\ell(\boldsymbol{p})\}_{\ell=1}^{L}$ where:

$$f_\ell(\boldsymbol{p}) = \frac{p_\ell + \varepsilon + \max\{0, \min\{z_\ell(\boldsymbol{p}), 1\}\}}{1 + L\varepsilon + \sum_{k=1}^{L} \max\{0, \min\{z_k(\boldsymbol{p}), 1\}\}}$$

- $\sum_{\ell=1}^{L} f_\ell(\boldsymbol{p}) = 1$ and $f_\ell(\boldsymbol{p}) \geq \frac{\varepsilon}{1+2L}$ $\forall \ell$
  - $\Rightarrow \boldsymbol{f}(\boldsymbol{p}) \in \Delta_\varepsilon$ for any $\boldsymbol{p} \in \Delta_\varepsilon$. The function maps Δ_ε onto itself.
- Each f_ℓ is also continuous, by the continuity of each z_ℓ and the denominator being bounded away from 0.
- $\boldsymbol{f}(\boldsymbol{p})$ is a continuous function mapping a compact, convex, non-empty set onto itself, so $\exists \boldsymbol{p}^\star$ s.t. $f(\boldsymbol{p}^\star) = \boldsymbol{p}^\star$.

Letting $\boldsymbol{\varepsilon} \longrightarrow \mathbf{0}$

- Now let $\varepsilon \longrightarrow 0$ and consider the associated sequence of fixed point price vectors $\boldsymbol{p}^n \longrightarrow \boldsymbol{p}$.
- The sequence $\boldsymbol{p}^n \in \mathbb{R}^L$ is bounded because $\boldsymbol{p}^n \in \Delta_\varepsilon \ \forall n$.
- Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence (Bolzano-Weierstrass theorem).
- Call the converged vector $\boldsymbol{p}^\star$.
- Because $\boldsymbol{p}^\star$ is in the simplex, $\boldsymbol{p}^\star \geq \mathbf{0}$ and $\boldsymbol{p} \neq \mathbf{0}$. We need to show that in fact $\boldsymbol{p}^\star \gg \mathbf{0}$.

Proving that $\boldsymbol{p}^\star \gg \mathbf{0}$

Because $f(\boldsymbol{p}^n) = \boldsymbol{p}^n$, every price vector in the sequence satisfies $(\forall \ell)$:

$$p_\ell^n \left[1 + L\varepsilon + \sum_{k=1}^{L} \max\left\{0, \min\left\{z_k\left(\boldsymbol{p}^n\right), 1\right\}\right\} \right] = p_\ell^n + \varepsilon + \max\left\{0, \min\left\{z_\ell\left(\boldsymbol{p}^n\right), 1\right\}\right\}$$

- Suppose $p_k^\star = 0$ for some good k. Then, as $p_k^n \to 0$:

$$\underbrace{p_k^n}_{\to 0} \underbrace{\left[L\varepsilon + \sum_{m=1}^{L} \max\left\{0, \min\left\{z_m\left(\boldsymbol{p}^n\right), 1\right\}\right\} \right]}_{\text{Positive, by property (v) and bounded due to the min}} =$$

$$\underbrace{\varepsilon}_{\to 0} + \underbrace{\max\left\{0, \min\left\{z_k\left(\boldsymbol{p}^n\right), 1\right\}\right\}}_{=1\text{, by property (v)}}$$

- LHS$\to 0$ but RHS$\to 1$. Therefore it must be that $\boldsymbol{p}^\star \gg \mathbf{0}$.

Last Step: Show that $\boldsymbol{f}(\boldsymbol{p}^\star) = \boldsymbol{p}^\star$ is an Equilibrium

- We now show that $\boldsymbol{f}(\boldsymbol{p}^\star) = \boldsymbol{p}^\star$ is an equilibrium ($\boldsymbol{z}(\boldsymbol{p}^\star) = \mathbf{0}$).
- The fixed point condition implies that (after $\varepsilon \to 0$):

$$p_\ell^\star \left[\sum_{k=1}^{L} \max\left\{0, \min\left\{z_k\left(\boldsymbol{p}^\star\right), 1\right\}\right\}\right] = \max\left\{0, \min\left\{z_\ell\left(\boldsymbol{p}^\star\right), 1\right\}\right\}$$

$$\underbrace{\sum_{\ell=1}^{L} z_\ell\left(\boldsymbol{p}^\star\right) p_\ell^\star}_{=0 \text{ by Walras' Law}} \underbrace{\left[\sum_{k=1}^{L} \max\left\{0, \min\left\{z_k\left(\boldsymbol{p}^\star\right), 1\right\}\right\}\right]}_{\text{Bounded due to the min}} = \sum_{\ell=1}^{L} z_\ell\left(\boldsymbol{p}^\star\right) \underbrace{\max\left\{0, \min\left\{z_\ell\left(\boldsymbol{p}^\star\right), 1\right\}\right\}}_{0 \text{ if } z_\ell(\boldsymbol{p}^\star)<0}$$

- The LHS is zero, so the RHS must be zero.
  - Can't have any $z_\ell(\boldsymbol{p}^\star) > 0$ because RHS must sum to zero and no term on the RHS can be negative, so we must have $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0}$.
  - Can't have any $z_\ell(\boldsymbol{p}^\star) < 0$ when $\boldsymbol{z}(\boldsymbol{p}^\star) \leq \mathbf{0}$ and $\boldsymbol{p}^\star \gg \mathbf{0}$ because of Walras' law: $\sum_{\ell=1}^{L} p_\ell z_\ell(\boldsymbol{p}^\star) = 0$.
  - Therefore the RHS is only zero if $\boldsymbol{z}(\boldsymbol{p}^\star) = \mathbf{0}$.

Arrow's Exceptional Case: Nonexistence of Equilibrium

- Consider the following example in the Edgeworth box:

$$u_1\left(x_{11}, x_{21}\right) = x_{11} + \sqrt{x_{21}}$$
$$u_2\left(x_{12}, x_{22}\right) = x_{22}$$

  with the initial endowment $\boldsymbol{\omega}_1 = (\bar{\omega}_1, 0)$ and $\boldsymbol{\omega}_2 = (0, \bar{\omega}_2)$.
- At $\boldsymbol{\omega}$, the slopes of both consumers' indifference curves are 0.
- The initial endowment is Pareto optimal, but there is no vector of prices that can sustain this allocation in equilibrium.
  - If $p_2 = 0$, both consumers demand an infinite amount of good 2.
  - If $p_1 = 0$, consumer 1 demands an infinite amount of good 1.
  - If $p_1 > 0$ and $p_2 > 0$, consumer 1 demands some of good 2 but consumer 2 is never willing to sell any.

Uniqueness of Walrasian Equilibria

- Certain conditions on preferences and/or the endowments can guarantee that there will be a unique equilibrium:
  1. Strict convexity and Pareto optimality of the initial endowment.
  2. Aggregate excess demand function satisfies WARP and all Y_j have CRS (only achieves convex set of equilibria).
  3. Aggregate excess demand function has the gross substitute property for all goods.
  4. If $\boldsymbol{Dz}(\boldsymbol{p})$ has full rank and is NSD.
- We will consider each of these cases in turn.
- Assume throughout that each consumer's preferences are continuous, strictly convex and strongly monotone and $\boldsymbol{\omega}_i \gg \mathbf{0}$.

Pareto Optimality of the Initial Endowment

Proposition

In a pure exchange economy, if $\boldsymbol{\omega}_i \gg \mathbf{0}$, $X_i = \mathbb{R}^L_+$, and preferences $\succeq_i$ satisfy continuity, strong monotonicity, and strict convexity for all i, then if $(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_I)$ is Pareto optimal, then $\boldsymbol{x}_i^\star = \boldsymbol{\omega}_i \ \forall i$ is the unique equilibrium allocation.

- $\boldsymbol{x}_i = \boldsymbol{\omega}_i \ \forall i$ is an equilibrium by the 2nd Welfare Theorem.
- Suppose $\boldsymbol{x}' \neq \boldsymbol{\omega}$ and $\boldsymbol{p}'$ is also an equilibrium.
- Because $\boldsymbol{x}'$ is an equilibrium, $\boldsymbol{x}_i' \succeq_i \boldsymbol{\omega}_i \ \forall i$.
- It also satisfies feasibility: $\sum_{i=1}^I \boldsymbol{x}_i' = \sum_{i=1}^I \boldsymbol{\omega}_i$.
- By strict convexity, $\boldsymbol{x}_i'' = \frac{1}{2}\boldsymbol{x}_i' + \frac{1}{2}\boldsymbol{\omega}_i$ satisfies $\boldsymbol{x}_i'' \succ_i \boldsymbol{\omega}_i \ \forall i$.
- Moreover, $\boldsymbol{x}_i''$ is feasible because:

$$\sum_{i=1}^I \boldsymbol{x}_i'' = \frac{1}{2}\sum_{i=1}^I \boldsymbol{x}_i' + \frac{1}{2}\sum_{i=1}^I \boldsymbol{\omega}_i = \sum_{i=1}^I \boldsymbol{\omega}_i$$

- So $\boldsymbol{x}''$ Pareto dominates $\{\boldsymbol{\omega}_i\}_{i=1}^I$, contradicting that it was Pareto optimal.

WARP and Uniqueness

- Suppose $Y \subset \mathbb{R}^L$ is a convex cone (constant returns).
  - If $\boldsymbol{y} \in Y$, then $\alpha \boldsymbol{y} \in Y \ \forall \alpha \geq 0$.
- If Y is a convex cone, then $\boldsymbol{p}$ is a Walrasian equilibrium iff:
  (i) $\boldsymbol{p} \cdot \boldsymbol{y} \leq 0 \ \forall \boldsymbol{y} \in Y$, and
  (ii) $\boldsymbol{z}(\boldsymbol{p}) \in Y$.
- The excess demand function $\boldsymbol{z}(\cdot)$ satisfies WARP if for any pair of price vectors $\boldsymbol{p}$ and $\boldsymbol{p}'$, we have:

$$\boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}(\boldsymbol{p}') \text{ and } \boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}') \leq 0 \text{ implies } \boldsymbol{p}' \cdot \boldsymbol{z}(\boldsymbol{p}) > 0$$

- Given this assumption on technology, we are interested if aggregate demand satisfying WARP implies uniqueness.

WARP Implies Set of Equilibrium Price Vectors is Convex

Proposition

Suppose that the excess demand function $\boldsymbol{z}(\cdot)$ is such that, for **any** constant returns convex technology Y, the economy formed by $\boldsymbol{z}(\cdot)$ and Y has a unique (normalized) equilibrium price vector. Then $\boldsymbol{z}(\cdot)$ satisfies WARP. Conversely, if $\boldsymbol{z}(\cdot)$ satisfies WARP then, for any constant returns technology Y, the set of equilibrium price vectors is convex.

- WARP is necessary but not sufficient for uniqueness, but it does give convexity.
- If the set of normalized equilibria is finite, then by convexity there can be at most one normalized price equilibrium.

Proof: $\Rightarrow$ Direction

Unique equilibrium with any convex cone $Y \Rightarrow$ Aggregate WARP:

- Suppose not (WARP was violated).
- Then $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}') \leq 0$ and $\boldsymbol{p}' \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$, with $\boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}(\boldsymbol{p}')$
- Consider the CRS convex $Y^\star$ given by:

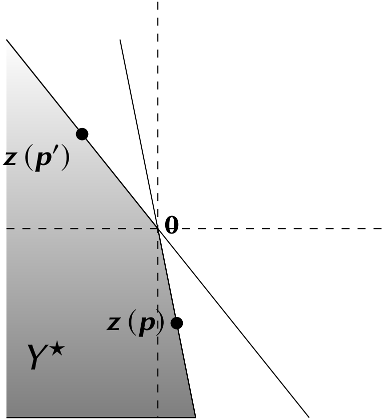
$$Y^\star = \left\{\boldsymbol{y} \in \mathbb{R}^L : \boldsymbol{p} \cdot \boldsymbol{y} \leq 0 \text{ and } \boldsymbol{p}' \cdot \boldsymbol{y} \leq 0\right\}$$

- But then both $\boldsymbol{p}$ and $\boldsymbol{p}'$ would be an equilibrium with this $Y^\star$ because:
  - $\boldsymbol{p} \cdot \boldsymbol{y} \leq 0$ and $\boldsymbol{p}' \cdot \boldsymbol{y} \leq 0\ \forall \boldsymbol{y} \in Y^\star$.
  - $\boldsymbol{z}(\boldsymbol{p}) \in Y^\star$ and $\boldsymbol{z}(\boldsymbol{p}') \in Y^\star$
  - $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$ by Walras' law, and similarly for $\boldsymbol{z}(\boldsymbol{p}')$.

$L = 2$ Example

WARP violated: $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}') \leq 0$, $\boldsymbol{p}' \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$ and $\boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}(\boldsymbol{p}')$ Convex cone production:

$$Y^{\star} = \left\{\boldsymbol{y} \in \mathbb{R}^{L} : \boldsymbol{p} \cdot \boldsymbol{y} \leq 0 \text{ and } \boldsymbol{p}' \cdot \boldsymbol{y} \leq 0\right\}$$



Proof: $\Leftarrow$ Direction

Aggregate WARP with any convex cone $Y \Rightarrow$ set of equilibrium $\boldsymbol{p}$ is convex:

1. Need to show that if $\boldsymbol{p}$ and $\boldsymbol{p}'$ are equilibria, then $\boldsymbol{p}^\alpha = \alpha\boldsymbol{p} + (1-\alpha)\,\boldsymbol{p}'$, $\alpha \in [0,1]$ is also an equilibrium.
2. $\boldsymbol{p}^\alpha \cdot \boldsymbol{y} = \alpha \underbrace{\boldsymbol{p} \cdot \boldsymbol{y}}_{\leq 0, \forall \boldsymbol{y} \in Y} + (1-\alpha) \underbrace{\boldsymbol{p}' \cdot \boldsymbol{y}}_{\leq 0, \forall \boldsymbol{y} \in Y} \leq 0, \forall \boldsymbol{y} \in Y.$
3. $\underbrace{0 = \boldsymbol{p}^\alpha \cdot \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right)}_{\text{Walras' law}} = \alpha\boldsymbol{p} \cdot \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right) + (1-\alpha)\,\boldsymbol{p}' \cdot \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right)$
4. Either $\boldsymbol{p} \cdot \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right) \leq 0$ or $\boldsymbol{p}' \cdot \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right) \leq 0$. Take $\boldsymbol{p} \cdot \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right) \leq 0$.
5. Because $\boldsymbol{z}(\boldsymbol{p}) \in Y$, we know from step 2 that $\boldsymbol{p}^\alpha \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$
6. If $\boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right)$, WARP with Step 4 would imply that $\boldsymbol{p}^\alpha \cdot \boldsymbol{z}(\boldsymbol{p}) > 0$, contradicting Step 5. Therefore we must have $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{z}\left(\boldsymbol{p}^\alpha\right)$, so $\boldsymbol{z}\left(\boldsymbol{p}^\alpha\right) \in Y$.
7. $\boldsymbol{p}^\alpha \cdot \boldsymbol{y} \leq 0\ \forall \boldsymbol{y} \in Y$ and $\boldsymbol{z}\left(\boldsymbol{p}^\alpha\right) \in Y$ imply $\boldsymbol{p}^\alpha$ is also an equilibrium.

The Gross Substitute Property

Definition

The function $\boldsymbol{z}(\cdot)$ has the *gross substitute* (GS) property if whenever $\boldsymbol{p}'$ and $\boldsymbol{p}$ are such that, for some ℓ, $p'_\ell > p_\ell$ and $p'_k = p_k$ for $k \neq \ell$, we have $z_k(\boldsymbol{p}') > z_k(\boldsymbol{p})$ for all $k \neq \ell$.

For small changes, the gross substitute property means:

- $\frac{\partial z_k(\boldsymbol{p})}{\partial p_\ell} > 0$ for all $k \neq \ell$.
- This means $\boldsymbol{Dz}(\boldsymbol{p})$ is positive off the diagonal.
- Because $\boldsymbol{z}(\boldsymbol{p})$ is HD0, $\boldsymbol{Dz}(\boldsymbol{p}) \cdot \boldsymbol{p} = \mathbf{0}$, so the diagonal of $\boldsymbol{Dz}(\boldsymbol{p})$ must be negative.

If every individual satisfies GS, then so does aggregate demand.

Two $L = 2$ Pure Exchange Examples

- Cobb-Douglas utility: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, $\alpha \in (0, 1)$.

$$\boldsymbol{z}(\boldsymbol{p}) = \left(\frac{\alpha (p_1\omega_1 + p_2\omega_2)}{p_1} - \omega_1, \frac{(1-\alpha)(p_1\omega_1 + p_2\omega_2)}{p_2} - \omega_2 \right)$$

$$\boldsymbol{D}\boldsymbol{z}(\boldsymbol{p}) = \begin{pmatrix} -\frac{\alpha p_2 \omega_2}{p_1^2} & \frac{\alpha \omega_2}{p_1} \\ \frac{(1-\alpha)\omega_1}{p_2} & -\frac{(1-\alpha) p_1 \omega_1}{p_2^2} \end{pmatrix}$$

  Positive off the diagonal $\Rightarrow$ Satisfies GS property (if $\omega_\ell > 0 \; \forall \ell$).

- Quasilinear utility: $u(x_1, x_2) = x_1 + 2\sqrt{x_2}$, where we assume $\boldsymbol{p} \cdot \boldsymbol{\omega} > 1/p_2^2$.

$$\boldsymbol{z}(\boldsymbol{p}) = \left(\frac{p_2}{p_1}\omega_2 - \frac{1}{p_1 p_2}, \frac{1}{p_2^2} - \omega_2 \right)$$

$\frac{\partial z_1(\boldsymbol{p})}{\partial p_2} = \frac{\omega_2}{p_1} + \frac{1}{p_1 p_2^2}$ and $\frac{\partial z_2(\boldsymbol{p})}{\partial p_1} = 0 \Rightarrow$ Violates GS property.

GS Implies Uniqueness in Exchange Economies

Proposition

An aggregate excess demand function $\boldsymbol{z}(\cdot)$ that satisfies the gross substitution property has at most one exchange equilibrium.

- Suppose $\boldsymbol{p}$ and $\boldsymbol{p}'$ were both equilibrium price vectors (and $\boldsymbol{p}'$ was not proportional to $\boldsymbol{p}$.)
- We need to show that $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{z}(\boldsymbol{p}') = \boldsymbol{0}$ is not possible.
- Let $m = \max_{\ell} \{p'_\ell / p_\ell\}$ (by strong monotonicity, $\boldsymbol{p} \gg \boldsymbol{0}$).
- For at least one good, $p'_k = mp_k$, and $\boldsymbol{z}(m\boldsymbol{p}) = \boldsymbol{0}$ by HD0.
- Now imagine lowering the price of each good $\ell \neq k$ sequentially from mp_ℓ to p'_ℓ.
  - By GS, the demand for good k will never increase.
  - The demand for good k decreases whenever $p'_\ell \neq mp_\ell$.
  - This happens at least once as $\boldsymbol{p}$ and $\boldsymbol{p}'$ are not proportional.

GS Uniqueness Proof with $L = 2$

- Suppose toward a contradiction that (p_1, p_2) and $\left(p_1', p_2'\right)$ where both equilibria with the vectors not proportional.
- Suppose wlog that $\frac{p_2'}{p_2} > \frac{p_1'}{p_1}$.
- Let $p_2' = mp_2$. From above we know that $p_1' < mp_1$.
- Because $\boldsymbol{z}(p_1, p_2)$ is HD0, $\boldsymbol{z}(mp_1, mp_2) = \mathbf{0}$.
- When we change prices from (mp_1, mp_2) to $\left(p_1', p_2'\right)$:
    - The price of good 2 doesn't change, but the price of good 1 falls.
    - GS implies that the demand for good 2 *decreases*.
    - But this means that $z_2\left(p_1', p_2'\right) < 0$, contradicting that $\left(p_1', p_2'\right)$ was an equilibrium.

Regular Economies

- Assume the $\boldsymbol{z}(\boldsymbol{p})$ satisfies properties (i)-(v) & is continuously differentiable.
- Normalize $p_L = 1$ and define $\widehat{\boldsymbol{z}}(\boldsymbol{p}) = (z_1(\boldsymbol{p}), \ldots, z_{L-1}(\boldsymbol{p}))$
- With this, $\boldsymbol{p} = (p_1, \ldots, p_{L-1}, 1)$ constitutes a Walrasian equilibrium iff $\widehat{\boldsymbol{z}}(\boldsymbol{p}) = \mathbf{0}$.

Definition

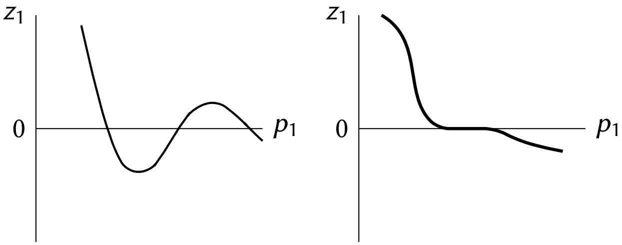
An equilibrium price vector $\boldsymbol{p}$ is *regular* if the $(L-1) \times (L-1)$ matrix of price effects $\boldsymbol{D}\widehat{\boldsymbol{z}}(\boldsymbol{p})$ is nonsingular.

Definition

If every normalized equilibrium price vector is regular, we say that the *economy is regular.*

Regular and Irregular Economies with $L = 2$

If $L = 2$, $\boldsymbol{D}\widehat{\boldsymbol{z}}\,(\boldsymbol{p})$ nonsingular $\Leftrightarrow \frac{\partial z_1(\boldsymbol{p})}{\partial p_1} \neq 0$



- $\frac{\partial z_1(\boldsymbol{p})}{\partial p_1} \neq 0$ at all equilibria
- Each equilibrium is regular
- Economy is regular
- All equilibria are locally isolated
- Finite (odd) number of equilibria

- $\frac{\partial z_1(\boldsymbol{p})}{\partial p_1} = 0$ at all equilibria
- No equilibrium is regular
- Economy is not regular
- No equilibrium is locally isolated
- Infinite number of equilibria

Index Analysis

Definition

Suppose that $\boldsymbol{p} = (p_1, \ldots, p_{L-1}, 1)$ is a regular equilibrium of the economy. Then we denote:

$$\text{index}\left(\boldsymbol{p}\right) = (-1)^{L-1} \operatorname{sgn}\left(\left|\boldsymbol{D}\widehat{\boldsymbol{z}}\left(\boldsymbol{p}\right)\right|\right)$$

where $\operatorname{sgn}\left(x\right) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$

In the left $L = 2$ example, the indices are $+1$, -1, and $+1$

The Index Theorem

For any regular economy, we have:

$$\sum_{\left\{\boldsymbol{p} \in \mathbb{R}^L_+ : \boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{0}, p_L = 1\right\}} \text{index}\left(\boldsymbol{p}\right) = 1$$

Index Analysis

- For regular economies, the number of equilibria is always odd.
- If $|\boldsymbol{D}\widehat{\boldsymbol{z}}(\boldsymbol{p})| < 0$ at all equilibria, then the equilibrium will be unique.
- The gross substitutes case is a special case of this:
  - $\boldsymbol{Dz}(\boldsymbol{p})$ is NSD whenever $\boldsymbol{z}(\boldsymbol{p}) = \mathbf{0}$ and has rank $L-1$. Therefore the determinant is negative, so its index is $+1$.
- Finally, it can be shown that *almost every* vector of initial endowments $(\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_I) \in \mathbb{R}^{LI}_{++}$, the economy defined by $\{\succeq_i, \boldsymbol{\omega}_i\}_{i=1}^{I}$ is regular.

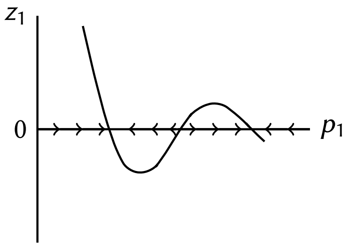
Stability: Price Tâtonnement

- Suppose at $t = 0$, the economy is out of equilibrium: $\boldsymbol{z}(\boldsymbol{p}) \neq \mathbf{0}$.
- Assume prices adjust over time according to:

$$\frac{dp_\ell}{dt} = c_\ell z_\ell(\boldsymbol{p}) \quad \forall \ell$$

  where $c_\ell > 0$ is the speed of adjustment.

- Example with $L = 2$:



Local and System Stability when $L = 2$

- Equilibrium relative prices $\frac{\bar{p}_1}{\bar{p}_2}$ are *locally stable* if, when $\frac{p_1(0)}{p_2(0)}$ is close to it, the dynamic trajectory causes relative prices to converge to $\frac{\bar{p}_1}{\bar{p}_2}$.
- Conversely, equilibrium relative prices $\frac{\bar{p}_1}{\bar{p}_2}$ are *locally totally unstable* if relative prices to diverge from $\frac{\bar{p}_1}{\bar{p}_2}$.
- If the excess demand function is downward-sloping at $\frac{\bar{p}_1}{\bar{p}_2}$ then the equilibrium is locally stable (and locally totally unstable if upward-sloping).
- There is *system stability* if for any initial position $\frac{p_1(0)}{p_2(0)}$, the corresponding trajectory of relative prices $\frac{p_1(t)}{p_2(t)}$ converges to some equilibrium arbitrarily closely as $t \to \infty$.

Normalizing Prices to a Unit Sphere

- Normalize prices such that $\sum_{\ell=1}^{L} p_\ell^2 = 1$
- Assume $c_\ell = c, \forall \ell$.
- As prices adjust, the Euclidian norm of the price vector changes according to:

$$\frac{d}{dt}\left(\sum_{\ell=1}^{L} p_\ell^2\left(t\right)\right) = \sum_{\ell=1}^{L} 2p_\ell\left(t\right)\frac{dp_\ell}{dt} = 2c\sum_{\ell=1}^{L} p_\ell\left(t\right) z_\ell\left(\boldsymbol{p}\right) = 0$$

  where the last equality is from Walras' law.
- Therefore prices are always on the unit sphere as they adjust.

Examples

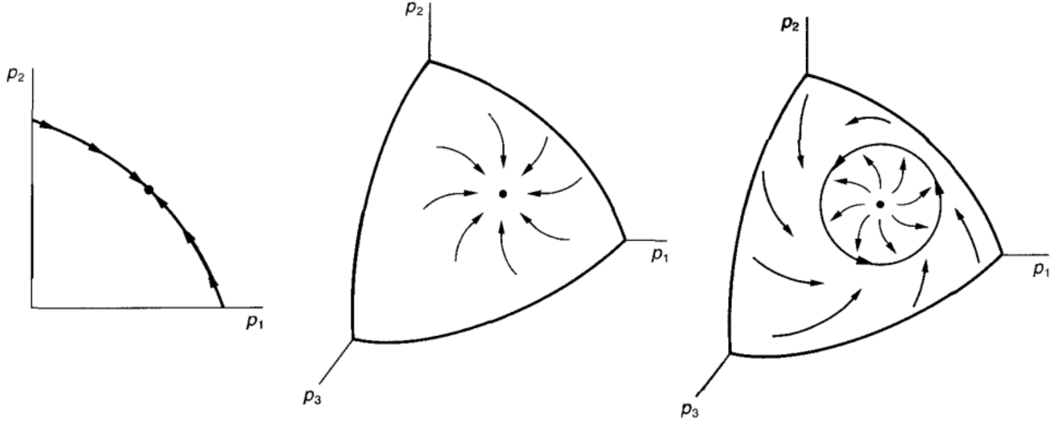


Image Source: Varian, Hal R. (2016) *Microeconomic analysis*

- In the first case, there is a unique stable equilibrium.
- In the second case, there is a unique stable equilibrium.
- In the third case, there is a unique totally unstable equilibrium.

WARP, GS and Globally Stability

- GS $\not\Rightarrow$ WARP and WARP $\not\Rightarrow$ GS.
- However, both properties imply the following:

$$\text{If } \boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{0} \text{ and } \boldsymbol{z}(\boldsymbol{p}') \neq \boldsymbol{0}\text{, then } \boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}') > 0$$

- WARP is defined as:

$$\text{If } \boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}(\boldsymbol{p}') \text{ and } \boldsymbol{p}' \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0\text{, then } \boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}') > 0$$

So if $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{0}$, then $\boldsymbol{p}' \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$, so $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}') > 0$.

- GS with $L = 2$, $p_2 = 1$ and $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{0}$.
  - GS with $p_1' > p_1$ implies $z_1(\boldsymbol{p}') < z_1(\boldsymbol{p}) = 0$.
  - GS with $p_1' < p_1$ implies $z_1(\boldsymbol{p}') > z_1(\boldsymbol{p}) = 0$.
  - Therefore $\left(p_1' - p_1\right) z_1(\boldsymbol{p}') < 0$. So:

$$\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}') = p_1 z_1(\boldsymbol{p}') + z_2(\boldsymbol{p}') > p_1' z_1(\boldsymbol{p}') + z_2(\boldsymbol{p}') \overset{Walras}{=} 0$$

Global Stability

The following proposition ensures that the WARP and GS cases we studied in the uniqueness section have a globally stable equilibrium:

Proposition

Suppose that $\boldsymbol{z}\left(\boldsymbol{p}^{\star}\right) = \boldsymbol{0}$ and $\boldsymbol{p}^{\star} \cdot \boldsymbol{z}(\boldsymbol{p}) > 0$ for every $\boldsymbol{p}$ not proportional to $\boldsymbol{p}^{\star}$. Then the relative prices of any solution trajectory of the differential equation $\frac{dp_\ell}{dt} = c_\ell z_\ell(\boldsymbol{p})$, with $c_\ell > 0 \ \forall \ell$ converge to the relative prices of $\boldsymbol{p}^{\star}$.

Proof

- Construct a Lyapunov function using the Euclidean distance function:

$$V(\boldsymbol{p}) = \sum_{\ell=1}^{L} \frac{1}{c_\ell} \left(p_\ell - p_\ell^\star\right)^2$$

- For $\boldsymbol{p}$ not proportional to $\boldsymbol{p}^\star$:

$$\begin{aligned}
\frac{dV(\boldsymbol{p})}{dt} &= 2 \sum_{\ell=1}^{L} \frac{1}{c_\ell} \left(p_\ell(t) - p_\ell^\star\right) \frac{dp_\ell(t)}{dt} \\
&= 2 \sum_{\ell=1}^{L} \frac{1}{c_\ell} \left(p_\ell(t) - p_\ell^\star\right) c_\ell z_\ell(p(t)) = -2\boldsymbol{p}^\star \cdot \boldsymbol{z}(\boldsymbol{p}(t)) < 0
\end{aligned}$$

- Because $\boldsymbol{p}^\star$ minimizes $V(\boldsymbol{p})$ and $\frac{dV(\boldsymbol{p}(t))}{dt} < 0 \; \forall \boldsymbol{p} \neq \boldsymbol{p}^\star$, by Lyapunov's Theorem, $\boldsymbol{p}^\star$ is globally stable.