General Equilibrium: Introduction

230333 Microeconomics 3 (CentER) – Part II
Tilburg University

230333 Part II: Syllabus Summary

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Topics: 7 lectures, covering:

- Partial equilibrium and partial equilibrium with market failures (externalities/public goods).
- General equilibrium: welfare, existence, uniqueness, stability.

Book: Mas-Colell, Whinston and Green
Assessment:

- Two problem sets (each worth 5% of final grade).
- Final exam (closed book, 80%): questions from parts 1 and 2.

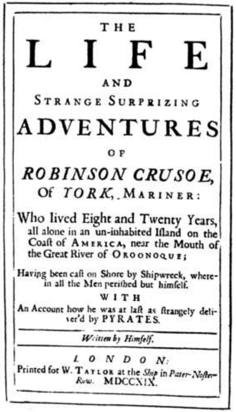
Definition of an Economy

- There are $I > 0$ consumers, $J > 0$ firms and L commodities in the economy.
- Each consumer i is characterized by a consumption set $X_i \subset \mathbb{R}^L$ and a complete and transitive preference relation $\succeq_i$ defined over X_i
- Each firm j is characterized by a nonempty and closed production set $Y_j \subset \mathbb{R}^L$
- There is an initial endowment $\bar{\boldsymbol{\omega}} = (\bar{\omega}_1, \ldots, \bar{\omega}_L) \in \mathbb{R}^L_+$ of each good.
- An *economy* can be summarized by:

$$\mathcal{E} = \left(\{(X_i, \succeq_i)\}_{i=1}^{I}, \{Y_j\}_{j=1}^{J}, \bar{\boldsymbol{\omega}} \right)$$

Robinson Crusoe by Daniel Defoe, 1719

The Life and Strange Surprizing Adventures of Robinson Crusoe, of York, Mariner: Who lived Eight and Twenty Years, all alone in an un-inhabited Island on the Coast of America, near the Mouth of the Great River of Oroonoque; Having been cast on Shore by Shipwreck, wherein all the Men perished but himself. With An Account how he was at last as strangely deliver'd by Pyrates. Written by Himself.



THE
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LONDON:
Printed for W. TAYLOR at the *Ship* in *Pater-Noster-Row*. MDCCXIX.

Example: A Day in the Life of Robinson Crusoe

- Robinson Crusoe is a lone castaway on the *Island of Despair*.
- There are two goods: coconuts and hours of leisure: $L = 2$.
- Crusoe is the only consumer on the island: $I = 1$.
- A coconut-producing firm is the only firm: $J = 1$.
- Crusoe can consume $x_{11} \geq 0$ coconuts but at most 24 hours of leisure per day:

$$X_1 = \mathbb{R}_+ \times [0, 24]$$

- Crusoe has preferences over X_1 given by $\succeq_1$.
- The firm's production technology is producing coconuts from leisure:

$$Y_1 = \left\{(y_{11}, y_{21}) \in \mathbb{R}^2 : y_{11} \leq f\,(-y_{21})\right\}$$

- The initial endowment is no coconuts and 24 hours of leisure:

$$\bar{\boldsymbol{\omega}} = (0, 24)$$

Feasible Allocations

Definition

An *allocation* $(\boldsymbol{x}, \boldsymbol{y}) = \left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_I, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_J\right)$ is a specification of a consumption vector $\boldsymbol{x}_i \in X_i$ for each consumer $i = 1, \ldots, I$ and production vector $\boldsymbol{y}_j \in Y_j$ for each firm $j = 1, \ldots, J$.

Definition

The allocation $\left(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_I, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_J\right)$ is *feasible* if

$$\sum_{i=1}^{I} x_{\ell i} = \bar{\omega}_\ell + \sum_{j=1}^{J} y_{\ell j} \quad \forall \ell = 1, \ldots, L$$

- Note: with equality because of free disposal.

Feasible Allocations on the Island

An allocation $(\boldsymbol{x}_1, \boldsymbol{y}_1) \in X_1 \times Y_1$ is feasible if:

$$\begin{aligned} \text{Coconuts: } x_{11} &= 0 + y_{11} && \text{with } x_{11} \geq 0 \\ \text{Leisure: } x_{21} &= 24 + y_{21} && \text{with } x_{21} \in [0, 24] \end{aligned}$$

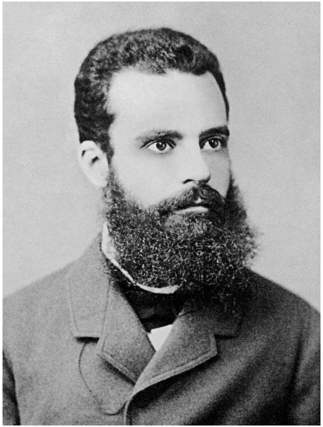
- Combining feasibility with the production technology:
  - Output of coconuts is

$$x_{11} = y_{11} \leq f\left(-y_{21}\right) = f\left(\underbrace{24 - x_{21}}_{\text{Labor hours}}\right)$$

(with equality under no disposal).

Vilfredo Federico Damaso Pareto (1848-1923)

- Born in Paris to the son of a Genoan marquis in exile. Educated at the Polytechnic University of Turin.
- Initially worked as a railway engineer.
- Was greatly enriched by an inheritance in 1898 and married a countess of Russian origins, who then ran away with their young cook.
- First to consider the consumer's problem as a constrained maximization problem of ordinal utility (as opposed to cardinal).
- Replaced the utilitarian view as maximizing the sum of utilities as an objective with the notion of Pareto optimality.
- He hinted towards the notions behind the first and second fundamental welfare theorems, without a formal proof.



Pareto Optimality

Denote the set of feasible allocations by:

$$\mathcal{A} = \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j : \sum_{i=1}^{I} \boldsymbol{x}_i = \bar{\boldsymbol{\omega}} + \sum_{j=1}^{J} \boldsymbol{y}_j \right\} \subset \mathbb{R}^{L(I+J)}$$

Definition

A feasible allocation $(\boldsymbol{x}, \boldsymbol{y})$ is *Pareto optimal* if there is no other feasible allocation $(\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{A}$ that *Pareto dominates* it, that is, if there is no feasible allocation $(\boldsymbol{x}', \boldsymbol{y}') \in \mathcal{A}$ such that $\boldsymbol{x}'_i \succeq_i \boldsymbol{x}_i$ for all i and $\boldsymbol{x}'_i \succ_i \boldsymbol{x}_i$ for some i.

Pareto Optimality on the Island of Despair

- Assume Crusoe's preferences over X_1 are continuous, convex and monotone and can be represented by:
$$u_1\left(x_{11}, x_{21}\right) = x_{11}x_{21}$$
- Assume the firm's production technology is:
$$Y_1 = \left\{\left(y_{11}, y_{21}\right) : 0 \leq y_{11} \leq \sqrt{-y_{21}} \text{ and } y_{21} \in [-24, 0]\right\}$$
  - z hours of labor can produce $\sqrt{z}$ coconuts.

Pareto Optimality on the Island of Despair

- Because $I = 1$, a Pareto optimal allocation $\boldsymbol{x}_1^\star$ maximizes $x_{11}x_{21}$ subject to:
  - $x_{11} \geq 0$ and $x_{21} \in [0, 24]$ (consumption set)
  - $y_{11} \leq \sqrt{-y_{21}}$ (production technology)
  - $x_{11} = y_{11}$ and $x_{21} = 24 + y_{21}$ (feasibility)
- Because utility is increasing in both goods, there is no disposal: $x_{11} = \sqrt{24 - x_{21}}$
- The problem becomes (combining all constraints):

$$\max_{x_{21} \in [0,24]} \left(\sqrt{24 - x_{21}}\right) x_{21}$$

- This is maximized at $x_{21} = 16$, which implies:

$$(x_{11}, x_{21}) = \left(\sqrt{8}, 16\right)$$
$$(y_{11}, y_{21}) = \left(\sqrt{8}, -8\right)$$

A Private Ownership Economy

- A market exists for each good and consumers and firms are price takers. Prices are given by $\boldsymbol{p} \in \mathbb{R}^L$.
- Each consumer i has an initial endowment $\boldsymbol{\omega}_i \in \mathbb{R}^L_+$, where $\bar{\omega}_\ell = \sum_{i=1}^{I} \omega_{\ell i} \; \forall \ell$.
- Each consumer i has a claim to a share $\theta_{ij} \in [0, 1]$ of the profits of firm j, where $\sum_{i=1}^{I} \theta_{ij} = 1 \; \forall j$.
- Consumer i's budget set with price vector $\boldsymbol{p} \in \mathbb{R}^L$ is then:

$$\mathcal{B}_i\left(\boldsymbol{p}\right) = \left\{ \boldsymbol{x}_i \in X_i : \boldsymbol{p} \cdot \boldsymbol{x}_i \leq \boldsymbol{p} \cdot \boldsymbol{\omega}_i + \sum_{j=1}^{J} \theta_{ij} \boldsymbol{p} \cdot \boldsymbol{y}_j \right\}$$

- A *private ownership economy* can be summarized by:

$$\mathcal{E}_p = \left(\left\{ (X_i, \succeq_i) \right\}_{i=1}^{I}, \left\{ Y_j \right\}_{j=1}^{J}, \left\{ \left(\boldsymbol{\omega}_i, \theta_{i1}, \ldots, \theta_{iJ} \right) \right\}_{i=1}^{I} \right)$$

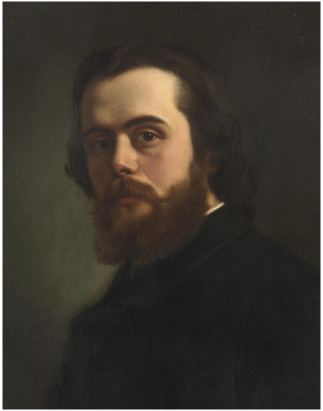
Robinson Crusoe's Private Ownership Economy

- There is a market for coconuts and labor.
- The price of a coconut is p_1 and the price of one hour of leisure is p_2.
- Crusoe owns the full initial endowment: $\boldsymbol{\omega}_1 = \bar{\boldsymbol{\omega}} = (0, 24)'$.
- Crusoe owns a 100% share in the coconut firm: $\theta_{11} = 1$.
- The firm hires Crusoe for labor and produces coconuts. The firm then sells Crusoe the coconuts.
- Crusoe's budget set is then all $\boldsymbol{x}_i \in X_i$ that satisfy:

$$\underbrace{p_1 x_{11} + p_2 x_{21}}_{\text{Cost of consumption}} \leq \underbrace{24 p_2}_{\text{Value of endowment}} + \underbrace{1 \times [p_1 y_{11} + p_2 y_{21}]}_{\text{Share of firm's profits}}$$

Marie-Esprit-Léon Walras (1834-1910)

- Born in Évreux, Upper Normandy and educated in École des Mines de Paris.
- Cofounder of the Lausanne School (with Pareto).
- One of the leading figures in the marginalist revolution.



Walrasian Equilibrium

Definition

Given a private ownership economy $\mathcal{E}_p$, an allocation $\left(\boldsymbol{x}^\star, \boldsymbol{y}^\star\right)$ and a price vector $\boldsymbol{p} \in \mathbb{R}^L$ constitute a *Walrasian equilibrium* if:

(i) For every j, $\boldsymbol{y}_j^\star$ maximizes profits in Y_j; that is $\boldsymbol{p} \cdot \boldsymbol{y}_j \leq \boldsymbol{p} \cdot \boldsymbol{y}_j^\star \ \forall \boldsymbol{y}_j \in Y_j$.

(ii) For every i, $\boldsymbol{x}_i^\star$ is maximal for $\succeq_i$ in the budget set $\mathcal{B}_i\left(\boldsymbol{p}\right)$

(iii) $\sum_{i=1}^I \boldsymbol{x}_i^\star = \bar{\boldsymbol{\omega}} + \sum_{j=1}^J \boldsymbol{y}_j^\star$

Solving for the Equilibrium: Firm's Problem

- The firm solves:

$$\max_{\{y_1 \in \mathcal{Y}_1\}} p_1 y_{11} + p_2 y_{21}$$

- If $\boldsymbol{p} \gg \mathbf{0}$, the firm will set $y_{11} = \sqrt{-y_{21}}$ (no disposal):

$$\max_{y_{21}} p_1 \sqrt{-y_{21}} + p_2 y_{21}$$

- First-order conditions:

$$-\frac{p_1}{2\sqrt{-y_{21}}} + p_2 = 0 \quad \Longrightarrow \quad y_{21} = -\frac{p_1^2}{4p_2^2} \quad \text{and} \quad y_{11} = \frac{p_1}{2p_2}$$

- Firm's profit function is then:

$$\pi_1(\boldsymbol{p}) = \frac{p_1^2}{2p_2} - \frac{p_1^2}{4p_2} = \frac{p_1^2}{4p_2}$$

Solving for the Equilibrium: Consumer's Problem

- The consumer solves:

$$\max_{\{x_{11}\geq 0, x_{21}\in[0,24]\}} \quad x_{11}x_{21}$$

subject to:

$$p_1 x_{11} + p_2 x_{21} \leq 24 p_2 + \frac{p_1^2}{4p_2}$$

- The budget constraint will bind, so:

$$x_{11} = \frac{p_2}{p_1}\left(24 - x_{21}\right) + \frac{p_1}{4p_2}$$

- The problem is then:

$$\max_{\{x_{21}\in[0,24]\}} \quad \left(\frac{p_2}{p_1}\left(24 - x_{21}\right) + \frac{p_1}{4p_2}\right) x_{21}$$

Solving for the Equilibrium: Consumer's Problem

- The problem again:

$$\max_{\{x_{21}\in[0,24]\}} \left(\frac{p_2}{p_1}(24 - x_{21}) + \frac{p_1}{4p_2}\right) x_{21}$$

- Taking first-order conditions (consider only interior solutions):

$$\frac{p_2}{p_1}(24 - x_{21}) + \frac{p_1}{4p_2} = \frac{p_2}{p_1} x_{21}$$

- Solving for x_{21}:

$$x_{21} = 12 + \frac{p_1^2}{8p_2^2}$$

- From the budget constraint, demand of coconuts is then:

$$x_{11} = \frac{p_2}{p_1}\left(24 - \left(12 + \frac{p_1^2}{8p_2^2}\right)\right) + \frac{p_1}{4p_2} = 12\frac{p_2}{p_1} - \frac{p_1}{8p_2} + \frac{p_1}{4p_2} = 12\frac{p_2}{p_1} + \frac{p_1}{8p_2}$$

Walrasian Equilibrium

- An allocation and price vector $\boldsymbol{p}$ constitute a Walrasian equilibrium if markets clear in all goods when firms and consumers are optimizing subject to their constraints.
- In equilibrium, total demand for coconuts equals total supply:

$$\underbrace{12\frac{p_2}{p_1} + \frac{p_1}{8p_2}}_{\text{Demand}} = \underbrace{\frac{p_1}{2p_2}}_{\text{Supply}} \implies \frac{p_1}{p_2} = \sqrt{32} = 2\sqrt{8}$$

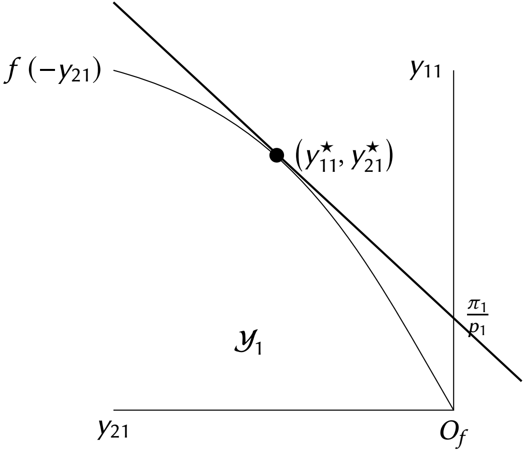
- At these prices, the market for leisure also clears:

$$x_{21} = \underbrace{24}_{\bar{\omega}_2} + y_{21} \implies \underbrace{12 + \frac{32}{8}}_{\text{Demand}} = \underbrace{24 - \frac{32}{4}}_{\text{Supply}} = 16$$

- Same as Pareto optimal outcome!
  - This is not a coincidence (1$^{\text{st}}$ Welfare Theorem).
- Any price vector where $\frac{p_1}{p_2} = 2\sqrt{8}$ is an equilibrium.
  - When normalizing prices, only 1 equilibrium.

Graphical Analysis: Firm's Problem

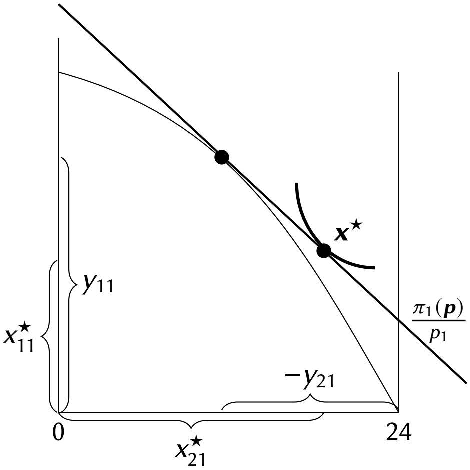
The firm's optimal choice of y_{21} is when the isoprofit line $\frac{\pi_1}{p_1} - \frac{p_2}{p_1} y_{21}$ is tangent to the production set:



The isoprofit line is found by solving $\pi_1 = p_1 y_{11} + p_2 y_{21}$ for y_{11}.

Graphical Analysis: Consumer's Problem

- The budget line is: $x_{11} = \frac{p_2}{p_1}(24 - x_{21}) + \frac{\pi_1(\boldsymbol{p})}{p_1}$.
- The slope is $-\frac{p_2}{p_1}$, and the intercept above $x_{21} = 24$ is $\frac{\pi_1(\boldsymbol{p})}{p_1}$, the same as the firm's isoprofit function. It therefore passes through firm's choice (y_{11}, y_{21}).



Graphical Analysis: Equilibrium

- In equilibrium the slope $-\frac{p_2}{p_1}$ adjusts so that markets clear: $x_{11} = y_{11}$ and $x_{21} = 24 + y_{21}$: