Extra Practice Questions - Dynamic Programming

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1 Short Horizon Cake Eating

Consider a consumer with initial wealth $W = 100$ who lives for $T = 3$ periods with discount factor $\beta = 0.9$ and utility function $u(c) = \ln(c)$. Time is discrete and runs for $t = 1, 2, 3$.

(a) Write down the consumer's sequential optimization problem with the appropriate constraint(s).

(b) Formulate this as a recursive (Bellman) problem. Clearly identify the state variable(s), control variable(s), and write the value function.

(c) Using either the Sequential or Recursive approach, obtain the Euler equation and derive the optimal consumption path $\{c_1, c_2, c_3\}$.

(d) Verify your solution obeys the resource constraint ("we can't eat more cake in total than we had at the start"). How do we know this resource constraint will hold with *equality* in this problem?

2 Envelope Condition Application

Consider the infinite horizon problem:

$$V(W) = \max_{c,W'} \{u(c) + \beta V(W')\} \quad \text{s.t.} \quad W' = W - c$$

(a) Take the first-order condition (FOC) with respect to c. You can substitute the constraint into the Bellman equation directly, BUT a Lagrangian will likely be more useful.

(b) Derive the envelope condition $V_W(W)$, the slope of the value function.

(c) Combine the FOC and envelope condition to obtain the Euler equation.

(d) Explain intuitively what the envelope condition represents in this context.

3 Investment with Productivity Shocks

A firm faces the following optimization problem:

$$V(\theta, K) = \max_{K'} \left\{ \theta K^{\alpha} - (K' - (1-\delta)K) - \frac{\phi}{2}(K' - (1-\delta)K)^2 + \beta\, \mathbb{E}[V(\theta', K')] \right\}$$

where θ follows a two-state Markov process (meaing only the current state matters for the distribution of future states tomorrow). The exogeneous state (business conditions) can take two values, high and low: $\theta \in \{\theta_L, \theta_H\}$ with transition matrix:

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} \quad \text{where you can think of each element as} \quad P_{ij} = P(j|i)$$

(a) Identify all state variables (2) and control variables (2) in this problem.

(b) What is the persistence of each business conditions state, that is, the probability that tomorrow's conditions are the same as today's, for each state θ_L and θ_H? Which state is more persistent? What does this imply about the firm's expectations (think about the duration of spells of good and bad times)?

(c) Write out explicitly what $\mathbb{E}[V(\theta', K')]$ means given the current state $\theta = \theta_L$.

(d) Derive the first-order condition for optimal investment.

(e) Interpret the role of the adjustment cost parameter ϕ in the investment decision.

4 Stationarity and Time Consistency

Consider the infinite horizon Bellman equation:

$$V(W) = \max_{W'}\{u(W - W') + \beta V(W')\}$$

(a) What does it mean for the value function to be "stationary"? Write this property mathematically.

(b) Suppose you solve this problem and obtain policy function $W' = f(W)$. If you start with $W_0 = 10$ and compute $W_1 = f(10)$, explain why applying the same function $f(\cdot)$ again gives the optimal W_2.

5 Marginal Q and Investment

In the adjustment cost model, we derived:

$$q = 1 + \phi I = \beta\,\mathbb{E}[\pi_{K'}(\theta', K') + (1-\delta)q']$$

(a) Provide an economic interpretation of marginal Q (what does q represent?).

(b) If current $q = 1.2$ and $\phi = 2$, calculate the optimal level of investment I.

(c) Suppose productivity is expected to be high next period with certainty ($\theta' = \theta_H$ with probability 1). Explain intuitively how this affects current investment through the expectation term.

(d) What happens to investment when $q < 1$? Is this economically sensible given the model structure?

6 Discretization for Numerical Solution

You want to solve the following problem numerically:

$$V(k) = \max_{k' \in [0, k_{max}]} \left\{ k^\alpha - k' + (1-\delta)k + \beta V(k') \right\}$$

with $\alpha = 0.3$, $\delta = 0.1$, $\beta = 0.96$, and $k_{max} = 50$.

(a) You want to set up a grid for capital with $N_k = 100$ points. Write the Matlab command to do this.

(b) Describe the structure of your value function iteration algorithm (you don't need to write actual code, just outline the steps).

(c) How would you check for convergence in your algorithm?

(d) What is a reasonable initial guess $V_0(k)$ to start the iteration? Justify your choice.

7 Reducing Choice Variables and Computation

Consider two formulations of the same problem:

Formulation A:

$$V(W) = \max_{c,W'}\{u(c) + \beta V(W')\} \quad \text{s.t.} \quad c + W' = W$$

Formulation B:

$$V(W) = \max_{W'}\{u(W - W') + \beta V(W')\}$$

(a) Explain the relationship between these two formulations.

(b) Solve Formulation A by setting up the Lagrangian with multiplier λ. Derive the FOCs and envelope condition.

(c) Solve Formulation B directly. Show that you obtain the same Euler equation as in part (b).

(d) Which formulation is computationally more efficient for numerical solution? Why? (Hint: what does the max operator mean computationally)

8 Bellman Operator and Convergence

Define the Bellman operator as a manipulation of an input function, which gives out another function according to:

$$\mathcal{T}[V](W) = \max_{c}\{u(c) + \beta V(W - c)\}$$

(a) Explain what it means for V^* to be a "fixed point" of the Bellman operator $\mathcal{T}$.

(b) How are we utilizing Blackwell's Sufficient Conditions when we do Value Function Iteration (VFI) numerically with (say) Matlab?

(c) If we start the VFI recursions with $V_{n=0}(z,k) =$ `zeros(Nz,Nk);` What is the first value function iteration doing? What is happening as iterations increase?