py-mathx-lab Documentation

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CONTENTS

Small, reproducible math experiments implemented in Python.

- **Audience:** curious engineers, students, and researchers
- **Idea:** each experiment is a self-contained runnable module with a short write-up
- **Goal:** a growing, searchable "lab notebook" of experiments

CHAPTER

ONE

START HERE

- *Mathematical experimentation* - what "experiments" in mathematics mean and how to read this repo
- *Experiments Gallery* - experiment gallery (IDs, tags, how to run)
- *Background* - mathematical background for experiments
- *Getting started* - install, setup, and run your first experiment
- *Development* - Makefile workflow, CI, coding conventions
- *References* - bibliography and reading list
- *PDF download* - download the PDF version of these docs

CHAPTER

TWO

RUN ONE EXPERIMENT

```
make uv-check
make venv
make install-dev
make run EXP=e001 ARGS="--out out/e001 --seed 1"
```

CHAPTER

THREE

LATEST

- – **E006** - *E006: Near Misses to Perfection*

3.1 Mathematical experimentation

Mathematical experimentation is the practice of using examples, computation, and visualization to discover structure, to generate conjectures, find counterexamples, estimate quantities, and build intuition that later supports (or refutes) formal arguments.

It is *not* "proof by computer output". Instead, experiments are a disciplined way to ask better questions and to stress-test ideas-especially now that modern computers make it easy to explore large search spaces, high-precision numerics, and rich visualizations.

This repository, **py-mathx-lab**, is a small "lab notebook" of such experiments: compact, reproducible, and readable.

3.1.1 What counts as an "experiment" in mathematics?

An experiment is a finite procedure that produces evidence about a mathematical claim or object. Typical outcomes:

- **Conjecture generation**: patterns suggest statements that might be true (or false).
- **Counterexample search**: systematic exploration tries to break a hypothesis early.
- **Quantitative exploration**: estimate constants, rates, limits, or distributions.
- **Model checking**: validate (or invalidate) approximations and heuristics.
- **Visualization**: reveal structure that is hard to see symbolically.

The modern viewpoint-where computation is a genuine part of mathematical discovery-is widely discussed under the name *experimental mathematics* ([BB05, BBC04, Bor05]). Some authors emphasize "plausible reasoning" supported by computation, paired with careful verification and eventual proof ([Bor09, BB08]). Other texts take a more problem-driven, exploratory style aimed at students and engineers ([LCD+03]), or present computation as a tool to formulate concrete research problems ([Arn15]).

3.1.2 Why computers change the game

Computers do not replace mathematical thinking-but they expand what is feasible to *inspect*:

- **Scale**: enumerate millions of cases (to find the first failure or build confidence).
- **Precision**: compute with high-precision floats or exact rationals/integers to avoid roundoff illusions.
- **Multiple lenses**: combine numerics, exact arithmetic, symbolic manipulation, and plotting.
- **Search**: automate discovery (parameter sweeps, optimization, random sampling, heuristics).
- **Reproducibility**: re-run the same pipeline with fixed seeds and pinned dependencies.

Used well, computation turns "I wonder if…" into "here is evidence, here are edge cases, and here is what we should prove next".

3.1.3 Experiments vs. proofs

A proof is the end of the story; an experiment is often the beginning.

Experiments are excellent for:

- **disproving** statements (one counterexample ends it),
- identifying **what is actually true** (after a naive conjecture fails),
- suggesting **lemmas** and **invariants** that make a proof possible.

But experiments can also mislead. Common failure modes:

- floating point error and catastrophic cancellation,
- plotting artifacts,
- "pattern matching" based on too few samples,
- unintentional selection bias ("I tried the cases that worked").

The goal is to use experiments to *reduce uncertainty*, not to hide it.

3.1.4 Targets for py-mathx-lab

py-mathx-lab aims to be a practical, long-lived collection of experiments with shared conventions:

1. **Reproducible runs**
   Each experiment is runnable as a module, writes results to a single output directory, and (when relevant) uses a fixed seed.
2. **Readable code**
   The code should be short, well-typed, and structured so readers can modify it.
3. **Useful artifacts**
   Each experiment should generate at least one of:
   - a figure
   - a table / summary statistics
   - a counterexample / witness object
   - a short narrative explaining what was learned
4. **Clear boundaries**
   An experiment write-up should state:
   - what is being tested,
   - what counts as "success" or "failure",
   - what might invalidate the result (precision limits, domain constraints, runtime limits).
5. **Traceability**
   Each page includes references to books/papers that motivated the work or explain the background.

3.1.5 Repository conventions for experiments

An experiment page should usually include:

- **Goal** (one paragraph)
- **How to run** (a command that works from the repo root)
- **Parameters** (including defaults and ranges, if swept)

- **Results** (figures/tables and a short interpretation)
- **Notes / pitfalls** (numerical caveats, surprising behavior)
- **References** (bibliography keys)

In code, prefer:

- deterministic outputs (fixed seeds, stable sorting),
- explicit configuration objects / CLI arguments,
- sanity checks (dimension checks, bounds checks, invariants),
- cross-checks (e.g., float vs. exact, two independent formulas).

3.1.6 Examples of good "experiment themes"

The experiment format supports many domains, for example:

- **Numerical analysis**: error landscapes, stability regions, conditioning, Monte Carlo integration.
- **Number theory**: continued fractions and convergents, integer sequences, modular patterns, primality heuristics.
- **Geometry/topology**: random point clouds, curvature approximations, combinatorial invariants.
- **Optimization/probability**: stochastic search behavior, concentration phenomena, empirical distributions.

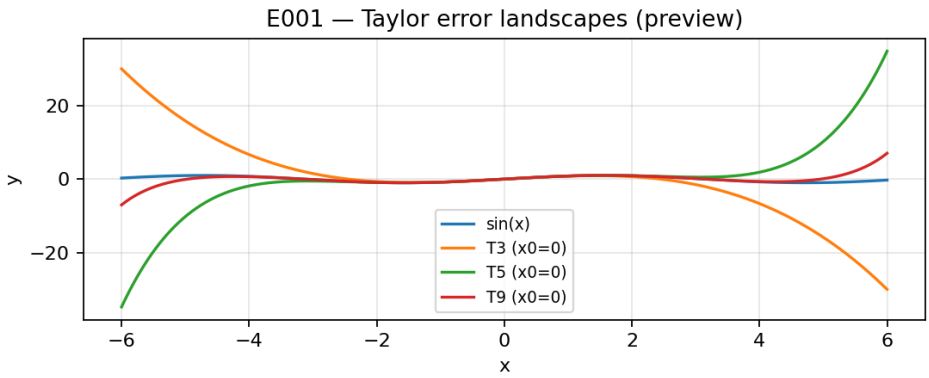
A concrete example of a rich, experiment-friendly topic is continued fractions, where it is natural to compute and plot convergents, partial quotient statistics, and periodicity phenomena ([BvdPSZ14]).

Next steps: start with *Getting started*, then browse the *Experiments Gallery* gallery, and use the tags to find topics.

3.2 Experiments Gallery

A compact, image-first overview of the experiments in **py-mathx-lab**.

3.2.1 E001 — Taylor Error Landscapes



Tags: analysis, numerics, visualization

Goal

Build intuition for Taylor truncation error by visualizing the absolute error landscape $|\sin(x) - T_n(x; x_0)|$ over a domain while varying:

- the polynomial degree n,
- the expansion center x_0.

Background (quick refresher)

If you want a short mathematical recap first, read: *Taylor series refresher*.

Research question

How does the approximation error of Taylor polynomials for $\sin(x)$ depend on the polynomial degree n and the expansion center x_0, over a fixed domain of x?

Concretely: for a chosen grid of (n, x_0), what does the error landscape $E_{n,x_0}(x) = |\sin(x) - T_n(x; x_0)|$ look like, and where do numerical artifacts start to dominate the truncation error?

Why this qualifies as a mathematical experiment

This page is not just a worked example or a derivation — it is an *experiment* in the sense of **experimental mathematics**: a finite, reproducible procedure that produces **evidence** about how a mathematical object behaves.

For E001, the object is the family of Taylor polynomials $T_n(x; x_0)$ for $\sin(x)$, and the observable is the error function $E_{n,x_0}(x) = |\sin(x) - T_n(x; x_0)|$. The experiment qualifies because it:

- **Explores a parameter space:** it varies degree n and center x_0 and inspects how the entire error landscape changes.
- **Generates testable conjectures:** e.g. "there is a widening low-error region around x_0 as n increases" and "improvement is not uniform across a fixed interval".
- **Searches for failure modes / edge cases:** large $|x - x_0|$, high degrees, and wide domains can expose numerical artifacts that *look* like truncation error but are actually floating-point limitations.
- **Produces artifacts that can be checked independently:** plots and parameter snapshots make it easy to compare runs, reproduce the same conditions, and verify that an observed pattern is not accidental.

The goal is to turn "Taylor series should be good near x_0" into *structured evidence* about **where** and **how** the approximation is good (or bad), which then informs what one would try to prove or bound formally.

- How does the truncation error $|\sin(x) - T_n(x; x_0)|$ behave as we move away from the center x_0?
- How many terms are needed to achieve a specific precision (e.g., 10^{-6}) over a fixed interval?
- Does increasing the degree n always improve the result everywhere in the domain?

Experiment design

- **Target function:** $\sin(x)$
- **Evaluation:** $x \in [-2\pi, 2\pi]$ (default)
- **Parameters:**
  - `degrees`: $\{1, 3, 5, 7, 9\}$
  - `centers`: $\{0, \pi/2, \pi\}$
- **Outputs:**
  - Plot of $f(x)$ vs $T_n(x)$

– Semi-log plot of $|f(x) - T_n(x)|$

How to run

```
make run EXP=e001 ARGS="--seed 1"
```

Artifacts are written under `out/e001/` (figures, parameters, and a short `report.md`).

What to expect

Qualitatively (and this is what the plots should confirm):

- near x_0, the higher degree reduces the error quickly,
- away from x_0, the approximation can degrade even for higher degrees,
- changing x_0 shifts the "low-error region".

Results

After running the experiment, include (or regenerate) the figures in the documentation. The canonical output location is `out/e001/`. For publishing, copy one representative "hero" image into `docs/_static/experiments/` (see "Gallery images" below).

Notes / pitfalls

- Use **log-scale** for absolute error plots to see the full dynamic range.
- Be careful interpreting relative error near zeros of $\sin(x)$.
- Huge domains and high degrees can expose floating-point artifacts that are not truncation error.

Extensions

- **Alternative functions:** Repeat the experiment for $exp(x)$ or $1/(1-x)$.
- **Relative error:** Plot $|(f - T_n)/f|$ instead of absolute error.
- **Automatic degree selection:** Find the minimal n such that error $< \epsilon$ on a given interval.

Gallery images (recommended)

To keep the experiment gallery attractive and stable:

1. run the experiment locally,
2. pick one representative output figure,
3. copy it into the repo under:

   `docs/_static/experiments/e001_hero.png`

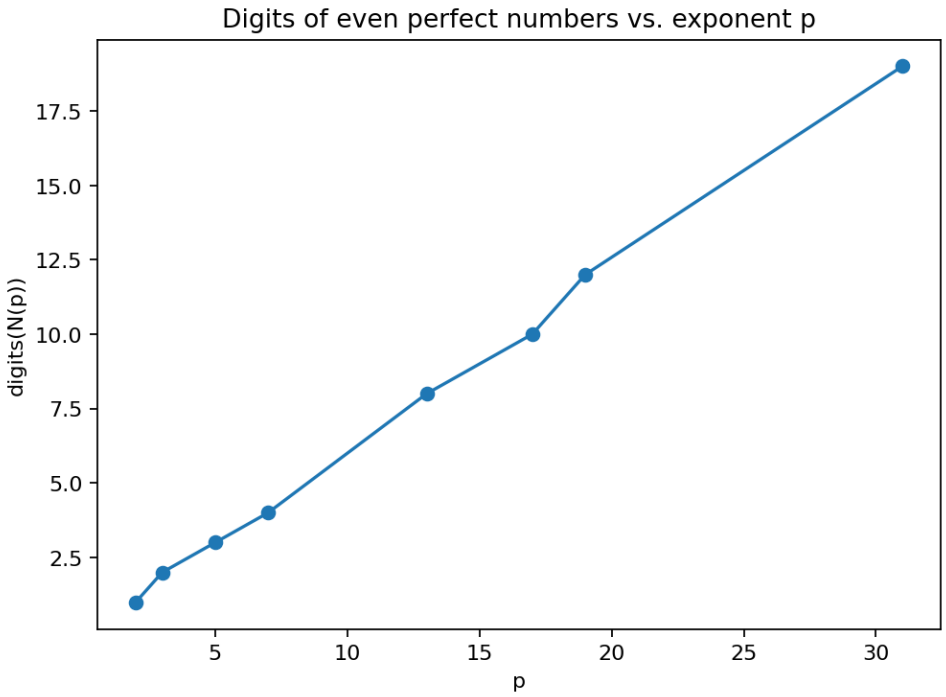
This allows the docs to show thumbnails without depending on generated `out/` artifacts.

References

See *References*. [BB05, Bor05, BB08]

3.2.2 E002: Even Perfect Numbers — Generator and Growth

Tags: number-theory, numerics, visualization



Highlights

- Generate even perfect numbers from known Mersenne exponents p.
- Plot digits and bit-length growth vs. p.
- Test the approximation $\log_{10}(N(p)) \approx 2p \log_{10}(2)$.

Goal

Generate **even perfect numbers** from known Mersenne prime exponents p and visualize how fast the numbers grow. Measure growth via **digit count**, **bit length**, and simple logarithmic approximations.

Research question

For

$$N(p) = 2^{p-1}(2^p - 1),$$

how do:

- digits(N),
- bit_length(N),
- $\log_{10}(N)$

scale with the exponent p?

How accurate is the approximation

$$\log_{10}(N(p)) \approx 2p \log_{10}(2)?$$

Why this qualifies as a mathematical experiment

The Euclid–Euler theorem tells us exactly what even perfect numbers look like, but it does not directly convey how quickly the objects become astronomically large. This experiment uses computation and visualization to build quantitative intuition and test simple asymptotic approximations.

Experiment design

Inputs

- A curated list of known Mersenne prime exponents, e.g. $p \in \{2, 3, 5, 7, 13, 17, 19, 31, \dots\}$.

Observables

For each exponent p:

- $N(p)$ as a Python integer
- digits$(N(p))$
- bit_length$(N(p))$
- approximation error:

$$\Delta(p) = \log_{10}(N(p)) - 2p \log_{10}(2)$$

Plots

- p vs. digits
- p vs. bit length
- p vs. $\Delta(p)$

How to run

From the repo root:

```
make run EXP=e002
```

or:

```
uv run python -m mathxlab.experiments.e002
```

Notes / pitfalls

- Avoid converting huge integers to decimal strings repeatedly in tight loops; compute digits using logs where possible.
- Use `int.bit_length()` for stable bit length (fast and exact).
- Keep the exponent list modest for fast docs builds; this is a growth experiment, not a search for new Mersenne primes.

Extensions

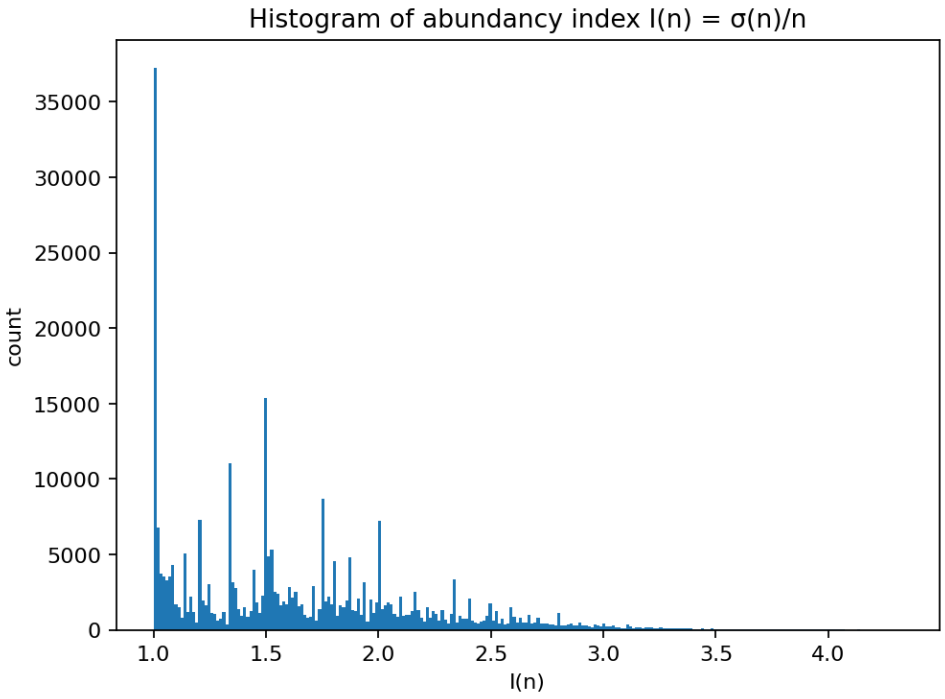
- Compare growth to 2^{2p} and quantify the relative gap.
- Add a "human scale" axis: compare digits$(N(p))$ to common benchmarks (atoms in the observable universe, etc.).
- Pull the exponent list from a small data file so the experiment can be updated without code changes.

References

See *References*.

[Cald., Voi98, OEISFInc25]

3.2.3 E003: Abundancy Index Landscape



Tags: number-theory, numerics, visualization

Highlights

- Compute $\sigma(1..N)$ via a divisor-sum sieve.
- Visualize the distribution of $I(n) = \sigma(n)/n$.
- Highlight perfect numbers as the razor-thin level set $I(n) = 2$.

Goal

Visualize how **rare** perfect numbers are by plotting the distribution of the **abundancy index**

$$I(n) = \frac{\sigma(n)}{n}$$

for integers $n \leq N$. Perfect numbers satisfy $I(n) = 2$.

Research question

For a given bound N:

- What does the empirical distribution of $I(n)$ look like?
- How frequently do we see values near 2?
- Where do the perfect numbers appear in the landscape?

Why this qualifies as a mathematical experiment

The divisor-sum function $\sigma(n)$ is highly structured but "spiky" and hard to intuit symbolically. Computational sweeps reveal qualitative structure (clusters, gaps, tails) and put perfect numbers into context.

Experiment design

Method: compute (1), ..., (N) via a divisor-sum sieve

Use the identity "each divisor contributes to its multiples":

- initialize an array `sigma[0..N]` with zeros
- for each `d = 1..N`:
  - add `d` to `sigma[k]` for all multiples `k = d, 2d, 3d, ...`

This runs in about $O(N \log N)$ time and avoids per-number factorization.

Observables

- $I(n) = \sigma(n)/n$
- distance to perfection: $|I(n) - 2|$
- classification:
  - deficient: $I(n) < 2$
  - perfect: $I(n) = 2$
  - abundant: $I(n) > 2$

Plots

- histogram of $I(n)$ (or of $I(n) - 1$)
- scatter plot of n vs. $I(n)$ (optionally with log-scale on n)
- highlight perfect numbers

How to run

```
make run EXP=e003
```

or:

```
uv run python -m mathxlab.experiments.e003
```

Notes / pitfalls

- $I(n)$ is rational. For classification, compare integers using $\sigma(n)$ and $2n$ rather than floats.
- For plots, floats are fine, but compute the "perfect" condition as `sigma[n] == 2*n`.
- Start with $N \leq 1\,000\,000$ to keep runtime and memory reasonable.

Extensions

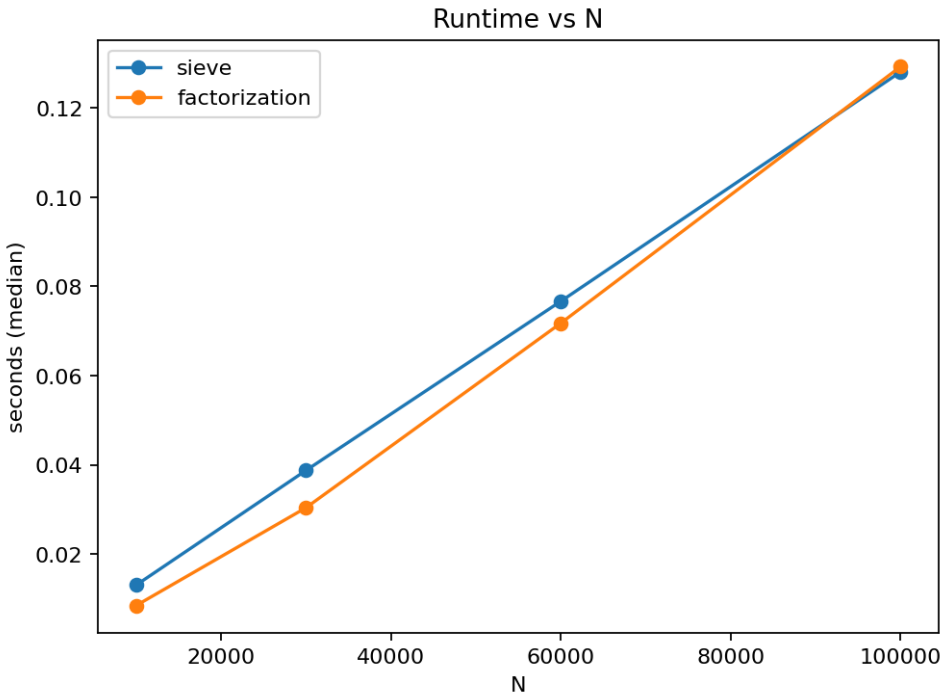
- Repeat for different ranges and overlay histograms.
- Plot the top-k values of $I(n)$ and compare to known extremal families.
- Explore the "near misses" set (feeds directly into E006).

References

See *References*.

[Voi98, Wei03, OEISFInc25]

3.2.4 E004: Computing (n) at Scale — Sieve vs. Factorization



Tags: number-theory, numerics, optimization

Highlights

- Benchmark bulk sieve vs. per-number factorization.
- Find runtime crossover points as N grows.
- Validate correctness on sampled inputs.

Goal

Benchmark two practical ways to compute the sum-of-divisors function:

1. **Sieve method:** compute all $\sigma(1..N)$ in one pass.
2. **Factorization method:** compute $\sigma(n)$ per-number from the prime factorization.

Research question

For a range of bounds N:

- which approach is faster?
- where is the crossover point?
- what are the memory tradeoffs?

Why this qualifies as a mathematical experiment

Both methods are mathematically equivalent, but performance depends on constants, caching, and implementation details. This is a quantitative exploration of algorithmic behavior grounded in number-theory structure.

Experiment design

Method A: divisor-sum sieve (bulk computation)

Compute `sigma[1..N]` by adding each divisor to its multiples (as in E003).

Method B: per-number factorization

Factor each n (e.g. using a precomputed prime list up to $\sqrt{N}$) and compute:

If

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

then

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Measurements

- wall-clock runtime vs. N (multiple trials, median)
- peak memory (rough estimate acceptable for v1)
- correctness cross-check: random sample where both methods agree

Outputs

- plot: runtime vs. N for both methods
- short table: N, runtime(A), runtime(B), speedup

How to run

```
make run EXP=e004
```

or:

```
uv run python -m mathxlab.experiments.e004
```

Notes / pitfalls

- Factorization becomes expensive quickly; keep the factorization method limited to moderate N.
- Use integer arithmetic for correctness checks (`sigma[n] == 2*n` etc.).
- Report both runtime and *effective throughput* (numbers processed per second).

Extensions

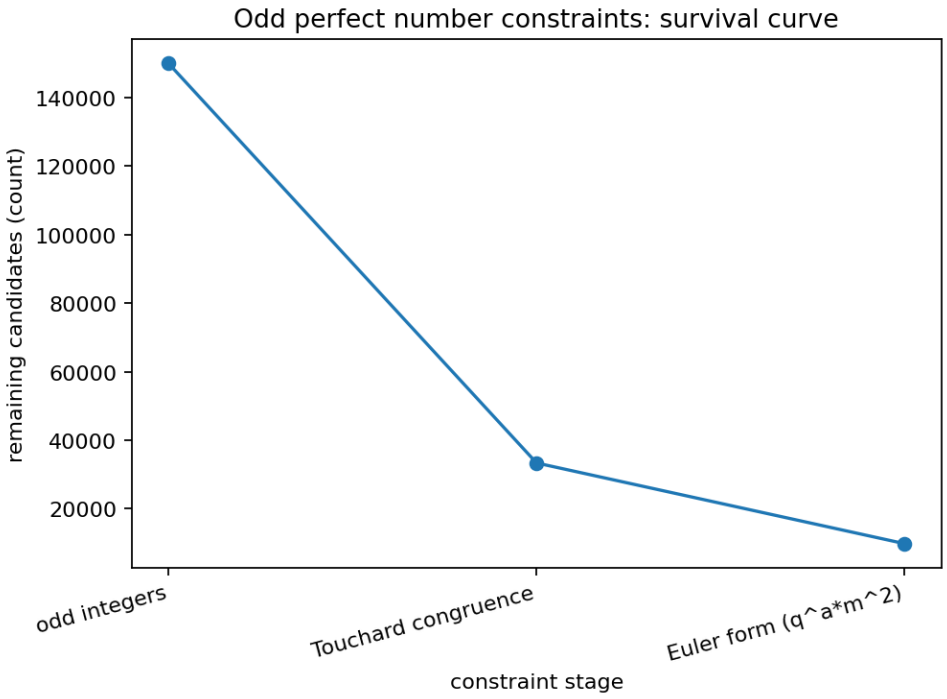
- Add a third method using smallest-prime-factor (SPF) sieve for fast factorization.
- Compare pure Python vs. `numpy` arrays for Method A.
- Turn the benchmark into a reusable utility for later experiments.

References

See *References*.

[Voi98, Wei03]

3.2.5 E005: Odd Perfect Numbers — Constraint Filter Pipeline



Tags: number-theory, search, visualization

Highlights

- Apply necessary constraints as staged filters on odd candidates.
- Plot survival curves (remaining candidates per constraint stage).
- Make the "open problem" constraints tangible at finite scales.

Goal

Demonstrate *why brute-force search for odd perfect numbers is unrealistic* by applying known **necessary conditions** as a step-by-step filter pipeline and visualizing how the candidate pool collapses.

Research question

Given odd integers up to a bound N:

- how many survive each constraint stage?
- which constraints are most "eliminating" in practice at small scales?
- what does a survival curve suggest about scalability?

Why this qualifies as a mathematical experiment

Odd perfect numbers are an open problem. Theorems provide constraints rather than a classification. Computation makes these constraints tangible by measuring their elimination power on finite candidate sets.

Experiment design

Candidate set

Start with odd integers $1 < n \leq N$.

Constraint stages (v1)

Use a conservative, well-known set of *necessary* conditions that can be checked mechanically:

1. **Euler form (shape constraint):** any odd perfect number must be of the form

$$n = q^{\alpha} m^2$$

   where q is prime, $\gcd(q, m) = 1$, and

$$q \equiv \alpha \equiv 1 \pmod 4.$$

2. **Congruence filter (Touchard-type):** odd perfect numbers satisfy strong congruence restrictions (implemented as one or two simple congruence checks supported by the references).
3. **Small-prime structure filters:** apply a few lightweight necessary conditions (e.g., reject candidates divisible by some specific small patterns if justified by the reference set).

For each stage, record "remaining candidates".

Output

- table: stage name, remaining count, elimination percentage
- plot: survival curve (stage index vs. remaining candidates)

How to run

```
make run EXP=e005
```

or:

```
uv run python -m mathxlab.experiments.e005
```

Notes / pitfalls

- Be explicit about what is *proved* vs. what is a heuristic. Only implement conditions that are truly necessary.
- At small N, some deep constraints won't show their full strength; the goal is the *pipeline idea*, not a record bound.
- Euler-form testing requires factorization; keep N small enough for trial division (v1).

Extensions

- Add stronger bounds/constraints from the modern literature and compare elimination power.
- Replace trial division with an SPF sieve to scale the Euler-form test.
- Report not just counts but also the distribution of remaining prime-factor patterns.

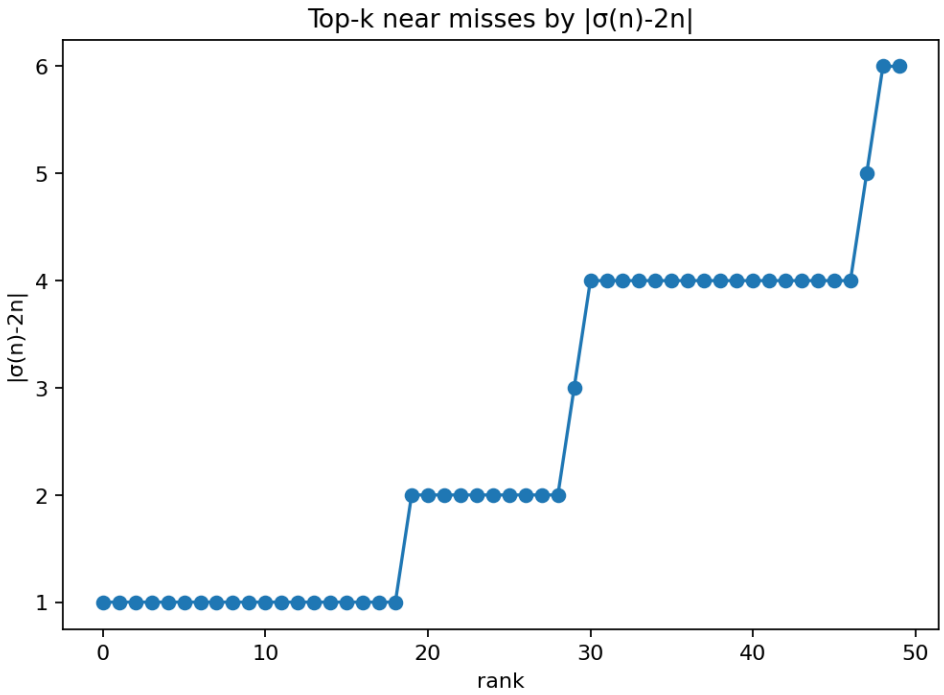
References

See *References*.

[Guy04, OR14, Sto24, Voi98]

3.2.6 E006: Near Misses to Perfection

Tags: number-theory, numerics, visualization



Highlights

- Search for n with $\sigma(n)$ unusually close to $2n$.
- Build leaderboards for absolute and relative deviation.
- Visualize how "near perfection" clusters by structure.

Goal

Find and visualize integers whose divisor sum is **unusually close** to the perfect condition $\sigma(n) = 2n$, without being perfect.

Research question

For integers $n \leq N$, which numbers minimize:

- absolute deviation:

$$D_1(n) = |\sigma(n) - 2n|$$

- relative deviation:

$$D_2(n) = \left|\frac{\sigma(n)}{n} - 2\right|?$$

Do "near misses" cluster in recognizable families (highly composite, abundant, etc.)?

Why this qualifies as a mathematical experiment

The perfect condition is a sharp equality. Studying the closest failures often reveals structure and suggests new questions (e.g., which multiplicative patterns drive $\sigma(n)$ toward $2n$).

Experiment design

Computation

- Compute $\sigma(1..N)$ via the divisor-sum sieve (as in E003).

- For each n, compute $D_1(n)$ and $D_2(n)$.
- Keep the top-k smallest deviations (excluding actual perfect numbers).

Outputs

- table: top-k near misses (with n, $\sigma(n)$, D_1, D_2)
- plot: n vs. $D_2(n)$ (log-scale on D_2 often helps)
- mark perfect numbers for reference

How to run

```
make run EXP=e006
```

or:

```
uv run python -m mathxlab.experiments.e006
```

Notes / pitfalls

- Use integer comparisons to identify perfect numbers (`sigma[n] == 2*n`).
- For D_2, floats are fine for plotting, but store exact rational values for ranking when possible (e.g., compare $|\sigma(n) - 2n|$ first, then normalize for reporting).
- Choose k small (e.g. 50 or 200) so the report stays readable.

Extensions

- Repeat for different N and compare stability of the "near miss" leaderboard.
- Add a second leaderboard restricted to odd n only.
- Compare near misses to known abundant/deficient classifications and prime factorizations.

References

See *References*.

[Voi98, Wei03, OEISFInc25]

3.3 Background

This section provides mathematical foundations for the experiments.

3.3.1 Perfect numbers refresher

This page is a *beginner-friendly* refresher for experiments about **perfect numbers**. You only need basic number theory facts (divisors, primes) to follow it.

Core definitions

For a positive integer n, let

- $d \mid n$ mean "d divides n",
- $\sigma(n) = \sum_{d \mid n} d$ be the **sum-of-divisors function**,
- $s(n) = \sum_{d \mid n,\ d<n} d = \sigma(n) - n$ be the sum of **proper** divisors.

A number n is **perfect** iff its proper divisors sum to itself:

$$n \text{ is perfect} \iff s(n) = n \iff \sigma(n) = 2n.$$

Examples:

- 6 is perfect because $1 + 2 + 3 = 6$.
- 28 is perfect because $1 + 2 + 4 + 7 + 14 = 28$.

Key theorem: all even perfect numbers (Euclid–Euler)

A classic result completely characterizes **even** perfect numbers:

Euclid–Euler theorem.
An integer n is an even perfect number **iff**

$$n = 2^{p-1}(2^p - 1)$$

where $2^p - 1$ is prime (a **Mersenne prime**).

So every known perfect number is generated from a Mersenne prime exponent p. This is the main "generator" you'll use in experiments. [Cald., Voi98]

Why σ matters (multiplicativity)

If n factors as

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

then

$$\sigma(n) = \prod_{i=1}^{k} \sigma\left(p_i^{a_i}\right) = \prod_{i=1}^{k} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

This formula turns "sum all divisors" into a fast computation once you know the prime factorization.

The big open question: odd perfect numbers

It is **unknown** whether any **odd** perfect numbers exist. A large literature proves *constraints* (congruences, size bounds, number of prime factors, etc.). Experiments can explore these constraints (and why they make brute-force search unrealistic). [Guy04, OR14, Sto24]

What experiments typically visualize

Typical "lab" questions you can turn into plots and tables:

- **Verification by computation:** compute $\sigma(n)$ (via factorization) and check $\sigma(n) = 2n$ for candidates.
- **Generator experiment:** produce even perfect numbers from known Mersenne exponents p and confirm perfection.
- **Growth:** number of digits / bit length of $2^{p-1}(2^p - 1)$ as a function of p.
- **Divisor-function behavior:** compare $\sigma(n)/n$ (abundancy index) for random n vs. perfect numbers.
- **Odd constraints (toy models):** test necessary conditions on odd n and see how restrictive they are.

For data (lists of perfect numbers and exponents), OEIS is a convenient reference. [OEISFInc25]

Practical numerical caveats

Even with correct math, computation has a few traps:

- **Factorization dominates.** Computing $\sigma(n)$ via divisors is slow unless you factor n. For large n, factorization becomes infeasible; prefer the Euclid–Euler generator for even perfect numbers.
- **Big integers are fine, but expensive.** Python integers won't overflow, but operations on huge numbers scale with the number of bits (so be mindful in loops and plotting).
- **Prime testing vs. proof.** Testing that $2^p - 1$ is prime is nontrivial for large p. For experiments, use a curated list of known Mersenne prime exponents (e.g., from OEIS / GIMPS) rather than trying to discover new ones from scratch. [MersenneResearchIncGIMPS24, MersenneResearchIncGIMPS25]
- **Be explicit about definitions.** Some sources define "perfect" via $\sigma(n) = 2n$; others via proper divisors. Use one convention consistently in code and docs.

References

See *References*.

[Cald., Sto24, Voi98, OEISFInc25]

3.3.2 Taylor series refresher

This page is a *beginner-friendly* refresher for experiments that use Taylor polynomials. You only need basic calculus (derivatives) to follow it.

Taylor polynomial

Assume f has enough derivatives near x_0 (this is true for $\sin(x)$, $\cos(x)$, polynomials, exponentials, etc.). The Taylor polynomial of degree n around x_0 is

$$T_n(x; x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k.$$

Intuition: $T_n(x; x_0)$ is the polynomial that matches $f(x_0)$ and the first n derivatives at x_0. It is usually accurate when x is close to x_0.

For $f(x) = \sin(x)$, the derivatives cycle, and around $x_0 = 0$ this becomes

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Truncation error and the remainder

The approximation error is the remainder

$$R_n(x; x_0) = f(x) - T_n(x; x_0).$$

Under standard conditions, Taylor's theorem gives a remainder representation. One common form is the (Lagrange) remainder:

$$R_n(x; x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1} \quad \text{for some } \xi \text{ between } x \text{ and } x_0.$$

This is the key qualitative message for experiments like E001:

- the factor $(x - x_0)^{n+1}$ makes the method **local** (good near x_0, potentially bad far away),
- increasing n helps most where $|x - x_0|$ is small.

What experiments typically visualize

In a numerical experiment, you often look at

- absolute error: $|R_n(x; x_0)|$
- relative error: $|R_n(x; x_0)|/|f(x)|$ (careful near zeros of f)

and plot them across a domain to see where the approximation is reliable.

Practical numerical caveats

Even when the mathematics are correct, computation can mislead:

- large $|x - x_0|$ and high n can produce huge intermediate terms,
- subtractive cancellation can reduce accuracy,
- floating-point rounding can dominate before the theoretical truncation error does.

A common "extension" experiment is to repeat the same plots using higher precision arithmetic to separate *truncation error* from *rounding error*.

Introductory reading

If you want a longer, *beginner-friendly* treatment (beyond this refresher), these are good starting points:

- A quick overview / definitions and examples: [Wikipediacontributors25d].
- A rigorous calculus textbook with a clean presentation of Taylor's theorem and remainders: [Apo91].
- A proof-oriented classic (slower, deeper): [Spi08].
- For the numerical viewpoint (truncation vs. rounding error): [BFB15].

References

See *References*.

[Apo91, BFB15, Spi08, Wikipediacontributors25d]

3.4 Getting started

This project uses an **uv-only** workflow and a small Makefile wrapper to run everything consistently.

3.4.1 Prerequisites

You need:

- **Python 3.14**
- **uv** on your PATH
- **GNU Make**

Windows notes

- Install GNU Make (e.g., via Chocolatey).
- You can run everything from `cmd.exe`, PowerShell, or from WSL.

3.4.2 Verify tools

From the repository root:

```
make uv-check
make python-check
make python-info
```

3.4.3 First-time setup

Create a virtual environment and install dependencies:

```
make venv
make install-dev
make install-docs
```

3.4.4 Run the full development chain

```
make final
```

`make final` runs:

- formatting (Ruff)
- linting (Ruff)
- type checking (mypy)
- tests (pytest)
- documentation (sphinx)

Documentation can be built separately via:

```
make docs
```

3.4.5 Run an experiment

Example (E001):

```
make run EXP=e001 ARGS="--seed 1"
```

See the experiment overview here: *Experiments Gallery*.

3.4.6 Build documentation locally

```
make docs
```

Output:

- `docs/_build/html`

3.4.7 Troubleshooting

`error: Failed to spawn: sphinx-build`

Sphinx is not installed in the uv environment.

Fix:

```
make install-docs
make docs
```

`make python-check` fails

The repo enforces Python **3.14**. Install Python 3.14, then recreate the environment:

```
make clean-venv
make venv
make install-dev
make install-docs
```

3.5 Development

This page describes the development workflow and the conventions used in this repository.

3.5.1 Workflow overview

Use the Makefile targets for everything. They wrap `uv run ...` so you do not need to activate a virtual environment manually.

Typical day-to-day:

```
make final
```

Documentation-only build:

```
make docs
```

Clean caches and build artifacts:

```
make clean
```

Remove the virtual environment (full reset):

```
make clean-venv
```

3.5.2 Makefile workflow

The Makefile is the **single entry point** for development tasks (env setup, quality checks, docs builds, and running experiments).

- Prefer `make final` before pushing.
- Prefer `make docs` to validate documentation changes.
- Use `make run EXP=<id>` to execute an experiment and write artifacts to `out/<id>/`.

Dependency groups

This summary is included from the Makefile documentation:

Your `pyproject.toml` defines dependency sets:

- Base runtime dependencies: `project.dependencies`
- Dev tools: `project.optional-dependencies.dev` (e.g. `ruff`, `mypy`, `pytest`)
- Docs tools: `project.optional-dependencies.docs` (e.g. `sphinx`, `sphinxcontrib-bibtex`)

The Makefile maps to these groups like this:

- `make install-all` → `uv sync` (base set from `uv.lock`)
- `make install-dev` → `uv sync --extra dev`
- `make install-docs` → `uv sync --extra docs`

- `make install` → `uv pip install -e .` (editable install)

  Recommendation: keep `uv.lock` committed. It is the reproducibility anchor for `uv sync`.

Run logs

The `run` target writes a per-run log file under:

- `out/<exp>/logs/run_<exp>_YYYYMMDD_HHMMSS.log`

On Windows, the Makefile calls a small PowerShell helper script (`scripts/run_experiment.ps1`) so that:

- the Makefile stays readable,
- logs are written as UTF-8,
- both the dependency sync (`uv sync --extra dev`) and the experiment output end up in the same log.

To enable DEBUG output **only** from this repository's code (`mathxlab.*`), run with `V=1`:

```
make run EXP=e001 ARGS="--seed 1" V=1
```

Common workflows

First-time setup (dev machine)

```
make uv-check
make python-check
make venv
make install-dev
```

Run checks locally (like CI)

```
make final
```

Build docs only

```
make docs
```

Run an experiment

```
make run EXP=e001 ARGS="--seed 1"
```

Use `V=1` to enable DEBUG logs from `mathxlab.*` only:

```
make run EXP=e001 ARGS="--seed 1" V=1
```

Full reference

For the complete target-by-target reference and troubleshooting, see makefile.

3.5.3 Formatting, linting, typing, tests

- Formatting: **Ruff formatter**
- Linting: **Ruff**
- Typing: **mypy**
- Tests: **pytest**

CI formatting behavior

In CI, formatting runs in check mode (`ruff format --check`). Locally it formats in place.

3.5.4 Experiment authoring guidelines

When adding a new experiment:

1. Add a new module under `mathxlab/experiments/`, e.g. `e002_...py`.
2. Prefer deterministic outputs:
   - `--seed` argument if randomness is involved
   - write results to a single `--out` directory
3. Keep the experiment runnable as a module:
   - `python -m mathxlab.experiments.e002`
4. Update the docs:
   - add a short entry to *Experiments Gallery*
   - optionally add a dedicated page under `docs/experiments/` later

3.5.5 Documentation

Docs are built with Sphinx + MyST.

Build locally:

```
make install-docs
make docs
```

Deployed website:

- GitHub Pages from the `docs` workflow

3.5.6 Contributing (high-level)

- Create a feature branch.
- Open a PR against `main`.
- CI must pass before merge.
- Keep PRs small and well-scoped.

3.6 References

3.7 PDF download

A PDF build of this documentation is generated by GitHub Actions and published alongside the HTML site.

- Download PDF

If the link is missing, it usually means you are viewing an older deployment or the PDF build step failed.

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