MA3002 Solid Mechanics & Vibration Exam - CheatSheet

Solid Mechanics & Vibration (Nanyang Technological University)

$\delta W = 0$ rigid bodies frictionless joint

$\delta W = \delta U$ find *unknown force* For **non-equilibrium** structure, LHS = W.D. by ext. force − W.D. by **net force**

$\delta W^* = \delta U^*$ find *deflection* at a point

$1 \cdot \Delta = \text{Loading expression} + \sum \frac{M_A m_A}{K_A} + \sum \frac{F_B f_B}{K_B}$ for elastic yielding support.

$\boxed{\theta \cdot \delta M}$ $\boxed{e \cdot \delta F}$

Engesser's First Theorem: $\frac{\partial U^*}{\partial P_i} = \Delta_i, U^* = f(P_1, P_2, P_3, \ldots, P_n)$

Castigliano's First Theorem: $\frac{\partial U}{\partial \Delta_i} = P_i, U = f(\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n)$

Castigliano's Second Theorem: $\frac{\partial U}{\partial P_i} = \Delta_i, U = f(P_1, P_2, P_3, \ldots, P_n)$

Special case of EFT, valid for linear elastic material only.

$\Delta_{dyn} = \Delta_{st}\left(1 \pm \sqrt{1 + \frac{2h}{\Delta_{st}}}\right); \Delta_{max} = \Delta_{st}\left(1 + \sqrt{1 + \frac{2h}{\Delta_{st}}}\right)$, no prior loading.

$1 \cdot \theta = \text{Loading expression}$ (Cross-section rotation) *angle in radian* $\vec{h} = h\hat{n}, \vec{\delta h} = \delta h\hat{n}$

Sign for each type of loading can *assign independently*.

Unit load method *confined to configuration of structure (supports), satisfy equilibrium.*

(a) Perform real load analysis on structure to find expressions for M, T, P, Q of *respective deformable member*, so that their actual deflection can be determined. Assign sign for each type of expression.

(b) Remove all real load (keep deflection), apply unit load at desired spot to find its deflection. Confined to structure's configuration, their supports. (δF (*unit load*) in δW^* induces $\delta P\ (m, t, p, q)$ in δU^*)

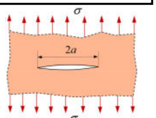
(c) Small displacement condition allows M, m, T, t, P, p, Q and q to be found from original configuration.

*Include ***EI*** and ***kx***.

U	δU^*	Type of loading
$\int_0^l \frac{P^2}{2EA}dx$	$\int_0^l \frac{Pp}{EA}dx$	Axial
$\int_0^l \frac{Q^2}{2GA}dx$	$\int_0^l \frac{Qq}{GA}dx$	Shear
$\int_0^l \frac{T^2}{2GJ}dx$	$\int_0^l \frac{Tt}{GJ}dx$	Torsional
$\int_0^l \frac{M^2}{2EI}dx$	$\int_0^l \frac{Mm}{EI}dx$	Bending
For linear elastic material, $U = U^*, \delta U = \delta U^*$		

1. LEFM (brittle, Griffith's): $U_s = (2a \times 1)2\gamma$ $\boxed{G = 2\gamma}$ $U_e = -\frac{\pi\sigma^2 a^2}{E}k$ Loss = Gain $G = \frac{\pi\sigma^2 a}{E}k$ $k = 1 - \nu^2$ or 1 Plane strain or p. stress

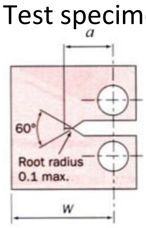
2. LEFM (brittle, Irwin's): $K = Y\sigma(\pi a)^{1/2}$ **LEFM, up to small yielding** $(EG_c)^{1/2} \equiv K_c$ $\left(\frac{EG_c}{1-\nu^2}\right)^{1/2} \equiv K_c$ Useful in deduce G_c if there is geometry factor in K_c, as **G is derived based on Y = 1**, central crack infinite width, fig beside

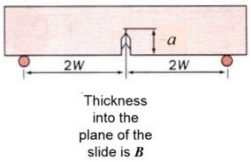


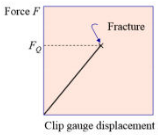
3. LEFM w/ plastic zone correction (moderate ductile): In ***estimating K/K$_c$*** value by taking into account small plastic zone,

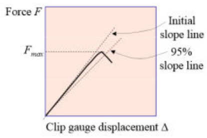
$K = \sigma(\pi a)^{1/2}$, $K_c = \sigma(\pi a)^{1/2}$ ⟶ $r_p = \frac{1}{2\pi}\left(\frac{K}{\sigma_Y}\right)^2$ plane stress $r_p = \frac{1}{6\pi}\left(\frac{K}{\sigma_Y}\right)^2$ plane strain ⟶ $K = \sigma(\pi a')^{1/2} = \sigma[\pi(a + r_p)]^{\frac{1}{2}}$, $K_c = \sigma(\pi(a + r_p))^{1/2}$ Corrected K, Conservative K (P. stress)

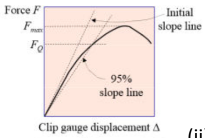
4. Experimental determination of K_{Ic} (Plain strain fracture toughness), small plastic zone at crack tip:
Test specimens: Compact tension test and bending test specimens.











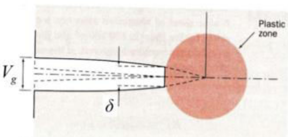
Bending: $K_Q = \frac{F_Q}{BW^{1/2}} f_1\left(\frac{a}{W}\right)$

Compact tension: $K_Q = \frac{F_Q}{BW^{1/2}} f_2\left(\frac{a}{W}\right)$

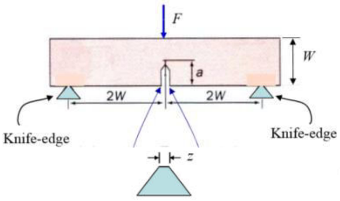
(i) $F_{max}/F_Q < 1.1$ (ii) $B, a, (W-a) \geq 2.5\left(\frac{K_Q}{\sigma_Y}\right)^2$, $0.45 < \left(\frac{a}{W}\right) < 0.55$

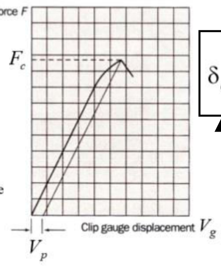
(i) Ensure **small crack tip** plastic zone.
(ii) Plane strain.

5. Crack-Tip Opening Displacement (COD) Method:



δ = crack tip opening displacement
V_g = clip gauge displacement





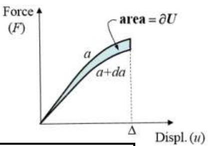
$\delta_c = \frac{K_c^2(1-\nu^2)}{2\sigma_Y E} + \frac{0.4(W-a)V_p}{0.4W + 0.6a + z}$ ⟶ $\delta_c = \frac{K_c^2}{\lambda\sigma_Y E}$, $\delta_c = \frac{K_{Ic}^2(1-\nu^2)}{\lambda\sigma_Y E}$ λ=1

$K_c = \frac{F_c}{BW^{1/2}} f_1\left(\frac{a}{W}\right)$

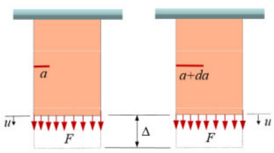
Conservative K_C (P. stress)

*For both 4 and 5, forces and displacements values are chosen such that crack growth is **just prior** to *propagate*.
For 4, crack tip is **sharp** (more or less), for 5, crack tip is **blunt**, and adjustment is made to find corrected K_c.

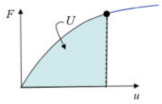
6. J-integral: $J = -\frac{1}{B}\frac{\partial U}{\partial a}$



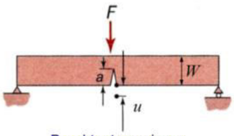
Linear material, $J_c = G_c$



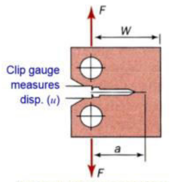
$J = \frac{2U}{B(W-a)}$



J_c can be obtained if pulled until onset of cracking



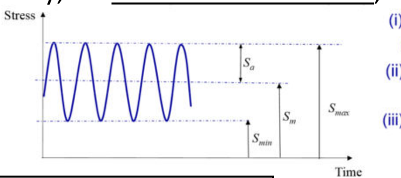
Bend test specimen



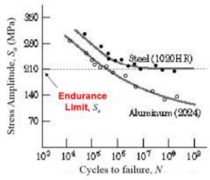
Compact tension specimen

7. In brittle, both plane stress and strain, LEFM fits. In moderate ductile, LEFM with plastic correction fits. As Irwin's SIF theory only *applicable up to moderate ductile*, adjustment must be made to find K_c for ductile material (COD test). J-Integral fits for all cases. K_c, J_c, and G_c are **material properties**, once these values are exceeded, crack is unstable and propagates.
*If K_c is given readily, for *moderate ductile case*, K_c is used solely to deduce a more accurate r_p.

8. Cyclic loading: Fatigue



(i) Reversed cycle (tensile-compressive loading)
(ii) Fluctuating cycle (zero-tensile loading)
(iii) Fully-reversed cycle (symmetric tensile-compressive loading)



Rotation Bending Fatigue test is used generally to produce S-N curve. Modification is made to S_e for different loading and conditions.
Endurance, fatigue life (N_f); Endurance limit, fatigue limit/strength (S_e); $S_a < S_e$ infinite life

9. S_e = Endurance limit; S_u = Ultimate Tensile Strength (or Tensile Strength)
$S_e = 0.5S_u$ for steels with $S_u < 690$ MPa
$S_e = 0.4S_u$ for aluminium alloys with $S_u < 131$ MPa
$S_e = 0.4S_u$ for copper alloys with $S_u < 96.5$ MPa
$S_e = S_{e'}$

$S_e = S_e' C_{size} C_{load} C_{surf\text{-}finish}$ S modified, $S_{e'}$ of bending

C_{size} (for circular rods) $= \begin{cases} 1.0 & \text{if } d \leq 8\text{ mm} \\ 1.189d^{-0.097} & \text{if } 8\text{ mm} \leq d \leq 250\text{ mm} \end{cases}$ for bending or torsional load
$= 1.0$ for axial load

C_{load} = Load parameter
$= 1.0$ for bending loads in the component
$= 0.7$ for axial loads in the component
$= 0.577$ for torsional loads in the component

1MPa = 0.1450377ksi, S_u, $C_{surf\text{-}fin}$

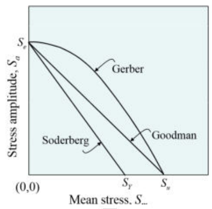
$S_e = \frac{S_e' C_{size} C_{load} C_{surf\text{-}finish} \cdots}{K_f}$ $K_f = 1 + q(K_t - 1)$ notches, grooves, holes, steps

10. Fatigue failure **occurrence** and safety factor for ***non-zero mean stress***: *S_e modified

$\frac{S_a}{(1 - S_m/S_u)} > S_e$ (Goodman) $\frac{S_a \times SF}{S_e} + \frac{S_m}{S_u} = 1$

$\frac{S_a}{[1 - (S_m/S_u)^2]} > S_e$ (Gerber) $\frac{S_a \times SF}{S_e} + \left(\frac{S_m}{S_u}\right)^2 = 1$

$\frac{S_a}{(1 - S_m/S_Y)} > S_e$ (Soderberg) $\frac{S_a \times SF}{S_e} + \frac{S_m}{S_Y} = 1$



11. $\frac{da}{dN} = C(\Delta K)^m = C\left(Y S_R\sqrt{\pi a}\right)^m$ $S_R \equiv S_{max} - S_{min}$ $S_R = S_{max}$ *Reversed cycle, only tensile part

$N_f = \frac{2}{C(YS_R)^m \pi^{m/2}(2-m)}\left(a_f^{1-m/2} - a_0^{1-m/2}\right)$

Units of C will give da/dN in m/cycle when ΔK is in MN m$^{-3/2}$

*a is half crack length for central crack (2a)

12. Variable amplitude loading: Miner's rule + Multi-axial loading fatigue failure (effective uni-axial conversion)

N_i from S-N $\frac{n_1}{N_1} + \frac{n_2}{N_2} + \frac{n_3}{N_3} + \cdots + \frac{n_k}{N_k} = 1$

$\bar{\sigma}_a = \frac{1}{\sqrt{2}}\sqrt{(\sigma_{xa} - \sigma_{ya})^2 + (\sigma_{ya} - \sigma_{za})^2 + (\sigma_{za} - \sigma_{xa})^2 + 6(\tau_{xya}^2 + \tau_{yza}^2 + \tau_{zxa}^2)}$ $\bar{\sigma}_m = \sigma_{xm} + \sigma_{ym} + \sigma_{zm}$

$S_{eq} \equiv \frac{\bar{\sigma}_a}{1 - \frac{\bar{\sigma}_m}{S_u}}$ Find N_f

1. Weight factor will not appear in EOM if oscillation is triggered at **equilibrium** position of respective masses.
2. Rayleigh's method:

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k_{eq}}{m_{eq}}}$$

Effective **mass** and **spring constant** at one POINT.

$$\frac{1}{2}m_{eq}\dot{x}^2 = \frac{1}{2}\int v(s)^2 dm_s$$

Equivalent mass is *lumped to the point where point mass is oscillating*, with speed $\dot{x}$.

Damping mechanisms include: Coulomb, viscous and hysteretic

Coulomb/Dry friction	Viscous	Hysteretic/material
Vibrating body rubbing on dry surfaces (surface irregularity), friction constant in magnitude, proportional to normal force.	Body vibrates in a fluid medium, due to resistance to laminar flow of viscous fluids, force proportional to velocity.	Internal friction (friction between grains) of solid, dependent on stress amplitude.
$F_d = \mu N$	$F_d = cv$	$F_d = hv/\omega$

* All forces are opposite to direction of motion of vibrating body.

3. Parallel plate viscous damper:

$$\tau = \mu\frac{\partial u}{\partial y} \qquad f = \tau A \qquad f = cv \text{ where } c = \frac{\mu A}{h}$$

4. Damping equation and its solutions:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$
$$m\ddot{x} + c\dot{x} + kx = 0$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \qquad \zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} \qquad c_c = 2\sqrt{km}$$

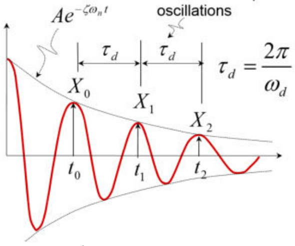
$$\zeta < 1 \quad x(t) = e^{-\zeta\omega_n t}(A\sin\omega_d t + B\cos\omega_d t) = X e^{-\zeta\omega_n t}\sin(\omega_d t + \psi)$$
$$\zeta = 1 \quad x(t) = (C_1 + C_2 t)e^{-\omega_n t}$$
$$\zeta > 1 \quad x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \qquad \lambda_1, \lambda_2 = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$$

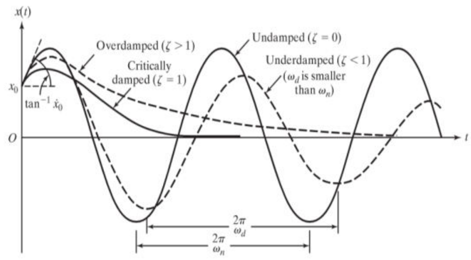
5. Logarithmic Decrement, $\boldsymbol{\delta}$:

$$\delta \equiv \ln\frac{X_{m-1}}{X_m} = \zeta\omega_n\tau_d$$

$$\zeta = \sqrt{\frac{\delta^2}{4\pi^2 + \delta^2}}$$

Period of damped oscillations





Overdamped: The system returns (exponentially decays) to equilibrium without oscillating.

Critically damped: The system returns to equilibrium as quickly as possible without oscillating.

Underdamped: The system oscillates (at reduced frequency compared to the *undamped* case) with the amplitude gradually decreasing to zero.

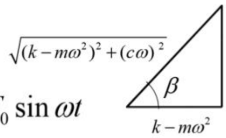
Undamped: The system oscillates at its natural resonant frequency (ω_n).

6. Response to Harmonic Excitation (Steady state response):

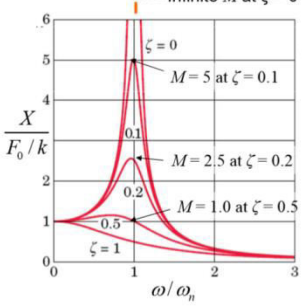
$$m\ddot{x} + c\dot{x} + kx = F_0\sin\omega t$$

$$x_p(t) = X\sin(\omega t - \phi)$$

$$X\sqrt{(k-m\omega^2)^2 + (c\omega)^2} \times \sin(\omega t - \phi + \beta) = F_0\sin\omega t$$



$$0 \le \beta \le \pi$$

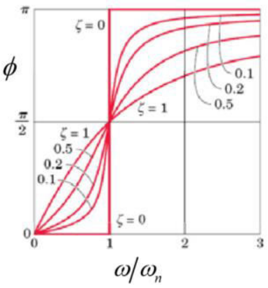


$$\frac{X}{F_0/k} = \frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} \qquad \phi$$

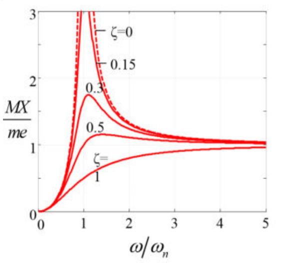
$$\left.\frac{X}{F_0/k}\right|_{\omega=\omega_p} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_{peak} = \omega_0\sqrt{1-2\zeta^2}$$

damped resonant frequency



$$\varphi = \tan^{-1}\frac{c\omega}{k-m\omega^2} = \tan^{-1}\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}$$



$$M\ddot{x} + c\dot{x} + kx = me\omega^2\sin\omega t \qquad X = \frac{me\omega^2}{\sqrt{(k-M\omega^2)^2 + (c\omega)^2}}$$

M inclusive of m, $\omega_n = \sqrt{\frac{k}{M}}$

$$\frac{MX}{me} = \frac{(\omega/\omega_n)^2}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2\zeta\,\omega/\omega_n)^2}}$$

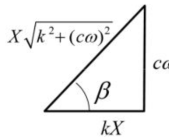
$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = \sqrt{(kY)^2 + (c\omega Y)^2}\ \sin(\omega t + \alpha)$$

$$x = X\sin(\omega t + \alpha - \phi) \qquad \alpha = \tan^{-1}\left(\frac{c\omega}{k}\right)$$

$$X = Y\sqrt{\frac{k^2 + (c\omega)^2}{(k-m\omega^2)^2 + (c\omega)^2}} \qquad \phi = \tan^{-1}\left(\frac{c\omega}{k-m\omega^2}\right)$$

7. Force transmission ratio (TR), absolute value:

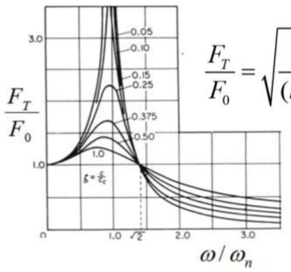
$$f_T(t) = kx + c\dot{x} = F_T\sin(\omega t - \varphi + \beta),\ F_T > 0$$



$$0 \le \beta \le \frac{\pi}{2}$$

$$\varphi = \tan^{-1}\frac{c\omega}{k-m\omega^2} = \tan^{-1}\frac{2\zeta\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

$$0 \le \varphi \le \pi$$



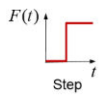
$$\frac{F_T}{F_0} = \sqrt{\frac{k^2 + (c\omega)^2}{(k-m\omega^2)^2 + (c\omega)^2}} = \sqrt{\frac{1 + (2\zeta\,\omega/\omega_n)^2}{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}$$

$$F_T = \sqrt{(kX)^2 + (c\omega X)^2}$$

$\omega > \sqrt{2}\omega_n$, $|F_T/F_0| < 1$

$\omega_n \downarrow$ or $\omega \uparrow$

Step loading (Transient):



*Take note of initial conditions

Force transmission reduction, achieve vibration isolation:

(a) Increase frequency ratio value such that $> \sqrt{2}$, keep force transmission ratio < 1. Increase driving frequency; Decrease natural frequency – k and m (mount machine on soft spring/rubber mat or on heavy block of granite)

(b) In case where frequency ratio value cannot be kept above $\sqrt{2}$, input damper to oscillation system, dissipate mechanical energy, reduce force transmitted.

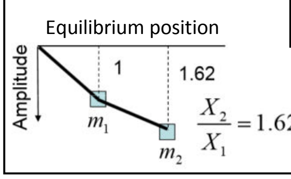
Continuous structure has mass and stiffness distributed continuously and coupled in space. The structure has many degrees of freedom, yet they are not free to move independently. They are coupled and exhibit a distinct shape when the structure resonates at natural frequency, and the resonant shape is known as mode shape.

8. Two degree of freedom (2DOF) systems (*small vibration* assumptions):

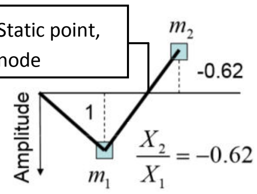
**Forced vibration* include Harmonic excitation, Base excitation etc. (Compare coefficient, *particular solution*)

Free vibration*: Natural frequencies and their ***respective* mode shapes, ***mode superposition*** due to initial v and s.

Solve det(matrix) = 0 to get the natural frequencies, then take ratio of the vibration amplitudes for *mode shapes*.



Static point, node



$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(1)} + \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)}\sin(\omega_1 t + \psi_1) + \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)}\sin(\omega_2 t + \psi_2)$$

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \end{Bmatrix} C_1\sin(\omega_1 t + \psi_1) + \begin{Bmatrix} \phi_{12} \\ \phi_{22} \end{Bmatrix} C_2\sin(\omega_2 t + \psi_2)$$

Initial condition, forces in equilibrium