Notes on 2nd order systems

Consider a 2nd order system such as a linear mass-spring-damper, with mass m, linear friction coefficient b and linear stiffness constant k, for which the following dynamics (Newton's law) hold:

$$f(t) = m\ddot{x}(t) + b\dot{x}(t) + kx(t)$$

This corresponds to a 2nd order Ordinary Differential Equation (ODE)

> NOTE: the **order** of an ODE is given the order of the highest time derivative. For Newton's law ($F = ma$) we always have the acceleration $a \equiv \ddot{x}$ which implies 2nd order dynamics.

Given a 2nd order ODE, the evolution of the system $x(t)$ is uniquely determined if we are given '*two pieces of information*' , typically position x_0 and velocity $v_0 \equiv \dot{x}_0$ at a given time t_0. This means that, at any time t, the '**state**' $s(t)$ of a system is fully captured by a pair

$$\boldsymbol{x}(t) \equiv \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$$

where we conveniently introduce a new '**velocity**' variable $v(t) \equiv \dot{x}(t)$.

State Space representation

Noting that $\dot{v} = \ddot{x}$, Newton's law can be rewritten two equations in two variables $x(t)$ and $v(t)$ but involving only first derivatives (instead of the second derivatives of the original formulation):

$$\begin{cases} \dot{x} & = & v \\ f & = & m\dot{v} + bv + kx \end{cases}$$

rearranging by keeping derivatives on the left hand side

$$\begin{cases} \dot{x} & = & v \\ \dot{v} & = & \frac{1}{m}(-kx - bv + f) \end{cases}$$

Or, equivalently, in matrix format

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \begin{bmatrix} x \\ v \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} f$$

i.e.

$$\boxed{\frac{d}{dt}\boldsymbol{x} = A\boldsymbol{x} + Bf}$$

AC Analysis

In AC Analysis, we drive systems with purely sinusoidal inputs, at a generic (radian) frequency ω and we simply look for solutions which are also purely sinusoidal, at the same frequency ω.

$$\cos\omega t \Rightarrow \boxed{\texttt{system}} \Rightarrow A_\omega \cos(\omega t + \phi_\omega)$$

> NOTE: the radian frequency ω, measured in [rad/sec], is related to the natural frequency f, measured in [Hz], simply by 2π factor, i.e.
>
> $$\omega = 2\pi f$$

Generalized Sinusoids via Complex Exponentials

Computationally, it is more efficient to consider **'generalized' sinusoids via complex exponentials**

$$e^{jx} := \cos x + j \sin x$$

where $j = \sqrt{-1}$.

> NOTE: Given a generalized signal $\tilde{s}(t) := S_0 e^{j\omega t}$, where $S_0 \in \mathbb{C}$ is a complex number (or **phasor** of the generalized signal) characterized by amplitude $|S_0|$ and phase $\angle S_0$, one can always retrieve a pure sinusoid by evaluating the real part
>
> $$\texttt{Real}(\tilde{s}(t)) = \texttt{Real}(S_0 e^{j\omega t}) = |S_0| \cos(\omega t + \angle S_0)$$

So, let's consider a 2nd order mechanical system driven by generalized sinusoidal force $\tilde{f}(t) \equiv F_0 e^{j\omega t}$ and look for generalized sinusoidal motions $\tilde{x}(t) \equiv X_0 e^{j\omega t}$

$$\tilde{f}(t) \equiv F_0 e^{j\omega t} \Rightarrow \boxed{\texttt{system}} \Rightarrow \tilde{x}(t) \equiv X_0 e^{j\omega t}$$

Note that taking derivatives of generalized sinusoids (i.e. complex exponentials) is particularly computationally straightforward, i.e.

$$\frac{d}{dt}\tilde{x}(t) = \frac{d}{dt}X_0 e^{j\omega t} = X_0 \frac{d}{dt}e^{j\omega t} = j\omega X_0 e^{j\omega t} = j\omega \tilde{x}(t)$$

so formally $\frac{d}{dt}$ can be replace by a $j\omega$ whenever dealing with generalized sinusoids. Therefore, Newton's law for generalized sinusoids simply becomes

$$\tilde{f}(t) = j\omega m(j\omega \tilde{x}(t)) + j\omega b\tilde{x}(t) + k\tilde{x}(t)$$

which, recalling that $j^2 = -1$, becomes

$$\tilde{f}(t) = (k - m\omega^2 + j\omega b)\tilde{x}(t)$$

Recalling that $\tilde{f}(t) = F_0 e^{j\omega t}$ and $\tilde{x}(t) = X_0 e^{j\omega t}$, one gets

$$F_0 \cancel{e^{j\omega t}} = (k - m\omega^2 + j\omega b)X_0 \cancel{e^{j\omega t}}$$

Frequency Response

Frequency response $H(j\omega)$ is a useful concept for linear systems and is usually defined as 'output' over 'input' ratio, which for a spring-mass-damper systems becomes:

$$H(j\omega) := \frac{\texttt{output}}{\texttt{input}} = \frac{\tilde{x}(t)}{\tilde{f}(t)} = \frac{X_0 \cancel{e^{j\omega t}}}{F_0 \cancel{e^{j\omega t}}} = \frac{1}{k - \omega^2 m + j\omega b}$$

A specific mechanical system is defined when values for $\{m, k, b\}$ are specified. So that $H(j\omega)$ is just a complex function of the independent variable ω.

The same system can be rewritten as

$$\boxed{H(j\omega) = \frac{1}{k - \omega^2 m + j\omega b} = \frac{k^{-1}}{1 - \frac{\omega^2}{\omega_0^2} + j\frac{\omega}{Q\omega_0}}}$$

where we conveniently define

- $\omega_0 := \sqrt{k/m}$, also known as **resonance frequency**, merasured in [rad/sec]
- $Q = \sqrt{mk/b^2}$, also know as **quality factor**, unitless

> to students:
>
> 1. verify that with this definition of ω_0 and Q, $H(j\omega)$ can be rewritten as above boxed equation
> 2. verify that Q is unitless

When $\omega = \omega_0$ the real part of the denominator of $H(j\omega)$ becomes zero and one has

$$H(j\omega_0) = -jk^{-1}Q$$

> **Note:** although not the analytical one, ω_0 is a good approximation for the point of maximum of $|H|$:
>
> - maximizing $|H|^2$ is equivalent to minimizing its inverse
>
> $$1/|H|^2 = (1 - \omega^2/\omega_0^2)^2 + Q^{-2}\omega^2/\omega_0^2$$
>
> - which, $\omega \simeq \omega_0$, can be approximated by $(1 - \omega^2/\omega_0^2)^2 + Q^{-2}$,
> - this approximation has a minimum at $\omega = \omega_0$