MA3004 Reference for Final 1

Mathematical Methods In Engineering (Nanyang Technological University)

(A) PDE: https://carousell.com/kevinvoo/

$\int x\sin(px)dx = -\frac{-\sin(px) + px\cos(px)}{p^2} + C$ | $\int x\cos(px)dx = \frac{\cos(px) + px\sin(px)}{p^2} + C$

1. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$ $\boxed{\textit{Laplace's equation}}$ 2. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}$ $\boxed{\textit{Wave equation}}$

3. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = f(x, y, z)$ (f is a given non-null real function of x, y and z) *inhomogeneous* PDE $\boxed{\textit{Poisson equation}}$

$f(x, y, z)$ does not contain the unknown function $\phi(x, y, z)$ or any of its partial derivatives.

*A partial differential equation is *homogeneous* if **every term of the PDE contains the dependent variable or one of its derivatives**, otherwise the PDE is inhomogeneous.

4. Second order **linear** PDE: the coefficients A, B, C, D, E, F and G in the PDE are functions of x_1 and x_2 (including constant functions)

$$A(x_1,x_2)\frac{\partial^2\phi}{\partial x_1^2} + 2B(x_1,x_2)\frac{\partial^2\phi}{\partial x_1\partial x_2} + C(x_1,x_2)\frac{\partial^2\phi}{\partial x_2^2} + D(x_1,x_2)\frac{\partial\phi}{\partial x_1} + E(x_1,x_2)\frac{\partial\phi}{\partial x_2} + F(x_1,x_2)\phi = G(x_1,x_2)$$

hyperbolic $AC - B^2 < 0$ for all x_1 and x_2 *parabolic* $AC - B^2 = 0$ for all x_1 and x_2 *elliptic* $AC - B^2 > 0$ for all x_1 and x_2 $AC - B^2$ different signs *mixed*

5. A PDE is to be solved in a physical domain defined by the set of all points in space where the **PDE holds**.
(BCs): Solving PDE subjected to conditions in which φ or expression involving φ and the spatial derivatives of φ is *suitably prescribed* at **every point** on the boundary of the physical solution domain.

6. Theorem for homogeneous linear PDE: linear superposition of solutions (**BCs superposition**)
"If $\phi = \phi_1$ and $\phi = \phi_2$ are solutions of a homogeneous linear PDE then so is $\phi = c_1\phi_1 + c_2\phi_2$ for any arbitrary real constants c_1 and c_2."

7. **ODE** review:
$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ $y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$ if $\lambda = \lambda_1$ and $\lambda = \lambda_2$ $y(x) = (A + Bx)e^{\lambda_1 x}$ has only one solution given by $\lambda = \lambda_1$ $y(x) = (A\cos(\beta x) + B\sin(\beta x))e^{\alpha x}$ $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$

8. **Fourier series** review: (a) Cosine (even) series (b) Sine (odd) series $f(x)$ is continuous in the interval $0 < x < L$

(a) $a_0 + \sum_{n=1}^{\infty} a_n\cos(\frac{n\pi x}{L}) = f(x)$ *for* $0 < x < L$ (b) $\sum_{n=1}^{\infty} b_n\sin(\frac{n\pi x}{L}) = f(x)$ *for* $0 < x < L$

$\boxed{a_0 = \frac{1}{L}\int_0^L f(x)dx, \quad a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx \ (n = 1, 2, \cdots)}$ $\boxed{b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx \ (n = 1, 2, \cdots)}$

9. In solving IBVP, (i) Method of *separation of variables* [γ] (ii) Apply *BCs*, summing up for general series solution (iii) *ICs*, FSPs.
**Requires the PDE to be homogeneous and linear, ensures that the sum of solutions is still a solution of PDE.

10. $\boxed{\gamma = 0, \gamma > 0 \text{ and } \gamma < 0}$, seeking for **non-trivial** solutions upon considering the BCs. $\gamma = p^2 > 0$ $\gamma = -p^2 < 0$

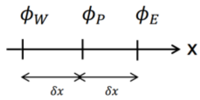
11. *Vibrating spring with **fixed end** (BCs, 1D wave): $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ $\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\{E_n\sin(\frac{n\pi ct}{L}) + F_n\cos(\frac{n\pi ct}{L})\}\sin(\frac{n\pi x}{L})}$

12. Method of solving in 9 requires the **BCs** to be of the form given by either one of the followings (homogeneous):
$\boxed{\textit{Dirichlet}}$ $\phi(0, t) = 0$ and $\phi(L, t) = 0$ for $t \geq 0$ $\boxed{\textit{Neumann}}$ $\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0$ and $\left.\frac{\partial\phi}{\partial x}\right|_{x=L} = 0$ for $t \geq 0$

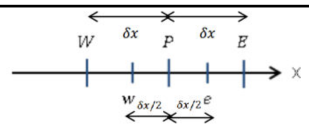
*Reformulation of PDE/BCs to desired form to use method 9. For instance, let $\phi(x,t) = v(x) + \psi(x,t)$ to reformulate BCs.

(C) CFD: [Numerical method of PDE]

1. **Central** differencing scheme:



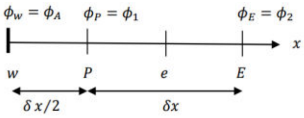
$\left(\frac{d\phi}{dx}\right)_P = \frac{\phi_E - \phi_W}{2\delta x}$



$\frac{\phi_E - 2\phi_P + \phi_W}{\delta x^2} = \left(\frac{d^2\phi}{dx^2}\right)_P$

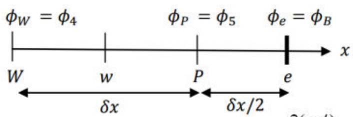
*Handling node 1 and 5,

Node 1 $\left(\frac{d\phi}{dx}\right)_e = \frac{\phi_E - \phi_P}{\delta x}$ $\left(\frac{d\phi}{dx}\right)_w = \frac{\phi_P - \phi_A}{\delta x/2}$



$\boxed{\left(\frac{d^2\phi}{dx^2}\right)_P = \frac{\phi_E - 3\phi_P + 2\phi_A}{\delta x^2}}$

Node 5



$\boxed{\left(\frac{d^2\phi}{dx^2}\right)_P = \frac{\phi_W - 3\phi_P + 2\phi_B}{\delta x^2}}$

2. General transport equation:
$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) = div(\Gamma\ grad\ \phi) + S_\phi$

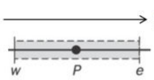
- Convection: Determined by the velocity of fluid particle (**u**)
- Diffusion determined by (Γ) *Diffusivity

* ϕ is variable of interest; S_ϕ is source/sink of quantity ϕ.

$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial}{\partial x}(\rho\phi u) + \frac{\partial}{\partial y}(\rho\phi v) + \frac{\partial}{\partial z}(\rho\phi w) = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\Gamma\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\Gamma\frac{\partial\phi}{\partial z}\right) + S_\phi$

3. FVM (Finite Volume Method): Domain is divided into finite CVs, equations **integrated over CVs** and Gauss' theorem applied.

$n = -1$ $A = A_w = 1$



$n = +1$ $A = A_e = 1$

(I) $\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$ (II) $\int_{\Delta V} SdV = \bar{S}\Delta V = \bar{S}\Delta x$ $^*\int_{\Delta V} SdV = C\int_{\Delta V}\phi dV = C\phi_P\Delta V$ $\boxed{\bar{S}\Delta V = S_u + S_p\phi_P}$

(III) $\int_{\Delta V}\frac{d}{dx}(\rho u\phi)dV = (\rho u\phi)_e - (\rho u\phi)_w$ $\boxed{\phi_e = \frac{\phi_P + \phi_E}{2}; \phi_w = \frac{\phi_W + \phi_P}{2}}$

4. Diffusion: $\left(\frac{\Gamma}{\delta x} + \frac{\Gamma}{\delta x} - S_P\right)\phi_P = \left(\frac{\Gamma}{\delta x}\right)\phi_W + \left(\frac{\Gamma}{\delta x}\right)\phi_E + S_u$

5. Diffusion-convection: **(CDS)**
$a_P\phi_P = a_E\phi_E + a_W\phi_W + S_u$
$a_P = a_E + a_W + (F_e - F_w) - S_P$

a_W	a_E
$D_w + \frac{F_w}{2}$	$D_e - \frac{F_e}{2}$

$F_w = (\rho u)_w$, $F_e = (\rho u)_e$
$D_w = \frac{\Gamma_w}{\delta x_{WP}}$, $D_e = \frac{\Gamma_e}{\delta x_{PE}}$
$\boxed{F_e/D_e = Pe_e < 2}$

6. Diffusion-convection (Transportiveness):
$\phi_e = \phi_P$ $\phi_w = \phi_W$ F>0
$\phi_e = \phi_E$ $\phi_w = \phi_P$ F<0
False diffusion at large ***Pe*** $\boxed{\text{Upwind}}$

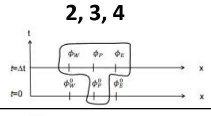
	a_W	a_E
$F_w > 0,\ F_e > 0$	$D_w + F_w$	D_e
$F_w < 0,\ F_e < 0$	D_w	$D_e - F_e$

Internal nodes 2, 3, 4

7. Temporal (diffusion):
$\int_t^{t+\Delta t}\int_{CV}\frac{\partial(\rho\phi)}{\partial t}dVdt = \int_t^{t+\Delta t}\int_{CV}\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)dVdt + \int_t^{t+\Delta t}\int_{CV}SdVdt$

$\int_{CV}\int_t^{t+\Delta t}\frac{\partial(\rho\phi)}{\partial t}dtdV = \rho\left(\phi_P - \phi_P^0\right)A\delta x$

$\boxed{\text{Implicit}}$ $\left(\rho\frac{\delta x}{\Delta t} + \frac{\Gamma}{\delta x} + \frac{\Gamma}{\delta x} - S_P\right)\phi_P = \left(\frac{\Gamma}{\delta x}\right)\phi_W + \left(\frac{\Gamma}{\delta x}\right)\phi_E + \rho\frac{\delta x}{\Delta t}\phi_P^0 + S_u$



$\Delta t < \rho\frac{(\delta x)^2}{2\Gamma}$ Explicit

$\boxed{\rho\left(\phi_P - \phi_P^0\right)\frac{\delta x}{\Delta t} = \left[\Gamma\left(\frac{\phi_E - \phi_P}{\delta x}\right) - \Gamma\left(\frac{\phi_P - \phi_W}{\delta x}\right)\right] + \left(S_u + S_P\phi_P\right)}$ $\boxed{\rho\left(\phi_P - \phi_P^0\right)\frac{\delta x}{\Delta t} = \left[\Gamma\left(\frac{\phi_E^0 - \phi_P^0}{\delta x}\right) - \Gamma\left(\frac{\phi_P^0 - \phi_W^0}{\delta x}\right)\right] + \left(S_u + S_P\phi_P^0\right)}$

8. Matrix problem [Jacobi and Gauss-Seidel]:
(Scarborough criterion)
One condition for convergence: diagonally dominant

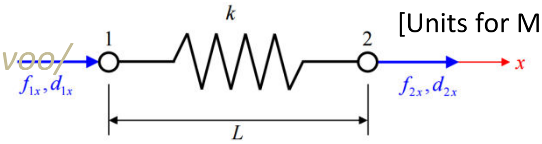
$\frac{\sum|a_{nb}|}{|a_P|} \leq 1$ for all nodes, < 1 for at least one node

...d to exploit these inherent differences of the error behaviour and use iterations on meshes of different size. The short-wavelength errors are effectively reduced on the finest meshes, whereas the long-wavelength errors decrease rapidly on the coarsest

Step 1: Fine grid iterations.
Step 2: Restriction
Step 3: Prolongation
Step 4: Correction and final iterations.

(B) FEM: [Units for Matrices, Preference of Global over Local matrices]

1. *Local*, **directions** defined (+ve ∀):



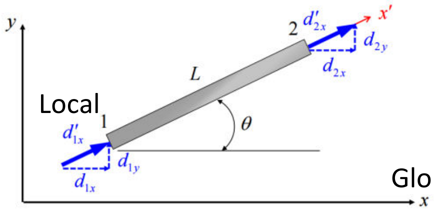
$$\begin{Bmatrix} f_{1x} \\ f_{2x} \end{Bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

2. Assemble spring elements from <u>local to global</u> (superposition) through **summing nodal forces** and <u>direct stiffness method</u>.

3. Some properties of stiffness matrix: (i) Determinant = 0 (Singular matrix); (ii) *Symmetric + banded* stiffness matrix.

4. Bar element: [<u>Plane</u> *truss*]

$$[\mathbf{k}] = \begin{bmatrix} k_b & -k_b \\ -k_b & k_b \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



$$\begin{Bmatrix} d'_{ix} \\ d'_{iy} \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} d_{ix} \\ d_{iy} \end{Bmatrix} \qquad \begin{Bmatrix} f'_{ix} \\ f'_{iy} \end{Bmatrix} = \begin{bmatrix} C & S \\ -S & C \end{bmatrix} \begin{Bmatrix} f_{ix} \\ f_{iy} \end{Bmatrix}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{2x} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} d'_{1x} \\ d'_{2x} \end{Bmatrix}$$

*Transformation of vectors in 2D [local (for individual) and global (whole structure) coordinates, for **displacements** and **forces**].

$$\begin{Bmatrix} d'_{1x} \\ d'_{1y} \\ d'_{2x} \\ d'_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \end{Bmatrix} \qquad \begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \begin{bmatrix} C & S & 0 & 0 \\ -S & C & 0 & 0 \\ 0 & 0 & C & S \\ 0 & 0 & -S & C \end{bmatrix} \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \end{Bmatrix}$$

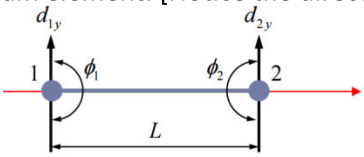
$$\{\mathbf{d}'\} = [\mathbf{T}]\{\mathbf{d}\} \qquad \{\mathbf{f}'\} = [\mathbf{T}]\{\mathbf{f}\}$$

$$\begin{Bmatrix} f'_{1x} \\ f'_{1y} \\ f'_{2x} \\ f'_{2y} \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} d'_{1x} \\ d'_{1y} \\ d'_{2x} \\ d'_{2y} \end{Bmatrix} \qquad [\mathbf{k}] = \frac{AE}{L}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}$$

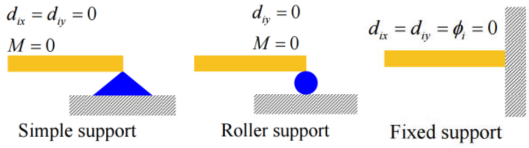
$$\{\mathbf{f}'\} = [\mathbf{k}']\{\mathbf{d}'\} \qquad [\mathbf{T}]\{\mathbf{f}\} = [\mathbf{k}'][\mathbf{T}]\{\mathbf{d}\} \qquad \{\mathbf{f}\} = [\mathbf{k}]\{\mathbf{d}\} \qquad [\mathbf{k}] = [\mathbf{T}]^T[\mathbf{k}'][\mathbf{T}]$$

*Assembly of bar elements in plane truss problem through <u>direct stiffness method</u>, using stiffness matrix of global coordinates.

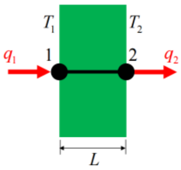
5. Beam element: [Notice the direction of *moment*, defined to be positive in <u>**counter-clockwise**</u> direction]



$$\begin{Bmatrix} f_{1y} \\ m_1 \\ f_{2y} \\ m_2 \end{Bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \end{Bmatrix}$$

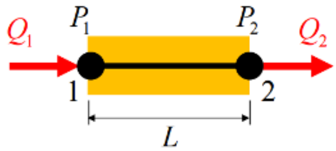


6. Heat conduction element (sign convention for in flux and out flux):



$$\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \frac{k_T A}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

7. Pipe (fluid flow) element (laminar, sign convention):



$$\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \frac{\pi D^4}{128\mu L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}$$

7. Principle of minimum potential energy: An **alternative method** used to derive the element equations and stiffness matrix. Of all the geometrically possible shapes that a body can assume, the true one, corresponding to the satisfaction of stable equilibrium of the body, is identified by a <u>minimum value</u> of the total potential energy.

$$\boxed{\Pi = U - \Omega}$$

internal strain energy U

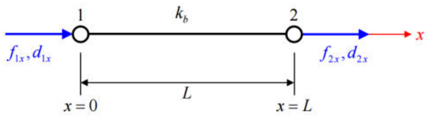
potential energy of external forces Ω

* Capacity of **internal forces** to do work through deformation.
* Capacity of forces such as **body forces, surface traction forces, and applied nodal forces** to do work through deformation of structure.

$$\delta\Pi = \frac{\partial\Pi}{\partial d_1}\delta d_1 + \frac{\partial\Pi}{\partial d_2}\delta d_2 + \ldots + \frac{\partial\Pi}{\partial d_n}\delta d_n = 0$$

$$\frac{\partial\Pi}{\partial d_i} = 0 \quad \text{or} \quad \frac{\partial\Pi}{\partial\{\mathbf{d}\}} = 0 \qquad \boxed{\delta\Pi = 0}$$

8. Using principle of minimum potential energy on <u>bar element</u>:



$$\boxed{U = \frac{A}{2}\int_0^L \sigma_x \varepsilon_x \, dx \qquad \Omega = f_{1x}d_{1x} + f_{2x}d_{2x}}$$

$$\varepsilon_x = \frac{du}{dx} \qquad \boxed{u = a_1 + a_2 x} \qquad u(x=0) = d_{1x} = a_1 \qquad u(x=L) = d_{2x} = a_1 + a_2 L$$

$$\{\varepsilon_x\} = \frac{d}{dx}\begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} \begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix} = [\mathbf{B}]\{\mathbf{d}\}$$

$$\{\sigma_x\} = [\mathbf{D}]\{\varepsilon_x\}$$

$$\Pi = \frac{A}{2}\int_0^L \{\sigma_x\}^T\{\varepsilon_x\}\,dx - \{\mathbf{d}\}^T\{\mathbf{f}\}$$

$$\Pi = \frac{A}{2}\int_0^L \{\mathbf{d}\}^T[\mathbf{B}]^T[\mathbf{D}]^T[\mathbf{B}]\{\mathbf{d}\}\,dx - \{\mathbf{d}\}^T\{\mathbf{f}\}$$

9. Elastic body in 2D Space: Potential energy approach

$$U = \frac{1}{2}\int_V \{\sigma\}^T\{\varepsilon\}\,dV \qquad \{\sigma\} = [\sigma_x \quad \sigma_y \quad \tau_{xy}]^T \qquad \boxed{dV = t\,dx\,dy} \qquad \{\varepsilon\} = [\varepsilon_x \quad \varepsilon_y \quad \gamma_{xy}]^T$$

$$\int_V \{\psi\}^T\{\mathbf{b}\}\,dV \qquad \{\mathbf{b}\} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$$ potential energy of body forces

$$\int_S \{\psi\}^T\{\mathbf{T}\}\,dS \qquad \{\mathbf{T}\} = \begin{Bmatrix} T_x \\ T_y \end{Bmatrix}$$ potential energy of surface forces

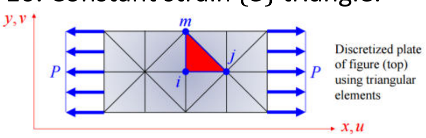
$$\{\psi\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = [\mathbf{N}]\{\mathbf{d}\}$$ $\{\psi\}$ displacement function

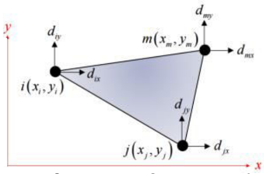
$$\{\psi\} = [\mathbf{N}]\{\mathbf{d}\} \qquad \{\varepsilon\} = [\mathbf{B}]\{\mathbf{d}\} \qquad \{\sigma\} = [\mathbf{D}][\mathbf{B}]\{\mathbf{d}\}$$

$$\boxed{\Pi = \frac{1}{2}\{\mathbf{d}\}^T\int_V [\mathbf{B}]^T[\mathbf{D}]^T[\mathbf{B}]\,dV\{\mathbf{d}\} - \{\mathbf{d}\}^T\int_V [\mathbf{N}]^T\{\mathbf{b}\}\,dV - \{\mathbf{d}\}^T\int_S [\mathbf{N}]^T\{\mathbf{T}\}\,dS}$$

$$\boxed{\frac{\partial\Pi}{\partial\{\mathbf{d}\}} = \left[\int_V [\mathbf{B}]^T[\mathbf{D}]^T[\mathbf{B}]\,dV\right]\{\mathbf{d}\} - \{\mathbf{f}\} = 0} \qquad \boxed{\{\mathbf{f}\} = \left[\int_V [\mathbf{B}]^T[\mathbf{D}]^T[\mathbf{B}]\,dV\right]\{\mathbf{d}\}}$$

10. Constant strain $\{\boldsymbol{\varepsilon}\}$ triangle:





$$\{\mathbf{d}\} = \begin{Bmatrix} \mathbf{d_i} \\ \mathbf{d_j} \\ \mathbf{d_m} \end{Bmatrix}$$

Using $\underline{u}$ as example, **plug** boundary conditions for displacement functions: $\begin{Bmatrix} d_{ix} \\ d_{jx} \\ d_{mx} \end{Bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$

$$\{\psi\} = \begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix} \qquad u = N_i d_{ix} + N_j d_{jx} + N_m d_{mx} \qquad v = N_i d_{iy} + N_j d_{jy} + N_m d_{my}$$

$$\{\mathbf{f}_b\} = \int_V [\mathbf{N}]^T\{\mathbf{b}\}\,dV \qquad \{\mathbf{f}_b\} = \frac{At}{3}\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} \qquad \{\mathbf{f}_S\} = \int_S [\mathbf{N}]^T\{\mathbf{T}\}\,dS \qquad dS = t\,ds$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix}$$

$*\{\varepsilon\} = [\mathbf{B}]\{\mathbf{d}\}$

11. <u>Shape function</u> allows evaluation of <u>**equivalent**</u> <u>nodal force</u> (of distributed load) for FE analysis, as **only nodal forces can be applied in FE**. The goal is to ensure equivalent load system has (*approx.*) the same strain energy as the distributed load system.

12. Rayleigh-Ritz Method, an approximate method (to *finite* DOFs) for <u>***continuous***</u> **system** with *infinite DOFs* (<u>poly. disp. function</u>).

$$U = \frac{A}{2}\int_0^L \sigma_x \varepsilon_x \, dx \qquad \boxed{U = \frac{A}{2}\int_0^L (E\varepsilon)\varepsilon\,dx = \frac{AE}{2}\int_0^L \left(\frac{du}{dx}\right)^2 dx}$$

*Unknown **coefficient** (Minimum potential energy),

*__BCs__ instead of nodal displacements

$$u = a_1 + a_2 x + a_3 x^2 \qquad \boxed{\frac{\partial\Pi}{\partial a_3} = 0}$$

13. Method of Weighted Residuals: (Solving BVPs, approximate/trial solution)

$$\tilde{u}(x) = c_1\phi_1(x) + c_2\phi_2(x) + c_3\phi_3(x) + \ldots$$

c_i are unknown coefficients

$\phi_i(x)$ are called trial functions

$$\int_{x=0}^{x=L} W_i(x)R(x)\,dx = 0 \;\Rightarrow\; F_i(c_1, c_2, c_3, \ldots)$$

weight functions $W_i(x)$ solved for the coefficients c_i

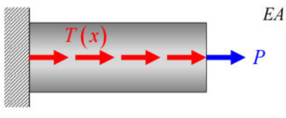
trial solutions

Have continuous (non-zero) derivatives up to the maximum order of derivatives appearing in the differential equation

Satisfy the boundary conditions of the problem

Galerkin's method : trial functions $\phi_i(x) = W_i(x)$

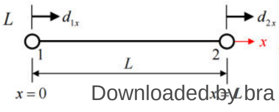
Bar Element: Galerkin's MWR



$$EA\frac{d^2u}{dx^2} + T(x) = 0, \quad 0 < x < L$$

$$u = 0, \quad \text{at } x = 0$$

$$EA\frac{du}{dx} = P, \quad \text{at } x = L$$



$$\begin{Bmatrix} R + \int_{x_1=0}^{x_2=L} N_1(x)T(x)\,dx \\ P + \int_{x_1=0}^{x_2=L} N_2(x)T(x)\,dx \end{Bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} d_{1x} \\ d_{2x} \end{Bmatrix}$$

$$\boxed{N_1 = 1 - \frac{x}{L} \qquad N_2 = \frac{x}{L}}$$

$*\tilde{u}(x) = N_1(x)d_{1x} + N_2(x)d_{2x}$

plane stress

$$[\mathbf{D}] = \frac{E}{1-\upsilon^2}\begin{bmatrix} 1 & \upsilon & 0 \\ \upsilon & 1 & 0 \\ 0 & 0 & \frac{1-\upsilon}{2} \end{bmatrix}$$

Extra for **CST (*10*)**

*Bar element

<u>8 = 13</u>

plane strain

$$[\mathbf{D}] = \frac{E}{(1+\upsilon)(1-2\upsilon)}\begin{bmatrix} 1-\upsilon & \upsilon & 0 \\ \upsilon & 1-\upsilon & 0 \\ 0 & 0 & \frac{1-2\upsilon}{2} \end{bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix}\begin{Bmatrix} d_{ix} \\ d_{jx} \\ d_{mx} \end{Bmatrix}$$

**1, 2, 3 -> 4, 5, 6*

**x -> y*

$$\alpha_i = x_j y_m - y_j x_m \qquad \alpha_j = y_i x_m - x_i y_m \qquad \alpha_m = x_i y_j - y_i x_j$$

$$\beta_i = y_j - y_m \qquad \beta_j = y_m - y_i \qquad \beta_m = y_i - y_j$$

$$\gamma_i = x_m - x_j \qquad \gamma_j = x_i - x_m \qquad \gamma_m = x_j - x_i$$