NANYANG TECHNOLOGICAL UNIVERSITY

# MA2011 MECHATRONICS SYSTEMS INTERFACING

## Tutorial 3

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**School of Mechanical and Aerospace Engineering**

# ASSESSMENT STRUCTURE

**Assessment (includes both continuous and summative assessment)**

| Component | Course LO Tested | Related Programme LO or Graduate Attributes | Weighting | Team/ Individual | Assessment rubrics |
|---|---|---|---|---|---|
| 1. Continuous Assessment 1 – Team Project Presentation | LO1-3 | *EAB SLO a, b, d, i* | 20% | Team | Appendix 1 |
| 2. Continuous Assessment 2 – Quiz 2 | LO4 | *EAB SLO a, b, d* | 20% | Individual | |
| 3. Final Examination – Restricted Open Book; 2.5hrs | LO1-4 | *EAB SLO a, b, c, d, e* | 60% | Individual | |
| Total | | | 100% | | |

# LINEAR SYSTEMS

Linear systems are of the form

$$\sum_{n=0}^{N} A_n \frac{d^n X_{out}}{dt^n} = \sum_{m=0}^{M} B_m \frac{d^m X_{in}}{dt^m}$$

where $X_{in}$ and $X_{out}$ are input and output variables, $A_n$ and $B_m$ are coefficients, $N$ is the order of the system.

# Q1

Which of the following is correct for the Characteristic Equations of the Linear System?

1) $$\sum_{n=1}^{N} A_n s^n = 0$$

2) $$\sum_{n=1}^{N} A_n s^n = 1$$

3) $$\sum_{n=0}^{N} A_n s^n = 0$$

4) $$\sum_{n=0}^{N} A_n s^n = 1$$

# ANSWER TO Q1

Which of the following is correct for the Characteristic Equations of the Linear System?

1) $$\sum_{n=1}^{N} A_n s^n = 0$$

2) $$\sum_{n=1}^{N} A_n s^n = 1$$

3)√ $$\sum_{n=0}^{N} A_n s^n = 0$$

4) $$\sum_{n=0}^{N} A_n s^n = 1$$

# Quadratics Equation Solving

$$ax^2 + bx + c = 0 \quad (a \neq 0)$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (\Delta = b^2 - 4ac \geq 0)$$

# Cubic or Quartic Equation Solving

Cubic equation
Quartic equation

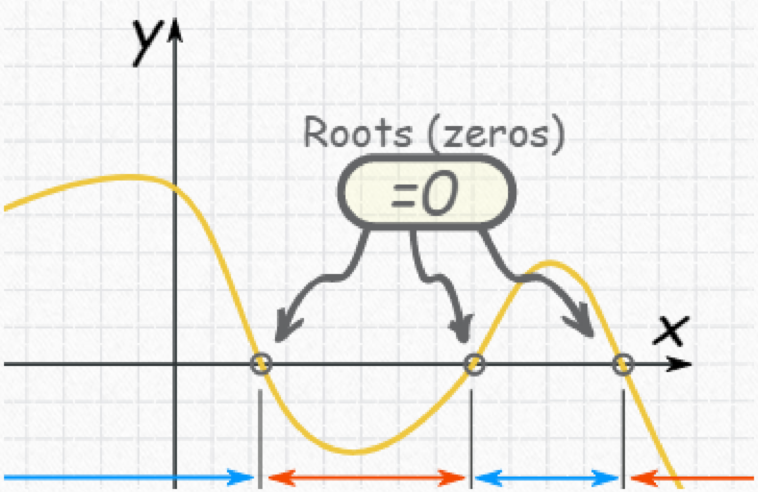
Explicit Form of Solution Available

# High Order Equation Solving

Quintic equation
High degree (N>5) equation

Galois Group Theory

$$f(s) = \sum_{n=0}^{N} A_n s^n, \quad A_N \neq 0$$



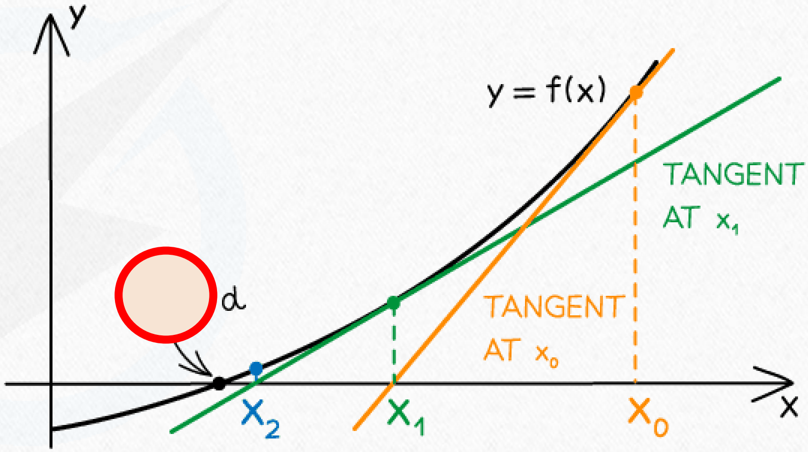

$f(s) = 0$

No Explicit Solution Available Normally, and Numerical Approach may be attempted

# Newton-Raphson Method

NEWTON–RAPHSON FORMULA

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



# STEP BY STEP LINEAR SYSTEMS SOLVING

**1: Look into the homogenous equation of Linear system:**

$$\sum_{n=0}^{N} A_n \frac{d^n X_{out}}{dt^n} = 0$$

**2: Find roots for the characteristic equation of Homogenous equation of Linear system**

$$\sum_{n=0}^{N} A_n S^n = 0$$

**3: Find homogenous or transit solutions $X_{out_g}$ of Homogenous System:**

$$\sum_{n=0}^{N} A_n \frac{d^n X_{out}}{dt^n} = 0$$

**4: Find a particular solution $X_{out_p}$ of Linear system**

$$\sum_{n=0}^{N} A_n \frac{d^n X_{out}}{dt^n} = \sum_{m=0}^{M} B_m \frac{d^m X_{in}}{dt^m}$$

**5: Find general solutions of Linear system**

$$X_{out} = X_{out_g} + X_{out_p}$$

# Q2

During a step function calibration, a first-order instrument is exposed to a step change of 100 units. If after 1.2 s the instrument indicates 80 units, estimate the instrument time constant. Estimate the error in the indicated value after 1.5 s. Assume $X_{out}(0) =$ 0 units and $K = 1$ unit/unit.

c $\tau = 0.75$ s; error at 1.5 s = 13.4 units

1st Order System $\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$

1: $\tau \frac{dX_{out}}{dt} + X_{out} = 0$

The Characteristic Equation of The Homogenous Equation of the 1st Order System

2: $$\tau s + 1 = 0$$

Step input

Since the root of this equation is $s = -1/\tau$, the **homogeneous** or **transient solution** is

3: $$X_{out_h} = C e^{-t/\tau}$$

where $C$ is a constant determined later by applying initial conditions. A **particular** or **steady state solution** resulting from the step input is

4: $$X_{out_p} = KA_{in}$$

The **general solution** is the sum of the homogeneous and particular solutions

5: $$X_{out}(t) = X_{out_h} + X_{out_p} = C e^{-t/\tau} + KA_{in}$$

## 5: Determine the coefficients using initial condition

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

$$X_{out}(t) = X_{out_h} + X_{out_p} = Ce^{-t/\tau} + KA_{in}$$

Applying the initial condition $X_{out}(0) = 0$ to this equation gives

$$0 = C + KA_{in}$$

thus,

$$C = -KA_{in}$$

so the resulting step response is

$$X_{out}(t) = KA_{in}(1 - e^{-t/\tau})$$

## 5: Determine the coefficients using initial condition (continue)

**Step input**

$$\tau \frac{dX_{out}}{dt} + X_{out} = KX_{in}$$

$$X_{out}(t) = X_{out_{tr}} + X_{out_{ss}} = Ce^{-t/\tau} + KA_{in}$$

$$X_{out}(t) = KA_{in}(1 - e^{-t/\tau})$$

**Given** $X_{out}(0) = 0$

SOLUTION:

A first order system subjected to a step function $A_{in}U_1(t)$ can be modelled as:

$$a_1 \frac{dX_{out}}{dt} + a_o X_{out} = b_o X_{in}$$

The solution is given as

$$X_{out}(t) = KA_{in} + (X_{out}(0) - KA_{in})e^{-t/\tau}$$

Substituting known variables gives:

$$X_{out}(t) = 100 + (0 - 100)e^{-t/\tau}$$

At $t = 1.2$ s, we have $X_{out}(1.2) = 80 = 100 - 100e^{-1.2/\tau}$.

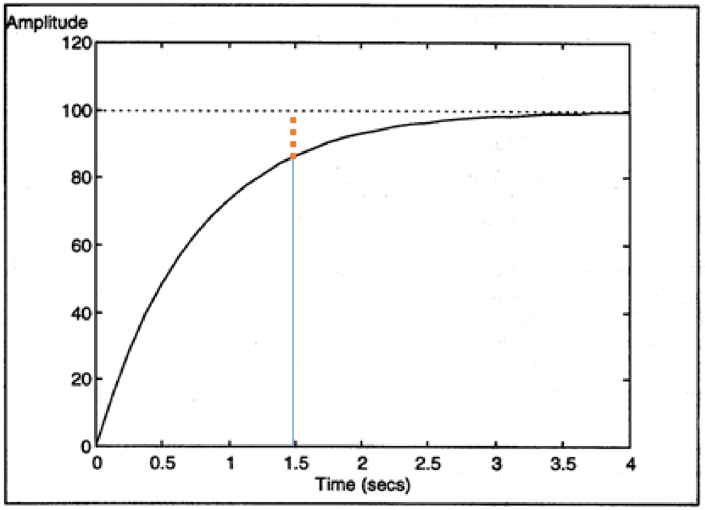
**Analysis:**

Thus, $e^{-1.2/\tau} = \dfrac{20}{100}$

$$\frac{-1.2}{\tau} = \ln(0.2) = -1.61$$

$$\therefore\ \tau \approx 0.75$$

At $t = 1.5$ s, error $= 100 - X_{out}(1.5) = 100e^{-1.5/0.75} = 13.4$ units.




Step response of system

Note that two parameters, i.e. $\tau$ and K, are needed to characterise the first order system. $\tau$ and K are system variables, and are **not** dependent on the input.

# Q3

A first-order instrument with a time constant of 2 s is to be used to measure a periodic input. If a dynamic error of 2% can be tolerated, determine the maximum frequency of a periodic input that can be measured.

($\omega_{max} = 0.1$ rad s$^{-1}$)

# ANSWER TO Q3

1. Magnitude Ratio or Normalized Amplitude Ratio: $\boldsymbol{M(\omega)}(Detail, see\ appendix)$

$$M(\omega) = \frac{1}{\sqrt{1+(\omega\tau)^2}}$$

2. Dynamic Error: $\boldsymbol{M(\omega)}$

$$\delta(\omega) = 1 - M(\omega)$$

3. Solution (next page)

# ANSWER TO Q3

For a first order system, the output of the system always lags (follows behind) the input and $M(\omega)$ must always be less than 1. So, for $\delta(\omega) \leq 0.02$, we have

$$1 \geq M(\omega) \geq 0.98$$

$$M(\omega) = \frac{1}{\sqrt{1+(\omega\tau)^2}}$$

Hence, $1 \geq \dfrac{1}{\sqrt{1+(\omega\tau)^2}} \geq 0.98$

i.e. $\dfrac{1}{0.98} \geq \sqrt{1+(\omega\tau)^2} \geq 1$

At $\tau = 2$ s, we find that $\boxed{0.1 \geq \omega \geq 0 \text{ rad s}^{-1}}$

So, $\omega_{max} = 0.1$ rad s$^{-1}$ or $f_{max} = 0.016$ Hz.

## Sinusoidal Forcing of a First-Order Process

For a first-order transfer function with gain $K$ and time constant $\tau$, the response to a general sinusoidal input, $x(t) = A\sin\omega t$ is:

$$y(t) = \frac{KA}{\omega^2\tau^2 + 1}\left(\omega\tau e^{-t/\tau} - \omega\tau\cos\omega t + \sin\omega t\right) \qquad (5\text{-}25)$$

Note that *y(t)* and *x(t)* are in deviation form. The *long-time response*, $y_l(t)$, can be written as:

$$y_\ell(t) = \frac{KA}{\sqrt{\omega^2\tau^2 + 1}}\sin(\omega t + \varphi) \text{ for } t \to \infty \qquad (13\text{-}1)$$

where:

$$\varphi = -\tan^{-1}(\omega\tau)$$

# ANSWER TO Q3 (APPENDIX)

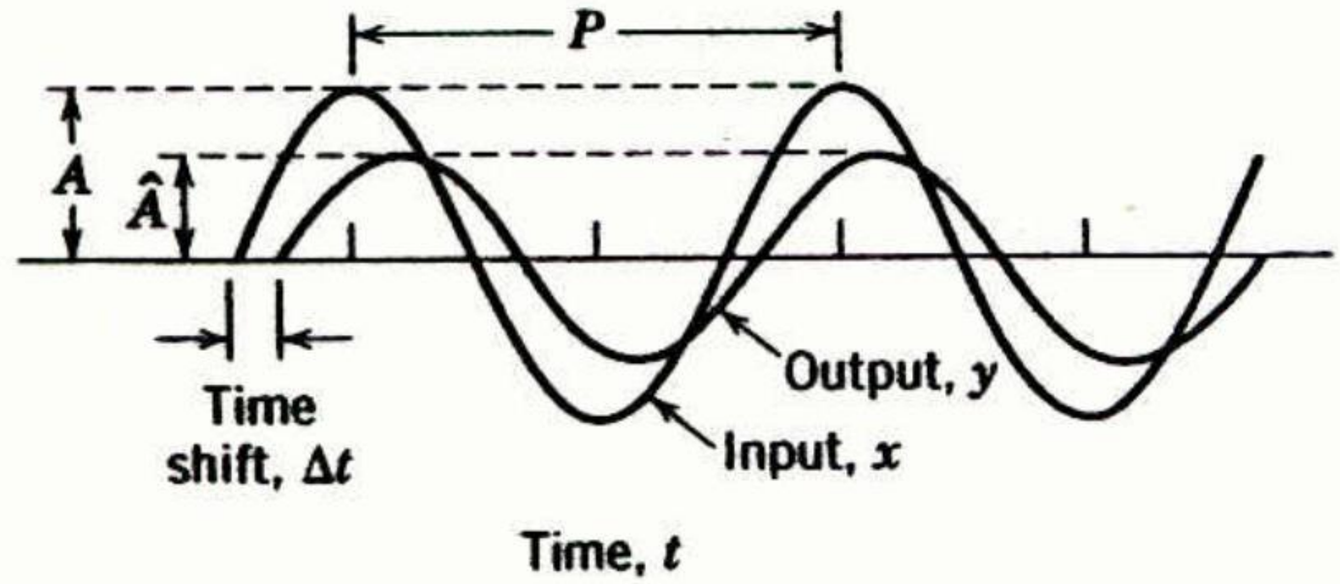



Figure 13.1 Attenuation and time shift between input and output sine waves ($K$= 1). The phase angle $\varphi$ of the output signal is given by $\varphi = -\text{Time shift} / P \times 360^{\circ}$, where $\Delta t$ is the (period) shift and $P$ is the period of oscillation.

# Frequency Response Characteristics of a First-Order Process

For $x(t) = A\sin\omega t,\ y_\ell(t) = \hat{A}\sin(\omega t + \varphi)$ as $t \to \infty$ where:

$$\hat{A} = \frac{KA}{\sqrt{\omega^2\tau^2 + 1}} \quad \text{and} \quad \varphi = -\tan^{-1}(\omega\tau)$$

1. The output signal is a sinusoid that has the same frequency, ω, as the input.signal, $x(t) = A\sin\omega t$.

2. The amplitude of the output signal, $\hat{A}$, is a function of the frequency ω and the input amplitude, $A$:

$$\hat{A} = \frac{KA}{\sqrt{\omega^2\tau^2 + 1}} \qquad (13\text{-}2)$$

3. The output has a phase shift, φ, relative to the input. The

# ANSWER TO Q3 (APPENDIX)

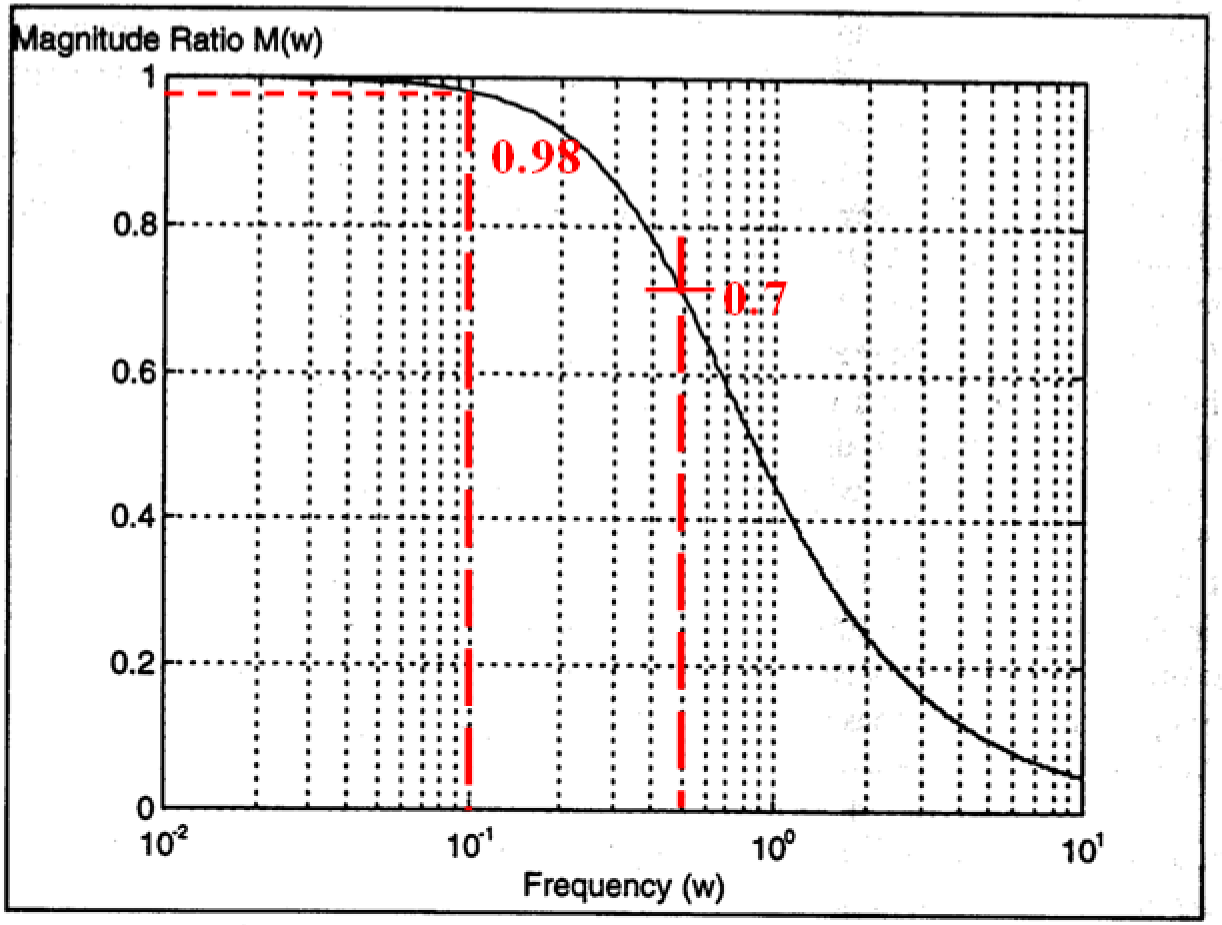
Dividing both sides of (13-2) by the input signal amplitude $A$ yields the *amplitude ratio* (AR)

$$\mathrm{AR} = \frac{\hat{A}}{A} = \frac{K}{\sqrt{\omega^2 \tau^2 + 1}} \qquad \text{(13-3a)}$$

which can, in turn, be divided by the process gain to yield the *normalized amplitude ratio* ($\mathrm{AR_N}$)

$$\mathrm{AR_N} = \frac{1}{\sqrt{\omega^2 \tau^2 + 1}} \qquad \text{(13-3b)}$$

# ANSWER TO Q3

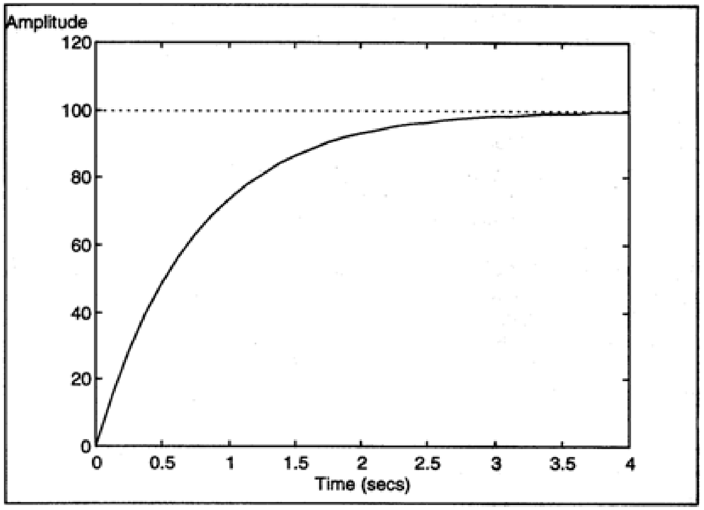



Magnitude ratio plot of system

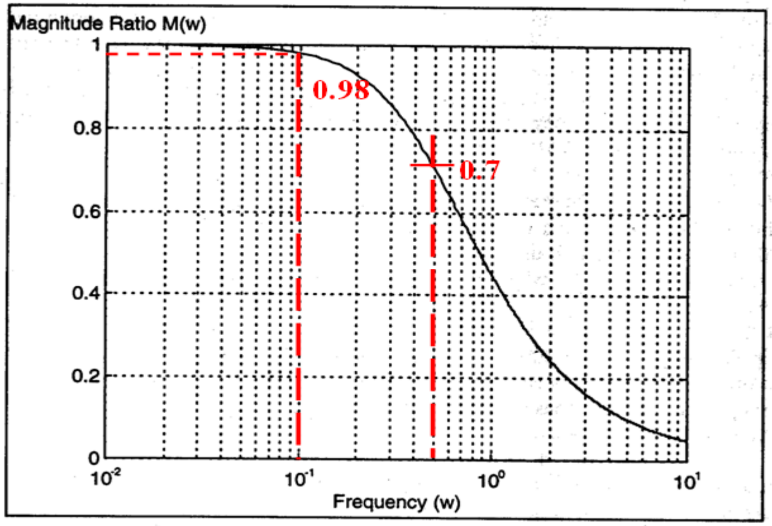
**Q2.** looks at the time response of a system, i.e. in the time domain.

**Q3.** looks at its frequency response, i.e. in the frequency domain. Both are often needed, especially for a more complicated system, to have a good understanding of the system.

**Plots.**




Step response of system




Magnitude ratio plot of system

# Q4

The output from a temperature system indicates a steady, time-varying signal having amplitude which varies between 30 and 40 °C, with a single frequency of 10 Hz. Express the output signal as a waveform equation, $T(t)$. If the dynamic error is to be less than 1%, what must the system time constant be? Assume that the sensitivity $K = 1$ and system is of first order.

($T(t) = 35 + 5\sin(20\pi t \pm \phi)$; $\tau \leq 2.27$ ms)

SOLUTION:

Amplitude varies between 30°C and 40°C, and frequency is 10 Hz.

1 Assume sensitivity $K = 1$ and system is first order.

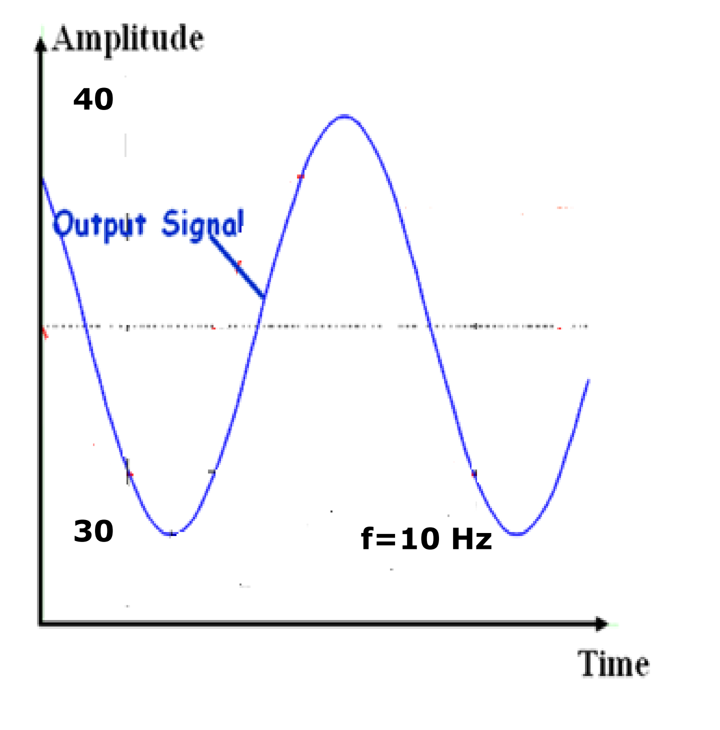
Output signal is $T(t) = \frac{40+30}{2} + \frac{40-30}{2}\sin(2\pi ft \pm \phi)$

$$= 35 + 5\ \sin(20\pi t \pm \phi)$$

2 $\phi$ is the unknown phase shift between input and output signal. For a 1st order system, the maximum phase shift is $\pi/2$ radians.

Dynamic error is $\delta(\omega) = 1 - M(\omega)$ and $M(\omega) = \frac{1}{\sqrt{1+(\omega\tau)^2}}$

And we know that $\delta \leq 0.01$, hence



# ANSWER TO Q4

$$1 \geq \frac{1}{\sqrt{1+(\omega\tau)^2}} \geq 0.99$$

At $\omega = 2\pi f = 20\pi$ rad s$^{-1}$, we find that

$$0 \leq \tau \leq 2.27 \text{ ms}$$