MA3005 Reference

Control Theory (Nanyang Technological University)

1. Final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$. Poles of $F(s)$ lie in left half s-plane and at most 1 pole on imaginary axis. [Steady state behaviour]
2. Initial value theorem: $f(0^+) = \lim_{s\to\infty} sF(s)$. No restriction on poles location, applicable for sinusoidal functions.
3. Linear System: Principle of superposition is applicable;

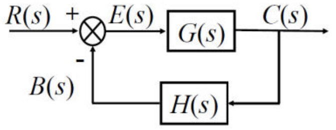
   Time-Invariant System: If, $u(t) \longrightarrow \boxed{G(s)} \longrightarrow y(t)$ Then, $u(t-\tau) \longrightarrow \boxed{G(s)} \longrightarrow y(t-\tau)$

4. Consider TF: $G(s) = \dfrac{(s+z_1)(s+z_2)}{s^2(s+p_1)(s+p_2)^3}$

   - $s = -p_1$ is a *simple* pole when $n = 1$ (power)
   - $s = 0$ is a pole of *multiplicity* 2
   - $s = -p_2$ is a pole of *multiplicity* 3

   $G(s) = 0$ at $-z_1$ and $-z_2$ (finite zeros)

   there are 4 infinite zeros

5. 

   Open-Loop TF $\dfrac{B(s)}{E(s)} = G(s)H(s)$

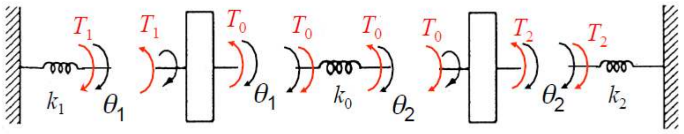
   Forward Loop TF $\dfrac{C(s)}{E(s)} = G(s)$

   For *negative* (*positive*) feedback system, (+ / -)

   $$\text{CLTF} = \frac{\text{Forward loop TF}}{1 \pm \text{Open loop TF}}$$

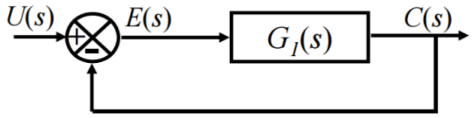
   For *unity* feedback system,

   $$\text{CLTF} = \frac{\text{Forward loop TF}}{1 + \text{Forward loop TF}}$$

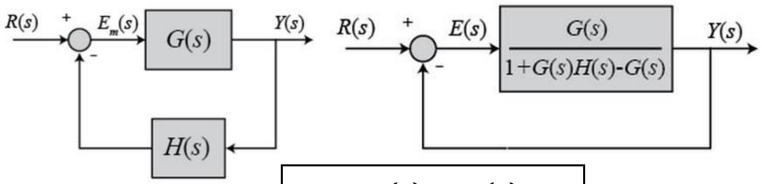
6. 

   Notice directions of forces and displacements; assumptions relating displacements

7. System error [Identify **desired** input and output]:



$$E(s) = U(s) - C(s) \qquad e_{ss}(t) = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\cdot E(s)$$



$$\boxed{E_m(s) \neq E(s)}$$

8. Natural response:

Stability, impulse input

(a) Stable

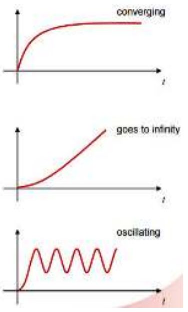
A system is called stable, if the output of the system converges to a particular value/state for a constant input.

(b) Unstable

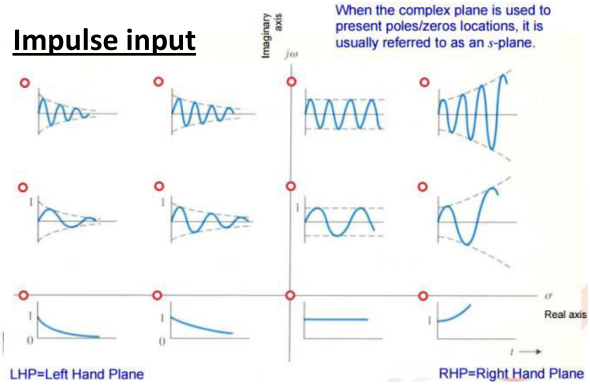
When a system is unstable, the output of the system goes to infinity for finite input (or constant input).

(c) Marginally Stable

A marginally stable system will have continues oscillatory output for a constant input.



Impulse input



When the complex plane is used to present poles/zeros locations, it is usually referred to as an s-plane.

Pole position & transient

9. Routh-Hurwitz (Closed Loop CE):

C.E.: $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = 0$

$a_0, a_1, \ldots, a_{n-1}, a_n > 0$ (or < 0)

None of the coefficient is missing

s^n	a_0	a_2	a_4	…
s^{n-1}	a_1	a_3	a_5	…
s^{n-2}	b_1	b_2	b_3	…
s^{n-3}	c_1	c_2	c_3	…
⋮	⋮	…	…	
s^1	α	0		
s^0	β	0		

From Characteristic Eqn.

Need to be computed

$$b_1 = \frac{a_1\cdot a_2 - a_0\cdot a_3}{a_1}, \quad b_2 = \frac{a_1\cdot a_4 - a_0\cdot a_5}{a_1}$$

$$c_1 = \frac{b_1\cdot a_3 - a_1\cdot b_2}{b_1}, \quad c_2 = \frac{b_1\cdot a_5 - a_1\cdot b_3}{b_1}$$

Refer to the 1st column:

- If there is no changing of sign ⇒ closed loop system is stable
- How many times the sign changes in the 1st column represents the number of poles in RHP

Special case

- If 1st element in any row is zero, but others are non-zero, replace 0 with ε > 0 and proceed as before
- If sign above zero (ε) is the same as below, it indicates that there are a pair of imaginary roots
- If sign above zero (ε) is opposite to that of below it, it indicates that there is a sign change ⇒ unstable system

Occurrence of entire row of zeros

i.e. $(-\sigma, +\sigma)$, or $(-j\omega, +j\omega)$ or $(-\sigma \pm j\omega, +\sigma \pm j\omega)$

Solution: Form auxiliary polynomial from the row immediately above all zeros row

10. $\dfrac{C(s)}{R(s)} = \dfrac{K}{Ts+1} = \dfrac{K/T}{s+1/T}$

   T is time constant

$$G(s) = \frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

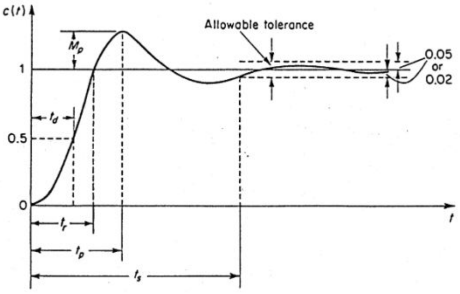
$0 < \zeta < 1$

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

$$\boxed{d^2(ramp) = d(step) = impulse}$$

Lower order approx.: Same gain + dominant pole

$$\frac{1000}{(s+20)(s^2+6s+25)} \approx \frac{50}{(s^2+6s+25)}$$

11. 

2nd order system with 0 < ζ < 1

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

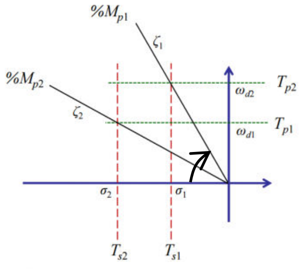
$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

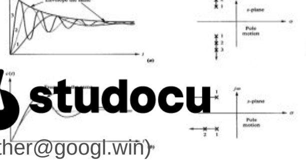
$$\%M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \times 100\%$$

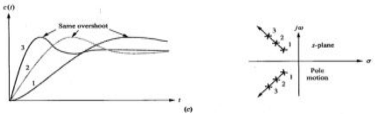
$$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n} \quad (2\%\ \text{criterion})$$

$$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n} \quad (5\%\ \text{criterion})$$

$$\zeta = \cos\beta \qquad \zeta = \frac{-\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}}$$







12.

Effects of *increasing* a parameter independently[14]

Parameter	Rise time	Overshoot	Settling time	Steady-state error	Stability[11]
K_p	Decrease	Increase	Small change	Decrease	Degrade
K_i	Decrease	Increase	Increase	Eliminate	Degrade
K_d	Minor change	Decrease	Decrease	No effect in theory	Improve if K_d small

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt + K_p T_d \frac{de(t)}{dt}$$

$$\frac{U(s)}{E(s)} = K_p\left(1 + \frac{1}{T_i s} + T_d s\right)$$

PI: eliminate steady state error.

PD: anticipate error to **increase damping**, improve transient + stability.

13.

Unity-feedback system

	Desired e_{ss}	Type 0		Type 1		Type 2	
Input	Steady-state error formula	Static error constant	Error	Static error constant	Error	Static error constant	Error
Step, $u(t) = 1$	$\frac{1}{1+K_p}$	$K_p =$ Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $u(t) = t$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v =$ Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $u(t) = \frac{1}{2}t^2$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a =$ Constant	$\frac{1}{K_a}$

$$K_p = \lim_{s\to 0} G(s) = G(0)$$

$$K_v = \lim_{s\to 0} sG(s)$$

$$K_a = \lim_{s\to 0} s^2 G(s)$$

$G(s)$ **Open loop** transfer function

Fix type 0, 1 or 2

14. Sketching of root locus and its procedure: $KN(s) + D(s) = 0$

Compute the poles (roots of D) and the zeros (roots of N)

$n_{\text{poles}} - n_{\text{zeros}}$ $\quad \sigma_a = \dfrac{\sum \text{poles} - \sum \text{zeros}}{n_{\text{poles}} - n_{\text{zeros}}}$

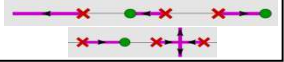
For each complex pole (resp. zero), compute the departure (resp. arrival) angle by

$$\alpha_{\text{dep}}(p) = \pi - \sum_{\text{poles } p_i} \alpha_{p_i \to p} + \sum_{\text{zeros } z_i} \alpha_{z_i \to p} \qquad \alpha_{\text{arr}}(z) = \pi + \sum_{\text{poles } p_i} \alpha_{p_i \to z} - \sum_{\text{zeros } z_i} \alpha_{z_i \to z}$$

break-in/break-out points $N'(s)D(s) - D'(s)N(s) = 0$

crossing of the imaginary axis $KN(j\omega) + D(j\omega) = 0$

"Every point of the real axis that is on the left of an odd number of poles/zeros belongs to the root locus."



$$\angle\frac{D(s_{\text{desired}})}{N(s_{\text{desired}})} = \angle(-K) = \pi \qquad K = \frac{\|D(s_{\text{desired}})\|}{\|N(s_{\text{desired}})\|}$$

Validity of 2nd order approximation for a particular value of ***K*** (CE & CLTF):

- non-dominant poles are far to the left;
- zeros are far to the left;
- zeros that are not far to the left are cancelled by poles

15. [Reshaping root locus] PD controller & Lead Compensator (improve transient behaviour by pass through s_{desired}): $t_s, M_p \downarrow$

$z_c < z_c^*$

PD: $K_D\left(s + \dfrac{K_P}{K_D}\right) = K(s + z_c)$ $\quad \angle(s_{\text{desired}} + z_c) = \angle D(s_{\text{desired}}) - \angle N(s_{\text{desired}}) - \pi$ $\quad K = \dfrac{\|D(s_{\text{desired}})\|}{\|s_{\text{desired}} + z_c\| \cdot \|N(s_{\text{desired}})\|}$

Lead: $K\dfrac{s+z_c}{s+p_c}$ $\quad \angle(s_{\text{desired}} + z_c) - \angle(s_{\text{desired}} + p_c) = \angle D(s_{\text{desired}}) - \angle N(s_{\text{desired}}) - \pi$ $\quad K = \dfrac{\|s_{\text{desired}} + p_c\| \cdot \|D(s_{\text{desired}})\|}{\|s_{\text{desired}} + z_c\| \cdot \|N(s_{\text{desired}})\|}$

PD controller = ideal lead compensator; PD controllers cannot be realized by passive circuits and are sensitive to noises; PD controller yield best steady-state behaviour. Decreasing ***z*$_c$** degrades performance of **lead compensators**. [e_{ss} increases (at same Q point), slower transient]

16. [Reshaping root locus] PI controller & Lag Compensator (reduce steady state error, yet no major change in root locus):

PI: $K_P + \dfrac{K_I}{s} = \dfrac{K(s + z_c)}{s}$ $\quad$ z_c is very small; transient behavior unchanged $\quad$ increases the type of a given system by 1 $\quad$ eliminates steady-state error

Lag: $K\dfrac{s+z_c}{s+p_c}$ $\quad$ z_c and p_c are very small close to zero; $p_c \ll z_c$ pole/zero cancellation $\quad$ steady-state error has been reduced by a factor $\frac{p_c}{z_c}$

K value kept the same before and after compensated, small tolerance on **desired roots**.

17. PID controller & Lead-Lag Compensator: forward transfer function $G(s)$

$$K_2\frac{s + z_{c2}}{s}K_1(s + z_{c1})G(s) \qquad K_2\frac{s + z_{c2}}{s + p_{c2}}K_1\frac{s + z_{c1}}{s + p_{c1}}G(s)$$

transients specifications

steady-state specifications

18. The steady-state response of a stable linear system with transfer function $G(s)$ to a sinusoidal input of the form $\sin(\omega t)$

$$\boxed{c_{ss}(t) = \|G(j\omega)\| \sin(\omega t + \angle G(j\omega))}$$

19. $20\log(\|G(j\omega)\|)$ Y-axis $\quad x = \log(\omega) \quad y = ax + b$ a straight line

corner frequency

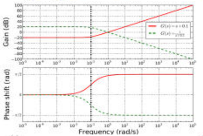
two asymptotes meet when $\omega = a$

Derivative $G(s) = s$ $\quad$ a straight line with slope +20 $\quad \angle G(j\omega) = \angle j = \pi/2$ $\quad$ **integral** $G(s) = 1/s$

First-order terms $G(s) = s + a$

$\omega \to 0,\ 20\log(\|j\omega + a\|) \simeq 20\log(a) \quad \angle(j\omega + a) \simeq \angle a = 0$

$\omega \to \infty,\ 20\log(\|j\omega + a\|) \simeq 20\log(\|j\omega\|) = 20\log(\omega) \quad \angle(j\omega + a) \simeq \angle j\omega = \pi/2$

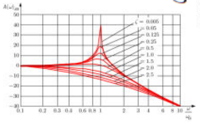


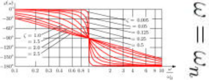
Second-order $\quad G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad G(j\omega) = \dfrac{1}{1 - \frac{\omega^2}{\omega_n^2} + 2j\zeta\frac{\omega}{\omega_n}}$

$\omega \to 0,\ G(j\omega) \to 1 \quad y = 0$

$\omega \to \infty,\ G(j\omega) \simeq -\frac{\omega_n^2}{\omega^2} \quad 20\log(\|G(j\omega)\|) \simeq 40(\log(\omega_n) - \log(\omega)) \quad y = -\pi$

ζ has no influence on the asymptotes $\;$ affects the transition around the corner frequency $\boxed{0.707}$





$\omega = \omega_n$

20. Bode plots of complex functions (Forward Loop TF, *not closed loop*):

Left part $\omega \simeq 0$ to $\omega = 0.1\omega_{\min}$	Middle part **Adjust* $\pm 360^0$ $\omega = 0.1\omega_{\min}$ to $\omega = 10\omega_{\max}$	Right part $\omega = 10\omega_{\max}$ to $\omega \simeq \infty$
plots will be represented by their left asymptotes	sampling and numerical evaluations sample more densely around corner frequencies	plots will be represented by their right asymptotes

asymptotes

$\omega \to 0$: system is of type k

$$G(s) = \frac{N(s)}{s^k D(s)} \qquad \angle G(j\omega) \simeq \angle(N(0)/D(0)) - k\angle(j\omega)$$

$N(0)/D(0) > 0 \quad y = -k\pi/2$

$N(0)/D(0) < 0 \quad y = \pi - k\pi/2$

$$20\log(\|G(j\omega)\|) \simeq 20\log\left(\frac{|N(0)|}{\omega^k |D(0)|}\right) = 20\log\left(\frac{|N(0)|}{|D(0)|}\right) - 20k\log(\omega)$$

$\omega \to \infty$: system is of order n

$$\angle G(j\omega) \simeq \angle A - n\angle(j\omega)$$

$A > 0 \quad y = -n\pi/2$

$A < 0 \quad y = \pi - n\pi/2$

$$G(s) = \frac{N(s)}{D(s)}, \quad \deg(D) - \deg(N) = n$$

$$20\log(\|G(j\omega)\|) \simeq 20\log\left(\frac{|A|}{\omega^n}\right) = 20\log|A| - 20n\log(\omega)$$

A is the ratio of the leading coefficient of N to D

21. Bode plot analysis:

Left asymptote's gradient for system type, **right asymptote's** gradient for system order.

Type-0 $\quad 20\log(|G(0)|) = 20\log K_p \quad K_p = 10^{\frac{y_{\text{asymptote}}}{20}} \quad e_{ss} = \dfrac{1}{1+K_p}$

Type-1 $\quad G(s) = \dfrac{N(s)}{sD(s)} \quad K_v = \dfrac{|N(0)|}{|D(0)|} \quad (x, y) = (0, 20\log(|N(0)/D(0)|)) \quad K_v = 10^{\frac{y_{\text{intersect}}}{20}} \quad e_{ss} = \dfrac{1}{K_v}$

Refer No.**13**

Nyquist stability criterion

TF is $G/(1+G)$ *Unity-Feedback System*

$\|G(j\omega)\| < 1$ when $\angle G(j\omega) = \pm\pi \mod 2\pi$

$\angle G(j\omega) > -\pi$ when $\|G(j\omega)\| = 1$

$G_M = -20\log\|G(j\omega^*)\| \quad \angle G(j\omega^*) = -\pi$

$\phi_M = \pi + \angle G(j\omega^*) \quad \|G(j\omega^*)\| = 1$