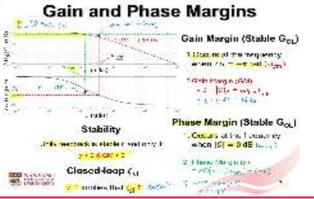
MA3005 Finals CS

Control Theory (Nanyang Technological University)

Time function $f(t)$	Laplace transform
A unit impulse	1
A unit step	$\frac{1}{s}$
t, a unit ramp	$\frac{1}{s^2}$
e^{-at}, exponential decay	$\frac{1}{s+a}$

Gain and Phase Margins



$$G(s) = \frac{s+1}{(s+2)^2(s+3)} = \frac{C_1}{(s+2)^2} + \frac{C_2}{(s+2)} + \frac{C_3}{s+3}$$

using heaviside theorem

$\frac{d}{dx}\left(\frac{u}{v}\right) = \left[v\frac{du}{dx} - u\frac{dv}{dx}\right] / v^2$

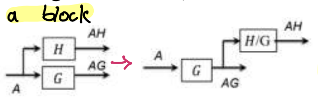
$\therefore C_1 = \left[\frac{-2+1}{-2+3}\right]_{s=-2} = -1$

$C_2 = \frac{d}{ds}\left[\frac{s+1}{s+3}\right]_{s=-2} = \frac{(s+3)-(s+1)}{(s+3)^2}\Big|_{s=-2} = 2$

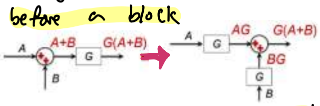
$C_3 = \left[\frac{-3+1}{(-3+2)^2}\right]_{s=-3} = -2$

moving a branch point before a block

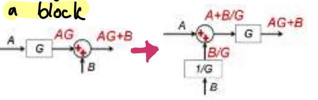
moving a branch point behind a block



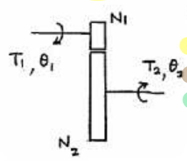
moving a summation point before a block



moving a summation point behind a block



4.1.5 Gear Trains



N = number of teeth
r = radius of gear

$r_1 N_2 = r_2 N_1$

$\theta_1 r_1 = \theta_2 r_2$

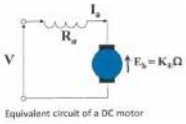
$T_1\theta_1 = T_2\theta_2$

$\therefore \frac{T_1}{T_2} = \frac{\theta_2}{\theta_1} = \frac{N_1}{N_2} = \frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = \frac{1}{n}$

(1) No of teeth & radii distance work done

(2) distance

(3) work done



$\frac{T_S}{V} = \frac{K_T}{R_a}$

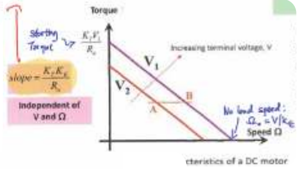
$T_S = \frac{K_T V}{R_a}$

Permanent Magnet DC Motor

Voltage equation:
$V = R_a I_a + K_E\Omega$ —— (1)

Armature current:
$I_a = (V - K_E\Omega)/R_a$ —— (2)

Torque:
$T = K_T I_a = \frac{K_T}{R_a}(V - K_E\Omega)$ - - - (3)



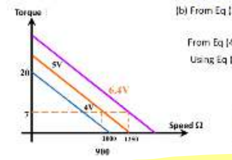
From Eq (3)
Starting torque: $T_S = (K_T V)/R_a$ —— (4)
No load speed: $\Omega_o = V/K_E$ —— (5)
K_T = torque constant (1 Nm A⁻¹)
K_E = Back emf constant (1 Vs rad⁻¹)
SI units

Using the above units
$K_T = K_E = K$ (motor constant)
(141.6 oz.in A⁻¹ = 104/7 VK rmp⁻¹) ← imperial units

The given motor has the speed-torque characteristics shown in Fig, R_a = 2 ohms
(a) If V_1 = 5V, calculate the motor no-load speed.
(b) Calculate the speed of the motor for V_1 = 5 V and T_L = 7 oz.-in
(c) What must be the input V_T to restore the motor speed calculated in part (a) assuming that the 7 oz-in. load is applied?

$T = K_T I_a = \frac{K_T}{R_a}(V - K_E\Omega)$ —— (3)

Starting torque: $T_s = (K_T V)/Ra$ --- (4)



(a) At no load speed,
Ω_o = 1000 rpm, V_1 = 4V
K_E = 4/1000, $[\Omega_o = V_1/K_E]$
If V_1 = 5V, then Ω_o = (5 x 1000)/4 = 1250 rpm
(b) From Eq (3), slope = $\frac{K_T K_E}{R_a}$ = 20/1000

From Eq (4) $T_S / V = K_T / R_a$ = 20 / 4 = 5
Using Eq (3) 7 = 5 x 5 – (20/1000) Ω,
Ω = 900 rpm

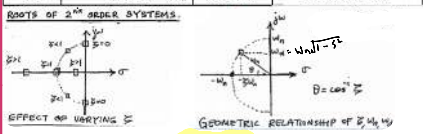
(c) Using (3) and (4)
7 = 5 x V_T – [(20/1000) x 1250]
V_T = 6.4V

k_T, k_E are constant, $T_S = \frac{k_T V}{R_a}$, $k_T = \frac{20 \times 2}{4} = 10$

slope $= \frac{10 \times 4/100}{2} = 0.02$

Damping Ratio	Roots (Poles)	System Behaviour	Nature of roots (Nature of poles)
$\zeta > 1$	$s_{1,2} = -\zeta\omega_n \pm \omega_d$	overdamped	Real & distinct $\omega_d = \omega_n\sqrt{\zeta^2-1}$
$\zeta = 1$	$s_{1,2} = -\omega_n = \omega_n$	critically damped	Real & equal
$\zeta < 1$	$s_{1,2} = -\zeta\omega_n \pm j\omega_d$	under-damped	Complex conjugate $\omega_d = \omega_n\sqrt{1-\zeta^2}$ $(-\zeta\omega_n = -a)$
$\zeta = 0$	$s_{1,2} = \pm j\omega_n = \omega_n$	undamped	Purely Imaginary

ROOTS OF 2ND ORDER SYSTEMS



poles $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ ← ω_d

$R(s) \rightarrow \boxed{\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \rightarrow C(s)$

Damping Ratio

$\zeta = \frac{C_a}{C_c}$ = actual damping in system / damping which will produce critical damping

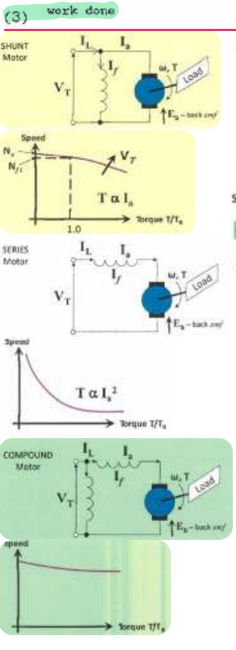
$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

For critical damping,
$c^2 - 4mk = 0$

For Mech system,

$C_a = C$, $C_c = 2\sqrt{mk} = 2m\omega_n$

$\zeta = \frac{c}{2\sqrt{mk}} = \frac{c}{2m\omega_n}$, $\omega_n = \sqrt{\frac{k}{m}}$



Shunt Motors

$V_T = E_b + I_a R_a$ ---- (1)

$V_T = k\phi\omega + I_a R_a$

$\omega = \frac{V_T}{K\phi} - \frac{I_a R_a}{K\phi} = \frac{V_T - I_a R_a}{K\phi}$

Speed Regulation $= \frac{N_o - N_{fl}}{N_f}$

Shunt Motor: $T = k\phi I_a$

Starting Current, $I_a = \frac{V_T - E_b}{R_a}$

Second Order Systems

$R(s) \rightarrow \boxed{\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \rightarrow C(s)$

$G(s)$

Find: $c(t)$ when $r(t)$ is a step input with magnitude p

DC gain of $G(s) = \lim_{s\to 0}\left(\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right) = k$

$C_{ss} = pk$ (product of DC gain and magnitude of step)

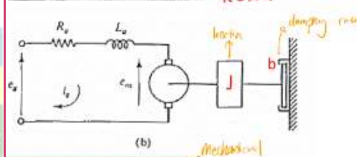
Poles of $G(s)$: $s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$

$\zeta > 1$ (overdamped) $s = a, b$

$\zeta = 1$ (critically damped) $s = -\zeta\omega_n$ (repeated pole)

$\zeta < 1$ (Underdamped) $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

Armature Controlled DC Motor — tested



$T(t) = J\ddot{\theta}(t) + B\dot{\theta}(t)$

$\therefore T(s) = s(Js + B)\theta(s)$

Armature loop:

$e_a = R_a i_a + L_a \dot{i}_a + e_m$

$\therefore E_a = (R_a + L_a s)I_a + E_m$

Torque α i_a

$T = K_t i_a$

$T = K_t I_a = (Js + b)\dot{\theta}(s)$

Counter emf voltage α shaft speed

$e_m = K_e\dot{\theta}$, $E_m = K_e s\theta$

Torque Eqn: $T = J\ddot{\theta} + b\dot{\theta}$, $T = Js^2\theta(s) + bs\theta(s)$

Armature loop: $e_a = R_a i_a + L_a \dot{i}_a + e_m$, $E_a = (R_a + L_a s)I_a + E_m$

Constant I_f source: $T_m = K_t i_a$, $T_m = K_t I_a$

Eliminating I_a, E_m, T

$$\frac{\theta}{E_a} = \frac{1/K_e}{s(T_a T_m s^2 + (T_m + \gamma T_a)s + \gamma + 1)}$$

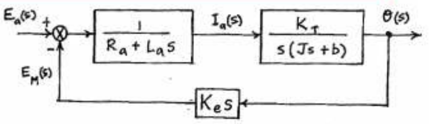
where $T_m = JR_a/(K_e K_t)$ - motor time constant

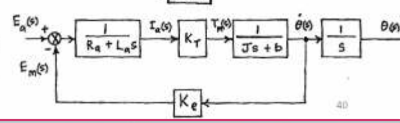
$T_a = L_a/R_a$ - armature time constant

$\gamma = BR_a/(K_e K_t)$ - damping factor

Assume armature inductance, L_a is small compared to armature resistance R_a

$$\frac{\theta_m(s)}{E_a(s)} = \frac{K_t / R_a J_m}{\left[s\left[s + \frac{1}{J_m}\left(b_m + \frac{K_t K_e}{R_a}\right)\right]\right]}$$





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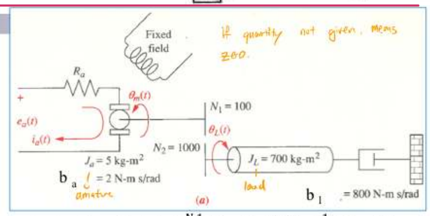
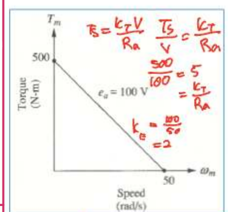
Find the transfer function of an armature-controlled dc motor and load.

Given the system and torque-speed curve, find the transfer function $\theta_L(s)/E_a(s)$.





$T_S = \frac{K_T V}{R_a}$, $\frac{T_S}{V} = \frac{K_T}{R_a}$, $\frac{500}{100} = 5 = \frac{K_T}{R_a}$, $K_e = \frac{100}{50} = 2$

Total inertia at the armature of the motor is: $J_m = J_a + J_L\left(\frac{N1}{N2}\right)^2 = 5 + 700\left(\frac{1}{10}\right)^2 = 12$

Total damping at the armature of the motor is: $b_m = b_a + b_L\left(\frac{N1}{N2}\right)^2 = 2 + 800\left(\frac{1}{10}\right)^2 = 10$

From torque speed curve: $T_{stall} = 500$, $\omega_{no\text{-}load} = 50$, $e_a = 100$

Hence the electrical constants: $\frac{K_t}{R_a} = \frac{T_{stall}}{e_a} = \frac{500}{100} = 5$, $K_e = \frac{e_a}{\omega_{no\text{-}load}} = \frac{100}{50} = 2$

will provide

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\frac{Kt}{Ra\,Jm}}{s\left[s + \frac{1}{J_m}\left(b_m + \frac{K_t K_e}{R_a}\right)\right]} = \frac{5/12}{s\left[s + \frac{1}{12}(10 + (5)(2))\right]} = \frac{0.417}{s(s+1.667)}$$

Since $N_1/N_2 = \frac{\theta_L(s)}{\theta_m(s)} = 1/10$, $\frac{\theta_L(s)}{E_a(s)} = \frac{0.0417}{s(s+1.667)}$

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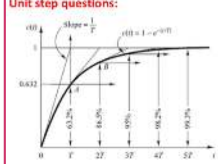
Important equations for first order system

Unit step questions:



2% settling time
= time that the system needs to be within 2% of the steady state
= 4T

5% settling time = 3T

Steady state value:
Using final value theorem:
$\lim_{t\to\infty} c(t) = \lim_{s\to 0} s \cdot \frac{1}{s} \cdot \frac{K}{Ts+1}$

$\lim_{t\to\infty} c(t) = K$

$\therefore c_{ss}(t) = K$

(DC gain)

FVT: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} s \cdot F(s)$

$\lim_{s\to 0} \cancel{s} \cdot \frac{1}{\cancel{s}} \cdot \frac{K}{Ts+1} \rightarrow \frac{K}{0+1}$

$\rightarrow K$

1st order (simple lag)

Denominator: max power of $s = 1$

Transfer function of 1st order

$\frac{C(s)}{R(s)} = \frac{k}{Ts+1} = \frac{k/T}{s + \frac{1}{T}}$

k: DC gain

T: time constant

Underdamped Systems

$\zeta < 1$

$R(s) \rightarrow \boxed{\frac{k\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \rightarrow C(s)$

R(s) is unit step input

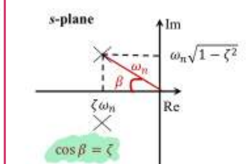
Poles $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ — ω_d

natural frequency

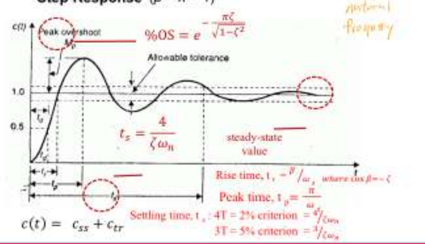
Transient $Ae^{-\zeta\omega_n t}\cos(\omega_n\sqrt{1-\zeta^2}t + \phi)$

damped natural frequency

Step Response ($p = k = 1$)



$\cos\beta = \zeta$



$c(t) = c_{ss} + c_{tr}$

% MP = percent overshoot

e.g. $\frac{2-0.5}{0.5}$

2 - peak

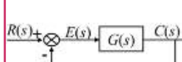
0.5 - css

2% settling time

$e^{-\sigma t} = e^{-4}$

$\sigma = \zeta\omega_n$

Steady-state Error



$E = R - C \rightarrow E = R - \frac{C}{R}R \rightarrow E = R\left(1 - \frac{G}{1+G}\right)$

$E(s) = \frac{R}{(1+G)}$

Unity-feedback system

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t) = 1$	$\frac{1}{s}$ $\frac{1}{1+K_p}$	K_p = Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $u(t) = t$	$\frac{1}{s^2}$ $\frac{1}{K_v}$	$K_v = 0$	∞	K_v = Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $u(t) = \frac{1}{2}t^2$	$\frac{1}{s^3}$ $\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	K_a = Constant	$\frac{1}{K_a}$

$K_p = \lim_{s\to 0} G(s) = G(0)$

$K_v = \lim_{s\to 0} sG(s)$

$K_a = \lim_{s\to 0} s^2 G(s)$

Input "Level" higher than System "level": $e_{ss} \rightarrow \infty$

Input "Level" equal to System "level": e_{ss} is non-zero

Input "Level" lower than System "level": $e_{ss} = 0$

Transient Response

$Ae^{-t} + Be^{-2t}$ (stable)

$A\cos(2t + \phi)$ (Partially)

$Ae^{t} + Be^{-2t}$ (unstable)

$s = a \pm jb$ (complex poles with real parts)

$Ae^{at}\cos(bt + \phi)$

step input e_{ss}

$k_p = \lim_{s\to 0} G(s) = G(s=0)$

$G(s) \to$ OLTF

Ramp input e_{ss}

$k_v = \lim_{s\to 0} sG(s)$

unit parabolic e_{ss}

$k_a = \lim_{s\to 0} s^2 G(s)$

e_{ss} are related to k_p, k_v, k_a

Increasing system type requires an integrator in the forward path but this will have destabilizing effect

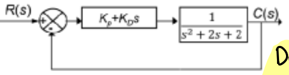
Designing a stable system with more than two integrators in the forward path is generally difficult

PD controller — Increase speed + stability — Transient Response

PI controller — Eliminate E_{ss} — Steady-state Response

↑ Good
— Neutral
↓ Bad

Controllers / Properties	K_p Proportional	$K_D s$ Derivative	$\frac{K_I}{s}$ Integral
Stability	↓	↑	↓
Steady-state error (E_{ss})	↑	—	↑
Speed	↑	↑	↓
Disturbance rejection	↑	—	↑
Noise rejection	—	↓	↑



Desired Transient Response

10% overshoot,
2% settling time of 0.5 s

Desired poles: $s = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$= -8 \pm 11j$

Determine the DESIRED closed-loop poles

Criterion 1: 10% OS

$e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} = 0.1$

$\frac{-\pi\zeta}{\sqrt{1-\zeta^2}} = \ln 0.1$

$\zeta = \sqrt{\frac{(\ln 0.1)^2}{(\ln 0.1)^2 + \pi^2}} = 0.59$

Criterion 2: 2% settling time

$\frac{4}{\zeta\omega_n} = 0.5$

$\frac{4}{(0.59)\omega_n} = 0.5$

$\omega_n = \frac{4}{(0.59)(0.5)} = 13.5$

CL poles define transient response

Harmonic input

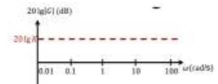
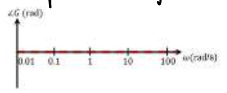
Input : $r = r_0 \cos(\omega t)$

$C_{ss} = (r_0 |G(s=j\omega)|)\cos[\omega t + \angle G(s=j\omega)]$

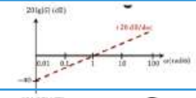
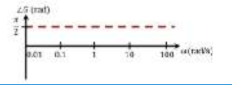
Bode plot

magnitude plot — phase plot

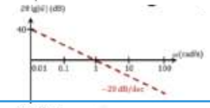
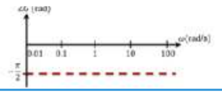
constant (k)

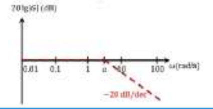
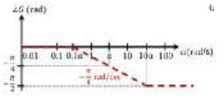
Differentiator (s)

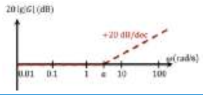
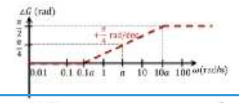
Integrator ($\frac{1}{s}$)

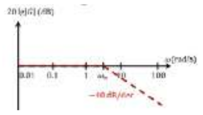
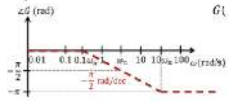
First order pole ($\frac{a}{s+a}$)

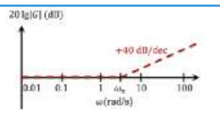
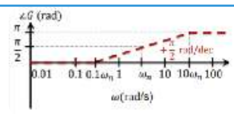
First order zero ($\frac{s+a}{a}$)

Second order pole $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Second order zero $\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$

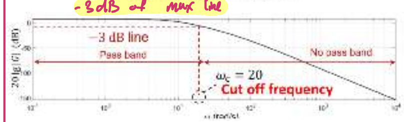



$\omega_R = \omega_n\sqrt{1-2\zeta^2}$ $\zeta < 0.1$ (sharp) $\zeta > 0.5$ (follow)

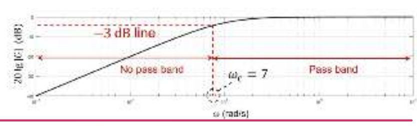
$0.3 < \zeta < 0.4$ (both)

Low-Pass Filter

-3dB of max line



High-Pass Filter



Root Locus Summary

1. Starting and ending points of the locus
   a. Starting points – open-loop POLES
   b. Ending points – open-loop ZEROS
2. Asymptotes (infinite open-loop ZEROS)
   a. Angles
   b. Location

   # = n - m (n: poles, m: zeros)
3. Locus on real axis
4. Break-in and break-out points
   a. Break-in (minimum K) (zero)
   b. Break-out (maximum K) (pole)
5. Departure and arrival angles
   (poles) (zeros)
6. Imaginary axis intersection

OL Poles [k=0]

OL zeros [k=∞]

n-m	∠s
1	180°
2	90°, 270°
3	±60°, 180°
4	±45°, ±135°

$\sigma_a = \frac{\Sigma_{i=1}^{n}(-p_i) - \Sigma_{j=1}^{m}(-z_j)}{n-m}$

$\angle s = \frac{\pi(1+2q)}{n-m}$

Let $\frac{dK}{ds} = 0$

Intersection of imaginary axis: $s = j\omega$ into CLTF CE

$\Sigma_{i=1}^{n}\angle(s+z_i) - \Sigma_{j=1}^{m}\angle(s+p_j) = \pm\pi$

Lead compensator

$G_c = k\frac{(s+z_c)}{(s+p_c)}$

Noise Amplification : $\lim_{\omega\to\infty} |G_c(s=j\omega)|$

$= \lim_{\omega\to\infty} \left|\frac{k(j\omega+z_c)}{(j\omega+p_c)}\right| = k$

Finite Amplification → Btr than PD controller

Designing slow PD first then slow for lead from PD

$p_c > z_c$

steps: 1) $k_D s + k_p = 14(s+15.1) = k_D(s+z_c')$

2) $z_c < z_c' \to z_c = 10$

lead compensator $= k\left(\frac{s+10}{s+p_c}\right)$

3) use CE to determine k & p_c
- sub desired poles into CE

PI controllers cannot be realized with passive components

Alternative : Lag compensator controller

$G_c = k\frac{(s+z_c)}{(s+p_c)}$ pole/zero cancellation

$p_c << z_c << 1$

$k_p = \lim_{s\to 0} G_c G = \lim_{s\to 0} k\frac{(s+z_c)}{(s+p_c)}G(s) = k\frac{z_c}{p_c}G(0)$

↑k $\frac{z_c}{p_c}$ factor

↓$e_{ss} = \frac{1}{1+k_p}$

Error reduces but not fully eliminated

PD controller

$k_D s + k_p = k_D\left(s + \boxed{\frac{k_p}{k_D}}\right) = G_c$ (z_c)

k_p & z_c change CL poles

Noise Amplification : $\lim_{\omega\to\infty} |G_c(s=j\omega)|$

$= \lim_{\omega\to\infty} |k_D(j\omega + z_c)| \to \infty$

Alternative controller: Lead compensator

PI controller

$k_p + \frac{k_I}{s} = k_p\frac{(s+z_c)}{s}$ $\left(\frac{k_I}{k_p}\right)$

PI controller eliminates steady-state error

PID controllers

$G_c = \underbrace{k_D(s+z_1)}_{PD}\underbrace{\frac{(s+z_2)}{s}}_{PI}$

Steps for PD controller:

Step 1 : Find Desired CL-poles

Step 2 : CLTF CE in terms of k_p & k_D

Step 3 : sub poles into CE

Step 4 : Solve Re & Im parts of CE to obtain k_p & k_D

Lead-lag Compensators

$G_c = \underbrace{\frac{k_1(s+z_1)}{(s+p_1)}}_{\text{lead compensator}} \cdot \underbrace{\frac{k_2(s+z_2)}{(s+p_2)}}_{\text{Lag compensator}}$

$z_1 < p_1$ $p_2 < z_2 << 1$

Gain & Phase Margins

Phase Margin (Stable G_{OL})

1. Occurs at the frequency when $|G| = 0$ dB (ω_{pm})
2. Phase Margin (γ) $= \angle G(s = j\omega_{pm}) - (-\pi)$

Gain Margin (Stable G_{OL})

1. Occurs at the frequency when $\angle G = -\pi$ rad (ω_{gm})
2. Gain Margin (GM) $= 0 - |G(s = j\omega_{gm})|_{dB}$

Stability

Unity feedback is stable if and only if:

$\gamma > 0$ & GM > 0

0dB $\angle G = -\pi + 180°$

Closed-loop ζ_{cl} γ ↑ implies that ζ_{cl} ↑

Bandwidth

The frequency when $|G| = -3$ dB.
We denote it as ω_{BW}

In general, higher bandwidth of the OL system → higher natural frequency of closed loop system (affects location of CL poles)

Closed-loop $\omega_{n,cl}$

ω_{BW} ↑ implies that $\omega_{n,cl}$ ↑

Closed-loop Poles

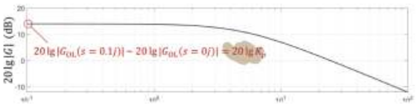
$s_{1,2} = -\zeta_{cl}\omega_{n,cl} \pm j\omega_{n,cl}\sqrt{1-\zeta_{cl}^2}$

Steady state errors

ess

Type 0

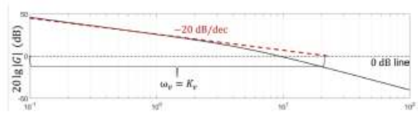
- Use magnitude at the smallest value of ω
- Let magnitude = 20 lg Kp to find Kp



$\frac{1}{1+K_p}$

Type 1

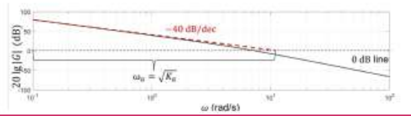
- $K_v = \omega_v$ where the asymptote intersects the horizontal axis



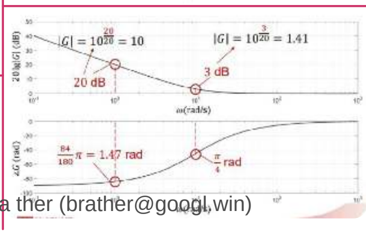
$\frac{1}{K_v}$

Type 2

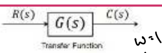
- $\sqrt{K_a} = \omega_a$ where the asymptote intersects the horizontal axis



$\frac{1}{K_a}$



Sys unstable if GM < 0



ω=1 ω=10

Compute the output if the input is: $r(t) = \cos t + 2\cos 10t$

$c(t) = |G(s=j)|\cos(t + \angle G(s=j))$

$+2|G(s=10j)|\cos(10t + \angle G(s=10j))$

$c(t) = 10\cos(t + 1.47) + 2.82\cos(10t + \frac{\pi}{4})$