NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2017-2018

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

November/December 2017 Time Allowed: 2 ½ hours

INSTRUCTIONS

1. This paper contains **THREE (3)** questions and comprises **SIX (6)** pages.
2. Answer **ALL** questions.
3. Marks for each question are as indicated.
4. This is a **RESTRICTED OPEN-BOOK** examination. You may bring in one double-sided A4 reference sheet.

1(a) If $w(x,y) = (ax + by + 1)^3$ (where a and b are constants) is a solution of the partial differential equation

$$\frac{\partial^2 w}{\partial y^2} + 10\frac{\partial^2 w}{\partial y \partial x} + 16\frac{\partial^2 w}{\partial x^2} = 0$$

at all points (x, y) on the Oxy plane, find the value(s) of b/a.

(6 marks)

(b) A partial differential equation in $\psi(r,z)$ is given by

$$r(\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^2 \psi}{\partial z^2} + \psi) + \frac{\partial \psi}{\partial r} = 0.$$

Apply the method of separation of variables on the partial differential equation above to obtain a pair of ordinary differential equations having an arbitrary constant in them. (**Note:** Let $\psi(r,z) = R(r)Z(z)$. Do <u>not</u> solve the ordinary differential equations.)

(6 marks)

Note: Question 1 continues on page 2.

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(c) Consider the boundary value problem defined by the partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{2}{(x+1)} \frac{\partial \phi}{\partial x} = 0 \text{ for } 0 < x < 3, \quad 0 < y < 2,$$

and the boundary conditions on the four sides of the rectangular solution domain as given by

$$\left.\begin{aligned} \phi(0, y) &= 0 \\ \phi(3, y) &= 0 \end{aligned}\right\} \text{ for } 0 < y < 2,$$

$$\left.\begin{aligned} \phi(x, 0) &= 0 \\ \phi(x, 2) &= (1+x)^{-1} \end{aligned}\right\} \text{ for } 0 < x < 3.$$

(i) Verify by direct substitution that the partial differential equation of the boundary value problem has a solution of the form

$$\phi(x, y) = A(1+x)^{-1}(e^{py} - e^{-py})\sin(px),$$

where A and p are arbitrary constants.

(6 marks)

(ii) Show that the first and the third boundary conditions are satisfied by the solution in part (i).

(3 marks)

(iii) For $A \neq 0$, find all positive values of p such that the second boundary condition is satisfied. Hence, from part (i), write down a set of solutions satisfying the first three boundary conditions.

(3 marks)

(iv) Use your answer in (iii) to derive a series solution for the boundary value problem.

(6 marks)

2(a) Figure 2(a) shows a structure which is made up of two equally-sized constant strain triangle elements. Element 1 consists of nodes 1, 2 and 3. Element 2 consists of nodes 1, 3 and 4. Nodes 1 and 4 are fixed to the wall. Equal forces F_{2x} and F_{3x} are applied to nodes 2 and 3 respectively. Assume that the material properties of the two constant strain triangle elements are identical and homogeneous.

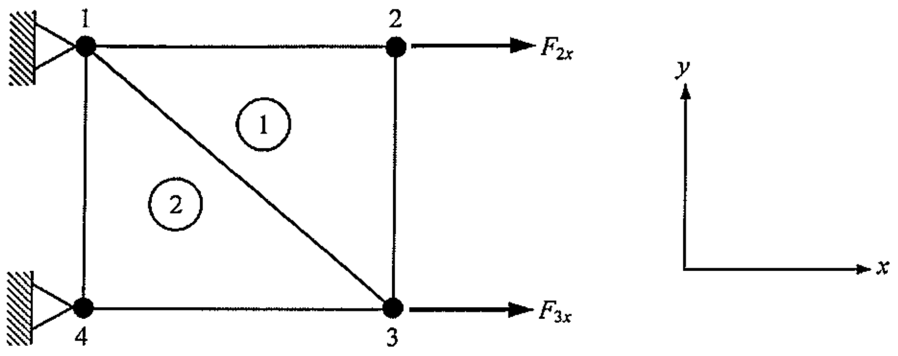

Figure 2(a)

After applying F_{2x} and F_{3x} on node 2 and 3 respectively, it was found that the x-component of the displacement at node 2 is larger than that at node 3. Explain why.

(3 marks)

(b) Figure 2(b) (not drawn to scale) shows a beam structure made up of three beam elements. Element 1 consists of nodes 1 and 2, element 2 consists of nodes 2 and 3, and element 3 consists of nodes 3 and 4. A concentrated load of P acts on node 4. The beam structure has a total length of $3L$, moment of inertia I and modulus of elasticity E.

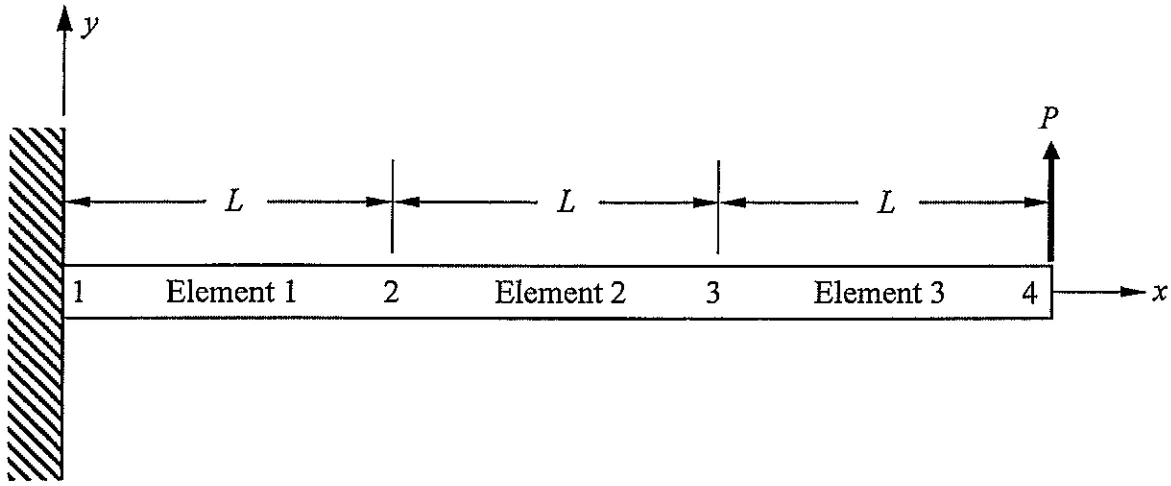

Figure 2(b)

Note: Question 2 continues on page 4.

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Assuming that the global stiffness matrix of the beam structure is given by

$$\begin{bmatrix} 4.7\times10^5 & 1.4\times10^6 & -4.7\times10^5 & 1.4\times10^6 & 0 & 0 & 0 & 0 \\ a & 5.6\times10^6 & -1.4\times10^6 & 2.8\times10^6 & 0 & 0 & 0 & 0 \\ -4.7\times10^5 & -1.4\times10^6 & b & 0 & -4.7\times10^5 & 1.4\times10^6 & 0 & 0 \\ 1.4\times10^6 & 2.8\times10^6 & 0 & 1.1\times10^7 & -1.4\times10^6 & 2.8\times10^6 & 0 & 0 \\ 0 & 0 & -4.7\times10^5 & -1.4\times10^6 & 9.3\times10^5 & 0 & -4.7\times10^5 & e \\ 0 & 0 & 1.4\times10^6 & c & 0 & 1.1\times10^7 & -1.4\times10^6 & 2.8\times10^6 \\ 0 & 0 & 0 & 0 & -4.7\times10^5 & -1.4\times10^6 & d & -1.4\times10^6 \\ 0 & 0 & 0 & 0 & 1.4\times10^6 & 2.8\times10^6 & -1.4\times10^6 & 5.6\times10^6 \end{bmatrix}$$

determine the values of a, b, c, d and e.

(10 marks)

(c) Figure 2(c) shows a plane truss structure made up of two elements. Element 1 comprises nodes 1 and 2, while element 2 comprises nodes 2 and 3. Nodes 1 and 3 are pivoted to the wall. An upward force of $P = 25$ kN is loaded on node 2. The length of each truss is $L = 2$ m. The cross-sectional area and the elastic modulus of each truss are 5×10^{-4} m^2 and 80 GPa respectively.

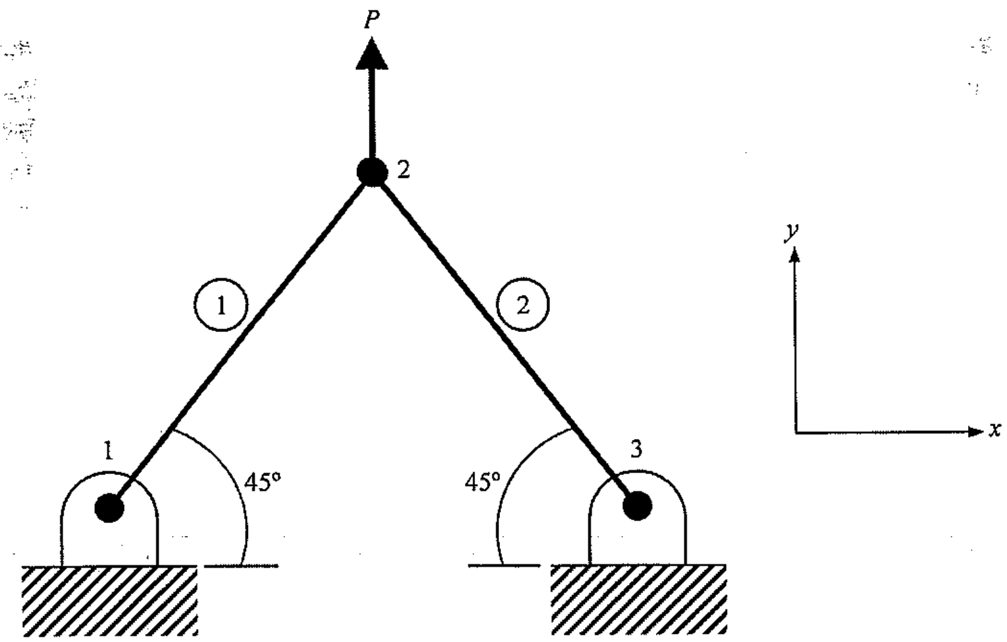

Figure 2(c)

Note: Question 2 continues on page 5.

(i) Determine the stiffness matrix for element 1 and 2.

(2 marks)

(ii) Assemble the global stiffness matrix.

(4 marks)

(iii) Formulate the global finite element equations.

(4 marks)

(iv) State the boundary conditions.

(4 marks)

(v) Formulate the reduced finite element equations.

(4 marks)

(vi) Determine the y-component of the displacement at node 2.

(4 marks)

3. Consider the one-dimensional convection and diffusion process for a variable ϕ described by the following equation:

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) + C\phi + G,$$

where the concentration ϕ is a function of the spatial coordinate x, and C and G are known constants. Assume that u (the velocity), ρ (the density) and Γ (the diffusion coefficient) are positive and known constants. You may assume a uniform cross section area of the geometry.

(a) For the ordinary differential equation above, use the finite volume method on the uniform mesh shown in Figure 3 together with the upwind scheme (use the central differencing method for the diffusion terms) to derive a discretized equation of the form

$$a_P \phi_P = a_W \phi_W + a_E \phi_E + S_u,$$

For only the internal nodes labelled 2, 3 and 4. State clearly the coefficients a_W, a_E and S_u and express a_P as $a_P = a_W + a_E - S_P$.

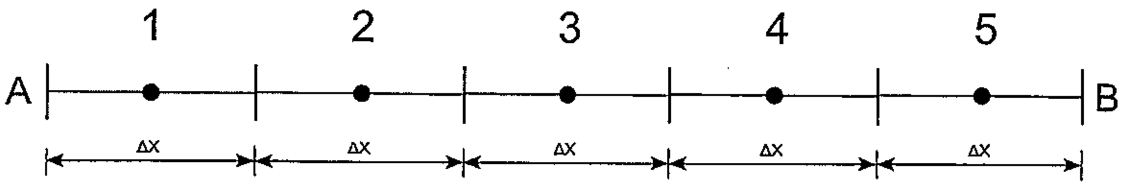

Figure 3

(10 marks)

Note: Question 3 continues on page 6.

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(b) Derive the coefficients a_W, a_E, S_u, S_P for node 1 (in Figure 3) if the boundary condition at boundary A is given by $\phi = \alpha$. Use the same scheme as in part 3(a).

(5 marks)

(c) Derive the coefficients a_W, a_E, S_u, S_P for node 5 (in Figure 3) if the boundary condition at boundary B is given by $\frac{d\phi}{dx} = \beta$. Use the same scheme as in part 3(a).

(5 marks)

(d) Set up the full numerical matrix problem by using the coefficients derived in parts (a), (b) and (c). Use the following values of the constants: $u = 2$, $\rho = 5$, $\Gamma = 4$, C=10, G=15, $\alpha = 1$ and $\beta = 2$. The distance between A and B in Figure 3 is 1. Assume that the units of all quantities are consistent.

(4 marks)

(e) Perform the first iteration on the numerical problem in part (d) using the Jacobi method. Take zero as the initial guess (iteration zero) for all the variables.

(4 marks)

(f) Change the boundary condition at boundary B to $\phi = \gamma$. Which elements in the matrix derived in part 3(d) will change? What are the new values of the elements if $\gamma = 5$?

(7 marks)

Note: Figure 3 appears on page 5.

END OF PAPER

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1(a) $w(x,y) = (ax+by+1)^3$

$\frac{\partial w}{\partial x} = 3(ax+by+1)^3(a)$ $\qquad$ $\frac{dw}{dy} = 3b(ax+by+1)^2$

$= 3a(ax+by+1)^2$ $\qquad$ $\frac{d^2w}{dy^2} = 6b^2(ax+by+1)^2$

$\frac{d^2w}{dx^2} = 6a^2(ax+by+1)$

$\frac{d^2w}{dydx} = 6ab(ax+by+1)$

$\therefore \frac{d^2w}{dy^2} + 10\frac{d^2w}{dydx} + 16\frac{d^2w}{dx^2} = 0$

$6b^2(ax+by+1) + 60ab(ax+by+1) + 96a^2(ax+by+1) = 0$

$ax+by+1=0$ $\qquad$ $6b^2+60ab+96a^2=0$

$b^2+10ab+16a^2=0$

$(b+8a)(b+2a)=0$

$b+8a=0$ $\qquad$ $b+2a=0$

$b=-8a$ $\qquad$ $b=-2a$

$\frac{b}{a}=-8$ # $\qquad$ $\frac{b}{a}=-2$ #

1 (b) $r\left(\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial z^2} + \psi\right) + \frac{\partial\psi}{\partial r} = 0$

let $\psi(r,z) = R(r)\,Z(z)$

$r\left[R''(r)Z(z) + R(r)Z''(z) + R(r)Z(z)\right] + R'(r)Z(z) = 0$

$rR''(r)Z(z) + rR(r)Z''(z) + rR(r)Z(z) + R'(r)Z(z) = 0$

divide everything by r

$R''(r)Z(z) + R(r)Z''(z) + R(r)Z(z) + \frac{R'(r)}{r}Z(z) = 0$

divide everything by $R(r)Z(z)$

$\frac{R''(r)}{R(r)} + \frac{Z''(z)}{Z(z)} + 1 + \frac{R'(r)}{rR(r)} = 0$

$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} = -\frac{Z''(z)}{Z(z)} - 1 = \gamma$

$\frac{R''(r)}{R(r)} + \frac{R'(r)}{rR(r)} = \gamma$ # $\frac{Z''(z)}{Z(z)} + 1 = -\gamma$ #

1 (c)(i) $\phi(x,y) = A(1+x)^{-1}(e^{py} - e^{-py})\sin(px)$

$\frac{\partial\phi}{\partial x} = -A(1+x)^{-2}(e^{py} - e^{-py})\sin(px) + A(1+x)^{-1}(e^{py} - e^{-py})p\cos(px)$

$\frac{\partial^2\phi}{\partial x^2} = 2A(1+x)^{-3}(e^{py} - e^{-py})\sin(px) - A(1+x)^{-2}(e^{py} - e^{-py})p\cos(px)$

$\quad - A(1+x)^{-2}(e^{py} - e^{-py})p\cos(px) - A(1+x)^{-1}(e^{py} - e^{-py})p\sin(px)$

$\frac{\partial\phi}{\partial y} = A(1+x)^{-1}(pe^{py} + pe^{-py})\sin(px)$

$= Ap(1+x)^{-1}(e^{py} + e^{-py})\sin(px)$

$\frac{\partial^2\phi}{\partial y^2} = Ap(1+x)^{-1}(pe^{py} - pe^{-py})\sin(px)$

$= Ap^2(1+x)^{-1}(e^{py} - e^{-py})\sin(px)$

Sub into PDE

$2A(1+x)^{-3}(e^{py} - e^{-py})\sin(px) - 2A(1+x)^{-2}(e^{py} - e^{-py})p\cos(px) - Ap^2(1+x)^{-1}(e^{py} - e^{-py})\sin(px)$

$+ Ap^2(1+x)^{-1}(e^{py} - e^{-py})\sin(px) + \frac{2}{x+1}\left[-A(1+x)^{-2}(e^{py} - e^{-py})\sin(px) + A(1+x)^{-1}(e^{py} - e^{-py})p\cos(px)\right]$

$= 2A(1+x)^{-3}(e^{py} - e^{-py})\sin(px) - 2A(1+x)^{-2}(e^{py} - e^{-py})p\cos(px) - Ap^2(1+x)^{-1}(e^{py} - e^{-py})\sin(px)$

$+ Ap^2(1+x)^{-1}(e^{py} - e^{-py})\sin(px) - 2A(1+x)^{-3}(e^{py} - e^{-py})\sin(px) + 2A(1+x)^{-2}(e^{py} - e^{-py})p\cos(px)$

$= 0$ (proved)

1(c)(ii) $\phi(x,y) = A(1+x)^{-1}(e^{py} - e^{-py})\sin(px)$

$\phi(0,y) = A(e^{py} - e^{-py})\sin(0) = 0$ # (shown)

$\phi(x,0) = A(1+x)^{-1}(e^{0} - e^{0})\sin(px) = 0$ # (shown)

1(c)(iii) $\phi(3,y) = A(\frac{1}{4})(e^{py} - e^{-py})\sin(3p)$

$= \frac{1}{4}A(e^{py} - e^{-py})\sin(3p) = 0$

$\therefore \sin(3p) = 0$

$3p = n\pi$

$p = \frac{n\pi}{3}$ for $n = 1, 2, 3, \ldots$

$\therefore \phi(x,y) = \sum_{n=1}^{\infty}(1+x)^{-1}(e^{\frac{n\pi y}{3}} - e^{-\frac{n\pi y}{3}})\sin(\frac{n\pi x}{3})$ #

1(c)(iv)

$\phi(x,2) = \sum_{n=1}^{\infty}A(1+x)^{-1}(e^{\frac{2n\pi}{3}} - e^{-\frac{2n\pi}{3}})\sin(\frac{n\pi x}{3}) = (1+x)^{-1}$

$\rightarrow \sum_{n=1}^{\infty}A(e^{\frac{2n\pi}{3}} - e^{-\frac{2n\pi}{3}})\sin(\frac{n\pi x}{3}) = 1$

use FSPII

$A(e^{\frac{2n\pi}{3}} - e^{-\frac{2n\pi}{3}}) = \frac{2}{3}\int_0^3 \sin(\frac{n\pi x}{3})dx$ #

2(a) Node 2 has a larger x displacement. There are 2 truss elements holding it instead of 3 truss elements at Node 3. Hence without the diagonal truss element it can displace more in the x direction.

(2)

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2(b) Global Stiffness Matrix

$$\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L & 0 & 0 & 0 & 0 \\ 6L & 4L^2 & -6L & 2L^2 & 0 & 0 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L & 0 & 0 \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 & 0 & 0 \\ 0 & 0 & -12 & -6L & 24 & 0 & -12 & 6L \\ 0 & 0 & 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

Compare this formula matrix to the given matrix in the question.

$a = 6L\left(\frac{EI}{L^3}\right)$

since $6L\left(\frac{EI}{L^3}\right) = 1.4\times10^6$

$a = 1.4\times10^6$ #

$b = 24\left(\frac{EI}{L^3}\right)$

since $24 = 9.3\times10^5$

$\therefore b = 9.3\times10^5$ #

$c = 2L^2\left(\frac{EI}{L^3}\right)$

since $2L^2\left(\frac{EI}{L^3}\right) = 2.8\times10^6$

$\therefore c = 2.8\times10^6$

$d = 12\left(\frac{EI}{L^3}\right)$

since $12\left(\frac{EI}{L^3}\right) = 4.7\times10^5$

$\therefore d = 4.7\times10^5$ #

Note: Alternative method is to work out E, I and L. You will obtain the same answers but more tedious.

2(c)(i) General formula

$$\frac{AE}{L}\begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta & -\cos^2\theta & -\sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta & -\sin\theta\cos\theta & -\sin^2\theta \\ -\cos^2\theta & -\sin\theta\cos\theta & \cos^2\theta & \sin\theta\cos\theta \\ -\sin\theta\cos\theta & -\sin^2\theta & \sin\theta\cos\theta & \sin^2\theta \end{bmatrix}$$

Element 1 stiffness matrix

$\theta = 45°$

$\cos^2\theta = 0.5$ $\sin^2\theta = 0.5$ $\sin\theta\cos\theta = 0.5$

$A = 5\times10^{-4}\,m^2$ $E = 80\times10^9\,Pa$ $L = 2m$

$\frac{AE}{L} = 20\times10^6$

$$20\times10^6\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \#$$

Element 2 stiffness matrix

$\theta = 135°$

$\cos^2\theta = 0.5$ $\sin^2\theta = 0.5$ $\sin\theta\cos\theta = -0.5$

$$20\times10^6\begin{bmatrix} 0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \#$$

2(c)(ii) Global Stiffness Matrix

$$20\times10^6\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 & 0 & 0 \\ 0.5 & 0.5 & -0.5 & -0.5 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0 & 1 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix}$$

(3)

2(c)(iii) Global Finite Element Equation

$$\begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \end{bmatrix} = 20\times10^{6} \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 & 0 & 0 \\ 0.5 & 0.5 & -0.5 & -0.5 & 0 & 0 \\ -0.5 & -0.5 & 1 & 0 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0 & 1 & 0.5 & -0.5 \\ 0 & 0 & -0.5 & 0.5 & 0.5 & -0.5 \\ 0 & 0 & 0.5 & -0.5 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} d_{1x} \\ d_{1y} \\ d_{2x} \\ d_{2y} \\ d_{3x} \\ d_{3y} \end{bmatrix}$$

2(c)(iv) $d_{1x}=0$, $d_{1y}=0$, $d_{3x}=0$, $d_{3y}=0$

$F_{2y} = P = 25$ kN

2(c)(v) Reduced finite element equation

$$\begin{bmatrix} F_{2x} \\ F_{2y} \end{bmatrix} = 20\times10^{6} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d_{2x} \\ d_{2y} \end{bmatrix}$$

2(c)(vi) $F_{2y} = 20\times10^{6} \times d_{2y}$

$d_{2y} = \frac{25000}{20\times10^{6}} = 0.00125$ m A

3. $\frac{d}{dx}(\rho u\phi) = \frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right) + C\phi + G$

$\int_{\Delta V}\frac{d}{dx}(\rho u\phi)\,dV = \int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV + \int_{\Delta V}[C\phi + G]\,dV$

$\int_A n(\rho u\phi)\,dA = \int_A n\left(\Gamma\frac{d\phi}{dx}\right)dA + C\phi\Delta V + G\Delta V$

$(\rho u\phi)_e - (\rho u\phi)_w = \left(\Gamma\frac{d\phi}{dx}\right)_e - \left(\Gamma\frac{d\phi}{dx}\right)_w + C\phi\Delta x + G\Delta x$

3 (a) Central differencing scheme for diffusion term for points 2, 3, 4

$\frac{\partial\phi}{\partial x}_e = \frac{\phi_E - \phi_P}{\partial x}$, $\frac{\partial\phi}{\partial x}_w = \frac{\phi_P - \phi_W}{\partial x}$

$\rho u\phi_e - \rho u\phi_w = \Gamma\left(\frac{\phi_E - \phi_P}{\partial x}\right) - \Gamma\left(\frac{\phi_P - \phi_W}{\partial x}\right) + C\phi_P\Delta x + G\Delta x$

Let $\rho u = F$, $\frac{\Gamma}{\partial x} = D$

$F\phi_e - F\phi_w = D\phi_E - D\phi_P + D\phi_W + C\phi_P\Delta x + G\Delta x$

Upwind differencing scheme for convection term for points 2, 3 and 4

$\phi_e = \phi_P$

$\phi_w = \phi_W$

$F\phi_P - F\phi_W = D\phi_E - D\phi_P - D\phi_P + D\phi_W + C\phi_P\Delta x + G\Delta x$

Rearrange the terms

$(2D + F - C\Delta x)\phi_P = (D + F)\phi_W + D\phi_E + G\Delta x$

$a_P = 2D + F - C\Delta x$

$a_W = D + F$

$a_E = D$

$S_u = G\Delta x$

$S_P = a_W + a_E - a_P$

$= D + F + D - 2D - F + C\Delta x$

$= C\Delta x$

(4)

3(b) For node 1, the only differences from 2,3,4 are:

$\left(\frac{d\phi}{dx}\right)_w = \frac{\phi_P - \phi_A}{\delta x/2}$ and $\phi_w = \phi_A$

$\rightarrow \rho u \phi_e - \rho u \phi_w = \Gamma\left(\frac{\phi_E - \phi_P}{\delta x}\right) - \Gamma\left(\frac{\phi_P - \phi_A}{\delta x/2}\right) + C\phi_P \Delta x + G\Delta x$

upwind differencing scheme of convection term

$\phi_e = \phi_P$

$\phi_w = \phi_A$

$F\phi_P - F\phi_A = D\phi_E - D\phi_P - 2D\phi_P + 2D\phi_A + C\phi_P\Delta x + G\Delta x$

Rearrange the terms

$(3D + F - C\Delta x)\phi_P = 0\,\phi_W + D\phi_E + G\Delta x + (2D+F)\phi_A$

$a_P = 3D + F - C\Delta x$

$a_W = 0$

$a_E = D$

$S_u = (2D+F)\phi_A + G\Delta x = (2D+F)\alpha + G\Delta x$

$S_P = a_W + a_E - a_P$

$= D - 3D - F + C\Delta x$

$= C\Delta x - 2D - F$

3(c) For node 5, the only differences from 2,3,4 are:

$\left(\frac{d\phi}{dx}\right)_e = \frac{\phi_B - \phi_P}{\delta x/2}$ and $\phi_B = \phi_e$

$\rightarrow \rho u \phi_e - \rho u \phi_w = \Gamma\left(\frac{\phi_B - \phi_P}{\delta x/2}\right) - \Gamma\left(\frac{\phi_P - \phi_W}{\delta x}\right) + C\phi_P \Delta x + G\Delta x$

upwind ~~fo~~ differencing of convection term

$\phi_e = \phi_P$

$\phi_w = \phi_W$

$F\phi_P - F\phi_W = 2D\phi_B - 2D\phi_P - D\phi_P + D\phi_W + C\phi_P\Delta x + G\Delta x$

Rearrange the terms

$(3D + F - C\Delta x)\phi_P = (D+F)\phi_W + 0\,\phi_E + 2D\phi_B + G\Delta x$

$a_P = 3D + F - C\Delta x$

$a_W = D + F$

$a_E = 0$

$S_u = 2D\phi_B + G\Delta x = 2\Gamma\frac{d\phi_B}{dx} + G\Delta x = 2\Gamma\beta + G\Delta x$

$S_P = -2D + C\Delta x$

3 (b) $\Delta x = \frac{1}{5} = 0.2$

$D = \frac{\Gamma}{\Delta x} = \frac{4}{0.2} = 20$

$F = \rho u = 5(2) = 10$

Node 2, 3, 4

$a_p = 2D + F - C\Delta x$	$a_W = D + F$	$a_E = D$	$S_u = G\Delta x$	$S_p = C\Delta x$
$= 2(20) + 10 - 10(0.2)$	$= 20 + 10$	$= 20$	$= 15(0.2)$	$= 10(0.2)$
$= 48$	$= 30$		$= 3$	$= 2$

Node 1

$a_p = 3D + F - C\Delta x$	$a_W = 0$	$a_E = D$	$S_u = (2D + F)\phi_A + G\Delta x$	$S_p = C\Delta x - 2D - F$
$= 3(20) + 10 - 2$		$= 20$	$= [2(20) + 10](1) + 3$	$= 2 - 2(20) - 10$
$= 68$			$= 53$	$= -48$

Node 5

$a_p = 3D + F - C\Delta x$	$a_W = D + F$	$a_E = 0$	$S_u = 2\Gamma B + G\Delta x$	$S_p = -2D + C\Delta x$
$= 3(20) + 10 - 2$	$= 20 + 10$		$= 2(4)(2) + 3$	$= -2(20) + 2$
$= 68$	$= 30$		$= 19$	$= -38$

This gives:

$$\begin{bmatrix} 68 & -20 & 0 & 0 & 0 \\ -30 & 48 & -20 & 0 & 0 \\ 0 & -30 & 48 & -20 & 0 \\ 0 & 0 & -30 & 48 & -20 \\ 0 & 0 & 0 & -30 & 68 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 53 \\ 3 \\ 3 \\ 3 \\ 19 \end{bmatrix}$$

Note: a_E and a_W is brought over to the left side of the equation, hence the $-$ve sign.

(5)

3(e) Expand the matrix in 3(d)

$68\phi_1 - 20\phi_2 = 53 \rightarrow \phi_1 = \frac{20\phi_2 + 53}{68}$

$-30\phi_1 + 48\phi_2 - 20\phi_3 = 3 \rightarrow \phi_2 = \frac{30\phi_1 + 20\phi_3 + 3}{48}$

$-30\phi_2 + 48\phi_3 - 20\phi_4 = 3 \rightarrow \phi_3 = \frac{30\phi_2 + 20\phi_4 + 3}{48}$

$-30\phi_3 + 48\phi_4 - 20\phi_5 = 3 \rightarrow \phi_4 = \frac{30\phi_3 + 20\phi_5 + 3}{48}$

$-30\phi_4 + 68\phi_5 = 19 \rightarrow \phi_5 = \frac{30\phi_4 + 19}{68}$

Iteration No.	0	1
ϕ_1	0	0.7794
ϕ_2	0	0.0625
ϕ_3	0	0.0625
ϕ_4	0	0.0625
ϕ_5	0	0.2794

3(f) Boundary condition B only affects node 5 S_u term

From 3(c)

$S_u = 20\phi_B + 9D2$

If $\phi_B = 7/25$

$S_u = 2(20)(5) + 3$

$= 203$ #

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

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NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 2 EXAMINATION 2017-2018

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

April/May 2018 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **THREE (3)** questions and comprises **SIX (6)** pages.
2. Answer **ALL** questions.
3. Marks for each question are as indicated.
4. This is a **RESTRICTED OPEN-BOOK** examination. You may bring in one double-sided A4 reference sheet

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1(a) If $u(x,y)=\sin(2x^2+y^2)$ is a solution of the partial differential equation

$$x^2\frac{\partial^2 u}{\partial y^2}+axy\frac{\partial^2 u}{\partial y\partial x}+y^2\frac{\partial^2 u}{\partial x^2}=bx\frac{\partial u}{\partial x}+2y\frac{\partial u}{\partial y}$$

at all points (x,y) on the Oxy plane, find the values of the constants a and b.

(6 marks)

(b) Apply the method of separation of variables on the partial differential equation

$$\frac{\partial \psi}{\partial x}+10\psi=\frac{\partial}{\partial y}((1+x^2)(1+y^2)\frac{\partial \psi}{\partial y})$$

to obtain a pair of ordinary differential equations having an arbitrary constant in them. (Let $\psi(x,y)=X(x)Y(y)$. Do <u>not</u> attempt to solve the ordinary differential equations you obtain.)

(6 marks)

Note: Question 1 continues on page 2.

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(c) Consider the initial-boundary value problem defined by the partial differential equation

$$2\frac{\partial \phi}{\partial t}+\frac{\partial^2 \phi}{\partial t^2}=\frac{\partial^2 \phi}{\partial x^2} \text{ for } 0<x<1 \text{ and } t>0,$$

together with the initial conditions

$$\left.\begin{array}{l} \phi(x,0)=2x+1 \\ \left.\dfrac{\partial \phi}{\partial t}\right|_{t=0}=10 \end{array}\right\} \text{ for } 0<x<1$$

and the boundary conditions

$$\left.\begin{array}{l} \phi(0,t)=1 \\ \phi(1,t)=3 \end{array}\right\} \text{ for } t>0$$

The unknown function to be determined is $\phi(x,t)$, where x and t are spatial and time coordinates respectively.

(i) Verify by direct substitution that

$$\phi(x,t)=2x+1+\sum_{n=1}^{\infty} A_n e^{-t}\sin((n^2\pi^2-1)^{1/2}t)\sin(n\pi x)$$

satisfies the partial differential equation in the initial-boundary value problem. (Assume that the coefficients A_n are constants and the series converges.)

(6 marks)

(ii) Which of the initial and boundary conditions stated above are directly satisfied by the function $\phi(x,t)$ given in part (i) above?

(6 marks)

(iii) Use part (i) to solve the initial-boundary value problem. (Derive explicit formula for A_n.)

(6 marks)

2(a) Figure 1 shows a uniform weightless beam ABC rigidly fixed at A, simply supported at B and roller supported against a wall at C. The roller support permits vertical deflection at C but prevents cross sectional rotation of the beam at C. A uniformly distributed load of intensity w (N/m) acts on portion AB and a vertical concentrated force P acts at point C. The beam has a uniform flexural rigidity EI (N.m^2). It is intended to analyse this beam using two 2-node beam elements (one for portion AB and the other for portion BC).

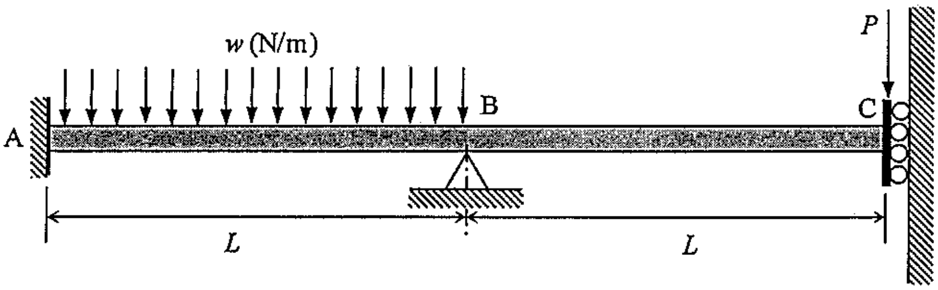

Figure 1

(i) Draw a neat sketch of the finite element model labelling clearly the element numbers, node numbers, nodal displacements and nodal forces. List all the boundary conditions.

(3 marks)

(ii) Generate element stiffness matrices and element load vectors. You may use any of the available methods/formulae to generate them. Obtain global equilibrium equations by assembling the element stiffness matrices, element load vectors and the concentrated load P. Apply boundary conditions to obtain a reduced system of global equalibrium equations.

(8 marks)

(iii) Taking numerical values L = 1 m, w = 1200 kN/m, P = 100 kN and EI = 100 kNm^2, solve the reduced set of global equilibrium equations obtained in part (ii) for the vertical displacement at C and vertical reaction force at B.

(6 marks)

(b) The distribution of temperature (T) due to heat conduction through a uniform rod of thermal conductivity k = 100 W.m^{-1}.°C^{-1}, cross sectional area A = 0.04 m^2 and length L = 1 m is described by the boundary value problem (BVP):

$$kA\frac{d^2T}{dx^2} + \sin \pi x = 0, \quad 0 < x < L \qquad (T \text{ in } ^\circ\text{C})$$

$$T = 0 \quad \text{at } x = 0$$

$$\frac{dT}{dx} = 0 \quad \text{at } x = L$$

It is intended to solve this BVP by Galerkin's method of weighted residual.

Note: Question 2 continues on page 4.

(i) From the three candidate trial solutions given below, choose the one that can satisfy all the three admissibility criteria and thereby construct the admissible trial solution.

$$\text{Trial solution 1: } \tilde{T} = c_0 + c_2 \sin \pi x$$

$$\text{Trial solution 2: } \tilde{T} = c_0 + c_1 x + c_2 \tan \pi x$$

$$\text{Trial solution 3: } \tilde{T} = c_0 + c_1 x + c_2 \sin \pi x$$

(4 marks)

(ii) Using the admissible trial solution constructed in part (i), solve the given BVP for the temperature distribution using Galerkin's method of weighted residual. You may use the following integration formula in your derivation:

$$\int_0^1 (\sin^2 \pi x + \pi x \sin \pi x)dx = \frac{3}{2}$$

(8 marks)

(c) A rod of length L = 1 m and cross sectional area A = 0.04 m^2 is hanging down vertically from a support as shown in Figure 2.

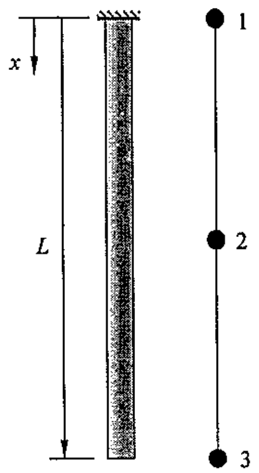

Figure 2

It is intended to model the rod with a single 3-node bar element using the following shape functions:

$$N_1 = \frac{(L-x)(L-2x)}{L^2};\; N_2 = \frac{4x(L-x)}{L^2};\; N_3 = -\frac{x(L-2x)}{L^2}$$

Using the formula given below, derive the element force vector caused by the self-weight (i.e., its own weight) of the bar. The self-weight can be modelled as a distributed tensile force $T = \rho g$ (N/m) where $\rho = 312$ kg/m is the mass per unit length of the bar and $g = 9.81$ m/s^2 is the acceleration due to gravity.

$$\left\{\mathbf{f}^{(e)}\right\} = \int_0^L [\mathbf{N}]^T \, T \, dx$$

(6 marks)

3(a) Name the four terms in the one-dimensional transport equation given by

$$\underbrace{\frac{\partial}{\partial t}(\rho\phi)}_{\mathrm{I}}+\underbrace{\frac{\partial}{\partial x}(\rho\phi u)}_{\mathrm{II}}=\underbrace{\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)}_{\mathrm{III}}+\underbrace{S(\phi)}_{\mathrm{IV}}$$

where the constants ρ, u and Γ are the density, velocity and diffusion coefficient respectively, the variable ϕ is a scalar function of the spatial coordinate x and time t and $S(\phi)$ is a given function of ϕ.

(5 marks)

(b) Consider the particular one-dimensional steady transport equation given by

$$\frac{d}{dx}(\rho\phi u)=\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right).$$

(i) Discretize the above equation in the form of $a_P\phi_p = a_E\phi_E + a_W\phi_W + S_u$, by using the central difference scheme on the uniform mesh shown in Figure 3 below.

(5 marks)

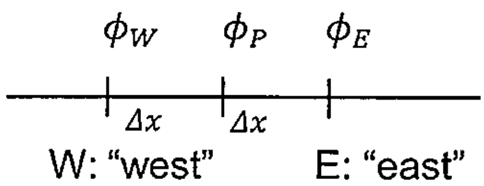

Figure 3

(ii) If $\rho = 1$, $u = 5$, $\Gamma = 1$ and $\Delta x = 0.5$, state whether your discretization in part (i) will yield a solution without wiggles.

(5 marks)

Note: Question 3 continues on page 6.

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(c) Consider the particular one-dimensional unsteady transport equation given by

$$\frac{\partial}{\partial t}(\rho\phi) = \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right).$$

Discretize the above equation on the uniform mesh in Figure 4 below by using

(i) the implicit time scheme in the form of $a_P\phi_p = a_E\phi_E + a_W\phi_W + a_P^0\phi_p^0 + S_u$.

(5 marks)

(ii) the explicit time scheme in the form of $a_P\phi_p = a_E^0\phi_E^0 + a_P^0\phi_P^0 + a_W^0\phi_W^0 + S_u$.

(5 marks)

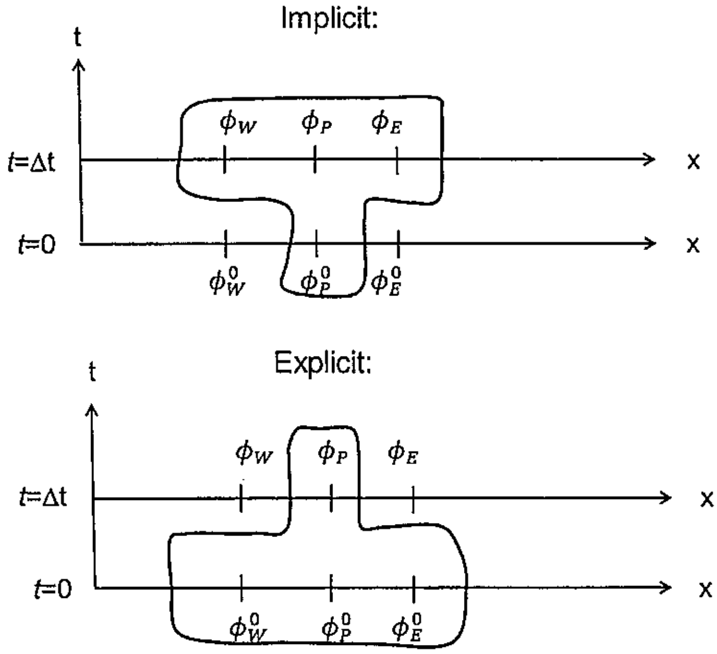


Figure 4

(d) Consider the system of equations

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.$$

(i) Solve the above the system of equations by using the Gauss-Seidel method (up to three iterations) with the initial guess of $\phi_1 = \phi_2 = 0$.

(5 marks)

(ii) Determine whether the iteration is converging or diverging by using the Scarborough criterion.

(5 marks)

END OF PAPER

MA3004 2017/18 Sem 2 (April/May 2018)

1 a) $\frac{\partial u}{\partial x} = 4x\cos(2x^2+y^2)$

$\frac{\partial u}{\partial y} = 2y\cos(2x^2+y^2)$

$\Big\}\ \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y} = -8xy\sin(2x^2+y^2)$

$\frac{\partial^2 u}{\partial x^2} = 4\cos(2x^2+y^2) - 16x^2\sin(2x^2+y^2)$

$\frac{\partial^2 u}{\partial y^2} = 2\cos(2x^2+y^2) - 4y^2\sin(2x^2+y^2)$

Substituting into PDE

$x^2(2\cos(2x^2+y^2) - 4y^2\sin(2x^2+y^2)) + axy(-8xy\sin(2x^2+y^2))$

$+ y^2(4\cos(2x^2+y^2) - 16x^2\sin(2x^2+y^2)) = bx(4x\cos(2x^2+y^2)) + 2y(2y\cos(2x^2+y^2))$

$(\Leftrightarrow)\ (2x^2+4y^2)\cos(2x^2+y^2) + (-4x^2y^2 - 8ax^2y^2 - 16x^2y^2)\sin(2x^2+y^2) = (4bx^2+4y^2)\cos(2x^2+y^2)$

Equating coefficient:

$\cos(2x^2+y^2) \rightarrow 2x^2+4y^2 = 4bx^2+4y^2 \rightarrow b = \frac{1}{2}$

$\sin(2x^2+y^2) \rightarrow -4x^2y^2 - 8ax^2y^2 - 16x^2y^2 = 0 \rightarrow a = -\frac{5}{2}$

b) Substituting $\psi = XY$ into the equation

$X'Y + 10XY = \frac{\partial}{\partial y}((1+x^2)(1+y^2)XY')$

$(X'+10X)Y = (1+x^2)X\frac{\partial}{\partial y}((1+y^2)Y') \rightarrow$ divide by $(1+x^2)XY$

$\frac{X'+10X}{(1+x^2)X} = \frac{1}{Y}\frac{\partial}{\partial y}((1+y^2)Y') = \gamma$

$\Rightarrow X'+10X = \gamma(1+x^2)X \Rightarrow X' + (10-\gamma(1+x^2))X = 0$

$\Rightarrow \frac{\partial}{\partial y}((1+y^2)Y') = \gamma Y \rightarrow \frac{\partial}{\partial y}((1+y^2)Y') - \gamma Y = 0$

c) $\frac{\partial \phi}{\partial t} = \sum_{n=1}^{\infty} A_n\left[(-e^{-t}\sin((n^2\pi^2-1)^{\frac{1}{2}}t) + e^{-t}(n^2\pi^2-1)^{\frac{1}{2}}\cos((n^2\pi^2-1)^{\frac{1}{2}}t)\right]\sin(n\pi x)$

$= \sum_{n=1}^{\infty} A_n\sin(n\pi x)\,e^{-t}\left[-\sin((n^2\pi^2-1)^{\frac{1}{2}}t) + (n^2\pi^2-1)^{\frac{1}{2}}\cos((n^2\pi^2-1)^{\frac{1}{2}}t)\right]$

$\frac{\partial^2 \phi}{\partial t^2} = \sum_{n=1}^{\infty} A_n\sin(n\pi x)\Big[-e^{-t}\left\{-\sin((n^2\pi^2-1)^{\frac{1}{2}}t) + (n^2\pi^2-1)^{\frac{1}{2}}\cos((n^2\pi^2-1)^{\frac{1}{2}}t)\right\}$

$+ e^{-t}\left\{-(n^2\pi^2-1)^{\frac{1}{2}}\cos((n^2\pi^2-1)^{\frac{1}{2}}t) - (n^2\pi^2-1)\sin((n^2\pi^2-1)^{\frac{1}{2}}t)\right\}\Big]$

$= \sum_{n=1}^{\infty} A_n\sin(n\pi x)\,e^{-t}\left[-2(n^2\pi^2-1)^{\frac{1}{2}}\cos((n^2\pi^2-1)^{\frac{1}{2}}t) + (2-n^2\pi^2)\sin((n^2\pi^2-1)^{\frac{1}{2}}t)\right]$

$\frac{\partial^2 \phi}{\partial x^2} = -\sum_{n=1}^{\infty} A_n e^{-t}\sin((n^2\pi^2-1)^{\frac{1}{2}}t)\,(n^2\pi^2)\sin(n\pi x) = RHS$

Substitute LHS

LHS: $\sum_{n=1} A_n\sin(n\pi x)e^{-t}\left[-2\sin((n^2\pi^2-1)^{\frac{1}{2}}t) + 2(n^2\pi^2-1)^{\frac{1}{2}}\cos((n^2\pi^2-1)^{\frac{1}{2}}t)\right]$

$+ \sum_{n=1} A_n\sin(n\pi x)e^{-t}\left[-2(n^2\pi^2-1)^{\frac{1}{2}}\cos((n^2\pi^2-1)^{\frac{1}{2}}t) + (2-n^2\pi^2)\sin((n^2\pi^2-1)^{\frac{1}{2}}t)\right]$

$= -\sum_{n=1} A_n\sin(n\pi x)\,e^{-t}\sin(n^2\pi^2-1)^{\frac{1}{2}}t\,(n^2\pi^2)$

① c) (i) Continued → LHS = RHS (q.e.d)

(ii) $\phi(x,0) = 2x+1 + \sum_{n=1}^{\infty} A_n e^0 \sin(0) \sin(n\pi x) = (2x+1)$ (satisfied)

$\frac{\partial \phi}{\partial t}\Big|_{t=0} = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, e^0 \left[-\sin(0) + (n^2\pi^2-1)^{\frac{1}{2}} \cos(0)\right]$

$= \sum_{n=1}^{\infty} A_n (n^2\pi^2-1)^{\frac{1}{2}} \sin(n\pi x)$ (not satisfied)

$\phi(0,t) = 2(0)+1+\sum_{n=1}^{\infty} A_n e^{-t} \sin\left((n^2\pi^2-1)^{\frac{1}{2}} t\right) \sin(0) = 1$ (satisfied)

$\phi(1,t) = 2(1)+1+\sum_{n=1}^{\infty} A_n e^{-t} \sin\left((n^2\pi^2-1)^{\frac{1}{2}} t\right) \sin(n\pi) = 3$ (satisfied)

(iii) From second initial condition

$10 = \sum_{n=1}^{\infty} A_n (n^2\pi^2-1)^{\frac{1}{2}} \sin(n\pi x)$ for $0<x<1$ → $L=1$

→ $A_n (n^2\pi^2-1)^{\frac{1}{2}} = \frac{2}{1}\int_0^1 (10) \sin\left(\frac{n\pi x}{1}\right) dx$

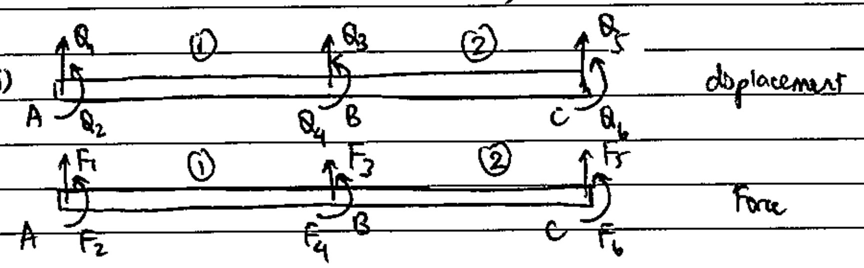
$A_n (n^2\pi^2-1)^{\frac{1}{2}} = 20\left[-\frac{1}{n\pi}\cos(n\pi x)\right]_0^1$

$= 20\left(-\frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}\cos(0)\right)$

$A_n = \frac{20}{n\pi (n^2\pi^2-1)^{\frac{1}{2}}}\left(1-\cos(n\pi)\right)$

$\therefore \phi(x,t) = 2x+1+\sum_{n=1}^{\infty} \frac{20(1-\cos(n\pi))}{n\pi(n^2\pi^2-1)^{\frac{1}{2}}} e^{-t} \sin\left((n^2\pi^2-1)^{\frac{1}{2}} t\right) \sin(n\pi x)$

② a) (i)



Boundary condition : $Q_1, Q_2, Q_3, Q_6 = 0$

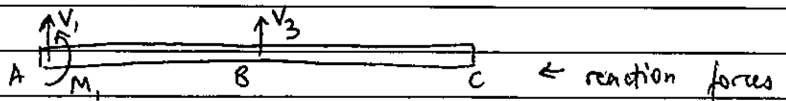
(ii) Bar AB → see chapter 5 pg 21 (Principle of minimum potential energy)

$$\{f\} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \\ f_4^{(1)} \end{bmatrix} = \begin{bmatrix} -wL/2 \\ -wL^2/12 \\ -wL/2 \\ wL^2/12 \end{bmatrix} \qquad (q = -w)$$

~~Bar AB~~

$$\cancel{\frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \begin{bmatrix} f_1^{(1)} \\ f_2^{(1)} \\ f_3^{(1)} \\ f_4^{(1)} \end{bmatrix}}$$

② a) (ii) Continued



← reaction forces

Global equation

$$\frac{EI}{L^3}\begin{bmatrix} \cancel{12} & \cancel{6L} & \cancel{-12} & \cancel{6L} & \cancel{0} & \cancel{0} \\ \cancel{6L} & \cancel{4L^2} & \cancel{-6L} & \cancel{2L^2} & \cancel{0} & \cancel{0} \\ \cancel{-12} & \cancel{-6L} & \cancel{12+12} & \cancel{6L-6L} & \cancel{-12} & \cancel{6L} \\ 6L & 2L^2 & -6L+6L & 4L^2+4L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ \cancel{0} & \cancel{0} & \cancel{6L} & \cancel{2L^2} & \cancel{-6L} & \cancel{4L^2} \end{bmatrix}\begin{bmatrix} \cancel{Q_1}^{\,0} \\ \cancel{Q_2}^{\,0} \\ \cancel{Q_3}^{\,0} \\ Q_4 \\ Q_5 \\ \cancel{Q_6}^{\,0} \end{bmatrix} = \begin{bmatrix} V_1 - wL/2 \\ M_1 - wL^2/12 \\ V_3 - wL/2 \\ wL^2/12 \\ -P \\ 0 \end{bmatrix}$$

Reduced equation

$$\frac{EI}{L^3}\begin{bmatrix} 8L^2 & -6L \\ -6L & 12 \end{bmatrix}\begin{bmatrix} Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} wL^2/12 \\ -P \end{bmatrix}$$

(iii) $$\frac{100\times10^3}{1^3}\begin{bmatrix} 8 & -6 \\ -6 & 12 \end{bmatrix}\begin{bmatrix} Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} 1200\times10^3 \times 1/12 \\ -100\times10^3 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -6 \\ -6 & 12 \end{bmatrix}\begin{bmatrix} Q_4 \\ Q_5 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$8Q_4 - 6Q_5 = 1 \rightarrow 16Q_4 - 12Q_5 = 2$

$-6Q_4 + 12Q_5 = -1 \rightarrow \underline{-6Q_4 + 12Q_5 = -1} \; +$

$10Q_4 = 1 \rightarrow Q_4 = \frac{1}{10}, \quad Q_5 = -\frac{1}{30}$

Vertical displacement at C $= -\frac{1}{30}$ m

From global equation

$$\frac{EI}{L^3}\left(-12\cancel{Q_1}^{\,0} - 6L\cancel{Q_2}^{\,0} + 24\cancel{Q_3}^{\,0} + (0)\cancel{Q_4}^{\,0} - 12Q_5 + 6L\cancel{Q_6}^{\,0}\right) = V_3 - \frac{wL}{2}$$

$$\frac{100\times10^3}{1}\left(-12\times -\frac{1}{30}\right) = V_3 - \frac{100\times10^3\times1}{2}$$

$$V_3 = 90\times10^3 \text{ N} = 90 \text{ kN}$$

b) (i) Domain $0 < x < 1$ $(L = 1)$

→ Must be continuous in the problem domain

→ $\tilde{T}_2$ is not admissible ($\tan(\pi x)$ is not continuous in $0 < x < 1$)

→ Must have continuous non zero function up to max order appearing in D.E

$$\frac{d^2\tilde{T}_1}{dx^2} = -\pi^2 C_2 \sin(\pi x), \quad \frac{d^2\tilde{T}_3}{dx^2} = -\pi^2 C_2 \sin(\pi x)$$

Both are admissible

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Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

② b) (i) Continued

→ Must satisfy boundary condition

$\tilde{T}_1(0) = C_0 + C_2 \sin(0) = C_0 = 0$

$\frac{d\tilde{T}_1}{dx} = \pi C_2 \cos(\pi x) \rightarrow \left.\frac{d\tilde{T}_1}{dx}\right|_{x=1} = \pi C_2 \cos \pi = -\pi C_2 = 0 \rightarrow C_2 = 0$

$\therefore \tilde{T}_1 = 0 + 0 \sin \pi x = 0 \rightarrow$ not admissible

$\tilde{T}_3(0) = C_0 + C_1(0) + C_2 \sin(0) = C_0 = 0$

$\frac{d\tilde{T}_3}{dx} = C_1 + \pi C_2 \cos(\pi x) \rightarrow \left.\frac{d\tilde{T}_3}{dx}\right|_{x=1} = C_1 + \pi C_2 \cos(\pi) = C_1 - \pi C_2 = 0$

$\rightarrow C_1 = \pi C_2$

$\therefore \tilde{T}_3 = 0 + (\pi C_2) x + C_2 \sin(\pi x)$

$= C_2 (\pi x + \sin \pi x) \rightarrow$ admissible ; $\phi_2(x) = \pi x + \sin \pi x$

(ii) $R(x) = kA \frac{d^2 T}{dx^2} + \sin \pi x$

$= 100 \times 0.04 \times (-\pi^2 C_2 \sin \pi x) + \sin \pi x$

$= -4\pi^2 C_2 \sin \pi x + \sin \pi x = (1 - 4\pi^2 C_2) \sin \pi x$

$\int_0^1 \phi_2(x) R(x)\, dx = 0$

$\int_0^1 (\pi x + \sin \pi x)(1 - 4\pi^2 C_2) \sin \pi x \, dx = 0$

$(1 - 4\pi^2 C_2) \int_0^1 (\pi x \sin \pi x + \sin^2 \pi x)\, dx = 0$

$(1 - 4\pi^2 C_2)\left(\frac{3}{2}\right) = 0 \rightarrow C_2 = \frac{1}{4\pi^2}$

$\therefore T(x) = \frac{1}{4\pi^2}(\pi x + \sin \pi x)$

c) $T = \rho g = 312 \times 9.81 = 3060.72 \ kg/s^2 = 3060.72 \ N/m$

$N_1 = (1-x)(1-2x) = 1 - 3x + 2x^2 \rightarrow \int_0^1 N_1\, dx = \int_0^1 (1 - 3x + 2x^2)\, dx$

$= \left[x - \frac{3}{2}x^2 + \frac{2}{3}x^3\right]_0^1 = \frac{1}{6}$

$N_2 = 4x(1-x) = 4x - 4x^2$

$\int_0^1 N_2\, dx = \int_0^1 (4x - 4x^2)\, dx = \left[2x^2 - \frac{4}{3}x^3\right]_0^1 = \frac{2}{3}$

$N_3 = -x(1-2x) = -x + 2x^2$

$\int_0^1 N_3\, dx = \int_0^1 (-x + 2x^2)\, dx = \left[-\frac{1}{2}x^2 + \frac{2}{3}x^3\right]_0^1 = \frac{1}{6}$

$$\{f^{(e)}\} = \int_0^1 \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} T\, dx = T \begin{bmatrix} \int_0^1 N_1\, dx \\ \int_0^1 N_2\, dx \\ \int_0^1 N_3\, dx \end{bmatrix} = 3060.72 \begin{bmatrix} \frac{1}{6} \\ \frac{2}{3} \\ \frac{1}{6} \end{bmatrix} = \begin{bmatrix} 510.12 \\ 2040.48 \\ 510.12 \end{bmatrix} N$$

③ a) I : temporal term (unsteady) → rate of increase of ϕ of fluid element

II : convection term → net rate of flow of ϕ out of fluid element

III : diffusion term → rate of increase of ϕ due to diffusion

IV : source term → rate of increase of ϕ due to sources

b) (i) $\int_{\Delta V} \frac{d}{dx}(\rho u \phi)dV = \int_{\Delta V} \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right)dV$

Gaussian theorem : $\int_A n(\rho u\phi)\, dA = \int_A n\left(\Gamma \frac{d\phi}{dx}\right) dA$

$(n \rho u \phi A)_w + (n \rho u \phi A)_e = \left(n\Gamma \frac{d\phi}{dx}A\right)_w + \left(n\Gamma \frac{d\phi}{dx}A\right)_e$

$\Rightarrow A_w = A_e = A$; $n_w = -1$; $n_e = +1$; ρ, u, Γ constant

$\rightarrow -\rho u \phi_w \cancel{A} + \rho u \phi_e \cancel{A} = -\Gamma\left(\frac{d\phi}{dx}\right)_w \cancel{A} + \Gamma\left(\frac{d\phi}{dx}\right)_e \cancel{A}$

$\Rightarrow \phi_w = \frac{\phi_W + \phi_P}{2}$; $\phi_e = \frac{\phi_P + \phi_E}{2}$

$\left(\frac{d\phi}{dx}\right)_w = \frac{\phi_P - \phi_W}{\Delta x}$; $\left(\frac{d\phi}{dx}\right)_e = \frac{\phi_E - \phi_P}{\Delta x}$

$\rightarrow -\frac{1}{2}\rho u(\phi_W + \phi_P) + \frac{1}{2}\rho u(\phi_E + \phi_P) = -\Gamma\left(\frac{\phi_P - \phi_W}{\Delta x}\right) + \Gamma\left(\frac{\phi_E - \phi_P}{\Delta x}\right)$

$$\underbrace{\left(\frac{2\Gamma}{\Delta x}\right)}_{a_P}\phi_P = \underbrace{\left(\frac{\Gamma}{\Delta x} - \frac{\rho u}{2}\right)}_{a_E}\phi_E + \underbrace{\left(\frac{\Gamma}{\Delta x} + \frac{\rho u}{2}\right)}_{a_W}\phi_W + \underbrace{0}_{S_u}$$

(ii) $\frac{\Gamma}{\Delta x} - \frac{\rho u}{2} = \frac{1}{0.5} - \frac{1 \times 5}{2} = 2 - 2.5 = -0.5 < 0 \rightarrow$ wiggles

c) $\underbrace{\int_t^{t+\Delta t}\int_{\Delta V} \frac{\partial}{\partial t}(\rho\phi)\, dV\, dt}_{I} = \underbrace{\int_t^{t+\Delta t}\int_{\Delta V} \frac{\partial}{\partial x}\left(\Gamma \frac{\partial \phi}{\partial x}\right) dV\, dt}_{II}$

I : $\int_t^{t+\Delta t}\int_{\Delta V} \frac{\partial}{\partial t}(\rho\phi)\, dV\, dt = \int_{\Delta V}\int_t^{t+\Delta t} \frac{\partial}{\partial t}(\rho\phi)\, dt\, dV$

$= \int_{\Delta V} \left[\rho\phi\right]_t^{t+\Delta t} dV$

$= \int_{\Delta V} \rho(\phi_P - \phi_P^0)\, dV = \rho A \Delta x (\phi_P - \phi_P^0)$

II : $\int_t^{t+\Delta t}\int_{\Delta V} \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) dV\, dt = \int_t^{t+\Delta t}\int_A n\left(\Gamma \frac{\partial \phi}{\partial x}\right) dA\, dt$

$= \int_t^{t+\Delta t} A\left[\Gamma\left(\frac{\partial\phi}{\partial x}\right)_e - \Gamma\left(\frac{\partial\phi}{\partial x}\right)_w\right] dt$

$= \int_t^{t+\Delta t} \Gamma A\left[\frac{\phi_E - \phi_P}{\Delta x} - \frac{\phi_P - \phi_W}{\Delta x}\right] dt$

(3) c) (ii) Explicit

$$\text{II}: \Gamma A \Delta t \left\{ \left[\frac{\phi_E^\circ - \phi_P^\circ}{\Delta x} \right] - \left[\frac{\phi_P^\circ - \phi_W^\circ}{\Delta x} \right] \right\}$$

Substitute back

$$\cancel{\rho A \Delta x (\phi_P - \phi_P^\circ) = \Gamma A \frac{\Delta t}{\Delta x} (\phi_E^\circ - 2\phi_P^\circ + \phi_W^\circ)} \text{ (divide by } \Delta t\text{)}$$

$$\left(\rho \frac{\Delta x}{\Delta t}\right) \phi_P = \left(\rho \frac{\Delta x}{\Delta t} - \frac{2\Gamma}{\Delta x}\right) \phi_P^\circ + \left(\frac{\Gamma}{\Delta x}\right) \phi_E^\circ + \left(\frac{\Gamma}{\Delta x}\right) \phi_W^\circ$$

(i) Implicit

$$\text{II}: \Gamma A \Delta t \left[\frac{\phi_E - \phi_P}{\Delta x} - \frac{\phi_P - \phi_W}{\Delta x} \right]$$

Substitute back

$$\cancel{\rho A \Delta x (\phi_P - \phi_P^\circ) = \Gamma A \frac{\Delta t}{\Delta x} (\phi_E - 2\phi_P + \phi_W)} \rightarrow \text{divide by } \Delta t$$

$$\left(\rho \frac{\Delta x}{\Delta t} + 2\frac{\Gamma}{\Delta x}\right) \phi_P = \left(\rho \frac{\Delta x}{\Delta t}\right) \phi_P^\circ + \left(\frac{\Gamma}{\Delta x}\right) \phi_E + \left(\frac{\Gamma}{\Delta x}\right) \phi_W$$

d) (i) $\phi_1 = -1 - 2\phi_2$

$\phi_2 = \frac{1}{4}(-3 - \phi_1)$

No	0	1	2	3
ϕ_1	0	−1	0	0.5
ϕ_2	0	−0.5	−0.75	−0.875

(ii)

$$\begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \begin{matrix} \rightarrow |a_p| = 1 \;;\; \Sigma |a_{nb}| = 2 \\ \rightarrow |a_p| = 4 \;;\; \Sigma |a_{nb}| = 1 \end{matrix}$$

1st equation: $\frac{\Sigma |a_{nb}|}{|a_p|} = \frac{2}{1} = 2 > 1 \rightarrow$ scarborough criterion not fulfilled $\rightarrow$ diverging

However, Gauss - Seidel method shows convergence

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. MAE Club and Campus supplies are not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

Should there be any mistake identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2018-2019

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

November/December 2018 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **THREE (3)** questions and comprises **FIVE (5)** pages.
2. Answer **ALL** questions.
3. Marks for each question are as indicated.
4. This is a **RESTRICTED OPEN-BOOK** examination. One double-sided A4 size reference sheet of paper is allowed.

1(a) If $w(x,y)=(1+e^{x+cy})^{-2}$ is a solution of the partial differential equation

$$\frac{\partial w}{\partial y}-\frac{\partial^2 w}{\partial x^2}=6(1-w)w$$

at all points (x,y) on the Oxy plane, find the value(s) of the constant c.

(9 marks)

(b) Letting $T(r,\theta)=R(r)\Theta(\theta)$, use the method of separation of variables on the partial differential equation

$$\frac{\partial^2 T}{\partial r^2}+\frac{2}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\cot(\theta)}{r^2}\frac{\partial T}{\partial \theta}=0$$

to obtain a pair of ordinary differential equations having an arbitrary constant in them. (Do not attempt to solve the ordinary differential equations.)

(6 marks)

Note: Question 1 continues on page 2.

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(c) Consider the boundary value problem defined by the partial differential equation in $u(x, y)$ given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{25}\frac{\partial^2 u}{\partial y^2} = 2 \text{ for } 0 < x < 1,\ 0 < y < 1,$$

together with the boundary conditions

$$\left.\begin{aligned} u(x,0) &= 0 \\ u(x,1) &= 0 \end{aligned}\right\} \text{for } 0 < x < 1,$$

$$\left.\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_{x=0} &= 0 \\ u(1,y) + \left.\frac{\partial u}{\partial x}\right|_{x=1} &= 25(y^2 - y + 2) \end{aligned}\right\} \text{for } 0 < y < 1.$$

(i) Verify by direct substitution that the partial differential equation has a solution of the form

$$u(x,y) = 25y(y-1) + \sum_{n=1}^{\infty} A_n (e^{n\pi x/5} + e^{-n\pi x/5}) \sin(n\pi y),$$

where A_n are constant coefficients.

(5 marks)

(ii) Show that the first three boundary conditions stated above are satisfied by the solution in part (i).

(4 marks)

(iii) Solve the boundary value problem stated above. (Derive explicit formula for A_n).

(6 marks)

2(a) The mesh in Figure 1(a) is made up of eleven (11) constant strain triangle elements and thirteen (13) nodes. Several problems are present in the mesh.

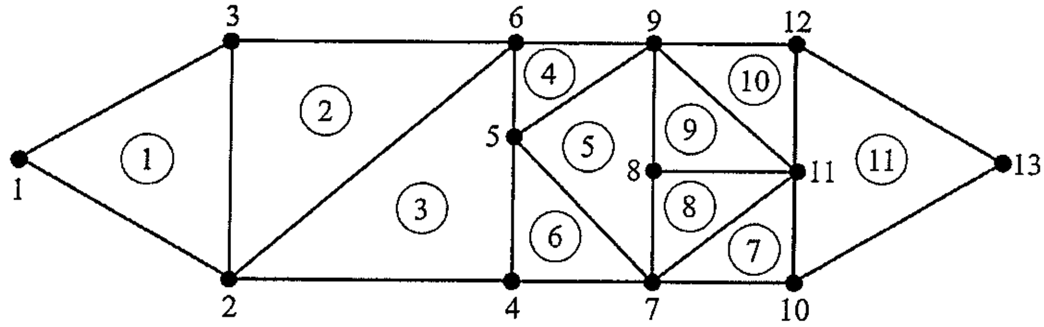

Figure 1(a)

(i) Identify the problems in the mesh and explain the origin of the mistake in each problem identified.

(9 marks)

(ii) List one disadvantage of using constant strain triangle element.

(2 marks)

(b) Figure 1(b) shows a structure made up of two (2) beam elements. Element 1 consists of node 1 and 2, and has a length of 2 m. Element 2 consists of node 2 and 3, and has a length of 1 m. Downward force $P = 2700$ N and moment $M = 4000$ N·m are applied to node 2. Node 1 is fixed to the wall, while node 3 is supported on a roller. The modulus of elasticity of the structure is 200 GPa and the moment of inertia is 4×10^{-5} m^4.

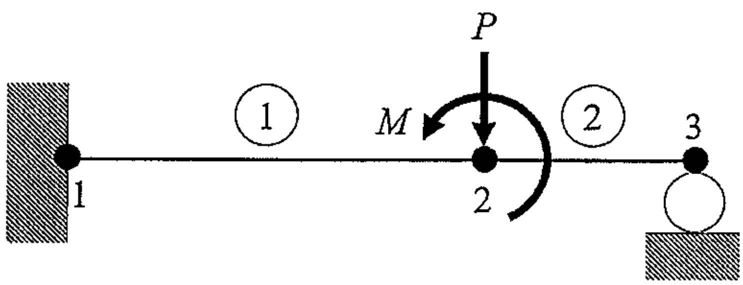

Figure 1(b)

Note: Question 2 continues on page 4.

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(i) Determine the element stiffness matrices of element 1 and 2.

(4 marks)

(ii) Determine the global stiffness matrix.

(6 marks)

(iii) Formulate the global finite element equation.

(6 marks)

(iv) State the boundary conditions.

(3 marks)

(v) Formulate the reduced finite element equation.

(3 marks)

(vi) Determine the displacement and rotation at node 2 if the rotation at node 3 is 1.125×10^{-4} rad.

(2 marks)

3. The one-dimensional steady convection-diffusion equation is given by

$$\frac{d}{dx}(\rho u \phi) = \frac{d}{dx}\left(\Gamma \frac{d\phi}{dx}\right) + S(\phi),$$

where ϕ is the concentration which depends on only the x coordinate, the velocity u, the density ρ and the diffusion coefficient Γ are given constants, and $S(\phi)$ is the source or sink.

(a) For the case where $S(\phi) = C\phi + G$, where C and G are constants, by integrating over the control volume in Figure 2 from w to e, use the finite volume method together with the central difference scheme on both the diffusion and convection terms to derive a discretised equation of the form

$$\alpha_P \phi_P = \alpha_E \phi_E + \alpha_W \phi_W + S_u,$$

where ϕ_W, ϕ_P and ϕ_E are the values of ϕ at the nodes W, P and E respectively. State clearly the coefficients α_P, α_E, α_W and S_u in terms of C, G, u, ρ, Γ and Δx, where $2\Delta x$ is the distance between the nodes W and E.

(The node P is the midpoint between W and E and both w and e are at a distance of $\frac{1}{2}\Delta x$ from P).

(8 marks)

Note: Question 3 continues on page 5.
Figure 2 appears on page 5.

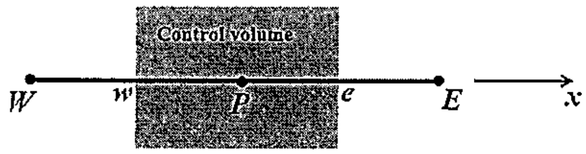

Figure 2

(b) If the values of the constants u, ρ, Γ, C, and G are 1, 7, 2, 2 and 3 respectively and the length Δx is 1, does the numerical solution obtained from the discretised equation in part (a) have wiggles? (Assume that the units of the given values are consistent.)

(4 marks)

(c) Using the data given in part (b), determine the range of the values that Δx can have if the discretised equation in part (a) is to give a stable numerical solution without wiggles?

(4 marks)

(d) Repeat part (a) by using the upwind scheme (for the convection term only). Consider both positive and negative values of u. State clearly the coefficients α_P, α_E, α_W and S_u. For each of the coefficients α_W and α_E, give a single expression that is valid for both positive and negative values of u.

(12 marks)

(e) Solve the linear algebraic system

$$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

numerically by using

(i) Gauss-Seidel method
(ii) Jacobi method

For each of the method, give only the first three iterations, excluding iteration number zero given by the initial guess of $\phi_1 = \phi_2 = 0$. Comment on the convergence of the numerical solution. Verify your comment by using the Scarborough criterion.

(7 marks)

End of Paper

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MA3004 PYP 2018-2019 S1

1a) $w = (1+e^{x+cy})^{-2}$

$$\frac{\partial w}{\partial y} = \frac{-2c\cdot e^{x+cy}}{(1+e^{x+cy})^2} = -2c\cdot e^{x+cy}\cdot w^{3/2}$$

$$\frac{\partial w}{\partial x} = \frac{-2\cdot e^{x+cy}}{(1+e^{x+cy})^3}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{-2\cdot e^{x+cy}(1+e^{x+cy})^3 + 6e^{2(x+cy)}(1+e^{x+cy})^2}{(1+e^{x+cy})^6}$$

$$= \frac{-2e^{x+cy}(1+e^{x+cy}) + 6e^{2(x+cy)}}{(1+e^{x+cy})^4}$$

$$= \left[2e^{x+cy}(2e^{x+cy}-1)\right]w^2$$

$$\frac{\partial w}{\partial y} - \frac{\partial^2 w}{\partial x^2} = w\left[-2cw^{1/2}e^{x+cy} - 2e^{x+cy}\cdot w(2\cdot e^{x+cy}-1)\right]$$

$$= w\left[\frac{-2c\cdot e^{x+cy}(1+e^{x+cy}) - 2e^{x+cy}(2\cdot e^{x+cy}-1)}{(1+e^{x+cy})^2}\right]$$

$$= w\left[\frac{e^{2(x+cy)}(-2c-4) + e^{x+cy}(-2c+2)}{(1+e^{x+cy})^2}\right]$$

$$= w\left[\frac{6(1+e^{x+cy})^2 + e^{2(x+cy)}(-2c-10) + e^{x+cy}(-2c-10) - 6}{(1+e^{x+cy})^2}\right]$$

$$= w\left[6 - \frac{6 - e^{2(x+cy)}(-2c-10) - e^{x+cy}(-2c-10)}{(1+e^{x+cy})^2}\right]$$

$$= 6w\left[1 - w\left(1 - \frac{1}{6}e^{2(x+cy)}(-2c-10) - \frac{1}{6}e^{x+cy}(-2c-10)\right)\right] = 6w[1-w]$$

$$1 - \frac{1}{6}e^{2(x+cy)}(-2c-10) - \frac{1}{6}e^{x+cy}(-2c-10) = 1$$

For the equation to be valid at all points (x,y),

$-2c-10 = 0$

$c = -5$

1b) $T(r,\theta) = R(r)\,\Theta(\theta)$

$\dfrac{\partial T}{\partial r} = R'(r)\,\Theta(\theta)$ $\qquad$ $\dfrac{\partial T}{\partial \theta} = R(r)\,\Theta'(\theta)$

$\dfrac{\partial^2 T}{\partial r^2} = R''(r)\,\Theta(\theta)$ $\qquad$ $\dfrac{\partial^2 T}{\partial \theta^2} = R(r)\,\Theta''(\theta)$

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\cdot\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\cot\theta}{r}\cdot\frac{\partial T}{\partial \theta} = 0$$

$$R''(r)\Theta(\theta) + \frac{2}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) + \frac{\cot\theta}{r^2}R(r)\Theta'(\theta) = 0$$

$$\frac{R''(r) + \frac{2}{r}R'(r)}{R(r)} = -\frac{\frac{1}{r^2}\Theta''(\theta) + \frac{\cot\theta}{r^2}\Theta'(\theta)}{\Theta(\theta)}$$

$$\frac{r^2R''(r) + 2rR'(r)}{R(r)} = -\frac{\Theta''(\theta) + \cot\theta\cdot\Theta'(\theta)}{\Theta(\theta)} = \gamma$$

(1) $r^2R''(r) + 2rR'(r) - \gamma R(r) = 0$

(2) $\Theta''(\theta) + \cot\theta\,\Theta'(\theta) + \gamma\,\Theta(\theta) = 0$

1c) i) $u(x,y) = 25y(y-1) + \sum_{n=1}^{\infty} A_n\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)$

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} A_n\cdot\frac{n\pi}{5}\left(e^{\frac{n\pi x}{5}} - e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} A_n\cdot\frac{n^2\pi^2}{25}\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)$$

$$\frac{\partial u}{\partial y} = 25(y-1) + 25y + \sum_{n=1}^{\infty} A_n\cdot n\pi\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\cos(n\pi y)$$

$$\frac{\partial^2 u}{\partial y^2} = 50 - \sum_{n=1}^{\infty} A_n\cdot n^2\pi^2\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{25}\frac{\partial^2 u}{\partial y^2} = \sum_{n=1}^{\infty} A_n\cdot\frac{n^2\pi^2}{25}\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y) + 2 - \sum_{n=1}^{\infty} A_n\cdot\frac{n^2\pi^2}{25}\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)$$

$= 2$ (Verified)

1c) ii) $u(x,y) = 25y(y-1) + \sum_{n=1}^{\infty} A_n\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)$

$u(x,0) = 25(0)(0-1) + \sum_{n=1}^{\infty} A_n\left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(0) = 0$ (Shown)

$u(x,1) = 25(1)(1-1) + \sum_{n=1}^{\infty} A_n\left(e^{\frac{n\pi x}{5}} + e^{\frac{n\pi x}{5}}\right)\sin(n\pi) = 0$ (Shown)

$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} A_n \cdot \frac{n\pi}{5}\left(e^{\frac{n\pi x}{5}} - e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)$

$\left.\frac{\partial u}{\partial x}\right|_{x=0} = \sum_{n=1}^{\infty} A_n \cdot \frac{n\pi}{5}\left(e^{0} - e^{0}\right)\sin(n\pi y) = 0$ (Shown)

iii) $u(1,y) + \left.\frac{\partial u}{\partial x}\right|_{x=1} = 25y(y-1) + \sum_{n=1}^{\infty} A_n \cdot \left(e^{\frac{n\pi}{5}} + e^{-\frac{n\pi}{5}}\right)\sin(n\pi y) + \sum_{n=1}^{\infty} A_n\left(e^{\frac{n\pi}{5}} - e^{-\frac{n\pi}{5}}\right)\sin(n\pi y)$

$25(y^2 - y + 2) = 25y(y-1) + 2\sum_{n=1}^{\infty} A_n \cdot e^{\frac{n\pi}{5}} \cdot \sin(n\pi y)$

$25 = \sum_{n=1}^{\infty} A_n \cdot e^{\frac{n\pi}{5}} \cdot \sin(n\pi y) \quad 0 < y < 1$

$A_n \cdot e^{\frac{n\pi}{5}} = \frac{2}{L}\int^{L} f(y)\sin\left(\frac{n\pi y}{L}\right) \cdot dy$

$= 2\int_0^1 25\sin(n\pi y) \cdot dy$

$= 2\left[-\frac{25}{n\pi}\cos(n\pi y)\right]_0^1 = 2\left(-\frac{25}{n\pi}(-1)^n + \frac{25}{n\pi}\right)$

$A_n = \frac{50\left(1-(-1)^n\right)}{n\pi \cdot e^{\frac{n\pi}{5}}}$

$\therefore u(x,y) = 25y(y-1) + \sum_{n=1}^{\infty}\left[\frac{50\left(1-(-1)^n\right)}{n\pi \cdot e^{\frac{n\pi}{5}}} \cdot \left(e^{\frac{n\pi x}{5}} + e^{-\frac{n\pi x}{5}}\right)\sin(n\pi y)\right]$

P(2)

2a) i) The mesh representation is not symmetric. As a result, if a force / bending moment is given in any nodes, the strain gradient will be big. However, Constant Strain Triangle always assumes that the element undergoes constant strain. Thus, the measurement using CST will not be accurate.

Also, element 3, 5, 11 have 4 nodes. However, CST representation only allows 3 elements. This is because the system is not discretised properly. Thus, some elements have 4 nodes.

ii) -) Only able to be used in areas where the strain gradient is small
-) Cannot be used in stress concentrated areas in the structure, such as edges of holes and corners
-) To increase the accuracy using CST, it will take longer time to solve the equation and find the inverse if 3 nodes triangular element is upgraded into 6 nodes triangular element

2b) i)

$$[k_1] = \frac{E_1 I_1}{L_1^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} = \frac{2\cdot 10^{11}\cdot 4\cdot 10^{-5}}{8}\begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix}$$

$$= 10^6 \begin{bmatrix} 12 & 12 & -12 & 12 \\ 12 & 16 & -12 & 8 \\ -12 & -12 & 12 & -12 \\ 12 & 8 & -12 & 16 \end{bmatrix} \text{N/m}$$

$$[k_2] = \frac{E_2 I_2}{L_2^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} = \frac{2\cdot 10^{11}\cdot 4\cdot 10^{-5}}{1}\begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

$$= 8\cdot 10^6 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \text{N/m}$$

ii)

$$[K] = 10^6 \begin{bmatrix} 12 & 12 & -12 & 12 & 0 & 0 \\ 12 & 16 & -12 & 8 & 0 & 0 \\ -12 & -12 & 12+8\cdot 12 & -12+8\cdot 6 & -96 & 48 \\ 12 & 8 & -12+8\cdot 6 & 16+8\cdot 4 & -48 & 16 \\ 0 & 0 & -96 & -48 & 96 & -48 \\ 0 & 0 & 48 & 16 & -48 & 32 \end{bmatrix} = 10^6 \begin{bmatrix} 12 & 12 & -12 & 12 & 0 & 0 \\ 12 & 16 & -12 & 8 & 0 & 0 \\ -12 & -12 & 96 & 36 & -96 & 48 \\ 12 & 8 & 36 & 48 & -48 & 16 \\ 0 & 0 & -96 & -48 & 96 & -48 \\ 0 & 0 & 48 & 16 & -48 & 32 \end{bmatrix} \text{N/m}$$

iii)

$$\begin{bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{bmatrix} = [k] \begin{bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{bmatrix} \qquad \begin{bmatrix} F_{1y} \\ M_1 \\ F_{2y} \\ M_2 \\ F_{3y} \\ M_3 \end{bmatrix} = 10^6 \begin{bmatrix} 12 & 12 & -12 & 12 & 0 & 0 \\ 12 & 16 & -12 & 8 & 0 & 0 \\ -12 & -12 & 96 & 36 & -96 & 48 \\ 12 & 8 & 36 & 48 & -48 & 16 \\ 0 & 0 & -96 & -48 & 96 & -48 \\ 0 & 0 & 48 & 16 & -48 & 32 \end{bmatrix} \begin{bmatrix} d_{1y} \\ \phi_1 \\ d_{2y} \\ \phi_2 \\ d_{3y} \\ \phi_3 \end{bmatrix}$$

iv) $d_{1y} = 0$

$\phi_1 = 0$

$d_{3y} = 0$

v)

$$\begin{bmatrix} \cancel{F_{1y}} \\ \cancel{M_1} \\ F_{2y} \\ M_2 \\ \cancel{F_{3y}} \\ M_3 \end{bmatrix} = 10^6 \begin{bmatrix} \cancel{12} & \cancel{12} & \cancel{-12} & \cancel{12} & \cancel{0} & \cancel{0} \\ \cancel{12} & \cancel{16} & \cancel{-12} & \cancel{8} & \cancel{0} & \cancel{0} \\ -12 & -12 & 96 & 36 & -96 & 48 \\ 12 & 8 & 36 & 48 & -48 & 16 \\ \cancel{0} & \cancel{0} & \cancel{-96} & \cancel{-48} & \cancel{96} & \cancel{-48} \\ 0 & 0 & 48 & 16 & -48 & 32 \end{bmatrix} \begin{bmatrix} \cancel{0} \\ \cancel{0} \\ d_{2y} \\ \phi_2 \\ \cancel{0} \\ \phi_3 \end{bmatrix}$$

$$\begin{bmatrix} F_{2y} \\ M_2 \\ M_3 \end{bmatrix} = 10^6 \begin{bmatrix} 96 & 36 & 48 \\ 36 & 48 & 16 \\ 48 & 16 & 32 \end{bmatrix} \begin{bmatrix} d_{2y} \\ \phi_2 \\ \phi_3 \end{bmatrix}$$

vi)

$$\begin{bmatrix} -2700 \\ 4000 \\ \cancel{0} \end{bmatrix} = 10^6 \begin{bmatrix} 96 & 36 & 48 \\ 36 & 48 & 16 \\ \cancel{48} & \cancel{16} & \cancel{32} \end{bmatrix} \begin{bmatrix} d_{2y} \\ \phi_2 \\ \cancel{1.125 \cdot 10^{-4}} \end{bmatrix}$$

$$-2700 = 10^6 (96 d_{2y} + 36 \phi_2 + 48 \cdot 1.125 \cdot 10^{-4})$$

$$-27 \cdot 10^{-4} = 96 d_{2y} + 36 \phi_2 + 54 \cdot 10^{-4}$$

$$-81 \cdot 10^{-4} = 96 d_{2y} + 36 \phi_2$$

$$-27 \cdot 10^{-4} = 32 d_{2y} + 12 \phi_2 \quad \ldots (1)$$

$$4 \cdot 10^3 = 10^6 (36 d_{2y} + 48 \phi_2 + 16 \cdot 1.125 \cdot 10^{-4})$$

$$4 \cdot 10^{-3} = 36 d_{2y} + 48 \phi_2 + 1.8 \cdot 10^{-3} \longrightarrow 11 \cdot 10^{-4} = 18 d_{2y} + 24 \phi_2 \quad \ldots (2)$$

Eliminate (1), (2)

$-59 \cdot 10^{-4} = 64 d_{2y} + 24\phi_2$

$11 \cdot 10^{-4} = 18 d_{2y} + 24\phi_2$

$-65 \cdot 10^{-4} = 46 d_{2y}$

$d_{2y} = -\frac{65}{46} \cdot 10^{-4}\, m$

$\phi_2 = 1.52 \cdot 10^{-4}\, rad$

3a) $\iiint \frac{\partial}{\partial x} (\rho U \phi) \cdot dV = \iiint \frac{\partial}{\partial x}\left(\Gamma \cdot \frac{d\phi}{dx}\right) \cdot dV + \iiint S \cdot dV$

$A \int \frac{\partial}{\partial x}(\rho U \phi) \cdot dx = A \cdot \int \frac{\partial}{\partial x}\left(\Gamma \cdot \frac{d\phi}{dx}\right) \cdot dx + A \int S \cdot dx$

$(\rho U \phi)_e - (\rho U \phi)_w = \Gamma\left(\frac{\phi_E - \phi_P}{\Delta x}\right) - \Gamma\left(\frac{\phi_P - \phi_W}{\Delta x}\right) + (G + C\phi_P)\Delta x$

$\rho U\left(\frac{\phi_E + \phi_P}{2}\right) - \rho U\left(\frac{\phi_P + \phi_W}{2}\right) = \Gamma\left(\frac{\phi_E - \phi_P}{\Delta x}\right) - \Gamma\left(\frac{\phi_P - \phi_W}{\Delta x}\right) + (G + C\phi_P)\Delta x$

$\left(2\frac{\Gamma}{\Delta x} - C \cdot \Delta x\right)\phi_P = \left(-\frac{1}{2}\rho U + \frac{\Gamma}{\Delta x}\right)\phi_E + \left(\frac{1}{2}\rho U + \frac{\Gamma}{\Delta x}\right)\phi_W + G \cdot \Delta x$

$\alpha_P = 2\frac{\Gamma}{\Delta x} - C \cdot \Delta x$

$\alpha_E = -\frac{1}{2}\rho U + \frac{\Gamma}{\Delta x}$

$\alpha_W = \frac{1}{2}\rho U + \frac{\Gamma}{\Delta x}$

$S_U = G \cdot \Delta x$

b) $Pe = \frac{F}{D} = \frac{\rho U}{\frac{\Gamma}{\Delta x}} = \frac{7 \cdot 1 \cdot 1}{2} = 3.5 > 2$ (will have wiggles)

c) $Pe < 2$

$\frac{\rho U}{\frac{\Gamma}{\Delta x}} < 2$

$\frac{7 \cdot 1 \cdot \Delta x}{2} < 2$

$0\,m < \Delta x < \frac{4}{7}\, m$

P(4)

d) $\iiint \frac{\partial}{\partial x}(\rho u\phi)\cdot dV = \iiint \frac{\partial}{\partial x}\left(\Gamma \cdot \frac{d\phi}{dx}\right)\cdot dV + \iiint S \cdot dV$

$A\int \frac{\partial}{\partial x}(\rho u\phi)\cdot dx = A\int \frac{\partial}{\partial x}\left(\Gamma\cdot\frac{d\phi}{dx}\right)\cdot dx + A\int S\cdot dx$

$(\rho u\phi)_e - (\rho u\phi)_w = \Gamma\left(\frac{\phi_E-\phi_P}{\Delta x}\right) - \Gamma\left(\frac{\phi_P-\phi_W}{\Delta x}\right) + (G + C\phi_P)\Delta x$

For: $u > 0$

$\rho u\phi_P - \rho u\phi_W = \Gamma\left(\frac{\phi_E-\phi_P}{\Delta x}\right) - \Gamma\left(\frac{\phi_P-\phi_W}{\Delta x}\right) + (G + C\phi_P)\Delta x$

$\left(\rho u + 2\frac{\Gamma}{\Delta x} - C\cdot\Delta x\right)\phi_P = \left(\frac{\Gamma}{\Delta x}\right)\phi_E + \left(\rho u + \frac{\Gamma}{\Delta x}\right)\phi_W + G\cdot\Delta x$

$\alpha_P = \rho u + 2\frac{\Gamma}{\Delta x} - C\cdot\Delta x$

$\alpha_E = \frac{\Gamma}{\Delta x}$

$\alpha_W = \rho u + \frac{\Gamma}{\Delta x}$

$S_u = G\cdot\Delta x$

For: $u < 0$

$\rho u\phi_E - \rho u\phi_P = \Gamma\left(\frac{\phi_E-\phi_P}{\Delta x}\right) - \Gamma\left(\frac{\phi_P-\phi_W}{\Delta x}\right) + (G + C\phi_P)\Delta x$

$\left(-\rho u + 2\frac{\Gamma}{\Delta x} - C\cdot\Delta x\right)\phi_P = \left(-\rho u + \frac{\Gamma}{\Delta x}\right)\phi_E + \left(\frac{\Gamma}{\Delta x}\right)\phi_W + G\cdot\Delta x$

$\alpha_P = -\rho u + 2\frac{\Gamma}{\Delta x} - C\cdot\Delta x$

$\alpha_E = -\rho u + \frac{\Gamma}{\Delta x}$

$\alpha_W = \frac{\Gamma}{\Delta x}$

$S_u = G\cdot\Delta x$

For: $u > 0$ & $u < 0$,

Let: $\langle x\rangle = x$, for $x \geq 0$

$= 0$, for $x < 0$

$\alpha_E = -\rho\langle -u\rangle + \frac{\Gamma}{\Delta x}$

$\alpha_W = \rho\langle u\rangle + \frac{\Gamma}{\Delta x}$

3e) i) $\phi_1 + 4\phi_2 = -1$

$\phi_1 = -1 - 4\phi_2 \ldots (1)$

$-\phi_1 - \phi_2 = 5$

$\phi_2 = -\phi_1 - 5 \ldots (2)$

0^{th} iteration : $\phi_1 = 0$

$\phi_2 = 0$

1^{st} iteration : $\phi_1 = -1$

$\phi_2 = -4$

2^{nd} iteration : $\phi_1 = 15$

$\phi_2 = -20$

3^{rd} iteration : $\phi_1 = 79$

$\phi_2 = -84$

Based on the numerical solution, ϕ_1 & ϕ_2 diverge since the difference between the value of ϕ_1 or ϕ_2 on the 2^{nd} and 3^{rd} iteration are greater than that on the 1^{st} and 2^{nd} iteration.

ii) 0^{th} iteration : $\phi_1 = 0$

$\phi_2 = 0$

1^{st} iteration : $\phi_1 = -1$

$\phi_2 = -5$

2^{nd} iteration : $\phi_1 = 19$

$\phi_2 = -4$

3^{rd} iteration : $\phi_1 = 15$

$\phi_2 = -24$

Based on the numerical solution, ϕ_1 can be convergent since the difference between the value on the 2^{nd} and 3^{rd} iteration is smaller than that on the 1^{st} and 2^{nd} iteration.

However, ϕ_2 is divergent since the difference between the value on the 2^{nd} and 3^{rd} iteration are greater than that on the 1^{st} and 2^{nd} iteration

P(5)

Using Scarborough criterion:

$$\begin{bmatrix} 1 & 4 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

For node ϕ_1 : $\frac{\sum |a_{nb}|}{|a_p|} = \frac{4}{1} = 4 > 1$ (Divergent)

For node ϕ_2 : $\frac{\sum |a_{nb}|}{|a_p|} = 1$

Since the numerical value does not follow the condition for convergence, by Scarborough criterion, the numerical solution diverges

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2019-2020

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

November/December 2019 Time Allowed: 2½ hours

INSTRUCTIONS

1. This paper contains **THREE** (3) questions and comprises **SIX** (6) pages.
2. Answer **ALL** questions.
3. Marks for each question are as indicated.
4. This is a **RESTRICTED-OPEN BOOK** examination. One double-sided A4 size reference sheet of paper is allowed.

1(a) If $w(x,y)=(y+ax^3)^{10}e^{bx^4-xy}$ is a solution of the partial differential equation

$$\frac{\partial w}{\partial x}+x^2\frac{\partial w}{\partial y}+cyw=0$$

at all points (x,y) on the Oxy plane, find the values of the constants a, b and c.

(6 marks)

(b) Letting $\phi(x,y)=X(x)Y(y)$, use the method of separation of variables on the partial differential equation

$$\frac{\partial}{\partial x}\left((1+y^2)(1+2x^2)\frac{\partial \phi}{\partial x}\right)+\frac{\partial}{\partial y}\left((1+y^2)(1+2x^2)\frac{\partial \phi}{\partial y}\right)=0$$

to obtain a pair of ordinary differential equations with an arbitrary constant in them. (Do not attempt to solve the ordinary differential equations.)

(6 marks)

Note: Question 1 continues on page 2.

(c) Consider the boundary value problem defined by the partial differential equation in $u(x,y)$ given by

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+2\frac{\partial u}{\partial x}=0 \text{ for } 0<x<1,\ 0<y<1,$$

together with the boundary conditions

$$\left.\begin{aligned} u(x,0)&=0\\ u(x,1)&=0\end{aligned}\right\}\text{for } 0<x<1,$$

$$\left.\begin{aligned} \left.\frac{\partial u}{\partial x}\right|_{x=0}&=0\\ u(1,y)&=10\end{aligned}\right\}\text{for } 0<y<1.$$

(i) Verify by direct substitution that the partial differential equation in the boundary value problem above has a solution of the form

$$u(x,y)=\sum_{n=1}^{\infty}(A_n e^{(-1+c_n)x}+B_n e^{(-1-c_n)x})\sin(n\pi y),$$

where $c_n=(1+n^2\pi^2)^{1/2}$ and A_n and B_n are any constant coefficients.

(8 marks)

(ii) Show that the first two boundary conditions stated above (for $0<x<1$) are satisfied by the solution in part (i).

(2 marks)

(iii) Use the third boundary condition to express B_n in terms of A_n.

(2 marks)

(iv) Use parts (i) and (iii) to derive an explicit series solution for the boundary value problem above.

(6 marks)

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2(a) Figure 1 shows a straight uniform bar AB of length L = 1 m and axial rigidity $EA = 50\text{x}10^6$ N hinged at A and connected to two springs of equal stiffness $k = 5\text{x}10^6$ N/m at B. The other ends of the springs are roller-supported as shown. A concentrated force $P = 10^6$ N is applied at B in a direction perpendicular to the axis of bar AB. It is intended to solve this problem by finite element method using a *truss element* to model the bar and *spring elements* to model the springs.

(i) Mark the element numbers, node numbers and nodal displacements on a sketch of the finite element model.

(3 marks)

(ii) Write the stiffness matrices, assemble the element stiffness matrices to obtain the global equilibrium equations, and obtain the reduced system of equations by applying the boundary conditions and loads.

(10 marks)

(iii) Solve the reduced system for the horizontal and vertical displacements at point B.

(5 marks)

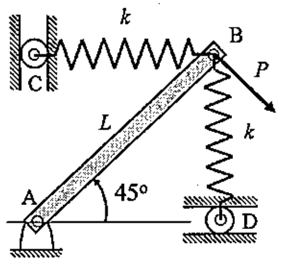

Figure 1

Note: Question 2 continues on page 4.

(b) Figure 2(a) shows a straight uniform beam of length L = 1 m fixed at the left end at A and carrying a linearly varying load of intensity $q(x) = 10x$. It is intended to model this problem using a beam element with the nodal degrees of freedom as defined in Figure 2(b).

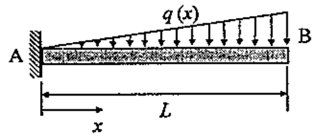

Figure 2(a)

Figure 2(b)

The shape functions of the beam element are as follows:

$$N_1 = (2x^3 - 3Lx^2 + L^3)/L^3; \quad N_2 = (x^3 - 2Lx^2 + L^2x)/L^2$$
$$N_3 = -(2x^3 - 3Lx^2)/L^3; \quad N_4 = (x^3 - Lx^2)/L^2,$$

where N_1, N_2, N_3 and N_4 correspond to the degrees of freedom w_1, θ_1, w_2 and θ_2, respectively, as defined in Figure 2(b).

Determine the lumped nodal forces and moments using the formula

$$\{f_q^e\} = \int_0^L [N]^T q(x)dx$$

and mark them on a sketch of the beam element.

(8 marks)

(c) It is intended to solve the following boundary value problem (BVP) by Galerkin's method of weighted residual:

$$\frac{d^3u}{dx^3} - 6x = 0, \quad 0 < x < 1$$
$$u = 0 \text{ at } x = 0$$
$$u = 1 \text{ at } x = 1$$

(i) Three trial solutions are given below:

$$\tilde{u} = c_0 + c_1x; \quad \tilde{u} = c_0 + c_1x + c_2x^2; \quad \tilde{u} = c_0 + c_1x + c_2x^3.$$

Choose the one that satisfies all the three admissibility criteria.

(4 marks)

(ii) Using your choice of trial solution in part (i), solve the above BVP for $u(x)$ by Galerkin's method of weighted residual.

(5 marks)

3(a) The one-dimensional general transport equation for a variable ϕ with constant velocity (u), positive constant fluid property ($\kappa > 0$), and no source term is given by:

$$\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = \kappa \frac{\partial^2 \phi}{\partial x^2}.$$

(i) Derive the discretized equation of the form $a_P\phi_P = a_E\phi_E + a_W\phi_W + a_P^0\phi_P^0 + S_u$, by using the implicit time scheme with uniform time step Δt for temporal discretization and the central differencing scheme with uniform space step Δx for spatial discretization, as shown Figure 3.

(10 marks)

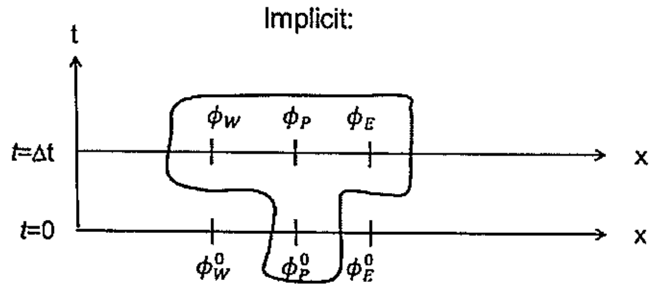

Figure 3

(ii) Derive the discretized equation again by changing only the scheme for spatial discretization in part 3(a)(i) to the upwind differencing scheme, for **both positive and negative velocities** ($u > 0$ and $u < 0$).

(10 marks)

Note: Question 3 continues on page 6.

(b) (i) Analyze whether the Scarborough criterion is fulfilled or not for the system of three linear algebraic equations below. If an iterative method such as the Gauss-Seidel method is used to solve this system of linear algebraic equations, do you expect the iterative solution to converge or diverge?

$$\begin{cases} x_1 + x_3 = -1 \\ x_1 + 2x_2 - x_3 = 1 \\ x_2 - 2x_3 = 0 \end{cases}.$$

(7 marks)

(ii) Solve the above system of equations by using **both Jacobi iterative method and Gauss-Seidel iterative method** with the initial guess of $x_1 = x_2 = x_3 = 0$. For each of the methods, give solutions from only the first three iterations.

(8 marks)

End of Paper

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1a) $w(x,y) = (y+ax^3)^{10} e^{bx^4 - xy}$

$$\frac{\partial w}{\partial x} = 10(y+ax^3)^9 (3ax^2) e^{bx^4-xy} + (4bx^3 - y) e^{bx^4-xy} (y+ax^3)^{10}$$
$$= (y+ax^3)^9 (e^{bx^4-xy}) \left[30ax^2 + (4bx^3-y)(y+ax^3)\right]$$

$$\frac{\partial w}{\partial y} = 10(y+ax^3)^9 e^{bx^4-xy} + (y+ax^3)^{10}(-x) e^{bx^4-xy}$$
$$= (y+ax^3)^9 (e^{bx^4-xy}) \left[10 + (y+ax^3)(-x)\right]$$

$$\frac{\partial w}{\partial x} + x^2 \frac{\partial w}{\partial y} + cyw = 0$$
$$(y+ax^3)^9 (e^{bx^4-xy}) \left[30ax^2 + (4bx^3-y)(y+ax^3)\right]$$
$$+ (y+ax^3)^9 (e^{bx^4-xy}) \left[10x^2 - x^3(y+ax^3)\right]$$
$$+ (y+ax^3)^9 (e^{bx^4-xy}) \left[cy(y+ax^3)\right] = 0$$

$$\underbrace{(y+ax^3)^9 (e^{bx^4-xy})}_{\neq 0} \left[30ax^2 + 10x^2 + (y+ax^3)\left[4bx^3 - y - x^3 + cy\right]\right] = 0$$

$$30ax^2 + 10x^2 + (y+ax^3)\left[4bx^3 - y - x^3 + cy\right] = 0$$

By comparing coefficient,

$30a + 10 = 0$	$4b - 1 = 0$	$c - 1 = 0$
$a = -\frac{1}{3}$	$b = \frac{1}{4}$	$c = 1$

P.S Qns 1a getting more tedious over the year, don't put so much time if little Mark cause time-consuming

1b) $\phi(x,y) = X(x)\,Y(y)$

$$\frac{\partial \phi}{\partial x} = X'(x)\,Y(y) \qquad \frac{\partial \phi}{\partial y} = X(x)\,Y'(y)$$

$$\frac{\partial}{\partial x}\left[(1+y^2)(1+2x^2)\frac{\partial \phi}{\partial x}\right] + \frac{\partial}{\partial y}\left[(1+y^2)(1+2x^2)\frac{\partial \phi}{\partial y}\right] = 0$$
$$\frac{\partial}{\partial x}\left[(1+y^2)(1+2x^2)X'(x)Y(y)\right] + \frac{\partial}{\partial y}\left[(1+y^2)(1+2x^2)X(x)Y'(y)\right] = 0$$
$$Y(y)(1+y^2)\frac{\partial}{\partial x}\left[(1+2x^2)X'(x)\right] + (1+2x^2)X(x)\frac{\partial}{\partial y}\left[(1+y^2)Y'(y)\right] = 0$$

Divide everything by $X(x)Y(y)(1+2x^2)(1+y^2)$

$$\frac{1}{X(x)(1+2x^2)}\frac{\partial}{\partial x}\left[(1+2x^2)X'(x)\right] = -\frac{1}{Y(y)(1+y^2)}\frac{\partial}{\partial y}\left[(1+y^2)Y'(y)\right] = \gamma$$

$$\frac{\partial}{\partial x}\left[(1+2x^2)X'(x)\right] - \gamma X(x)(1+2x^2) = 0 \quad (1)$$
$$\frac{\partial}{\partial y}\left[(1+y^2)Y'(y)\right] + \gamma Y(y)(1+y^2) = 0 \quad (2)$$

1(c i) $u(x,y) = \sum_{n=1}^{\infty} \left(A_n e^{(-1+C_n)x} + B_n e^{(-1-C_n)x} \right) \sin(n\pi y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \left[A_n(-1+C_n) e^{(-1+C_n)x} + B_n(-1-C_n) e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} \left[A_n (C_n-1)^2 e^{(-1+C_n)x} + B_n(-1-C_n)^2 e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$$= \sum_{n=1}^{\infty} \left[A_n (C_n^2 - 2C_n + 1) e^{(-1+C_n)x} + B_n (C_n^2 + 2C_n + 1) e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$$\frac{\partial u}{\partial y} = \sum_{n=1}^{\infty} n\pi \left(A_n e^{(-1+C_n)x} + B_n e^{(-1-C_n)x} \right) \cos(n\pi y)$$

$$\frac{\partial^2 u}{\partial y^2} = \sum_{n=1}^{\infty} -n^2\pi^2 \left(A_n e^{(-1+C_n)x} + B_n e^{(-1-C_n)x} \right) \sin(n\pi y)$$

$$C_n = (1+n^2\pi^2)^{1/2} \rightarrow C_n^2 = 1+n^2\pi^2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} \left[A_n (C_n^2 - \cancel{2C_n} + 1) e^{(-1+C_n)x} + B_n (C_n^2 + \cancel{2C_n} + 1) e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$$+ \sum_{n=1}^{\infty} \left[-n^2\pi^2 A_n e^{(-1+C_n)x} - n^2\pi^2 B_n e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$$+ \sum_{n=1}^{\infty} \left[2(C_n-1) A_n e^{(-1+C_n)x} - 2(C_n+1) B_n e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$$= \sum_{n=1}^{\infty} \left[A_n (C_n^2 + 1 - n^2\pi^2 - 2) e^{(-1+C_n)x} + B_n (C_n^2 + 1 - n^2\pi^2 - 2) e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$$= \sum_{n=1}^{\infty} \left[A_n (1 + \cancel{n^2\pi^2} - \cancel{n^2\pi^2} - 1) e^{(-1+C_n)x} + B_n (1 + \cancel{n^2\pi^2} - \cancel{n^2\pi^2} - 1) e^{(-1-C_n)x} \right] \sin(n\pi y)$$

$= 0$ (shown)

1(c ii) $\sin 0 = \sin(n\pi) = 0$, n is an integer

$$u(x,0) = \sum_{n=1}^{\infty} \left(A_n e^{(-1+C_n)x} + B_n e^{(-1-C_n)x} \right) \sin(0)$$

$= 0$ (shown)

$$u(x,1) = \sum_{n=1}^{\infty} \left(A_n e^{(-1+C_n)x} + B_n e^{(-1-C_n)x} \right) \sin(n\pi)$$

$= 0$ (shown)

1(c iii) $\left.\frac{\partial u}{\partial x}\right|_{x=0} = \sum_{n=1}^{\infty} \left[A_n(-1+C_n)e^0 + B_n(-1-C_n)e^0 \right] \underbrace{\sin(n\pi y)}_{\neq 0} = 0$

$$A_n(-1+C_n) + B_n(-1-C_n) = 0$$

$$B_n = \frac{A_n(C_n-1)}{C_n+1}$$

1c(iv) $u(1,y) = \sum_{n=1}^{\infty} \left(A_n e^{-1+C_n} + B_n e^{-1-C_n}\right)\sin(n\pi y) = 10$

$$A_n e^{-1+C_n} + B_n e^{-1-C_n} = 2\int_0^1 10\sin(n\pi y)\,dy$$

$$A_n\left[e^{-1+C_n} + \frac{C_n-1}{C_n+1}e^{-1-C_n}\right] = 20\int_0^1 \sin(n\pi y)\,dy$$

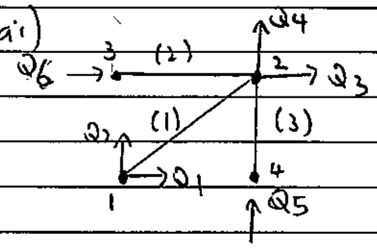
$$= 20\left[-\frac{1}{n\pi}\cos(n\pi y)\right]_0^1$$

$$= \frac{20}{n\pi}\left[1-(-1)^n\right]$$

$$A_n = \frac{\frac{20}{n\pi}\left[1-(-1)^n\right]}{e^{-1+C_n} + \frac{C_n-1}{C_n+1}e^{-1-C_n}}$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{\frac{20}{n\pi}\left[1-(-1)^n\right]}{e^{-1+C_n} + \frac{C_n-1}{C_n+1}e^{-1-C_n}}\left[e^{(-1+C_n)x} + \frac{C_n-1}{C_n+1}e^{(-1-C_n)x}\right]\sin(n\pi y)$$

2a(i)



2a(ii) $K^{(1)}$ (bar AB) *,

reference node 1, $\theta = 45°$

$l = \cos 45 = \frac{\sqrt{2}}{2}$ $\qquad l^2 = m^2 = lm = 0.5$

$m = \sin 45 = \frac{\sqrt{2}}{2}$

$$K^{(1)} = \frac{AE}{L}\begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

$$= 10^6 \begin{array}{c} \begin{bmatrix} 25 & 25 & -25 & -25 \\ 25 & 25 & -25 & -25 \\ -25 & -25 & 25 & 25 \\ -25 & -25 & 25 & 25 \end{bmatrix} \begin{matrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{matrix} \\ \begin{matrix} Q_1 & \quad Q_2 & \quad Q_3 & \quad Q_4 \end{matrix} \end{array}$$

$$K^{(2)} = 10^6 \begin{array}{c} \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{matrix} Q_3 \\ Q_6 \end{matrix} \\ \begin{matrix} Q_3 & Q_6 \end{matrix} \end{array}$$

$$K^{(3)} = 10^6 \begin{array}{c} \begin{matrix} Q_5 & Q_4 \end{matrix} \\ \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \begin{matrix} Q_5 \\ Q_4 \end{matrix} \end{array}$$

Assembling all stiffness Matrix,

$$k^{(T)} = 10^6 \begin{array}{c} \\ \left[\begin{array}{cccccc} 25 & 25 & -25 & -25 & 0 & 0 \\ 25 & 25 & -25 & -25 & 0 & 0 \\ -25 & -25 & 30 & 25 & 0 & -5 \\ -25 & -25 & 25 & 30 & -5 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 & 0 & 5 \end{array}\right] \\ \begin{array}{cccccc} Q_1 & Q_2 & Q_3 & Q_4 & Q_5 & Q_6 \end{array} \end{array} \begin{array}{c} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ \\ \end{array}$$

$Q_1 = Q_2 = Q_5 = Q_6 = 0$, $F_1 = R_{1x}$ $F_2 = R_{1y}$ $F_3 = P\cos 45$ $F_4 = -P\sin 45$ $F_5 = R_{4y}$ $F_6 = R_{3x}$

$$10^6 \begin{bmatrix} 25 & 25 & 25 & 25 & 0 & 0 \\ 25 & 25 & -25 & -25 & 0 & 0 \\ -25 & -25 & 30 & 25 & 0 & -5 \\ -25 & -25 & 25 & 30 & -5 & 0 \\ 0 & 0 & 0 & -5 & 5 & 0 \\ 0 & 0 & -5 & 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix}$$

$$10^6 \begin{bmatrix} 30 & 25 \\ 25 & 30 \end{bmatrix} \begin{bmatrix} Q_3 \\ Q_4 \end{bmatrix} = 10^6 \begin{bmatrix} \cos 45 \\ -\sin 45 \end{bmatrix}$$

* F_4 is negative cause downward force as ↑→ defined as positive

2aiii) Solving by Cramer's rule or simultaneous,

$Q_3 = 0.1414\,m$

$Q_4 = -0.1414\,m$

2b) $L = 1\,m$, $q(x) = 10x$

$N_1 q(x) = (2x^3 - 3x^2 + 1)(10x)$
$= 20x^4 - 30x^3 + 10x$

$N_3 q(x) = (-2x^3 + 3x^2)(10x)$
$= -20x^4 + 30x^3$

$N_2 q(x) = (x^3 - 2x^2 + x)(10x)$
$= 10x^4 - 20x^3 + 10x^2$

$N_4 q(x) = (x^3 - x^2)(10x)$
$= 10x^4 - 10x^3$

$$\{f^e_q\} = -\int_0^1 [N]^T q(x)\,dx$$

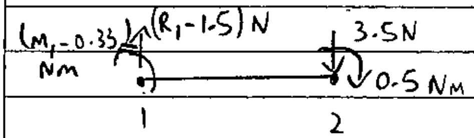
P.S. sign negative cause downward force, plese verify with professor

$$= -\int_0^1 \begin{bmatrix} 20x^4 - 30x^3 + 10x \\ 10x^4 - 20x^3 + 10x^2 \\ -20x^4 + 30x^3 \\ 10x^4 - 10x^3 \end{bmatrix} dx$$

$$= -\begin{bmatrix} \left[\frac{20}{5}x^5 - \frac{30}{4}x^4 + 5x^2\right]_0^1 \\ \left[\frac{10}{5}x^5 - \frac{20}{4}x^4 + \frac{10}{3}x^3\right]_0^1 \\ \left[-\frac{20}{5}x^5 + \frac{30}{4}x^4\right]_0^1 \\ \left[\frac{10}{5}x^5 - \frac{10}{4}x^4\right]_0^1 \end{bmatrix}$$

$$= -\begin{bmatrix} 1.5 \\ 0.33 \\ 3.5 \\ -0.5 \end{bmatrix} =$$

$$= \begin{bmatrix} -1.5 \\ -0.33 \\ -3.5 \\ 0.5 \end{bmatrix}$$



2c i) All three are polynomial and have continuous within problem domain. $(n\bar{u}=2(1)0)$
However only $\hat{u} = c_0 + c_1 x + c_2 x^3$ has continuous derivatives up to $\frac{d^3u}{dx^3}$

$\hat{u} = c_0 + c_1 x + c_2 x^3$

when $x=0$, $u=0$

$0 = c_0 + 0$

$\therefore c_0 = 0$

$\hat{u} = c_1 x + c_2 x^3$

when $x=1$, $u=1$

$1 = c_1 + c_2$

$c_1 = 1 - c_2$

$\therefore \hat{u} = (1-c_2)x + c_2 x^3$

$= c_2(x^3 - x) + x$

$\phi_1 = x^3 - x$

2c ii) Let $R(x) = \frac{d^3u}{dx^3} - 6x$

$\frac{d\hat{u}}{dx} = 3c_2x^2 - c_2 + 1$

$\frac{d^2\hat{u}}{dx^2} = 6c_2 x$

$\frac{d^3\hat{u}}{dx^3} = 6c_2$

$\int_0^1 \phi_1(x) R(x)\, dx = 0$

$\int_0^1 (x^3 - x)(6c_2 - 6x)\, dx = 0$

$\int_0^1 (6c_2x^3 - 6c_2x + 6x^2 - 6x^4)\, dx = 0$

$\left[\frac{6}{4}c_2x^4 - 3c_2x^2 + 2x^3 - \frac{6}{5}x^5\right]_0^1 = 0$

$-1.5c_2 + 0.8 = 0$

$c_2 = \frac{8}{15}$

$\therefore \hat{u} = \frac{8}{15}(x^3 - x) + x$

3a i) $\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x} = k\frac{\partial^2 \phi}{\partial x^2}$

$$\int_{CV}\int_{t}^{t+\Delta t}\left(\frac{\partial \phi}{\partial t} + u\frac{\partial \phi}{\partial x}\right)dt\,dV = \int_{CV}\int_{t}^{t+\Delta t} k\frac{\partial^2 \phi}{\partial x^2}\,dt\,dV$$

$$(\phi_P - \phi_P^{\,o})\frac{\Delta x}{\Delta t} + u(\phi_e - \phi_w)\Delta t = k\left(\frac{d\phi}{dx}\right)_e - k\left(\frac{d\phi}{dx}\right)_w$$

$$(\phi_P - \phi_P^{\,o})\frac{\Delta x}{\Delta t} + u\Delta t\left(\frac{\phi_P + \phi_E}{2} - \frac{\phi_P + \phi_W}{2}\right) = \frac{k}{\Delta x}(\phi_E - \phi_P - \phi_P + \phi_W)$$

$$(\phi_P - \phi_P^{\,o})\frac{\Delta x}{\Delta t} + \frac{u\Delta t}{2}(\phi_E - \phi_W) = \frac{k}{\Delta x}(\phi_E - 2\phi_P + \phi_W)$$

$$\phi_P\left(\frac{\Delta x}{\Delta t} + 2\frac{k}{\Delta x}\right) = \left(\frac{k}{\Delta x} - \frac{u\Delta t}{2}\right)\phi_E + \left(\frac{k}{\Delta x} + \frac{u\Delta t}{2}\right)\phi_W + \frac{\Delta x}{\Delta t}\phi_P^{\,o}$$

$a_P = \frac{\Delta x}{\Delta t} + 2\frac{k}{\Delta x}$ $\qquad a_P^{\,o} = \frac{\Delta x}{\Delta t}$

$a_E = \frac{k}{\Delta x} - \frac{u\Delta t}{2}$ $\qquad S_u = 0$

$a_W = \frac{k}{\Delta x} + \frac{u\Delta t}{2}$

3a ii) Positive velocity, $u > 0$:

$$(\phi_P - \phi_P^{\,o})\frac{\Delta x}{\Delta t} + u\Delta t(\phi_e - \phi_w) = \frac{k}{\Delta x}(\phi_E - 2\phi_P + \phi_W)$$

$$(\phi_P - \phi_P^{\,o})\frac{\Delta x}{\Delta t} + u\Delta t(\phi_P - \phi_W) = \frac{k}{\Delta x}(\phi_E - 2\phi_P + \phi_W)$$

$$\phi_P\left(\frac{\Delta x}{\Delta t} + u\Delta t + \frac{2k}{\Delta x}\right) = \frac{k}{\Delta x}\phi_E + \left(\frac{k}{\Delta x} + u\Delta t\right)\phi_W + \frac{\Delta x}{\Delta t}\phi_P^{\,o}$$

$a_P = \frac{\Delta x}{\Delta t} + u\Delta t + 2\frac{k}{\Delta x}$ $\qquad S_u = 0$

$a_E = \frac{k}{\Delta x}$ $\qquad a_P^{\,o} = \frac{\Delta x}{\Delta t}$

$a_W = \frac{k}{\Delta x} + u\Delta t$

Negative velocity, $u < 0$:

$$(\phi_P - \phi_P^{\,o})\frac{\Delta x}{\Delta t} + u\Delta t(\phi_e - \phi_w) = \frac{k}{\Delta x}(\phi_E - 2\phi_P + \phi_W)$$

$$(\phi_P - \phi_P^{\,o})\frac{\Delta x}{\Delta t} + u\Delta t(\phi_E - \phi_P) = \frac{k}{\Delta x}(\phi_E - 2\phi_P + \phi_W)$$

$$\phi_P\left(\frac{\Delta x}{\Delta t} - u\Delta t + 2\frac{k}{\Delta x}\right) = \left(\frac{k}{\Delta x} - u\Delta t\right)\phi_E + \frac{k}{\Delta x}\phi_W + \frac{\Delta x}{\Delta t}\phi_P^{\,o}$$

$a_P = \frac{\Delta x}{\Delta t} - u\Delta t + 2\frac{k}{\Delta x}$ $\qquad a_P^{\,o} = \frac{\Delta x}{\Delta t}$

$a_E = \frac{k}{\Delta x} - u\Delta t$ $\qquad S_u = 0$

$a_W = \frac{k}{\Delta x}$

3bi)

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$|a_p| = 1 \quad \Sigma|a_{nb}| = 1$

$|a_p| = 2 \quad \Sigma|a_{nb}| = 1 + 1 = 2$

$|a_p| = 2 \quad \Sigma|a_{nb}| = 1$

First eqn: $\frac{\Sigma|a_{nb}|}{|a_p|} = 1 \leq 1$

Second eqn: $\frac{\Sigma|a_{nb}|}{|a_p|} = \frac{2}{2} \leq 1$

Third eqn: $\frac{\Sigma|a_{nb}|}{|a_p|} = \frac{1}{2} < 1$

$\therefore$ Scarborough fulfilled and the iterative solution will be converging

3bii)

$x_1 = -1 - x_3$

$x_2 = \frac{1 - x_1 + x_3}{2}$

$x_3 = \frac{x_2}{2}$

Jacobi

Iteration	0	1	2	3
x_1	0	-1	-1	-1.25
x_2	0	0.5	1	1.125
x_3	0	0	0.25	0.5

Gauss - Seidel

Iteration	0	1	2	3
x_1	0	-1	-1.5	-1.75
x_2	0	1	1.5	1.75
x_3	0	0.5	0.75	0.875

NANYANG TECHNOLOGICAL UNIVERSITY

SEMESTER 1 EXAMINATION 2020-2021

MA3004 – MATHEMATICAL METHODS IN ENGINEERING

November/December 2020 Time Allowed: 1 hour

INSTRUCTIONS

1. This paper contains **FOUR (4)** questions and comprises **THREE (3)** pages.
2. Answer **ALL** questions.
3. All questions carry equal marks.
4. This is an **OPEN-BOOK E-EXAMINATION**.

1(a) Find all values of the constants a, b and c such that the function

$$T(x,y,z) = 8x^3y + axy^3 + e^{bz} + e^{2y}\sin(z)$$

is a solution of the partial differential equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + c\frac{\partial^2 T}{\partial z^2} = 16e^{bz}$$

at all points (x,y,z) in space.

(17 marks)

(b) By letting $\phi = \ln(\psi)$, where ϕ and ψ are twice partially differentiable functions of x and y, rewrite the partial differential equation

$$\frac{\partial}{\partial x}\left(e^{\phi}\frac{\partial \phi}{\partial x}\right) + 3\frac{\partial}{\partial y}\left(e^{\phi}\frac{\partial \phi}{\partial y}\right) = 0$$

as a partial differential equation in ψ.

(8 marks)

2(a) Figure 1 shows a two-spring assembly wherein nodes 1 and 3 are pulled towards supports A and B, respectively, by 0.1 m and 0.3 m so that the nodes can be welded to the respective supports. Determine the displacement at node 2 and reaction forces at nodes 1 and 3 by modelling each spring as a spring element.

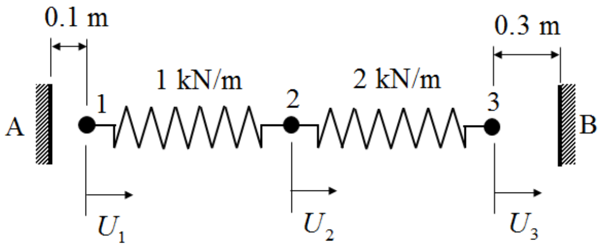


Figure 1

(10 marks)

(b) Figure 2 shows a weightless uniform beam AB of length L and bending rigidity EI rigidly fixed at end A and supported on a spring of stiffness $k = 12EI/L^3$ at end B. The beam carries a vertical load of P at B as shown.

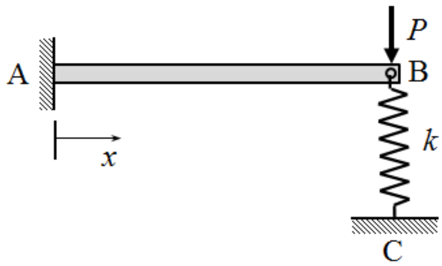


Figure 2

Modelling the structure using a single beam element and a spring element, determine the vertical deflection and cross-sectional rotation at B by solving the finite element equations.

(15 marks)

3. A property T is transported by a one-dimensional convection and diffusion process governed by the partial differential equation

$$\frac{\partial T}{\partial t}+\frac{\partial}{\partial x}\left(\alpha T-\beta\frac{\partial T}{\partial x}\right)=0,$$

where $\alpha > 0$ and $\beta > 0$ are positive constants. The property T is a function of the Cartesian coordinate x and the time coordinate t.

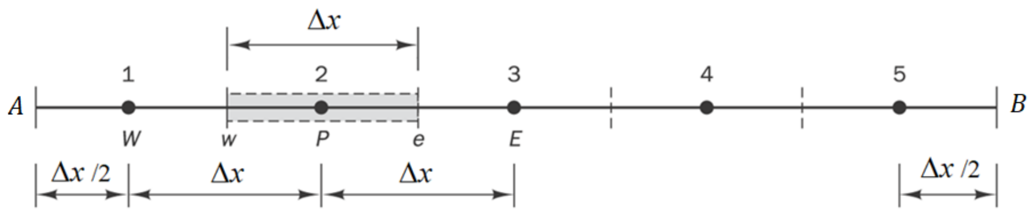


Figure 3

Use the uniform mesh shown in Figure 3 together with the implicit time scheme with time step Δt and the upwind differencing scheme (central difference discretization of diffusion term and upwind difference discretization of convection term) with space step Δx to derive discretized equations of the form $a_P T_p = a_E T_E + a_W T_W + a_P^0 T_p^0 + S_u$ for:

(a) nodes labelled 2, 3 and 4, and (15 marks)

(b) node 1 if the boundary condition at A is given by $T|_A = 0$. (10 marks)

4(a) Analyze the condition for each of the following systems of linear equations below by using the Scarborough criterion. State which system has a converging iterative solution and which system has a diverging iterative solution.

(System A) $\begin{cases} T_1 + T_2 = 1 \\ T_1 - 2T_2 = 4 \end{cases}$ (System B) $\begin{cases} T_1 + T_2 = 1 \\ 2T_1 - T_2 = -4 \end{cases}$

(15 marks)

(b) Solve each of the systems of linear equations above by using the Gauss-Seidel iterative method with the initial guess of $T_1 = T_2 = 0$. Consider only the first three iterations.

(10 marks)

End of Paper

Date No.

1.a $T(x,y,z) = 8x^3y + axy^3 + e^{bz} + e^{2y}\sin z$

$\frac{\partial T}{\partial x} = 24x^2y + ay^3$ $\qquad \frac{\partial^2 T}{\partial x^2} = 48xy$

$\frac{\partial T}{\partial y} = 8x^3 + 3axy^2 + \sin z \cdot e^{2y} \cdot 2$ $\qquad \frac{\partial^2 T}{\partial y^2} = 6axy + 2 \cdot 2 \cdot \sin z \cdot e^{2y}$

$\frac{\partial T}{\partial z} = b \cdot e^{bz} + e^{2y} \cdot \cos z$ $\qquad \frac{\partial^2 T}{\partial z^2} = b^2 \cdot e^{bz} - e^{2y} \cdot \sin z$

$48xy + 6axy + 4\sin z e^{2y} + c(b^2 \cdot e^{bz} - e^{2y}\sin z) = 16 \cdot e^{bz}$

$xy(48+6a) + e^{2y}\left((4-c)\sin z\right) + c \cdot b^2 \cdot e^{bz} = 16 \cdot e^{bz}$

$$\begin{cases} 48+6a=0 \\ 4-c=0 \\ c \cdot b^2 = 16 \end{cases} \quad \begin{cases} a=-3 \\ c=4 \\ b=\pm 2 \end{cases} \Rightarrow (a,b,c) = \begin{matrix}(-3,2,4) \\ (-3,-2,4)\end{matrix}$$

(b) $\phi = \ln(\psi)$

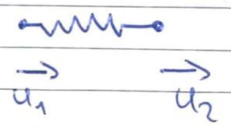
$$\frac{\partial}{\partial x}\left(e^{\phi}\frac{\partial \phi}{\partial x}\right) + 3\frac{\partial}{\partial y}\left(e^{\phi}\frac{\partial \phi}{\partial y}\right) = 0$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial(\ln\psi)}{\partial x} = \frac{1}{\psi} \cdot \frac{\partial \psi}{\partial x}$$

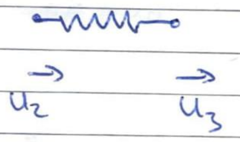
$\frac{\partial}{\partial x}\left(e^{\ln\psi} \cdot \frac{1}{\psi} \cdot \frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x^2}$, Likewise, $\frac{\partial}{\partial y}\left(e^{\phi}\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial y^2}$

$\Rightarrow$ Ans: $\frac{\partial^2 \psi}{\partial x^2} + 3 \cdot \frac{\partial^2 \psi}{\partial y^2} = 0$

(2,a)



$$\begin{array}{c|cc} & 1 & 2 \\ 1 & k_1 & -k_1 \\ 2 & -k_1 & k_1 \end{array} \quad \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$



$$\begin{array}{c|cc} & 2 & 3 \\ 2 & k_2 & -k_2 \\ 3 & -k_2 & k_2 \end{array} \quad \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} f_2^{(2)} \\ f_3^{(2)} \end{Bmatrix}$$

$k_1 = 1000\ N/m$
$k_2 = 2000\ N/m$
$u_1 = -0{,}1 m$
$u_3 = 0{,}3 m$

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 1000 & -1000 & 0 \\ 2 & -1000 & 3000 & -2000 \\ 3 & 0 & -2000 & 2000 \end{array} \quad \begin{bmatrix} 1000 & -1000 & 0 \\ -1000 & 3000 & -2000 \\ 0 & -2000 & 2000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix}$$

$$\begin{cases} 1000u_1 - 1000u_2 = F_1 \\ -1000u_1 + 3000u_2 - 2000u_3 = 0 \\ -2000u_2 + 2000u_3 = F_3 \end{cases} \Rightarrow \begin{cases} -100 - 1000u_2 = F_1 \\ 100 + 3000u_2 - 600 = 0 \\ -2000u_2 + 600 = F_3 \end{cases}$$

$u_2 = \frac{1}{6} m$

$F_1 = -100 - 1000 \cdot \frac{1}{6} = -266{,}7\ N$

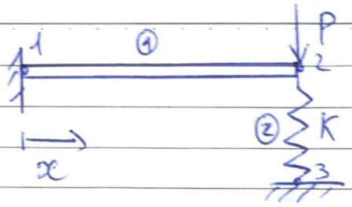
$F_3 = -2000 \cdot \frac{1}{6} + 600 = 266{,}7\ N$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. Campus supplies is not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

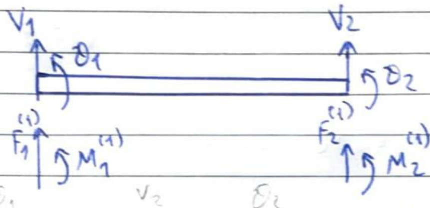
Should there be any mistakes identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

Date No.

(b)

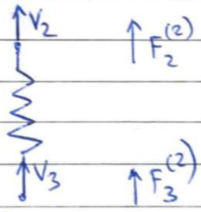


Element 1:



$$\frac{EI}{L^3}\begin{array}{c|cccc|} & V_1 & \theta_1 & V_2 & \theta_2 \\ V_1 & 12 & 6L & -12 & 6L \\ \theta_1 & 6L & 4L^2 & -6L & 2L^2 \\ V_2 & -12 & -6L & 12 & -6L \\ \theta_2 & 6L & 2L^2 & -6L & 4L^2 \end{array}\begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1^{(1)} \\ M_1^{(1)} \\ F_2^{(1)} \\ M_2^{(1)} \end{Bmatrix}$$

Element 2:



$$\frac{12EI}{L^3}\begin{array}{c|cc|} & V_3 & V_2 \\ V_3 & 1 & -1 \\ V_2 & -1 & 1 \end{array} \cdot \cancel{\begin{Bmatrix} F_3^{(2)} \\ F_2^{(2)} \end{Bmatrix}} \begin{Bmatrix} V_3 \\ V_2 \end{Bmatrix} = \begin{Bmatrix} F_3^{(2)} \\ F_2^{(2)} \end{Bmatrix}$$

$$\frac{EI}{L^3}\begin{array}{c|ccccc|} & V_1 & \theta_1 & V_2 & \theta_2 & V_3 \\ V_1 & 12 & 6L & -12 & 6L & 0 \\ \theta_1 & 6L & 4L^2 & -6L & 2L^2 & 0 \\ V_2 & -12 & -6L & 24 & -6L & -12 \\ \theta_2 & 6L & 2L^2 & -6L & 4L^2 & 0 \\ V_3 & 0 & 0 & -12 & 0 & 12 \end{array}\begin{Bmatrix} V_1 \\ \theta_1 \\ V_2 \\ \theta_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} R_1 \\ M_1 \\ -P \\ 0 \\ R_3 \end{Bmatrix}$$

From Boundary conditions, $V_1 = 0$, $\theta_1 = 0$, $V_3 = 0$

$$\frac{EI}{L^3}\begin{bmatrix} 24 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} V_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \end{Bmatrix}$$

$$\begin{cases} \frac{EI}{L^3}\left(24V_2 - 6L\theta_2\right) = -P \\ \frac{EI}{L^3}\left(-6LV_2 + 4L^2\theta_2\right) = 0 \Rightarrow 6\cancel{L}\cdot V_2 = 4L^{\cancel{2}}\cdot\theta_2 \end{cases}$$

$$V_2 = \frac{2}{3}\theta_2 \cdot L$$

$$\frac{EI}{L^3}\left(24\cdot\frac{2}{3}\theta_2\cdot L - 6L\cdot\theta_2\right) = -P$$

$$\frac{EI}{L^3}\left(10L\theta_2\right) = -P \Rightarrow \theta_2 = \frac{-PL^2}{10EI} \text{ (Rotation, CW)}$$

$$V_2 = \frac{2}{3}\cdot\left(\frac{-PL^2}{10EI}\right)L = \frac{-PL^3}{15EI} \text{ (Vertical deflection, downwards)}$$

(3) $\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}\left(\alpha T \cancel{+} - \beta\frac{\partial T}{\partial x}\right) = 0$

(a) $$\int_t^{t+\Delta t}\int_{CV}\frac{\partial T}{\partial t}\cdot dv\cdot dt + \alpha\cdot\int_t^{t+\Delta t}\int_{CV}\frac{\partial T}{\partial x}\cdot dv\cdot dt - \beta\cdot\int_t^{t+\Delta t}\int_{CV}\frac{\partial^2 T}{\partial x^2} = 0$$

•) First term: $\int_{CV}(\phi - \phi^\circ)\,dv = (\phi_P - \phi_P^\circ)\cdot A\cdot\Delta x = (T_P - T_P^\circ)A\Delta x$

•) Second term: $\alpha\cdot\int_t^{t+\Delta t}(T_e - T_w)\cdot A\cdot dt = \alpha\cdot A\cdot(T_e - T_w)\cdot\Delta t$

•) Third term: $\beta\cdot\int_t^{t+\Delta t}\left(\frac{\partial T}{\partial x}\bigg|_e - \frac{\partial T}{\partial x}\bigg|_w\right)\cdot A\cdot dt = \beta\cdot A\left(\frac{\partial T}{\partial x}\bigg|_e - \frac{\partial T}{\partial x}\bigg|_w\right)\Delta t$

By upwind differencing scheme, $u>0$ $(a>0)$ $\Rightarrow$ $T_e = T_P$, $T_w = T_W$

Date No.

For nodes 2,3,4;

$$\left.\frac{\partial T}{\partial x}\right|_e = \frac{T_E - T_P}{\Delta x} \quad \text{and} \quad \left.\frac{\partial T}{\partial x}\right|_w = \frac{T_P - T_W}{\Delta x}$$

$$\underset{(T_P - T_P^\circ)}{(\cancel{\phi_P - \phi_P^\circ})} \Delta x + \alpha (T_P - T_W) \Delta t - \beta \left(\frac{T_E - 2T_P + T_W}{\Delta x} \right) \Delta t = 0$$

$$T_P \left(\Delta x + \alpha \Delta t + \frac{2\beta}{\Delta x} \Delta t \right) = T_E \left(\frac{\beta \cdot \Delta t}{\Delta x} \right) + T_W \left(\frac{\beta \Delta t}{\Delta x} + \alpha \Delta t \right)$$
$$+ T_P^\circ \cdot \Delta x$$

(b) For node 1,

$$\left.\frac{\partial T}{\partial x}\right|_e = \frac{T_2 - T_1}{\Delta x} \qquad \left.\frac{\partial T}{\partial x}\right|_w = \frac{T_1 - T_0}{\Delta x / 2}$$

$$T_e = T_1 \qquad T_w = T_A = 0 \qquad T_0 = T_A = 0$$

$$(T_1 - T_1^\circ) \Delta x + \alpha (T_1) \Delta t - \beta \cdot \left(\frac{T_2 - 3T_1 + 2T_0}{\Delta x} \right) \Delta t = 0$$

$$T_1 \underbrace{\left(\Delta x + \alpha \Delta t + \frac{3\beta \Delta t}{\Delta x} \right)}_{a_P} = T_2 \underbrace{\left(\frac{\beta \cdot \Delta t}{\Delta x} \right)}_{a_E} + T_1^\circ \cdot \underbrace{\Delta x}_{a_P^\circ}$$

$a_W = 0$

$S_u = 0$

DISCLAIMER: *The solutions are done by students who scored A or above in this subject. Campus supplies is not liable or responsible for any errors in the contents of these solutions. Students are advised to take the solutions as a guide rather than absolute answers to exam paper.*

Should there be any mistakes identified, please proceed to the Facebook link encoded in the QR code to feedback or submit correct answers. The link is: http://bit.ly/2lW2C32

Date No.

(4.a) For system A, $\begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \end{Bmatrix}$

$$\left.\begin{array}{l} \frac{1}{1} = 1 \leq 1 \\ \\ \frac{1}{|-2|} = \frac{1}{2} < 1 \end{array}\right\} \text{Convergent}$$

For system B, $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -4 \end{Bmatrix}$

$$\frac{1}{1} = 1 \leq 1$$

$$\frac{2}{|-1|} = 2 > 1 \Rightarrow \text{Divergent}$$

(b)

For system A, $\begin{cases} T_1 = 1 - T_2 \\ T_2 = (T_1 - 4)/2 \end{cases}$

1st iteration:

$$T_1 = 1 - 0 = 1$$

$$T_2 = \frac{1-4}{2} = -\frac{3}{2}$$

2nd iteration:

$$T_1 = 1 - \left(-\frac{3}{2}\right) = 2{,}5$$

$$T_2 = \frac{2{,}5-4}{2} = -\frac{3}{4}$$

3rd iteration:

$$T_1 = 1 - \left(-\frac{3}{4}\right) = \frac{7}{4} \qquad T_2 = \frac{\frac{7}{4}-4}{2} = -\frac{9}{8}$$

Date No.

For system B, $\begin{cases} T_1 = 1 - T_2 \\ T_2 = 2T_1 + 4 \end{cases}$

$1^{\underline{st}}$ iteration : $T_1 = 1$

$T_2 = 2 \cdot 1 + 4 = 6$

$2^{\underline{nd}}$ iteration: $T_1 = 1 - 6 = -5$

$T_2 = 2 \cdot (-5) + 4 = -6$

$3^{\underline{rd}}$ iteration: $T_1 = 1 - (-6) = 7$

$T_2 = 2 \cdot 7 + 4 = 18$