MA2011: W3 DC motors

```
date
```

```
ans =
'28-Mar-2022'
```

State-Space Representation

The order of a system depends on the highest derivative present, in mechanical systems we typically have second-order derivatives due to the acceleration in Netwon's law $f(t) = m\ddot{x} + b\dot{x} + kx$ which we can rewrite simply as

$$\ddot{x} = \frac{-b\dot{x} - kx + f(t)}{m}$$

However, a different way to present a system is to augment its state, e.g. define

$v := \dot{x}$ which also means $\dot{v} = \ddot{x}$

and rewrite Newton's law as a pair of equations

- $\dot{x} = v$
- $\dot{v} = \dfrac{-bv - kx - f(t)}{m}$

or in matrix format, by defining an augmented state vector $Y = [x \;\; v]^T$

$$\frac{d}{dt}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} f(t)$$

$$\frac{d}{dt}X = A.X + B.f(t)$$

```
syms x(t) v(t) f(t)
syms m b k real

% augmented state
X = [x; v]
```

X(t) =

$$\begin{pmatrix} x(t) \\ v(t) \end{pmatrix}$$

```
dXdt = diff(X)
```

dXdt(t) =

$$\begin{pmatrix} \frac{\partial}{\partial t}\, x(t) \\ \frac{\partial}{\partial t}\, v(t) \end{pmatrix}$$

```
A = [0 1; -k/m -b/m]
```

A =

$$\begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{pmatrix}$$

```
B = [0; 1/m]
```

B =

$$\begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$$

```
myNewton = dXdt == A* X + B*f
```

myNewton(t) =

$$\begin{pmatrix} \frac{\partial}{\partial t}\, x(t) = v(t) \\ \frac{\partial}{\partial t}\, v(t) = \frac{f(t)}{m} - \frac{b\, v(t)}{m} - \frac{k\, x(t)}{m} \end{pmatrix}$$

numerical analysis - try: help ss

```
m=1, b=0.1, k=1
```

```
m = 1
b = 0.1000
k = 1
```

```
A = [0 1; -k/m -b/m]
```

```
A = 2×2
         0    1.0000
   -1.0000   -0.1000
```

```
B = [0; 1/m]
```

```
B = 2×1
     0
     1
```

```
mySystem = ss(A, B, [1 0], 0)
```

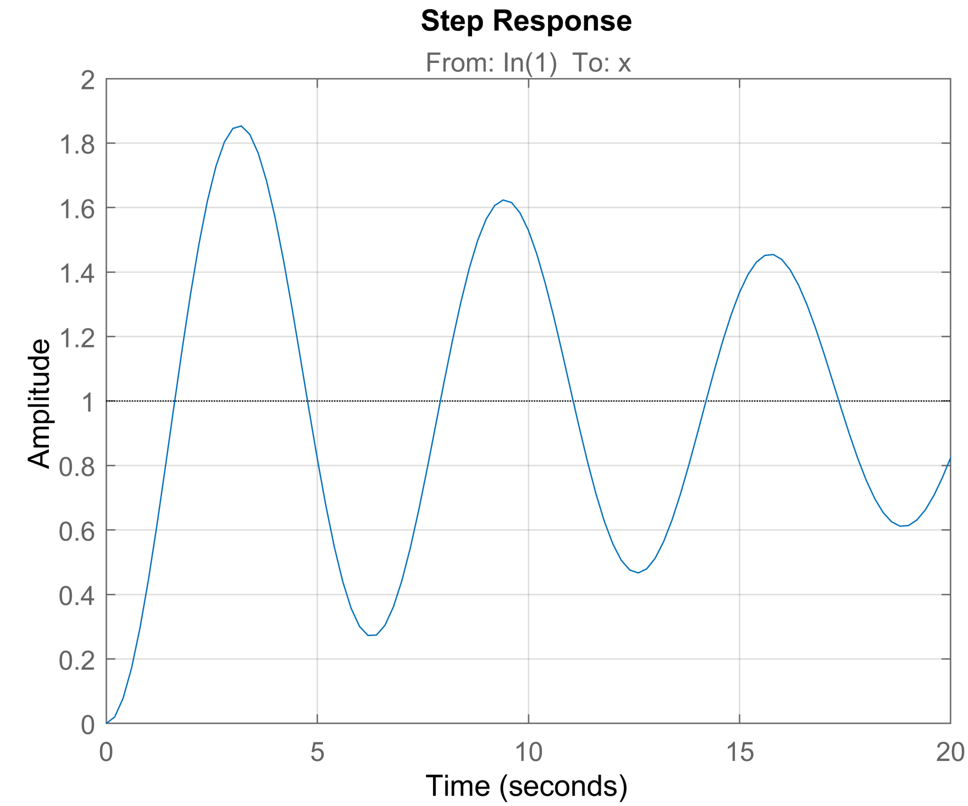
```
mySystem =

  A =
         x1    x2
   x1     0     1
   x2    -1  -0.1
```

```
  B = 
       u1
   x1   0
   x2   1
 
  C = 
       x1  x2
   y1   1   0
 
  D = 
       u1
   y1   0
 
Continuous-time state-space model.
```

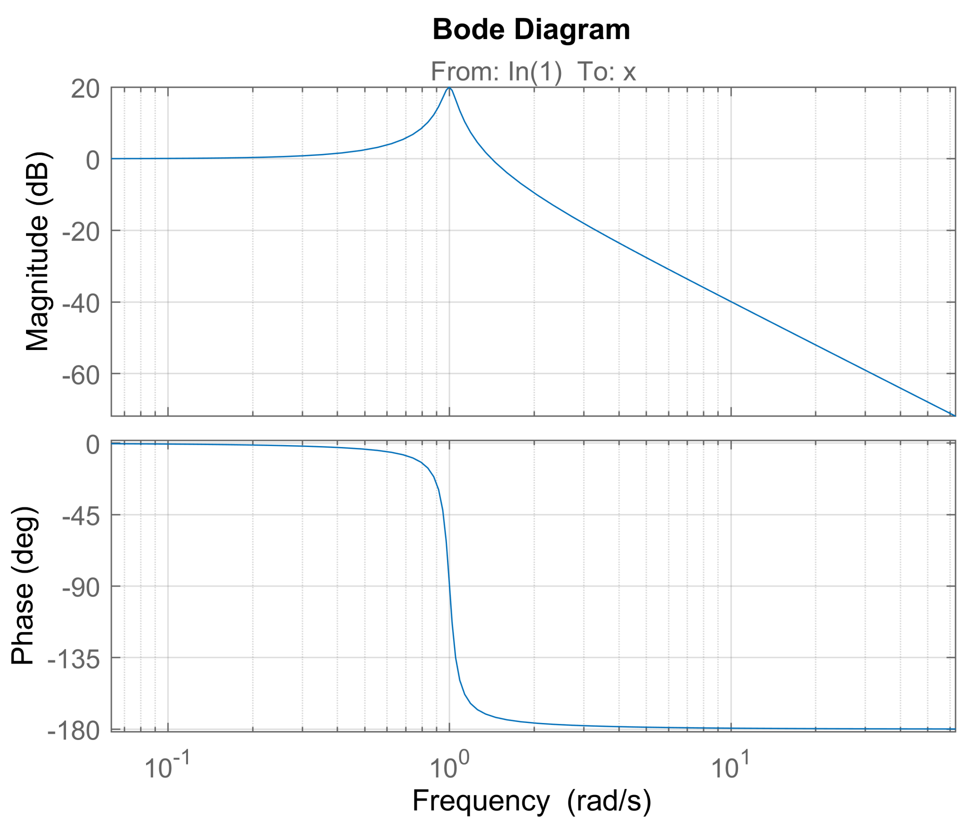
```
mySystem.OutputName = {'x'};
mySystem.StateName = {'x', 'v'}
```

```
mySystem =
 
  A = 
         x     v
   x     0     1
   v    -1  -0.1
 
  B = 
      u1
   x   0
   v   1
 
  C = 
      x  v
   x  1  0
 
  D = 
      u1
   x   0
 
Continuous-time state-space model.
```

```
figure
step(mySystem, 20); grid on
```




```
figure
bode(mySystem, {2*pi*1e-2, 2*pi*10}); grid on
```



Block Diagram Representation

a general ODE written as

```
myNewton
```

```
myNewton(t) =
```

$$\begin{pmatrix} \frac{\partial}{\partial t}\, x(t) = v(t) \\ \frac{\partial}{\partial t}\, v(t) = \frac{f(t)}{m} - \frac{b\, v(t)}{m} - \frac{k\, x(t)}{m} \end{pmatrix}$$

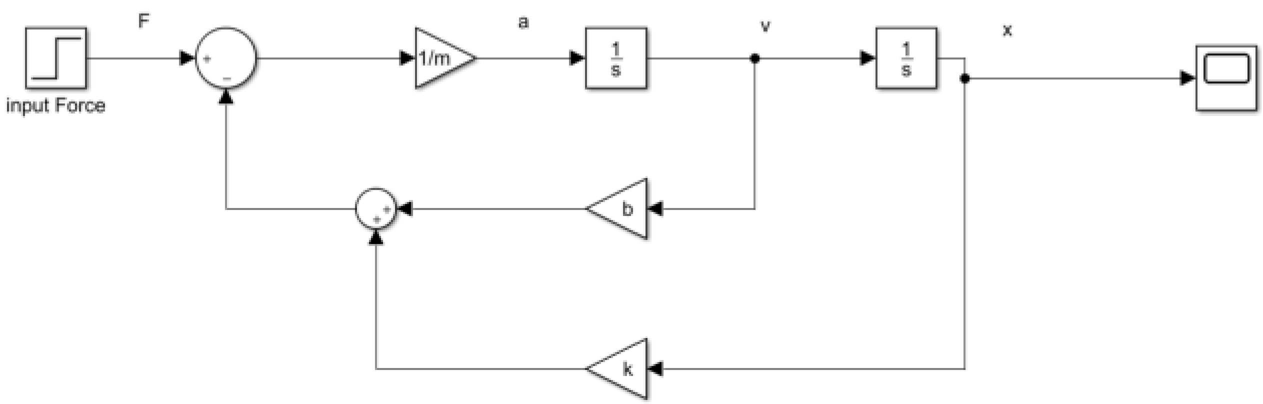
```
m =1, b = 0.1, k = 1
```

```
m = 1
b = 0.1000
k = 1
```

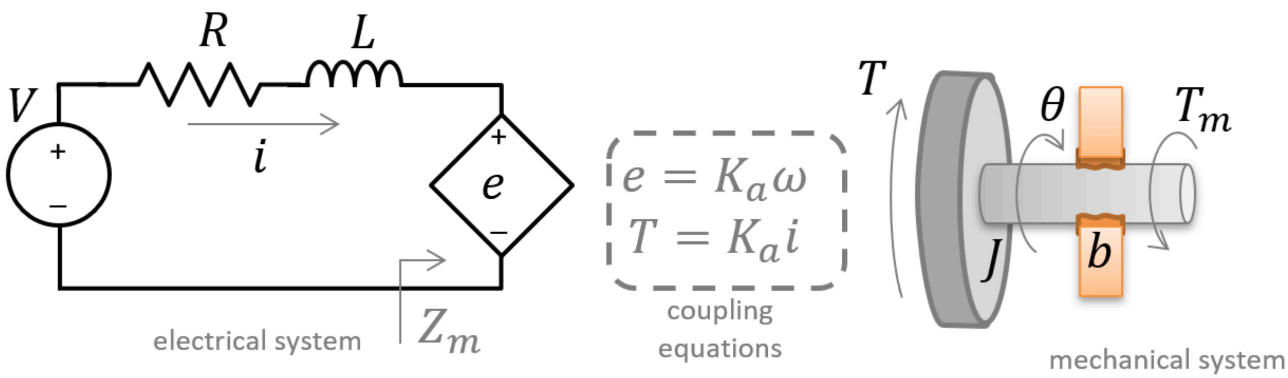
a = 1/m * (F(t) - b*v - k*x)

v = dx / dt
a = dv / dt

m=1, b=1, k=1



DC Motor - State-Space representation



- Mechanical System $J\frac{d\omega}{dt} + b\,\omega = T - T_m$
- Electrical system $V - e = Ri + L\frac{di}{dt}$

$$\frac{d}{dt}\begin{bmatrix}\theta\\ \omega\\ i\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & -\frac{b}{J} & \frac{K_a}{J}\\ 0 & -\frac{K_a}{L} & -\frac{R}{L}\end{bmatrix}\begin{bmatrix}\theta\\ \omega\\ i\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}V - \begin{bmatrix}0\\ \frac{1}{J}\\ 0\end{bmatrix}T_m$$

$$u \rightarrow \boxed{\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}} \rightarrow y$$

Analog driving

Consider a motor with a following

- physical parameters

```
% PHYSICAL PARAMETERS
J = 3.2284E-6;                    % rotor inertia
b = 3.5077E-6;                    % mech. damping
Ke = 0.0274;                      % e.m.f constant
R = 4;                            % electric resistance
L = 2.75E-6;                      % electric inductance
```

- A matrix

```
A=[ 0 1 0;
    0 -b/J  Ke/J;
```

```
    0 -Ke/L -R/L;]
```

```
A = 3×3
10^6 ×
         0    0.0000         0
         0   -0.0000    0.0085
         0   -0.0100   -1.4545
```

- A matrix eigenvalues

```
eig(A)'
```

```
ans = 1×3
10^6 ×
         0   -0.0001   -1.4545
```

- B input matrix

```
B=[ 0; 0; 1/L]
```

```
B = 3×1
10^5 ×
         0
         0
    3.6364
```

- output matrix (3×3 Identity)

```
C=eye(3); % y=[theta; omega; i];
```

- State Space (SS) system definition

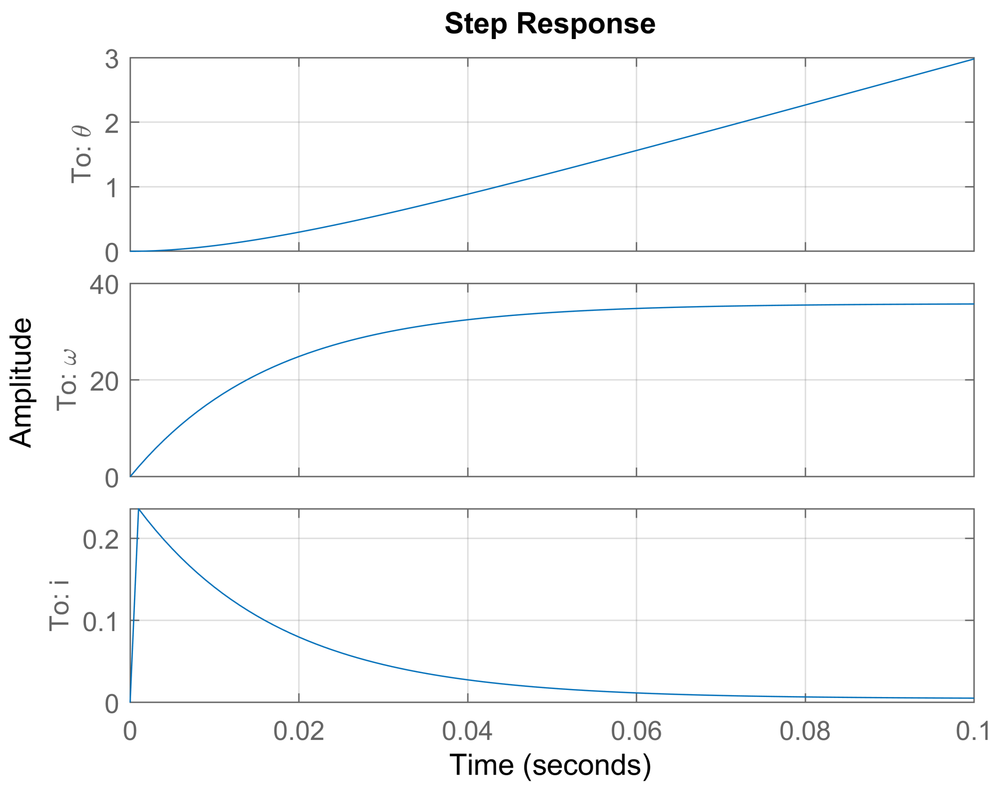
```
myDCM = ss(A,B,C,0);
myDCM.OutputName = {'\theta', '\omega', 'i'};
myDCM.StateName = {'\theta', '\omega', 'i'}
```

```
myDCM =

  A = 
             \theta      \omega           i
   \theta          0           1           0
   \omega          0      -1.087        8487
   i               0       -9964  -1.455e+06

  B = 
                u1
   \theta          0
   \omega          0
   i       3.636e+05

  C = 
           \theta  \omega       i
   \theta       1       0       0
   \omega       0       1       0
   i            0       0       1

  D = 
```

```
          u1
   \theta   0
   \omega   0
   i        0

Continuous-time state-space model.
```

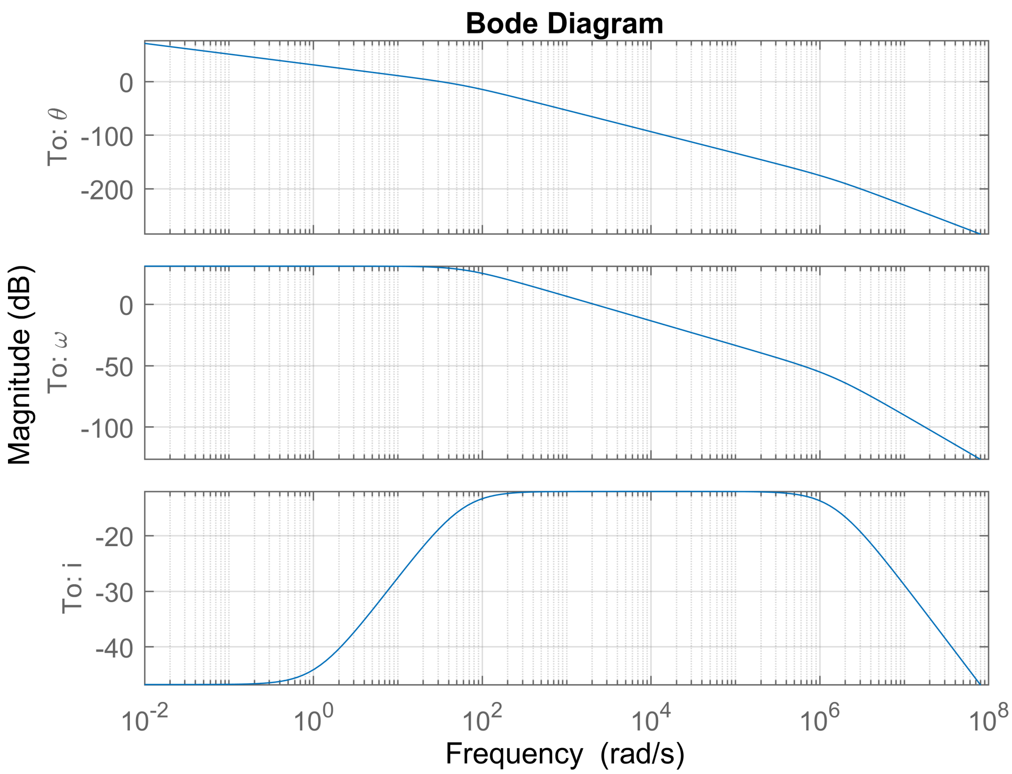
- system step response

```
step(myDCM, 0.1), grid on
```



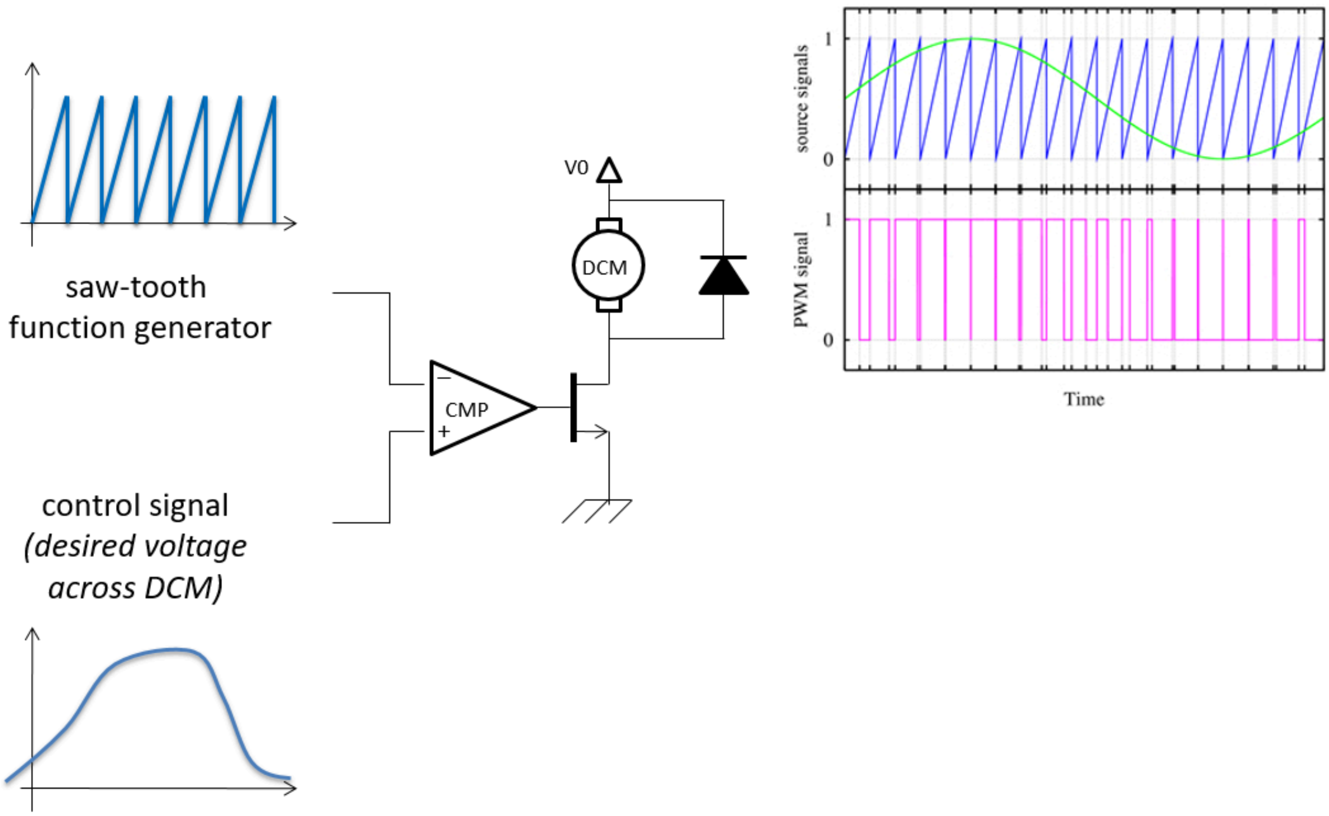
- system Bode plot

```
bodemag(myDCM), grid on
```



PWM: Pulse Width Modulation

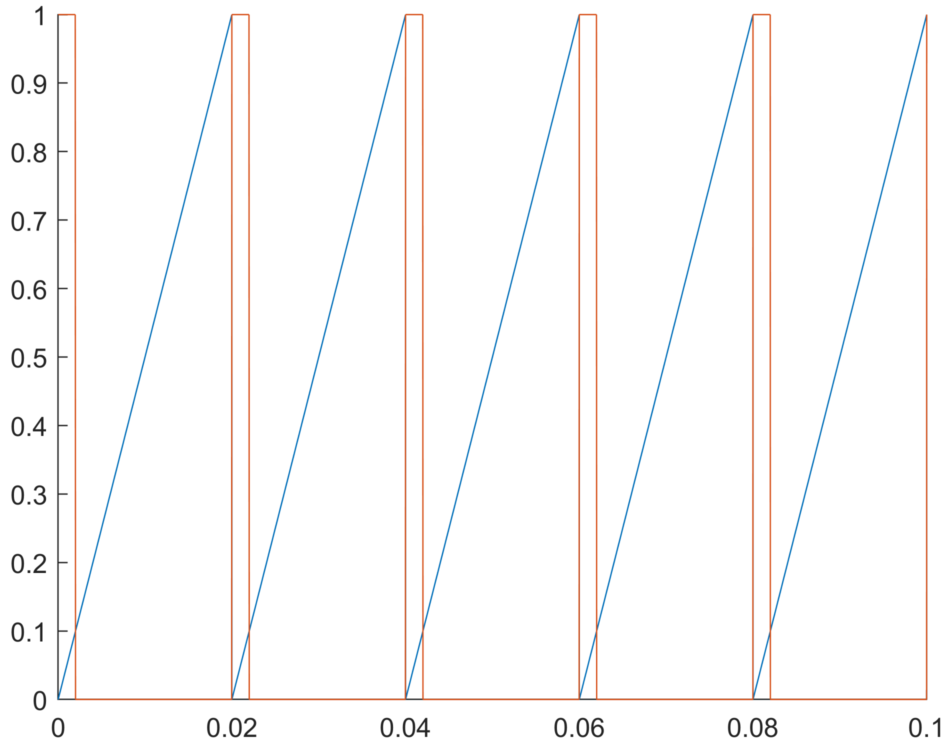
Driving loads such as a DC motor or a heater (which is basically just a high-power resistor) with analog devices, such as a transitor in active region, is usually very inefficient.



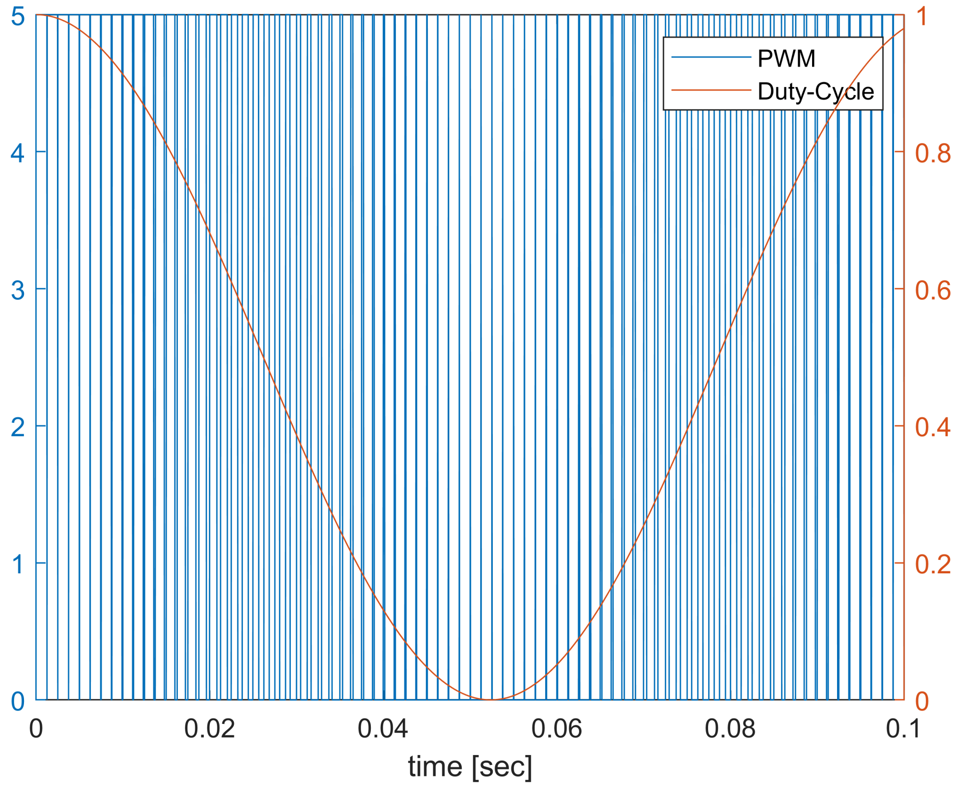

```
t = 0: 1e-6 : 0.1;

PWM = @(duty, f0, time) duty > mod(f0*time , 1)     %% this function generates a PWM
```

```
PWM = function_handle with value:
    @(duty,f0,time)duty>mod(f0*time,1)
```

```
figure
hold on
f0 = 50;
plot(t, mod(f0*t, 1))
plot (t, PWM(.1, f0, t))
```

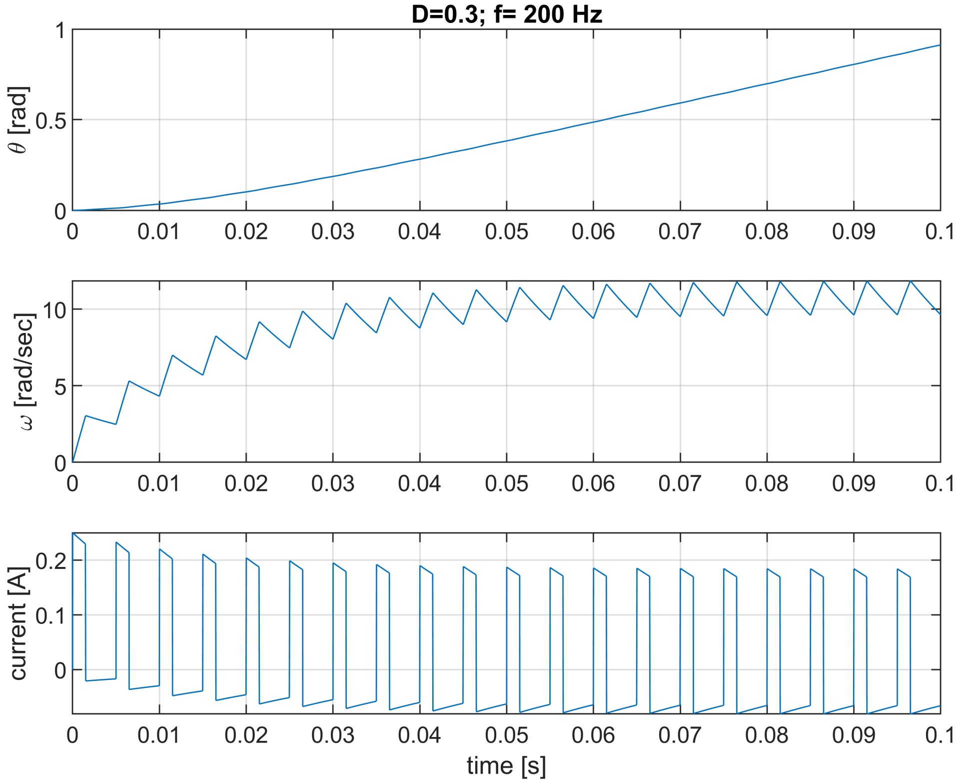


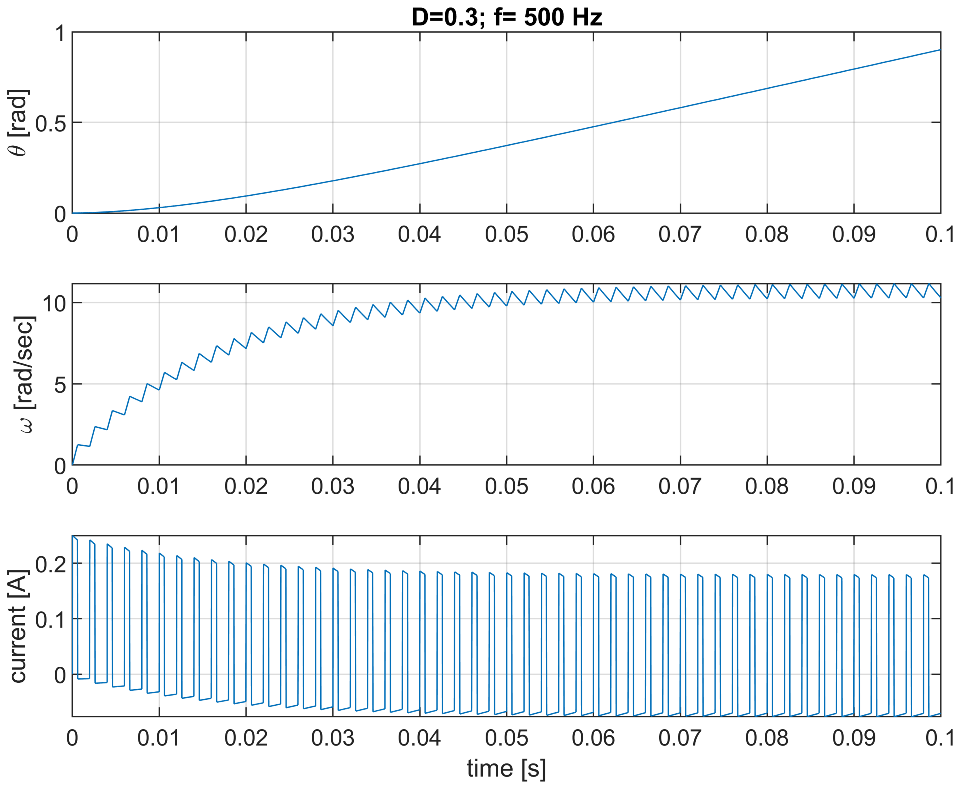

```
figure
modulating_signal = cos(30*t).^2;
plotyy(t, 5*PWM(modulating_signal , 800, t), t, modulating_signal )
legend('PWM', 'Duty-Cycle')
xlabel('time [sec]')
```

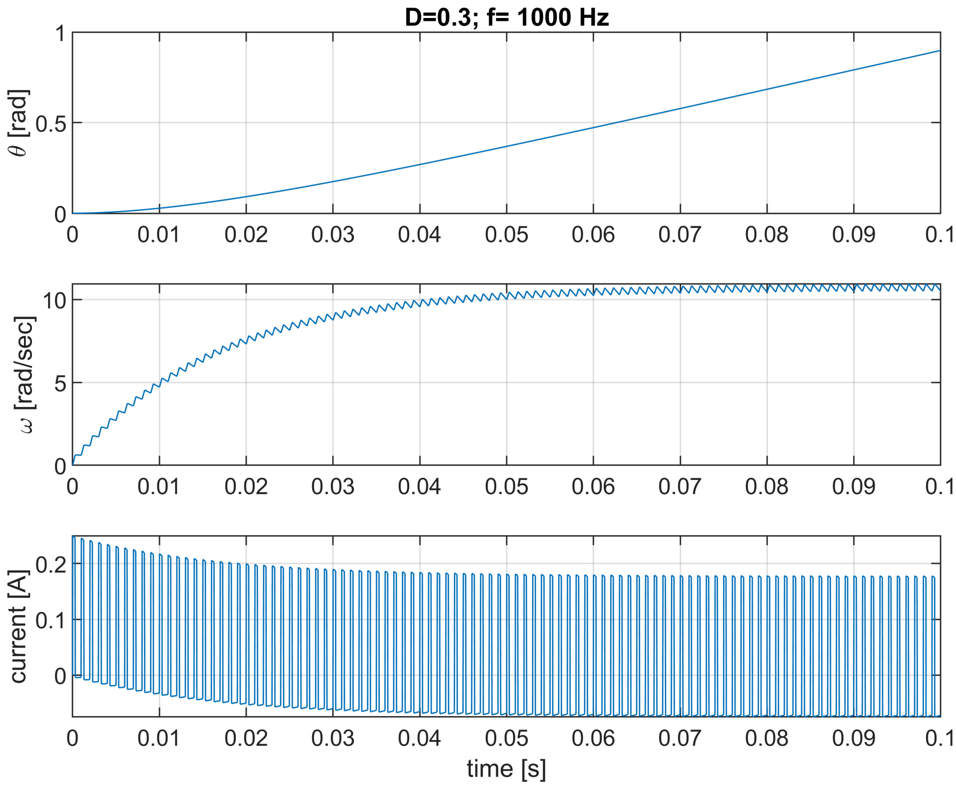



```
% now use the PWM to drive your motor
```

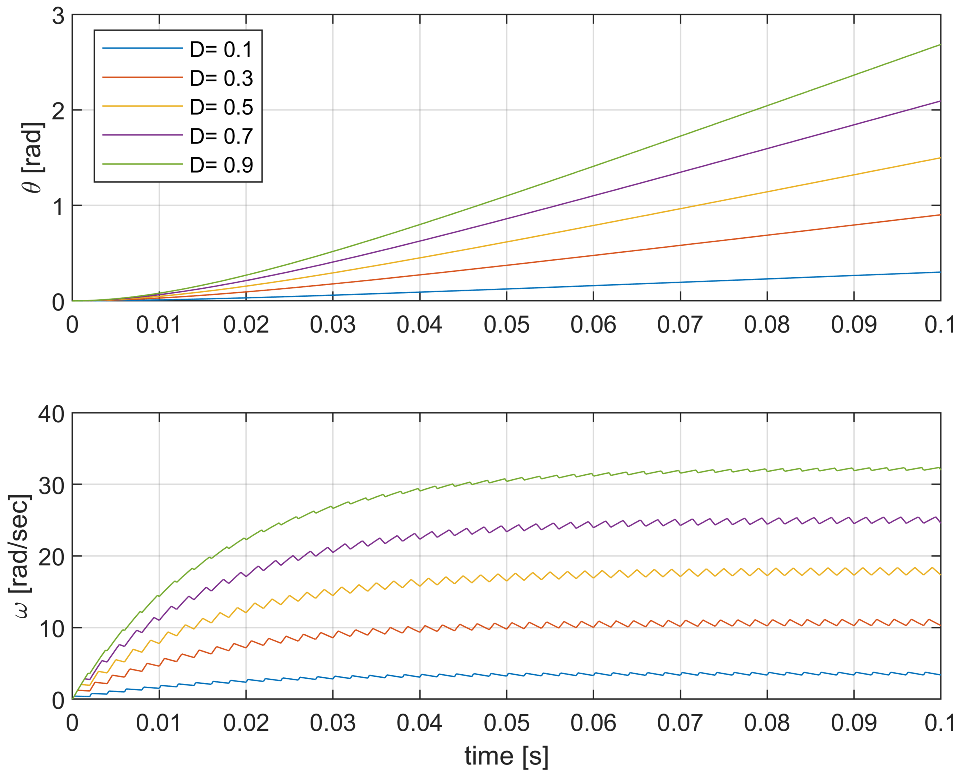
```
for freq= [200 500 1000]
    duty = 0.3;
    y= lsim(myDCM, 1*PWM(duty, freq, t) ,t);
    theta = y(:,1);
    omega = y(:,2);
    current = y(:,3);
figure;
    subplot(3,1,1)
    plot(t, theta); ylabel('\theta [rad]'), grid on
    title(['D=' num2str(duty) '; f= ' num2str(freq) ' Hz'])
    subplot(3,1,2)
    plot(t, omega); ylabel('\omega [rad/sec]'), grid on
    subplot(3,1,3)
    plot(t, current); ylabel('current [A]'), grid on
    xlabel('time [s]')
end
```


D=0.3; f= 200 Hz
θ [rad]
ω [rad/sec]
current [A]
time [s]

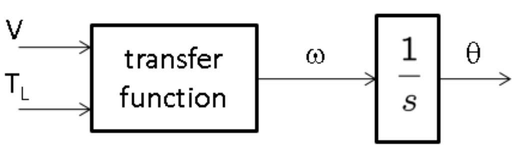

D=0.3; f= 500 Hz
θ [rad]
ω [rad/sec]
current [A]
time [s]



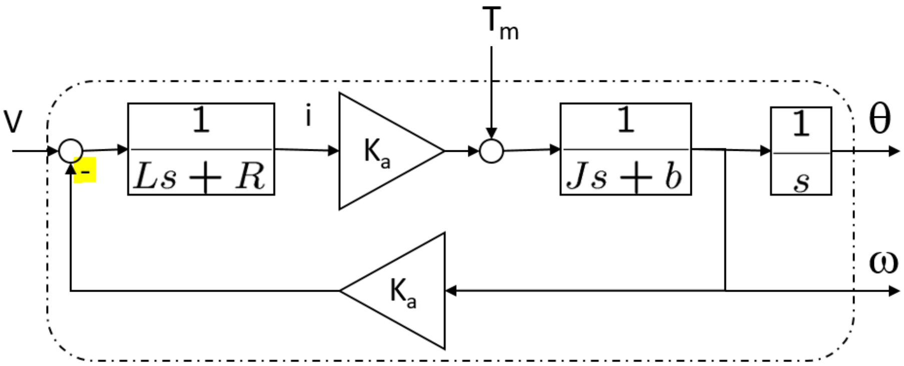

```
figure
myLegend = {};
for duty= [.1 .3 .5 .7 .9]
    myLegend = {myLegend{:}, ['D= ' num2str(duty)]};
    y= lsim(myDCM, 1*PWM(duty, 500, t) ,t);
    theta = y(:,1);
    omega = y(:,2);
    current = y(:,3);
    subplot(2,1,1)
    plot(t, theta); ylabel('\theta [rad]'), grid on
    hold on
    legend(myLegend, 'Location','northwest')
    subplot(2,1,2)
    plot(t, omega); ylabel('\omega [rad/sec]'), grid on
    xlabel('time [s]')
    hold on
end
```



DC Motor: block diagram



$$\begin{cases} V = Ri + L\dfrac{di}{dt} + K_a\,\omega \\ J\dfrac{d\omega}{dt} + b\,\omega = T_m + K_a\,i \end{cases} \xrightarrow{\text{Laplace}} \begin{cases} V - K_a\,\omega = (R + Ls)\,i \\ (Js + b)\,\omega = T_m + K_a\,i \end{cases}$$



T_m
V
-
$\frac{1}{Ls+R}$
i
K_a
$\frac{1}{Js+b}$
$\frac{1}{s}$
θ
ω
K_a