NANYANG TECHNOLOGICAL UNIVERSITY

MA2011 MECHATRONICS SYSTEMS INTERFACING

Tutorial 4

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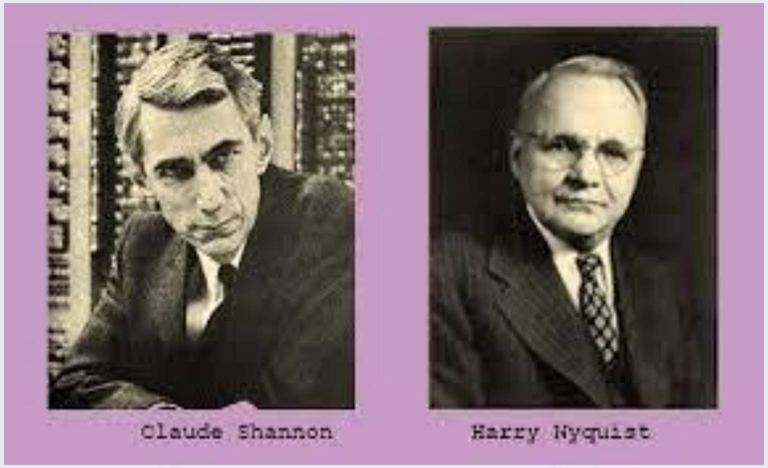
Sampling



Shannon-Nyquist Theorem

- We need to sample a digital signal at a rate more than two times the maximum frequency component in the signal to retain all frequency components.

- In other words, to faithfully represent the analog signal, the digital samples must be taken at a frequency f_s, such that

$$f_s > 2f_{max}$$

f_s is called sampling rate, and f_{max} is called Nyquist frequency

Sampling

$$F(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

$$\boldsymbol{F(t) \backsimeq C_0 + \sum_{n=1}^{N} A_n \cos(n\omega_0 t) + \sum_{n=1}^{N} B_n \sin(n\omega_0 t)}$$

- If we approximate a signal by a truncated Fourier series, the maximum frequency component is the highest harmonic frequency. Then the time interval between the digital samples is

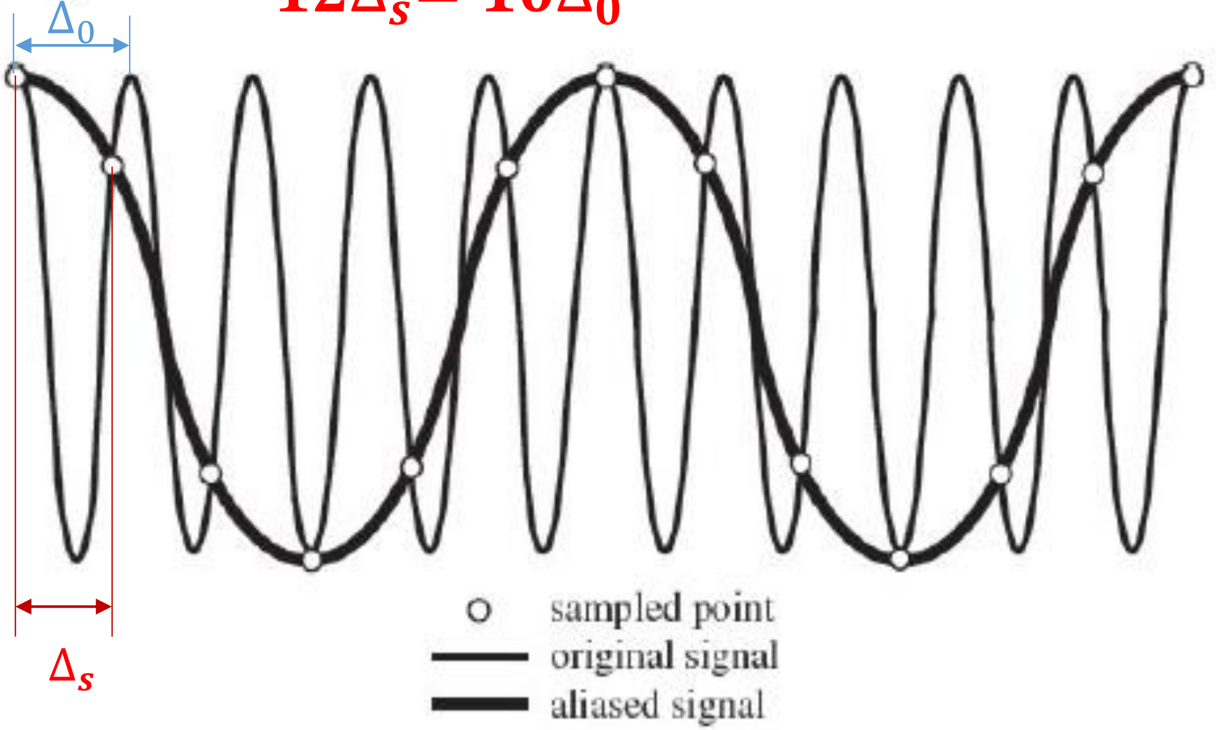
$\Delta t = 1/\ f_s$

ALIASING

If sampling rate is too low, we obtain wrong digital signal. This is called alaising.

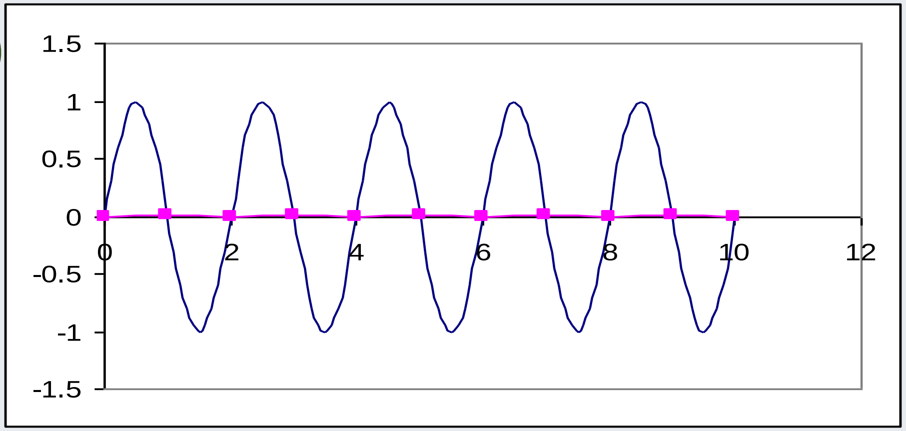
Example: 12 equally spaced samples are taken over 10 cycles of an analog signal, so $\boxed{f_s = 1.2 f_0}$ with f_0 frequency of the analog signal. Since $\boxed{f_s < 2f_0,}$ digitized signal does not describe the original signal accurately.

$$12\Delta_s = 10\Delta_0$$

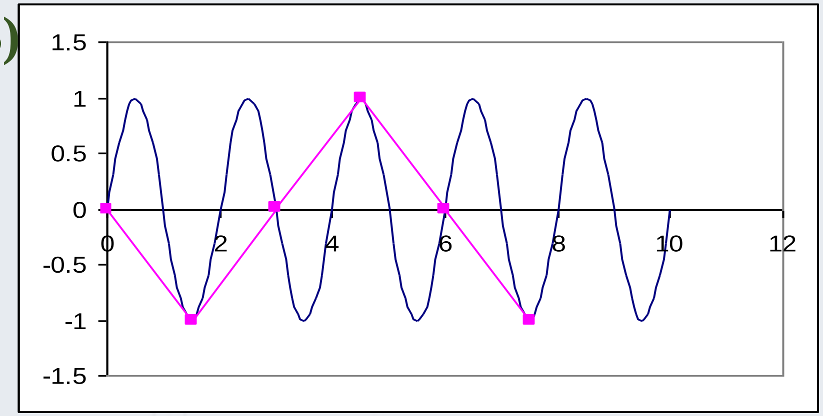


DISCUSSION Q1

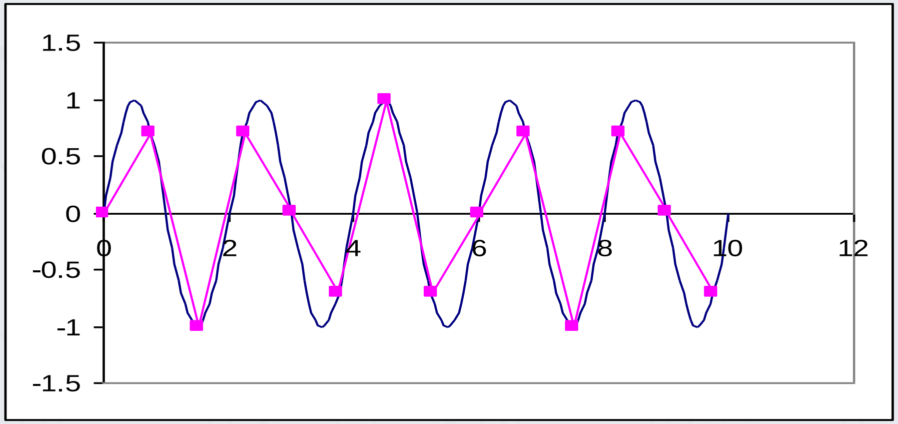
A)



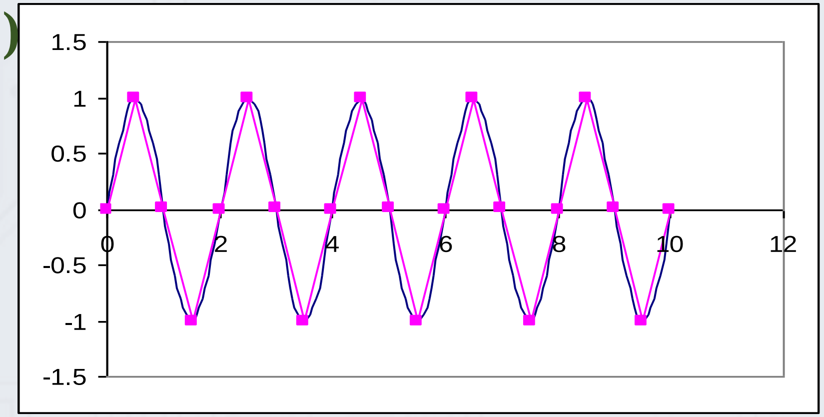
B)



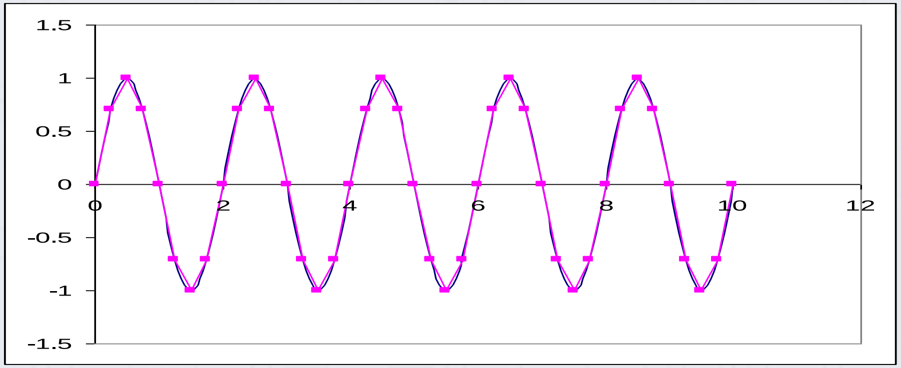
C)



D)



E)



Sampling time interval Δ_s =?

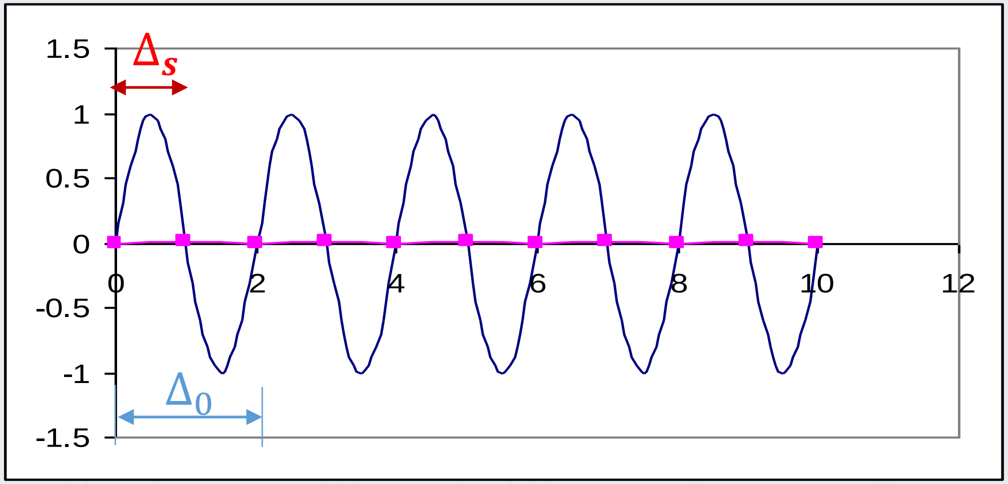
Sampling rate f_s=?

Nyquist Frequency f_{max}=?

Wave Period Δ_0 =? Frequency f_0 = ?

Sampling rate > 2*Wave Frequency?

POOR SAMPLING



Sampling time interval **Δs** =?

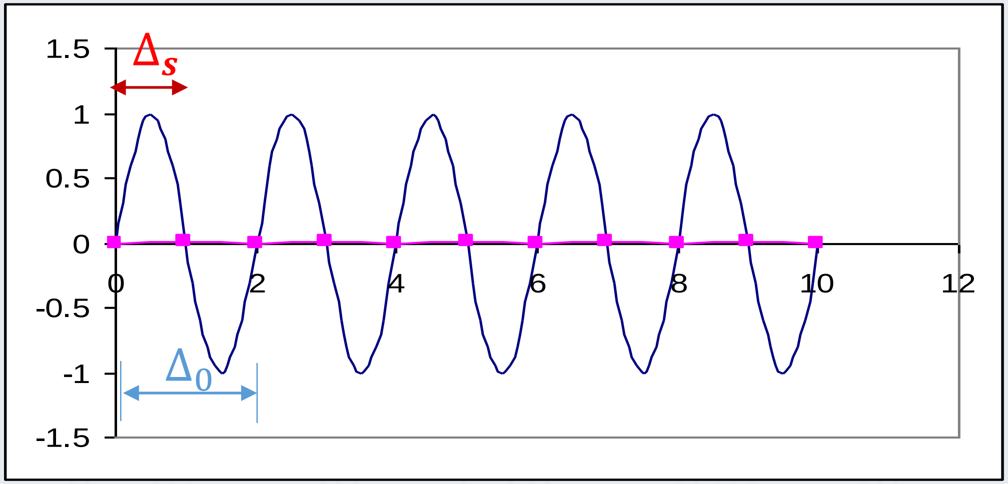
Sampling rate f_s=?

Wave Period Δ_0 =? Nyquist Frequency f_{max} = Wave Frequency f_0 = ?

Sampling rate > 2 * Nyquist Frequency ?

DISCUSSION Q1(A)

POOR SAMPLING



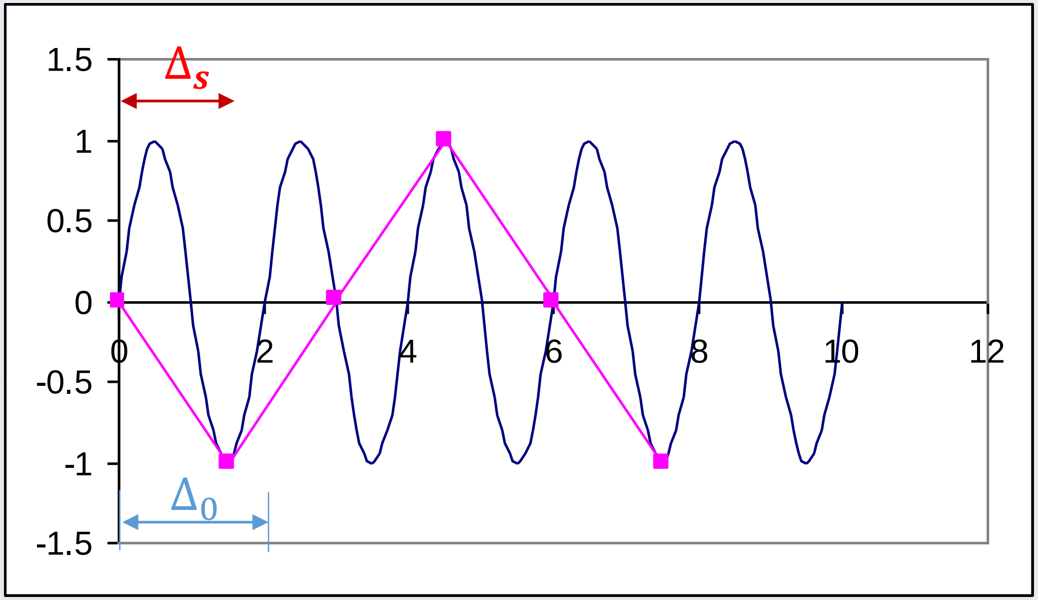
Sampling time interval Δs =2s/2=1s

Sampling rate f_s=1/ Δs=1Hz

Δ_0=2s, Nyquist Frequency f_{max} =Wave Frequency f_0 =1/ Δ_0 =1/2Hz

Sampling rate (1 Hz) = 2 * Nyquist Frequency (1 Hz)

POOR SAMPLING



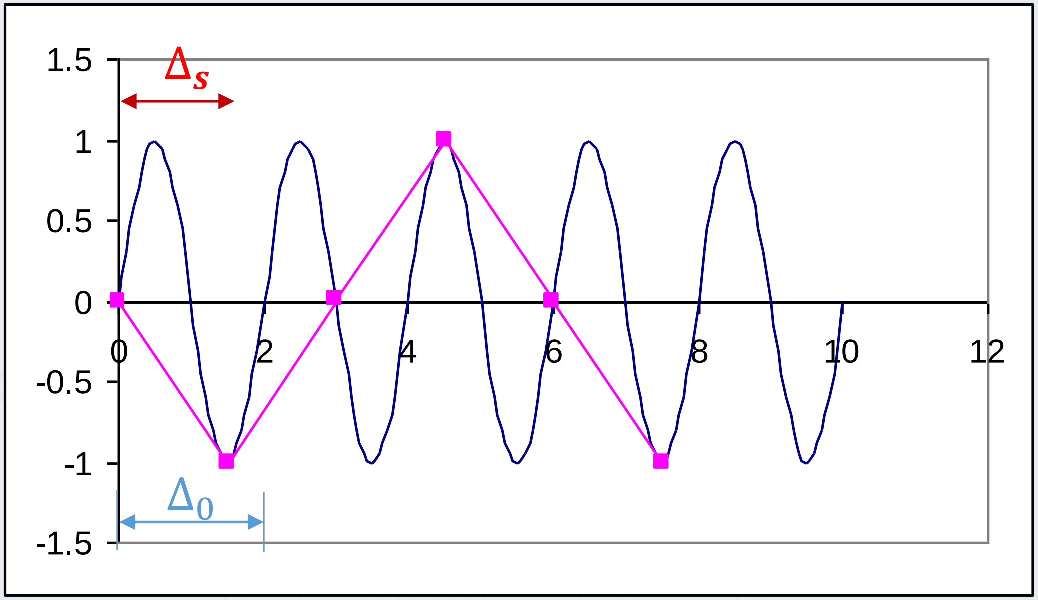
Sampling time interval Δs =?

Sampling rate f_s=?

Δ_0=? Nyquist Frequency f_{max} = Wave Frequency f_0 = ?

Sampling rate > 2*Nyquist Frequency ?

POOR SAMPLING



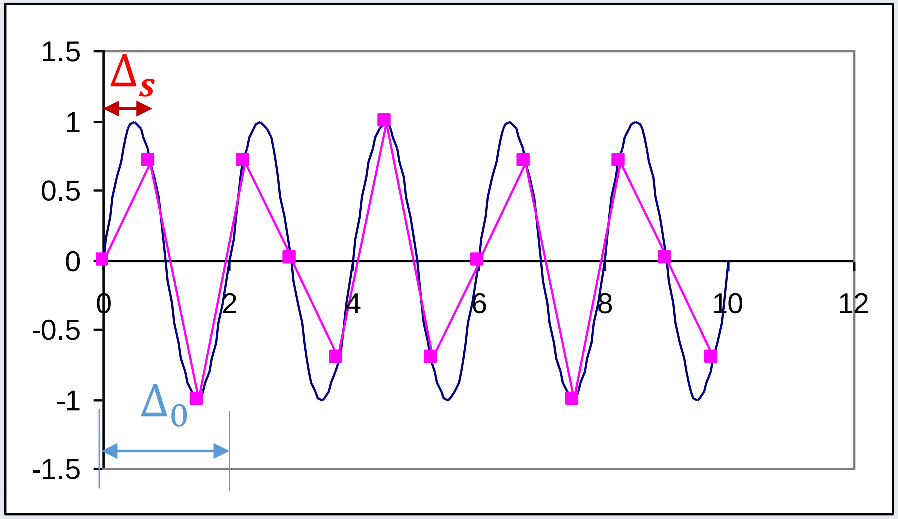
Sampling time interval Δs =6/4s=1.5s

Sampling rate f_s=1/ Δs=2/3 Hz

Δ_0=2s , Nyquist Frequency f_{max} = Wave Frequency f_0 =1/ Δ_0= 1/2Hz

Sampling rate (2/3 Hz) < 2 * Nyquist Frequency (1 Hz)

HIGHER SAMPLING FREQUENCY



Sampling time interval Δs =?

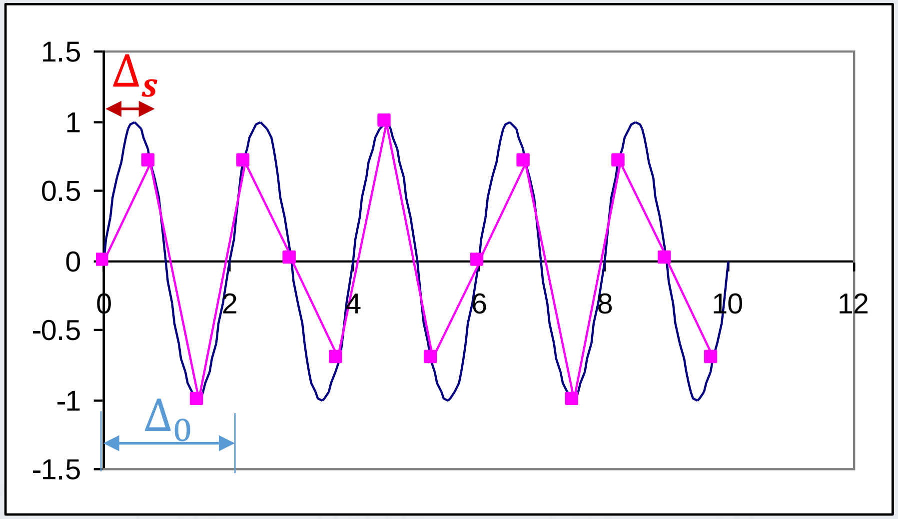
Sampling rate f_s=?

Δ_0=? Nyquist Frequency f_{max} = Wave Frequency f_0 = ?

Sampling rate > 2*Nyquist Frequency ?

DISCUSSION Q1(C)

HIGHER SAMPLING FREQUENCY



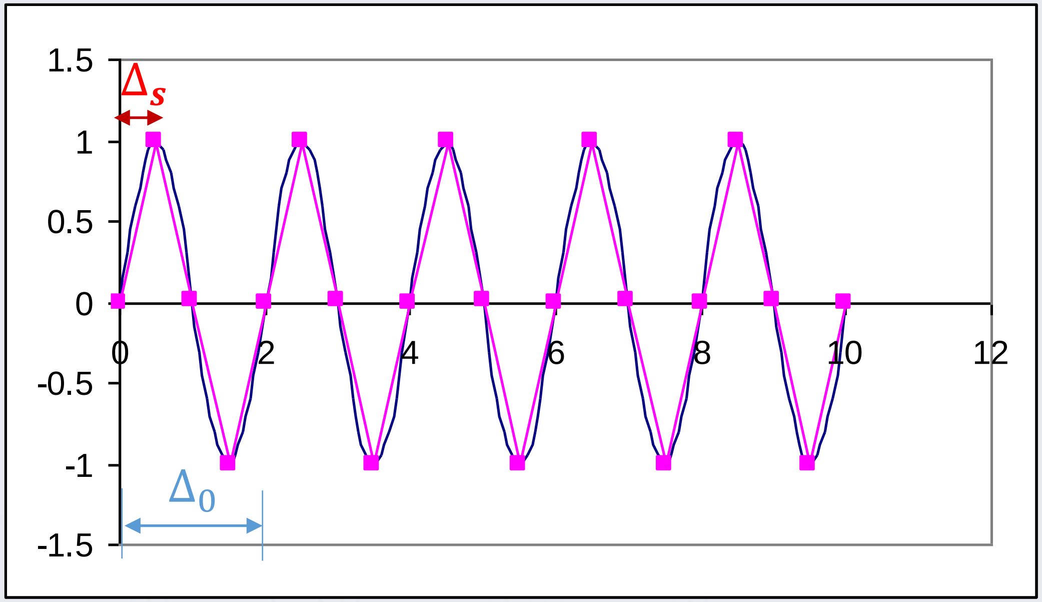
Sampling time interval Δs =6/8s

Sampling rate f_s=1/ Δs=4/3 Hz

Δ_0=2s , Nyquist Frequency f_{max} = Wave Frequency f_0 =1/ Δ_0**=** 1/2Hz

Sampling rate (4/3 Hz)= **8/3** * Nyquist Frequency > **2** * Nyquist Frequency

GETTING BETTER



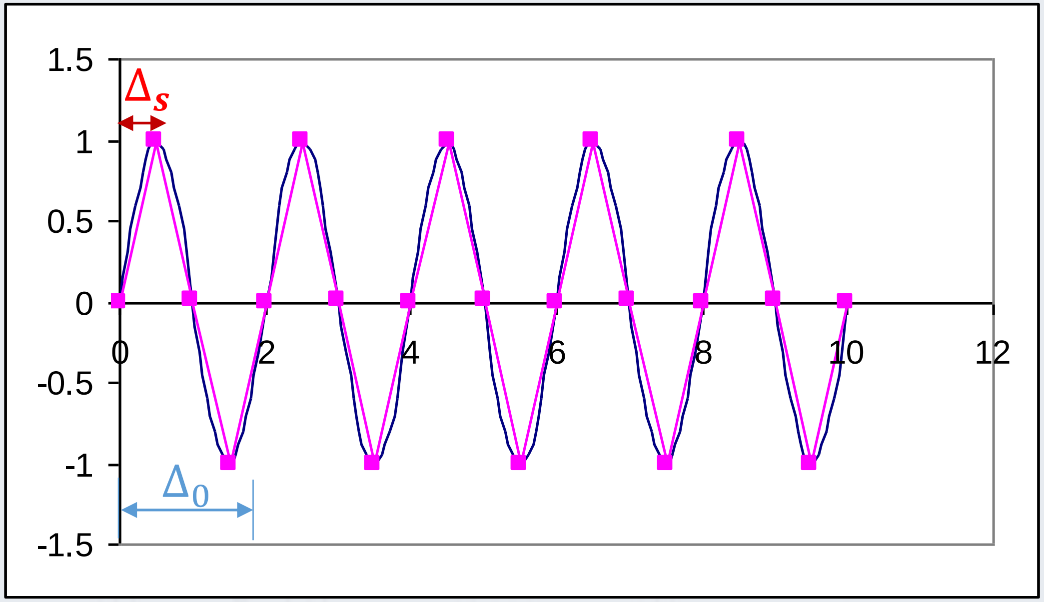
Sampling time interval **Δs** =?

Sampling rate f_s=?

Δ_0**=?** Nyquist Frequency f_{max} =Wave Frequency f_0 = ?

Sampling rate > 2* Nyquist Frequency ?

GETTING BETTER



Sampling time interval Δs =2/4s

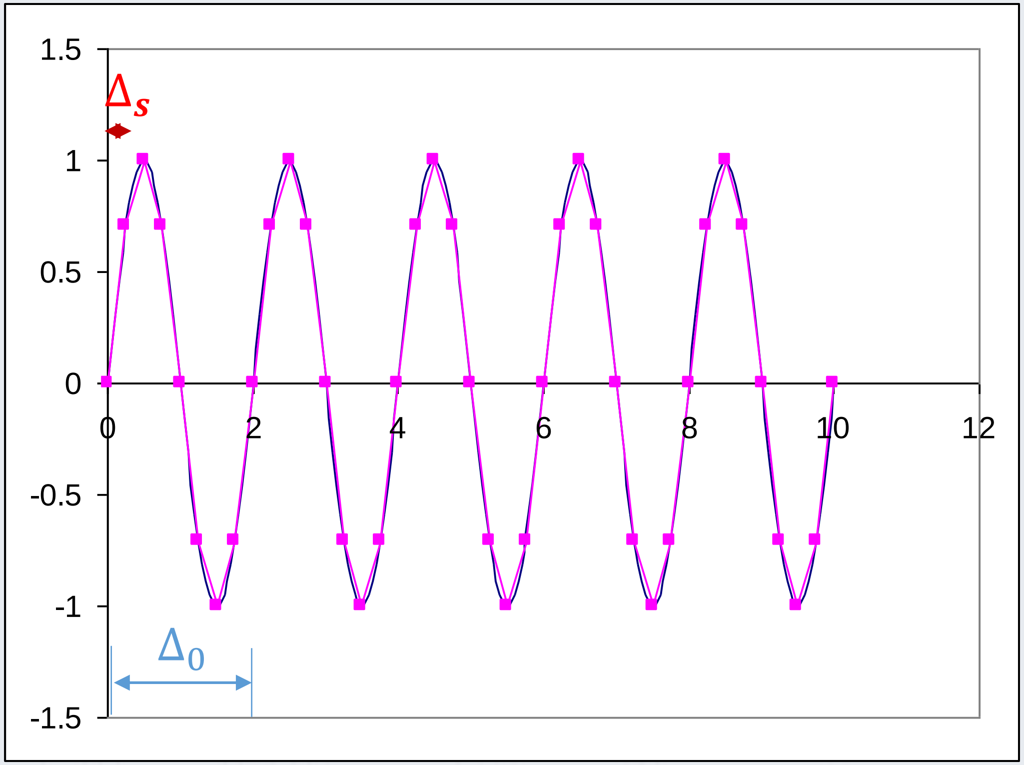
Sampling rate f_s=1/ Δt=2 Hz

Δ_0**=2s ,** Nyquist Frequency f_{max} =Wave Frequency f_0 =1/ Δ_0**=** 1/2Hz

Sampling rate (2 Hz) = **4** * Nyquist Frequency

DISCUSSION Q1(E)

GOOD SAMPLING



Sampling time interval **Δs** =**?**

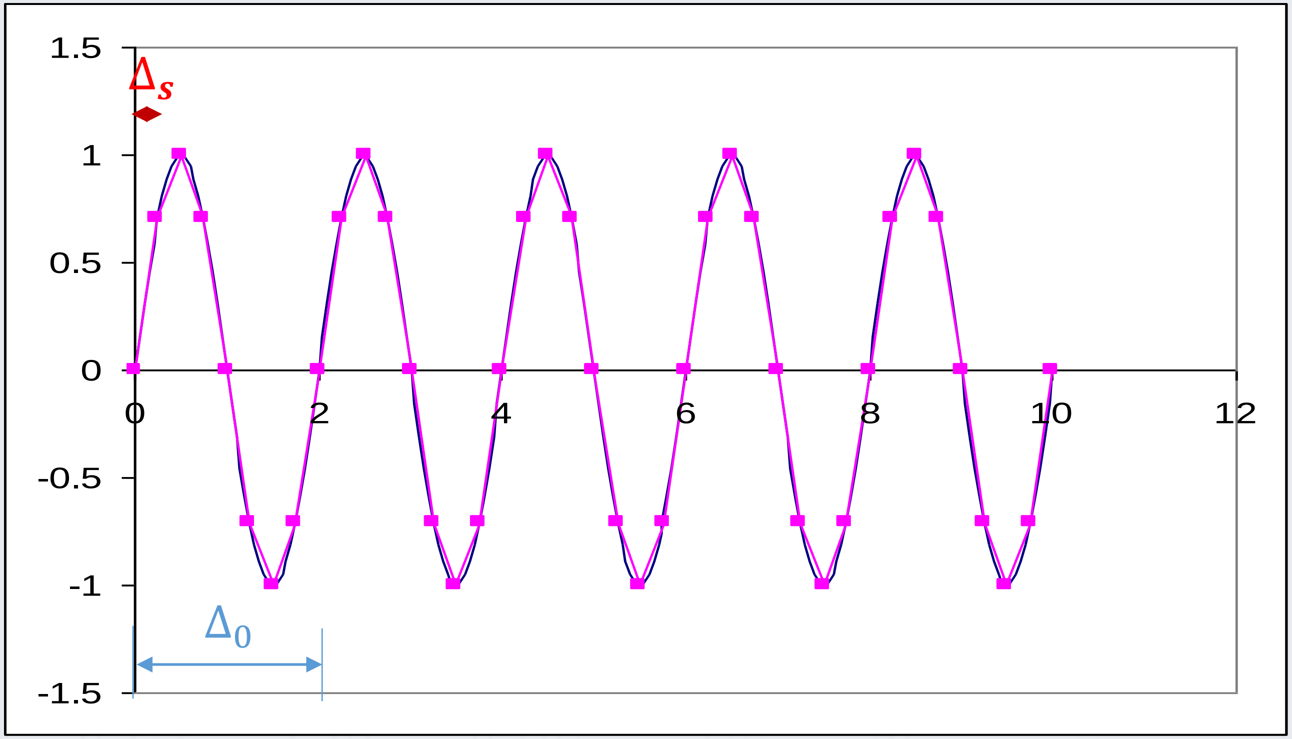
Sampling rate f_s=?

Δ_0**=?** Nyquist Frequency f_{max} =Wave Frequency f_0 = ?

Sampling rate > **2*** Nyquist Frequency ?

DISCUSSION Q1(E)

GOOD SAMPLING

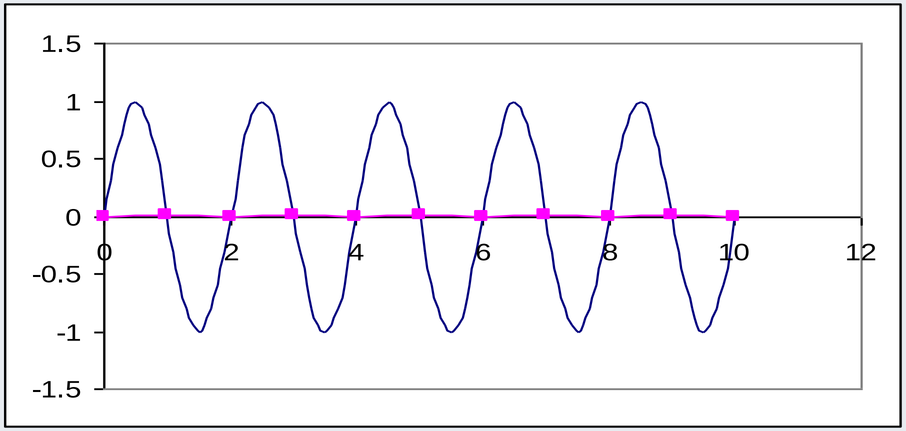


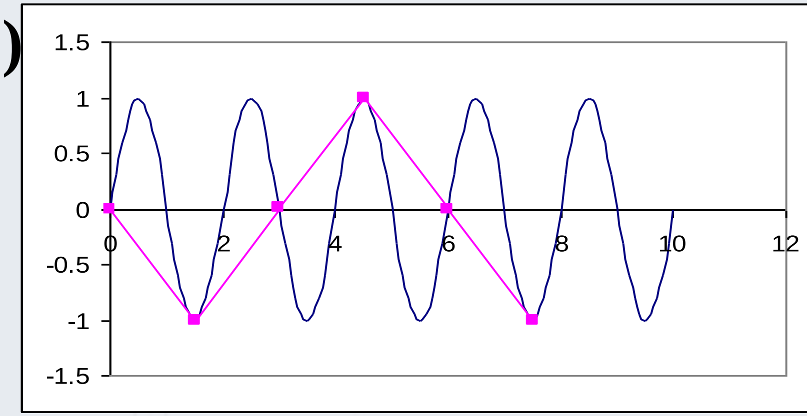
Sampling time interval Δs =2/8=0.25s

Sampling rate f_s=1/ Δs=4 Hz

Δ_0**=2s,** Nyquist Frequency f_{max} =Wave Frequency f_0 =1/ Δ_0**=** 1/2Hz

Sampling rate (4 Hz)= **8** * Nyquist Frequency

1)


2)


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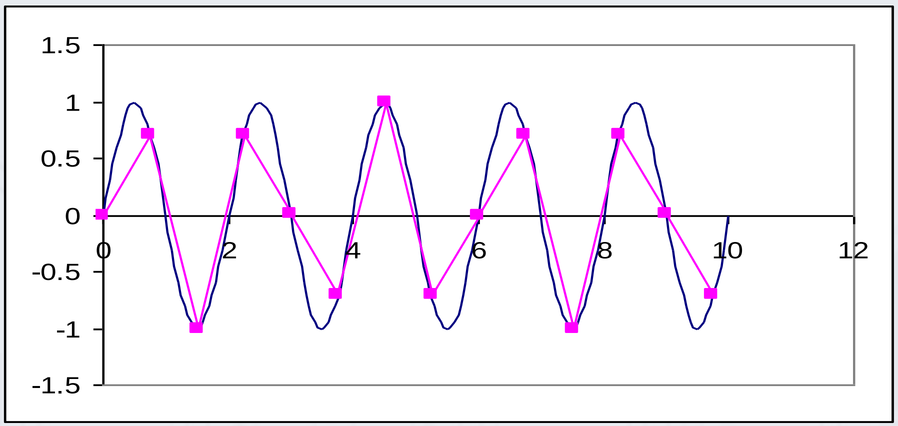
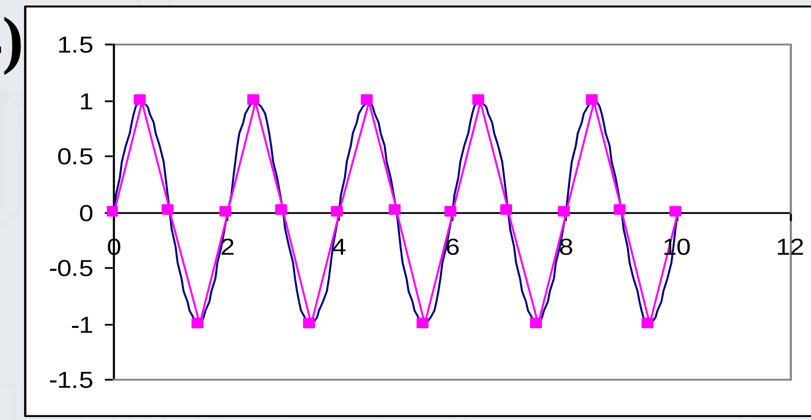
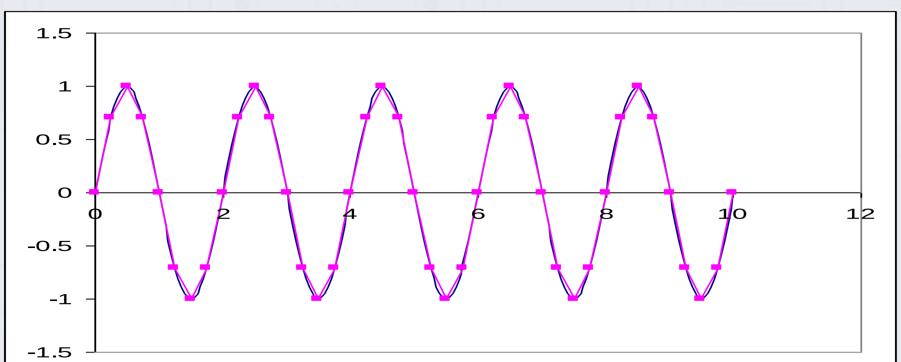


4)


5)


	1	2	3	4	5
Δs	1 s	1.5s	0.667s	0.5s	0.25s
$\Delta_0/2$	2/2s	2/2s	2/2s	2/2s	2/2s

Sampling interval < Wave period/2?

ANSWER TO Q1

RECOMMENDED SAMPLING

Sampling rate =**10** * Nyquist Frequency

Q2

$$F(t) = C_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} B_n \sin(n\omega_0 t)$$

$n\omega_0$

Consider the continuous signal:

$$\sum_n \frac{4}{(2n-1)\pi} sin\left[\frac{2\pi(2n-1)t}{10}\right]$$

$\frac{2n-1}{10}\,2\pi$

1) What would be an appropriate sampling rate to use in sampling this signal if is to filtered at and above 2 Hz before sampling?
2) What are the alias frequencies of the filtered signal at this sampling rate?

($f_k > 4$ Hz; many possible solutions)

ANSWER TO Q2

$$\omega_n = 2\pi\frac{2n-1}{10},\ f_n = \frac{2n-1}{10}$$

The n^{th} term of the series has a frequency $f_n = (2n - 1)/10$.

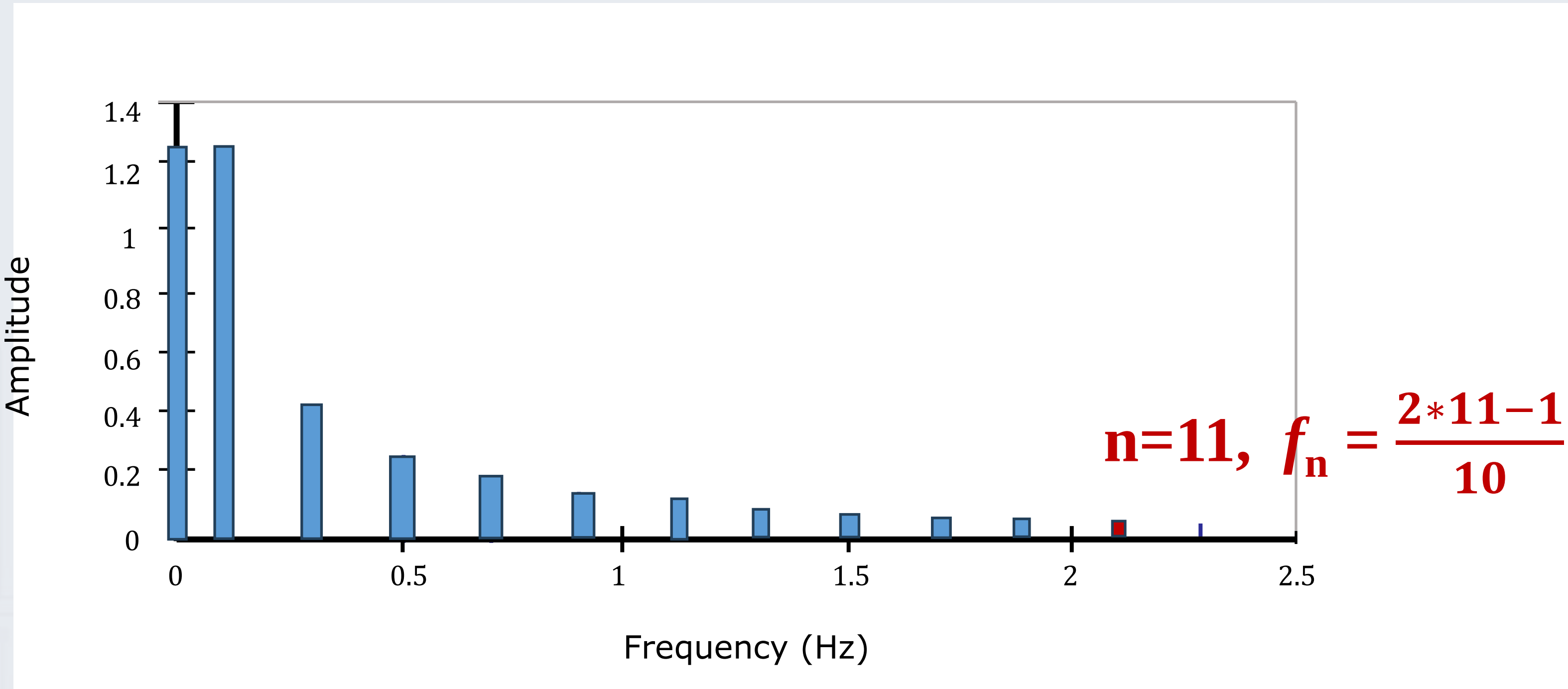
The fundamental frequency $(n = 1)$ is $f_1 = f_0 = 0.1$ Hz.

n	0	1	2	3	4	5	6	7	8	9	10
$f_n = \frac{2n-1}{10}$	1/10	1/10	3/10	5/10	7/10	9/10	11/10	13/10	15/10	17/10	19/10
Amplitude $\frac{4}{(2n-1)\pi}$	$\frac{4}{\pi}$	$\frac{4}{\pi}$	$\frac{4}{3\pi}$	$\frac{4}{5\pi}$	$\frac{4}{7\pi}$	$\frac{4}{9\pi}$	$\frac{4}{11\pi}$	$\frac{4}{13\pi}$	$\frac{4}{15\pi}$	$\frac{4}{17\pi}$	$\frac{4}{19\pi}$
~	1.27	1.27	0.43	0.25	0.18	0.14	0.12	0.10	0.09	0.08	0.07

1) What would be an appropriate sampling rate to use in sampling this signal if is to filtered at and above 2 Hz before sampling?

SOLUTION:

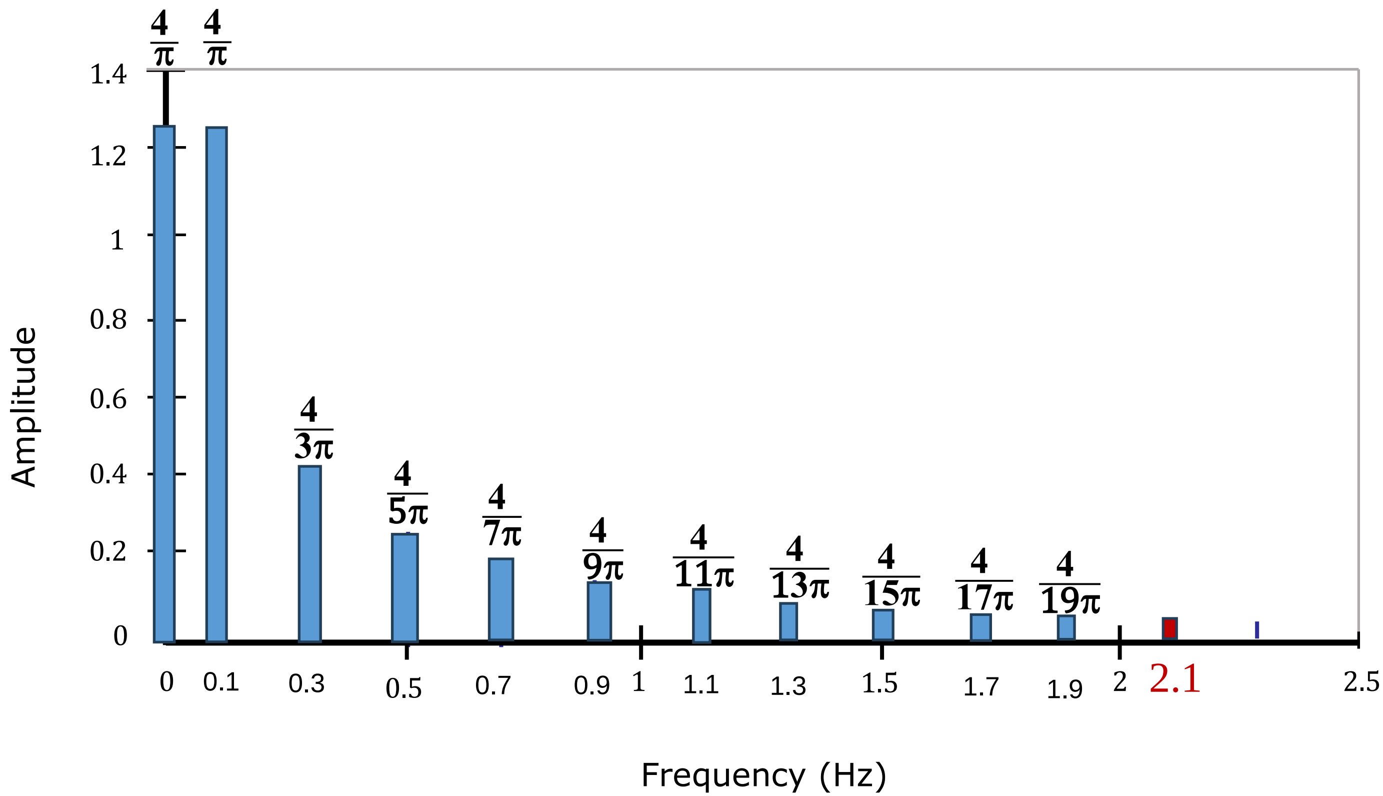
This signal has a frequency spectrum as shown:



$$n=11, \; f_n = \frac{2*11-1}{10} = 2.1$$

Filtering the signal at and above 2 Hz limits this series representation of y(t) to 10 terms.

ANSWER TO Q2



Amplitude
1.4
1.2
1
0.8
0.6
0.4
0.2
0
$\frac{4}{\pi}$
$\frac{4}{\pi}$
$\frac{4}{3\pi}$
$\frac{4}{5\pi}$
$\frac{4}{7\pi}$
$\frac{4}{9\pi}$
$\frac{4}{11\pi}$
$\frac{4}{13\pi}$
$\frac{4}{15\pi}$
$\frac{4}{17\pi}$
$\frac{4}{19\pi}$
0
0.1
0.3
0.5
0.7
0.9
1
1.1
1.3
1.5
1.7
1.9
2
2.1
2.5
Frequency (Hz)

ANSWER TO Q2

2) What are the alias frequencies of the filtered signal at this sampling rate?

According to Shannon-Nyquist Theorem:

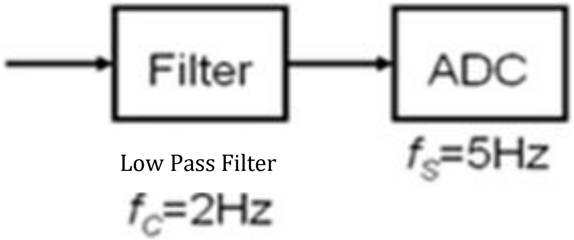
Nyquist Frequency f_{max} = 2 Hz, need to set sampling rate f_s > 4 Hz.

The alias frequencies of the filtered signal depend on the sampling rate. Let us say we select $\boldsymbol{f_s}$ **=** 5 Hz.

Alias frequencies can be found using

$f_a = \pm f_n + i f_s$ for i = 1 to ∞

e.g. f_a = -0.1+5 =4.9 f_a = 0.1+5 =5.1



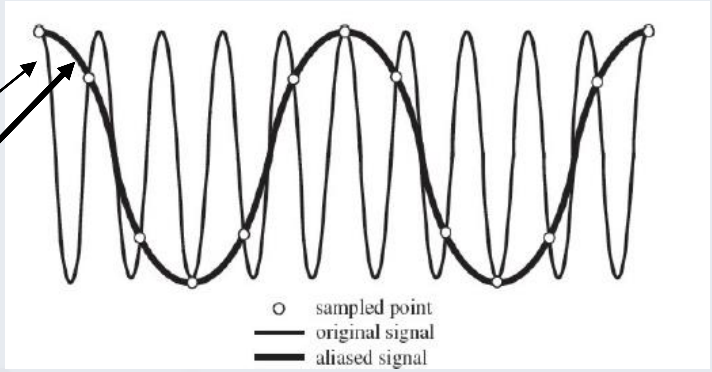
n	f_n	i=1	i=2	i=3	i=4	…
1	0.1	4.9, 5.1	9.9, 10.1	24.9, 15.1	19.9, 20.1	Etc.
2	0.3	4.7, 5.3	9.7, 10.3	14.7, 15.3	19.7, 20.3	Etc.
3	0.5	4.5, 5.5	9.5, 10.5	14.5, 15.5	19.5, 20.5	Etc.
4	0.7	4.3, 5.7	9.3, 10.7	Etc.		
5	0.9	4.1, 5.9	9.1, 10.9	Etc.		
6	1.1	3.9, 6.1	8.9, 11.1	Etc.		
7	1.3	3.7, 6.3	8.7, 11.3	Etc.		
8	1.5	3.5, 6.5	Etc.			
9	1.7	3.3, 6.7	Etc.			
10	1.9	3.1, 6.9	Etc.			

Objective function: |i*fs – fn| with variable i=1, 2, 3 ….

Shaparenko, B. and Cimbala, J. M., Int. J. *Mech. Engr Education*, Vol. 39, No. 3, pp. 195-199, 2012

ALIAS FREQUENCIES

$f_a = \pm f_n + i * f_s, for\ i = 1, 2, 3 \ldots$? **(Proof)**



Given two signals $x(t)$ and $y(t)$:

$$x(t) = Acos(2\pi f_n t)$$
$$y(t) = Acos(2\pi f_a t)$$

We sample these two signals at sampling rate of f_s:

$$x[n] = Acos(2\pi f_n \frac{n}{f_s})$$

$$y[n] = Acos(2\pi f_a \frac{n}{f_s})$$

In the next few steps, we prove that $y[n] = x[n]$, in the case of $f_a = i * f_s - f_n$.

$$y[n] = Acos\left[2\pi(i * f_s - f_n)\frac{n}{f_s}\right]$$

$$= Acos\left[2\pi i * f_s * \frac{n}{f_s} - 2\pi f_n \frac{n}{f_s}\right]$$

$$= Acos[2\pi i * n - 2\pi f_n \frac{n}{f_s}]$$

$$= Acos(-2\pi f_n \frac{n}{f_s})$$

$$= Acos(2\pi f_n \frac{n}{f_s})$$

$$= x[n]$$

Same for the case of $f_a = i * f_s + f_n$. Proof is done.

Q3

a) How many bits are needed to represent the number 756? Without using a calculator, convert 756 to binary

(10 bits; 10 1111 0100)

b) An 8-bit digital-to-analog converter has an output range of 0 to 5V. What is its resolution? Estimate the voltage output if the input code has the decimal value of 32.

(0.0195V; 0.625V)

ANSWER TO Q3(A)

SOLUTION:

a) How many bits are needed to represent the number 756? Without using a calculator, convert 756 to binary

(10 bits; 10 1111 0100)

$$2^n \geq 756$$

$$n \geq \frac{\log_{10} 756}{\log_{10} 2} = \frac{2.878}{0.301}$$

$$= 9.56 \rightarrow 10$$

Thus 10 bits are required.

756 / 2 = 378 remainder **0**
378 / 2 = 189 remainder **0**
189 / 2 = 94 remainder **1**
94 / 2 = 47 remainder **0**
47 / 2 = 23 remainder **1**
23 / 2 = 11 remainder **1**
11 / 2 = 5 remainder **1**
5 / 2 = 2 remainder **1**
2 / 2 = 1 remainder **0**
1 / 2 = 0 remainder **1**

Thus $756_{10} = 10\ 1111\ 0100_2$

ANSWER TO Q3(B)

b) An 8-bit digital-to-analog converter has an output range of 0 to 5V. What is its resolution? Estimate the voltage output if the input code has the decimal value of 32.

(0.0195V; 0.625V)

SOLUTION:

$$\text{Resolution } Q = \frac{(V_{max} - V_{min})}{2^n} = \frac{5}{2^8} = 0.01953125\text{V}$$

An input of 32 will then give an output of

$$V_{out} = 32 \times 0.01953 = 0.625\text{V}$$

COMMENT: Maximum input code is 1111 1111 = 255. Therefore the maximum $V_{out} = 255 \times 0.01953125 = 4.980\text{V}$, not 5V!

Error = 5V - 4.98V ~ **<u>0.01953125V</u>**

Note: Q=(Vmax-Vmin)/(2^n-1) when 2^n is fairly large the difference for the results can be ignored both in terms of resolution and output.

THANK YOU