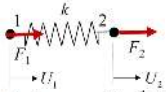
Math CheatSheet

Mathematical Methods In Engineering (Nanyang Technological University)

3.2 Stiffness Matrix of Spring Element by Principle of MPE



Strain energy: $U = \frac{1}{2}k(U_2 - U_1)^2$

Work done: $W = F_1U_1 + F_2F_2$

Total potential energy:(TPE) U W

$$\Pi \equiv U - W = \frac{1}{2}k(U_2 - U_1)^2 - (F_1U_1 + F_2F_2)$$

Principle of MPE: $\delta\Pi = 0$

$$\delta\Pi \equiv \frac{\partial\Pi}{\partial U_1}\delta U_1 + \frac{\partial\Pi}{\partial U_2}\delta U_2 = 0$$

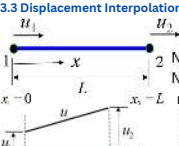
$$\Rightarrow \frac{\partial\Pi}{\partial U_1} = 0 \Rightarrow kU_1 - kU_2 = F_1 \text{ ---------Eqn 1}$$

$$\Rightarrow \frac{\partial\Pi}{\partial U_2} = 0 \Rightarrow -kU_1 + kU_2 = F_2 \text{ -------Eqn 2}$$

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$ <<<Matrix form (eqn 1 & 2)

TPE (matrix form): $\Pi = \frac{1}{2}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}^T \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} - \begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix}^T \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$

3.3 Displacement Interpolation: 2-Node Bar Element



$$u = \alpha_1 + \alpha_2 x$$

Node 1: $x = x_1, u = u_1$

Node 2: $x = x_2, u = u_2$

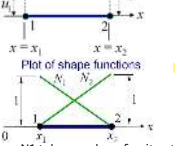
Displacement interpolation:

$$u = N_1u_1 + N_2u_2$$

$$u = [N]\{u^{(e)}\}$$

$$[N] \equiv [N_1 \quad N_2]$$

$$\{u^{(e)}\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$



Shape functions:

$$N_1 = \frac{x_2 - x}{x_2 - x_1}$$

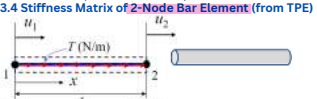
$$N_2 = \frac{x - x_1}{x_2 - x_1}$$

- N1 takes a value of unity at node 1 and 0 at node 2;

N2 takes a value of unity at node 2 and 0at node 1

- Sum of shape functions is equal to 1: N1 + N2 = 1

3.4 Stiffness Matrix of 2-Node Bar Element (from TPE)



Step1: Potential energy expression

$$U = \frac{1}{2}\int_0^L \sigma\varepsilon A dx = \frac{1}{2}\int_0^L E\varepsilon^2 A dx$$

$$W = \int_0^L uT dx$$

eqn 1

$$\Pi \equiv U - W = \frac{1}{2}\int_0^L E\varepsilon^2 A dx - \int_0^L uT dx$$

u - displacement strain; $\varepsilon = du/dx$ - strain $\sigma = E\varepsilon$

T - distributed tensile load intensity (function of x)

Step2: Element displacement interpolation

$$u = N_1u_1 + N_2u_2 \quad [N] \equiv [N_1 \quad N_2]$$

$$u = [N]\{u^{(e)}\} \quad \{u^{(e)}\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$ from section 3.3

Step3: Element strain

$$\varepsilon = [B]\{u^{(e)}\} \quad [B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} \end{bmatrix}$$

$$\{u^{(e)}\} = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

Step4: Potential energy in matrix form

Rewriting eqn 1,

$$\Pi = \frac{1}{2}\int_0^L \varepsilon^T E\varepsilon A dx - \int_0^L u^T T dx$$

$$\Pi = \frac{1}{2}\int_0^L \{u^{(e)}\}^T [B]^T E[B]\{u^{(e)}\} A dx - \int_0^L \{u^{(e)}\}^T [N]^T T dx$$

$$\Pi = \frac{1}{2}\{u^{(e)}\}^T \left(\int_0^L [B]^T E[B] A dx\right)\{u^{(e)}\} - \{u^{(e)}\}^T \left(\int_0^L [N]^T T dx\right)$$

$$\Pi = \frac{1}{2}\{u^{(e)}\}^T [K^{(e)}]\{u^{(e)}\} - \{u^{(e)}\}^T \{f^{(e)}\}$$ <<< general form

$$u^T = \{u^{(e)}\}^T [N]^T \qquad [K^{(e)}] = \int_0^L [B]^T E[B] A dx$$

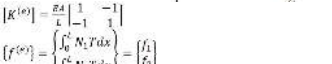
$$\varepsilon^T = \{u^{(e)}\}^T [B]^T \qquad \{f^{(e)}\} = \int_0^L [N]^T T dx$$

[K] - **element stiffness matrix**

{f} - **element force vector** (due to distributed tensile force)

Step5: Obtaining element stiffness matrix & force vector

Force applied on 2-node bar element >>>

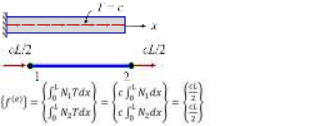


$$[K^{(e)}] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\{f^{(e)}\} = \begin{Bmatrix} \int_0^L N_1 T dx \\ \int_0^L N_2 T dx \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

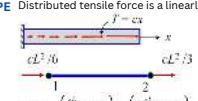
2 Cases of element force vector

Distributed tensile force is a constant (T = c):



$$\{f^{(e)}\} = \begin{Bmatrix} \int_0^L N_1 T dx \\ \int_0^L N_2 T dx \end{Bmatrix} = \begin{Bmatrix} cL/2 \\ cL/2 \end{Bmatrix}$$

Distributed tensile force is a linearly varying force (T = cx):

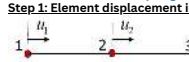


$$\{f^{(e)}\} = \begin{Bmatrix} \int_0^L N_1 T dx \\ \int_0^L N_2 T dx \end{Bmatrix} = \begin{Bmatrix} c\int_0^L N_1 x dx \\ c\int_0^L N_2 x dx \end{Bmatrix} = \begin{Bmatrix} \frac{cL^2}{6} \\ \frac{cL^2}{3} \end{Bmatrix}$$

3 NODE BAR

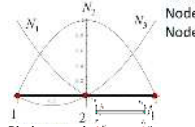
3.5 Stiffness Matrix of ~~Spring Element~~ by Principle of MPE

Step 1: Element displacement interpolation



CANNOT think as 2 2-node bar elements

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$



Node 1: $x = x_1, u = u_1$

Node 2: $x = x_2, u = u_2$

Node 3: $x = x_3, u = u_3$

Shape functions:

$$N_1 = \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)}$$

$$N_2 = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)}$$

$$N_3 = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)}$$

Displacement in x:

$$u = N_1u_1 + N_2u_2 + N_3u_3 \quad [N] \equiv [N_1 \quad N_2 \quad N_3]$$

$$u = [N]\{u^{(e)}\} \quad \{u^{(e)}\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Step 2: Element strain

$$\varepsilon = [B]\{u^{(e)}\} \quad [B] = \begin{bmatrix} \frac{dN_1}{dx} & \frac{dN_2}{dx} & \frac{dN_3}{dx} \end{bmatrix}$$

$$\{u^{(e)}\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Step 3: Obtaining element stiffness matrix & force vector

For constant values of E, A and T,

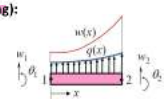
$$[K^{(e)}] = \int_0^L [B]^T E[B] A dx = \frac{EA}{L}\begin{bmatrix} \frac{7}{3} & -\frac{8}{3} & \frac{1}{3} \\ -\frac{8}{3} & \frac{16}{3} & -\frac{8}{3} \\ \frac{1}{3} & -\frac{8}{3} & \frac{7}{3} \end{bmatrix}$$

$$\{f^{(e)}\} = \int_0^L [N]^T T dx\, T = \begin{Bmatrix} \int_0^L N_1 dx \\ \int_0^L N_2 dx \\ \int_0^L N_3 dx \end{Bmatrix} = \begin{Bmatrix} \frac{TL}{6} \\ \frac{2TL}{3} \\ \frac{TL}{6} \end{Bmatrix}$$

3.6 Stiffness Matrix of 2-node Beam Element (from TPE)

Strain energy (from bending):

$$U_b = \frac{1}{2}\int_0^L EI\left(\frac{d^2w}{dx^2}\right)^2 dx$$



w = vertical displacement

q = distributed force (N/m)

Work done (by distributed force):

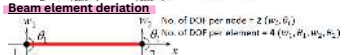
$$W = \frac{1}{2}\int_0^L qw dx$$

TPE:

$$\Pi = U_b - W$$

$$\Pi = \frac{1}{2}\int_0^L \left(\frac{d^2w}{dx^2}\right)^T EI\left(\frac{d^2w}{dx^2}\right) dx - \int_0^L w^T q dx$$ eqn 1

Beam element deriation



^^2-node Euler-Bernoulli beam element

Step1: Displacement interpolation

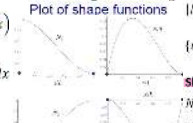
$$w = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3$$

$$\theta = \frac{dw}{dx} = \alpha_2 + 2\alpha_3 x + 3\alpha_4 x^2$$

Node 1: $x = x_1, w = w_1, \frac{dw}{dx} = \theta_1$

Node 2: $x = x_2, w = w_2, \frac{dw}{dx} = \theta_2$

Plot of shape functions



$$|N| = [N_1 \quad N_2 \quad N_3 \quad N_4]$$

$$\{u^e\} = \begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

Shape functions:

$$N_1 = \frac{(2x^3 - 3Lx^2 + L^3)}{L^3}$$

$$N_2 = \frac{(x^3 - 2Lx^2 + L^2x)}{L^2}$$

$$N_3 = \frac{-(2x^3 - 3Lx^2)}{L^3}$$

$$N_4 = \frac{(x^3 - Lx^2)}{L^2}$$

$$w = [N]\{u^e\}$$

Step2: Second Derivative expression

$$\frac{d^2w}{dx^2} = \frac{d^2([N]\{u^e\})}{dx^2} = \frac{d^2[N]}{dx^2}\{u^e\} = [B]\{u^e\}$$

$$[B] = \frac{d^2[N]}{dx^2} = \begin{bmatrix} \frac{d^2N_1}{dx^2} & \frac{d^2N_2}{dx^2} & \frac{d^2N_3}{dx^2} & \frac{d^2N_4}{dx^2} \end{bmatrix}$$

Step3: Potential energy

Rewriting eqn 1,

$$\Pi = \frac{1}{2}\{u^e\}^T \left(\int_0^L [B]^T EI[B] dx\right)\{u^e\} - \{u^e\}^T \int_0^L [N]^T q dx$$

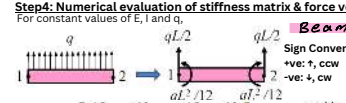
$$\Pi = \frac{1}{2}\{u^e\}^T [K^e]\{u^e\} - \{u^e\}^T \{f_Q^e\}$$ <<< general form

$$[K^e] = \int_0^L [B]^T EI[B] dx \qquad w^T = \{u^e\}^T [N]^T$$

$$\{f_Q^e\} = \int_0^L [N]^T q dx \qquad \left(\frac{d^2w}{dx^2}\right)^T = \{u^e\}^T [B]^T$$

Step4: Numerical evaluation of stiffness matrix & force vector

For constant values of E, I and q,



Beam

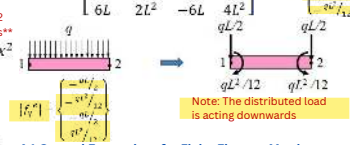
Sign Conventions:

+ve: ↑, ccw

-ve: ↓, cw

$$[K^e] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$\{f_b^e\} = \begin{Bmatrix} \frac{wL}{2} \\ \frac{wL^2}{12} \\ \frac{wL}{2} \\ -\frac{wL^2}{12} \end{Bmatrix}$$



$$|f_t^e| = \begin{Bmatrix} -\frac{wL}{2} \\ -\frac{wL^2}{12} \\ -\frac{wL}{2} \\ \frac{wL^2}{12} \end{Bmatrix}$$

Note: The distributed load is acting downwards

4.1 General Expressions for Finite Element Matrixes

$$\Pi = \frac{1}{2}\int_V \{\varepsilon\}^T\{\sigma\} dV - \int_V \{u\}^T\{b\} dV - \int_S \{u\}^T\{\bar{t}\} dS$$ eqn 1

Displacement vector (m): $\{u\} = \begin{Bmatrix} u \\ v \end{Bmatrix}$

Stress vector (N/m²): $\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$

Body force vector (N/m³): $\{b\} = \begin{Bmatrix} b_x \\ b_y \end{Bmatrix}$

Strain vector (no units): $\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$

Surface force vector (N/m²): $\{\bar{t}\} = \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix}$

Step1: Element displacement interpolation

$$u = N_1u_1 + N_2u_2 + N_3u_3 + \cdots + N_nu_n$$

$$v = N_1v_1 + N_2v_2 + N_3v_3 + \cdots + N_nv_n$$

$$\{u\} = |N|\{u^{(e)}\}$$ -------eqn 2

$$[N] = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix}$$

$$\{u^{(e)}\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_n \\ v_n \end{Bmatrix}$$

n = number of nodes in element [N] = Shape function matrix

Step2: Element strain expression

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{Bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}\begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\{\varepsilon\} = |B|\{u^{(e)}\}$$ -------eqn 3

$$|B| = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \frac{\partial N_2}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & 0 & \frac{\partial N_2}{\partial y} & \cdots & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial y} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

[B] = Strain-displacement matrix

Step3: Stress-strain expression

For plane stress/plane strain problems, stress vector and strain vector are related as:

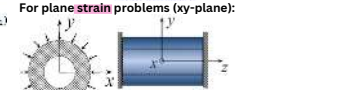
$$\{\sigma\} = [D]\{\varepsilon\}$$

$$\{\sigma\} = [D][B]\{u^e\}$$ -------Eqn 4

[D] = Constitutive/material property matrix

For plane stress problems (xy-plane):

*A **thin** subjected to in-plane loading (xy-plane)*



$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

E = Young's modulus

Small thickness ν = Poisson's ratio

For plane strain problems (xy-plane):

L = constant

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

Step4: Identify element stiffness matrix & element load vectors

Sub eqn 2, 3, 4 into eqn 1,

$$\Pi = \frac{1}{2}\{u^e\}^T\left(\int_V [B]^T[D][B] dV\right)\{u^e\} - \{u^e\}^T\left(\int_V [N]^T\{b\} dV^e\right) - \{u^e\}^T\left(\int_S [N]^T\{\bar{t}\} dS\right)$$

$$[K^e] = \int_V [B]^T[D][B] dV$$ <<< Element stiffness matrix

$$\{f_b^e\} = \int_V [N]^T\{b\} dV^e$$ <<< Element load vector due to body force

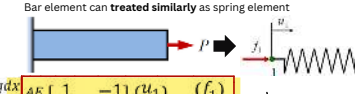
$$\{f_{\bar{t}}^e\} = \int_S [N]^T\{\bar{t}\} dS$$ <<< Element load vector due to surface force

dv = tdxdy (infinitesimal volume) t = element thickness

dS = tdy (infinitesimal area on boundary surface)

2.1 Bar Element

Bar element can **treated similarly** as spring element



$$\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix}$$

$\frac{AE}{L} = \frac{P}{\delta} = k$

4.2 Bilinear Rectangular Element

Step1: Element displacement interpolation

$$x_1 = -a, x_2 = a, x_3 = a, x_4 = -a$$

$$y_1 = -b, y_2 = -b, y_3 = b, y_4 = b$$

Thickness = t

$$u = N_1u_1 + N_2u_2 + N_3u_3 + N_4u_4$$

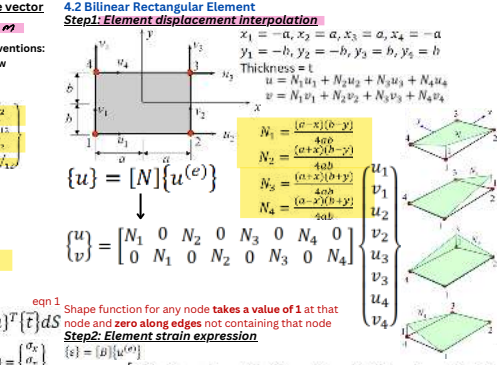
$$v = N_1v_1 + N_2v_2 + N_3v_3 + N_4v_4$$

$$N_1 = \frac{(a-x)(b-y)}{4ab}$$

$$N_2 = \frac{(a+x)(b-y)}{4ab}$$

$$N_3 = \frac{(a+x)(b+y)}{4ab}$$

$$N_4 = \frac{(a-x)(b+y)}{4ab}$$



$$\{u\} = [N]\{u^{(e)}\}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

Shape function for any node **takes a value of 1** at that node and **zero along edges** not containing that node

Step2: Element strain expression

$$\{\varepsilon\} = [B]\{u^{(e)}\}$$

$$[B] = \frac{1}{4ab}\begin{bmatrix} (b-y) & 0 & (b-y) & 0 & (b+y) & 0 & (b+y) & 0 \\ 0 & -(a-x) & 0 & -(a+x) & 0 & (a+x) & 0 & (a-x) \\ -(a-x) & -(b-y) & -(a+x) & (b-y) & (a+x) & (b+y) & (a-x) & -(b+y) \end{bmatrix}$$

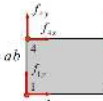
Step3: Element stiffness matrix calculation

$$[K^e] = \int_V [B]^T[D][B] dV$$

$$= \int_{A^e} [B]^T[D][B] t dA^e$$

$$= \int_{-b}^{b}\int_{-a}^{a} [B]^T[D][B] t dx dy$$

$$\int_{-a}^{a}\int_{-b}^{b} N_i dx dy = ab$$



Step4: Element load vector due to body force

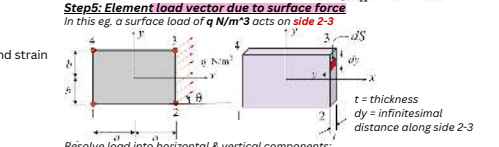
$$\{f_b^e\} = \int_V [N]^T\{b\} dV$$

$$= \int_{A^e} [N]^T\{b\} t dx dy$$

$$\{f_b^e\} = \int_{A^e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix}\begin{Bmatrix} b_x \\ b_y \end{Bmatrix} t dx dy = \begin{Bmatrix} tb_x\int_{A^e} N_1 dx dy \\ tb_y\int_{A^e} N_1 dx dy \\ tb_x\int_{A^e} N_2 dx dy \\ tb_y\int_{A^e} N_2 dx dy \\ tb_x\int_{A^e} N_3 dx dy \\ tb_y\int_{A^e} N_3 dx dy \\ tb_x\int_{A^e} N_4 dx dy \\ tb_y\int_{A^e} N_4 dx dy \end{Bmatrix} = \begin{Bmatrix} tb_xab \\ tb_yab \\ tb_xab \\ tb_yab \\ tb_xab \\ tb_yab \\ tb_xab \\ tb_yab \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix}$$

Step5: Element load vector due to surface force

*In this eg. a surface load of **q N/m^3** acts on **side 2-3***



t = thickness

dy = infinitesimal distance along side 2-3

Resolve load into horizontal & vertical components:

$$\{\bar{t}\} = \begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} = \begin{Bmatrix} q\cos\theta \\ q\sin\theta \end{Bmatrix}$$

$$\{f_{\bar{t}}^e\} = \int_S [N]^T\{\bar{t}\} dS$$

$$= \int_S [N]^T\{\bar{t}\} t dy$$

Along side 2-3, x = a

$$N_1 = \frac{(a-a)(b-y)}{4ab} = 0$$

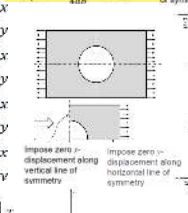
$$N_2 = \frac{(a+a)(b-y)}{4ab} = \frac{(b-y)}{2b}$$

$$N_3 = \frac{(a+a)(b+y)}{4ab} = \frac{(b-y)}{2b}$$

$$N_4 = \frac{(a-a)(b+y)}{4ab} = 0$$

Impose zero x-displacement along vertical line of symmetry

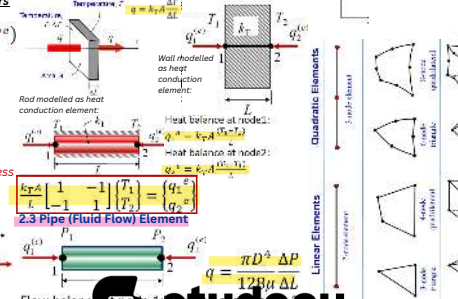
$$\begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ f_{4x} \\ f_{4y} \end{Bmatrix} = \int \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \bar{t}_x \\ \bar{t}_y \end{Bmatrix} t dy = \begin{Bmatrix} 0 \\ 0 \\ tb\bar{t}_x \\ tb\bar{t}_y \\ tb\bar{t}_x \\ tb\bar{t}_y \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ f_{2x} \\ f_{2y} \\ f_{3x} \\ f_{3y} \\ 0 \\ 0 \end{Bmatrix}$$



Impose zero y-displacement along vertical line of symmetry

Impose zero y-displacement along horizontal line of symmetry

2.2 Heat Conduction Element



Rod modelled as heat conduction element:

Wall modelled as heat conduction element:

Heat balance at node1:

Heat balance at node2:

$$\frac{k_TA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \end{Bmatrix}$$

2.3 Pipe (Fluid Flow) Element

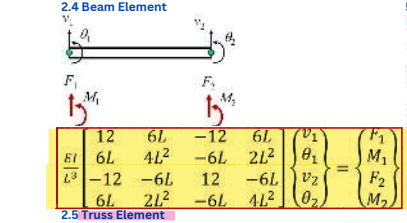
Flow balance at node 1:

Flow balance at node 2:

$$\frac{\pi D^4}{128\mu L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \begin{Bmatrix} q_1^e \\ q_2^e \end{Bmatrix}$$

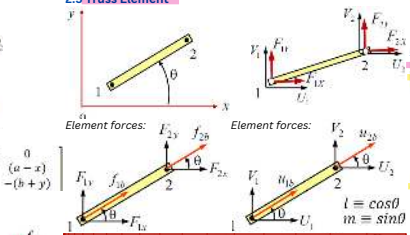
$$q = \frac{\pi D^4}{128\mu}\frac{\Delta P}{\Delta L}$$

4.3 Beam Element



$$\begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ M_1 \\ F_2 \\ M_2 \end{Bmatrix}$$

2.5 Truss Element



Element forces:

Element forces:

$$t = \cos\theta \qquad m = \sin\theta$$

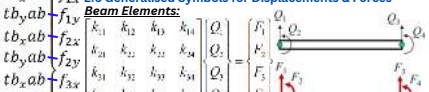
$$\frac{AE}{L}\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{Bmatrix} U_1 \\ V_1 \\ U_2 \\ V_2 \end{Bmatrix} = \begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix}$$

Note: *Displacements and forces of reference node will be labelled first. (In this eg. Node 1 is taken as reference)*

θ is measured from the positive x-axis of that node

2.6 Generalised Symbols for Displacements & Forces

Beam Elements:



$$\begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Q_1 is y-displacement

Q_2 is z-rotation

Q_3 is y-motion

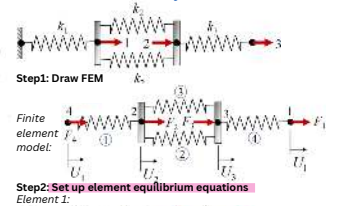
Truss Elements:

$$\frac{AE}{L}\times\begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Spring Elements:

$$\begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

1.5 Finite Element Assembly Process



Step1: Draw FEM

Finite element model:

Step2: Set up element equilibrium equations

Element 1:

$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \end{Bmatrix} = \begin{Bmatrix} f_1^{(1)} \\ f_2^{(1)} \end{Bmatrix}$$

Element 2:

$$\begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} f_1^{(2)} \\ f_2^{(2)} \end{Bmatrix}$$

Element 3:

$$\begin{bmatrix} k_3 & -k_3 \\ -k_3 & k_3 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(3)} \\ f_2^{(3)} \end{Bmatrix}$$

Element 4:

$$\begin{bmatrix} k_4 & -k_4 \\ -k_4 & k_4 \end{bmatrix}\begin{Bmatrix} U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} f_1^{(4)} \\ f_2^{(4)} \end{Bmatrix}$$

Step3: Set up and assemble global element stiffness matrix

Sum of internal forces = external force acting on same node

$$f_1^{(1)} = F_1$$

$$f_2^{(1)} + f_1^{(2)} + f_1^{(3)} = F_2$$

$$f_2^{(2)} + f_1^{(4)} = F_3$$

$$f_2^{(3)} + f_2^{(4)} = F_4$$

Global element stiffness matrix:

$$\begin{bmatrix} k_1 & -k_1 & 0 & 0 \\ -k_1 & k_1+k_2+k_3 & -k_3 & -k_4 \\ 0 & -k_3 & k_3 & 0 \\ 0 & -k_4 & 0 & k_4 \end{bmatrix}\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Apply boundary conditions:

5.1 Theory Stuff

Avoid distorted element shapes *Mesh density affects accuracy:*

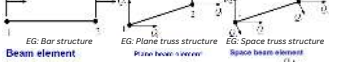
- Aspect ratio **close to one** (better results)
- Exploit symmetry of geometry
- Conditions:
  - Symmetric geometry
  - Symmetric Loads
  - Symmetric BCs
- *Coarse mesh -> less accurate*
- *Use **fine mesh** in **regions of stress concentrations** (holes, grooves)*
- *Perform **mesh convergence study***

5.2 Typical 1-D Elements

Bar element

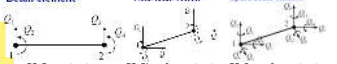
EG: Bar structure EG: Plane truss structure EG: Space truss structure

Beam element



EG: Beam structure EG: Plane frame structure EG: Space frame structure

Conduction element (1-D)



EG: Bar structure subjected to heating EG: Plane frame subjected to heating EG: Plane frame subjected to heating

5.3 Typical 2-D Elements

Plane element:

Plane triangular element (plane stress/ plane strain)

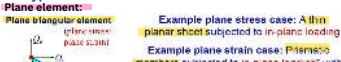
Example plane stress case: A thin planar sheet subjected to in-plane loading

Example plane strain case: Prismatic members subjected to in-plane loading with end faces restrained

Plane quadrilateral element (plane stress/ plane strain)

Plate element:

Triangular plate element



Quadrilateral plate element

3 DOF per node

Degrees of freedom are shown only for node 1.

Similar degrees of freedom apply to other nodes as well.

Example:

Thin plates subjected to out-of-plane loading

Q_1 is z-displacement

Q_2 is x-rotation

Q_3 is y-rotation

Shell element:

Shell element = + Plane stress element + Plate element

Triangular shell element

Quadrilateral shell element

6 DOF per node (Degrees of freedom are shown only for node 1.)

3 translations along three axes and 3 rotations about the three axes.

In-plane rotations are suppressed during formulation/derivation of stiffness matrix.

Common on degrees of freedom are same as for 3 in-plane element

EG. Rectangular shell elements for flat shell structure/curved shell

5.4 Some Examples

EG 1:Stepped cantilever bar subjected to **tensile/compressive loading**

- **Bar element model** for computing **axial displacement only**
- **3D solid element model** compute **displacements & stresses**

EG 2:Hollow cylinder subjected to **bending load**

- **Beam element model** for **nodal displacements only**
- **Shell element model** if cylinder is **thin (d > 20t)** [compute **deformations & stresses**]
- **3D solid element model** if cylinder is **thick (d < 20t)** [compute **deformations & stresses**]

5.5 Major Stages in Practical FEA

Preprocessing

- *Building the geometrical model (CAD model)*
- *Meshing the geometrical model*
- *Applying B.Cs and loads*

Solution

- *Choosing type of analysis (Static, Modal, Harmonic, etc)*
- *Solving finite element equations*

Postprocessing

- *Plotting deformed shape*
- *Plotting contour plots of displacements and stresses/strains*
- *Animating deformed shape/modeshape*
- *Listing/probing the numerical values of results (e.g. displacement, stress/strain, reactions)*

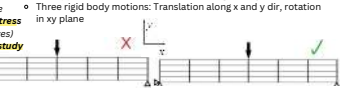
5.5 Practical Aspects of FEA

Accuracy of results depends on the how accurately we model:

- Geometrical details
  - Ignore small holes, cutouts and logos if **located away from region of interest**
- Boundary Loads
  - Simplified BCs can apply if **region of interest is away from BCs**
- Loads
  - Simplified loads can apply if **region of interest is away from location of loads**

Apply boundary conditions:

- Minimum necessary boundary condition
  - BCs applied must be able to **restrain all rigid body motion**
  - Three rigid body motions: Translation along x and y dir, rotation in xy plane



Partial Differential Equations

Linear

PDE in ϕ is **linear** if **ALL its terms** containing **ϕ** or a **partial derivative of ϕ** is in linear form:

cϕ or **c × (a partial derivative of ϕ)**

coefficient c is either a constant or function of the independent variables of ϕ (x, y and t)

Homogeneous

PDE in ϕ is **homogeneous** if **ALL non-zero terms** contain either **ϕ** or a **partial derivative of ϕ**

Theorem of superposition of solutions

If ϕ_1 and ϕ_2 are **two solutions** of a **homo-linear PDE** so is $\alpha\phi_1 + \beta\phi_2$ for all real numbers of α and β

Review on ODE

General Solution: / Characteristic equation:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \quad y = e^{\lambda x} \quad a\lambda^2 + b\lambda + c = 0$$

Case (a): a = 0, b ≠ 0 *(1st order ODE)*

$$b\frac{dy}{dx} + cy = 0$$

Solving bλ + c = 0, we get **λ=-c/b**

General solution: $y = Ae^{-cx/b}$ *A is an arbitrary constant*

Case (b): a ≠ 0, b² - 4ac > 0 *(2nd order ODE)*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Solving aλ² + bλ + c = 0, we get **2 distinct real solutions:**

$\lambda = \lambda_1$ & $\lambda = \lambda_2$

General solution: $y = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

A & B are arbitrary constants

Case (c): a ≠ 0, b² - 4ac = 0 *(2nd order ODE)*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Solving aλ² + bλ + c = 0, we get **one solution: λ = -b/(2a)**

General solution: $y = (Ax + B)e^{-bx/(2a)}$

A & B are arbitrary constants

Case (d): a ≠ 0, b² - 4ac < 0 *(2nd order ODE)*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

Solving aλ² + bλ + c = 0, we get **two distinct complex solutions:**

λ = α + jβ & λ = α - jβ

General solution:

$$y = e^{\alpha x}(A\cos(\beta x) + B\sin(\beta x))$$

A & B are arbitrary constants

Review on Fourier Series

FSP 1 *(cosine series)*

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) \quad \text{equal to } f(x) \text{ for } 0 < x < L$$

$a_0 = \frac{1}{L}\int_0^L f(x)dx$ and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ for n = 1,2,3,...

FSP 1 *(sine series)*

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{equal to } f(x) \text{ for } 0 < x < L$$

$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ for n = 1,2,3,...

Solving IBVP for Vibrating String Problem

Solve $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ for $0 < x < L$ and $t > 0$

subject to $u(x,0) = G(x)$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ for $0 < x < L$ (ICs)

and $u(0,t) = 0$ and $u(L,t) = 0$ for $t > 0$ (BCs)

Step 1: Method of separation of variables

Let $u(x,t) = X(x)T(t)$.

Working out the partial derivatives in the PDE:

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2X}{dx^2}\cdot T(t) = X''(x)T(t)$$

$$\frac{\partial^2 u}{\partial t^2} = X(x)\cdot\frac{d^2T}{dt^2} = X(x)T''(t)$$

Substituting into the PDE:

$$X''(x)T(t) = \frac{1}{c^2}X(x)T''(t)$$

Divide both sides by $X(x)T(t)$:

$$\frac{X''(x)}{X(x)} = \frac{1}{c^2}\frac{T''(t)}{T(t)} = \gamma$$

function of x / function of t / arbitrary constant

We obtain a pair of ODEs:

$X''(x) - \gamma X(x) = 0$ ----eqn1

$T''(t) - \gamma c^2 T(t) = 0$ --eqn2

Step 2: Solving ODEs

The ODEs (eqn 1 and 2) are solved to find **non-trivial solutions** that satisfies BCs

Consider **3 cases**

Case γ = 0

ODEs: ***X''(x) = 0 & T''(t) = 0***

Integrate twice,

X(x) = Ax + B T(t) = Ct + D *(A,B,C,D are arbitrary constants)*

u(x, t) = (Ax+B)(Ct+D)

BC #1: u(0, t) = 0 → *B(Ct + D) = 0*

To satisfy BC #1 and get non-trivial solution, **B = 0**

BC #2: u(L, t) = 0 → (AL)(Ct + D) = 0

To satisfy BC #2 and get non-trivial solution, AL = 0 → **A = 0**

for γ = 0, u(x, t) = 0 **(trivial solution)**

Case γ = p²

ODEs: ***X''(x) - p²X(x) = 0 & T''(t) - p²c²T(t) = 0***

Let $X(x) = e^{\lambda x}$,
$X''(x) = \lambda^2 e^{\lambda x}$
$\lambda^2 e^{\lambda x} - p^2 e^{\lambda x} = 0$
$(\lambda^2 - p^2)e^{\lambda x} = 0$
$\lambda = \pm p$
∴ $X(x) = Ae^{px} + Be^{-px}$
$T(t) = Ce^{pct} + De^{-pct}$ *(A,B,C,D are arbitrary constants)*

$$u(x,t) = (Ae^{px} + Be^{-px})(Ce^{pct} + De^{-pct})$$

BC #1: u(0, t) = 0 → $(A + B)(Ce^{pct} + De^{-pct}) = 0$

BC #2: u(L, t) = 0 → $(Ae^{pL} + Be^{-pL})(Ce^{pct} + De^{-pct}) = 0$

To satisfy BCs and get non-trivial solution,

$(A + B) = 0$ and $(Ae^{pL} + Be^{-pL}) = 0$

∴ **A = 0 and B = 0**

for γ = p², u(x, t) = 0 (trivial solution)

Case γ = -p²

ODEs: ***X''(x) + p²X(x) = 0 & T''(t) + p²c²T(t) = 0***

Let $X(x) = e^{\lambda x}$,
$X''(x) = \lambda^2 e^{\lambda x}$
$\lambda^2 e^{\lambda x} + p^2 e^{\lambda x} = 0$
$(\lambda^2 + p^2)e^{\lambda x} = 0$
$\lambda = jp$ or $-jp$
∴ $X(x) = A\sin(px) + B\cos(px)$
$T(t) = C\sin(pct) + D\cos(pct)$ *(A,B,C,D are arbitrary constants)*

u(x, t) = [Asin(px) + Bcos(px)][Csin(pct) + Dcos(pct)]

BC #1: u(0, t) = 0 → (B)[Csin(pct) + Dcos(pct)] = 0

To satisfy BC #1 & get non-trivial solution, **B = 0**

u(x, t) = [Esin(pct) + Fcos(pct)]sin(pL), (E = AC & F = AD)

BC #2: u(L, t) = 0 → [Esin(pct) + Fcos(pct)]sin(pL) = 0

To satisfy BC#2 & get non-trivial solution, **sin(pL) = 0**

*where pL = nπ for n = 0, ±1, ±2, ±3, ... → **p = L/nπ***

$u_n(x,t) = \left(E_n\sin\left(\frac{n\pi ct}{L}\right) + F_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$ *for n = 1, 2, 3, ...*

E_n and F_n are arbitrary constants

Step3: Sum up non-trivial solutions satisfying BCs

$$u(x,t) = \sum_{n=1}^{\infty}\left(E_n\sin\left(\frac{n\pi ct}{L}\right) + F_n\cos\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

Step4: Tackling ICs

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\frac{n\pi c}{L}\left(E_n\cos\left(\frac{n\pi ct}{L}\right) - F_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

IC #2: $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \Rightarrow \sum_{n=1}^{\infty}\frac{n\pi c}{L}E_n\sin\left(\frac{n\pi x}{L}\right) = 0$ *for 0 < x < L*

To satisfy IC # 2, **E_n = 0**

$$\therefore u(x,t) = \sum_{n=1}^{\infty}F_n\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

IC #1: u(x, 0) = G(x) for 0 < x < L

$$\sum_{n=1}^{\infty}F_n\sin\left(\frac{n\pi x}{L}\right) = G(x) \quad \text{for } 0 < x < L$$

^This is FSP2

$F_n = \frac{2}{L}\int_0^L G(x)\sin\left(\frac{n\pi x}{L}\right)dx$ for n = 1,2,3,...

Find F_n then **substitute** into series solution for final solution

$$G(x) = \begin{cases} \frac{x}{10} & \text{for } 0 < x \le \frac{L}{2} \\ \frac{L-x}{10} & \text{for } \frac{L}{2} < x < L \end{cases}$$

<<<Initial shape of string

$$F_n = \frac{2}{L}\int_0^L G(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_0^{L/2}\frac{x}{10}\sin\left(\frac{n\pi x}{L}\right)dx + \frac{2}{L}\int_{L/2}^{L}\frac{L-x}{10}\sin\left(\frac{n\pi x}{L}\right)dx$$

$$F_n = \frac{2L\sin(\frac{n\pi}{2})}{5n^2\pi^2}$$

Final answer:

$$u(x,t) = \sum_{n=1}^{\infty}\frac{2L\sin(\frac{n\pi}{2})}{5n^2\pi^2}\cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Solving IBVP for 1D time dependent linear heat conduction with thermally insulated boundary

Solve $\frac{\partial^2\phi}{\partial x^2} = \frac{\rho c}{k}\frac{\partial\phi}{\partial t}$ for $0 < x < L$ and $t > 0$

subject to $\phi(x,0) = f(x)$ for $0 < x < L$ (IC)

and $\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0$ and $\left.\frac{\partial\phi}{\partial x}\right|_{x=L} = 0$ for $t > 0$ (BCs)

Step 1: Method of separation of variables

Let $\phi(x,t) = X(x)T(t)$.

$X''(x) - \gamma X(x) = 0$ ---eqn1

$T'(t) - \frac{\gamma k}{\rho c}T(t) = 0$ --eqn2

Step2: Solving ODEs

Case γ = 0

ODEs: ***X''(x) = 0 & T'(t) = 0***

Integrate twice for X''(x), once for T'(t),

X(x) = Ax + B T(t) = C *(A,B,C are arbitrary constants)*

ϕ(x, t) = (Ax+B)(C) = (Dx+E) (AC = D, BC = E)

$\frac{\partial\phi}{\partial x} = D$

To satisfy BCs at x = 0 & x = L, **D = 0**

No restriction on constant E, ∴ there is **non-trivial solution** for γ = 0

$\phi_0(x,t) = E_0$ *(E_0 is an arbitrary constant)*

Case γ = p²

ODEs: ***X''(x) - p²X(x) = 0 & T''(t) - p²c²T(t) = 0***

$$\phi(x,t) = (De^{px} + Ee^{-px})e^{p^2kt/(\rho c)}$$

$$\frac{\partial\phi}{\partial x} = p(De^{px} - Ee^{-px})e^{p^2kt/(\rho c)}$$

Satisfying BCs at x = 0 & x = L,

Take **D - E = 0 & $De^{pL} - Ee^{-pL} = 0$ → D = 0 & E = 0**

∴ for γ = p2, ϕ(x, t) = 0 (trivial solution)

Case γ = -p²

X''(x) - p²X(x) = 0 & T''(t) - p²c²T(t) = 0

$\phi(x,t) = (D\sin(px) + E\cos(px))e^{-p^2kt/(\rho c)}$

$\frac{\partial\phi}{\partial x} = p(D\cos(px) - E\sin(px))e^{-p^2kt/(\rho c)}$

BC #1: $\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0 \to D = 0$

To satisfy BC #1 & get non-trivial solution, **sin(pL) = 0**

where pL = nπ for n = 0, 0, ±1, ±2, ±3, ... → $p = \frac{n\pi}{L}$

$\phi_n(x,t) = E_n\cos\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2kt/(\rho cL^2)}$ *for n = 1, 2, 3*

E_n is an arbitrary constant

Step3: Sum up non-trivial solutions satisfying BCs

$$\phi(x,t) = E_0 + \sum_{n=1}^{\infty}E_n\cos\left(\frac{n\pi x}{L}\right)e^{-n^2\pi^2kt/(\rho cL^2)}$$

Step4: Tackling ICs

IC #1: *ϕ(x, 0) = f(x) for 0 < x < L*

$$\phi(x,0) = E_0 + \sum_{n=1}^{\infty}E_n\cos\left(\frac{n\pi x}{L}\right) = f(x) \text{ for } 0 < x < L$$

^This is FSP1

$E_0 = \frac{1}{L}\int_0^L f(x)dx,\ E_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ for n = 1, 2, 3, ...

Reformulating IBVP (non-homo BCs)

Example 1

Solve $\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial t}$ for $0 < x < 2$ and $t > 0$

subject to $\phi(x,0) = 2x$ for $0 < x < 2$ (IC)

and $\phi(0,t) = 1$ and $\phi(2,t) = 5$ for $t > 0$ (BCs)

Let $\phi(x,t) = U(x,t) + \psi(x,t)$

Let $U(x,t) = V(x)$

PDE ($\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t}$) is reduced to: $\frac{d^2V}{dx^2} = 0$

BCs [U(0,t) = 1 & U(2,t) = 5] become **V(0) = 1 & V(2) = 5**

Integrate $\frac{d^2V}{dx^2} = 0$ twice → **V(x) = Ax + B** *(A & B are arbitrary constants)*

∴ From BCs, **A = 2 & B = 1 → U(x,t) = 2x + 1**

Thus, $\phi(x,t) = (2x+1) + \psi(x,t)$

$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2}{\partial x^2}((2x+1) + \psi(x,t)) = \frac{\partial^2\psi}{\partial x^2}$

$\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial t}((2x+1) + \psi(x,t)) = \frac{\partial\psi}{\partial t}$

New PDE:

Sub the above into original PDE equa. $\frac{\partial^2\psi}{\partial x^2} = \frac{\partial\psi}{\partial t}$

New BC#1: $\phi(0,t) = 1$; $(2(0)+1) + \psi(0,t) = 1 \to \psi(0,t) = 0$

New BC#2: $\phi(2,t) = 5$; $(2(2)+1) + \psi(2,t) = 5 \to \psi(2,t) = 0$

New IC#1: $\phi(x,0) = 2x$ for $0 < x < 2$; $(2x+1) + \psi(x,0) = 2x \to \psi(x,0) = -1$

Reformulating IBVP (non-homo PDE & BC)

Solve $\frac{\partial^2\phi}{\partial x^2} + \frac{\pi^2}{4}\cos\left(\frac{\pi x}{2}\right) = 4\frac{\partial\phi}{\partial t}$ for $0 < x < 1$ and $t > 0$

subject to $\phi(x,0) = 1$ for $0 < x < 1$ (IC)

and $\phi(0,t) = 0$ and $\phi(1,t) = 2$ for $t > 0$ (BCs)

Let $\phi(x,t) = U(x,t) + \psi(x,t)$

Let $U(x,t) = V(x)$

PDE ($\frac{\partial^2 U}{\partial x^2} + \frac{\pi^2}{4}\cos\left(\frac{\pi x}{2}\right) = 4\frac{\partial U}{\partial t}$) is reduced to: $\frac{d^2V}{dx^2} = -\frac{\pi^2}{4}\cos\left(\frac{\pi x}{2}\right)$

BCs [ϕ(0,t) = 0 & ϕ(1,t) = 2] become V(0) = 0 & V(1) = 0

Integrate $\frac{d^2V}{dx^2} = -\frac{\pi^2}{4}\cos\left(\frac{\pi x}{2}\right)$ twice → $V(x) = \cos\left(\frac{\pi x}{2}\right) + Ax + B$

∴ From BCs, **A = 3 & B = -1** → $U(x,t) = \cos\left(\frac{\pi x}{2}\right) + 3x - 1$

Thus, $\phi(x,t) = \cos\left(\frac{\pi x}{2}\right) + 3x - 1 + \psi(x,t)$

$\frac{\partial^2\phi}{\partial x^2} = -\frac{\pi^2}{4}\cos\left(\frac{\pi x}{2}\right) + \frac{\partial^2\psi}{\partial x^2}$, $\frac{\partial\phi}{\partial t} = \frac{\partial\psi}{\partial t}$

New PDE: $\frac{\partial^2\psi}{\partial x^2} = 4\frac{\partial\psi}{\partial t}$

New IC#1: $\phi(x,0) = 1$ for $0 < x < 1 \to \psi(x,0) = 2 - \cos\left(\frac{\pi x}{2}\right) - 3x$

New BC#1: $\phi(0,t) = 0 \to \psi(0,t) = 0$

New BC#2: $\phi(1,t) = 2 \to \psi(1,t) = 0$

Reformulating IBVP (non-homo PDE & BC)

Solve $\frac{\partial^2\phi}{\partial x^2} - 6xe^{-t} = 4\frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial t}$ for $0 < x < 1$ and $t > 0$

subject to $\phi(x,0) = x^2(x-1)$ and $\left.\frac{\partial\phi}{\partial t}\right|_{t=0} = 0$ for $0 < x < 1$ (ICs)

and $\phi(0,t) = 0$ and $\phi(1,t) = e^{-t}$ for $t > 0$ (BCs)

Let $\phi(x,t) = U(x,t) + \psi(x,t)$

Let $U(x,t) = e^{-t}V(x)$

PDE ($\frac{\partial^2\phi}{\partial x^2} - 6xe^{-t} = 4\frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial t}$) is reduced to: $\frac{d^2V}{dx^2} = 6x$

BCs [$\phi(0,t) = 0$ & $\phi(1,t) = e^{-t}$] become **V(0) = 0 & V(1) = 1**

Integrate $\frac{d^2V}{dx^2} = 6x$ twice → $V(x) = x^3 + Ax + B$

∴ From BCs, **A = 0 & B = 0** → $U(x,t) = x^3e^{-t}$

Thus, $\phi(x,t) = x^3e^{-t} + \psi(x,t)$

$\frac{\partial^2\phi}{\partial x^2} = -6xe^{-t} + \frac{\partial^2\psi}{\partial x^2}$, $\frac{\partial\phi}{\partial t} = -x^3e^{-t} + \frac{\partial\psi}{\partial t}$, $\frac{\partial^2\phi}{\partial t^2} = x^3e^{-t} + \frac{\partial^2\psi}{\partial t^2}$

New PDE: $\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial t^2} + \frac{\partial\psi}{\partial t}$

New IC#2: $\left.\frac{\partial\phi}{\partial t}\right|_{t=0} = 0$ *for 0 < x < 1* $\to \left.\frac{\partial\psi}{\partial t}\right|_{t=0} = x^3$

New BC#1: $\phi(0,t) = 0 \to \psi(0,t) = 0$

New BC#2: $\phi(1,t) = e^{-t} \to \psi(1,t) = 0$

New IC#1: $\phi(x,0) = x^2(x-1)$ for $0 < x < 1 \to \psi(x,0) = -x^2$

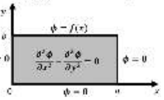
Solving BVP

Solve $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$ for $0 < x < a$, $0 < y < b$

subject to $\phi(x,0) = 0$ and $\phi(x,b) = f(x)$ for $0 < x < a$ (BCs)

and $\phi(0,y) = 0$ and $\phi(a,y) = 0$ for $0 < y < b$

Start with the BCs with **most zeros**



Step1: Separation of variables

Let $\phi(x,y) = X(x)Y(y)$.

$X''(x) + \gamma X(x) = 0$ --eqn1

$Y''(y) - \gamma Y(y) = 0$ --eqn2

Step2: Satisfy BCs (2 BCs)

Case γ = 0

ODEs: ***X''(x) = 0 & Y''(y) = 0***

Integrate twice → ***X(x) = Ax + B & Y(y) = Cy + D***

ϕ(x, y) = (Ax+B)(Cy+D)

BC #3: ϕ(0, y) = 0 → **B**(Cy+D) = 0 → **B = 0**

BC #4: ϕ(a, y) = 0 → A(a)(Cy+D) = 0 → **A = 0**

trivial solution

Case γ = -p²

ODEs: ***X''(x) - p²X(x) = 0 & Y''(y) + p²Y(y) = 0***

Solving: $X(x) = Ap^x + Be^{-px}$ & $Y(y) = C\sin(py) + D\cos(py)$

$\phi(x,y) = (Ae^{px} + Be^{-px})(C\sin(py) + D\cos(py))$

BC #3: $\phi(0,y) = 0 \to (A + B)[C\sin(py) + D\cos(py)] = 0$

BC #4: $\phi(a,y) = 0 \to (Ap^a + Bp^a)(C\sin(py) + D\cos(py)) = 0$

∴ **A = 0 & B = 0** → trivial solution

Case γ = p²

ODEs: ***X''(x) + p²X(x) = 0 & Y''(y) - p²Y(y) = 0***

Solving: $X(x) = A\sin(px) + B\cos(px)$ & $Y(y) = Ce^{py} + De^{-py}$

$\phi(x,y) = (A\sin(px) + B\cos(px))(Ce^{py} + De^{-py})$

BC #3: $\phi(0,y) = 0 \to B(C\sin(py) + D\cos(py)) = 0 \to B = 0$

$\phi(x,y) = (Ee^{py} + Fe^{-py})\sin(px)$ (E = AC, F = AD)

BC #4: $\phi(a,y) = 0 \to (Ee^{py} + Fe^{-py})\sin(pa) = 0$

To satisfy BC #2 & get non-trivial solution, **sin(pa) = 0**

where pa = nπ, for n = 0, ±1, ±2, ±3, ...

$\phi_n(x,y) = (E_ne^{n\pi y/a} + F_ne^{-n\pi y/a})\sin\left(\frac{n\pi x}{a}\right)$ *for n = 1, 2, 3*

Step3: Sum up non-trivial solutions

$$\phi(x,y) = \sum_{n=1}^{\infty}(E_ne^{n\pi y/a} + F_ne^{-n\pi y/a})\sin\left(\frac{n\pi x}{a}\right)$$

Step4: Satisfy Remaining BCs

BC #1: $\phi(x,0) = 0 \to \sum_{n=1}^{\infty}(E_n + F_n)\sin\left(\frac{n\pi x}{a}\right) = 0 \to F_n = -E_n$

$\therefore \phi(x,y) = \sum_{n=1}^{\infty}E_n(e^{n\pi y/a} - e^{-n\pi y/a})\sin\left(\frac{n\pi x}{a}\right)$

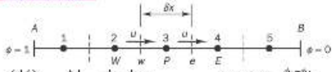
BC #2: $\phi(x,b) = f(x)$ *for 0 < x < a*

$\phi(x,b) = \sum_{n=1}^{\infty}E_n(e^{n\pi b/a} - e^{-n\pi b/a})\sin\left(\frac{n\pi x}{a}\right) = f(x)$ *for 0 < x < a*

^This is FSP2

$E_n(e^{n\pi b/a} - e^{-n\pi b/a}) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$ for n = 1, 2, 3, ...

Discretization



$$\left(\frac{d\phi}{dx}\right)_i \approx \frac{\Delta\phi}{\Delta x} = \frac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}}$$

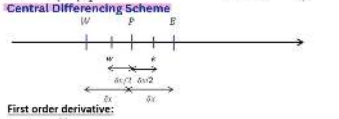
$$\phi_i = \frac{\phi_i - \phi_{i-1}}{x_i - x_{i-1}}$$

Point 1: $\phi_i = \frac{\phi_1 - \phi_A}{\delta x/2}$

Points 2-5: $\phi_P = \frac{\phi_P - \phi_W}{\delta x}$

Point B: $\phi_B = \frac{\phi_B - \phi_5}{\delta x/2}$

Central Differencing Scheme



First order derivative:

Equation: $\frac{d\phi}{dx} = \phi$

$\left(\frac{d\phi}{dx}\right)_P = \phi_P$ *(for all points on grid)*

$\left(\frac{d\phi}{dx}\right)_P = \frac{\delta\phi}{\delta x} = \frac{\phi_E - \phi_W}{2\delta x}$ *(central differencing using "W" & "E")*

Second order derivative:

Equation: $\frac{d^2\phi}{dx^2} = \phi$

$\left(\frac{d^2\phi}{dx^2}\right)_P = \left(\frac{d}{dx}\frac{d\phi}{dx}\right)_P = \phi_P$ *(for all points on grid)*

General transport equation

$$\frac{\partial(\rho\phi)}{\partial t} + div(\rho\phi\mathbf{u}) = div(\Gamma\, grad\,\phi) + S_\phi$$

Rate of increase of ϕ of fluid element + Net rate of flow of ϕ out of fluid element = Rate of increase of ϕ due to diffusion + Rate of increase of ϕ due to sources

Integral form

- The key step in the finite volume method is to integrate the transport equation and then to apply the divergence theorem, which for a vector a states:

$$\int_{CV} div\,\mathbf{a}\,dV = \int_A \mathbf{n}\cdot\mathbf{a}\,dA$$

- This leads to the following general conservation equation in integral form:

$$\frac{\partial}{\partial t}\left(\int_{CV}\rho\phi\,dV\right) + \int_A \mathbf{n}\cdot(\rho\phi\mathbf{u})dA = \int_A \mathbf{n}\cdot(\Gamma\,grad\,\phi)dA + \int_{CV}S_\phi\,dV$$

Rate of increase of ϕ + Net rate of decrease of ϕ due to convection across boundaries = Net rate of increase of ϕ due to diffusion across boundaries + Net rate of creation of ϕ

- This is the actual form of the transport equation solved by a finite volume based CFD program.

The first term: $\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV$

DIFFUSION EQUATION :

3D: $div(\Gamma\, grad\,\phi) + S_\phi = 0$

1D: $\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right) + S = 0$

Apply Gauss' theorem: $\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV = \int_A n\left(\Gamma\frac{d\phi}{dx}\right)dA =$

Evaluate the integral (summation over two points): $\left(n\Gamma\frac{d\phi}{dx}A\right)_w + \left(n\Gamma\frac{d\phi}{dx}A\right)_e =$

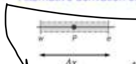
Insert the value of n: $(-1)\left(\Gamma\frac{d\phi}{dx}A\right)_w + (+1)\left(\Gamma\frac{d\phi}{dx}A\right)_e =$

Rewrite: $\Gamma_e\left(\frac{d\phi}{dx}\right)_e(A_e) - \Gamma_w\left(\frac{d\phi}{dx}\right)_w(A_w) =$

Insert the value of A: $\Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$

$\left(\frac{d\phi}{dx}\right)_e = \frac{T_B - T_P}{\delta x/2}$ $\left(\frac{d\phi}{dx}\right)_w = \frac{T_P - T_W}{\delta x}$ $\phi_e = \phi_B$ $\phi_w = \frac{\phi_W + \phi_P}{2}$

- Alternative derivation of the integral valid only in 1D:



$$\Delta V = A\Delta x \Rightarrow \int_{\Delta V} dV = A\int_{\Delta x} dx = A\int_w^e dx$$

$\int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV = A\int_{\Delta x}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dx =$

$A\int_w^e\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dx = A\left[\Gamma\frac{d\phi}{dx}\right]_w^e =$

$A\left[\left(\Gamma\frac{d\phi}{dx}\right)_e - \left(\Gamma\frac{d\phi}{dx}\right)_w\right] = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$

$\int S\,dV = \bar{S}\Delta V = \bar{S}A\Delta x$

if S const → $S\delta x$

$S = S_u + S_P\phi_P$

if const $B\delta x$; if var $C\delta x$

δx_{WP} δx_{wP} δx_{Pe} δx_{PE}

finite volume + central diff for diff eq

$$\Gamma_e A_e\left(\frac{\phi_E - \phi_P}{\delta x_{PE}}\right) - \Gamma_w A_w\left(\frac{\phi_P - \phi_W}{\delta x_{WP}}\right) + S_u + S_P\phi_P = 0$$

For our simple geometry we have:

$$A_e = A_w = A = 1;\quad \Gamma_e = \Gamma_w = \Gamma;\quad \delta x_{PE} = \delta x_{WP} = \delta x$$

So the equation becomes:

$$\Gamma\left(\frac{\phi_E - \phi_P}{\delta x}\right) - \Gamma\left(\frac{\phi_P - \phi_W}{\delta x}\right) + S_u + S_P\phi_P = 0$$

Collecting terms for each unknown ϕ_P ϕ_E ϕ_W:

$$\left(\frac{\Gamma}{\delta x} + \frac{\Gamma}{\delta x} - S_P\right)\phi_P = \left(\frac{\Gamma}{\delta x}\right)\phi_W + \left(\frac{\Gamma}{\delta x}\right)\phi_E + S_u$$

Node 234 — gen for diff eq

$a_P\phi_P = a_W\phi_W + a_E\phi_E + S_u$

$a_P = a_W + a_E - S_P$

$a_E = \Gamma/\delta x$

$a_W = \Gamma/\delta x$

$a_P = 2\Gamma/\delta x - S_P$

$S = B + C\phi$

$S_u = B\delta x$

$S_P = C\delta x$

$\frac{d}{dx}\Gamma\frac{d\phi}{dx} + S = 0$

Node 2/3/4

$\left(\frac{d\phi}{dx}\right)_e = \frac{T_E - T_P}{\delta x}$

$\left(\frac{d\phi}{dx}\right)_w = \frac{T_P - T_W}{\delta x}$

Node 1

$\left(\frac{d\phi}{dx}\right)_e = \frac{\phi_E - \phi_P}{\delta x}$

$\left(\frac{d\phi}{dx}\right)_w = \frac{2\phi_P - 2\phi_W}{\delta x}$ (ϕ_A)

Node 5 :

$\frac{2\phi_E - 2\phi_P}{\delta x}$ (ϕ_B)

$\frac{\phi_P - \phi_W}{\delta x}$

General Temporal Diffusion Equation

$\frac{\partial(\rho\phi)}{\partial t} = div(\Gamma\,grad\,\phi) + S_\phi$ → $\frac{\partial(\rho\phi)}{\partial t} = \frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right) + S$

Integrate over CV and time:

$$\int_t^{t+\Delta t}\int_{\Delta V}\frac{\partial(\rho\phi)}{\partial t}dVdt = \int_t^{t+\Delta t}\int_{\Delta V}\frac{\partial}{\partial x}\left(\Gamma\frac{\partial\phi}{\partial x}\right)dVdt + \int_t^{t+\Delta t}\int_{\Delta V}S\,dVdt$$

First term:

$\int_{\Delta V}\int_t^{t+\Delta t}\frac{\partial(\rho\phi)}{\partial t}dtdV$ (swap time integration and volume integration)

$= \int_{\Delta V}[\rho(\phi - \phi^0)]dV$

$= \rho(\phi_P - \phi_P^0)\Delta V = \rho(\phi_P - \phi_P^0)A\Delta x$

$\Delta V = A\Delta x$

Second term:

$\int_t^{t+\Delta t}\left[\left(\Gamma A\frac{\partial\phi}{\partial x}\right)_e - \left(\Gamma A\frac{\partial\phi}{\partial x}\right)_w\right]dt$

$= \left[\left(\Gamma A\frac{\partial\phi}{\partial x}\right)_e - \left(\Gamma A\frac{\partial\phi}{\partial x}\right)_w\right]\Delta t$

Third Term:

$\int_t^{t+\Delta t}\bar{S}\Delta V dt$ → i: $(S_u + S_P\phi_P)dt$; e: $(S_u + S_P\phi_P^0)dt$

Implicit scheme:

$\left(\frac{\partial\phi}{\partial x}\right)_e = \frac{\phi_E - \phi_P}{\delta x}$ and $\left(\frac{\partial\phi}{\partial x}\right)_w = \frac{\phi_P - \phi_W}{\delta x}$ (taking nodes 2,3,4 as reference)

Explicit scheme:

$\left(\frac{\partial\phi}{\partial x}\right)_e = \frac{\phi_E^0 - \phi_P^0}{\delta x}$ and $\left(\frac{\partial\phi}{\partial x}\right)_w = \frac{\phi_P^0 - \phi_W^0}{\delta x}$ (taking nodes 2,3,4 as reference)

Δx and δx can be used interchangeably

Third term:

$\int_t^{t+\Delta t}\bar{S}\Delta V\,dt$

Implicit scheme:

$= (S_u + S_P\phi_P)\Delta t$

Explicit scheme:

$= (S_u + S_P\phi_P^0)\Delta t$

Therefore, putting them together:

Implicit scheme:

$\rho(\phi_P - \phi_P^0)A\Delta x - \left[\Gamma A\left(\frac{\phi_E - \phi_P}{\delta x}\right) - \Gamma A\left(\frac{\phi_P - \phi_W}{\delta x}\right)\right]\Delta t + (S_u + S_P\phi_P)\Delta t$

$\rho(\phi_P - \phi_P^0)\frac{\Delta x}{\Delta t} - \left[\Gamma\left(\frac{\phi_E - \phi_P}{\delta x}\right) - \Gamma\left(\frac{\phi_P - \phi_W}{\delta x}\right)\right] + (S_u + S_P\phi_P)$

Explicit scheme:

$\rho(\phi_P - \phi_P^0)A\Delta x = \left[\Gamma A\left(\frac{\phi_E^0 - \phi_P^0}{\delta x}\right) - \Gamma A\left(\frac{\phi_P^0 - \phi_W^0}{\delta x}\right)\right]\Delta t - (S_u + S_P\phi_P^0)\Delta t$

$\rho(\phi_P - \phi_P^0)\frac{\Delta x}{\Delta t} - \left[\Gamma\left(\frac{\phi_E^0 - \phi_P^0}{\delta x}\right) - \Gamma\left(\frac{\phi_P^0 - \phi_W^0}{\delta x}\right)\right] + (S_u + S_P\phi_P^0)$

Collect and rearrange as follows:

$a_P\phi_P = a_W\phi_W + a_E\phi_E + a_P^0\phi_P^0 + S_u$

To get coefficients a_P, a_W, a_E, a_P^0, S_u

ϕ_P^0 decides conditions

S_P/S_u dc.

$\Delta t < \rho\frac{\delta x^2}{2\Gamma}$

Condition to use explicit scheme:

Coefficient a_P^0 ... $\Delta t < \rho\frac{(\delta x)^2}{2\Gamma}$

Iterative Tec...

The following ... ll use the same matrix

$$\begin{bmatrix}2 & 1 & 1\\ 1 & 3 & 1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} = \begin{bmatrix}7\\ 2\\ 5\end{bmatrix}$$

$2x_1 + x_2 + x_3 = 7$
$-x_1 + 3x_2 - x_3 = 2$
$x_1 - x_2 + 2x_3 = 5$
➡
$x_1 = \frac{7 - x_2 - x_3}{2}$
$x_2 = \frac{2 + x_1 + x_3}{3}$
$x_3 = \frac{5 - x_1 + x_2}{2}$

Jacobi Method

Iteration number	0	1	2	3	4	5	...	17
x_1	0	3.5000	1.9167	1.6250	1.2292	1.1563	...	1.0000
x_2	0	0.6667	2.6667	1.6667	2.1667	1.9167	...	2.0000
x_3	0	2.5000	1.0833	2.8750	2.5208	2.9688	...	3.0000

Gauss-Seidel Model

x^0 = start value (iteration 0)

x^1 = iteration 1

x^n = iteration n

$$x_1^1 = \frac{7 - x_2^0 - x_3^0}{2}$$

$$x_2^1 = \frac{2 + x_1^1 + x_3^0}{3}$$

$$x_3^1 = \frac{5 - x_1^1 - x_2^1}{2}$$

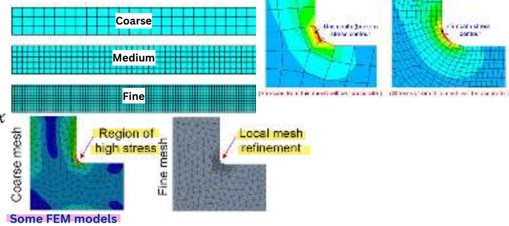
Iteration number	0	1	2	3	4	5	...	13
x_1	0	3.5000	1.7500	1.3333	1.1181	1.0475	...	1.0000
x_2	0	1.8333	1.8056	1.9537	1.9761	1.9922	...	2.0000
x_3	0	1.6667	2.5278	2.8102	2.9290	2.9724	...	3.0000

Condition for Convergence

Scarborough criterion

$\frac{\sum|a_{nb}|}{|a_P|}$ ≤ 1 for all nodes; < 1 for at least one node

Mesh

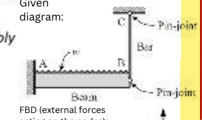


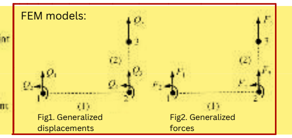
Some FEM models

ALWAYS draw them as one **DO NOT SEPARATE** into elements

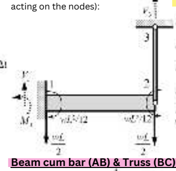
Beam (AB) & bar (BC) element:

Given diagram:





FBD (external forces acting on the nodes):



$F_1 = V_1 - \frac{wL}{2}$

$F_2 = M_1 - \frac{wL^2}{2}$

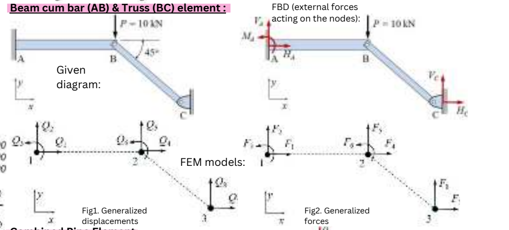
$F_3 = -\frac{wL}{2}$

$F_4 = \frac{wL^2}{2}$

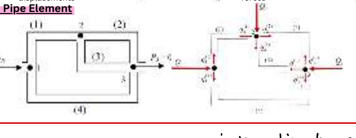
$F_5 = V_3$

V_1, M_1, V_3 are reaction forces

Beam cum bar (AB) & Truss (BC) element :



Combined Pipe Element



WEEK 11

STEADY 1D CONV+ D

$$\frac{d}{dx}(\rho u\phi) = \frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)$$

$F_e\phi_e - F_w\phi_w = D_e(\phi_E - \phi_P) - D_w(\phi_P - \phi_W)$

F = ρu D = Γ/δx

The integrated equation: $\int_{\Delta V}\frac{d}{dx}(\rho u\phi)dV = \int_{\Delta V}\frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right)dV$

Apply the Gauss' theorem: $\int_A n(\rho u\phi)dA = \int_A n\left(\Gamma\frac{d\phi}{dx}\right)dA$

Evaluate the integral at the two endpoints: $(n\rho u\phi A)_e + (n\rho u\phi A)_w = \left(n\Gamma\frac{d\phi}{dx}A\right)_e + \left(n\Gamma\frac{d\phi}{dx}A\right)_w$

Insert the value of n: $(-1)(\rho u\phi A)_w + (+1)(\rho u\phi A)_e = (-1)\left(\Gamma\frac{d\phi}{dx}A\right)_w + (+1)\left(\Gamma\frac{d\phi}{dx}A\right)_e$

Rewrite: $(\rho u\phi)_e(A_e) - (\rho u\phi)_w(A_w) = \Gamma_e\left(\frac{d\phi}{dx}\right)_e(A_e) - \Gamma_w\left(\frac{d\phi}{dx}\right)_w(A_w)$

Insert the value of A(=1): $(\rho u\phi)_e - (\rho u\phi)_w = \Gamma_e\left(\frac{d\phi}{dx}\right)_e - \Gamma_w\left(\frac{d\phi}{dx}\right)_w$

integration by parts

$u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]$

ILATE: inv trig, log, alg, trig, exp

ln(A) = B

$e^B = A$

The central differencing scheme

- Applied on the equation with source term $\frac{d}{dx}(\rho u\phi) = \frac{d}{dx}\left(\Gamma\frac{d\phi}{dx}\right) + S$

$\phi_e = \frac{\phi_E + \phi_P}{2}$ $\phi_w = \frac{\phi_P + \phi_W}{2}$

NODE 2,3,4

NODE 1 - RHS

$\left.\frac{d\phi}{dx}\right)_e = \frac{\phi_E - \phi_P}{\delta x}$

$\left.\frac{d\phi}{dx}\right)_w = \frac{2\phi_P - 2\phi_W}{\delta x}$

LHS

$\phi_e = \frac{\phi_P + \phi_E}{2}$

$\phi_w = \phi_A$

NODE 5 - RHS

$\frac{2\phi_E - 2\phi_P}{\delta x}$

$\frac{\phi_P - \phi_W}{\delta x}$

$\phi_e = \phi_B$

$\phi_w = \frac{\phi_W + \phi_P}{2}$

Pe = 2 is required for a bounded solution (with no wiggles).

The upwind differencing scheme

Central (What we used previously): $\phi_e = \frac{\phi_P + \phi_E}{2}$, $\phi_w = \frac{\phi_W + \phi_P}{2}$ F>0

Central (Identical for F>0 and F<0): $\phi_e = \frac{\phi_P + \phi_E}{2}$, $\phi_w = \frac{\phi_W + \phi_P}{2}$ F<0