# 102S207 CS10L01

Engineering Experiments (Me) (Nanyang Technological University)

**Theorem 2.5**

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if the reduced row echelon form of $A$ is $I_n$.

***Proof*** First, suppose that $A$ is invertible.

$$\text{For } \mathbf{v} \in \mathcal{R}^n,\ A\mathbf{v} = \mathbf{0} \Rightarrow A^{-1}A\mathbf{v} = A^{-1}\mathbf{0} = \mathbf{0} \Rightarrow \mathbf{v} = \mathbf{0}$$
$$\Rightarrow \text{rank } A = n \text{ by Theorem 1.8 (d)(f)}$$
$$\Rightarrow \text{reduced row echelon form of } A \text{ is } I_n \text{ since } A \text{ is } n \times n$$

So, the reduced row echelon form of $A$ must equal $I_n$.

## Theorem 2.5

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if the reduced row echelon form of $A$ is $I_n$.

***Proof*** Conversely, suppose that the reduced row echelon form of $A$ equals $I_n$.

Therefore, $A$ is invertible.

## Theorem 2.5

Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if the reduced row echelon form of $A$ is $I_n$.

Examples:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 4 & 8 \end{bmatrix}$$

has the reduced row echelon form $I_n \Rightarrow$ invertible.

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

has the reduced row echelon form $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ not invertible.

## Algorithm for Matrix Inversion

Let $A$ be an $n \times n$ matrix. Use elementary row operations to transform $\begin{bmatrix} A & I_n \end{bmatrix}$ into the form $\begin{bmatrix} R & B \end{bmatrix}$, where $R$ is a matrix in reduced row echelon form. Then either

(a) $R = I_n$, in which case $A$ is invertible and $B = A^{-1}$; or

(b) $R \neq I_n$, in which case $A$ is not invertible.

***Proof***

(a) $[R\ B] = P[A\ I_n] = [PA\ PI_n] = [PA\ P]$

(b) By Thm 2.5

# Algorithm for Matrix Inversion

Let $A$ be an $n \times n$ matrix. Use elementary row operations to transform $\begin{bmatrix} A & I_n \end{bmatrix}$ into the form $\begin{bmatrix} R & B \end{bmatrix}$, where $R$ is a matrix in reduced row echelon form. Then either

(a) $R = I_n$, in which case $A$ is invertible and $B = A^{-1}$; or

(b) $R \neq I_n$, in which case $A$ is not invertible.

## Example:

$$\begin{bmatrix} A & I_3 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 6 & 0 & 1 & 0 \\ 3 & 4 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & -2 & -1 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & -1 & -7 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right]$$

$A^{-1}$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -20 & 6 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -16 & 4 & 3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 7 & -2 & -1 \end{array}\right] = \begin{bmatrix} I_3 & B \end{bmatrix}$$

## Algorithm for Computing $A^{-1}B$

Let $A$ be an invertible $n \times n$ matrix and $B$ be an $n \times p$ matrix. Suppose that the $n \times (n+p)$ matrix $\begin{bmatrix} A & B \end{bmatrix}$ is transformed by means of elementary row operations into the matrix $\begin{bmatrix} I_n & C \end{bmatrix}$ in reduced row echelon form. Then $C = A^{-1}B$.

***Proof*** $\begin{bmatrix} I_n & C \end{bmatrix} = P\begin{bmatrix} A & B \end{bmatrix} = \begin{bmatrix} PA & PB \end{bmatrix}$

Example:

$$\begin{bmatrix} A & B \end{bmatrix} = \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 2 & 5 & 1 & 1 & 3 \\ 2 & 4 & 1 & 0 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & -3 & 5 \\ 0 & 0 & -1 & -4 & 4 \end{array}\right]$$

Clearly, it is easier to compute $A^{-1}B$ directly than to compute $A^{-1}$ first and get $A^{-1}B$ afterwards.

$$\rightarrow \left[\begin{array}{ccc|cc} 1 & 2 & 1 & 2 & -1 \\ 0 & 1 & -1 & -3 & 5 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|cc} 1 & 2 & 0 & -2 & 3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 0 & -4 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 4 & -4 \end{array}\right]$$

$A^{-1}B$

## Theorem 2.6 (Invertible Matrix Theorem)

Let $A$ be an $n \times n$ matrix. The following statments are equivalent:

(a) $A$ is invertible.

(b) The reduced row echelon form of $A$ is $I_n$.

(c) The rank of $A$ equals $n$.

(d) The span of the columns of $A$ is $\mathcal{R}^n$.

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^n$.

(f) The nullity of $A$ equals zero.

(g) The columns of $A$ are linearly independent.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(i) There exists an $n \times n$ matrix $B$ such that $BA = I_n$

(j) There exists an $n \times n$ matrix $C$ such that $AC = I_n$

(k) $A$ is a product of elementary matrices.

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***Proof***

(a) ⇔ (b): Theorem 2.5.

(d) ⇔(e): By definition of span and matrix-vector products.

(b) ⇔ (c): rank($A$) = the number of nonzero rows of reduced row echelon form of $A$, $I_n = n$.

(a) ➔ (e) $\mathbf{x} = A^{-1}\mathbf{b}$.

(e) ➔ (a) $A\mathbf{x} = \mathbf{b}$ has at least a solution for any $\mathbf{b}$ in $R^n$.

Let $A = PR$, where $R$ is the reduced row echelon form.

Suppose $A$ is NOT invertible, then $R$ is not $I_n$, and the last row of $R$ is zero.

## Theorem 2.6 (Invertible Matrix Theorem)

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(a) $A$ is invertible.

(b) The reduced row echelon form of $A$ is $I_n$.

(c) The rank of $A$ equals $n$.

(f) The nullity of $A$ equals zero.

(g) The columns of $A$ are linearly independent.

***Proof***

(c) ⇔ (f) Nullity = $n$ – rank($A$) = $n$- $n$ = 0. (# of free variables = 0)

(g) ⇔ (c) Thm 1.8 (a)(d).

(g) ⇔ (f) Thm 1.8 (a)(c).

(g) ⇔ The columns of R are linearly independent. They are all pivot columns.⇔R=In ⇔(b).

## Theorem 2.6 (Invertible Matrix Theorem)

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(a) $A$ is invertible.

(b) The reduced row echelon form of $A$ is $I_n$.

(g) The columns of $A$ are linearly independent.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(k) $A$ is a product of elementary matrices.

***Proof***

(a) ➔ (h): $\mathbf{x} = A^{-1}\,\mathbf{0} = \mathbf{0}$;

(h) ⇔ (g): By definitions of l.i. and matrix-vector product.

(b) ➔ (k) $I_n = E_k \dots E_2\, E_1\, A$ ➔ $E_k^{-1} = E_{k-1} \dots E_2 E_1 A$ ➔ $A = E_1^{-1} E_2^{-1} \dots E_k^{-1}$

(k) ➔ (a) $A =$ is invertible (Thm 2.2 b)

## Theorem 2.6 (Invertible Matrix Theorem)

Let $A$ be an $n \times n$ matrix. The following statments are equivalent:

(a) $A$ is invertible.

(b) The reduced row echelon form of $A$ is $I_n$.

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^n$.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(i) There exists an $n \times n$ matrix $B$ such that $BA = I_n$

(j) There exists an $n \times n$ matrix $C$ such that $AC = I_n$

***Proof*** (a) ➔ (i) obvious. (a)➔(j) obvious (by def.)

(i) ➔ (h) ➔ (a) : Let $\mathbf{x}$ be any vector in $\mathcal{R}^n$ such that $A\mathbf{x} = \mathbf{0}$.
Then $\mathbf{x} =$

(j) ➔ (e) ➔ (a): Let $\mathbf{b}$ be any vector in $\mathcal{R}^n$ and let $\mathbf{v} = C\mathbf{b}$.
Then $A\mathbf{v} =$

## Theorem 2.6 (Invertible Matrix Theorem)

Let $A$ be an $n \times n$ matrix. The following statments are equivalent:

(a) $A$ is invertible.

(b) The reduced row echelon form of $A$ is $I_n$. $R = I_n$

(c) The rank of $A$ equals $n$. $\text{rank}(A) = n$

(d) The span of the columns of $A$ is $\mathcal{R}^n$. $\text{span}\,\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\} = \mathcal{R}^n$

(e) The equation $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$ in $\mathcal{R}^n$. $\forall \mathbf{b} \in \mathcal{R}^n, \exists \mathbf{x} \in \mathcal{R}^n s.t. A\mathbf{x} = \mathbf{b}$

(f) The nullity of $A$ equals zero. $\text{nullity}(A) = 0$

(g) The columns of $A$ are linearly independent. $\{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ is l.i.

(h) The only solution of $A\mathbf{x} = \mathbf{0}$ is $\mathbf{0}$.

(i) There exists an $n \times n$ matrix $B$ such that $BA = I_n$

(j) There exists an $n \times n$ matrix $C$ such that $AC = I_n$

(k) $A$ is a product of elementary matrices. $A = E_1 E_2 \cdots E_k$

Note that though only one of $BA = I_n$ or $AC = I_n$ is needed to show the invertibility of $A$, the matrix $A$ has to be square.

There are nonsqaure matrices $A$ and $C$ for which the product AC is an identity matrix. For instance, let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 \\ -1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Then $AC =$

Of course, $A$ and $C$ are not invertible.

## Homework Set for Section 2.4

Section 2.4: Problems 1, 8, 14, 19, 21, 22, 27, 28, 29, 31, 32.