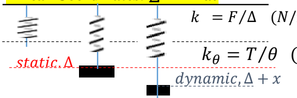
MP3002 Cheatsheet EXAM 2013 2014 SEM2

Solid Mechanics & Vibration (Nanyang Technological University)

Linear Coordinates: $\sum F = m\ddot{x}$



$k = F/\Delta \quad (N/m)$

$k_\theta = T/\theta \quad (Nm/rad)$

Equilibrium State: $\sum F = m(0)$
$mg - k\Delta = 0; \quad \Delta = mg/k$

Dynamic State: $\sum F = m\ddot{x}$
$mg - k(\Delta + x) = m\ddot{x}; \quad m\ddot{x} + kx = 0$

$EOM: \ddot{x} + \frac{k}{m}x = 0; \quad \omega_{n(rad/s)} = \sqrt{k/m}$

Conservation of Energy: $E = PE + KE$

$[mgh + 1/2\,(mv^2)]_1 = [mgh + 1/2\,(mv^2)]_2$

Max height (energy storage), PE=max, KE=0

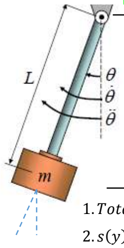
At lowest level, KE=max, PE=0

Pendulum, $PE = mgh, \quad KE = (1/2)mv^2$

Spring, $PE = (1/2)k(\Delta + x)^2 - mgx$ (ignore static∆) **IMPULSE:** F=ma, $F = m\left(\frac{dv}{dt}\right)$, $\int Fdt = (mv)_2 - (mv)_1$

$[mv]_1 = [mv]_2$ – (linear) (angular) $(I\omega)$

Angular Coordinates: $\sum M = J\ddot{\theta}$



EOM: $\sum M_0 = J_0$

$-mgLsin\theta = mL^2\ddot{\theta}$

$mL^2\ddot{\theta} + mgLsin\theta = 0$

$\ddot{\theta} + \frac{g}{L}sin\theta = 0;$

$f_{n(Hz)} = (1/2\pi)\sqrt{g/L}$

Lumped Mass System

1. $Total\ KE = KE_{mass} + KE_{spring}$ **Linear

2. $s(y) = x(t)\left(\frac{y}{L}\right); \ \dot{s} = \dot{x}\left(\frac{y}{L}\right)$

3. $KE_{spring} = \frac{1}{2}\int \dot{s}(y)^2 dm = \frac{1}{2}\int_0^L \frac{\dot{s}^2 m}{L} dy$

4. $KE_{spring} = \frac{1}{2}\int_0^L \left(\frac{y}{L}\dot{x}\right)^2 \frac{m}{L} dy = \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$

5. $KE_{equiv.} = \frac{1}{2}m_{eq}\dot{v}^2 = \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$

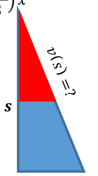
1. $x = L\theta;\ \dot{x} = L\dot{\theta};\ v = s\dot{\theta}$ **Rotating (pendulum/bar)

2. $J_o = \frac{1}{2}\int v^2 dm = \frac{1}{2}\int_0^L \left(\frac{\dot{x}}{L}s\right)^2 \left(\frac{m}{L}\right) ds$

3. $J_o = \frac{1}{2}\left(\frac{m}{L}\right)\left(\frac{s^3\dot{\theta}^2}{3}\right) = \frac{mL^2}{3}$

4. $J_o = \frac{1}{2}\int_0^L \left(\frac{y}{L}\dot{x}\right)^2 \frac{m}{L} dy = \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$

5. $J_o = \frac{1}{2}m_{eq}\dot{v}^2 = \frac{1}{2}\left(\frac{m_s}{3}\right)\dot{x}^2$



$v = \dot{x}, s = L$

$\frac{v}{s} = \frac{\dot{x}}{L} = \frac{\dot{\theta}L}{L}$

$x(t) = Asin\omega_n t + Bcos\omega_n t$

1. $x(t)$ or $\theta(t) = Xsin(\omega t + \emptyset);$
2. $\dot{x}(t)$ or $\dot{\theta}(t) = \omega Xcos(\omega t + \emptyset);$
3. $\ddot{x}(t)$ or $\ddot{\theta}(t) = -\omega^2 Xsin(\omega t + \emptyset);$

Natural freq (rad/s): $\omega = \sqrt{\frac{k}{m}}$ or $\sqrt{\frac{g}{L}}$

$sin(A \pm B) = sinAcosB \pm cosAsinB$

$cos(A \pm B) = cosAcosB \mp sinAsinB$

$tan(A \pm B) = \frac{tanA \pm tanB}{1 \mp tanAtanB}$

$asin\theta \pm bcos\theta = \sqrt{a^2 + b^2}sin(\theta \pm \beta)$

$acos\theta \pm bsin\theta = \sqrt{a^2 + b^2}cos(\theta \mp \beta)$

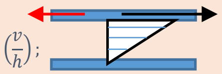
$\beta = \tan^{-1}(b/a)$

$2sinAcosB = sin(A + B) + sin(A - B)$ *if opposite (-)

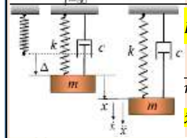
$2cosAcosB = cos(A + B) + cos(A - B)$ *-sin(-)

Damped Free Vibration: reduce amplitude of oscillation through the viscous fluid within the damper (3 mode of damping: Dry friction takes into consideration of surface irregularities; Hysteretic damping caused by internal solid friction $f = hv/\omega$; viscous damping, shearing effect of fluid vibration $F = (c)(v)$ **NOTE c = Ns/m

$\tau\left(\frac{N}{m^2}\right) = \mu\left(\frac{du}{dy}\right) = \mu\left(\frac{v}{h}\right);$



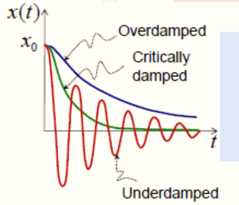
$shear\ (damp)force\ f = \tau A = \mu\left(\frac{A}{h}\right)v = cv$



$EOM: \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0;$

$\frac{c}{m} = \frac{c}{c_c}\frac{c_c}{m} = \zeta\left(\frac{2\sqrt{km}}{m}\right) = 2\zeta\omega_n$

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$



Damp ratio,

$\zeta = C/C_{critical}$

Critical Damping,

$C_{critical} = 2\sqrt{km}$

Critically Damped (Fast non-oscillatory): $\zeta = 1$

No oscillation, from initial, system restores close to equilibrium in minimal time. (I.e. door closer, recoiled guns) $**C = C_{critical}$

$x(t) = (C_1 + C_2 t)e^{-\omega_n t}$

$\dot{x}(t) = -\omega_n(C_1 + C_2 t)e^{-\omega_n t} + C_2 e^{-\omega_n t}$

$\ddot{x}(t) = +\omega_n^2(C_1 + C_2 t)e^{-\omega_n t} + C_2 e^{-\omega_n t} - \omega_n C_2 e^{-\omega_n t}$

Over-Damped (Exponential Decay): $\zeta > 1$

System approach equilibrium with no damp. (I.e. mistuned door closer) $**C > C_{critical}$

$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$;

$\dot{x}(t) = \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t}$

$\ddot{x}(t) = \lambda_1^2 C_1 e^{\lambda_1 t} + \lambda_2^2 C_2 e^{\lambda_2 t}$

$\lambda_{1,2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1})$

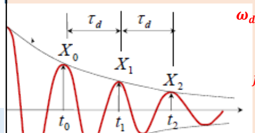
Under Damped (Decay In oscillation): $\zeta < 1$

Oscillates about equilibrium position with decay amplitude. (I.e. mechanical system) $**C < C_{critical}$

$x(t) = e^{-\zeta\omega_n t}(Asin\omega_d t + Bcos\omega_d t)$

$\dot{x}(t) =$

$\ddot{x}(t) =$



$\omega_{damped} = \omega_n\sqrt{1 - \zeta^2}$

$\omega_{damped} < \omega_n$

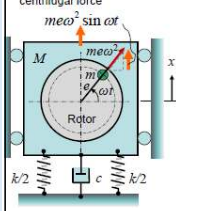
$frequency, f = \frac{1}{\tau_d}$

Logarithmic Decry, $\delta = Ln\left(\frac{X_{m-1}}{X_m}\right) = \zeta\omega_n\tau_d$

$\zeta = \sqrt{\left(\frac{\delta^2}{4\pi^2 + \delta^2}\right)};$

$\frac{X_0}{X_1} = \frac{X_1}{X_2}\frac{X_2}{X_3} = \left(\frac{X_1}{X_2}\right)^2 \qquad * note\ \frac{X_1}{X_2} = \frac{X_2}{X_3}$

Vertical component of centrifugal force



$\omega_n = \sqrt{\frac{k}{M}}$

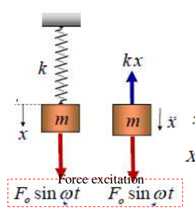
$M\ddot{x} + c\dot{x} + kx = me\omega^2 sin\omega_{drive}t$

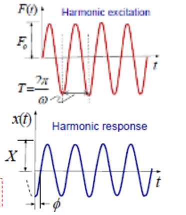
$X = \frac{me\omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}}$

$\frac{MX}{me} = \frac{(\omega^2/\omega_n^2)^2}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$

Forced Periodic Motion: REASONANCE

$\omega_{driving} = \omega_{natural}$ *Force excitation





Excited (Input): $F(t) = F_0 sin\omega_{driving}t$

$EOM: m\ddot{x} + kx = F_0 sin\omega_{drive}t$

$x(t)$ or $\theta(t) = Xsin(\omega t - \emptyset);$

$\dot{x}(t)$ or $\dot{\theta}(t) = \omega Xcos(\omega t - \emptyset);$

$\ddot{x}(t)$ or $\ddot{\theta}(t) = -\omega^2 Xsin(\omega t - \emptyset);$

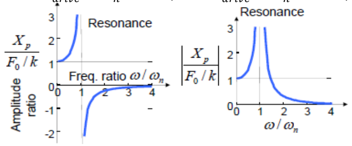
$(-m\omega^2 + k)Xsin(\omega t - \emptyset) = F_0 sin\omega_{drive}t$

Compare amplitude: $X(-m\omega^2 + k) = F_0$

$X = \frac{F_0}{-m\omega^2 + k} = \frac{F_0}{1 - m\omega^2/k} = \frac{F_0}{1 - \omega^2/\omega_n^2};$

Compare phase: $sin(\omega t - \emptyset) = sin\omega_{drive}t$; $\emptyset = 0$

Phase: $\omega_{drive} < \omega_n \quad \emptyset = 0, \qquad \omega_{drive} > \omega_n \quad \emptyset = \pi,$



$x(t)$ or $\theta(t) = \frac{F_0}{m(\omega_n^2 - \omega^2)}sin(\omega t - \emptyset)$

Damping on Forced Vibration: reduce amplitude and prevent mechanical system from fatigue by a damper $m\ddot{x} + c\dot{x} + kx = F_0 sin\omega_{drive}t$

$(-m\omega^2 + k)Xsin(\omega t - \emptyset) + c\omega Xcos(\omega t - \emptyset) = F_0 sin\omega_{drive}t$

$X\sqrt{(-m\omega^2 + k)^2 + (c\omega)^2}sin(\omega t - \emptyset + \beta) = F_0 sin\omega_{drive}t$

Compare amplitude: $X\sqrt{(-m\omega^2 + k)^2 + (c\omega)^2} = F_0$

$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$

$\frac{X}{F_0/k} = \frac{1}{\sqrt{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}}$

$Q_{uality} = \frac{X}{F_0/k} = \frac{1}{2\zeta};\ at\ resonance$

Compare phase: $sin(\omega t - \emptyset + \beta) = sin\omega_{drive}t$

$\emptyset = \beta = \tan^{-1}\frac{c\omega}{k - m\omega^2} = \tan^{-1}\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n^2}$

Unbalanced: centrifugal forces giving vertical vibration and poor road contact. Due to misalignment, non-uniform thickness (i.e. washing machine, ceiling fan)

M: total mass of system

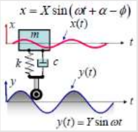
m: unbalanced mass

Based Excitation: analysis for rough road or earthquakes

$\sum F = m\ddot{x}$

$-k(x - y) - c(\dot{x} - \dot{y}) = m\ddot{x}$

$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}\ (EOM)$



$x(t)$ or $\theta(t) = Xsin(\omega t - \emptyset);$

$\dot{x}(t)$ or $\dot{\theta}(t) = \omega Xcos(\omega t - \emptyset);$

$\ddot{x}(t)$ or $\ddot{\theta}(t) = -\omega^2 Xsin(\omega t - \emptyset);$

$y(t)$ or $\theta(t) = Ysin(\omega t - \emptyset);$

$\dot{y}(t)$ or $\dot{\theta}(t) = \omega Ycos(\omega t - \emptyset);$

$\ddot{y}(t)$ or $\ddot{\theta}(t) = -\omega^2 Ysin(\omega t - \emptyset);$

$m\ddot{x} + c\dot{x} + kx = \sqrt{(kY)^2 + (c\omega Y)^2}\,sin(\omega t + \alpha)$

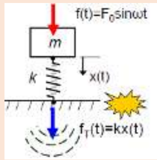
$\alpha = \tan^{-1}\left(\frac{c\omega}{k}\right)$

Force Transmission and Vibration Isolation: isolate the vibration source. Uses isolation pads or spring, an insertion of resilient members. Damper are used to dissipate mechanical energy in prevention of damaging the system. Reduce the impact to the ground else the building will shake.

$f_T(t) = kx(t)$

$m\ddot{x} + c\dot{x} + kx = F_0 sin\omega t$

$X = \frac{F_0}{k - m\omega^2}$



$f_T(t) = kXsin(\omega t - \varphi)$

$f_T(t) = \frac{kF_0}{k - m\omega^2}sin(\omega t - \varphi)$

Force transmission amplitude w/o damping:

$TR = \left|\frac{F_T}{F_0}\right| = \left|\frac{kF_0}{F_0(k - m\omega^2)}\right| = \left|\frac{1}{\left(1 - \left(\frac{m}{k}\right)\omega^2\right)}\right| = \left|\frac{1}{(1 - \omega^2/\omega_n^2)}\right|$

Consider with damping:

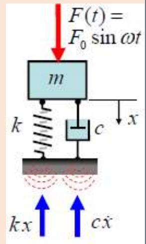
$f_T(t) = kx(t) + c\dot{x}(t) = k(Xsin\omega t) + c(\omega Xcos\omega t)$

$F_T(t) = \sqrt{(k)^2 + (c\omega)^2}Xsin(\omega t + \varphi)$

$m\ddot{x} + c\dot{x} + kx = F_0 sin\omega t$

$X = \frac{F_0}{\sqrt{(-\omega m^2 + k)^2 + (c\omega)^2}}$

Amplitude: $F_T(t) = \frac{F_0\sqrt{(k)^2 + (c\omega)^2}}{\sqrt{(-m\omega^2 + k)^2 - (c\omega)^2}}$



$TR = \left|\frac{F_T}{F_0}\right| = \left|\frac{F_0\sqrt{(k)^2 + (c\omega)^2}}{F_0\sqrt{(-m\omega^2 + k) + (c\omega)^2}}\right| = \left|\frac{\sqrt{(k)^2 + (c\omega)^2}}{\sqrt{(-m\omega^2 + k)^2 + (c\omega)^2}}\right| = \sqrt{\frac{1 + (2\zeta(\omega/\omega_n))^2}{(1 - (\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2}} < 1$

**Note: To avoid disturbance to surrounding the force transmitted from a vibrating machine is desired to be less than the source excitation force (F_0)

Transient Vibration & Initial Condition: (temporary, impulse, sudden excitation. i.e. step load or drop test)

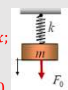
Free vibration (no load): $m\ddot{x} + kx = 0 \qquad x(0) = x_0$

Equation: $x(t) = Acos\omega_n t + Bsin\omega_n t \qquad \dot{x}(0) = \dot{x}_0$

Transient vibration (excited by load): $m\ddot{x} + kx = F_0$

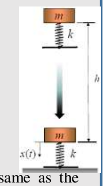
Equation: $x(t) = Acos\omega_n t + Bsin\omega_n t + F_0/k$

METHOD: $x(t) = Acos\omega_n t + Bsin\omega_n t + F_0/k$



1. $x(t = 0) = 0 = Acos\omega_n(0) + Bsin\omega_n(0) + F_0/k;$
2. $A = -(F_0/k)$
3. $\dot{x}(t) = \frac{dx}{dt} = -\omega_n Asin\omega_n(t) + \omega_n Bcos\omega_n(t) + 0$
4. $\dot{x}(0) = 0 = -\omega_n Asin\omega_n(0) + \omega_n Bcos\omega_n(0)$
5. $B = 0$
6. $x(t) = (F_0/k)(1 - cos\omega_n t)$

Drop Test (excited by load): $m\ddot{x} + kx = F_0$



$m\ddot{x} + kx = mg$

$x(t) = Acos\omega_n t + Bsin\omega_n t + mg/k$

In[illegible]tion:

$x(0) = 0$

[illegible] $\dot{x}(0) = \sqrt{2gh}$ Do the same as the Method listed and solve for constants.

Free Vibration of Two Degree Freedom Systems: (2 reference coordinates)

Case A: Pendulum $\sum M_0 = J_0\ddot{\theta}$

$ml^2\ddot{\theta}_1 = -mgsin\theta_1 - kl^2(\theta_1 - \theta_2)$

$ml^2\ddot{\theta}_1 + (mg + kl^2)\theta_1 - kl^2\theta_2 = 0$

$ml^2\ddot{\theta}_2 = -mgsin\theta_2 - kl^2(\theta_2 - \theta_1)$

$ml^2\ddot{\theta}_2 + (mg + kl^2)\theta_2 - kl^2\theta_1 = 0$

General

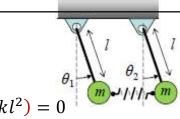
$\theta_{1,2}(t) = \Theta sin\omega t$

$\dot{\theta}_{1,2}(t) = \omega\Theta cos\omega t$

$\ddot{\theta}_{1,2}(t) = -\omega^2\Theta sin\omega t$

$\begin{bmatrix} ml^2 & 0 \\ 0 & ml^2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} mg + kl^2 & -kl^2 \\ -kl^2 & mg + kl^2 \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$\left(\begin{bmatrix} -\omega^2 ml^2 & 0 \\ 0 & -\omega^2 ml^2 \end{bmatrix} + \begin{bmatrix} mg + kl^2 & -kl^2 \\ -kl^2 & mg + kl^2 \end{bmatrix}\right)\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}sin\omega t = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$



Natural Frequencies:

$\begin{vmatrix} -\omega^2 ml^2 + mg + kl^2 & -kl^2 \\ -kl^2 & -\omega^2 ml^2 + mg + kl^2 \end{vmatrix} = 0$

$(-\omega^2 ml^2 + mg + kl^2)(-\omega^2 ml^2 + mg + kl^2) - (-kl^2)(-kl^2) = 0$

$(-\omega^2 ml^2 + mg + kl^2)^2 - (-kl^2)^2 = 0 \ \rightarrow (a^2 - b^2) = (a + b)(a - b) = 0$

$(-\omega^2 ml^2 + mg)\ (-\omega^2 ml^2 + mg + 2kl^2) = 0$

EQ1: $(-\omega^2 ml^2 + mg) = 0 \qquad$ EQ2: $(-\omega^2 ml^2 + mg + 2kl^2) = 0$ solved

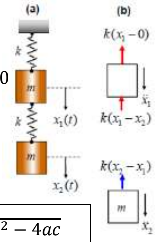
Case B: springs $\sum F = m\ddot{x}$

$m\ddot{x}_1 = -k(x_1 - 0) - k(x_1 - x_2)$

$m\ddot{x}_1 + k(x_1) + k(x_1) - k(x_2) = 0$

$m\ddot{x}_2 = -k(x_2 - x_1)$

$m\ddot{x}_2 + k(x_2) - k(x_1) = 0$



$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

Natural Frequencies:

$\begin{vmatrix} -\omega^2 m + 2k & -k \\ -k & -\omega^2 m + k \end{vmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}sin\omega t = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$(-\omega^2 m + 2k)(-\omega^2 m + k) - (-k)^2 = 0$

$\omega^4 m^2 - \omega^2 mk - 2\omega^2 mk + k^2 = 0$

$\omega^4 m^2 - 3\omega^2 mk + k^2 = 0$

$\omega^4 m^2 - 3\omega^2 mk + k^2 = 0$

$\omega^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\omega_1 = 3.7\ rad/s$

$\omega_2 = 9.7\ rad/s$

MODE SHAPES: look for the free end. Coordinate II

E (1): $-\omega^2 m + 2k(X_1) - k(X_2) = 0 \ \rightarrow (X_2/X_1) = \frac{-\omega^2 m + 2k}{k} = 1.62$

E (2): $-k(x_1) - \omega^2 m + k(x_2) = 0 \ \rightarrow (X_2/X_1) = \frac{k}{-\omega^2 m + k} = -0.62$

Degree of freedom at Coordinate 2, Mode E (1), $\emptyset_{2,1} = 1.62$

Degree of freedom at Coordinate 2, Mode E (2), $\emptyset_{2,2} = -0.62$

Super positioning:

$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} \emptyset_{11} & \emptyset_{12} \\ \emptyset_{21} & \emptyset_{22} \end{bmatrix}\begin{Bmatrix} C_1 sin(\omega_1 t + \varphi_1) \\ C_2 sin(\omega_2 t + \varphi_2) \end{Bmatrix} = \begin{bmatrix} \emptyset_{11} & \emptyset_{12} \\ \emptyset_{21} & \emptyset_{22} \end{bmatrix}\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$

Initial Conditions:

$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix}\begin{Bmatrix} C_1 sin(\omega_1 t + \varphi_1) \\ C_2 sin(\omega_2 t + \varphi_2) \end{Bmatrix} = \begin{bmatrix} \emptyset_{11} & \emptyset_{12} \\ \emptyset_{21} & \emptyset_{22} \end{bmatrix}\begin{Bmatrix} q_1(t) \\ q_2(t) \end{Bmatrix}$

Displacement:

$\begin{Bmatrix} x_1(0) \\ x_2(0) \end{Bmatrix} = \begin{Bmatrix} 10 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix}\begin{Bmatrix} C_1 sin(\omega_1(0) + \varphi_1) \\ C_2 sin(\omega_2(0) + \varphi_2) \end{Bmatrix}$

Velocity:

$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.62 & -0.62 \end{bmatrix}\begin{Bmatrix} \omega_1 C_1 cos(\omega_1(0) + \varphi_1) \\ \omega_2 C_2 cos(\omega_2(0) + \varphi_2) \end{Bmatrix}$

Cramer's Rule:

$C_1 cos(\varphi_1) = 0; \quad \varphi_1 = \frac{\pi}{2}$

$C_1 sin(\varphi_1) = \frac{\begin{vmatrix} 10 & 1 \\ 0 & -0.62 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1.62 & -0.62 \end{vmatrix}} = 2.77$

$C_2 sin(\varphi_2) = \frac{\begin{vmatrix} 1 & 10 \\ 1.62 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1.62 & -0.62 \end{vmatrix}} = 7.23$

$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1.62 \end{bmatrix}2.77sin\left(\omega_1 t + \frac{\pi}{2}\right) + \begin{bmatrix} 1 \\ -0.62 \end{bmatrix}7.23sin\left(\omega_1 t + \frac{\pi}{2}\right)$

$q_1(t) = 2.77sin\left(\omega_1 t + \frac{\pi}{2}\right); \ q_2(t) = 7.23sin\left(\omega_1 t + \frac{\pi}{2}\right)$

** **If there is a forcing input analyze with the same method.. Apply newton's 2nd law and do from beginning.**

Principle of Virtual Displacement ($\delta W_{PVD}=0$)

External Force *Virtual displacement $P_1\delta\Delta_1+P_2\delta\Delta_2=0$

(i) Determine the external applied force P_1, P_2 ...

(ii) Displacement (X), (Y) travelled by force (use trigo or Pythagoras)

(iii) Find virtual $\delta x=\frac{dx}{dy}$ or $\delta y=\frac{dy}{dx}$; chose either by reference one

Note: if the force is opposing the direction, (-'ve)

Principle of Virtual Work ($\delta W_{external}=\delta U_{internal}$)

i.e. $F\delta x+mg\delta x=F\delta e$; $F\delta x+mg\delta x=ke\delta e$

(i) $\delta W_{external}$ = same analysis as PVD

(ii) Determine the spring force, $\delta U=F\delta e$ $\quad F_{spring}=ke_{extension}$

(iii) Determine the extension spring, e = stretched – unstretched

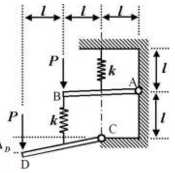
(iv) Find virtual $\delta e=\frac{de}{d\theta \text{ or } dx}$ Note: Torsion Spring: $K=\frac{M \text{ or } Torque}{\theta}$

Principle of Virtual Complementary Work ($\delta W^{*}_{external}=\delta U^{*}_{internal}$)

i.e. $\Delta_A\delta P_A+\Delta_B\delta P_B=e_1\delta F_1+e_2\delta F_2$

$$\Delta_A\delta P_A+\Delta_B\delta P_B=\frac{F_1}{k_1}\delta F_1+\frac{F_2}{k_2}\delta F_2$$

Determine the forces by Moment:



$\Delta_D\delta P_2+\Delta_B\delta P_1=\frac{F_1}{k_1}\delta F_1+\frac{F_2}{k_2}\delta F_2$

At Lever C: $M_C=0$ (Real Force)

$2LP_2-F_2L=0;\quad F_2=2P_2$

(Virtual Force): Differentiate real Force

$\delta F_2=2\delta P_2$

Since question state for deflection at D, so apply unit load $P_2=1$

$\#\delta F_2=2(1)$

At Lever A: $M_A=0$ (Real Force)

$2LP_1+2F_2L=F_1L;\quad$ Founded: $F_2=2P_2$

$2LP_1+4P_2L=F_1L;\quad 6LP=F_1L;\quad F_1=6P\;$: For $P_1=P_2$

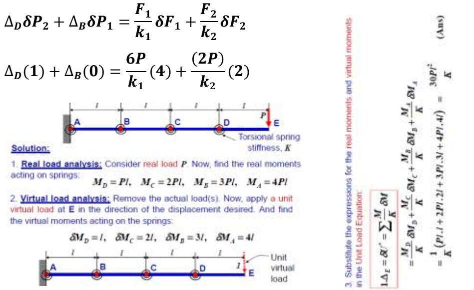
(Virtual Force): Differentiate real Force

$2L\delta P_1+2\delta F_2L=\delta F_1L\;\rightarrow \delta P_1=0$ (unit load at D)

$2\delta F_2L=\delta F_1L$; $\quad 2(2\delta P_2)L=\delta F_1L$; $(4\delta P_2)=\delta F_1$; $\quad 4(1)=\delta F_1$

$\Delta_D\delta P_2+\Delta_B\delta P_1=\frac{F_1}{k_1}\delta F_1+\frac{F_2}{k_2}\delta F_2$

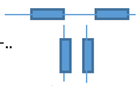
$\Delta_D(1)+\Delta_B(0)=\frac{6P}{k_1}(4)+\frac{(2P)}{k_2}(2)$

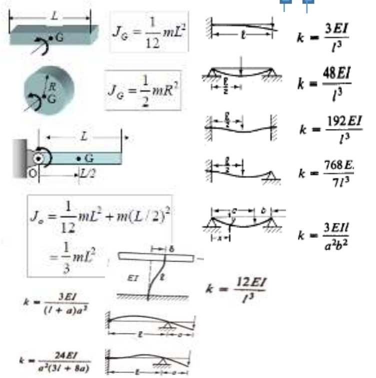


Springs: $K=\frac{M}{\theta}=K=\frac{Torque}{\theta}=K=\frac{F}{e}$ $\quad$ Mass Moment of inertia

series $k_{effec}=k_1+k_2+..$

parallel $\frac{1}{k_{effec}}=\frac{1}{k_1}+\frac{1}{k_2}+..$





Unit Load Method: Elastic Material ($\Delta_i(1)=\delta U^*$) *Note: thin beam, axial and direct shear = 0

$$\Delta_i(1)=\left(\int_0^l\frac{P(p)}{EA}dx\right)_{ax}+\left(\int_0^l\frac{M(m)}{EI}dx\right)_{Ben}+\left(\int_0^l\frac{Q(q)}{GA}dx\right)_{she}+\left(\int_0^l\frac{T(t)}{GJ}dx\right)_{Tors}+\left(\frac{Ff}{K}\right)_{sp}+\left(\frac{Mm}{K}\right)_{spr}$$

$$\Delta_{MAX}=\Delta_{Static}\left[1+\sqrt{1+\frac{2h}{\Delta S}}\right]_{impact}$$

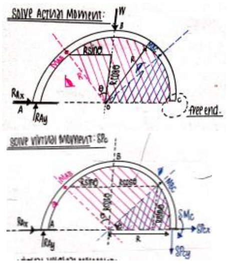
NOTE: If it involve an angular analysis, $dx=Rd\theta=radian$

Steps:

(i) (M) Real Load Analysis $\sum M=0$ and $\sum F_y=0$ solve

(ii) (m) Virtual load Analysis: only consider the force (δP_i) at point of deflection

Example: Determine the vertical and horizontal deflection at C:



$$\Delta_{vertical\,(C)}(1)=\int_{AB}\frac{M(m)}{EI}dx+\int_{BC}\frac{M(m)}{EI}dx$$

Real Load Analysis: $\sum M_{AB}=0$; $W(R\sin\theta)$ and $\sum M_{BC}=0$ (free end);

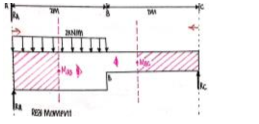
Virtual (Vertical) Load Analysis: (Put imaginary F at where u want Δ

$m_{AB\_Y}=(R+R\sin\theta)\delta P_{CY}=(R+R\sin\theta)\cdot 1$

$m_{BC\_Y}=(R-R\cos\theta)\delta P_{CY}=(R-R\cos\theta)\cdot 1$

$$\Delta_{vertical\,(C)}(1)=\int_0^{\frac{\pi}{2}}\frac{WR\sin\theta(R+R\sin\theta)}{EI}(Rd\theta)+\int_0^{\frac{\pi}{2}}\frac{0(R-R\cos\theta)}{EI}(Rd\theta)$$

IMPT to Mark the reference axis to analyze the case: $\quad$ NOTE: Analyze of same direction (M), (m)

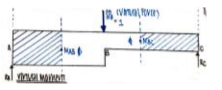


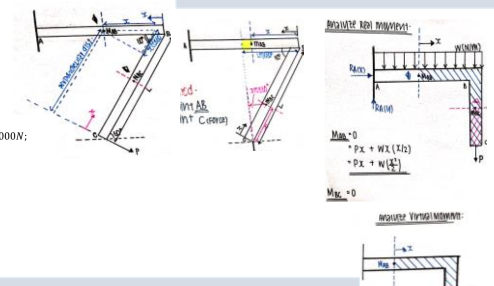
$\sum M_c=0; 4R_A=(2000(2)*(2+1))=0;\; R_A=3000N;$

$\sum F_y=0;\; R_A+R_c=(2000)(2);\; R_c=1000N$

$\sum M_{AB}=0=R_Ax-2000x\left(\frac{x}{2}\right)=3000x-1000x^2$

$\sum M_{BC}=0=R_cx=1000x$



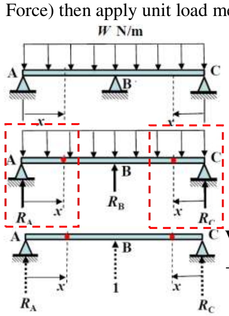


Put finger at point Moment point. Put Pen at Force

$1.\Delta_i=\delta U^*=\sum e\,\delta F=\sum\frac{F}{K}\delta F$ $\quad$ $1.\Delta_i=\delta U^*=\sum\theta\,\delta M=\sum\frac{M}{K}\delta M$

Statically Indeterminate (Degree of Indeterminacy= no. of unknown support reaction – 3 = 1)

Solve for the hinted force using unit load method, Equate to (0) and obtain (i.e. Horizontal Force) then apply unit load method for vertical component and solve for deflection.



Moment Analysis: $\sum M_A(CW)=0$

$W(L)\left(\frac{L}{2}\right)-R_B\left(\frac{L}{2}\right)-R_C(L)=0,$

$R_C=W\left(\frac{L}{2}\right)-\left(\frac{LR_B}{2}\right)$

Force Analysis: $\sum F_Y=0$

$W(L)=R_A+R_B+R_c$

$R_A=WL-R_B-R_c$

$R_A=WL-R_B-\left(W\left(\frac{L}{2}\right)-\left(\frac{LR_B}{2}\right)\right)$

REAL Load Analysis:

$\sum M_{AB}(CW)=R_A(x)-(Wx)(x/2)$

$\sum M_{BC}(CW)=-R_C(x)+(Wx)(x/2)$

Virtual Moment Analysis: $\sum m_A(CW)=0$

$-\delta P_B\left(\frac{L}{2}\right)=R_B(L);\quad R_c=-\frac{\delta P_B}{2},$

Virtual Force Analysis: $\sum F_y=0$

$R_A+R_c+\delta P_B=0,$

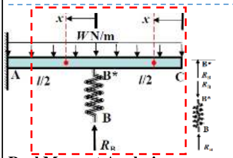
$R_A=1-R_c=-\frac{\delta P_B}{2}$

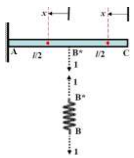
Virtual Load Analysis:

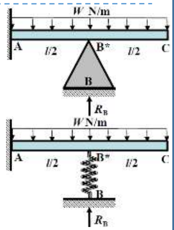
$\sum m_{AB}(CW)=R_A(x)=-\frac{\delta P_B}{2}(x)=-\frac{x}{2}$

$\sum m(CW)=-R_c(x)=\frac{\delta P_B}{2}(x)=\frac{x}{2}$

Exclude the support Forces at A







Real Moment Analysis:

$\sum M_{AB}(CW)=W\left(\frac{L}{2}+x\right)\left(\frac{1}{2}\left(\frac{L}{2}+x\right)\right)-R_B(x)$

$\sum M_{BC}(CW)=W(x)\left(\frac{x}{2}\right)$

Spring Force (Compression)

$F_B=-R_B$

Virtual Analysis:

$\sum m_{AB}(CW)=\delta R_B(x)=x$

$\sum m_{BC}(CW)=0$

Spring Force (tensile)

$f_B=+\delta P_B=1$

Linear Elastic Fracture: (Brittle):

Plane stress: k = 1
Plane strain: $k=(1-v^2)$

Griffith: (J/m^2) (only brittle)

SCF, $k_t=\frac{\sigma_{max}}{\sigma_{norm}}=1+2\left(\frac{a}{b}\right)$

$\sigma_{axial}=F/A$
$\sigma_{bending}=My_{N.A}/I$

$G_{strain\,E.release\,rate}=\frac{\pi a\sigma^2}{E}k$

Irwin's: SIF ($Nm^{-3/2}$) (ductile, brittle)

$K_c=\sigma(\pi a)^{1/2}Y,$ $\quad$ Edge: $Y=1.12$

$K_{Ic}=\sigma_{max}(\pi a_{critical})^{1/2}Y_{geometry}$ $\quad$ Penny: $Y=2/\pi$

Fracture toughness (compact tension), K_{Ic}

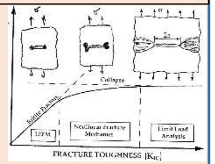
$K_Q=\frac{F_{Peak}}{BW^{\frac{1}{2}}}f_1\left(\frac{a}{W}\right)$

$B, a, (W-a)\geq 2.5\left(\frac{K_Q}{\sigma_{yield}}\right)^2_{geometry}$

$0.45<\left(\frac{a}{W}\right)<0.55_{pre-crack}$

Irwin-Griffith

$K_c=\left(\frac{EG_c}{k_{stress/strain}}\right)^{\frac{1}{2}}$



J-Integral is a line contour surrounding the crack tip, It's the energy release per unit area of crack ext (J/m^2)

$J_c\equiv G_c;(EG_c)^{\frac{1}{2}}=K_c$

$J=\int_c\left\{wdy-T\left(\frac{\partial u}{\partial x}\right)\right\}ds$

$J=-\frac{1}{B}\frac{\partial U}{\partial a}=-\frac{P^2}{2B}\left(\frac{\partial C}{\partial a}\right);$

$J=\frac{2U}{B(W-a)};\quad U=Joules/m^2$

Plastic Zone Consideration

1. $K_c=\sigma(\pi a'_{effective})^{1/2}Y$

$a'_{effective}=a+r_{plastic}$

$r_{p,(stress)}=\frac{1}{2\pi}\left(\frac{K}{\sigma_{yield}}\right)^2 \qquad r_{p,(strain)}=\frac{1}{2\pi}\left(\frac{K}{\sqrt{3}\sigma_{yield}}\right)^2$

Plastic zone valid if (2a)/(diameter plastic) < 10

2. Crack Opening Displacement (CTOD)

$\delta_{critical}=\delta_{elastic}+\delta_{plastic};\quad \lambda$=1 or 2

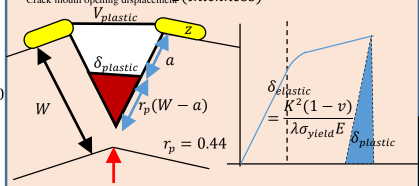
$\delta_{critical}=\frac{K_c^2}{\lambda\sigma_YE}$

$=\left(\frac{K_c^{\,2}(1-v^2)}{\lambda\sigma_{yield}E}\right)_{elastic}+\left(\frac{v_p\left(r_p(w-a)\right)}{r_p(w-a)+a+z}\right)_{plastic}$

$K_c=\frac{F_{Peak}}{BW^{\frac{1}{2}}}f_1\left(\frac{a}{W}\right)\rightarrow$ sub inside elastic

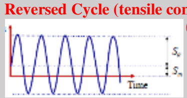
3. Cylindrical Questions: (pressure vessels)

$\sigma_\theta=\sigma_{permissible}=\frac{(Pressure)(diamter)}{2(thickness)}$



Reversed Cycle (tensile compressive loading)

$0<S_{Mean}>0$

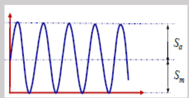


$S_{amplitude}=\frac{(\sigma_{max}-\sigma_{min})}{2}$

$S_{range}=\sigma_{max}-\sigma_{min}$

Fluctuating Cycle (zero tensile loading)

$S_{Mean}=S_{amplitude}$

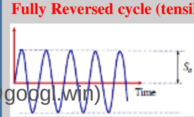


$S_{Mean}=\frac{(\sigma_{max}+\sigma_{min})}{2}$

$S_{ratio}=\frac{\sigma_{min}}{\sigma_{max}}$

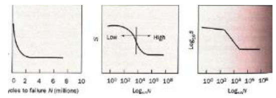
Fully Reversed cycle (tensile compressive loading)

$S_{Mean}=0$



Specimen of high fatigue cycles (long life to failure N > 10^3 and is under low nominal stress.

S-N Curve (Stress amplitude Vs Number of cycles): Stress amplitude must be below endurance stress (limit) or fatigue strength to ensure it doesn't fail by fatigue to achieve infinite life. A. **HIGH cycle fatigue** has high frequency and low amplitude stress or strain. Cycles is 10^4 and above to fail. (i.e. air wings, crankshaft, suspension springs). **LOW cycle fatigue** is low frequency and high amplitude strain or stress. Failure N < 10^3. Plastic deformation (i.e. fuselage)



Modification factor for Endurance Limit

$S_{e(modif)}=S_{e(test)}C_{size}C_{load}C_{surf\,finish}$

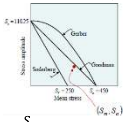
$C_{size}=1\,(axial, rod\; d<8mm)\,1.189d^{-0.097}$

$C_{load}=1\,(bend), 0.7(axial), 0577\,(torsion)$

$Factor\ of\ safety\ (against\ failure)=\frac{Endurance\ limit\ (modified)}{Stress\ amplitude}$

Fatigue Strength Reduction Factor (K_f), notches, groves causes geometric discontinuity giving stress concentration which affect endurance limit. Hence,

$S_{e(corrected)}=\frac{S_{e(test)}C_{size}C_{load}C_{surf\,finish}}{K_f};$



$q_{notch\ sensitivity\ factor\ (0<q<1)}=\frac{K_f-1}{K_t-1};\; K_t=elastic\ SCF$

Effect of Mean Stress:

Goodman: $\frac{S_a}{S_e}+\frac{S_m}{S_u}>1$ Gerber: $\frac{S_a}{S_e}+\left(\frac{S_m}{S_u}\right)^2>1$; Soderberg : $\frac{S_a}{S_e}+\frac{S_m}{S_{yield}}>1$

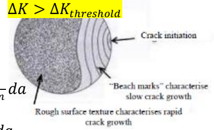
Given fluctuating axial load, $10^5+10^4\sin 50t$ $\;10^5$: mean stress and 10^4 amplitude stress

Mechanism of Fatigue. Crack initiation: stress value close to endurance limit, slip bands are formed due to shear effect, with cyclic loading the slips expands and two layers are formed known as cracks. Crack Propagation, in tensile load, the stress will stretch the crack tip and causes a plastic deformation.but during compressive, the crack is squeezed to shut off. Analysis of types of fatigue. (i) Crack initiated, (ii) beach marks represent cyclic loading slow crack growth (iii) rough surface, rapid crack growth leading to failure.

Crack Growth Rate: PARIS'S

To use Paris's crack propagation, check $\Delta K>\Delta K_{threshold}$

$\frac{da(mm)}{dN(cycles)}=A(\Delta K)^m$



$\int dN=\int\frac{1}{A(\Delta\sigma)^m(\pi)^{m/2}(\Delta a)^{m/2}(Y)^m}da$

$=\frac{1}{A(\Delta\sigma)^m(\pi)^{m/2}(Y)^m}\int_{a_{initi}}^{a_{crit}}(\Delta a)^{-\left(\frac{m}{2}\right)}da$

$N(cycles)=\frac{2}{A(\Delta\sigma)^m(\pi)^{\frac{m}{2}}(Y)^m(m-2)}\left((\Delta a)^{-\left(\frac{m-2}{2}\right)}\right)_{a_{final}}^{a_{initial}}$

$$N(cycles)=\frac{2}{A(\Delta\sigma)^m(\pi)^{\frac{m}{2}}(Y)^m(m-2)}\left(\frac{1}{a_{initial}^{\left(\frac{m-2}{2}\right)}-a_{final}^{\left(\frac{m-2}{2}\right)}}\right)$$

MINER's Cumulative Fatigue Damage, Note, n is operation of cycles taken base on load fluctuation and N is taken from SN curve for its no of cycles to failure

$D=\frac{n_1}{N_1}+\frac{n_2}{N_2}+\frac{n_3}{N_3}\ldots$ and $D=1\,(failure)$

No of hours to failure $=\frac{1}{D}blocks*hours$

Multiple axial loading, check using the Goodman, Soderberg etc.

$\overline{\sigma_{amplitude}}=\frac{1}{\sqrt{2}}\left(\sqrt{(\sigma_{xa}-\sigma_{ya})^2+(\sigma_{ya}-\sigma_{za})^2+(\sigma_{za}-\sigma_{xa})^2+6(\tau_{xya}^2+\tau_{yza}^2+\tau_{zxa}^2)}\right)$

$\overline{\sigma_{mean}}=\sigma_{xm}+\sigma_{ym}+\sigma_{zm}$

Math formula:

$csc=1/sin$	$sec=1/cos$	$cot=1/tan$	$tan=sin/cos$
$sin2A=2sinAcosA$		$cos2A=1-2\sin^2A$	
$\sec^2A=1+\tan^2A$		$\sin^2A+\cos^2A=1$	
$\frac{d}{dx}sinu=cosu\left(\frac{du}{dx}\right)$	$\frac{d}{dx}cosu=-sinu\left(\frac{du}{dx}\right)$	$\frac{d}{dx}tan=\sec^2u\left(\frac{du}{dx}\right)$	
$\frac{d}{dx}\sec(u)=\sec(u)\tan(u)\left(\frac{du}{dx}\right)$		$\frac{d}{dx}cot=-\csc^2u\left(\frac{du}{dx}\right)$	
$\frac{d}{dx}\csc(u)=-csc(u)\cot(u)\left(\frac{du}{dx}\right)$		$\frac{d}{dx}\left(\frac{1}{sinu}\right)=\frac{1}{\sqrt{1-U^2}}\left(\frac{du}{dx}\right)$	
$\frac{d}{dx}\left(\frac{1}{cosu}\right)=-\frac{1}{\sqrt{1-U^2}}\left(\frac{du}{dx}\right)$		$\frac{d}{dx}\left(\frac{1}{tan}\right)=\frac{1}{1+u^2}\left(\frac{du}{dx}\right)$	
$\frac{d}{dx}(e^u)=e^u\left(\frac{du}{dx}\right)$		$\frac{d}{dx}\left(\frac{1}{sinu}\right)=\frac{1}{\sqrt{1-U^2}}\left(\frac{du}{dx}\right)$	