3005 Cheatsheet - Summary Control Theory

MA3005 Control Theory Reference Sheet (Name: Seat Number:)

Laplace transforms and partial fraction table

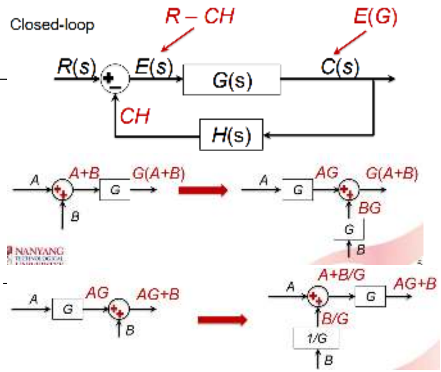
$f(t)$	$F(s)$
Impulse $\delta(t)$	1
Step $u_s(t)$	$1/s$
t^n	$n!/s^{n+1}$
$t^{n-1}e^{-at}/(n+1)!$	$1/(s+a)^n$
$f(t)e^{-at}$	$F(s+a)$
$f(t-t_0)u_s(t-t_0)$	$F(s)e^{-st_0}$
$\frac{d^n f(t)}{dt^n}$	$s^nF(s)-s^{n-1}f(0^-)-\dots f^{(n-1)}(0^-)$
$\sin(at)$	a/s^2+a^2
$\cos(at)$	s/s^2+a^2
$\lim_{t\to\infty} f(t)$ (if stable)	Final value $=\lim_{s\to 0} sF(s)$
$\lim_{t\to 0} f(t)$ (if exists)	Initial value $=\lim_{s\to\infty} sF(s)$

$\frac{1}{(b-a)}\left(e^{-at}-e^{-bt}\right)$	$\frac{1}{(s+a)(s+b)}$
$\frac{1}{ab}\left[1+\frac{1}{(a-b)}\left(be^{-at}-ae^{-bt}\right)\right]$	$\frac{1}{s(s+a)(s+b)}$
$\frac{c-s_1}{s_2-s_1}\cdot e^{-s_1t}-\frac{c-s_2}{s_2-s_1}\cdot e^{-s_2t}$	$\frac{s+c}{(s+s_1)(s+s_2)}$
$\frac{\omega_n}{\sqrt{1-\zeta^2}}\cdot e^{-\zeta\omega_n t}\cdot\sin\left(\omega_n\sqrt{1-\zeta^2}\,t\right), \zeta<1$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
$\frac{-1}{\sqrt{1-\zeta^2}}\cdot e^{-\zeta\omega_n t}\cdot\sin\left(\omega_n\sqrt{1-\zeta^2}\,t-\phi\right)$, $\tan(\phi)=\frac{\sqrt{1-\zeta^2}}{\zeta}, \zeta<1$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
$1-\frac{1}{\sqrt{1-\zeta^2}}\cdot e^{-\zeta\omega_n t}\cdot\sin\left(\omega_n\sqrt{1-\zeta^2}\,t+\phi\right)$, $\tan(\phi)=\frac{\sqrt{1-\zeta^2}}{\zeta}, \zeta<1$	$\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\dots+\frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\dots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}$

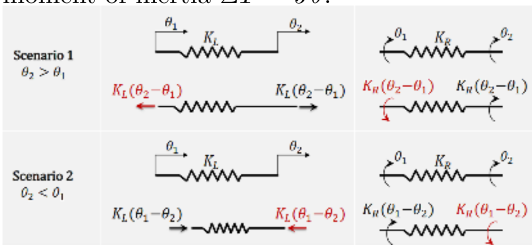
Block diagram

Multiply blocks together. Swap summing junctions.



Mechanical systems

Separate mechanical elements. Analyse forces on elements. Use FBD to write down motion equation. Take Laplace transform. Spring: $F=k\theta$, damper $F=c\dot{\theta}$, moment of inertia $\Sigma T=J\ddot{\theta}$.



Responses and error

Error $E(s)=\text{Input}(s)-\text{Output}(s)$
DC gain is steady state response to unit step input.

Stability criterion

Consider CE $s^4+8s^3+32s^2+80s+100$. Note: not stable if signs are different in CE already.

s^4:	1	32	100
s^3:	8	80	
s^2:	$\frac{8\times 32-1\times 80}{8}$	$\frac{8\times 100}{8}$	

Stable system if no sign change and no special case for leftmost column. If sign change, unstable. Reverse coefficients and resolve if stuck. (Case 1) If first element of any row is 0, replace with small ϵ. If no sign change, then marginally stable and pair of roots on Im. Else, unstable. (Case 2) Whole row of 0s, differentiate (d/ds) the last non-zero row and form auxiliary polynomial. Replace the zero row with it. This case has complex conjugate pair.

First order system: $(K/\tau)/(s+1/\tau)$

The time constant τ is time taken to drop 63.2%, 4τ is 98.2%. For unit ramp response, at steady state the error is τ.

Second order system: $(K\omega_n^2)/(s^2+2\zeta\omega_n s+\omega_n^2)$

Attenuation (real component) $\sigma=\zeta\omega_n$
Poles: $s=-\zeta\omega_n\pm\omega_n\sqrt{\zeta^2-1}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}$
Complex plane angle $\beta=\cos^{-1}\zeta$
Damped frequency (imaginary component) $\omega_d=\omega_n\sqrt{1-\zeta^2}$
Maximum overshoot percentage (e.g. 20% MP is 1.2 times SS) $M_p=100\exp(-\sigma\pi/\omega_d)$
2% settling time $t_2=4/\zeta\omega_n$, 5% settling time $t_5=3/\zeta\omega_n$
Delay time is time to reach half of steady state. Rise time to reach 100% of steady state $t_r=\pi-\beta/\omega_d$
Peak time to reach 1st peak and is period of damped oscillation $t_p=\pi/\omega_d$
Damping ratio $\zeta=\frac{-\ln(M_p/100)}{\sqrt{\pi^2+\ln^2(M_p/100)}}$

Static error constants (unity feedback)

As system type increases, steady state error reduces, stability reduces. $G(s)$ is open loop NOT CLOSED LOOP. $K_p=\lim_{s\to 0}G(s)$, $K_v=\lim_{s\to 0}sG(s)$, $K_a=\lim_{s\to 0}s^2G(s)$. Cells in this table give steady state error:

Type	Unit step in	Unit ramp in	Unit parabolic in
0	$1/(1+K_p)$	∞	∞
1	0	$1/K_v$	∞
2	0	0	$1/K_a$

Root loci (for K controller, symmetric about Re(s))

Transfer function form $KG(s)/(1+KG(s)H(s))$, note: denominator is CE
(1.0) Starts at open loop poles (x), ends at open loop zeroes (o).
(2.0) If there are m zeroes and n poles, then there are (n-m) infinity open loop zeroes.
(2.1) Asymptote angle: $\angle s=\pi(1+2q)/(n-m)$ for $q=0,\dots n-m$

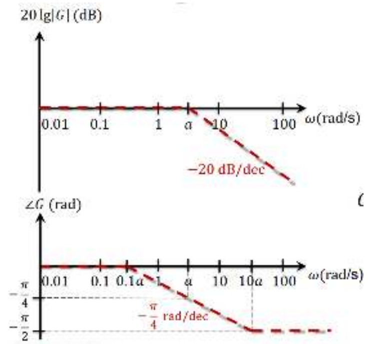
(2.2) Asymptote location:
$\sigma_a = \sum_i^n(-p_i) - \sum_j^m(-z_j)/(n-m)$. **Eg: For pole (s+3), you put in -3 here**.
(3.0) To find locus on real axis, label pole or zero on real axis from right to left 1, 2, 3... In between 1-2, 3-4, ... there is locus!
(3.1) To find break points, find asymptote. Then use $dK/ds = 0$ where K is gain from CE.
(3.2) In-out determined from geometry. In for min K, Out for max K.
(4.0) Phase criterion for departure/arrival angles:
$\sum_i^m(\angle s + z_i) - \sum_j^n((\angle s + p_i)) = \pm\pi$
(5.0) To find intersection with imaginary axis, substiute $s = j\omega$ into CE. Then form real part and imaginary parts of CE. Both parts are equal to 0. This yields simultaneous equations to solve.

PID and Compensators

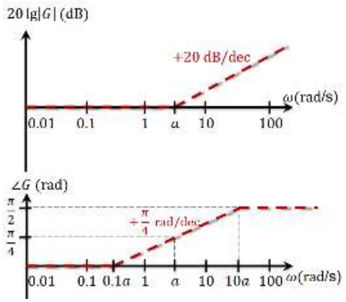
For lead compensators $K(s+z_c)/(s+p_c)$: design PD first with $K_D(s+z_c')$ and $p_c > z_c$. Specify $z_c' > z_c$. Then use CE to find gain K and pole p_c
PD controller: $K_D(s+z)$, use $s - \zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ to find desired roots. Find CE, solve Im then Re component $= 0$ to get z and K_D
PI controller: $K_D(s+z)/s$, pick small z to cancel zero.
PID controller: $[K_D(s+z_1)(s+z_2)]/s$.
Lag compensators: $K(s+z_c)/(s+p_c)$ and $p_c << z_c << 1$.
Lead-lag: $[K_1(s+z_1)/(s+p_1)][K_2(s+z_2)/(s+p_2)$
Proportional don't stabilise. PD, lead, PID, lead-lag can usually stabilise with high gain, PI and lag only add pole at origin to upgrade system type, not stabilise.

Bode plot sketch

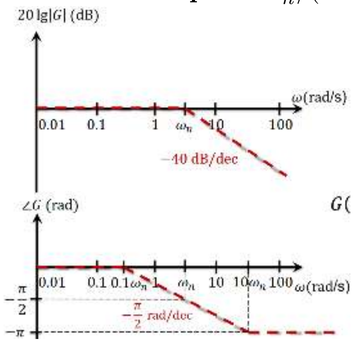
Harmonic steady state: $r_0|G(j\omega)|\cos(\omega t + \angle G(j\omega))$
Angle combination:
$\sum_{i=1}^n \tan^{-1}(\omega/z_i) - \sum_{j=1}^m \tan^{-1}(\omega/p_i)$
Magnitude combination: $20\lg|G| =$
$20\lg K + \sum_{i=1}^n 20\lg(\sqrt{\omega^2 + z_i^2}) + \sum_{j=1}^m 20\lg(\sqrt{\omega^2 + p_j^2})$
Constant: magnitude $20\lg|G|$, no phase.
Differentiator: +20 dB/dec upward sloping line. Find magnitude intercept with $20\lg|G|$. Phase is CONSTANT $\pi/2$.
Integrator: -20 dB/dec downward sloping line. Find magnitude intercept with $20\lg|G|$. Phase is CONSTANT $-\pi/2$.
First order pole: $a/(s+a)$



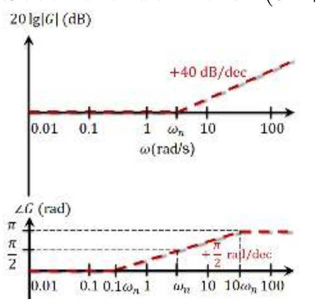
First order zero: $(s+a)/(a)$



Second order pole: $\omega_n^2/(s^2 + 2\omega_n\zeta s + \omega_n^2)$



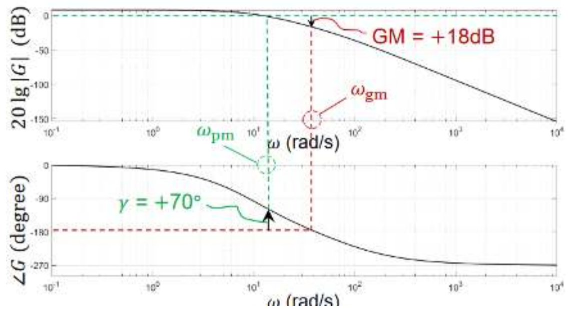
Second order zero: $(s^2 + 2\omega_n\zeta s + \omega_n^2)/\omega_n^2$



For second order system, smaller damping ζ means sharper resonance peak which means steeper drop in $\angle G$. Resonance frequency is $\omega_r = \omega_n\sqrt{1-2\zeta^2}$. **Bode plot analysis based on RESONANCE FREQUENCY.**

Bode plot analysis

Gain margin at ω_G at $\angle G = -\pi$, is $|0 - G(j\omega_G)|$
Phase margin at ω_P at $|G| = 0\,dB$, is $|\angle G(j\omega_P) - (-\pi)|$



Unity feedback is stable if both phase and gain margins are positive.
Bandwidth at ω_B at $(X - 3\,dB)$, and X is initial gain.

$$\gamma = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}};\ \text{or}\ \zeta = \frac{\tan(\gamma)}{2\left(\sqrt[4]{1+\tan^2(\gamma)}\right)}$$

$$\omega_b = \omega_n\sqrt{\left(1-2\zeta^2\right) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

Freq. at intercept at 0 dB/dec line gives K_P. Note $20\lg(K_P)$ Freq. at intercept at -20 dB/dec line gives K_v. Freq. at intercept at -40 dB/dec line gives K_a^2.