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Mathematical Methods In Engineering (Nanyang Technological University)

MA3004 Mathematical Methods in Engineering

Part I – Partial Differential Equations

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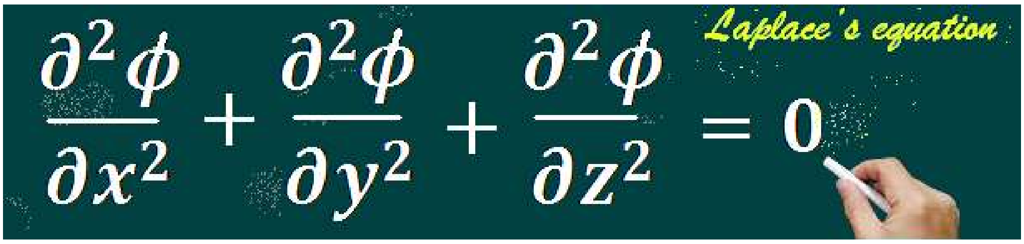


Reference books

1. Erwin Kreyszig, *Advanced Engineering Mathematics*, 10th Edition, Wiley.

2. James Brown and Ruel Churchill, *Fourier Series and Boundary Value Problems*, 8th Edition, McGraw-Hill.

Contents

What is a partial differential equation? Why is it important?

A partial differential equation (PDE) is an equation containing partial derivative(s) of unknown (yet to be determined) function(s) of two or more variables.

An example of a famous PDE in one unknown function $\phi(x, y, z)$ (a function of three Cartesian coordinates x, y and z) is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

The PDE above contains the second order partial derivatives $\dfrac{\partial^2\phi}{\partial x^2}$, $\dfrac{\partial^2\phi}{\partial y^2}$ and $\dfrac{\partial^2\phi}{\partial z^2}$. Known as the *Laplace's equation*, it arises in the formulation of many problems in physical and engineering sciences.

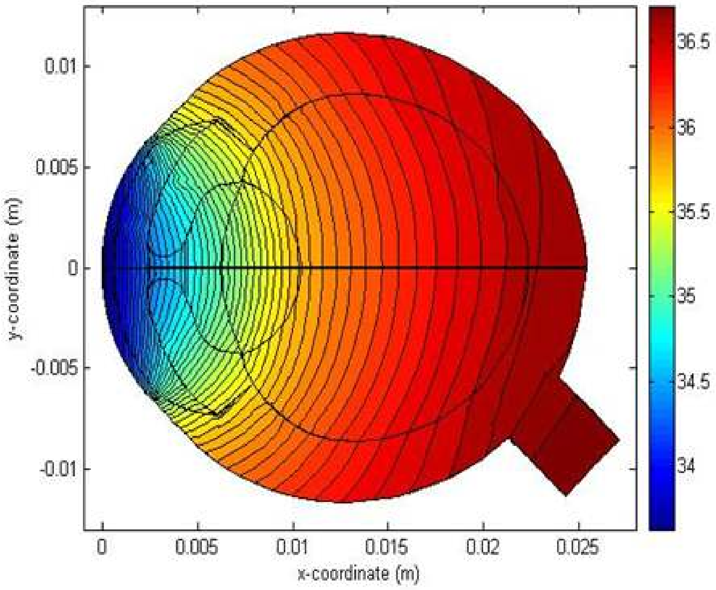
Here is a trivial matter concerning notation. The partial derivatives $\dfrac{\partial^2\phi}{\partial x^2}$, $\dfrac{\partial^2\phi}{\partial y^2}$ and $\dfrac{\partial^2\phi}{\partial z^2}$ are also denoted by ϕ_{xx}, ϕ_{yy} and ϕ_{zz} respectively. The Laplace's equation above may also be written as

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0.$$

In the linear theory of heat conduction, if $\phi(x, y, z)$ is the steady state temperature at the point (x, y, z) in a solid then, under certain assumptions, the conservation of energy requires $\phi(x, y, z)$ to satisfy the Laplace's equation everywhere inside the solid.

Shown below is the temperature contour on a cross section of the human eye, which is obtained by solving the Laplace's equation. Details on how the Laplace's equation is solved to calculate the temperature inside the human eye are given in the following research article.

E. H. Ooi, W. T. Ang and E. Y. K. Ng, Bioheat transfer in the human eye: A boundary element approach, *Engineering Analysis with Boundary Elements* **31** (2007) 494-500.



Another example where the Laplace's equation occurs is in the flow of an ideal fluid. In such a fluid flow, the principle of conservation of mass requires that the velocity potential $\phi(x, y, z)$ satisfies the Laplace's equation everywhere in the fluid.

In engineering science, the unknown functions in the PDEs are usually functions of coordinates of points in space, such as Cartesian coordinates x, y and z, and possibly also time t. The PDEs are essentially mathematical statements of the laws of physics which govern the system under consideration (for example, conservation of energy) $-$ the PDEs are derived from physical laws.

A few other well known PDEs in physical and engineering sciences are given below.

Diffusion equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \alpha\frac{\partial\phi}{\partial t} \text{ } (\alpha \text{ is a positive real constant}) \qquad (1)$$

Here ϕ is a function of the Cartesian coordinates x, y and z and time t.

The transportation of matter in space as a result of random molecular motions may be governed by the diffusion equation, with ϕ being the concentration of the diffusing substance at the point (x, y, z) at time t. The diffusion equation arises from the conservation of mass.

The diffusion equation is also known as the *heat equation*, as it may also be used to describe heat flow due to random molecular motions. In heat conduction, the PDE is derived from the conservation of energy and ϕ gives the temperature. For steady state heat conduction, ϕ is independent of t and (1) reduces to the Laplace's equation which we have mentioned earlier on.

Wave equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} \text{ } (c \text{ is a positive real constant}) \qquad (2)$$

In the propagation of sound wave in air, $\phi(x, y, z, t)$ is the air pressure at the point (x, y, z) at time t and c is the wave speed (speed of sound).

The wave equation may also be used to describe the vibration of an elastic membrane or a drum skin. If (x, y) denotes a point on the membrane when it (the membrane) is in equilibrium, the displacement of the point at time t is given by $\phi(x, y, t)$. Under certain assumptions, the displacement $\phi(x, y, t)$ satisfies the two-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. \tag{3}$$

Similarly, under certain assumptions, the vibration of an elastic string is governed by the one-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2}. \tag{4}$$

Here, when the string is in equilibrium, it is horizontal, lying on the x axis. The coordinate x refers to a point on the string when it is in equilibrium and $\phi(x, t)$ is the vertical displacement of the point x from its equilibrium position at time t. More on this later on.

Later on, we will learn how the one-dimensional wave equation in (4) can be derived from Newton's law of motion for a vibrating string and how the partial differential equation can be solved.

Helmholtz equation

If $\phi(x, y, z, t)$ in (2) is given by

$$\phi(x, y, z, t) = \psi(x, y, z) e^{i\omega t} \ (i = \sqrt{-1} \text{ and } \omega \text{ a real constant}),$$

we obtain what is called the Helmholtz equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{\omega^2}{c^2} \psi = 0. \tag{5}$$

Poisson equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = f(x, y, z) \tag{6}$$

(f is a given non-null real function of x, y and z)

Note that the definition of Poisson equation does not include the null function $f(x, y, z) = 0$. If $f(x, y, z) = 0$, the PDE in (6) is the Laplace's equation.

The Poisson equation has a *non-zero* term, namely $f(x, y, z)$, that does not contain the unknown function $\phi(x, y, z)$ or any of its partial derivatives. Because of this, the Poisson equation is said to be an *inhomogeneous* (*non-homogeneous*) PDE. The Laplace's equation and the PDEs in (1), (2), (3), (4) and (5) do not have such non-zero terms $-$ they are *homogeneous* PDEs.

PDEs for plane elastostatics

Plane elastostatic deformations of an isotropic material may be governed by the two simultaneous PDEs in two unknown functions $u(x, y)$ and $v(x, y)$:

$$\begin{aligned}(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + (\lambda + \mu)\frac{\partial^2 v}{\partial x \partial y} + \mu\frac{\partial^2 u}{\partial y^2} &= 0, \\ (\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + (\lambda + \mu)\frac{\partial^2 u}{\partial x \partial y} + \mu\frac{\partial^2 v}{\partial x^2} &= 0. \end{aligned} \tag{7}$$

Here λ and μ are the Lamé constants which may be expressed in terms of the Young modulus and Poisson ratio of the material, and the unknown functions $u(x, y)$ and $v(x, y)$ are respectively the x and y components of the displacement at the point (x, y) in the solid.

A PDE in ϕ is said to be of order N if the highest order of the partial derivatives of ϕ that appear in the PDE is N. Up to now, all the PDEs we have mentioned above are of order 2 (second order). Many PDEs in engineering applications are of order 2.

Here are two PDEs of an order other than 2 − one in solid mechanics and another in non-classical theory of heat conduction.

Biharmonic equation

The simultaneous PDEs in (7) may be reduced to a single PDE of the fourth order. The fourth order PDE is what is known as the two-dimensional *biharmonic equation* given by

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0. \tag{8}$$

The unknown function $\phi(x, y)$ in the biharmonic equation is known as the "Airy stress function" in solid mechanics − it is related to the partial derivatives of $u(x, y)$ and $v(x, y)$ in (7) in a certain way. We will not go into details here.

PDE in microscale heat conduction

If microscale phenomena are taken into consideration in the conduction of heat in solids, the temperature distribution $\phi(x, y, z, t)$ is governed by a PDE more complicated than the heat equation in (1). Under certain assumptions, the governing PDE for microscale heat conduction is a third order PDE of the form

$$\begin{aligned} & \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \tau_1\left(\frac{\partial^3 \phi}{\partial x^2 \partial t} + \frac{\partial^3 \phi}{\partial y^2 \partial t} + \frac{\partial^3 \phi}{\partial z^2 \partial t}\right) \\ = \; & \alpha\left(\frac{\partial \phi}{\partial t} + \tau_2 \frac{\partial^2 \phi}{\partial t^2}\right) \qquad (\alpha,\ \tau_1 \text{ and } \tau_2 \text{ are constants}). \end{aligned} \tag{9}$$

In the alternative notation for partial derivatives of multivariable functions, the microscale heat equation in (9) may be rewritten as

$$\phi_{xx} + \phi_{yy} + \phi_{zz} + \tau_1(\phi_{xxt} + \phi_{yyt} + \phi_{zzt}) = \alpha(\phi_t + \tau_2\phi_{tt}).$$

Note that $\phi_{xxt} = \phi_{txx} = \phi_{txt}$, $\phi_{yyt} = \phi_{tyy} = \phi_{tyt}$ and $\phi_{zzt} = \phi_{tzz} = \phi_{tzt}$ if ϕ is partially differentiable thrice with respect to its independent variables.

Homogeneous and nonhomogeneous linear PDEs and nonlinear PDEs

A PDE in ϕ is said to be linear if all its terms that contain ϕ or a partial derivative of ϕ can be written in the form given by either "(coefficient) $\times\phi$" or "(coefficient) $\times$ (a partial derivative of ϕ)", with the "coefficient" being a constant or a function of only the independent variables of ϕ (e.g. x and t). For a linear PDE, the "coefficient" should be independent of ϕ and its partial derivatives.

If a linear PDE has a non-zero term that does not contain ϕ or a partial derivative of ϕ, it is said to be inhomogeneous or nonhomogeneous. Otherwise, it is said to be homogeneous.

All the PDEs we have seen so far above (Laplace's equation, diffusion equation and so on) are linear. In all the linear PDEs above, the coefficients of the unknown function and its partial derivatives are constants. With the exception of Poisson equation, all of them are homogeneous. Below are some more examples of linear PDEs:

(a) $\dfrac{\partial\phi}{\partial x} + x\dfrac{\partial\phi}{\partial y} - 2x - 2xy = 0$ (nonhomogeneous linear PDE)

(b) $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + e^{-x-y}\phi = 1$ (nonhomogeneous linear PDE)

(c) $(x^2+y^2+z^2)(\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}) + 2x\dfrac{\partial\phi}{\partial x} + 2y\dfrac{\partial\phi}{\partial y} + 2z\dfrac{\partial\phi}{\partial z} = 0$

(homogeneous linear PDE)

If ϕ is a function of two variables denoted by x_1 and x_2 [x_i may represent a spatial coordinate such as the Cartesian coordinates x, y and z, or the time coordinate t, e.g. $(x_1, x_2) = (x, t)$ or $(x_1, x_2) = (x, y)$], then second order linear PDEs in $\phi(x_1, x_2)$ may be written in the general form:

$$\begin{aligned}A(x_1,x_2)\frac{\partial^2\phi}{\partial x_1^2} + 2B(x_1,x_2)\frac{\partial^2\phi}{\partial x_1\partial x_2} + C(x_1,x_2)\frac{\partial^2\phi}{\partial x_2^2}\\ +D(x_1,x_2)\frac{\partial\phi}{\partial x_1} + E(x_1,x_2)\frac{\partial\phi}{\partial x_2} + F(x_1,x_2)\phi = G(x_1,x_2).\end{aligned}$$

In general, the coefficients A, B, C, D, E, F and G in the PDE are functions of x_1 and x_2 (including constant functions).

Depending on the coefficients A, B and C, second order linear PDEs of the form above may be classified as *hyperbolic*, *parabolic* and *elliptic* types as follows.

1. If $AC - B^2 < 0$ for all x_1 and x_2 then the PDE is of the *hyperbolic* type.

2. If $AC - B^2 = 0$ for all x_1 and x_2 then the PDE is of the *parabolic* type.

3. If $AC - B^2 > 0$ for all x_1 and x_2 then the PDE is is of the *elliptic* type.

4. If $AC - B^2$ has different signs for different values of x_1 and x_2 then the PDE is of *mixed* type.

Examples

1. For the one-dimensional diffusion equation $\phi_{xx} = \alpha\phi_t$ $(\alpha > 0)$, if we take $x_1 = x$ and $x_2 = t$, then $A = 1$ and $B = C = 0$. Here $AC - B^2 = 0$. Thus, the one-dimensional diffusion equation is of the parabolic type.

2. For the one-dimensional wave equation $\phi_{xx} = \dfrac{1}{c^2}\phi_{tt}$ $(c > 0)$, if we take $x_1 = x$ and $x_2 = t$, then $A = 1$, $B = 0$ and $C = -c^{-2}$. This gives $AC - B^2 = -c^2 < 0$. Thus, the one-dimensional wave equation is of the hyperbolic type.

3. The two-dimensional Laplace's equation is of the elliptic type, since $AC - B^2 = 1 > 0$ with $A = C = 1$ and $B = 0$.

4. For the Tricomi equation $y\phi_{xx} + \phi_{yy}$, we find that $AC - B^2 = y$. Thus, the Tricomi equation is of mixed type, being hyperbolic in the region $y < 0$ and elliptic in the region $y > 0$.

The above classification of the second order linear PDEs has a mathematical purpose and it may be generalised to include other more general forms of PDEs in unknown functions of more than two independent variables. We will not go into further details on this here.

Nonlinear PDEs are PDEs that are not linear. Below are some examples of nonlinear PDEs:

(a) $\dfrac{\partial\phi}{\partial x} + 2\phi\dfrac{\partial\phi}{\partial y} = 0$

(b) $\phi(\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2}) + (\dfrac{\partial\phi}{\partial x})^2 + (\dfrac{\partial\phi}{\partial y})^2 = \dfrac{\partial\phi}{\partial t}$

(c) $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \sin(\phi)$

(d) $(\frac{\partial^2\phi}{\partial x^2})^5 + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$

Solutions of PDEs

Given a PDE in ϕ, we are interested in finding functions ϕ that satisfy it, that is, we are interested in solving it.

For the purpose of illustration, consider the following first order PDE in $\phi(x, y)$:

$$\frac{\partial\phi}{\partial x} = x + y. \tag{10}$$

This PDE may be easily solved for ϕ. To find ϕ, we just "un-do" the partial differentiation of ϕ with respect to x by treating y as a "constant" − we perform an indefinite integration with respect to x on both side of the PDE, that is,

$$\int \frac{\partial\phi}{\partial x} dx = \int (x + y) dx.$$

It follows that

$$\phi = \frac{1}{2}x^2 + yx + F(y), \tag{11}$$

where $F(y)$ is an arbitrary function of y. No matter what the function $F(y)$ is, (10) is satisfied by (11).

The example above shows that the solutions of a given PDE contain arbitrary functions. Recall that the general solution of an ordinary differential equation (ODE) contains arbitrary constants. PDEs are more complicated − their general solutions can be expressed in terms of arbitrary functions.

Example

By direct substitution, verify that each of the following functions is a solution of the two-dimensional Laplace's equation in $\phi(x, y)$.

(a) $\phi(x, y) = x^3 - 3xy^2 + x^2 - y^2$

(b) $\phi(x, y) = \ln((x-1)^2 + (y-3)^2)$ except at $(x, y) = (1, 3)$

(c) $\phi(x, y) = e^{-x}(\cos(y) - \sin(y))$

(d) $\phi(x, y) = \dfrac{x}{x^2 + y^2}$ except at $(x, y) = (0, 0)$

Solution:

This is nothing more than an exercise in partial differentiation of functions of several variables. We will show how the verification may be done for only one of the functions above − we will do part (d) only.

Differentiating $\phi(x, y) = \dfrac{x}{x^2 + y^2}$ partially twice with respect x:

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{(x^2 + y^2)(1) - x(2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \\
\Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} &= \frac{(x^2 + y^2)^2(-2x) - 2(y^2 - x^2)(2x)(x^2 + y^2)}{(x^2 + y^2)^4} \\
&= \frac{(x^2 + y^2)(2x^3 - 6xy^2)}{(x^2 + y^2)^4}
\end{aligned}$$

Differentiating partially ϕ with respect to y twice:

$$\begin{aligned}\frac{\partial\phi}{\partial y} &= -\frac{2xy}{(x^2+y^2)^2} \\ &\Rightarrow \frac{\partial^2\phi}{\partial y^2} = -\frac{(x^2+y^2)^2(2x)-(2xy)(2)(2y)(x^2+y^2)}{(x^2+y^2)^4} \\ &\qquad\quad = -\frac{(x^2+y^2)(2x^3-6xy^2)}{(x^2+y^2)^4}\end{aligned}$$

It follows that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2} = \frac{(x^2+y^2)(2x^3-6xy^2)}{(x^2+y^2)^4} - \frac{(x^2+y^2)(2x^3-6xy^2)}{(x^2+y^2)^4} = 0,$$

that is, $\phi(x,y) = \dfrac{x}{x^2+y^2}$ is a solution of the two-dimensional Laplace's equation everywhere, except at $(x,y)=(0,0)$ where ϕ is not well defined.

From the example above, it is obvious that many different solutions of various forms exist for the two-dimensional Laplace's equation. Many more different solutions, as many as we wish, can be written down.

Initial and boundary conditions: Boundary value problems and initial boundary value problems

In a physical problem governed by a PDE, the PDE is to be solved together with prescribed condition(s). If the physical problem is properly (well) posed, exactly *only one* solution PDE can be found to satisfy the prescribed condition(s), that is, the physical problem has a unique solution.

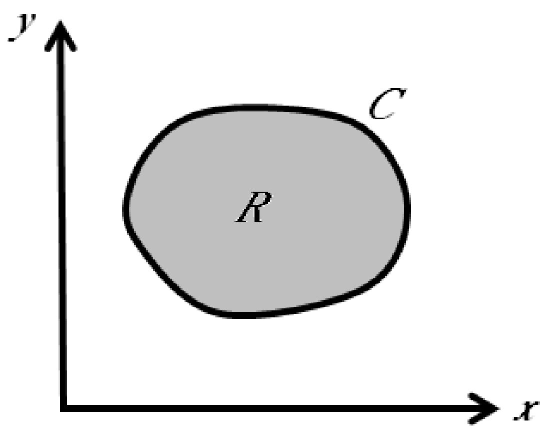
Recall that the arbitrary constant(s) in the general solution of an ODE is (are) determined by prescribed condition(s), that is, the condition(s) is (are) required to solve an ODE for a specific (particular) solution.

A PDE is to be solved in a physical domain defined by the set of all points in space where the PDE holds.

Examples

1. If we are solving the one-dimensional wave equation (4) for the displacement $\phi(x, t)$ of a vibrating string, then the physical solution domain is the interval $0 < x < L$ (on the x axis) where the string lies when it is in equilibrium. The endpoints of the string at $x = 0$ and $x = L$ define the boundary of the solution domain.

2. If we are solving the two-dimensional wave equation (3) for the displacement $\phi(x, y, t)$ of a vibrating membrane, then the physical solution domain is the two dimensional region R (on the Oxy plane) where the membrane lies when it is in equilbirum. In the figure below, R is bounded by the curve C. The curve C is the boundary of the solution domain R.



3. If we are solving the three-dimensional Laplace's equation for the steady state temperature $\phi(x, y, z)$ inside a spherical solid, then the physical solution domain is the set of all points inside the three-dimensional spherical region occupied by the solid. The boundary of the solution domain is the spherical surface bounding the spherical region.

A PDE in ϕ is solved subject to conditions in which ϕ or an expression involving ϕ and the spatial derivatives of ϕ is suitably prescribed at every point on the boundary of the physical solution domain. Such conditions are known as *boundary conditions* (BCs). BCs have physical meaning − they are prescribed based on physical consideration.

Examples

For continuity in discussion, refer to the corresponding examples listed on page 15.

1. For the vibrating string, if the string is fixed at its end points, that is, if the endpoints are not allowed to move, then the boundary conditions are $\phi(0, t) = 0$ and $\phi(L, t) = 0$ for all time $t \geq 0$.

2. For the vibrating membrane, if the boundary of the membrane is fixed (just like a drum skin is fixed along the edge of the drum), then the required boundary condition is given by $\phi(x, y, t) = 0$ at all points (x, y) on C and for all time $t \geq 0$.

3. Assume that the solid occupies the spherical region $x^2 + y^2 + z^2 < a^2$. If the upper portion of the spherical boundary where $x^2 + y^2 + z^2 = a^2$,

$z > 0$, is thermally insulated, and if the steady state temperature ϕ is known on the remaining part of the spherical boundary, then the three-dimensional Laplace's equation is to be solved for the temperature $\phi(x, y, z)$ inside the sphere subject to the boundary conditions

$$x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z} = 0 \text{ on } x^2 + y^2 + z^2 = a^2,\ z > 0,$$

$$\phi = f(x, y, z) \text{ on } x^2 + y^2 + z^2 = a^2,\ z < 0,$$

where $f(x, y, z)$ gives the known boundary temperature on the lower portion of the spherical boundary.

A boundary value problem (BVP) is a mathematical problem that requires solving a PDE in a physical domain subject to prescribed conditions on the boundary of the domain.

For time independent problems − those where the unknown functions in the governing PDEs do not change with time − the BCs alone, if properly prescribed, are sufficient to ensure that the problems are well posed having unique solutions. A large part of theoretical work on PDEs is devoted to analysing whether a given BVP is well posed or not. The Gauss-Ostrogradski theorem in your second year mathematics course may be used to prove that a BVP governed by the Laplace's equation is well posed with a unique solution if the unknown function ϕ is specified at all points on the boundary of the solution domain. Analyses for proving that BVPs are well posed having unique solutions are, in general, difficult and are beyond the scope of this course. In this course, we are mainly concerned with mathematical methods for solving BVPs which are assumed to be well posed with unique solutions.

For time dependent problems, such as those governed by the diffusion and the wave equations, the BCs alone are not sufficient to ensure unique solutions. Additional condition(s) specifying the state of the system under consideration at initial time $t = 0$ is (are) required. Such conditions are known as *initial conditions* (ICs).

If the unknown function ϕ is partially differentiated only once with respect to time t in the PDE, such as in the diffusion or heat equation, then the initial condition is to specify ϕ at all points in the solution domain at time $t = 0$.

If the unknown function ϕ is partially differentiated twice with respect to time t in the PDE, such as in the wave equation and the PDE for microscale heat equation, then the ICs require specifying both ϕ and $\dfrac{\partial \phi}{\partial t}$ at all points in the solution domain at time $t = 0$. For the vibrating string problem, such ICs imply that the displacement and the velocity of the string are known at all points in the solution domain at time $t = 0$; for the microscale heat conduction, the temperature and the rate of change of the temperature per unit time are known at all points in the solid at time $t = 0$.

We refer to a BVP to be solved with initial condition(s) as an initial BVP (IBVP).

In general, the degree of difficulty involved in solving a BVP or IBVP depends on the governing PDE, the geometry of the solution domain and the conditions imposed on the required solution.

Example

Consider the BVP governed by the two-dimensional Poisson equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=1 \text{ in the square region } 0<x<1,\ 0<y<1,$$

together with the BCs

$$\left.\begin{array}{rcll}\phi & = & 0 & \text{on } x=0 \\ \phi & = & \frac{1}{2}+\cos(\pi y) & \text{on } x=1\end{array}\right\} \text{ for } 0<y<1,$$

$$\frac{\partial\phi}{\partial y}=0 \text{ on } y=0 \text{ and } y=1 \text{ for } 0<x<1.$$

Assume that this BVP is well posed with a unique solution. Verify by direct substitution that its solution is

$$\phi(x,y)=\frac{1}{2}x^2+\frac{\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}.$$

Solution:

Firstly, check that $\phi(x,y)=\frac{1}{2}x^2+\frac{\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}$ satisfies the governing PDE.

Partially differentiating the given function ϕ :

$$\begin{array}{rcl}\frac{\partial\phi}{\partial x} & = & x+\frac{\pi\cosh(\pi x)\cos(\pi y)}{\sinh(\pi)},\ \frac{\partial^2\phi}{\partial x^2}=1+\frac{\pi^2\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}, \\ \frac{\partial\phi}{\partial y} & = & -\frac{\pi\sinh(\pi x)\sin(\pi y)}{\sinh(\pi)},\ \frac{\partial^2\phi}{\partial y^2}=-\frac{\pi^2\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}.\end{array}$$

It follows that

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=1+\frac{\pi^2\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}-\frac{\pi^2\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}=1,$$

that is, the given Poisson equation is satisfied.

Secondly, check that all the boundary conditions are satisfied.

On the vertical sides of the square region:

$$\left.\begin{aligned} \phi(0,y) &= \frac{1}{2}0^2 + \frac{\sinh(0)\cos(\pi y)}{\sinh(\pi)} = 0 \\ \phi(1,y) &= \frac{1}{2} + \frac{\sinh(\pi)\cos(\pi y)}{\sinh(\pi)} = \frac{1}{2} + \cos(\pi y) \end{aligned}\right\} \text{ as given.}$$

On the horizontal sides:

$$\left.\begin{aligned} \left.\frac{\partial\phi}{\partial y}\right|_{y=0} &= -\frac{\pi\sinh(\pi x)\sin(0)}{\sinh(\pi)} = 0 \\ \left.\frac{\partial\phi}{\partial y}\right|_{y=1} &= -\frac{\pi\sinh(\pi x)\sin(\pi)}{\sinh(\pi)} = 0 \end{aligned}\right\} \text{ as given.}$$

Thus, we verify that the given function $\phi(x,y) = \frac{1}{2}x^2 + \frac{\sinh(\pi x)\cos(\pi y)}{\sinh(\pi)}$ is the solution of the BVP.

A theorem for homogeneous linear PDE:

Linear superposition of solutions

"If $\phi = \phi_1$ and $\phi = \phi_2$ are solutions of a homogeneous linear PDE then so is $\phi = c_1\phi_1 + c_2\phi_2$ for any arbitrary real constants c_1 and c_2."

Let us prove the above theorem for the general homogeneous linear second order PDE

$$\begin{aligned} A(x_1,x_2)\frac{\partial^2\phi}{\partial x_1^2} + 2B(x_1,x_2)\frac{\partial^2\phi}{\partial x_1\partial x_2} + C(x_1,x_2)\frac{\partial^2\phi}{\partial x_2^2} \\ +D(x_1,x_2)\frac{\partial\phi}{\partial x_1} + E(x_1,x_2)\frac{\partial\phi}{\partial x_2} + F(x_1,x_2)\phi = 0. \end{aligned} \tag{12}$$

Since ϕ_1 and ϕ_2 are solutions of the general PDE above, we can write

$$\begin{aligned} A(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_1^2} + 2B(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_1\partial x_2} + C(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_2^2} \\ +D(x_1,x_2)\frac{\partial\phi_1}{\partial x_1} + E(x_1,x_2)\frac{\partial\phi_1}{\partial x_2} + F(x_1,x_2)\phi_1 = 0, \end{aligned} \tag{13}$$

and

$$A(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_1^2}+2B(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}+C(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_2^2}$$
$$+D(x_1,x_2)\frac{\partial\phi_2}{\partial x_1}+E(x_1,x_2)\frac{\partial\phi_2}{\partial x_2}+F(x_1,x_2)\phi_1=0. \qquad (14)$$

Is $\phi=c_1\phi_1+c_2\phi_2$ also a solution of the PDE? To answer this question, we substitute $\phi=c_1\phi_1+c_2\phi_2$ into the left hand side of the general PDE in (12) to see if we can recover the right hand side.

$$\begin{aligned}
&A(x_1,x_2)\frac{\partial^2}{\partial x_1^2}[c_1\phi_1+c_2\phi_2]+2B(x_1,x_2)\frac{\partial^2}{\partial x_1\partial x_2}[c_1\phi_1+c_2\phi_2]\\
&+C(x_1,x_2)\frac{\partial^2}{\partial x_2^2}[c_1\phi_1+c_2\phi_2]+D(x_1,x_2)\frac{\partial}{\partial x_1}[c_1\phi_1+c_2\phi_2]\\
&+E(x_1,x_2)\frac{\partial}{\partial x_2}[c_1\phi_1+c_2\phi_2]+F(x_1,x_2)[c_1\phi_1+c_2\phi_2]
\end{aligned}$$

$$\begin{aligned}
= \quad & c_1A(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_1^2}+c_2A(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_1^2}\\
&+2c_1B(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_1\partial x_2}+2c_2B(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_1\partial x_2}\\
&+c_1C(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_2^2}+c_2C(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_2^2}\\
&+c_1D(x_1,x_2)\frac{\partial\phi_1}{\partial x_1}+c_2D(x_1,x_2)\frac{\partial\phi_2}{\partial x_1}\\
&+c_1E(x_1,x_2)\frac{\partial\phi_1}{\partial x_2}+c_2E(x_1,x_2)\frac{\partial\phi_2}{\partial x_2}\\
&+c_1F(x_1,x_2)\phi_1+c_2F(x_1,x_2)\phi_2
\end{aligned}$$

$$
\begin{aligned}
&= c_1\{A(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_1^2} + 2B(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_1 \partial x_2} + C(x_1,x_2)\frac{\partial^2\phi_1}{\partial x_2^2} \\
&\quad +D(x_1,x_2)\frac{\partial\phi_1}{\partial x_1} + E(x_1,x_2)\frac{\partial\phi_1}{\partial x_2} + F(x_1,x_2)\phi_1\} \\
&\quad +c_2\{A(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_1^2} + 2B(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_1 \partial x_2} + C(x_1,x_2)\frac{\partial^2\phi_2}{\partial x_2^2} \\
&\quad +D(x_1,x_2)\frac{\partial\phi_2}{\partial x_1} + E(x_1,x_2)\frac{\partial\phi_2}{\partial x_2} + F(x_1,x_2)\phi_2\} \\
&= 0 \text{ because of (13) and (14).}
\end{aligned}
$$

Thus, $\phi = c_1\phi_1 + c_2\phi_2$ is also a solution of the PDE in (12).

What we will do in Part I of MA3004

Our scope of studies here on PDEs is a modest one. We will learn how to use the method of separation of variables together with Fourier series to solve some relatively simple BVPs and IBVPs in engineering science. We will not look into theoretical questions concerning existence and uniqueness of solutions − the BVPs and IBVPs under consideration here are well posed with unique solutions.

In general, it may be difficult, if not impossible, to derive explicit solutions in analytical forms for BVPs or IBVPs. One may have to resort to numerical methods such as the finite element method for approximate solutions. The remaining parts of the course look at particular numerical methods for PDEs.

Tutorial 1

1. Find values of the constants a and b such that $\phi(x,y) = x^4 + ax^2y^2 + by^4$ satisfies the two-dimensional Laplace's equation at all points (x,y) on the Oxy plane.

2. If $\psi(x, y, z) = (x^2 + y^2 + z^2)^p$ satisfies the three-dimensional Laplace's equation everywhere in space except perhaps at $(x, y, z) = (0, 0, 0)$, what are the possible values that the constant p can have?

3. Use your knowledge of solving ODEs to find the general solution of each of the following PDEs in $u(x, y)$:

(a) $\dfrac{\partial u}{\partial y} = -u$ [*Hint.* What is the general solution $\dfrac{dp}{dt} = -p(t)$?]

(b) $\dfrac{\partial^2 u}{\partial x \partial y} = -\dfrac{\partial u}{\partial x}$ [*Hint.* Use part (a).]

Additional problems

4. A solid occupies the annular region $r_0^2 < x^2 + y^2 < r_1^2$, where r_0 and r_1 are given constants such that $0 < r_0 < r_1$. The temperature at the point (x, y) inside the solid is given by $\phi(x, y) = A \ln(x^2 + y^2) + B$, where A and B are arbitrary constants.

(a) Show that the temperature satisfies the Laplace's equation in the solid.

(b) Determine A and B such that ϕ satisfies the BCs

$$\begin{aligned} \phi(x, y) &= \phi_0 \text{ on } x^2 + y^2 = r_0^2, \\ \phi(x, y) &= \phi_1 \text{ on } x^2 + y^2 = r_1^2, \end{aligned}$$

where ϕ_0 and ϕ_1 are given constants.

5. Consider the PDE

$$a\frac{\partial^2 \phi}{\partial x^2} + 2b\frac{\partial^2 \phi}{\partial x \partial y} + c\frac{\partial^2 \phi}{\partial y^2} = 0,$$

where a, b and c are constants. If $c \neq 0$ and the PDE admits solutions of the form $\phi(x,y) = f(x+\tau y)$, where f is an arbitrary single variable function which is twice differentiable and τ is a constant, show:

(a) τ has only one real value if $b^2 - ac = 0$,

(b) τ has two distinct real values if $b^2 - ac = 0 > 0$,

(c) τ has two distinct complex values if $b^2 - ac < 0$.

6. Consider the PDE in $T(x,y)$ given by

$$\frac{\partial}{\partial x}((ax+by+c)^2\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}((ax+by+c)^2\frac{\partial T}{\partial y}) = 0$$

(a, b and c are constants)

in some region R on the Oxy plane where $ax+by+c>0$. If $T(x,y) = (ax+by+c)^{-1}\phi(x,y)$ is a solution of the PDE above in R, show that $\phi(x,y)$ satisfies the two-dimensional Laplace's equation in R.

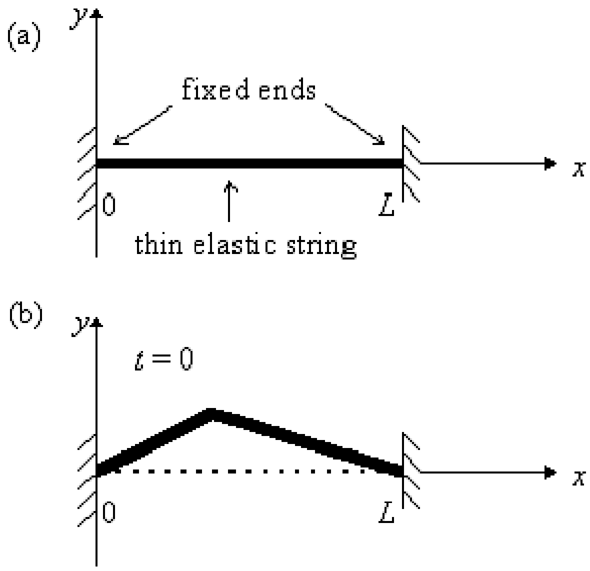
7. Consider the nonlinear PDE in $T(x,y,z)$ given by

$$T(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}) + (\frac{\partial T}{\partial x})^2 + (\frac{\partial T}{\partial y})^2 + (\frac{\partial T}{\partial z})^2 = 0.$$

If $T(x,y,z) = \sqrt{\Theta(x,y,z)}$,where $\Theta(x,y,z)$ is a function such that $\Theta(x,y,z) \geq 0$ and $\Theta(x,y,z)$ is partial differentiable at least twice with respect to the Cartesian coordinates x, y and z, show that $\Theta(x,y,z)$ is a solution of the Laplace's equation.

The problem of a vibrating string

Consider a very thin elastic string of length L with its end fixed as shown in the figures below. The figure in (a) shows the string in its motionless equilibrium state. In the equilibrium state, the string lies along the x-axis from $x = 0$ to $x = L$. At time $t = 0$, we "pluck" a point on the string *slightly* upward from the x axis as shown in the figure in (b) and then release the string from rest. The string vibrates. Assuming that the only force influencing the vibration is the tension in the string, we are interested in studying the motion of the string.



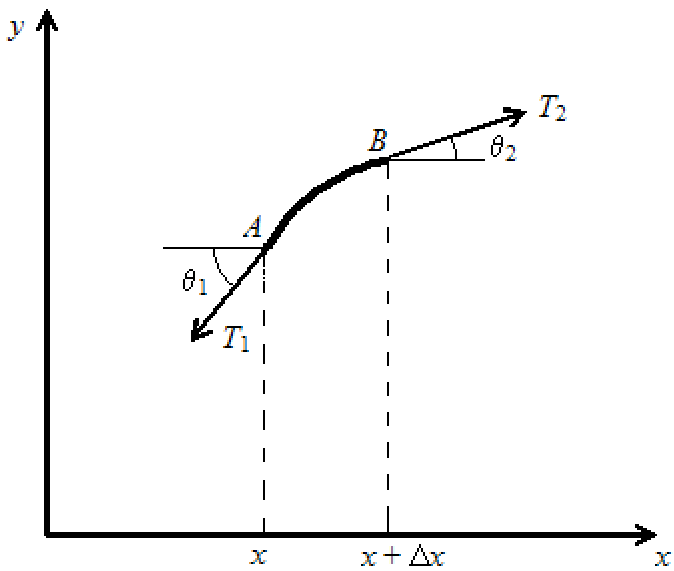
PDE for the vibrating string and d'Alembert's solution

Let u be the vertical upward displacement of the string from the x-axis. (If the string is above the x-axis, $u > 0$. If it is below, $u < 0$.) In the figure in (a) showing the string in its equilibrium state, $u = 0$ at all points on the string. It is obvious from the figure in (b) that u varies along the x-

axis. When the string vibrates, u also changes with time $t \geq 0$. Thus, u is a function of x and t. We may think of x as a point on the string when there is no force acting on the string (that is, when the string is in equilibrium).

All motion obeys Newton's law of motion. The vibrating string is no exception. We will now show that, for the vibrating string, Newton's law of motion gives rise to the one-dimensional wave equation containing $u(x, t)$ as an unknown function.

Consider a very small piece of the string from x to $x + \Delta x$ at an arbitrary time t as shown in the figure below. The tensile forces in the string at the points A and B have magnitudes T_1 and T_2 respectively. Also, the tensile forces at A and B are tangential to the string at those points.



A small piece of the string in motion.

We assume that the piece of string in the figure above does not move horizontally. Thus, there is no net horizontal force acting on it, that is, $T_1 \cos\theta_1 - T_2 \cos\theta_2 = 0$. The net upward force acting on the piece of string is given by $T_2 \sin\theta_2 - T_1 \sin\theta_1$. Note that we assume that the only force acting on the piece of string is due to the tension in the string. All other forces such as gravitational force are negligible.

According to Newton's law of motion, for the piece of string, we may write

$$(\text{mass}) \times (\text{upward acceleration}) = (\text{net upward force}).$$

If the 'density' (mass per unit length) of the string is a constant given by ρ and $\partial u/\partial x$ is sufficiently small then the mass of the piece of string may be given by $\rho\Delta x$ at any time. Thus, the left hand side of the relation above becomes

$$\rho\Delta x \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=\xi}$$

where $x < \xi < x + \Delta x$. (We may think of ξ as the center of mass of the piece of string.)

Newton's law of motion now becomes

$$\rho\Delta x \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=\xi} = T_2 \sin\theta_2 - T_1 \sin\theta_1.$$

From $T_1 \cos\theta_1 - T_2 \cos\theta_2 = 0$, that is, $T_2 = T_1 \cos\theta_1 / \cos\theta_2$, the equation above may be rewritten as

$$\rho \Delta x \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=\xi} = T_1 \cos\theta_1 \left[\tan\theta_2 - \tan\theta_1\right] = T\left[\tan\theta_2 - \tan\theta_1\right]$$

if we let $T = T_1 \cos\theta_1 = T_2 \cos\theta_2$.

At any fixed time t, the slopes of the string at the points A and B are given by $\dfrac{\partial u}{\partial x}$ evaluated at x and $x+\Delta x$ respectively. Geometrically, the slopes at A and B are also given by $\tan\theta_1$ and $\tan\theta_2$ respectively as the tension forces shown in the figure above are tangential to the string at A and B.

We may now write

$$\begin{aligned} \rho \Delta x \left.\frac{\partial^2 u}{\partial t^2}\right|_{x=\xi} &= T\Big(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\Big) \\ \Rightarrow \quad \frac{\rho}{T}\left.\frac{\partial^2 u}{\partial t^2}\right|_{x=\xi} &= \frac{\Big(\left.\frac{\partial u}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial u}{\partial x}\right|_{x}\Big)}{\Delta x}. \end{aligned}$$

Letting $\Delta x \to 0^+$, we find that the equation above leads to

$$\frac{\rho}{T}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

If we let $c = \sqrt{T/\rho}$, we obtain the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}. \tag{15}$$

If the vibration of the string is very small, the tension T and hence c may be assumed to be constant.

Note that the derivation of the PDE above is independent of whether the ends of the string are fixed or not. The PDE holds irrespective of what we do to the string at its ends or at time $t = 0$.

It is possible to give a physical interpretation to the constant c. The constant c is the wave speed. It gives the speed at which waves travel along the string. This is shown below by examining the motion of a simple wave pulse as it moves along a string of infinite length.

Let us look for a solution of (15) that is of the form $u(x,t) = f(x + pt)$, where p is a constant (see Problem 5(b) in Tutorial 1). If we let $\xi = x + pt$ and differentiate $u(x,t) = f(x + pt)$ partially twice with respect to x or t (by using chain rule), we obtain

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{df}{d\xi} \cdot \frac{\partial \xi}{\partial x} = \frac{df}{d\xi}, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{d^2 f}{d\xi^2} \cdot \frac{\partial \xi}{\partial x} = \frac{d^2 f}{d\xi^2}, \\
\frac{\partial u}{\partial t} &= \frac{df}{d\xi} \cdot \frac{\partial \xi}{\partial t} = p\frac{df}{d\xi}, \\
\frac{\partial^2 u}{\partial t^2} &= p\frac{d^2 f}{d\xi^2} \cdot \frac{\partial \xi}{\partial t} = p^2\frac{d^2 f}{d\xi^2}.
\end{aligned}$$

Substituting the above into (15), we find that

$$(1 - \frac{p^2}{c^2})\frac{d^2 f}{d\xi^2} = 0,$$

which is true for any arbitrary f if $(1 - \frac{p^2}{c^2}) = 0$. Thus, $p = \pm c$.

Thus, $u = f(x + ct)$ and $u = g(x - ct)$ are solutions of the one-dimensional wave equation. Since the one-dimensional wave equation is a homogeneous linear PDE, we may write

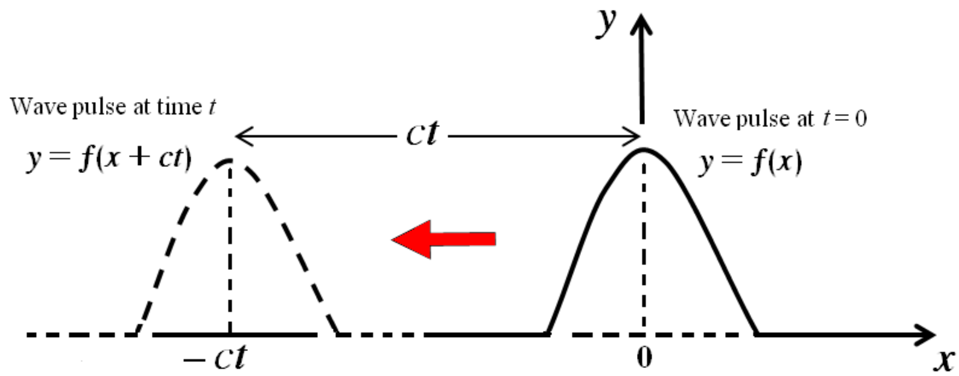
$$u(x,t) = f(x+ct) + g(x-ct)$$

as a solution of the one-dimensional wave equation. Such a solution is known as the *d'Alembert's solution*.

To show that c is the speed of a wave pulse travelling along a string lying along the x axis, let us first ignore the term $g(x-ct)$ and write

$$u(x,t) = f(x+ct). \tag{16}$$

From (16), we find that $u(x,0) = f(x)$, that is, the shape of the string is given by $y = f(x)$ at time $t = 0$ as shown in the figure below. Some time t later, the shape of the string is given by $y = f(x+ct)$, which is the initial shape (at $t = 0$) shifted to the left by a distance of ct. This tell us that the wave pulse with the initial shape $f(x)$ moves to the left by a distance of ct within the period of time t. Thus, c is the speed of the wave pulse.



Note that if we take $u(x,t) = g(x-ct)$ (instead of $u(x,t) = f(x+ct)$) we would be describing a wave travelling with speed c to the right. Thus, the d'Alembert's solution describes a linear superposition of waves travelling to the left and right of the string.

IBVP for the vibrating string problem

For the vibrating string problem stated on page 24, the string is fixed at the ends $x = 0$ and $x = L$. Thus, $u(0,t) = u(L,t) = 0$ for $t \geq 0$.

The initial displacement of the string is also known. (We are the one "plucking" a point on the string slightly upward from the x axis at time $t = 0$, to give the string its initial profile. Thus, we write $u(x,0) = G(x)$ for $0 < x < L$, where $G(x)$ which gives the initial shape of the string is a known function satisfying $G(0) = G(L) = 0$. The upward velocity of the string is given by $\dfrac{\partial u}{\partial t}$ and the upward acceleration by $\dfrac{\partial^2 u}{\partial t^2}$. The string is released from rest at time $t = 0$. Thus, $\dfrac{\partial u}{\partial t} = 0$ at $t = 0$.

From the discussion above, the IBVP to be solved for finding the displacement $u(x,t)$ of the vibrating string problem can be stated as follows.

"Solve the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \text{ for } 0 < x < L \text{ and } t > 0,$$

subject to the initial and boundary conditions

$$\begin{aligned} u(x,0) &= G(x) \;\text{ and }\; \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \;\text{ for } 0 < x < L \text{ (ICs)} \\ u(0,t) &= 0 \;\text{ and }\; u(L,t) = 0 \;\text{ for } t \geq 0 \text{ (BCs).”} \end{aligned}$$

We will study a mathematical method for solving the IBVP above. The method of solution requires some knowledge of ODEs and Fourier series. A brief review on ODEs and Fourier series (just on what is needed to solve the IBVP) is given below.

Review on ODEs

Consider the homogeneous second order linear ODE in $y(x)$:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$
$$(a,\ b \text{ and } c \text{ are given real constants}). \tag{17}$$

The characteristic equation of (17) is

$$a\lambda^2 + b\lambda + c = 0. \tag{18}$$

To find the general solution of (17), solve the characteristic equation (18) for λ.

1. For $a = 0$ and $b \neq 0$ (homogeneous first order linear ODE), $\lambda = -c/b$ and the general solution of (17) is

   $$y(x) = Ae^{-cx/b},$$

   where A is an arbitrary constant.

2. For $a \neq 0$ (homogeneous second order linear ODE), if $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are two distinct real numbers of (18) then the general solution of (17) is

$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x},$$

where A and B are arbitrary constants.

3. For $a \neq 0$, if (18) has only one solution given by $\lambda = \lambda_1$ then the general solution of (17) is

$$y(x) = (Ax + B)e^{\lambda_1 x},$$

where A and B are arbitrary constants.

4. For $a \neq 0$, if $\lambda = \lambda_1$ and $\lambda = \lambda_2$ are complex numbers given by $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ (β is a non-zero real number) then the general solution of (17) is

$$y(x) = (A\sin(\beta x) + B\cos(\beta x))e^{\alpha x},$$

where A and B are arbitrary constants.

Examples

For each of the following cases, solve the ODE subject to the given conditions:

(a) $y''(x) + y'(x) - 2y(x) = 0$, $y(0) = 1$, $y'(0) = 0$

(b) $y''(x) - 2y'(x) + y(x) = 0$, $y(0) = 0$, $y(1) = 1$

(c) $y''(x) + 9y(x) = 0$, $y(0) = y(\frac{\pi}{2}) = 1$

Solution:

To solve each of the ODE subject to the given conditions, find the general solution of the ODE first. The arbitrary constants in the general solution must be given particular values to ensure that the given conditions are satisfied.

(a) The characteristic equation gives $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0$. We find that $\lambda = 1$ and $\lambda = -2$ are solutions of the characteristic equation. Thus, the general solution of the ODE is

$$y(x) = Ae^x + Be^{-2x}$$

which gives

$$y'(x) = Ae^x - 2Be^{-2x}$$

From the conditions,

$$A + B = 1 \text{ and } A - 2B = 0 \Rightarrow A = \frac{2}{3} \text{ and } B = \frac{1}{3}.$$

Thus, the required solution is $y(x) = \frac{2}{3}e^x + \frac{1}{3}e^{-2x}$.

(b) The characteristic equation gives $\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$, that is, $\lambda = 1$ is the only solution of the characteristic equation. Thus, the general solution of the ODE is

$$y(x) = (Ax + B)e^x.$$

Applying the given conditions, we obtain $B = 0$ and $A = e^{-1}$. Thus, the required solution is $y = xe^{x-1}$.

(c) The characteristic equation is $\lambda^2 + 9 = 0$ which gives $\lambda = 3i$ and $\lambda = -3i$. The general solution is

$$y(x) = (A\sin(3x) + B\cos(3x))e^{0x} = A\sin(3x) + B\cos(3x).$$

Applying the conditions, we obtain $B = 1$ and $A = -1$. Thus, the required solution is $y(x) = \cos(3x) - \sin(3x)$.

Review on Fourier series

FSP I *Can we represent $f(x)$ in the interval $0 < x < L$ by a cosine (even) Fourier series , that is, can we find constants a_n $(n = 0, 1, 2, \cdots)$ such that*

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) = f(x) \textit{ for } 0 < x < L?$$

The answer is yes, if $f(x)$ is continuous in the interval $0 < x < L$, and the constants a_n are given by

$$\begin{aligned} a_0 &= \frac{1}{L}\int_0^L f(x)dx, \\ a_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx \ (n = 1, 2, \cdots). \end{aligned}$$

For details on how the formulae for a_n come about, refer to the subtopic on "Even and odd functions: Half-range expansions" under the topic "Fourier series" in Chapter 11 in Kreyszig.

FSP II *Can we represent $f(x)$ in the interval $0 < x < L$ by a sine (odd) Fourier series , that is, can we find constants b_n $(n = 1,\ 2,\ \cdots)$ such that*

$$\sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) = f(x) \text{ for } 0 < x < L\ ?$$

The answer is yes, if the function $f(x)$ is continuous in the interval $0 < x < L$, and the constants b_n are given by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})dx\ (n = 1, 2, \cdots).$$

<u>*Examples*</u>

1. *Represent the function $f(x) = x$ for $0 < x < 1$ in terms of an even Fourier series.*

Here $L = 1$. From FSP I:

$$\begin{aligned} a_0 &= \int_0^1 x dx = \frac{1}{2}, \\ a_n &= 2\int_0^1 x\cos(n\pi x)dx = \frac{2(\cos(n\pi) - 1)}{n^2\pi^2}\ (n = 1, 2, \cdots). \end{aligned}$$

Thus, the required cosine Fourier series is

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(\cos(n\pi) - 1)}{n^2\pi^2}\cos(n\pi x).$$

If we like, we can check if the series is correctly derived by replacing ∞ in the series with a large integer and evaluate approximately the truncated series for selected values x within the interval $0 < x < 1$ to see if the approximate value of the series is close to $f(x) = x$ for $0 < x < 1$. For example,

$$\frac{1}{2} + \sum_{n=1}^{20} \frac{2(\cos(n\pi) - 1)}{n^2\pi^2} \cos(\frac{n\pi}{4}) = 0.249\,931\,893\,7 \simeq \frac{1}{4}.$$

2. *Repeat the exercise above using an odd Fourier series.*

As before, $L = 1$. From FSP II:

$$b_n = 2\int_0^1 x \sin(n\pi x)dx = -\frac{2\cos(n\pi)}{n\pi} = -(-1)^n \frac{2}{n\pi} \ (n = 1, 2, \cdots).$$

Thus, the required sine Fourier series is

$$-\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x).$$

If we check by replacing ∞ in the series with 20 in the series, we find that

$$-\frac{2}{\pi}\sum_{n=1}^{20} \frac{(-1)^n}{n} \sin(\frac{n\pi}{4}) = 0.256\,582\,818\,8.$$

In this example, using only the first 20 terms, the approximate value from the sine Fourier series is less accurate than the corresponding approximate value from the cosine Fourier series. Can you see why this is expected? If we use more terms in the sine Fourier series, the approximate value gets closer to 1/4 as expected. For example,

$$-\frac{2}{\pi}\sum_{n=1}^{500} \frac{(-1)^n}{n} \sin(\frac{n\pi}{4}) = 0.251\,318\,405\,2,$$

which is better but still not as good as the approximate value from the cosine Fourier series with only 20 terms.

Are we ready now for the solution of the IBVP for the vibrating string?

Solution of the IBVP for the vibrating string

Consider the vibrating string problem stated on page 24. The IVBP for the problem is stated on page 31. We solve the IBVP in four steps.

1. We apply the so called *method of separation of variables* on the PDE to obtain a set of ODEs.
2. We solve the ODEs in Step 1 to find solutions that satisfy all the BCs. We find infinitely many solutions.
3. Since the PDE is homogeneous linear, we sum up all the solutions in Step 2 to form a general series solution for the PDE.
4. We fit the ICs into the general series solution in Step 3. This leads to solving a Fourier series problem (FSP).

Step 1: Method of separation of variables

The PDE to solve here is the one-dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

We assume that $u(x,t)$ can be written in the form

$$u(x,t) = X(x)T(t),$$

where $X(x)$ and $T(t)$ are to be determined.

It follows that

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= X''(x)T(t) \\ \frac{\partial^2 u}{\partial t^2} &= X(x)T''(t).\end{aligned}$$

Substitution into the one-dimensional wave equation gives

$$\frac{1}{c^2}X(x)T''(t) = X''(x)T(t)$$

which may be rearranged to give (after dividing both sides of the equation above by $X(x)T(t)$)

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)}.$$

Now the left hand side of the equation is a function of time t alone, while the right hand side is a function of x alone. Since x and t are independent variables, a function of t can only be equal to a function of x only if both functions are constant. Thus, we write

$$\frac{1}{c^2}\frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \gamma \text{ (an arbitrary constant called “separation constant”)}$$

which leads to the the second-order linear ODEs

$$\begin{aligned}T''(t) - \gamma c^2 T(t) &= 0 \\ X''(x) - \gamma X(x) &= 0.\end{aligned}$$

Such an approach which reduces a PDE into ODEs is called the *method of separation of variables.*

Step 2: Solving ODEs and dealing with the BCs first

To solve the ODEs, consider three possibilities for the separation constant γ, that is, $\gamma = 0$, $\gamma > 0$ and $\gamma < 0$. We now deal with each of these possibilities separately, taking into consideration the BCs $u(0, t) = 0$ and $u(L, t) = 0$. As we see below, for the vibrating string with fixed ends, the first two possibilities give rise to only the trivial solution $u = 0$.

1. If $\gamma = 0$ the two linear ODEs become

   $$T''(t) = 0 \quad \text{and} \quad X''(x) = 0.$$

   Solving the ODEs, we obtain

   $$T(t) = At + B \quad \text{and} \quad X(x) = Cx + D$$

   where A, B, C and D are arbitrary constants. Thus, we obtain

   $$u(x, t) = (At + B)(Cx + D).$$

   To ensure that the BCs $u(0, t) = 0$ and $u(L, t) = 0$ are satisfied for all $t \geq 0$, we have to choose $D = 0$ and $C = 0$. This leads to the trivial solution $u = 0$.

2. For $\gamma > 0$, we may write $\gamma = p^2$ with p being a positive real number. The ODEs become

   $$\begin{aligned} T''(t) - p^2c^2T(t) &= 0 \\ X''(x) - p^2X(x) &= 0. \end{aligned}$$

The general solutions of the ODEs are

$$\begin{aligned} T(t) &= Ae^{pct} + Be^{-pct} \\ X(x) &= Ce^{px} + De^{-px} \end{aligned}$$

where A, B, C and D are arbitrary constants. Thus,

$$u(x,t) = [Ae^{pct} + Be^{-pct}][Ce^{px} + De^{-px}].$$

The BCs $u(0,t) = 0$ and $u(L,t) = 0$ give rise to

$$C + D = 0 \text{ and } e^{pL}C + e^{-pL}D = 0.$$

As $p \neq 0$, we find that $C = D = 0$. Once again, we obtain the trivial solution $u = 0$.

3. For $\gamma < 0$, we may write $\gamma = -p^2$ with p being a positive real number. The ODEs become

$$\begin{aligned} T''(t) + p^2c^2T(t) &= 0 \\ X''(x) + p^2X(x) &= 0. \end{aligned}$$

Solving the ODEs, we obtain the general solutions

$$\begin{aligned} T(t) &= A\sin(pct) + B\cos(pct) \\ X(x) &= C\sin(px) + D\cos(px) \end{aligned}$$

where A, B, C and D are arbitrary constants. Thus, we obtain

$$u(x,t) = [A\sin(pct) + B\cos(pct)][C\sin(px) + D\cos(px)].$$

The BC $u(0,t) = 0$ implies that $D = 0$ and hence

$$u(x,t) = [E\sin(pct) + F\cos(pct)]\sin(px)$$

where $E = AC$ and $F = BC$ are arbitrary constants. The other BC $u(L,t) = 0$ gives rise to

$$[E\sin(pct) + F\cos(pct)]\sin(pL) = 0 \quad \text{for all } t \geq 0.$$

If we set $E = 0$ and $F = 0$ to satisfy the above, we end up having the trivial solution again! The only way to avoid having again $u = 0$ is to let

$$\sin(pL) = 0.$$

Thus, p is chosen to be given by

$$p = \frac{n\pi}{L} \text{ for } n = 1, 2, 3, \cdots.$$

There are infinitely many values which p can take to satisfy the BCs $u(0,t) = 0$ and $u(L,t) = 0$. When $p = \pi/L$, let us denote the solution by $u_1(x,t)$, that is,

$$u_1(x,t) = \{E_1\sin(\frac{\pi ct}{L}) + F_1\cos(\frac{\pi ct}{L})\}\sin(\frac{\pi x}{L})$$

where E_1 and F_1 are arbitrary constants. When $p = 2\pi/L$, let us denote the solution by $u_2(x,t)$, that is,

$$u_2(x,t) = \{E_2\sin(\frac{2\pi ct}{L}) + F_2\cos(\frac{2\pi ct}{L})\}\sin(\frac{2\pi x}{L})$$

where E_2 and F_2 are arbitrary constants. In general,

$$u_n(x,t) = \{E_n\sin(\frac{n\pi ct}{L}) + F_n\cos(\frac{n\pi ct}{L})\}\sin(\frac{n\pi x}{L}) \text{ for } n = 1, 2, 3, \cdots. \tag{19}$$

To summarise, using the method of separation of variables, we find a set of functions $u_n(x,t)$, all of which satisfy the one-dimensional wave equation and the BCs $u(0,t) = 0$ and $u(L,t) = 0$. We now show how these functions may be used to construct the solution $u(x,t)$ of the IBVP for the vibrating string.

Note that in the above discussion we do not include $n = 0, -1, -2, \cdots$. It is not necessary to consider negative n as it gives solutions that are of the same form as those in (19). For example, if we let $n = -2$ to obtain

$$\begin{aligned} u_{-2}(x,t) &= \{E_{-2}\sin(\frac{-2\pi ct}{L}) + F_{-2}\cos(\frac{-2\pi ct}{L})\}\sin(\frac{-2\pi x}{L}) \\ &= \{E_{-2}\sin(\frac{2\pi ct}{L}) + (-F_{-2})\cos(\frac{2\pi ct}{L})\}\sin(\frac{2\pi x}{L}) \end{aligned}$$

we find that $u_{-2}(x,t)$ has exactly the same form as $u_2(x,t)$. The case $n = 0$ gives a trivial solution.

Step 3: Forming series solution

Since the one-dimensional wave equation is a homogeneous linear PDE, we can apply the theorem on page 20, that is, the sum of all the solutions in (19) is still a solution of the governing PDE of the vibrating string problem. Thus, to construct a solution for the vibrating string problem, we may sum up all the non-trivial solutions that satisfy the BCs to obtain the series solution

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty}\{E_n\sin(\frac{n\pi ct}{L}) + F_n\cos(\frac{n\pi ct}{L})\}\sin(\frac{n\pi x}{L}).$$

The above series solution not only satisfies the one-dimensional wave equation but also the BCs $u(0,t) = 0$ and $u(L,t) = 0$ for all $t \geq 0$. (It is easy to check this $-$ just let $x = 0$ and $x = L$ in the series.) The solution may be regarded as general in the sense that it contains arbitrary constants E_n and F_n (infinitely many of them).

Step 4: Dealing with the ICs

To solve the vibrating string problem, we now only have to ensure that the initial conditions are satisfied by choosing constant coefficients E_n and F_n in the series solution above carefully.

We deal with the simpler IC first, that is, $\dfrac{\partial u}{\partial t} = 0$ at $t = 0$. Differentiating u above partially with respect to t, we obtain

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\sum_{n=1}^{\infty}\{E_n \sin(\frac{n\pi ct}{L}) + F_n \cos(\frac{n\pi ct}{L})\}\sin(\frac{n\pi x}{L}) \\
&= \sum_{n=1}^{\infty}\frac{\partial}{\partial t}\{E_n \sin(\frac{n\pi ct}{L}) + F_n \cos(\frac{n\pi ct}{L})\}\sin(\frac{n\pi x}{L}) \\
&= \sum_{n=1}^{\infty}\frac{n\pi c}{L}\{E_n \cos(\frac{n\pi ct}{L}) - F_n \sin(\frac{n\pi ct}{L})\}\sin(\frac{n\pi x}{L}).
\end{aligned}$$

The initial condition $\dfrac{\partial u}{\partial t} = 0$ at $t = 0$ is satisfied if

$$\sum_{n=1}^{\infty}\frac{n\pi c}{L}E_n \sin(\frac{n\pi x}{L}) = 0.$$

To satisfy this, we choose $E_n = 0$ for $n = 1, 2, 3, \cdots$.

It follows that the displacement u can now be rewritten as

$$u(x,t) = \sum_{n=1}^{\infty}F_n \cos(\frac{n\pi ct}{L})\sin(\frac{n\pi x}{L}).$$

The only condition left now to be satisfied is the initial condition $u(x, 0) = G(x)$ for $0 < x < L$. This condition is satisfied if we choose the constants F_n

in such a way that

$$\sum_{n=1}^{\infty} F_n \sin(\frac{n\pi x}{L}) = G(x) \quad \text{for } 0 < x < L.$$

For certain functions $G(x)$, finding F_n is a relatively simple exercise. For example, if $G(x) = 2\sin(3\pi x/L) - 3\sin(5\pi x/L)$, then we find that

$$\sum_{n=1}^{\infty} F_n \sin(\frac{n\pi x}{L}) = 2\sin(\frac{3\pi x}{L}) - 3\sin(\frac{5\pi x}{L}) \quad \text{for } 0 < x < L$$

which may be rewritten as

$$\begin{aligned} & F_1 \sin(\frac{\pi x}{L}) + F_2 \sin(\frac{2\pi x}{L}) + F_3 \sin(\frac{3\pi x}{L}) \\ & + F_4 \sin(\frac{4\pi x}{L}) + F_5 \sin(\frac{5\pi x}{L}) + F_6 \sin(\frac{6\pi x}{L}) + \cdots \\ = \; & 2\sin(\frac{3\pi x}{L}) - 3\sin(\frac{5\pi x}{L}) \quad \text{for } 0 < x < L. \end{aligned}$$

By comparing the left and the right hand sides, it is obvious that $F_3 = 2$, $F_5 = -3$ and $F_n = 0$ for $n = 1, 2, 4, 6, 7, 8, \cdots$.

Thus, if the initial shape of the string is given by $G(x) = 2\sin(3\pi x/L) - 3\sin(5\pi x/L)$ for $0 < x < L$, the displacement is given by

$$u(x,t) = 2\cos(\frac{3\pi ct}{L})\sin(\frac{3\pi x}{L}) - 3\cos(\frac{5\pi ct}{L})\sin(\frac{5\pi x}{L}).$$

How do we choose F_n for a general $G(x)$? How do we choose F_n such that

$$\sum_{n=1}^{\infty} F_n \sin(\frac{n\pi x}{L}) = G(x) \quad \text{for } 0 < x < L?$$

The answer lies in FSP II (the Fourier series problem on page 35)!

For a general $G(x)$, FSP II gives

$$F_n = \frac{2}{L}\int_0^L G(x)\sin(\frac{n\pi x}{L})dx.$$

To summarise, the displacement for the vibrating string is given by

$$u(x,t) = \sum_{n=1}^{\infty} F_n \cos(\frac{n\pi kt}{L})\sin(\frac{n\pi x}{L})$$

with

$$F_n = \frac{2}{L}\int_0^L G(x)\sin(\frac{n\pi x}{L})dx$$

if it has fixed ends at $x = 0$ and $x = L$, has initial shape given by $u(x,0) = G(x)$ and is initially at rest.

Example

At time $t = 0$, the string is at rest and has shape given by the function

$$G(x) = \begin{cases} x/10 & \text{if } \; 0 < x < L/2 \\ (L-x)/10 & \text{if } \; L/2 \leq x < L. \end{cases}$$

Find the displacement $u(x,t)$ of the string.

Solution:

Finding F_n, we obtain

$$\begin{aligned} F_n &= \frac{2}{L}\int_0^L G(x)\sin(\frac{n\pi x}{L})dx \\ &= \frac{2}{L}[\int_0^{L/2} \frac{x}{10}\sin(\frac{n\pi x}{L})dx + \int_{L/2}^{L} \frac{(L-x)}{10}\sin(\frac{n\pi x}{L})dx] \\ &= \begin{cases} \frac{2L}{5n^2\pi^2}\sin(\frac{1}{2}n\pi) & \text{if } \; n = 1,3,5,\cdots \\ 0 & \text{if } \; n = 2,4,6,\cdots . \end{cases} \end{aligned}$$

The required displacement is

$$\begin{aligned} u(x,t) &= \sum_{n=1,3,5,\cdots}^{\infty} \frac{2L}{5n^2\pi^2} \sin(\frac{1}{2}n\pi) \cos(\frac{n\pi ct}{L}) \sin(\frac{n\pi x}{L}) \\ &= \sum_{m=0}^{\infty} \frac{2L(-1)^m}{5[2m+1]^2\pi^2} \cos(\frac{[2m+1]\pi ct}{L}) \sin(\frac{[2m+1]\pi x}{L}). \end{aligned}$$

To derive the series on the second line, we let $n = 2m + 1$. Since $n = 1, 3, 5, \cdots$, we find that m has values $0, 1, 2, \cdots$ and $\sin(\frac{1}{2}n\pi) = \sin(\frac{1}{2}[2m+1]\pi) = (-1)^m$. (Alternatively, we can also choose to let $n = 2m - 1$. In this case, m has values $1, 2, 3, \cdots$ and $\sin(\frac{1}{2}n\pi) = \sin(\frac{1}{2}[2m-1]\pi) = (-1)^{m+1}$. The second line has to be modified appropriately.)

According to the solution above, the string vibrates forever. This is not surprising, as we have not introduced any damping mechanism (for example, air resistance) into the physics of the problem. If damping is introduced, the governing PDE is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t},$$

where b is the damping coefficient (a positive real number). Solutions such that $u(x,t) \to 0$ as $t \to \infty$ can be obtained for the damped wave equation.

The vibrating string problem may have other ICs or BCs. For example, if the string is initially horizontal then, to produce vibration, we have to give each and every point on the string an initial velocity. In this case, the ICs are

$$u(x,0) = 0 \text{ and } \left.\frac{\partial u}{\partial t}\right|_{t=0} = V(x) \quad \text{for } 0 < x < L.$$

It may be possible to modify the the method of solution shown above to deal with other initial and boundary conditions.

Tutorial 2

1. Use the method of separation of variables to reduce each of the following PDEs to a set of two ODEs (with an arbitrary separation constant):

   (a) $\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} + b\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ (b and c constants) [Let $u(x,t) = X(x)T(t)$]

   (b) $\dfrac{\partial^2 \psi}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial \psi}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 \psi}{\partial \theta^2} = 0$ [Let $\psi(r,\theta) = R(r)\Theta(\theta)$]

2. Solve
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < 1 \text{ and } t > 0,$$
subject to
$$\begin{aligned} u(x,0) &= 0 \text{ and } \left.\frac{\partial u}{\partial t}\right|_{t=0} = x(x-1) \text{ for } 0 < x < 1, \\ u(0,t) &= 0 \text{ and } u(1,t) = 0 \text{ for } t \geq 0. \end{aligned}$$
[*Guide.* You may start from from the series solution in Problem 6 below in "Additional problems" with $L = 1$ and $c = 1$. Since the series solution satisfies the PDE and the BCs, all you need to do is: use the ICs to determine the constant coefficients E_n and F_n. Start with the simpler IC first. You may use (without proof) the integral formula:
$$\begin{aligned} \int x(x-1)\sin(px)dx &= -\frac{1}{p^3}\{p^2x^2\cos(px) - 2\cos(px) - 2px\sin(px) \\ &\quad + p\sin(px) - p^2 x\cos(px)\} + C. \end{aligned}$$
]

3. Solve again Problem 2, but with the ICs replaced by
$$\left.\begin{aligned} u(x,0) &= \sin(2\pi x) + 3\sin(4\pi x) \\ \left.\frac{\partial u}{\partial t}\right|_{t=0} &= \pi(2\sin(\pi x) - 3\sin(3\pi x)) \end{aligned}\right\} \text{ for } 0 < x < 1.$$

Additional problems

4. If b and c in Problem 1(a) are positive real numbers, find the general solution of the ODE in $T(t)$ for the case where the corresponding ODE in $X(x)$ has solutions that are expressed in terms of real sine and cosine functions. Show that the general solution of the ODE in $T(t)$ vanishes to zero as time t tends to infinity.

5. Use the method of separation of variables to reduce each of the following PDEs to a set of two ODEs (with an arbitrary separation constant):

   (a) $\frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial^2\phi}{\partial y^2} + \phi = 0$ [Let $\phi(x,y) = X(x)Y(y)$]

   (b) $\frac{\partial}{\partial x}(x^2y^4\frac{\partial T}{\partial x}) + \frac{\partial}{\partial y}(x^2y^4\frac{\partial T}{\partial y}) = 0$ [Let $T(x,y) = X(x)Y(y)$]

6. As derived during lecture, the vibrating string with fixed ends admits a series solution of the form

$$u(x,t) = \sum_{n=1}^{\infty}\{E_n\sin(\frac{n\pi ct}{L}) + F_n\cos(\frac{n\pi ct}{L})\}\sin(\frac{n\pi x}{L}),$$

   where E_n and F_n are constant coefficients. Identify the solutions above as the d'Alembert's solution, that is, rewrite the solutions in the form

$$u(x,t) = f(x+ct) + g(x-ct).$$

Heat conduction in solids

The above method of solution for the vibrating string problem, which involves the separation of variables and Fourier series, may be extended to solve certain heat conduction problems. We will study here the extension of the method to a few specific problems concerning heat conduction in solids.

The linear theory of heat conduction in a nutshell is given below.

If $\phi(x, y, z, t)$ denotes the temperature at a point (x, y, z) inside a solid at time t then under certain assumptions the classical linear theory of heat conduction requires that $\phi(x, y, z, t)$ satisfies the PDE (the three-dimensional heat equation)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \frac{\rho c}{\kappa} \frac{\partial \phi}{\partial t} \text{ inside the solid,}$$

where ρ, c and κ (positive real constants) are respectively the density, specific heat capacity and thermal conductivity of the solid.

The PDE may be derived from the principle of conservation of energy by using the Gauss-Ostrogradski (divergence) theorem. We will not look into this here. The derivation may be found in some textbooks on heat conduction.

A quantity of interest in heat conduction is the normal heat flux across the boundary of the solid. If a unit normal (unit magnitude perpendicular) vector to the boundary is given by $\underline{\mathbf{n}} = n_x \underline{\mathbf{i}} + n_y \underline{\mathbf{j}} + n_z \underline{\mathbf{k}}$, then the normal heat flux across the boundary in the direction of $\underline{\mathbf{n}}$ is given by

$$-\kappa \underline{\mathbf{n}} \bullet \underline{\nabla} \phi = -\kappa (n_x \frac{\partial \phi}{\partial x} + n_y \frac{\partial \phi}{\partial y} + n_z \frac{\partial \phi}{\partial z}) \text{ at points on the boundary.}$$

Roughly speaking, the normal heat flux is the thermal energy per unit area of the boundary per unit time.

The normal heat flux across a thermally insulated boundary is zero, that is,

$$n_x \frac{\partial \phi}{\partial x} + n_y \frac{\partial \phi}{\partial y} + n_z \frac{\partial \phi}{\partial z} = 0 \text{ on a thermally insulated boundary.}$$

A one-dimensional time-dependent heat conduction problem

IBVP for a particular problem

Consider now the case where the temperature ϕ is a function of only x and t. For such a case, the temperature $\phi(x,t)$ in the solid obeys the PDE

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\rho c}{\kappa} \frac{\partial \phi}{\partial t},$$

and the region occupied by the solid is defined by $0 < x < L$, that is, the solid occupies the region between the two planes $x = 0$ and $x = L$.

A unit normal vector to the boundary of the solid (defined by the planes $x = 0$ and $x = L$) is given by $\underline{\mathbf{n}} = \underline{\mathbf{i}}$. Thus, if the plane $x = 0$ (for example) is thermally insulated, then the boundary condition on $x = 0$ can be written as

$$\left. \frac{\partial \phi}{\partial x} \right|_{x=0} = 0 \text{ for } t \geq 0.$$

For a start, let us consider the following relatively simple IBVP:

"Solve the PDE

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\rho c}{\kappa} \frac{\partial \phi}{\partial t} \text{ for } 0 < x < L \text{ and } t > 0,$$

subject to the initial and boundary conditions

$$\begin{aligned} \phi(x, 0) &= \phi_0 \text{ (constant) for } 0 < x < L \text{ (IC)}, \\ \phi(0, t) &= 0 \ \text{ and } \ \phi(L, t) = 0 \ \text{ for } t \geq 0 \text{ (BCs).''} \end{aligned}$$

Note that, unlike the vibrating string problem, there is only one IC, that is, only ϕ is specified throughout the region $0 < x < L$ at $t = 0$, since the highest order of the partial derivative of ϕ with respect to time t is one in the governing PDE.

The method of solution we have studied earlier on for the one-dimensional vibrating string problem can be applied to solve the IBVP above as follows.

Step 1: Method of separation of variables

If we let $\phi(x, t) = X(x)T(t)$, the PDE gives

$$X''(x)T(t) = \frac{\rho c}{\kappa} X(x)T'(t).$$

On dividing both sides of the above equation by $X(x)T(t)$, we obtain

$$\frac{X''(x)}{X(x)} = \frac{\rho c}{\kappa} \frac{T'(t)}{T(t)} = \gamma,$$

where γ is the separation constant.

Step 2: Dealing with the BCs first

As in the case of the vibrating string problem, the BCs $\phi(0,t) = 0$ and $\phi(L,t) = 0$ give only trivial solution $\phi = 0$ if the separation constant γ is zero or positive.

For $\gamma = -p^2$ (p is a positive real number), we obtain the ODEs

$$X''(x) + p^2 X(x) = 0 \text{ and } T'(t) = -\frac{p^2\kappa}{\rho c}T(t),$$

with general solutions

$$\begin{aligned} T(t) &= Ae^{-p^2\kappa t/(\rho c)}, \\ X(x) &= C\sin(px) + D\cos(px), \end{aligned}$$

where A, C and D are arbitrary constants.

It follows that we can write

$$\phi(x,t) = (E\sin(px) + F\cos(px))e^{-p^2\kappa t/(\rho c)},$$

where $E = AC$ and $F = AD$ are arbitrary constants.

The BC $\phi(0,t) = 0$ gives

$$Fe^{-p^2\kappa t/(\rho c)} = 0.$$

Thus, $F = 0$.

With $F = 0$ and $E \neq 0$ (to avoid another trivial solution), the BC $\phi(L,t) = 0$ gives

$$\sin(pL) = 0.$$

Thus, p is given by

$$p = \frac{n\pi}{L} \text{ for } n = 1,\ 2,\ \cdots .$$

As in the case of the vibrating string problem, we do not include $n = 0$ in the above, as it merely gives $\phi(x,t) = 0$ (a trivial solution) for the BCs here. We do not include negative n as it eventually leads to the same set of solutions given by positive n.

For every value of n in the equation above, we have a non-trivial solution of the PDE, that is,

$$\phi_n(x,t) = E_n \sin(\frac{n\pi x}{L})e^{-n^2\pi^2\kappa t/(L^2\rho c)} \text{ for } n = 1, 2, \cdots ,$$

are solutions of PDE satisfying the BCs at $x = 0$ and $x = L$.

<u>*Step 3: Forming series solution*</u>

Since the PDE is homogeneous linear, the theorem on page 20 is valid and we can write

$$\phi(x,t) = \sum_{n=1}^{\infty} E_n \sin(\frac{n\pi x}{L})e^{-n^2\pi^2\kappa t/(L^2\rho c)}.$$

It is easy to verify that the BCs $\phi(0,t) = 0$ and $\phi(L,t) = 0$ are satisfied in the series solution above.

<u>*Step 4: Dealing with the IC to derive the final solution*</u>

The IC $\phi(x,0) = \phi_0$ for $0 < x < L$ is the only condition left to be satisfied. Applying the IC into the series solution above, we obtain

$$\sum_{n=1}^{\infty} E_n \sin(\frac{n\pi x}{L}) = \phi_0 \text{ for } 0 < x < L.$$

From FSP II, we obtain

$$\begin{aligned} E_n &= \frac{2}{L}\phi_0 \int_0^L \sin(\frac{n\pi x}{L}) dx \ (n = 1, 2, \cdots) \\ &= -2\phi_0 \frac{\cos(n\pi) - 1}{n\pi} \ (n = 1, 2, \cdots). \end{aligned}$$

Thus, the required final solution of the IBVP is

$$\phi(x,t) = -2\phi_0 \sum_{n=1}^{\infty} \frac{\cos(n\pi) - 1}{n\pi} \sin(\frac{n\pi x}{L}) e^{-n^2\pi^2\kappa t/(L^2\rho c)}.$$

Note that the exponential function $e^{-n^2\pi^2\kappa t/(L^2 c)}$ decays to zero very quickly as n increases. Thus, the series converges quickly and can be computed accurately by retaining relatively few terms in the series.

From the solution above, we find that $\phi \to 0$ as $t \to \infty$, that is, in the long run, the temperature $\phi(x,t)$ tends to 0 at all points in the region $0 < x < L$.

IBVP for a one-dimensional heat conduction problem with thermally insulated boundary

Consider the following IBVP:

"Solve the PDE

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\rho c}{\kappa} \frac{\partial \phi}{\partial t} \text{ for } 0 < x < L \text{ and } t > 0,$$

subject to the initial and boundary conditions

$$\begin{aligned}
\phi(x,0) &= f(x) \text{ for } 0 < x < L \text{ (IC)},\\
\left.\frac{\partial\phi}{\partial x}\right|_{x=0} &= 0 \text{ and } \left.\frac{\partial\phi}{\partial x}\right|_{x=L} = 0 \text{ for } t \geq 0 \text{ (BCs)},
\end{aligned}$$

where $f(x)$ is a suitably given function giving the initial temperature distribution in the solution domain."

Note that the BCs $\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0$ and $\left.\frac{\partial\phi}{\partial x}\right|_{x=L} = 0$ imply that the planes $x = 0$ and $x = L$ are thermally insulated (that is, heat cannot flow through the planes).

Following closely the four steps used in solving the earlier IBVPs, we solve the IBVP here as follows.

Letting $\phi(x,t) = X(x)T(t)$, we obtain the ODEs

$$X''(x) - \gamma X(x) = 0 \text{ and } T'(t) = \frac{\gamma\kappa}{\rho c}T(t),$$

where γ is the separation constant.

For the case where $\gamma = p^2$ (p is a positive real number), we obtain

$$\begin{aligned}
\phi(x,t) &= (Ve^{px} + We^{-px})e^{p^2\kappa t/(\rho c)}\\
&\Rightarrow \frac{\partial\phi}{\partial x} = p(Ve^{px} - We^{-px})e^{p^2\kappa t/(\rho c)},
\end{aligned}$$

where V and W are arbitrary constants. For the above to satisfy the given BCs, we require

$$V - W = 0 \text{ and } Ve^{pL} - We^{-pL} = 0.$$

For $p \neq 0$, this leads to $V = W = 0$, hence the trivial solution $\phi(x,t) = 0$.

For the case where $\gamma = 0$, we obtain

$$\phi(x,t) = R + Qx \Rightarrow \frac{\partial \phi}{\partial x} = Q,$$

where R and Q are arbitrary constants. For the above to satisfy the given BCs, $Q = 0$. The constant R remains arbitrary. Thus, for $\gamma = 0$, we obtain $\phi(x,t) = R$. Note that, in the earlier IBVP where the BCs are $\phi(0,t) = 0$ and $\phi(L,t) = 0$, we obtain the trivial solution $\phi(x,t) = 0$ for $\gamma = 0$. Here, since R is an arbitrary constant, $\phi(x,t) = R$ is not necessarily trivial. We cannot ignore this solution later on. Let us denote this solution by $\phi_0(x,t) = F_0$, where F_0 is an arbitrary constant.

Lastly, for $\gamma = -p^2$ (p is a positive real number), we obtain

$$\begin{aligned} \phi(x,t) &= (E\sin(px) + F\cos(px))e^{-p^2\kappa t/(\rho c)}, \\ &\Rightarrow \frac{\partial \phi}{\partial x} = p(E\cos(px) - F\sin(px))e^{-p^2\kappa t/(\rho c)}, \end{aligned}$$

where E and F are arbitrary constants. The BC $\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = 0$ gives $E = 0$, while the BC $\left.\frac{\partial \phi}{\partial x}\right|_{x=L} = 0$ gives

$$\sin(pL) = 0 \Rightarrow p = \frac{n\pi}{L} \text{ for } n = 1, 2, \cdots .$$

Thus, we obtain the non-trivial solutions

$$\phi_n(x,t) = F_n \cos(\frac{n\pi x}{L})e^{-n^2\pi^2\kappa t/(L^2\rho c)} \text{ for } n = 1, 2, \cdots ,$$

where F_1, F_2, $\cdots$ are constants. The above solutions satisfy the BCs of the IBVP.

If we sum up all the non-trivial solutions above (not forgetting $\phi_0(x,t) = F_0$ from $\gamma = 0$),we obtain the series solution

$$\phi(x,t) = F_0 + \sum_{n=1}^{\infty} F_n \cos(\frac{n\pi x}{L}) e^{-n^2\pi^2\kappa t/(L^2\rho c)}. \tag{20}$$

It is easy to check that the above series solution still satisfies the BCs of the IBVP. So, only the IC remains to be satisfied. Applying the IC, we obtain

$$F_0 + \sum_{n=1}^{\infty} F_n \cos(\frac{n\pi x}{L}) = f(x) \text{ for } 0 < x < L.$$

The answer to finding $F_0,\ F_1,\ F_2, \cdots$ is in FSP I on page 35, that is,

$$\begin{aligned} F_0 &= \frac{1}{L}\int_0^L f(x)dx, \\ F_n &= \frac{2}{L}\int_0^L f(x)\cos(\frac{n\pi x}{L})dx \ (n = 1, 2, \cdots). \end{aligned} \tag{21}$$

The required solution of the IBVP here, as given by (20) together with (21), is such that ϕ tends to the constant F_0 as time increases, that is, the thermal energy eventually distributes itself out uniformly. From (21), we see that the constant F_0 gives the average value of the initial temperature.

Inhomogeneous PDEs and BCs

The method of solution above for solving the IBVPs for the vibrating string and the one-dimensional time-dependent heat conduction problems requires the governing PDE to be homogeneous and linear, as the method makes use of the theorem on page 20 to sum up solutions of the PDE. According to the theorem, the sum of the solutions is still a solution of the PDE only if the PDE is homogeneous and linear.

The method of solution also requires the BCs to be of the form given by either

$$\phi(0,t) = 0 \text{ and } \phi(L,t) = 0 \text{ for } t \geq 0, \tag{22}$$

or

$$\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = 0 \quad \text{and} \quad \left.\frac{\partial \phi}{\partial x}\right|_{x=L} = 0 \quad \text{for } t \geq 0. \tag{23}$$

In the mathematics literature, a BVP governed by a PDE with the unknown function ϕ specified everywhere on the boundary of the solution domain (such as the BCs in (22)) is called a *Dirichlet* problem. If $\underline{\mathbf{n}} \bullet \underline{\nabla} \phi$ ($\underline{\mathbf{n}}$ is a unit normal vector to the boundary) is specified everywhere on the boundary, such as in (23), the BVP is called a *Neumann* problem.

The BCs in (22) and (23) are said to be *homogeneous*, as ϕ and $\underline{\mathbf{n}} \bullet \underline{\nabla} \phi$ are given the value 0 at all points on the boundary.

In some cases, even though the PDE in a IBVP is not homogeneous linear or the BCs are not of the form mentioned above, we may still be able to use the method of solution here to solve the IBVP if we can find a way to reformulate the IBVP as one governed by a homogeneous linear PDE and homogeneous BCs of the form (22) or (23).

If the unknown function in the IBVP is $\phi(x,t)$, we may *attempt* to reformulate the IBVP as follows. Let $\phi(x,t) = U(x,t) + \psi(x,t)$, where $U(x,t)$

is any solution of the PDE satisfying *only* the BCs of the IBVP. Note that $U(x,t)$ does not have to satisfy the ICs. If we can find $U(x,t)$, we can then recast or reformulate the IBVP in terms of $\psi(x,t)$. If the IBVP in $\psi(x,t)$ is governed by a homogeneous linear PDE with BCs of the form (22) or (23), then we can solve for $\psi(x,t)$.

<u>*Examples*</u>

1. *Solve*

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\rho c}{\kappa}\frac{\partial \phi}{\partial t} \text{ for } 0 < x < L \textit{ and } t > 0,$$

subject to the initial and boundary conditions

$$\begin{aligned} \phi(x,0) &= \phi_0 \textit{ (constant) for } 0 < x < L \textit{ (ICs)} \\ \phi(0,t) &= \phi_1 \textit{ (constant) and } \phi(L,t) = \phi_2 \textit{ (constant)} \\ &\qquad\qquad\qquad\qquad \textit{for } t > 0 \textit{ (BCs).} \end{aligned}$$

Solution:

To reformulate the IBVP in terms of homogeneous PDE and BCs, we let $\phi(x,t) = U(x,t) + \psi(x,t)$.

Since $U(x,t)$ is any solution of the PDE, we require $U(x,t)$ to satisfy

$$\frac{\partial^2 U}{\partial x^2} = \frac{\rho c}{\kappa}\frac{\partial U}{\partial t}.$$

Furthermore, $U(x,t)$ satisfies the BCs, that is,

$$U(0,t) = \phi_1 \text{ and } U(L,t) = \phi_2.$$

The PDE does not contain any nonhomogeneous term, as it is homogeneous linear. The BCs are in general nonhomogeneous, since ϕ_1 and ϕ_2 are not necessarily 0. Since ϕ_1 and ϕ_2 are constants which do not change with time, why not just look for $U(x,t)$ that is independent of time, as $U(x,t)$ does not have to satisfy the ICs?

So, we let $U(x,t) = V(x)$. If we substitute this into the PDE in $U(x,t)$ above, we find that

$$\frac{\partial^2}{\partial x^2}(V(x)) = \frac{\rho c}{\kappa}\frac{\partial}{\partial t}(V(x)),$$

which can be rewritten as the ODE

$$\frac{d^2V}{dx^2} = 0.$$

The ODE abive is easy to solve − its general solution is given by $v(x) = px + q$, where p and q are arbitrary constants. From the BCs of the given IBVP, we require $v(x)$ to satisfy

$$v(0) = \phi_1 \text{ and } v(L) = \phi_2.$$

From the above, we find that $q = \phi_1$ and $p = (\phi_2 - \phi_1)/L$. Thus,

$$\phi(x,t) = \frac{(\phi_2 - \phi_1)}{L}x + \phi_1 + \psi(x,t). \tag{24}$$

If we substitute (24) into the governing PDE, we obtain

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\rho c}{\kappa}\frac{\partial\psi}{\partial t}, \tag{25}$$

that is, the same homogeneous linear PDE in the earlier problem but with the unknown function ϕ replaced by ψ.

The BCs become homogeneous:

$$\psi(0,t) = 0 \text{ and } \ \psi(L,t) = 0 \text{ for } t > 0. \tag{26}$$

The IC becomes:

$$\psi(x,0) = \phi_0 - \phi_1 \ - \frac{(\phi_2 - \phi_1)}{L}x \quad for\ 0 < x < L. \tag{27}$$

We will now solve the IBVP defined by (25), (26) and (27). This IBVP has the same governing PDE and BCs as the IBVP on on page 51. The only slight difference in the IBVP here and the one on page 51 is in the initial condition where $\psi(x, 0)$ is given by a linear function of x (instead of a constant).

Following closely the analysis for the IBVP on page 51, we obtain

$$\psi(x,t) = \sum_{n=1}^{\infty} E_n \sin(\frac{n\pi x}{L}) e^{-n^2\pi^2\kappa t/(L^2\rho c)}. \tag{28}$$

as a series solution of (25) which satisfies the BCs in (26).

The only remaining condition that needs to be satisfied is the IC in (27). From (28), the IC in (27) gives

$$\sum_{n=1}^{\infty} E_n \sin(\frac{n\pi x}{L}) = \phi_0 - \phi_1 - \frac{(\phi_2 - \phi_1)}{L} x \quad \text{for } 0 < x < L.$$

From FSP II on page 35, we obtain

$$\begin{aligned} E_n &= \frac{2}{L} \int_0^L (\phi_0 - \phi_1 - \frac{(\phi_2 - \phi_1)}{L} x) \sin(\frac{n\pi x}{L}) dx \\ &= \frac{2(\phi_2 - \phi_0)\cos(n\pi) + \phi_0 - \phi_1}{n\pi} \end{aligned}$$

Thus, the required solution is

$$\begin{aligned} \phi(x,t) &= \frac{(\phi_2 - \phi_1)}{L} x + \phi_1 \\ &\quad + \sum_{n=1}^{\infty} \frac{2(\phi_2 - \phi_0)\cos(n\pi) + \phi_0 - \phi_1}{n\pi} \sin(\frac{n\pi x}{L}) e^{-n^2\pi^2\kappa t/(L^2\rho c)}. \end{aligned}$$

2. *Solve*

$$\frac{\partial^2\phi}{\partial x^2} + \pi^2 \sin(\pi x) = \frac{\partial \phi}{\partial t} \text{ for } 0 < x < 1 \text{ and } t > 0,$$

subject to the initial and boundary conditions

$$\begin{aligned} \phi(x,0) &= \sin(\pi x) \textit{ for } 0 < x < 1, \\ \phi(0,t) &= 0 \textit{ and } \phi(1,t) = 1 \textit{ for } t > 0. \end{aligned}$$

Solution:

The PDE is inhomogeneous and so is the BC $\phi(1,t) = 1$.

To reformulate the IBVP in terms of homogeneous PDE and BCs, let $\phi(x,t) = U(x,t) + \psi(x,t)$,where $U(x,t)$ is to be determined from

$$\frac{\partial^2 U}{\partial x^2} + \pi^2 \sin(\pi x) = \frac{\partial U}{\partial t},\ U(0,t) = 0,\ U(1,t) = 1.$$

Since the BCs and the nonhomogenous term (that is, $\pi^2 \sin(\pi x)$) do not change with time, why not just look for $U(x,t)$ that is independent of time, as $U(x,t)$ does not have to satisfy the ICs?

So, we let $U(x,t) = V(x)$. If we substitute this into the PDE and BCs in $U(x,t)$ above, we find that

$$\frac{d^2V}{dx^2} + \pi^2 \sin(\pi x) = 0,\ V(0) = 0,\ V(0) = 1.$$

If we rewrite the ODE in $v(x)$ as

$$\frac{d^2V}{dx^2} = -\pi^2 \sin(\pi x),$$

and integrate it twice, we obtain the general solution

$$v(x) = \sin(\pi x) + px + q,$$

where p and q are arbitrary constants.

The condition $v(0)$ gives $q = 0$,while $v(1) = 1$ gives $p = 1$. Thus, $\phi(x, t) = \sin(\pi x) + x + \psi(x, t)$ and

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= \pi \cos(\pi x) + 1 + \frac{\partial \psi}{\partial x}, \\
\frac{\partial^2 \phi}{\partial x^2} &= -\pi^2 \sin(\pi x) + \frac{\partial^2 \psi}{\partial x^2}, \\
\frac{\partial \phi}{\partial t} &= \frac{\partial \psi}{\partial t}.
\end{aligned}$$

It is clear that the governing PDE can be rewritten as

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{\partial \psi}{\partial t}.$$

The BCs in terms of $\psi(x, t)$ are given by

$$\begin{aligned}
\phi(0, t) &= 0 \Rightarrow \psi(0, t) = 0, \\
\phi(1, t) &= 1 \Rightarrow \psi(1, t) = 0.
\end{aligned}$$

The IC in terms of $\psi(x, t)$ is given by

$$\begin{aligned}
\phi(x, 0) &= \sin(\pi x) \\
&\Rightarrow \phi(x, 0) = \sin(\pi x) + x + \psi(x, 0) = \sin(\pi x) \\
&\Rightarrow \psi(x, 0) = -x.
\end{aligned}$$

Using the analysis for the IBVP on page 51 with $\rho c/\kappa = 1$ and $L = 1$, we obtain

$$\psi(x, t) = \sum_{n=1}^{\infty} E_n \sin(n\pi x) e^{-n^2\pi^2 t},$$

where E_n are coefficients to be obtained from

$$\sum_{n=1}^{\infty} E_n \sin(n\pi x) = -x \text{ for } 0 < x < 1.$$

From FSP II on 35, we obtain

$$\begin{aligned} E_n &= -2\int_0^1 x\sin(n\pi x)dx \\ &= \frac{2\cos(n\pi)}{n\pi}. \end{aligned}$$

Thus, the required solution is

$$\phi(x,t) = \sin(\pi x) + x + \sum_{n=1}^{\infty} \frac{2\cos(n\pi)}{n\pi}\sin(n\pi x)e^{-n^2\pi^2 t}.$$

3. *Consider the IBVP defined by the PDE*

$$\frac{\partial^2\phi}{\partial x^2} - 6xe^{-t} = \frac{\partial^2\phi}{\partial t^2} + \frac{\partial\phi}{\partial t} \text{ for } 0 < x < 1 \text{ and } t > 0,$$

and the initial and boundary conditions

$$\begin{aligned} \phi(x,0) &= x^3 - x^2 \textit{ and } \left.\frac{\partial\phi}{\partial t}\right|_{t=0} = 0 \textit{ for } 0 < x < 1, \\ \phi(0,t) &= 0 \textit{ and } \phi(1,t) = e^{-t} \textit{ for } t > 0. \end{aligned}$$

Reformulate the IBVP as one governed by a homogeneous PDE with homogeneous BCs.

Solution:

To reformulate the IBVP in terms of homogeneous PDE and BCs, let $\phi(x,t) = U(x,t) + \psi(x,t)$,where $U(x,t)$ is to be determined from

$$\frac{\partial^2 U}{\partial x^2} - 6xe^{-t} = \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} \text{ for } 0 < x < 1 \text{ and } t > 0,$$

and

$$U(0,t) = 0 \text{ and } U(1,t) = e^{-t} \text{ for } t > 0.$$

Since the nonhomogeneous terms in the PDE and the second BC contain the factor e^{-t}, let us try looking for $U(x,t)$ in the form $U(x,t) = V(x)e^{-t}$. If we substitute $U(x,t) = V(x)e^{-t}$ into the PDE and BCs above, we obtain

$$\frac{d^2V}{dx^2} = 0, V(0) = 0, V(1) = 1.$$

The ODE may be integrated twice to obtain $V(x) = x^3 + Ax + B$. Applying the BCs, we find that $A = 0$ and $B = 0$. Thus, $V(x) = x^3$.

To reformulate the given IBVP, we use the substitution

$$\phi(x,t) = x^3e^{-t} + \psi(x,t).$$

We find that

$$\begin{aligned} \frac{\partial^2\phi}{\partial x^2} &= 6xe^{-t} + \frac{\partial^2\psi}{\partial x^2}, \\ \frac{\partial\phi}{\partial t} &= -x^3e^{-t} + \frac{\partial\psi}{\partial t}, \\ \frac{\partial^2\phi}{\partial t^2} &= x^3e^{-t} + \frac{\partial^2\psi}{\partial t^2}. \end{aligned}$$

Substituting the results above into the PDE in ϕ, we obtain

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial t^2} + \frac{\partial\psi}{\partial t}.$$

From the initial conditions, we find that

$$\begin{aligned} x^3 + \psi(x,0) &= x^3 - x^2 \Rightarrow \psi(x,0) = -x^2, \\ -x^3 + \left.\frac{\partial\psi}{\partial t}\right|_{t=0} &= 0 \Rightarrow \left.\frac{\partial\psi}{\partial t}\right|_{t=0} = x^3. \end{aligned}$$

From the boundary conditions, we find that

$$\begin{aligned} 0^3e^{-t} + \psi(0,t) &= 0 \Rightarrow \psi(0,t) = 0, \\ 1^3e^{-t} + \psi(1,t) &= e^{-t} \Rightarrow \psi(1,t) = 0. \end{aligned}$$

Hence, the reformulated IBVP is as stated below.

Solve

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial t^2} + \frac{\partial\psi}{\partial t} \text{ for } 0 < x < 1 \text{ and } t > 0,$$

and the initial and boundary conditions

$$\begin{aligned} \psi(x,0) &= -x^2 \textit{ and } \left.\frac{\partial\psi}{\partial t}\right|_{t=0} = x^3 \textit{ for } 0 < x < 1, \\ \psi(0,t) &= 0 \textit{ and } \psi(1,t) = 0 \textit{ for } t > 0. \end{aligned}$$

A two-dimensional steady state heat conduction problem

IBVP for a particular problem

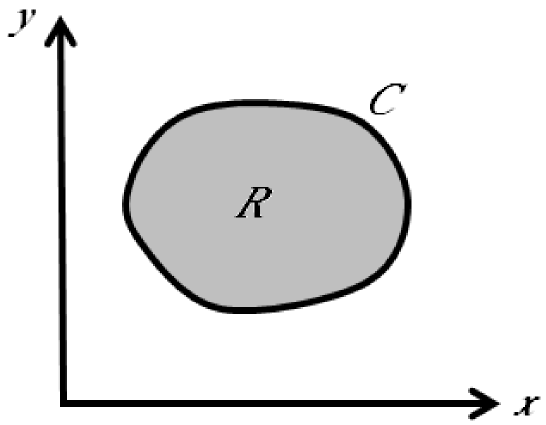
Recall the three-dimensional heat equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \frac{\rho c}{\kappa}\frac{\partial\phi}{\partial t}.$$

If the temperature ϕ is time independent and does not change with the z coordinate, the heat equation above reduces to the two-dimensional Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0,$$

to be solved in a two-dimensional region R on the Oxy plane subject to suitably prescribed conditions on the boundary of R. The boundary of R is a curve on the Oxy plane. See figure below.



A particular two-dimensional steady state heat conduction problem that can be solved by using the method of separation of variables and Fourier series is mathematically stated below.

"Solve the PDE

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

in the rectangular region $0 < x < a,\ 0 < y < b$,

subject to the boundary conditions

$$\left.\begin{array}{r}\phi(0,y) = 0\\ \phi(a,y) = 0\end{array}\right\}\ \text{for } 0 < y < b,$$

$$\left.\begin{array}{r}\phi(x,0) = 0\\ \phi(x,b) = f(x)\end{array}\right\}\ \text{for } 0 < x < a,$$

where $f(x)$ is a given function."

As the function ϕ in the PDE is independent of time, no IC is involved. The above is a BVP where the temperature ϕ is specified on all the four sides of the rectangular region. The temperature is prescribed to be 0 on three of the sides.

We will extend the four step method of solution we have used earlier on to solve the vibrating string problem and the one-dimensional non-steady heat conduction problem to solve the BVP above.

Step 1: Method of separation of variables to obtain a pair of ODEs

If we let $\phi(x,y) = X(x)Y(y)$, the two-dimensional Laplace's equation gives

$$X''(x)Y(y) + X(x)Y''(y) = 0.$$

On dividing both sides of the above equation by $X(x)Y(y)$and re-arranging the terms in the equation, we obtain

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \gamma,$$

where γ is the separation constant.

We obtain the ODEs

$$\begin{aligned} X''(x) - \gamma X(x) &= 0, \\ Y''(y) + \gamma Y(y) &= 0. \end{aligned}$$

Step 2: Dealing with the homogeneous BCs on $x = 0$ and $x = a$

The pair of homogeneous BCs on the two vertical sides $x = 0$ and $x = a$, that is, $\phi(0, y) = 0$ and $\phi(a, y) = 0$ for $0 < y < b$, are analogous to the ICs in the vibrating string problem on page 31. We solve the ODEs in Step 1 subject to $\phi(0, y) = 0$ and $\phi(a, y) = 0$ for $0 < y < b$.

We can rule out $\gamma = 0$ as it gives $\phi(x, y) = (p_0 x + q_0)(p_1 y + q_1)$, where p_0, q_0, p_1 and q_1 are constants. If we attempt to satisfy the BCs $\phi(0, y) = 0$ and $\phi(a, y) = 0$ for $0 < y < b$, we end up with $p_0 = q_0 = 0$ and hence we obtain the trivial solution $\phi = 0$ which cannot satisfy $\phi(x, b) = f(x)$ for $0 < x < a$ (unless $f(x) = 0$).Thus, $\gamma = 0$ is not acceptable.

What about γ being a positive constant? If $\gamma = p^2$, where p is a non-zero real number, then

$$\begin{aligned}
X''(x) - p^2 X(x) &= 0 \quad \Rightarrow X(x) = Ae^{px} + Be^{-px} \\
Y''(y) + p^2 Y(y) &= 0 \quad \Rightarrow Y(y) = D\cos(py) + E\sin(py).
\end{aligned}$$

that is, $\phi(x, y) = (Ae^{px} + Be^{-px})(D\cos(py) + E\sin(py))$,where A, B, C and D are constants. If we attempt to satisfy the BCs on the vertical sides, we have to choose A and B to satisfy $A + B = 0$ and $Ae^{pa} + Be^{-pa} = 0$. This leads to $A = 0$ and $B = 0$ and hence $\phi(x, y) = 0$ (again!). Hence, γ being a positive constant is also not acceptable.

If γ is negative, that is, $\gamma = -p^2$, where p is a non-zero real number, then

$$\begin{aligned}
X''(x) + p^2 X(x) &= 0 \quad \Rightarrow X(x) = A\cos(px) + B\sin(px) \\
Y''(y) - p^2 Y(y) &= 0 \quad \Rightarrow Y(y) = Ce^{py} + De^{-py}.
\end{aligned}$$

Hence, we obtain

$$\phi(x,y) = (A\cos(px) + B\sin(px))(Ce^{py} + De^{-py}),$$

where A, B, C and D are constants.

The BC on the left vertical side, that is, $\phi(0,y) = 0$ for $0 < y < b$, gives

$$A(Ce^{py} + De^{-py}) = 0 \text{ for } 0 < y < b \quad \Rightarrow A = 0.$$

The BC on the right vertical side, that is, $\phi(a,y) = 0$ for $0 < y < b$, gives

$$B\sin(pa)(Ce^{py} + De^{-py}) = 0 \text{ for } 0 < y < b.$$

We cannot choose $B = 0$ as this leads to the undesirable solution $\phi(x,y) = 0$. Thus, we have:

$$\sin(pa) = 0 \Rightarrow p = \frac{n\pi}{a} \text{ for } n = 1, 2, \cdots .$$

We are on familiar ground now!

For each value of n above, we have a solution denoted by

$$\phi_n(x,y) = (E_n e^{n\pi y/a} + F_n e^{-n\pi y/a})\sin(\frac{n\pi x}{a}),$$

where E_n and F_n are arbitrary constants.

Step 3: Forming series solution

We sum up all the non-trivial solutions in Step 2 to obtain

$$\phi(x,y) = \sum_{n=1}^{\infty}(E_n e^{n\pi y/a} + F_n e^{-n\pi y/a})\sin(\frac{n\pi x}{a}).$$

Since the Laplace's equation is homogeneous linear, the series is still a solution of the PDE. We can easily verify that BCs at $x = 0$ and $x = a$ are also satisfied by the series solution.

Step 4: Dealing with the remaining BCs on the horizontal sides

Let us tackle the BC on the boundary horizonatl side first, that is, $\phi(x, 0) = 0$ for $0 < x < a$. From the series solution in Step 3, we find that

$$\sum_{n=1}^{\infty}(E_n + F_n)\sin(\frac{n\pi x}{a}) \text{ for } 0 < x < a \Rightarrow E_n = -F_n.$$

The solution series can now be reduced to

$$\phi(x, y) = \sum_{n=1}^{\infty} E_n(e^{n\pi y/a} - e^{-n\pi y/a})\sin(\frac{n\pi x}{a}).$$

The only BC left to be satisfied is the inhomogeneous BC $\phi(x, b) = f(x)$ for $0 < x < a$. If we apply this BC to the series solution, we obtain

$$\sum_{n=1}^{\infty} E_n(e^{n\pi b/a} - e^{-n\pi b/a})\sin(\frac{n\pi x}{a}) = f(x) \text{ for } 0 < x < a.$$

If we think of $E_n(e^{n\pi b/a} - e^{-n\pi b/a})$ as the coefficient b_n in the Fourier series in FYP II on page 35, we obtain

$$E_n = \frac{2}{a(e^{n\pi b/a} - e^{-n\pi b/a})}\int_0^a f(x)\sin(\frac{n\pi x}{a})dx \ (n = 1, 2, \cdots).$$

Thus, for a given $f(x)$, we can evaluate E_n, obtaining a complete solution for the BVP under consideration.

Tutorial 3

1. If the influence of gravitational force is taken into consideration, the vibrating string problem considered during lecture is governed by the inhomogeneous second order linear PDE
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \frac{g}{c^2},$$
where the constant g is the acceleration due to gravity. An IBVP of interest is to solve the above inhomogeneous PDE in $u(x,t)$ subject to to the initial and boundary conditions
$$\begin{aligned} u(x,0) &= 0 \ \text{ and } \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \ \text{ for } 0 < x < L, \\ u(0,t) &= 0 \ \text{ and } \ u(L,t) = 0 \ \text{ for } t \geq 0. \end{aligned}$$
By letting $u(x,t) = v(x) + \psi(x,t)$, where $v(x)$ is the time independent solution of the inhomogeneous PDE satisfying $v(0) = 0$ and $v(L) = 0$, recast the IBVP above in terms of the function $\psi(x,t)$ (find $v(x)$ and give the governing PDE, BCs and ICs in terms of $\psi(x,t)$). (You are not required to solve the IBVP.)

2. Consider the following BVP:

"Solve the two-dimensional Poisson's equation
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2y$$
in the square region $0 < x < 1, \ 0 < y < 1$,

subject to the boundary conditions
$$\left.\begin{aligned} \phi(0,y) = 1 \\ \phi(1,y) = 1 \end{aligned}\right\} \text{ for } 0 < y < 1,$$
$$\left.\begin{aligned} \phi(x,0) &= 1 \\ \phi(x,1) &= x(1-x) \end{aligned}\right\} \text{ for } 0 < x < 1."$$

(a) Verify by direct substitution that $v(x, y) = 1 + x(1 - x)y$ is a particular solution of the Poisson's equation above.

(b) By letting $\phi(x, y) = v(x, y) + \psi(x, y)$, where $v(x, y)$ is as given in part (a), recast the BVP in terms of $\psi(x, y)$. (You are not required to solve the BVP.)

3. Solve the PDE

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

in the square region $0 < x < 1,\ 0 < y < 1$,

subject to the boundary conditions

$$\left.\begin{array}{r} \phi(0, y) = 0 \\ \phi(1, y) = 0 \end{array}\right\} \text{ for } 0 < y < 1,$$

$$\left.\begin{array}{r} \left.\dfrac{\partial \phi}{\partial y}\right|_{y=0} = 0 \\ \phi(x, 1) = 2x \end{array}\right\} \text{ for } 0 < x < 1.$$

[*Note.* During lecture, we have solved a similar BVP with the same PDE and the same BCs on $x = 0$ and $x = 1$ (see the section "A two-dimensional steady state heat conduction problem" in lecture note). In solving that BVP, we have derived solutions satisfying the PDE and the BCs on $x = 0$ and $x = 1$. You may use those solutions here without deriving them from scratch. You may also find useful the integration formula:

$$\int x \sin(px) dx = -\frac{-\sin(px) + px\cos(px)}{p^2} + C.\ ]$$

<u>*Additional problems*</u>

4. Solve the PDE

$$\frac{\partial^2\phi}{\partial x^2} = \frac{\partial\phi}{\partial t} \text{ for } 0 < x < 1 \text{ and } t > 0,$$

subject to

$$\begin{aligned} \phi(x,0) &= 3x \text{ for } 0 < x < 1, \\ \left.\frac{\partial\phi}{\partial x}\right|_{x=0} &= 1 \text{ and } \left.\frac{\partial\phi}{\partial x}\right|_{x=1} = 1 \text{ for } t > 0. \end{aligned}$$

[*Note.* Let $\phi(x,t) = v(x) + \psi(x,t)$. Choose *any* function $v(x)$ that satisfies the PDE and the conditions $v'(0) = 1$ and $v'(1) = 1$. Recast the IBVP in terms of $\psi(x,t)$. You do not have to start from scratch. You may use any result derived during lecture and the integration formula

$$\int x\cos(px)dx = \frac{\cos(px) + px\sin(px)}{p^2} + C. \qquad]$$

5. Use the method of separation of variables to find a set of non-trivial solutions $\phi(x,t)$ of the PDE

$$2\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2}$$

such that

$$\left.\frac{\partial\phi}{\partial x}\right|_{x=0} = 0 \text{ and } \phi(1,t) = 0.$$

6. Consider the following BVP:

"Solve the two-dimensional Laplace's equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

in the square region $0 < x < 1,\ 0 < y < 1$, subject to the boundary conditions

$$\left.\begin{array}{r}\phi(0,y)=1\\ \phi(1,y)=-1\end{array}\right\}\text{ for }0\quad<\quad y<1,$$

$$\left.\begin{array}{r}\left.\dfrac{\partial\phi}{\partial y}\right|_{y=0}=0\\ \phi(x,1)=1\end{array}\right\}\text{ for }0\quad<\quad x<1.\text{"}$$

By letting $\phi(x,y)=v(x)+\psi(x,y)$, where $v(x)$ is to be obtained by solving the ODE $\dfrac{d^2v}{dx^2}=0$ subject to $v(0)=1$ and $v(1)=-1$, recast the BVP above in terms of $\psi(x,y)$.(You are not required to solve the BVP.)