Briefing 1-MA3004(2) - nanananna

Mathematical Methods In Engineering (Nanyang Technological University)

MA3004 in Week 1

General information

Part I: Partial Differential Equations (PDEs) (3 weeks)
Part II: Finite Elements Method (FEM) (5-6 weeks)
Part III: Computational Fluid Dynamics (CFD) (4-5 weeks)

CA1 (20%) in Week 5
CA2 (20%) after midterm break
Final exam (60%) Restricted open-book (oneA4 cheat sheet)

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Part I: Partial differential equations (PDEs)

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All the learning materials for Part I are posted in the section "Part I on PDEs" in the course website at NTULearn.

Go through Video Lessons 1 and 2 or read pages 1-24 of the 76 page PDF containing the lecture notes. Try questions in Tutorial 1.

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Important Points in Video Lesson 1

What is a partial differential equation (PDE)?

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \qquad \text{Laplace's equation}$$

A PDE is an equation containing one or more partial derivatives of an unknown real function of two or more real variables.

Laws of physics can be expressed using PDEs.

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Order of a PDE

The order of a PDE is the order of the highest order partial derivative of the unknown function in the PDE.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

PDE of order 2 (2nd order PDE)

$$\frac{\partial}{\partial x}\left(p \frac{\partial^2 p}{\partial t^2}\right) = \frac{\partial p}{\partial x} \Rightarrow p \frac{\partial^3 p}{\partial x \partial t^2} + \frac{\partial p}{\partial x} \cdot \frac{\partial^2 p}{\partial t^2} = \frac{\partial p}{\partial t}$$

PDE of order 3 (3rd order PDE)

$$\left(\frac{\partial u}{\partial x}\right)^{10} + u \frac{\partial u}{\partial y} = 1$$

PDE of order 1 (1st order PDE)

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Solutions of a PDE

Consider a PDE in ϕ.

A function f is said to be a solution of the PDE if the LHS of the PDE equals the RHS of the PDE when we replace ϕ by f in the LHS and RHS.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\phi = 0$ is a solution.

$\phi = 1$ is a solution.

$\phi = 2x + y$ is a solution.

$\phi = x^2 - y^2$ is a solution.

$\phi = e^x \cdot \sin(y) + 3e^{2y}\cos(x)$ is a solution.

A PDE can have infinitely many solutions.

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Linear and nonlinear PDEs

Consider a PDE in ϕ. The PDE is said to be linear if all its terms that contain ϕ or a partial derivative of ϕ can be written in the form

$$c\phi \quad \text{or} \quad c \times (\text{a partial derivative of } \phi)$$

where the coefficient c is either a constant or a function of the independent variables of ϕ (e.g. x and y).

If the PDE is linear, c should not contain ϕ and its partial derivatives.

Otherwise, the PDE is nonlinear.

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Examples of linear and nonlinear PDEs

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$ Linear

$$\frac{\partial}{\partial x}\left(e^{2x}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(e^{2x}\frac{\partial u}{\partial y}\right) = 1$$

$$\Rightarrow e^{2x}\frac{\partial^2 u}{\partial x^2} + 2e^{2x}\frac{\partial u}{\partial x} + e^{2x}\frac{\partial^2 u}{\partial y^2} = 1$$ Linear

$$p\frac{\partial^2 p}{\partial x \partial y} + \frac{\partial^2 p}{\partial x^2} + 3p = 0$$ Nonlinear

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \sin(\phi)$$ Nonlinear

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Homogeneous linear PDEs

A linear PDE in ϕ is said to be homogeneous if all its non-zero terms contain ϕ or a partial derivative of ϕ.

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \qquad \text{Homogeneous}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + 1 + x^2 = 0 \qquad \text{Non homogeneous}$$

If $\phi = 0$ is a solution of a linear PDE then the PDE is homogeneous. Otherwise, it is nonhomogeneous.

inhomogeneous

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Theorem for superposition of solutions of homogeneous linear PDEs

If ϕ_1 and ϕ_2 are solutions of a homogeneous linear PDE then $\alpha\phi_1 + \beta\phi_2$ is also a solution of the PDE for <u>all</u> real numbers α and β.

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Important Points in Video Lesson 2

Initial boundary value problems (IBVPs)

In engineering science, PDEs are solved subject to suitably given conditions.
For time-dependent problems, initial conditions (ICs) and boundary conditions (BCs) are needed.
Time-dependent problems that require solving PDEs subject to ICs and BCs are called initial-boundary value problems (IBVPs).

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Boundary value problems (BVPs)

For time-independent problems, only BCs are needed.

Problems that require solving PDEs subject to BCs are called boundary value problems (BVPs).

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An example of IBVP for a vibrating string

Solve $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ for $0 < x < L$ and $t > 0$

Subject to

$u(x,0) = G(x)$ and $\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$ for $0 < x < L$

(ICs)

$u(0,t) = 0$ and $u(L,t) = 0$ for $t > 0$

(BCs)

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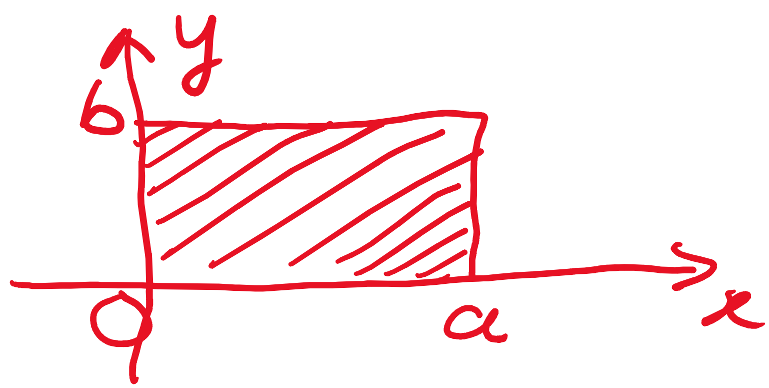
An example of BVP for a 2D time-independent heat conduction problem

Solve $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$ for $0 < x < a, 0 < y < b$

$T = T(x, y)$

subject to:

$T(0,y) = 0$ and $T(a,y) = 0$ for $0 < y < b$



$T(x,0) = 0$ and $T(x,b) = f(x)$ for $0 < x < a$

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What you will learn in Part I of MA3004

You will learn how the method of separation of variables together with Fourier series can be used to solve some IBVPs and BVPs.

Prior knowledge of ordinary differential equations (ODEs) and Fourier series will be needed.

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Review on ODEs

Constructing general solutions for $y = y(x)$

$$b\frac{dy}{dx} + cy = 0$$

$b \neq 0$ $\quad$ b and c are constants

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$a \neq 0$ $\quad$ a, b and c are constants

1st and 2nd order homogeneous linear ODEs

To construct the general solution, let $y(x) = e^{\lambda x}$, λ constant

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Review on ODEs

Find the general solution of

$$3\frac{dy}{dx} - 2y = 0 \quad \cdots\cdots \quad (*)$$

Let $y(x) = e^{\lambda x} \Rightarrow \frac{dy}{dx} = \lambda e^{\lambda x}$

Substitute into $(*)$:

$$3\lambda e^{\lambda x} - 2e^{\lambda x} = 0 \Rightarrow e^{\lambda x}(3\lambda - 2) = 0 \Rightarrow 3\lambda - 2 = 0$$
$$\Rightarrow \lambda = 2/3$$

The required general solution:

$$y(x) = Ae^{2x/3}.$$

A is an arbitrary constant

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Review on ODEs

Find the general solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0.$$

Let $y(x) = e^{\lambda x} \Rightarrow \frac{dy}{dx} = \lambda e^{\lambda x} \Rightarrow \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$

Substitute into ODE:

$$e^{\lambda x}(\lambda^2 + 3\lambda + 2) = 0 \Rightarrow \lambda^2 + 3\lambda + 2 = 0$$

$$\Rightarrow (\lambda + 1)(\lambda + 2) = 0$$

$$\Rightarrow \lambda = -1, \ \lambda = -2$$

The required general Solution is

$$y(x) = Ae^{-x} + Be^{-2x}$$

A and B are arbitrary constants

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Review on ODEs

Find the general solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0.$$

Let $y(x) = e^{\lambda x} \Rightarrow \frac{dy}{dx} = \lambda e^{\lambda x} \Rightarrow \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$

$\Rightarrow e^{\lambda x}(\lambda^2 + 4\lambda + 4) = 0 \Rightarrow \lambda^2 + 4\lambda + 4 = 0$

$\Rightarrow (\lambda + 2)^2 = 0$

$\Rightarrow \lambda = -2$ is the only solution

The required general solution is $y(x) = (Ax + B)e^{-2x}$

A and B are arbitrary constants

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Review on ODEs

Find the general solution of $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 13y = 0$.

Let $y(x) = e^{\lambda x} \Rightarrow \frac{dy}{dx} = \lambda e^{\lambda x} \Rightarrow \frac{d^2y}{dx^2} = \lambda^2 e^{\lambda x}$

$\Rightarrow e^{\lambda x}(\lambda^2 - 4\lambda + 13) = 0$

$\Rightarrow \lambda^2 - 4\lambda + 13 = 0$

$\Rightarrow \lambda = 2 \pm 3i$

The required general solution is:

$y(x) = e^{2x}\left(A\cos(3x) + B\sin(3x)\right)$

A and B are arbitrary constants

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Review on Fourier series

Fourier series Problem I (FSP I)

Let $f(x)$ be a continuous function over $0 < x < L$.

Can we make

$$a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = f(x) \text{ for } 0 < x < L ?$$

YES, if we take

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx \quad \text{and} \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

for $n = 1, 2, \cdots$

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Review on Fourier series

Fourier Series Problem II (FSP II)

Let $f(x)$ be a continuous function over $0 < x < L$.

Can we make

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x) \text{ for } 0 < x < L?$$

YES, if we let

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \text{ for } n = 1, 2, \ldots$$

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Past year examination question S1 2015-16

1. (a) Find values of the constant A such that the function $u(x,y) = A\,y^{1/2}(1+x^2)^{1/2}$ is a solution of the nonlinear partial differential equation

$$\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} - x = 0$$

at all points (x,y) where $y > 0$.

$$\frac{\partial u}{\partial x} = A y^{1/2}\left(\frac{1}{2}\right)(1+x^2)^{-1/2}(2x) = \frac{Axy^{1/2}}{(1+x^2)^{1/2}}$$

$$\frac{\partial u}{\partial y} = A\left(\frac{1}{2}\right)y^{-1/2}(1+x^2)^{1/2} = \frac{A(1+x^2)^{1/2}}{2y^{1/2}}$$

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Past year examination question S1 2015-16

$$\frac{\partial u}{\partial x} = A y^{1/2} \left(\frac{1}{2}\right)\left(1+x^2\right)^{-1/2}(2x) = \frac{A x y^{1/2}}{(1+x^2)^{1/2}}$$

$$\frac{\partial u}{\partial y} = A\left(\frac{1}{2}\right) y^{-1/2}\left(1+x^2\right)^{1/2} = \frac{A\left(1+x^2\right)^{1/2}}{2 y^{1/2}}$$

LHS of PDE

$$= \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} - x = \frac{A x y^{1/2}}{(1+x^2)^{1/2}}\cdot\frac{A(1+x^2)^{1/2}}{2 y^{1/2}} - x$$

RHS of PDE

$$= A^2 x - x = (A^2 - 1)x = 0 \quad \text{for all } x$$

$$\text{if } \frac{A^2}{2} - 1 = 0 \Rightarrow A = \pm\sqrt{2}$$