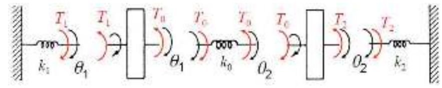
MA3005 Cheat Sheet

Control Theory (Nanyang Technological University)

1. Final value theorem: $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$. Poles of $F(s)$ lie in left half s-plane and at most 1 pole on imaginary axis. [Steady state behaviour]
2. Initial value theorem: $f(0^+) = \lim_{s\to\infty} sF(s)$. No restriction on poles location, applicable for sinusoidal functions.



$f(t)$	$F(s) = \mathcal{L}[f(t)]$	
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2+a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2+a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2-a^2}$	$s > \lvert a\rvert$
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2-a^2}$	$s > \lvert a\rvert$
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2+b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2+b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \frac{b}{(s-a)^2-b^2}$	$s - a > \lvert b\rvert$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2-b^2}$	$s - a > \lvert b\rvert$

Routh-Hurwitz (Closed Loop CE):

C.E.: $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$ $a_0, a_1, \dots, a_{n-1}, a_n > 0$ (or < 0) None of the coefficient is missing

s^n	a_0	a_2	a_4	…
s^{n-1}	a_1	a_3	a_5	…
s^{n-2}	b_1	b_2	b_3	…
s^{n-3}	c_1	c_2	c_3	…
⋮	⋮	…	…	
s^1	α	0		
s^0	β	0		

From Characteristic Eqn.

Need to be computed

$$b_1 = \frac{a_1\cdot a_2 - a_0\cdot a_3}{a_1}, \quad b_2 = \frac{a_1\cdot a_4 - a_0\cdot a_5}{a_1}$$
$$c_1 = \frac{b_1\cdot a_3 - a_1\cdot b_2}{b_1}, \quad c_2 = \frac{b_1\cdot a_5 - a_1\cdot b_3}{b_1}$$

Refer to the 1st column:
- If there is no changing of sign $\Rightarrow$ closed loop system is stable
- How many times the sign changes in the 1st column represents the number of poles in RHP

Special case

E.g

s^3	1	1
s^2	2	2
s	0	

- If 1st element in any row is zero, but others are non-zero, replace 0 with $\varepsilon > 0$ and proceed as before
- If sign above zero (ε) is the same as below, it indicates that there are a pair of imaginary roots = Marginally Stable
- If sign above zero (ε) is opposite to that of below it, it indicates that there is a sign change $\Rightarrow$ unstable system

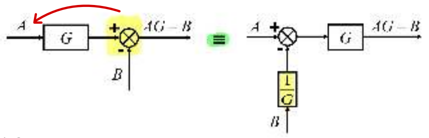
Occurrence of entire row of zeros

i.e. $(-\sigma, +\sigma)$, or $(-j\omega, +j\omega)$ or $(-\sigma \pm j\omega, +\sigma \pm j\omega)$

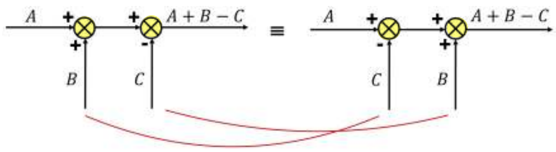
Solution: Form auxiliary polynomial from the row immediately above all zeros row

Block diagram reduction

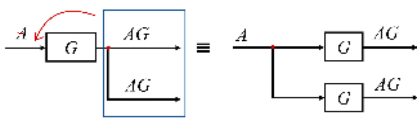
1) Moving summing point ahead of block:



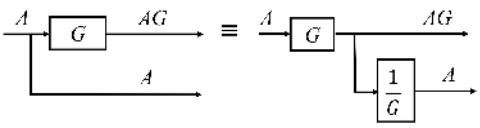
2) Consecutive summation blocks are swappable:



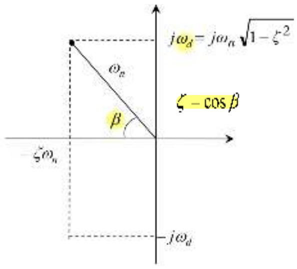
3) Moving a branch point ahead of a block



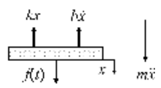
4) Moving a branch point behind a block



Poles: $s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
$= -\sigma \pm j\omega_d$

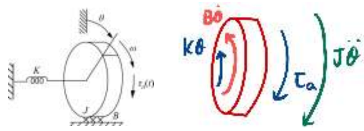


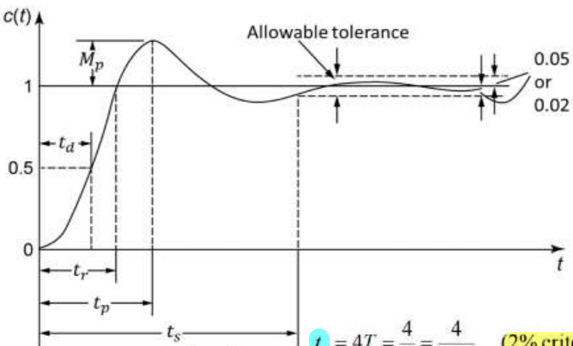
$$\sum F = f(t) - kx - b\dot{x} = m\ddot{x}$$
$$\Rightarrow m\ddot{x} + b\dot{x} + kx = f(t)$$



$\tau_a - K\theta - B\dot{\theta} = J\ddot{\theta}$

$$J\ddot{\theta} + B\dot{\theta} + K\theta = \tau_a$$





$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(\frac{\omega_d}{-\sigma}\right) = \frac{\pi - \beta}{\omega_d} = \frac{\pi - \cos^{-1}\zeta}{\omega_d}$$

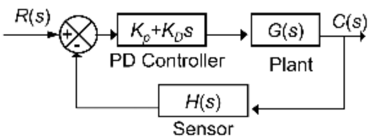
$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$ (2% criterion)

$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$ (5% criterion)

$\zeta = \cos\beta$ $\quad \zeta = \frac{-\ln(M_p)}{\sqrt{\pi^2 + \ln^2(M_p)}}$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d} \quad \%M_p = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}\times 100\% = e^{-\frac{\pi\sigma}{\omega_d}}\times 100\%$$

PD Controller: $G_c(s) = K_d(s + z_c)$

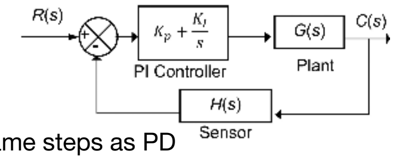


1. Determine desired closed loop poles
2. Sub Kd(S+Zc) into controller and find CLTF
3. Express CE=0 in terms of K and Zc and substitute desired poles and solve

Lead Compensator: $G_c(s) = K\frac{(s+z_c)}{(s+p_c)}$, $p_c > z_c$

Choose: $z_c \leq z_c'$ of PD controller

PI Controller: $G_c(s) = \frac{K_p(s+z_c)}{s}$



Same steps as PD controller but sub K(S+Zc)/S into controller instead for step 2

Unity-feedback system

Input	Desired e_{ss}: Steady-state error formula	Type 0: Static error constant	Type 0: Error	Type 1: Static error constant	Type 1: Error	Type 2: Static error constant	Type 2: Error
Step, $u(t) = 1$ $\frac{1}{s}$	$\frac{1}{1+K_p}$	K_p = Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $u(t) = t$ $\frac{1}{s^2}$	$\frac{1}{K_v}$	$K_v = 0$	∞	K_v = Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $u(t) = \frac{1}{2}t^2$ $\frac{1}{s^3}$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	K_a = Constant	$\frac{1}{K_a}$

$$K_p = \lim_{s\to 0} G(s) = G(0)$$
$$K_v = \lim_{s\to 0} sG(s)$$
$$K_a = \lim_{s\to 0} s^2 G(s)$$

Lag Compensator:

$$G_c(s) = K\frac{(s+z_c)}{(s+p_c)}$$
$$p_c \ll z_c \ll 1$$

For PID/Lead-Lag, just multiply the equations and then sub into controller

Root Locus Sketching

The number of branches equals to the order of CL
Ending points ($K = \infty$) are finite GH or infinite zeros
Root Locus symmetrical about the real axis

1. Starting and ending points of the locus
   a. Starting points – open-loop GH poles
   b. Ending points open-loop GH zeros
2. Locus on real axis: Segments on the real axis to the left of odd number real-axis open-loop poles and zeros are parts of root locus.
3. Break-in and break-out points
   a. Break-in (minimum K)
   b. Break-out (maximum K)

   CE = 0, express K in terms of S, then $\frac{dK}{ds} = 0$
4. Imaginary axis intersection: Set $s = j\omega$ in CE
5. Asymptotes (infinite open-loop zeros)
   a. Angles
   b. Location

   n = no. of poles, m = no. of zeroes

Asymptote angle:
$$\angle s = \frac{\pi(1+2q)}{[n-m]}, q = 0, \dots, n-m-1$$

Real axis intercept:
$$\sigma_a = \frac{\sum GH \text{ finite poles} - \sum GH \text{ finite zeros}}{n-m}$$

6. Departure and arrival angles (for complex poles/zeros)

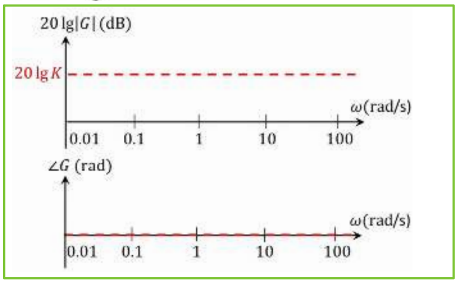
Departure angle:
$$\sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j) = -\pi$$

Arrival angle:
$$\sum_{i=1}^{m}\angle(s+z_i) - \sum_{j=1}^{n}\angle(s+p_j) = \pi$$

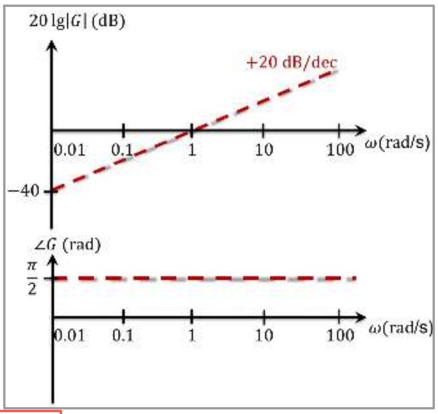
Bode Plot Sketching

1. Separate transfer function into various terms (E.g Constant, Differentiator, Integrator, 1st order pole etc)

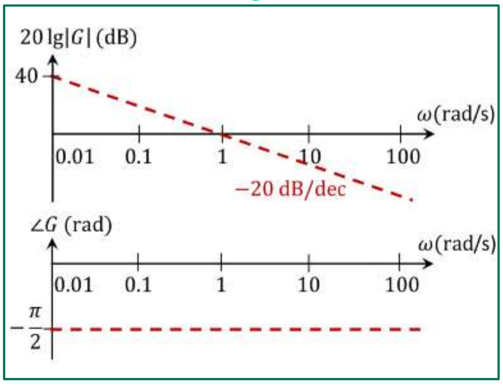
Plotting constant K



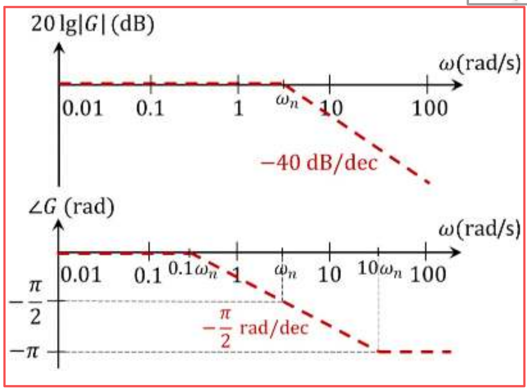
Plotting Differentiator S



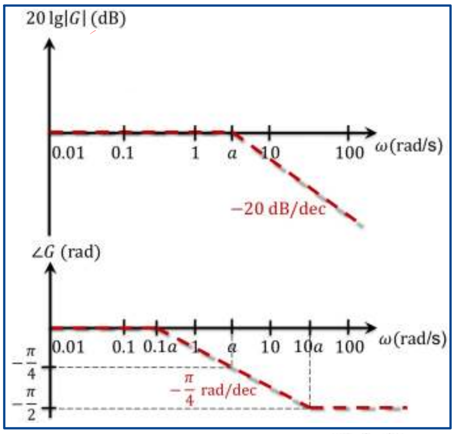
Plotting integrator $\frac{1}{s}$

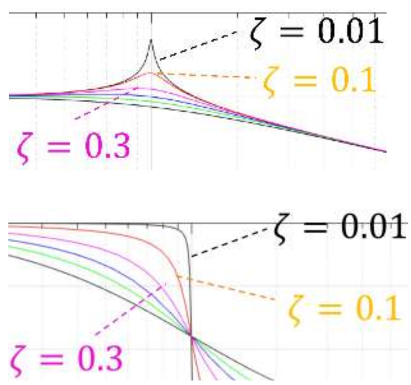


Plotting second order pole $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

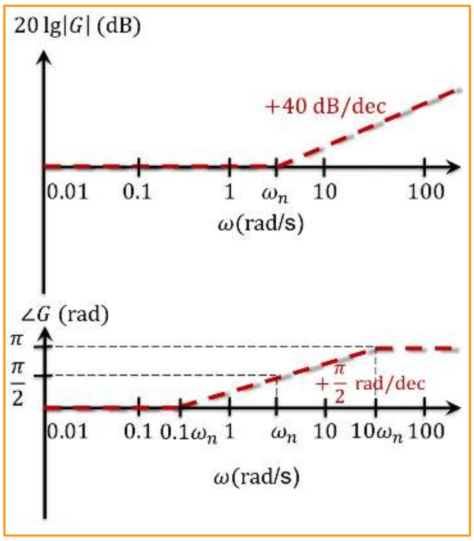


Plotting First Order Pole $\frac{a}{s+a}$

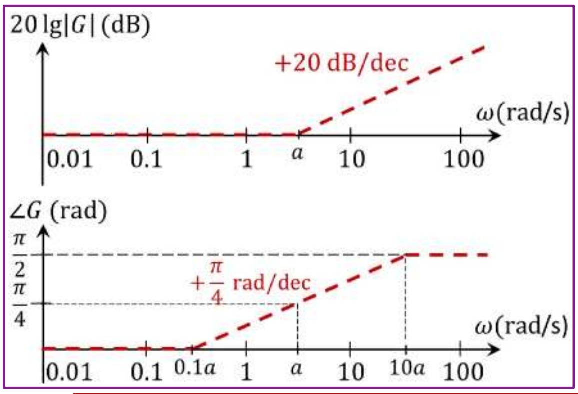




Plotting Second Order Zero $\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$



Plotting first order zero $\frac{s+a}{a}$



Bode Plot Analysis

Steady State Response (Look at magnitude plot)

1. Type 0 (No initial slope)

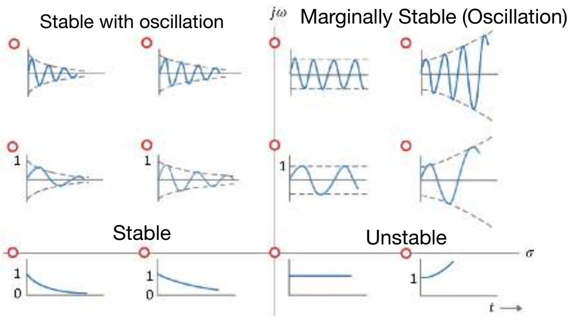
For step input, starting Y axis value = 20logKp. Find Kp and sub into formula for Ess = 1/1+Kp

2. Type 1 (Intial slope is -20db/dec)

For ramp input, draw -20db/dec line, intersecting 0db line. Wn value at point of intersection = Kv.
Sub Kv back into Ess = 1/Kv

3. Type 2 (Initial slope is -40db/dec)

For parabolic inputs, draw -40db/dec line, intersecting 0db line. Wn value at point of intersection = $\sqrt{K_a}$
Find Ka and sub back into Ess = 1/Ka



Bode Plot Analysis

Approximating Transfer Function

1. Check initial slope for Differentiators(+20db/dec) Integrators(-20db/dec)
2. Find the Corner Frequencies (0.1 Wn, Wn and 10Wn), and assume Damping Factor (0.01 for sharp, 0.3 for smooth), if theres resonance theres 2nd order pole
3. Look at slope after 10Wn, if slope changes, might have 1st order zero(E.g -60db/dec to -40db/dec)
4. Look at slope right after Wn, if it is consistent, if not somewhere before resonance theres another term(E.g first order pole)

Read Wr from graph, then use formula to find Wn

$$\omega_r = \omega_n\sqrt{1-2\zeta^2}$$

Example to find K

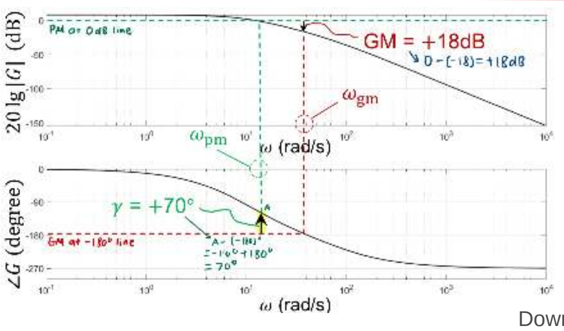
Changes depending on starting X axis value

Low Frequency Gain Analysis

Starting Y axis value

$$20\lg\left|\lim_{s\to j} G(s)\right| = 16.5 \rightarrow \left|\lim_{s\to j} G(s)\right| = 10^{\frac{16.5}{20}}$$

$$\rightarrow \left|\left(\frac{K}{j}\right)\frac{161}{-1+7.6j+161}\right| = 10^{\frac{16.5}{20}} \rightarrow K = 6.6$$



Stability

Unity feedback is stable if and only if:

$\gamma > 0$ & GM > 0

If entire Magnitude plot below 0db y = infinity
If entire Phase plot above 180 deg, GM = infinity

lim s→j = 10