Análisis II - Matemática 3 - Análisis Matemático II

Segundo Parcial - 7 de Julio de 2018

1	2	3	4
B/B	B⁻	B⁻	R

CALIF.
A

TEMA A

[redacted]

Justificar todas las respuestas y escribir con prolijidad. Duración 4 horas.

1. Considerar la ecuación $y'' + y' + y = t^2 + 1$.

   (a) Hallar todas las soluciones reales de la ecuación homogénea asociada.

   (b) Dar la solución de la ecuación con datos iniciales: $y(0) = -2$, $y'(0) = 0$.

2. Hallar $n > 0$ tal que $\mu(x, y) = \dfrac{1}{(xy)^n}$ sea un factor integrante de la ecuación

$$(*) \qquad \left(\frac{xy}{\cos^2(x+y^2)} + xy^2\right) dx + \left(\frac{2xy^2}{\cos^2(x+y^2)} + x^2 y\right) dy = 0$$

   A partir de μ resolver la ecuación $(*)$ y encontrar implícitamente la solución tal que $y(\pi/4) = 0$.

3. Hallar la solución general de la ecuación:

$$y''(x) - \frac{3}{x} y'(x) + \frac{4}{x^2} y(x) = x^3,$$

   sabiendo que $y_1(x) = x^2$ es solución de la ecuación homogénea asociada.

4. Considerar, para $\alpha > 0$, el sistema:

$$\begin{cases} x_1' &= -2x_1 - 4x_2 \\ x_2' &= -\alpha^2 x_1 - 2x_2 \end{cases}$$

   (a) Encontrar todos los valores de α para los cuales el origen es el único punto de equilibrio del sistema y además sea asintóticamente estable.

   (b) Realizar el diagrama de fases del sistema con $\alpha = 2$. ¿Cómo es el comportamiento asintótico en el origen?

Rocío Bernardini 1

(1) $y'' + y' + y = t^2 + 1$

a) Homogéneo asociado: $y'' + y' + y = 0$

Propongo solución: $y(t) = e^{\lambda t} \leadsto y'(t) = \lambda e^{\lambda t}$
la meto en la ecuación $y''(t) = \lambda^2 e^{\lambda t}$

$\Rightarrow e^{\lambda t}[\lambda^2 + \lambda + 1] = 0$ pero $e^{\lambda t} \neq 0$

$\Rightarrow \lambda^2 + \lambda + 1 = 0$

$\lambda_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

CA
$\omega^2 = -3$
$\omega = \sqrt{3} i$

$\leadsto$ la solución es de la forma

$y(t) = c_1 e^{(-1/2 + \frac{\sqrt{3}}{2} i) t} + c_2 e^{(-1/2 - \frac{\sqrt{3}}{2} i) t}$

quiero las soluciones reales $\Rightarrow$

$y(t) = c_1 e^{-\frac{t}{2}} [\cos(\sqrt{3}/2\, t) + i \operatorname{sen}(\sqrt{3}/2\, t)] + c_2 e^{-t/2} [\cos(\sqrt{3}/2\, t) - i \operatorname{sen}(\sqrt{3}/2\, t)]$

$= (c_1 + c_2) e^{-t/2} \cos(\sqrt{3}/2\, t) + i (c_1 - c_2) e^{-t/2} \operatorname{sen}(\sqrt{3}/2\, t)$

pero $c_2 = \overline{c_1}$ $c_1 = a + bi$

$\Rightarrow c_1 + c_2 = 2\operatorname{Re}(c_1)$ $c_1 - c_2 = 2\operatorname{Im}(c_1) i$

$\Rightarrow 2\operatorname{Re}(c_1) e^{-t/2} \cos(\sqrt{3}/2\, t) + i\, 2\operatorname{Im}(c_1) i\, e^{-t/2} \operatorname{sen}(\sqrt{3}/2\, t)$

$\underbrace{2\operatorname{Re}(c_1)}_{k_1} e^{-t/2} \cos(\sqrt{3}/2\, t) \underbrace{- 2\operatorname{Im}(c_1)}_{k_2} e^{-t/2} \operatorname{sen}(\sqrt{3}/2\, t)$

$\boxed{k_1, k_2 \in \mathbb{R}}$

$\Rightarrow$ $\boxed{y(t) = k_1 e^{-t/2} \cos(\sqrt{3}/2\, t) + k_2 e^{-t/2} \operatorname{sen}(\sqrt{3}/2\, t)}$

b) Propongo:

$$y_p(t) = At^2 + Bt + C$$

$$y_p'(t) = 2At + B$$

$$y_p''(t) = 2A$$

meto en la ecuación inhomogénea

$$2A + 2At + B + At^2 + Bt + C = t^2 + 1$$

$$At^2 + (2A+B)t + (2A+B+C) = t^2 + 1$$

$$\Rightarrow \begin{cases} \boxed{A = 1} \\ 2A + B = 0 \rightsquigarrow \boxed{B = -2} \\ 2A + B + C = 1 \end{cases}$$

$$2 - 2 + C = 1 \rightsquigarrow \boxed{C = 1}$$

$$\Rightarrow \boxed{y_p(t) = t^2 - 2t + 1}$$

Sol gral:

$$\Rightarrow y(t) = k_1 e^{-t/2} \cos\left(\tfrac{\sqrt{3}}{2}t\right) + k_2 e^{-t/2} \operatorname{sen}\left(\tfrac{\sqrt{3}}{2}t\right) + t^2 - 2t + 1$$

pero sé que $y(0) = -2$ e $y'(0) = 0$

$$\Rightarrow y'(t) = -\frac{k_1}{2} e^{-t/2} \cos\left(\tfrac{\sqrt{3}}{2}t\right) - \frac{\sqrt{3}}{2} k_1 e^{-t/2} \operatorname{sen}\left(\tfrac{\sqrt{3}}{2}t\right) +$$

$$- \frac{k_2}{2} e^{-t/2} \operatorname{sen}\left(\tfrac{\sqrt{3}}{2}t\right) + \frac{\sqrt{3}}{2} k_2 e^{-t/2} \cos\left(\tfrac{\sqrt{3}}{2}t\right) + 2t - 2$$

$$\rightsquigarrow y(0) = k_1 + 1 = -2 \rightsquigarrow \boxed{k_1 = -3}$$

$$y'(0) = -\frac{k_1}{2} + \frac{\sqrt{3}}{2} k_2 - 2 = 0$$

$$\frac{3}{2} - 2 + \frac{\sqrt{3}}{2} k_2 = 0$$

Solución al P.V.I

$$\frac{\sqrt{3}}{\cancel{2}} k_2 = \frac{1}{\cancel{2}} \rightsquigarrow \boxed{k_2 = \frac{1}{\sqrt{3}}}$$

$$\Rightarrow \boxed{y(t) = -3 e^{-t/2} \cos\left(\tfrac{\sqrt{3}}{2}t\right) + \frac{e^{-t/2}}{\sqrt{3}} \operatorname{sen}\left(\tfrac{\sqrt{3}}{2}t\right) + t^2 - 2t + 1}$$

Rocío Bernardini

(2) $n>0 / \mu(x,y) = \frac{1}{(xy)^n}$ Sea factor integrante de

$$\overbrace{\left(\frac{xy}{\cos^2(x+y^2)} + xy^2\right)}^{P} dx + \overbrace{\left(\frac{2xy^2}{\cos^2(x+y^2)} + x^2y\right)}^{Q} dy = 0$$

$$P_y = \frac{x\cos^{\cancel{2}}(x+y^2) - xy\,2\cancel{\cos(x+y^2)}\,\mathrm{Sen}(x+y^2)\,2y}{\cos^{\cancel{4}3}(x+y^2)} + 2xy =$$

$$= \frac{x\left[\cos(x+y^2) - 4y^2\,\mathrm{Sen}(x+y^2)\right]}{\cos^3(x+y^2)} + 2xy$$

$$Q_x = \frac{2y^2\cos^{\cancel{2}}(x+y^2) - 2xy^2\,2\cancel{\cos(x+y^2)}\,\mathrm{Sen}(x+y^2)}{\cos^{\cancel{4}3}(x+y^2)} + 2xy =$$

$$= \frac{2y^2\left[\cos(x+y^2) - 2x\,\mathrm{Sen}(x+y^2)\right]}{\cos^3(x+y^2)} + 2xy$$

llamo $^*\beta = x+y^2$ solo en los argumentos !! Calculo auxiliar

$$P_y = \frac{x}{\cos^2\beta} - \frac{4xy^2\,tg\,\beta}{\cos^2\beta} + 2xy$$

$$Q_x = \frac{2y^2}{\cos^2\beta} - \frac{4xy^2\,tg\,\beta}{\cos^2\beta} + 2xy$$

$$\left[Q_x - P_y = \frac{2y^2 - x}{\cos^2\beta} \underset{*}{=} \frac{2y^2 - x}{\cos^2(x+y^2)}\right]$$

(*) Quiero $(\mu P)_y = (\mu Q)_x$

$$\mu_y P + \mu P_y = \mu_x Q + \mu Q_x$$

$$\mu_y P - \mu_x Q = \mu(Q_x - P_y)$$

$$\varphi(x,y) = (xy)^n$$

$$\mu' = \frac{d\mu}{d\varphi}$$

$$\mu_y = \frac{d\mu}{d\varphi} \cdot \frac{d\varphi}{dy} = \mu' \cdot n\, x^n y^{n-1}$$

$$\mu_x = \frac{d\mu}{d\varphi} \cdot \frac{d\varphi}{dx} = \mu'\, n\, x^{n-1} y$$

$$\Rightarrow \mu_y P - \mu_x Q = \mu \cdot (Q_x - P_y)$$

$$n\mu' \left[P x^n y^{n-1} - Q x^{n-1} y^n \right] = \mu \left(\frac{2y^2 - x}{\cos^2(x+y^2)} \right)$$

(ANULADO)

$$\frac{n\mu'}{\mu} = \frac{Q_x - P_y}{(xy)^n \left[P y^{-1} - Q x^{-1} \right]} \overset{*}{=} \frac{Q_x - P_y}{(xy)^n \cdot \left[-(Q_x - P_y) \right]} = \frac{-1}{(xy)^n} = \frac{-1}{\varphi(x,y)}$$

(*) CA:

$$\frac{P}{y} = P y^{-1} = \frac{x}{\cos^2(x+y^2)} + xy$$

$$\frac{Q}{x} = Q x^{-1} = \frac{2y^2}{\cos^2(x+y^2)} + xy$$

$$P y^{-1} - Q x^{-1} = \frac{x - 2y^2}{\cos^2(x+y^2)} = \frac{-(2y^2 - x)}{\cos^2(x+y^2)} = -(Q_x - P_y)$$

$$\Rightarrow \frac{n\mu'}{\mu} = \frac{-1}{\varphi} \Rightarrow \int \frac{n\, d\mu}{\mu} = -\int \frac{d\varphi}{\varphi}$$

$k, c \in \mathbb{R}$

$$n \ln \mu = -\ln \varphi + c \rightsquigarrow$$

$$\rightsquigarrow (e^{\ln \mu})^n = e^c \frac{1}{\varphi} \rightsquigarrow \mu^n = \frac{K}{\varphi} = \frac{K}{(xy)^n} = \mu^n$$

$$\Rightarrow \mu = \frac{(K)^{1/n}}{xy}$$

(ANULADO)

NOTA

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uso la forma

$$\mu_y P - \mu_x Q = \mu(Q_x - P_y) \qquad \mu = \frac{1}{(xy)^n}.$$

$$\mu_y = \frac{-n}{x^n}\frac{1}{y^{n+1}}$$

$$-\frac{n}{(xy)^{n+1}}(n x^n y^{n-1}) P - \left(-\frac{n}{(xy)^{n+1}}(n x^{n-1} y^n) Q\right) =$$

$$= \frac{1}{(xy)^n}\left(\frac{2y^2 - x}{\cos^2(x+y^2)}\right)$$

$$-\frac{n^2 \cancel{x^n y^{n}} y^{-1}}{\cancel{(xy)^n}(xy)} \cdot P + \frac{n^2 x^{-1} \cancel{x^n y^n}}{\cancel{(xy)^n}\, xy} Q = \frac{1}{(xy)^n}\left(\frac{2y^2 - x}{\cos^2(x+y^2)}\right)$$

$$\frac{n^2 Q}{x^2 y} - \frac{n^2 P}{x y^2} = \frac{1}{(xy)^n}\left(\frac{2y^2 - x}{\cos^2(x+y^2)}\right)$$

$$\frac{n^2}{x^2 y}\left(\frac{2xy^2}{\cos^2(x+y^2)}\right) + \frac{n^2 \cancel{x^2 y}}{\cancel{x^2 y}} - \left[\frac{n^2}{xy^2}\left(\frac{xy}{\cos^2(x+y^2)}\right) + \frac{n^2}{\cancel{xy^2}}\cancel{xy^2}\right] = \frac{2y^2 - x}{(xy)^n \cos^2 \beta}$$

$$\frac{2yn^2}{x\cos^2(x+y^2)} - \frac{n^2}{y\cos^2(x+y^2)} = \frac{2y^2 - x}{(xy)^n \cos^2(x+y^2)}$$

$$\frac{2y^2 n^2 - x n^2}{xy\,\cancel{\cos^2(x+y^2)}} = \frac{2y^2 - x}{(xy)^n\,\cancel{\cos^2(x+y^2)}}$$

$$n^2 \frac{\cancel{(2y^2 - x)}}{xy} = \frac{\cancel{2y^2 - x}}{(xy)^n}$$

$$\boxed{n = 1} \text{ RTA}$$

SIGUE ATRÁS

$$\frac{n^2}{xy} - \frac{1}{(xy)^n} = 0$$

$$\frac{(xy)^{n-1} n^2 - 1}{(xy)^n} = 0 \to (xy)^{n-1} n^2 = 1$$

pero $n=1$ lo tiene que cumplir

$$\mu(x,y) = \frac{1}{xy}$$

$$\int \mu P \, dx = \int \frac{1}{\cos^2(x+y^2)} + y \, dx = tg(x+y^2) + yx + \underbrace{h(y)}_{h(y)} = F(x,y)$$

<u>CA</u>: $$\int \frac{1}{\cos^2(x+y^2)} dx = \int \frac{1}{\cos^2(v)} dv = tg\, v + C = tg(x+y^2) + C$$

$v = x + y^2$

$dv = dx$

$$(tg\, v)' = \left(\frac{\operatorname{sen} v}{\cos v}\right)' = \frac{\cos^2 v - (-\operatorname{sen}^2 v)}{\cos^2 v} = \frac{1}{\cos^2 v}$$

$F_y = \int \mu Q$

$$F_y = \cancel{\frac{2y}{\cos^2(x+y^2)}} + \cancel{x} + h'(y) = \cancel{\frac{2y}{\cos^2(x+y)}} + \cancel{x} = Q$$

$$\Rightarrow h'(y) = 0 \rightsquigarrow h(y) = C$$

$$\Rightarrow F(x,y) = tg(x+y^2) + xy + C = K \qquad k - c = b$$

$$tg(x+y^2) + xy = b$$

$$tg\left(\frac{\pi}{4} + 0\right) + \cancel{\frac{\pi}{4} y(\pi/4)} = b \quad (y(\pi/4) = 0)$$

$$\rightarrow \boxed{b = 1}$$

$$\Rightarrow \boxed{F(x,y) = tg(x+y^2) + xy = 1}$$

bellow 60

constant ?

dapoi

Dabaz

hhrtw

poo ob

tobw

sowcos.

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(3) $y'' - \frac{3}{x} y' + \frac{4}{x^2} y = x^3$

(*) sé que $y_1(x) = x^2$ es sol de y_H

$y_1'(x) = 2x$

$y_1''(x) = 2$

Propongo: $y(x) = \varphi(x)\, x^2$

$y'(x) = \varphi'(x)\, x^2 + 2\varphi(x)\, x$

$y''(x) = \varphi''(x)\, x^2 + 4\varphi'(x)\, x + 2\varphi(x)$

(*) sí: $2 - \frac{3}{\cancel{x}} \cdot 2\cancel{x} + \frac{4}{\cancel{x^2}} \cancel{x^2} = 0$ ✓

$\Rightarrow x^3 = \varphi''(x)\, x^2 + 4\varphi'(x)\, x + 2\varphi(x) - \frac{3}{x}\left[\varphi'(x)\, x^2 + 2\varphi(x)\, x\right] + \frac{4}{x^2}\left[\varphi(x) x^2\right]$

$\varphi''(x)\, x^2 + \varphi'(x)\left[\underbrace{4x - 3x}_{x}\right] + \varphi(x)\left[\cancel{2 - 6 + 4}\right]^{0} = x^3$

llamo $u = \varphi'(x)$

$u' = \varphi''(x)$

$x^2 u' + x u = x^3 \Rightarrow$ Sol gral: $u_G = u_H + u_P$

resuelvo el homogéneo asociado

$x^2 u' + x u = 0$

$\frac{u'}{u} = -\frac{1}{x}$

$\int \frac{du}{u} = -\int \frac{dx}{x} \Rightarrow \ln|u| = -\ln|x| + c$

$|u| = \left(e^{\ln|x|}\right)^{-1} e^{c}$ >0

$u_H = \frac{K}{x}$

Propongo:

$u_p = a x^3 + b x^2 + c x + d$

$u_p' = 3a x^2 + 2b x + c$

$x^2\left[3a x^2 + 2b x + c\right] + x\left[a x^3 + b x^2 + c x + d\right] = x^3$

$$3a x^4 + 2b x^3 + c x^2 + a x^4 + b x^3 + c x^2 + d x = x^3$$

$$4a x^4 + 3b x^3 + 2c x^2 + d x = x^3$$

$$\begin{cases} 4a = 0 \rightsquigarrow a = 0 \\ 3b = 1 \rightsquigarrow b = 1/3 \\ 2c = 0 \rightsquigarrow c = 0 \\ d = 0 \rightsquigarrow d = 0 \end{cases}$$

$\Rightarrow$ $\boxed{u_p(x) = \frac{1}{3} x^2}$

$\Rightarrow$ $\boxed{u_G(x) = \frac{k}{x} + \frac{1}{3} x^2} = \varphi'(x)$

$$\int \frac{k}{x} + \frac{1}{3} x^2 \, dx = K \ln|x| + \frac{1}{9} x^3 + C_1 = \varphi(x)$$

$\Rightarrow y(x) = \varphi(x) y_1(x)$ $\quad y_1(x) = x^2$

RTA:

$\Rightarrow$ $\boxed{y_2(x) = K x^2 \ln|x| + \frac{1}{9} x^5 + C_1 x^2}$ $\quad K, C_1 \in \mathbb{R}$

4) $\alpha > 0$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -2 & -4 \\ -\alpha^2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$A = \begin{pmatrix} -2 & -4 \\ -\alpha^2 & -2 \end{pmatrix} \quad P_A(\lambda) = \lambda^2 + 4\lambda - (4 - 4\alpha^2) =$$

(*) en λ ... en ω

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{16 + 16(1-\alpha^2)}}{2} \; (A) \; \frac{-2 \pm \sqrt{16[2-\alpha^2]}}{2} \Rightarrow$$

$$\lambda_{1,2} = -2 \pm 2\sqrt{2-\alpha}$$ estudio por casos.

NOTA

ASINTÓTICAMENTE ESTABLE

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HOJA N° 5

FECHA

CASO A: $2-\alpha > 0 \leadsto \boxed{\alpha < 2} \leadsto$ ~~[illegible]~~

$\Rightarrow \alpha \in (0,2)$ (A)

tomo $\alpha = 1$

$\Rightarrow \lambda_{1,2} = -2 \pm 2$ $\lambda_1 = 0$, $\lambda_2 = -4$ → [illegible]

busco autovectores

$\lambda_1 = 0$: $\left(\begin{array}{cc|c} -2 & -4 & 0 \\ -1 & -2 & 0 \end{array}\right)$ $\left.\begin{array}{l} x + 2y = 0 \\ x = -2y \end{array}\right\}$ $v_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$\lambda_2 = -4$: $\left(\begin{array}{cc|c} 2 & -4 & 0 \\ -1 & 2 & 0 \end{array}\right)$ $\left.\begin{array}{l} -x + 2y = 0 \\ x = 2y \end{array}\right\}$ $v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x(t) = Ae^{-4t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + B\begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$x(0) = \begin{pmatrix} 2A - 2B \\ A + B \end{pmatrix}$

CASO B: $2-\alpha = 0 \leadsto \alpha = 2$

$\lambda_{1,2} = -2 = \lambda$ raíz doble

busco autovectores

$\left(\begin{array}{cc|c} 0 & -4 & 0 \\ -2 & 0 & 0 \end{array}\right) \overset{\text{triangulo}}{\leadsto} \left(\begin{array}{cc|c} -2 & 0 & 0 \\ 0 & 4 & 0 \end{array}\right)$

$x(t) = Ae^{-2t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + Ae^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x(0) = \begin{pmatrix} A \\ A \end{pmatrix}$

CASO C: ~~$\alpha > \alpha$~~ ~~$\alpha > 2$~~ $\alpha < 2$

$2 - \alpha < 0 \rightsquigarrow \alpha > 2$ $\qquad \lambda_{1,2} = -2 \pm 2\sqrt{\alpha - 2}\, i$

NOTA