Experiencing Physics Draft Activity: Exploration of the Pendulum
Instructor Materials

1 What This Document Is

This is a partial draft of an activity that would go late in the Experiencing Physics 1 semester, written by Walter as a prototype implementation of the Experiencing Physics ethos. It contains a full outline of a three-week module (what students will do and what we hope they learn from it), along with a draft of the student-facing materials for the first half of it. (Once we decide on computational and data-analysis platforms, of course, we will add specifics of those.)

I have not yet written draft materials for the second half (even though that is critical to the activity!), but I have outlined what they might look like.

As any prototype, this is a design that I imagine we will iterate on.

2 Philosophy

This module implements the Experiencing Physics ethos in an exploration of simple (and not-so-simple!) harmonic motion. That is:

- It is good for students to experience phenomena they cannot yet explain fully
- It is critical for students to get practice constructing explanations on their own
- This ethos can extend to all elements of the sense-making process – theory, modeling, experiment, numerics, visualization, etc. We can ask them to come up with theory on their own, too!
- Instructors can be most effective as "guardrails" and as metacognitive framers – we make sure they don't waste too much time as they explore, and help them make sense of their explorations after the fact
- Visualization, communication, and collaboration are essential skills throughout

3 Prerequisites and Objectives

This activity is designed for late in the semester, as either the fourth or fifth out of a five-unit class. It assumes that the students already have some skills. By this point the students should be able to:

1. **Experimental skills:**
   (a) Measure a time series and write it to a data file
   (b) Measure position (or angle) vs. time, either from a direct sensor or video analysis, and write it to a data file
   (c) Reduce and plot data
   (d) Fit data to a model and describe the quality of the fit (qualitatively)
2. **Computational skills:**
   (a) Use the Euler-Cromer algorithm to evolve a system in time (e.g. $\ddot{x} = f(x)$)
   (b) Describe the impact of integration stepsize on simulation accuracy
   (c) Animate simulation results
3. **Theoretical skills:** (taught in parallel in 215)
   (a) Knowledge of the work-energy theorem and the conservation of energy
   (b) Knowledge of the notion of torque (maybe)
   (c) Knowledge of the small-angle approximation to $\sin\theta$

In this unit students will learn:

1. **Experimental skills:**
   (a) Gain further practice proposing, interrogating, and validating models of phenomena
   (b) Handle situations where a model is correct in one limit, but incorrect in another
   (c) Use log/log plots to visualize data that span many orders of magnitude.
2. **Numerical skills:**
   (a) Numerical solution of equations of motion (continuing from before)
   (b) "Numerical experimental design": how do we make use of numerics (varying parameters etc.) to explore physics?

(c) Disentangle physical signals from simulation artifacts in numerical data

3. **Theoretical skills:** (taught in parallel in 215)
   (a) Introduction to differential equations (we did this in 211 last year and it was fantastic!)
   (b) What to do when confronted by a model we cannot solve: introduction to "perturbative reasoning"
   (c) Solve the SHO differential equation and understand its properties

4 Overview

In this module, students will explore the phenomena that a pendulum exhibits. This is predominantly simple harmonic motion, so the students will discover many of the major features of simple harmonic motion. They will learn to solve the differential equation that leads to SHM, both numerically (using a computer) and analytically (using the small-angle approximation $\sin\theta \approx \theta$). When they compare it to their experimental data, they will discover that their theoretical model works well for small angles but diverges at larger angles; the culprit is of course the small-angle approximation. They will look at their numerical results that do not require this approximation, and compare them to their empirical ones.

This module spans six 80-minute course periods. A rough allocation of that time:

- **Week 1 Day 1:** Introduction; first empirical measurements; framing of what interesting questions we can explore. Framing of different limits. Take a first data set to use for baseline analysis.
- **Week 1 Day 2:** Theoretical modeling; arrival at the differential equation. Discussion of small-angle approximation and introduction to "perturbative reasoning". Arrival at SHM solution.
- **Weekend 1 Homework:** Fit this solution to empirical data. Come to class next week with your parameters of best fit, and a possible explanation of your results and why models did/didn't fit.
- **Week 2 Day 1:** Class discussion of fit results. Stock-taking: what have we explained? What can we not yet explain? Begin implementing numerical simulations. (This should not be too hard for them given the expected prerequisites.)
- **Week 2 Midweek Homework:** Complete numerical simulations if not done in class.

- **Week 2 Day 2:** Compare numerical simulations of pendulum to analytic result at high and low amplitudes; compare numerical simulations to experimental data at high and low amplitudes. Frame new question: "How can we understand the dependence of period on amplitude?" Decide on new measurements to make – what amplitudes, needed precision, and how to achieve it – and make them.

- **Weekend 2 Homework:** Generate data of period vs. amplitude for small-angle analytic result, numerical simulation, and empirical data. Play with plotting them; come to class next week with a few plots to show that tell the story of your data.

- **Week 3 Day 1:** Brief student presentations from part of the class. Introduction of log/log plots. Replot data in log/log form. Discussion of perturbative reasoning, the notion of models and their limits of validity, and the question of "what is the true behavior?" in comparing theory/numerics/experiment.

- **Week 3 Day 2:** Presentations from other groups. Extensions (driving? Damping?)

5 Day 1: Introduction (Instructor's Notes)

Give students materials to make pendula. The ideal here is rigid light rods with a mass on the end – things that can be swung at angles greater than π. But balls and strings work too, so long as we have a way to measure their period to high precision such as a photogate and a way to measure their motion, using either automated motion analysis or a direct sensor. More instructor's notes are included in italics inside the student materials.

6 Introduction to the Pendulum

Few things are as fundamental to physics as **vibrations**, since they are so common in nature. Whenever a system has the property that **a disturbance in one direction creates an acceleration in the other direction**, it oscillates back and forth. This is true for physical objects, like the pendulums we will explore here. But many other systems in physics have this same property, and these are both some of the most useful to humanity in our technology and some of the most critical for our understanding of the universe.

Regular vibrations of the atoms in crystals, operating on the same principle, have replaced pendulums in our clocks. Light and radio waves are just traveling oscillations of electric and magnetic fields. Electric circuits can oscillate just like pendulums, and they are the basis of tuners in radios. Sound is the oscillation of air. Stable systems in quantum mechanics – which obey many of the same equations as the pendula we will study – are those that oscillate in regular patterns. All of these phenomena have close similarities, and so the principles behind the pendulum you will study here will transfer to many other fields of physics.

In this module, you will:

1. Experiment with a pendulum to determine its properties
2. Develop a physical model to explain as many of those properties as possible
3. Use pencil-and-paper mathematics – which here involves some compromises! – to look at the predictions of this model.
4. Compare these predictions to your experimental data
5. Explore the idea of *limits of validity*: when is your model more valid? When is it less valid?
6. Use computational methods – which involves a *different* set of compromises! – to make another set of predictions from your model, and compare them to your experimental data
7. Iterate on both your models and your experimental data to incrementally increase your understanding, proposing and answering new questions that arise

6.1 Exploration

You should have all the things you need to make a simple pendulum. Make one, and fiddle with it for a little while. What properties does it have that you can measure?

Record in your lab notebook:

- How you constructed your pendulum, in enough detail that someone else could replicate what you did
- The parameters that you can vary (i.e. what different sorts of pendula can you construct?)
- A qualitative description of its motion – what does it do? Even though you are describing things qualitatively, you can still make these descriptions precise. How does its motion depend on its properties?

- Your group's ideas about *what properties you could measure quantitatively* that a physical model of the pendulum should be able to explain.

  You should include in your description the *parameters* that you can vary: remember that at heart an experiment involves changing something in a system and seeing how its behavior changes. In short: what things can you measure, and what things can you change that might affect them?

  Also, describe some "first guesses" about how what properties these things might have.

Instructor's notes: This previous activity should take about thirty minutes. We want them to have a good long think about this last question, and they should have multiple conversations with teaching staff during the process. We want them to arrive at at least the following:

- *It swings back and forth, with the pendulum bob moving along an arc of constant radius from the point of attachment*
- *If it is released from rest it reaches the same height when it swings back (or maybe a slightly lower height)*
- *Its motion is a function of one variable: $\theta(t)$. We can measure this. We might expect $\theta(t)$ to be sinusoidal (it goes back and forth).*
- *We can measure the period and it appears to be longer when the string is longer.*
- *We can vary the length of the string and the mass of the bob; when swinging it, we can also vary the initial angle*

6.2 Day 1: Measurement

Remember that the only real test of a physical model is whether its predictions agree with experiment. Before we come up with a physical model, we need some experimental data to compare it with!

Also remember that physical models have *limits of validity*. For instance, we could model the motion of a falling object as "It accelerates relative to the ground at a constant value g, where g is around 9.80 m/s^2.". This model seems to be very good – if we stay in Syracuse! But a falling object near the Equator will fall at around 9.78 m/s^2, while an object far from Earth's surface will fall much more slowly – or fall toward another celestial body entirely! Likewise, we will learn that $\vec{F} = m\vec{a}$ is only valid for objects moving much less than the speed of light, and needs to be modified for very fast objects.

It is often the case that physical models do a good job of predicting the behavior of a system in one regime, but are less accurate in another. As you will see, the pendulum is no exception.

Working with your group, devise a series of measurements of the things you decided on previously that will let you validate any physical model of your pendulum. You should discuss:

- What things will you measure?
- How will you measure them?
- How will you ensure that your measurements are as precise as they can be?
- What things will you vary?
- What limits do you need to make sure you explore?
- What sorts of things can you measure with high precision that will serve as a good probe of the validity of a model?

Once you decide on a plan, invite your TA or coach over to discuss your experimental design. Then make your measurements. Make sure you document everything that someone would need to interpret your data, and include that documentation alongside the data in the shared data directory for the course.

Instructor's notes: Students need to measure, at least, angle vs. time for a small angle and a large angle; they will use this later to conclude that the small-angle behavior is very close to sinusoidal, but the large-angle behavior is not. You should encourage them to focus many of their measurements on the swing period, as a very sensitive probe of a particular model. They should vary the mass, the string length, and the starting amplitude. They should change only one variable at a time, obviously.

If they want to probe something else – say, the long-term decay behavior – that's fine too.

It might be worthwhile if one group, using a more sophisticated setup with motion capture equipment, focuses on getting angle vs. time data, while other groups focus on making many different period vs. whatever measurements. This will depend on how labor-intensive doing angle vs. time is, and how much time there is to take data. So long as the data are of high quality, collaboration between groups is good! It will encourage people to document well.

7 Homework 1

Instructor's notes: this homework assignment should be given only after they make their *measurements, so as not to take away their agency in choosing what to measure. It should take them only about thirty minutes.*

Your group and the other groups will have collected a great deal of data: observations of the motion of the pendulum as it swings back and forth, and how its period τ depends on the mass m of the bob, the length L of the string, and the maximum angle A (the amplitude) of its swing.

Before next class:

1. Examine the plots of the swing angle θ vs. the time t. What mathematical form do they appear to have? This will be very helpful later when constructing models, since you can focus on models that predict this form.
2. Working together with your group, each person should come up with a data visualization strategy (a graph or chart) that illustrates *one* trend you see in the period vs. amplitude, mass, or length data. Send your chart, along with a brief summary of the conclusions it supports, to the class commons.

8 Day 2: Modeling

Instructor's notes: First, the class should spend ten minutes discussing their findings and everyone else's.

We now have a pretty good idea of the behavior pendulums exhibit. In today's class, your group will use the things you know about mechanics to construct a model that will be able to predict some of that behavior.

8.1 Determining the Equation of Motion

Instructor's notes: We want them to arrive at

$$\ddot{\theta} = \alpha = -\frac{g}{L}\sin\theta.$$

in this section. There are two ways they can do this – one uses the ideas of torque, while the other uses linear dynamics with a coordinate system aligned with the string. Which approach you take will depend on whether the students have seen the idea of torque yet. Regardless, they should spend a little while arriving at the equation above.

In the previous discussion, you should have arrived at a conclusion about how the mass of a pendulum bob affects the motion of the pendulum. Does this match your data from last time? Discuss this with your group briefly.

Earlier in our class, we have concerned ourselves with applying kinematics to situations when *acceleration is constant*. This is because the calculus is very easy:

$$\begin{aligned} \text{acceleration} &= \text{constant} & a(t) &= a \\ \text{velocity} &= \text{integral of acceleration} & v(t) &= \int a\,dt = at + v_0 \\ \text{position} &= \text{integral of velocity} & x(t) &= \int (at + v_0)\,dt = \frac{1}{2}at^2 + v_0 t + x_0 \end{aligned}$$

This allows us to calculate everything about the motion of objects where the force, and thus the acceleration, does not change over time.

Even if the acceleration changes with time in a known way – say, $a(t) = \beta t$ – you can take two integrals of that function and find other kinematics equations.

For instance, in a situation where the acceleration $a(t) = \beta t$:

$$\begin{aligned} &\text{acceleration} & a(t) &= \beta t \\ \text{velocity} &= \text{integral of acceleration} & v(t) &= \int a\,dt = \frac{1}{2}\beta t^2 + v_0 \\ \text{position} &= \text{integral of velocity} & x(t) &= \int \left(\frac{1}{2}\beta t^2 + v_0\right)\,dt = \frac{1}{6}\beta t^3 + v_0 t + x_0 \end{aligned}$$

For our pendulum, you previously arrived at an equation for the angular acceleration. (Remember that calculus, and thus kinematics, works the same way for angular variables.)

Above, we saw that if you know the *acceleration as a function of time*, you can take two integrals to find the position as a function of time (the thing we want). Discuss with your group why the above approach will not work here, and call your TA/coach over to join your discussion.

Instructor's notes: they should say something like "here α is a function of θ, which is itself an unknown function of time, instead of a known function of t, so our problem is recursive.

Since we can't use the thing we already know, we will have to figure something else out! Our problem is that the acceleration is now a function of *position.* We know what to do if we know $\alpha(t)$ – but now we know $\alpha(\theta)$:

We don't quite know what to do with this! The biggest problem is that x – an unknown function of time – appears on the right-hand side, so *both* sides of this equation have things we do not know.

The easiest approach is to replace α with its definition as "the second derivative of θ with respect to time", and then ask "what kind of function for $\theta(t)$ can I put in here that will work?"

With your group, write down an equation that relates the second derivative of the angle $\frac{d^2\theta}{dt^2}$ to the angle itself. This sort of equation, relating a function to its derivatives, is called a *differential equation.*

These equations relate a function to its second derivative. We need to find some function $\theta(t)$ that satisfies the following:

"When I take the second derivative of $\theta(t)$, I get the sine of that function back, but with a negative sign, and multiplied by $\frac{g}{L}$"

Since this function is supposed to describe the swinging of a pendulum, you might draw inspiration from your observations of $\theta(t)$ for a real pendulum.

Talk with your group for a few minutes. What sorts of functions might have this property? Would they be simple, or would they be complicated?

Instructor's notes: Of course, there is no analytic solution here. Let them stew on this for just a little bit.

8.2 Approximating the Equation of Motion

The differential equation you've arrived at has no solution. This may seem like a disaster... but it turns out that physics and engineering are full of differential equations that don't have solutions you can write down! Much of the skill in physics is in *making wise approximations*, and thinking carefully about the regimes in which they apply. (There's that idea of "models with limited regimes of validity" again!)

This section is the heart of this module, and we will return to it and its consequences many times in the next few classes: *what do we do when our models cannot be solved?* We cannot give up; we must instead do something clever. Often in physics, we go through the following series of steps:

1. Analyze a physical system and arrive at *a model we cannot solve* (we are here!)
2. Think carefully about what *limiting regimes* there are in this physical system, and which ones you are interested in
3. Make an *approximation* to the model that is only valid in one of those limits, but that makes it simpler to solve
    - This approximation it, itself, another model!
4. Solve the approximate model and make a prediction for how the system will behave in that regime

Think again about your differential equation:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

Discuss with your group:

- What mathematical part of the differential equation makes it very hard to solve?
- Is there a *certain regime* where that part of the differential equation can be approximated as something else that is simpler?

Once you have an idea, call your TA or coach over to discuss. What approximation will you make, and in what regime do you expect it to be valid?

8.3 Solving the Approximate Model

You have now arrived at a simpler differential equation that you *can* solve. In situations like this, the easiest way to figure it out is to just *guess the answer!* The way to do this is a two-step process:

1. Consider "template" functions that have unknown parameters in them – for instance, a generic exponential might be $\theta(t) = Ae^{\beta t}$, with A and β unknown – and find one with the right properties.
2. Plug your template into the equation $\frac{d}{dt^2}\theta(t) = -\frac{g}{L}\sin\theta(t)$ and figure out what the unknown parameters are

If you are stuck, here are some template functions you might consider:

$\theta(t) = A\tan(\Omega t)$	$\theta(t) = Ae^{\beta t}$	$\theta(t) = Ae^{-\beta t}$	$\theta(t) = A\sqrt{\beta t + \gamma t^2}$
$\theta(t) = A\cos(\Omega t)$	$\theta(t) = A + Bt + Ct^2$	$\theta(t) = A + Bt + Ct^2 + Dt^3$	$\theta(t) = A\sqrt{\beta t}$
$\theta(t) = A\sin(\Omega t)$	$\theta(t) = A\ln\frac{t}{\tau}$	$\theta(t) = \frac{\tau}{t}$	$\theta(t) = A\frac{t+B}{t-C}$

Instructor's notes: They likely will not need this table. We did this activity in 211 without having them observe a sinusoid first as an exploration of SHM, so I gave them the chart. The whole exercise was quite successful. They will likely figure out that both sine and cosine functions are solutions. We will talk about converting this later.

Now, we need to find out what the unknown variables (A, B, C, D, β, τ, ω) might be in your template function(s). Take two derivatives of your function(s). For instance, the second derivative of $Ae^{\beta t}$ is $A\beta^2 e^{\beta t}$.

Substitute your function back into the equation you started with: $\frac{d^2\theta}{dt^2}(t) = -\frac{g}{L}\,\theta(t)$. Then solve for your unknowns (for instance, what is Ω?)

Now, what about A? Determine what value or values that A could have. Remember that we are figuring out what happens when a pendulum swings back and forth, and this equation is supposed to describe all possible ways that it could oscillate.

8.4 Interpreting the Constants and Determining the Period

For this part, focus only on the *cosine* solution. We will see how to handle the fact that there are two solutions later.

One important skill in physics is the ability to look at an equation and translate it to a statement about the physical world. Considering your solution

$$\theta(t) = A\cos\Omega t,$$

discuss with your group:

1. ... what sort of motion you expect this to describe

2. ... what the physical interpretation of A and Ω are

3. ... whether the formula for Ω you got in the last section makes sense (how would the pendulum behave if you made the string longer? What if you made it shorter? What if you took it to the Moon?)

Your empirical measurements were of the period τ of the pendulum, but it's not one of your constants. Which one of these constants relates to τ? See if you can find this relationship. (It may be useful to plot $y(x) = \cos \Omega x$ in Desmos for different values of Ω, and to remember that one cycle is 2π radians.)

These constants have names:

- A is called the *amplitude*
- Ω is called the *angular frequency*

8.5 From Two Equations to One

You have just discovered that both

$$\begin{aligned} \theta(t) &= A\cos\Omega t \text{ and} \\ \theta(t) &= B\sin\Omega t \end{aligned}$$

solve the differential equation (approximately) describing the pendulum. This should make sense: the only difference between sine and cosine is "where they start" – both functions have the same shape but begin at a different point in the oscillation at $t = 0$.

Any sum of these also solves the equation, so a general solution is

$$\theta(t) = A\cos\Omega + B\sin\Omega$$

This is sort of messy to think about. But all it does is encode the possibility of starting a pendulum at any point in its swing at $t = 0$.

To see this, we need to use a trig identity to rewrite this sum of sine plus cosine using a trig identity:

$$\theta(t) = A\cos\Omega t + B\sin\Omega t = C\sin(\Omega t + \phi)$$

Here ϕ is called the *phase*, which just tells us "where in the cycle the pendulum starts". We can use $\theta(t) = C\sin(\Omega t + \phi)$ from now on. (The trig identity lets us find C and ϕ from A and B, but we don't really care. It's enough to know that $C\sin(\Omega t + \phi)$ is a general formula for a sinusoid.)

8.6 Taking Stock

In today's class, you have "done theoretical physics" in the most traditional way:

1. Construct a physical model of a system that you hope will predict its behavior
2. Try to predict its consequences with a pencil and paper and get stuck
3. Approximate that physical model with a simpler one that is valid in a certain *limiting regime*
4. Analyze that approximate model and examine its consequences

Look at your solution to the differential equation above with your group. What does it predict about...

1. ... a general description of how a pendulum moves
2. ... how a pendulum's period depends on its mass
3. ... how a pendulum's period depends on its length
4. ... how a pendulum's period depends on the amplitude that it is swung at

Instructor's notes: the last of these is subtle. The model predicts that the period is $2\pi\sqrt{L/g}$ *regardless of amplitude – but we expect this to be only true for small amplitudes. This will be important next week.*

Think back to the approximation that you made in constructing your model. When do you expect your model to be more accurate? When do you expect it to be less accurate?

Now, consider the *observations* made by your group and others. (You can also make some new ones, if you want to tinker with the apparatus!) Discuss:

1. ... how you would describe the real pendulum's motion
2. ... how its period depends on its mass
3. ... how its period depends on its length
4. ... how its period depends on the amplitude that it is swung at

Does your model do a good job of predicting the real pendulum's properties?

9 Homework: Fitting Data

For homework over the weekend, fit your observations of the pendulum's "angle vs. time" curve to the form that we predicted, doing at least one curve with low amplitude and one with high amplitude.

Does this form fit your data? Are the fit parameters what you expect?

Your group should prepare two plots (showing your data and your best-fit curves) that support your conclusions, and summarize those conclusions briefly in a few sentences. At least 24 hours before the next class, upload these to the class shared space.

Then, after everyone has uploaded theirs, you should look at other people's plots and draw some conclusions from them.

Instructor's notes: For $A = \pi/2$, the maximum angle that can be done with string rather than a rigid bar for the pendulum, Ω will differ from $\Omega_0 \equiv \sqrt{\frac{g}{L}}$ by 12% or so. But the data still appear to be sinusoidal. Differences will start to be visible as θ_0 increases above 2.5 radians.

IMPORTANT NOTE: Python's general purpose curve fitter is very temperamental when fitting sinusoids unless the guess frequency is very close to the best fit value. This is because the parameter space has many shallow local minima and the fitter will mistake them from the deep global minimum. If we are using `scipy.optimize.curve-fit`*, then students need to be warned about this.*

10 Day 3: Evaluating analytical model; creating simulation

10.1 Taking stock

Instructor's notes:

TODO: Students should share and discuss their fit results, and discuss – both qualitatively and with reference to quantitative data – how good of a job their model does at explaining the behavior of the pendulum. The instructor should steer the conversation toward asking about the high-amplitude results.

10.2 Numerical simulation

TODO: Point out that even though we cannot analytically solve our model without the approximation, we can do it numerically. Remind them of procedure and how to use whatever visualization tools we are using. Students should work on implementing this.

11 Day 4: Evaluating numerical results; refocus on high amplitudes

TODO: our numerical model is supposed to be better than our analytic approximation at high amplitudes; let's look at them

11.1 Comparing numerical model to analytic result

TODO: how do these compare at high amplitudes? At low ones?

11.2 Comparing numerical model to empirical data

TODO: how do these compare at high amplituides? At low ones?

Frame the detailed question: "The analytical model fails to predict what happens at high amplitudes. Does the numerical model do a good job of describing those – to high precision?"

11.3 Experimental design, again

TODO: students design detailed series of measurements of period vs. amplitude to explore that question in more depth now that we have framed it

11.4 Taking data

TODO: students make detailed (20 trials) measurements of period vs. amplitude, going for as high of precision as possible

12 Homework over weekend 2: data visualization

TODO: students make plots of period vs. amplitude for both models and the data, and plan to present their conclusions + plots to class.

13 Day 5: Log/log plots; data storytelling

TODO: students present their stuff. Need for log/log plots should emerge organically. Replot data in this form. Discussion within groups and in the class about perturbative reasoning, the idea of models and limits of validity, and this issue of "what is actually true/fiducial/the thing against which we compare?" (analytical model has clear limits, numerical model is better but has limits of a different sort, empirical data is noisy and includes effects we are *not* trying to understand). Conversation about the nature of experiment and the process of isolating physical laws so they can be studied in a reductive way.

14 Day 6: TBD

Discussion of dissipation and exponential decay? Discussion of use of pendula as gravitometers and error analysis?