Recitation Exercises

Week 14 Day 1

In this penultimate exercise of the semester, you will study *transmissions* – the assemblages of gears that transmit torque from a motor to the machine that it turns. For instance, the motor might be the engine of a car or the legs of a cyclist; the "machine" is the drive wheel applying traction to the ground. This works the same for all sorts of machines that use rotary motion to transmit power.

This exercise is a little different than others: it is designed to let you explore a very useful sort of device we see around us. We intend for you to work actively with your coaches and TA's here, so ask us questions!

In all of these cases, the motor applies a torque to the driveshaft, which is connected to a machine that applies an equal and opposite torque. Thus, the motor delivers power to the machine. For this problem, the motor will always be spinning at a constant angular velocity, so $\sum \tau = 0$.

Motors have two limitations:

- They are limited in the torque that they can apply. For instance, for a human riding a bicycle, there is only so much force they can apply to the pedals.
- They are limited in their angular velocity. For instance, a person riding a bicycle can only spin their legs so fast.

We will see how we can partially overcome these limitations using gears. Let's think about this in an idealized case of an electric motor spinning a machine, and then apply it to a person riding a bike. (Perhaps the machine is a circular saw used to cut wood, or the motor spinning the propeller on a quadcopter drone, or a water pump...)

Suppose our motor's limits are as follows:

- Maximum torque: $\tau_{\text{max}} = 100\ \text{N} \cdot \text{m}$ to the driveshaft
- Maximum angular velocity: $\omega_{\text{max}} = 50$ rad/sec

Let's imagine a situation where the motor is always applying maximum torque, and see what happens to the machine the motor is driving.

1. What is the maximum power that the motor can deliver to the machine? *(Hint: For translational motion, $P = \vec{F} \cdot \vec{v}$. What is the analogous formula for rotation?)*

   Fill in the first row of the table on the back; note that since the motor and machine are connected directly, the torque and angular velocity of the motor are the same as those of the machine.

   Since $P = \tau\omega$, we know $P_{max} = \tau_{max}\omega_{max} = 5000$ W.

   The key idea of this whole module is that we only get maximum power from the motor when it is applying its maximum torque at its maximum angular velocity. We cannot get more torque or angular velocity than this, and we cannot get the maximum power out when the angular velocity is less than the maximum.

2. However, in general, machines need to run at different speeds; for instance, cars can drive at many different speeds, and the driver may want high power at any speed.

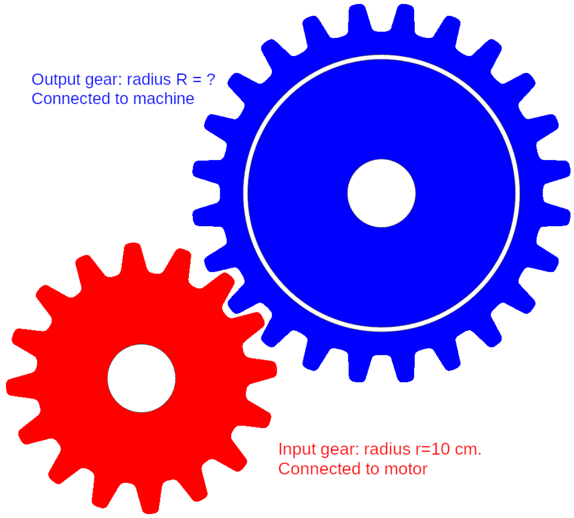
   Suppose that the operator of the machine wants to run it at low speed – say, at 25 rad/s. Can the motor still deliver the same power in this case? If not, how much power can it deliver? Complete the second row of the table on the back.

   Here since ω is only half as much (25 rad/s), we only get 2500 W out.

   This is simple, but the point here is to construct a baseline without a transmission that we will compare to.

The motor is simply *unable* to rotate any faster than 50 rad/sec. For instance, running the machine at $\omega = 100$ rad/s is impossible with the motor alone. Since the machine operator may want to run the machine at any speed, and will likely want the most power from the motor at *any* speed, they construct a transmission out of gears. Transmissions like this are a constant companion to motors in engineering.

In this figure, the motor is connected to the red gear with $r_{in} = 10$ cm; the machine is connected to the blue gear. We will first think about how this transmission works using a single blue gear with a radius of $R_{out} = 20$ cm in order to understand the principles at work here; then, we will think about the advantages of *shifting* gears, as in a bicycle or car transmission.



Here everything is rotating at a constant angular velocity, although the angular velocities of the two gears aren't necessarily the same. This means that the motor applies a clockwise torque to the red gear; the blue gear applies an equal and opposite counterclockwise torque to it.

3. Newton's third law applies to the forces that the two gears exert on each other: the two gears push on each other with equal and opposite forces. Given this, determine the relationship between τ_{motor} (the torque the motor applies to the red gear) and τ_{machine} (the torque the blue gear applies to the machine) in terms of r_{in} and R_{out}. Record this formula on the back page. Note that the ratio between r_{in} and R_{out} appears in this formula: this is called the *gear ratio*, and is critically important here.

   This is the critical result of this whole thing and the students should spend some time thinking carefully here. The critical idea is that the **force** is the same, so the **torque** is the product of that force with the radii. This means:

$$\begin{aligned} F =& \tau_{\text{motor}}/r_{in} \\ \tau_{\text{machine}} =& F R_{out} \\ \tau_{\text{machine}} =& \tau_{motor} \frac{R_{out}}{r_{in}} \end{aligned}$$

   In other words, the larger the output gear compared to the input gear, the larger the torque delivered to the machine.

4. The velocities of the gears' teeth must also be the same as they turn. Given this, determine the relationship between ω_{motor} (the speed at which the motor and red gear turn) and ω_{machine} (the speed at which the blue gear and the machine turn). Again, you should have a result that depends on the gear ratio. Record this formula on the back page; call your coach or TA over to check your result.

   This is the second critical result and the students will again need to think carefully. Since v_T is the same for both gears, but $\omega = v_T/r$, using a larger radius for the output gear results in a *lower* output angular velocity.

   In other words,

$$\omega_{\text{machine}} = \tau_{motor}\frac{r_{in}}{R_{out}}$$

   i.e. the inverse relationship for above. You cannot get something for nothing; "gearing down" to increase torque also decreases angular velocity.

5. Suppose that the motor is running at its maximum angular velocity and torque, and the output (blue) gear has $R = 20$ cm. Calculate the angular velocity of the machine and the torque and power supplied to the machine. Enter those in your data table.

   The critical idea here is that the gear ratio is 2:1 – the output gear is twice the radius of the input gear. This will increase torque by a factor of two but decrease angular velocity by a factor of two, based on the relationships they derived previously.

   So $\omega_{out} = 25$ rad/s and $\tau_{\text{out}} = 200\ \text{N} \cdot \text{m}$ – angular velocity goes down, torque goes up. But critically the power is unchanged – 25 rad/s $\times$ 200 $\text{N} \cdot \text{m} = 5000$ W.

6. How does this power compare to the maximum power that the motor could deliver to the machine at this angular velocity without the transmission?

   Now they should see the point of all this. Without using the transmission, decreasing the angular velocity by half decreases the power by half as well. But using the transmission we can still get the full 5 kW of power out of the motor even if the machine is turning at a slower speed.

7. Suppose that the engineer designing the machine wants to be able to run the machine at an angular velocity of $\omega = 100$ rad/s. (Perhaps it is a quadcopter rotor that needs to spin very quickly.) This would be simply impossible without a transmission, since the motor can only turn at 50 rad/s. What radius should the output gear have so the machine can spin at $\omega = 100$ rad/s, and how much torque could be delivered to the output gear in this case? Record these parameters in the next row of your data table.

   Here, we need a *smaller* output gear than our input gear. If we want ω to go up by a factor of two, the output gear should be smaller by a factor of two:

$$R_{out} = 5\,\text{cm}$$

   But this doesn't happen for free. Here the torque goes down by a factor of two (same force pushing on a gear half the size) – now $\tau =$ only 50 newton-meters.

8. Suppose that you now need to use the same motor to generate an extremely large torque to run a different machine. (Perhaps you are trying to lift something extremely heavy: you are not terribly concerned with how *quickly* you lift it, you just need to generate a very large force.) If you need an output torque of $\tau_{\text{machine}} = 1000\text{N} \cdot \text{m}$, what radius should the output gear have? Enter this in your data table.

   Now we take it to extremes just to illustrate to students the power of this whole thing. We need ten times the torque the motor can provide. How can we do that? Simple – just use an output gear ten times as big as the input gear: $R_{out} = 100$ cm.

   The angular velocity drops by a factor of ten to 5 rad/s, but for many engineering applications this is okay, like the one above. But the power is the same: 5 kW. This illustrates how the appropriate choice of gear ratios will let you get the maximum power out of the motor regardless of what you need the machine to do.

9. A bicycle uses a chain to connect the input and output gears, but the principle is the same: the rider's legs are limited in both their torque and their angular velocity. They want to be able to apply maximum power to the output gear (connected to the rear wheel) for a range of angular velocities – whether this is to climb a steep hill at low speed, or to go as fast as possible on flat ground.

   On many bicycles the rider can change the radius of both input and output gears. Discuss how this allows the rider to deliver maximum power to the bicycle's wheel at any speed. What combination of gears does a rider want when they are climbing a steep hill at low speed? What combination do they want when they are going very fast on flat ground?

   This relates what they're doing to their own experience, if they've ridden a bike. Here the "input gear" is the front one connected to the pedals, and the "output gear" is the back one connected to the wheel. We know that a gear ratio with a large output gear and small input gear produces high torque but low angular velocity on the output gear); this is the hill climbing gear ("granny gear", as cyclists call it), where you need maximum torque but it's okay if you don't go very fast.

   Conversely a large front gear and a small back gear gives a larger output angular velocity but low output torque – this is where you need to go as fast as possible on flat ground or downhill.

10. As you've seen, with appropriate choice of gear sizes, a transmission lets a motor (or a human) produce either extremely large torque or extremely high angular velocity. Is a transmission able to increase the amount of *power* that a motor (or cyclist) can produce? Why or why not?

    No – they should have gotten this earlier. Since $P = \tau\omega$, and changing the gear ratio makes τ go up and ω go down, the product stays the same. We would expect this, since energy is conserved.

Torque from motor (100 $N \cdot m$ max)	**ω of motor (50 rad/s max)**	**Radius of input gear (connected to motor)**	**Radius of output gear (connected to machine)**	**τ to machine ($N \cdot m$)**	**ω of machine (rad/s)**	**Power to machine (watts)**
100	***50 (matches machine, no transmission)***	— No gears, motor connected directly —		***100 (matches motor, no transmission)***	50	***5000 ($P = \tau\omega$)***
100	***25 (must match machine; motor runs at half speed)***	— No gears, motor connected directly —		***100 (still matches motor)***	25	***2500 (only get half power since half speed)***
impossible!	impossible!	— No gears, motor connected directly —		impossible!	100	impossible!
100	***50 (running at full speed)***	10 cm	20 cm	***200 (!) (gear ratio 2:1 – "high τ low ω")***	***25 (gear ratio 2:1)***	***5000 (!) Note we get full power even at half speed***
100	***50 (running at full speed)***	10 cm	5 cm	***50 (gear ratio 2:1 – "low τ high ω")***	***100 (we couldn't do this at all w/o gears)***	***5000 again***
100	50	10 cm	***100 cm (need a gear ratio of 1:10 to get output torque = 10x input torque)***	1000 (in an extreme "low speed high torque" situation)	5	5000 (note power is always the same here)

Relationship between
τ_{motor} (red gear) and
τ_{machine} (blue gear)

Relationship between
ω_{motor} (red gear) and
ω_{machine} (blue gear)