

# Moment of inertia

Physics 211  
Syracuse University, Physics 211 Spring 2022  
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April 14, 2022

# Announcements

- Homework 8 will be posted later today
- It will be due next Wednesday
- You will have recitation tomorrow as usual

## Exam 3

We are still grading Exam 3; we won't have it ready for you by Friday.

However, it looks like students' performance on this exam was somewhat lower than on the previous two exams, despite this material being simpler and with fewer nuances than the material on Exams 1 and 2.

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How many of you also had a calculus exam on the same day?

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To be clear:

- 1 Our standards haven't changed: we expect you to understand work, energy, power, and the conservation of momentum
- 2 We are going to make additional space and time for you to learn this material
- 3 ... and we are going to give you another chance to demonstrate your knowledge of it.

In short:

- Everyone will have an opportunity to redeem low grades on this exam, even beyond what we're already doing
- ... but you'll need to do some work toward that end.

## Exam 3: learning the material

Part of Homework 8 will be to do Exam 3 again, as homework. *(You don't need to do this if you earned 80% or higher.)*

This will let you earn a fraction of credit back – the lower your original score, the more credit you can earn back. *(Because of the limitations on TA's time, they will be grading this somewhat quickly and not looking in detail for partial credit: we expect you to understand things fully the second time around!)*

The rest of HW8 will be shorter to allow you to do that.

I will hold extra help hours Monday from 2:30-5pm in the Physics Clinic to help people learn this material.

## Final exam: earning credit back

I mentioned earlier that everyone will have a second opportunity to demonstrate mastery of material that they didn't have the first time around.

Here's how this will work:

- The final exam will have material from throughout the semester
- If you do better on the material from Unit 1, 2, or 3 on the final exam than you did on Exam 1, 2, or 3...
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- This will be structured to ensure that anyone with a passing understanding of the material at the end of the semester passes.

## “I’m worried about my grade!” and other FAQs

The median grade for Exams 1 and 2 was consistent with the class median being a B+ or even A-, given the lenient grading policy.

The median grade for Exam 3 will be lower, but most students will get a B or better in the course.

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### Q: Am I going to pass this class?

A1: If you have been going to recitation and doing your homework, almost certainly.

- Remember 50% is passing, and homework/recitation attendance is free credit
- If your exam grades are very low, you should identify the core skills you need to develop and work on them before the final

A2: If you have been earning low scores on the exams *and* not doing your homework/coming to recitation, maybe not



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A1: Yes, but you can calculate it better than I can.

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## Q: I missed an exam for [good reason X], what do I do?

A1: We will arrange for you to take an abbreviated makeup exam either on the day of the final or at another time during finals week.

I am completely saturated with email about missed exams. I will answer it as quickly as I can, but I have too much mail right now.

## A final note:

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I often have students disclose mental health problems, personal issues, issues with the university administration, and worse to me.

I don't go look up their grades before doing whatever I can for them.

# Taking stock: what have we learned?

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**Unit 4:** Rotation

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**Unit 3:** Conserved quantities – energy and momentum

**Unit 4:** Rotation

It turns out that rotational dynamics is mostly a mirror of what you have learned already!

Translational Idea	What it means
Position $\vec{s}$ Velocity $\vec{v}$ Acceleration $\vec{a}$	Where is the thing? How is it moving? How is its motion changing?
Kinematics: $\vec{s}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}_0t + \vec{s}_0$	How does accel relate to position and velocity?
Force $\vec{F}$ Mass $m$ Newton's second law $\vec{F} = m\vec{a}$	What pushes on the thing? How hard is the thing to move? How do forces make things move?
Kinetic energy $KE = \frac{1}{2}mv^2$ Work $W = \vec{F} \cdot \Delta\vec{s}$ Power $P = \vec{F} \cdot \vec{v}$	Energy associated with speed How do forces change objects' speed? At what rate do forces change objects' energy?
Momentum $\vec{p} = m\vec{v}$	The “persistence” of an object's motion

Translation	Rotation
Position $\vec{s}$ Velocity $\vec{v}$ Acceleration $\vec{a}$	Angle $\theta$ Angular velocity $\omega$ Angular acceleration $\alpha$
Kinematics: $\vec{s}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}_0t + \vec{s}_0$	$\theta(t) = \frac{1}{2}\alpha t^2 + \omega_0t + \theta_0$
Force $\vec{F}$ Mass $m$ Newton's second law $\vec{F} = m\vec{a}$	Torque $\tau$ Rotational inertia $I$ Newton's second law for rotation $\tau = I\alpha$
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The analogue of mass is called “moment of inertia” (letter  $I$ )

- More massive things are harder to turn, but that’s only part of it
- The mass *distribution* matters, too
- The further the mass is from the center, the harder it will be to turn
- The moment of inertia depends on the *average squared distance from the center*



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$$I = MR^2$$

(if all the mass is the same distance from the center)  
(our demo rods; hoops; rings; bike wheels)

# Moment of inertia: why?

To see why  $I = M \langle r^2 \rangle$ , let's consider the kinetic energy of a spinning object.

The kinetic energy of a single “point mass” moving in a circle is  $\frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$ , where  $r$  is its distance from the center.

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For an extended object, we simply add up the energy of all the moving particles:

$$I = \int r^2 dm = M \langle r^2 \rangle$$

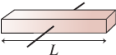
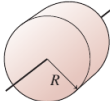
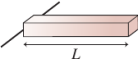
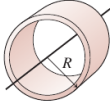
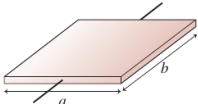
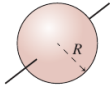
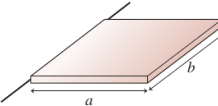
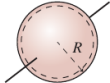
i.e. the moment of inertia is just the total mass times the average squared distance from the axis.

# Moment of inertia, other things

What about the moment of inertia of other objects?

Requires calculus in general; here are some common ones

**TABLE 12.2** Moments of inertia of objects with uniform density

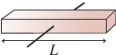
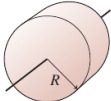
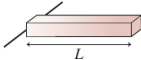
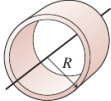
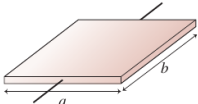
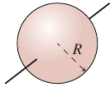
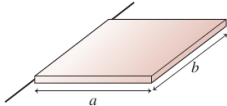
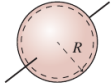
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Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		$MR^2$
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
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In general:  $I = \lambda MR^2$

We will always give you  $I$  if it's not 1 (i.e. not a ring etc.)

# What about rotational kinetic energy?

Translation	Rotation
Position $\vec{s}$ Velocity $\vec{v}$ Acceleration $\vec{a}$	Angle $\theta$ Angular velocity $\omega$ Angular acceleration $\alpha$
Kinematics: $\vec{s}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}_0t + \vec{s}_0$	$\theta(t) = \frac{1}{2}\alpha t^2 + \omega_0t + \theta_0$
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An object with moment of inertia  $I$   
rotating at angular velocity  $\omega$  has  
rotational kinetic energy

$$KE_{rot} = \frac{1}{2}I\omega^2$$



## An example

How much energy does this bicycle wheel store when I make it spin?

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What if it were a solid cylinder?

## Another example

Remember the “Atwood machine”?

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What happens if we remove one of the weights?

## Another example

Remember the “Atwood machine”?

What happens if we remove one of the weights?

What happens if the pulley isn't light?

# The relationship between linear and rotational motion

What's the acceleration of an object traveling in circular motion?

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$$a = \omega^2 r \quad \text{toward the center}$$

$$a = v^2/r \quad \text{toward the center}$$

Why do we have two different formulae? This came from the relationship:

$$v = \omega r$$

If an object rotates at angular velocity  $\omega$ , a point a distance  $r$  from the center moves at speed  $v$ .

Suppose I wrap a string around a solid cylinder with mass  $M$  and radius  $r$ , and let a mass  $m$  hang from the string.

How fast is the falling mass traveling when it hits the ground if it starts from a height  $h$ ?

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$$(\text{initial KE}) + (\text{work done by gravity}) = (\text{final KE})$$

$$(\text{initial rotational KE}) + (\text{initial translational KE}) + (\text{work done by gravity}) = \\ (\text{final rotational KE}) + (\text{final translational KE})$$

Which object will reach the bottom of the ramp faster?

A: The wooden one

B: The one with the mass located near the middle

C: The one with the mass located near the edge

D: A tie between A and B

E: A tie between B and C

# Rotation plus translation

In general, rotation and translation are separate; we can study each separately.

Example: this bike wheel

- Its position is given by some function  $\vec{s}(t)$ : “where is it at some time  $t$ ?”
- Its angle is given by some other function  $\theta(t)$ : “which way is the reference point pointing at some time  $t$ ?”
- The angle has the familiar derivatives: angular velocity  $\omega$ , angular acceleration  $\alpha$

Recall that points along the edge of a rotating object move at a speed  $v_{\text{edge}} = \omega r$ .

## Example: rolling without slipping

Sometimes the translational and rotational motion are linked.

“How fast do the tires on a car turn?”

→ Static friction means that the bottom piece of the wheel doesn't move

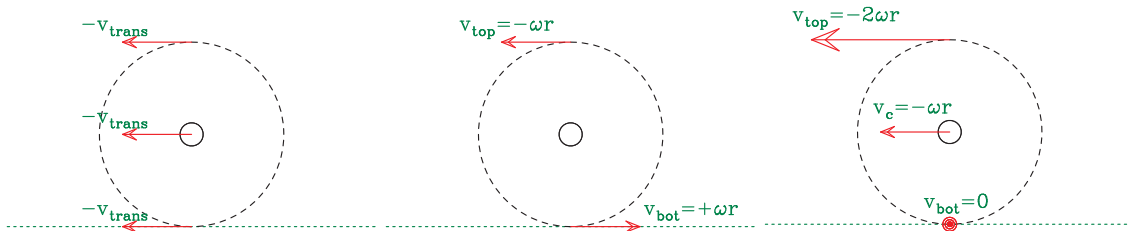
- If a wheel is turning counterclockwise at angular velocity  $\omega$ :
  - the top moves at  $v_{\text{top}} = -\omega r$  (left)
  - the bottom moves at  $v_{\text{bot}} = \omega r$  (right)
- This means that the velocity of the axle must be equal and opposite to  $v_{\text{bot}}$
- Thus, the car must be moving at  $v_{\text{axle}} = -\omega r$  (left).

Let's look at a diagram.

So: if the wheels turn counterclockwise at  $\omega$ :

- The axle moves at a velocity  $-\omega r$  (left);
- The top of the wheels move at a velocity  $v_{\text{axle}} + v_{\text{top}} = -\omega r - \omega r = -2\omega r$ ;
- The bottom of the wheels move at a velocity  $v_{\text{axle}} + v_{\text{bot}} = -\omega r + \omega r = 0$ .

# Rolling without slipping



Translation + Rotation = Rolling



## The “rolling constraint”

If an object rolls forward on an edge of radius  $r$ ,

$$v = \omega r$$

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Common algebra pattern:

$$K E_{rot} = \frac{1}{2} I \omega^2$$

$$K E_{rot} = \frac{1}{2} \lambda m r^2 \omega^2$$

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