Motion with constant acceleration

Physics 211
Syracuse University
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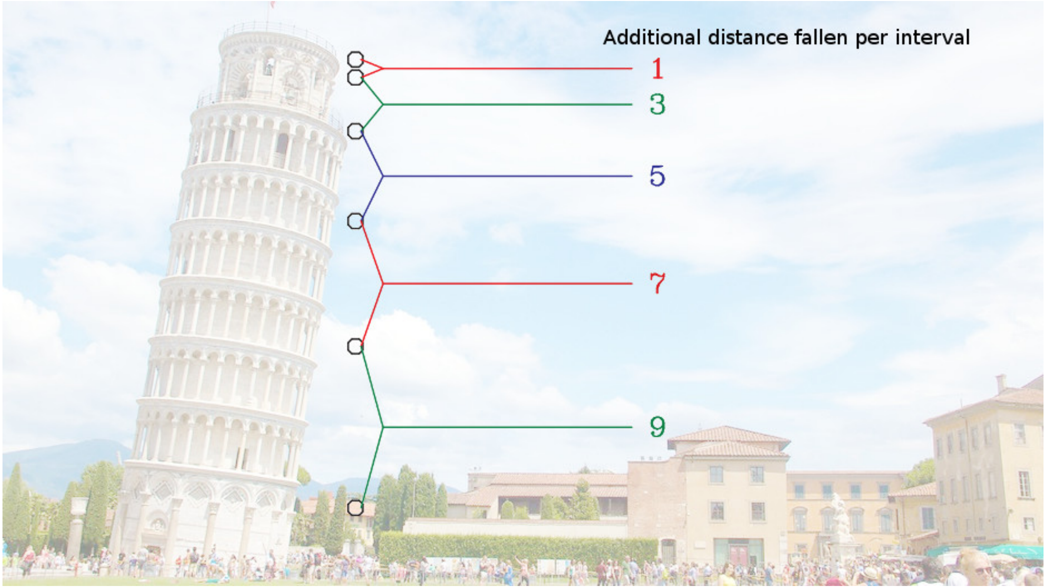
January 18, 2023

The beginning: Free fall

My purpose is to set forth a very new science dealing with a very ancient subject. There is, in nature, perhaps nothing older than motion, concerning which the books written by philosophers are neither few nor small nevertheless I have discovered by experiment some properties of it which are worth knowing and which have not hitherto been either observed or demonstrated....

So far as I know, no one has yet pointed out that the distances traversed, during equal intervals of time, by a body falling from rest, stand to one another **in the same ratio as the odd numbers beginning with unity.**

–Galileo Galilei, *Dialogues and Mathematical Demonstrations Concerning Two New Sciences*, 1638 (translated into English by some old dead guy)


Additional distance fallen per interval
1
3
5
7
9

The beginning: Free fall

I have discovered something new about something very old – motion. Philosophers have written piles of thicc books about it for centuries. But by doing *experiments* I've learned something they've never figured out.

None of them have figured out that a falling object travels one unit of distance, then three, then five, then seven, etc., in equal amounts of time.

–Galileo Galilei, *Dialogues and Mathematical Demonstrations Concerning Two New Sciences*, 1638 (in modern English)

Empiricism

The highest authority in science is what we measure.

If we're seeking truth about nature, the starting point *must be* what we see.

Galileo made this observation about falling objects. He *didn't* know why.

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Our job for today:

- Come up with a model that might explain falling
- Use mathematics to explore the things that our model predicts
  - Does our model predict Galileo's observation?
- See what else our model could also explain besides falling objects

Reminders:

- Webpage: `https://walterfreeman.github.io/phy211/`
  - Syllabus, homework, etc. are all there
- Homework 0 is a brief survey including about your math skills
- Homework 1 will be posted today and will be due next Thursday or Friday in recitation

“Ask a Physicist”

There are a lot of cool things in physics that go beyond mechanics.

If you’ve got questions you’d like me to address, send them in and I’ll answer them!

- What are gravity waves?
- Is there a smallest distance in the universe? *(This is my research actually!)*
- How is physics used in medicine? Or computer design? Or environmental science? Or the military?
- What’s the Large Hadron Collider for?
- How does a touchscreen work?
- How do 3D movies work?
- What is the Higgs boson?
- How is physics used in video games?
- How does a nuclear reactor work?
- How do we simulate physics using computers?
- How does a supercomputer work (and do this faster)?

Homework tips

Your first “real” homework assignment is due next Friday.

- Make use of words, pictures, and algebra (not just algebra!) in your reasoning
- We’re interested in how you think, not just the answer – you must explain what you’re doing
- Physical values need to be given with units (“4 meters”, not “4”)
- Leave variables in until the very end
- Paper is cheap – don’t cramp yourself!

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  - Email: wafreema@syr.edu
  - Discord server
  - Office hours
  - The Physics Clinic
  - Recitations

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  - In class!

Office hours

In the Physics Clinic:

- Tuesdays: 2:00-4:00 PM
- Fridays: 9:00 AM-11:00 AM (may change)
- Other times announced (if homework is due Friday, I may hold Thursday office hours)

or by appointment.

Outside these times you might find me in the Clinic or in my office in room 215.

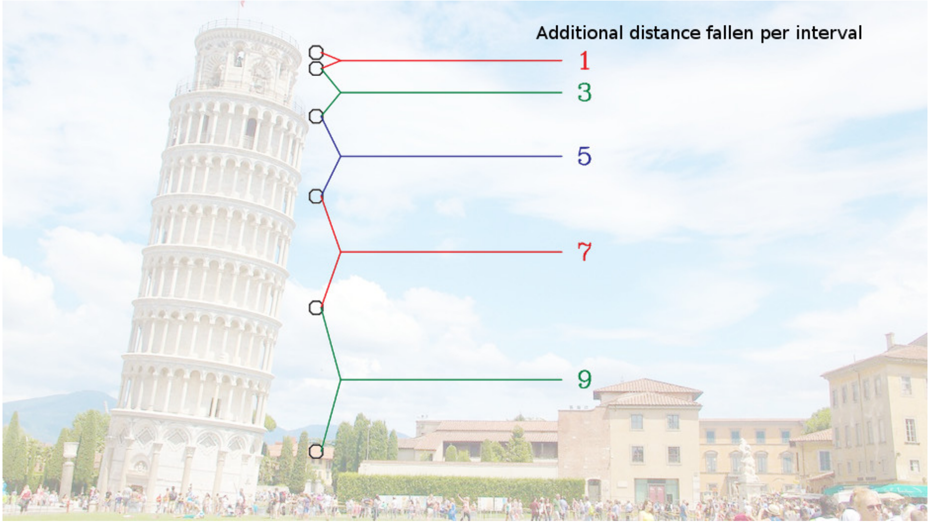
The beginning: describing motion (1-D)

Recall that at first, we are only concerned with describing motion.

- Most fundamental question: “where is the object I’m talking about?”
- Quantify position using a “number line” marked in meters:
  - Choose one position to be the origin (“zero”) – anywhere will do
  - Choose one direction to be positive
  - Measure everything relative to that
  - Can measure in any convenient units: centimeters, meters, kilometers...

- You’re used to this already, perhaps:
  - Mile markers on highways
  - Yard lines in American football

The beginning: Free fall



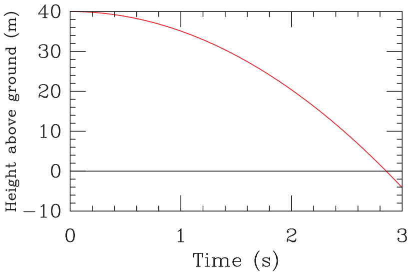
Galileo observed this, but can we explain it?

Equations of motion

Complete description of motion: "Where is my object at each point in time?"

This corresponds to a mathematical function. Two ways to represent these. Suppose I drop a ball off a building, putting the origin at the ground and calling "up" the positive direction:

Graphical representation



Algebraic representation

$$y(t) = (40\,\mathrm{m}) - Ct^2$$

(C is some number; we'll learn what it is in a bit)

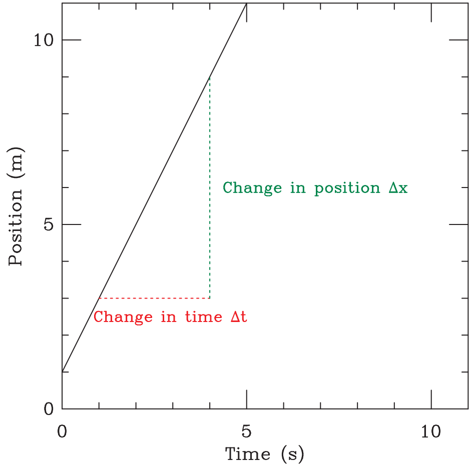
Both let us answer questions like "When does the object hit the ground?"

→ ... the curve's x-intercept

→ ... when $y(t) = 0$

Velocity: how fast position changes

The slope of the position vs. time curve has a special significance. Here's one with a constant slope:



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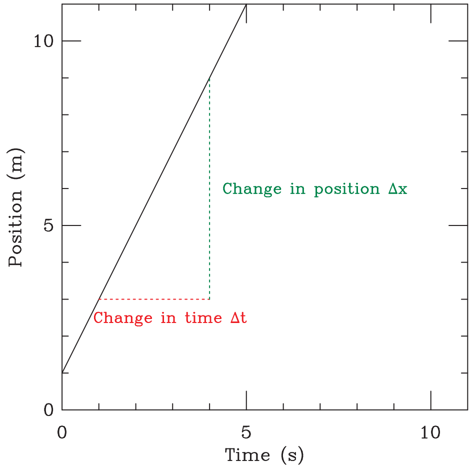

Slope is $\frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{2\,\text{m}}{1\,\text{s}} = 2$ meters per second (positive; it could well be negative!)

Velocity: how fast position changes

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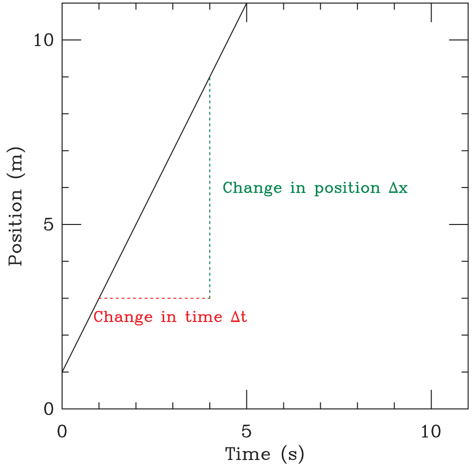
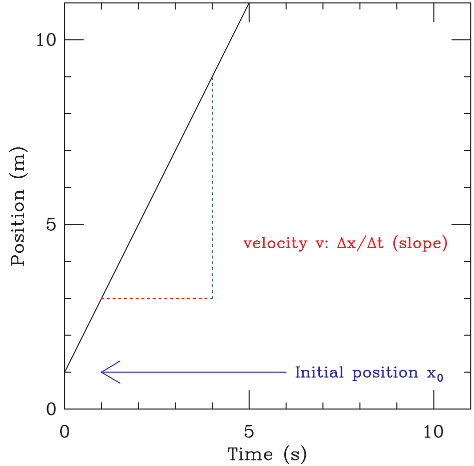


Slope is $\frac{\text{rise}}{\text{run}} = \frac{\Delta x}{\Delta t} = \frac{2\,\text{m}}{1\,\text{s}} = 2$ meters per second (positive; it could well be negative!)

→ The slope here – change in position over change in time – is the **velocity**! Note that it can be positive or negative, depending on which way the object moves.

Constant-velocity motion: connecting graphs to algebra

If an object moves with constant velocity, its position vs. time graph is a line:



We know the equation of a straight line is is $x = mt + b$ (using t and x as our axes).

- m is the slope, which we identified as the velocity
- b is the vertical intercept, which we recognize as the value of x when $t = 0$

We can thus change the variable names to be more descriptive:

$$x(t) = vt + x_0 \text{ (constant-velocity motion)}$$

Going from “equations of motion” to answers

$x(t) = vt + x_0$ is called an *equation of motion*; in this case, it is valid for constant-velocity motion.

It gives you the same information as a position vs. time graph, but in algebraic form.

To solve real problems, we need to be able to translate physical questions into algebraic statements:

- “If a car starts at milepost 30 and drives at 50 mph, where is it an hour later?”

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To solve real problems, we need to be able to translate physical questions into algebraic statements:

- "If a car starts at milepost 30 and drives at 50 mph, where is it an hour later?"
  - Using $x(t) = x_0 + vt$, with $x_0 = 30\,\mathrm{mi}$ and $v = 50\frac{\mathrm{mi}}{\mathrm{hr}}$, calculate x at $t = 1\,\mathrm{hr}$

Asking the right questions

"I drop an object from a height h. When does it hit the ground?" How do I do this? (Take $x_0 = h$ and upward to be positive.)

Remember, we want to ask a question in terms of our physical variables. This question has the form:

"What is __________ when __________ equals __________?"

Fill in the blanks.

A: v, x, 0

B: t, x, h

C: x, t, 0

D: t, x, 0

E: x, v, 0

Asking the right questions

"At what location do two moving objects meet?"

A: "At what time does $x_1 = x_2$?"
B: "At what time does $v_1 = v_2$?"
C: "What is x_1 at the time when $x_1 = x_2$?"
D: "What is x_1 when $t_1 = t_2$?"

Velocity, acceleration, and calculus

Constant-velocity motion: $x(t) = x_0 + vt$

- Came from looking at the equation of a line
- We can understand this in a different framework, too:

- Velocity is the rate of change of position
  - Graphical representation: Velocity is the slope of the position vs. time graph
  - Mathematical language: Velocity is the derivative of position

We know we need to know about acceleration (“F=ma”) – what is it?

- Acceleration is the rate of change of velocity

Position, velocity, and acceleration

$$\text{Position} \xrightarrow[]{\substack{\text{(take the derivative)}\\ \text{take the rate of change of}}} \text{Velocity}$$

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Kinematics: how does acceleration affect movement?

Newton's law $a = F/m$ tells us that *acceleration* – the second derivative of position – is what results from forces.

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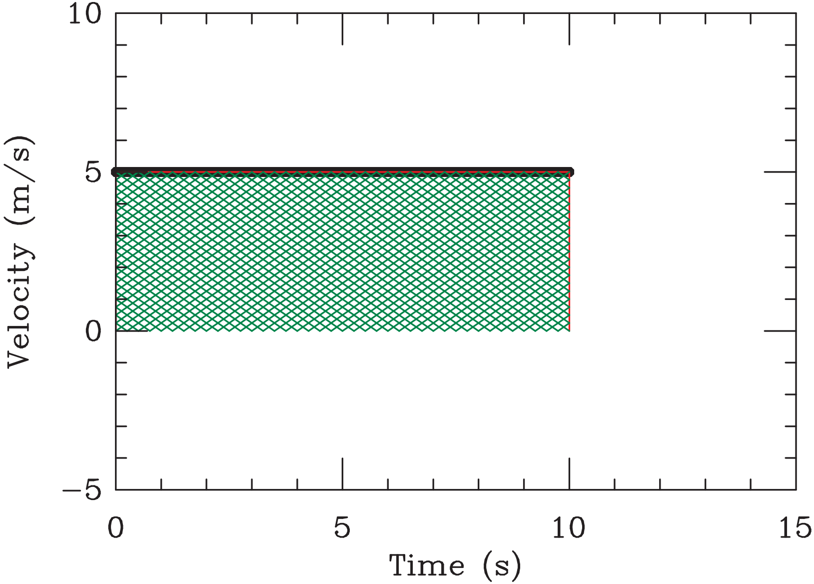
(You can approximate this as $g = 10\text{m/s}^2$ unless you need high precision.)

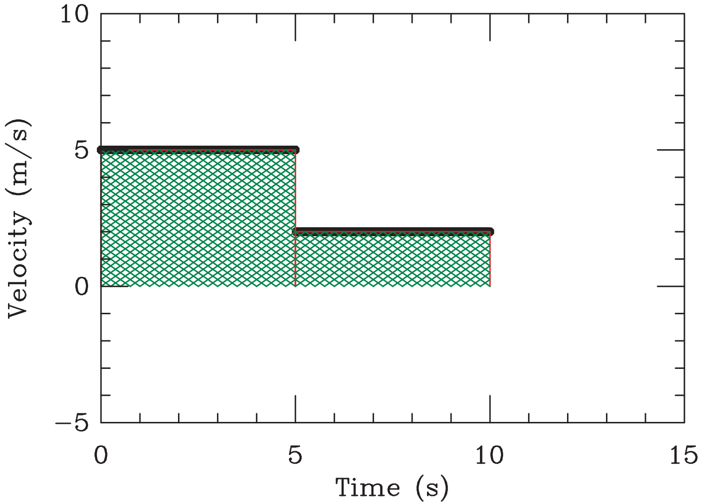
Some calculus

If velocity is the rate of change of position,
why is the area under the v vs. t curve equal to displacement?



We know $\Delta s = vt$. What is that here? What's the area of the shaded region?

Some calculus



Now what is Δs? What is the area of the shaded region?

What's the area of the shaded region?

A: 25 m

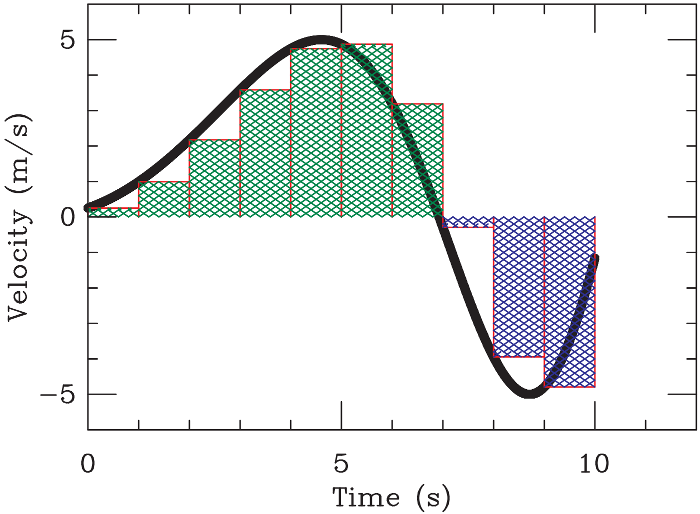
B: 50 m

C: 35 m

D: 45 m

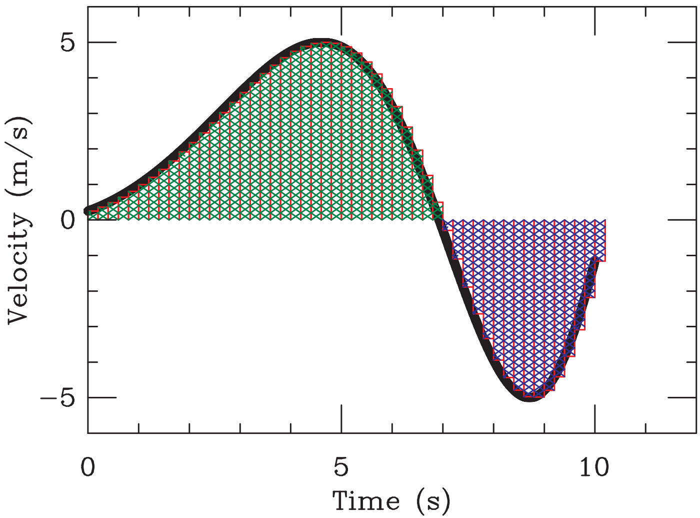
A calculus review

What if the velocity is more complicated?



Does this work? How do we fix it?

A calculus review



The area between the t-axis and the velocity curve is the distance traveled.
(The area below the t-axis counts negative: "the thing is going backwards"

In calculus notation: $\int v(t)\,dt = \delta x = x(t) - x_0$

Position, velocity, and acceleration

$$\text{Position} \quad \underset{\substack{\text{take the area under the curve of} \\ \text{(take the integral of)}}}{\overset{\substack{\text{(take the derivative of)} \\ \text{take the rate of change of}}}{\rightleftarrows}} \quad \text{Velocity}$$

Position, velocity, and acceleration

(take the derivative of)
take the rate of change of

(derivative of)
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Position　　Velocity　　Acceleration

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Constant acceleration

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Plan of attack:

- We know what the acceleration curve looks like (it's flat). How do we get the velocity curve?

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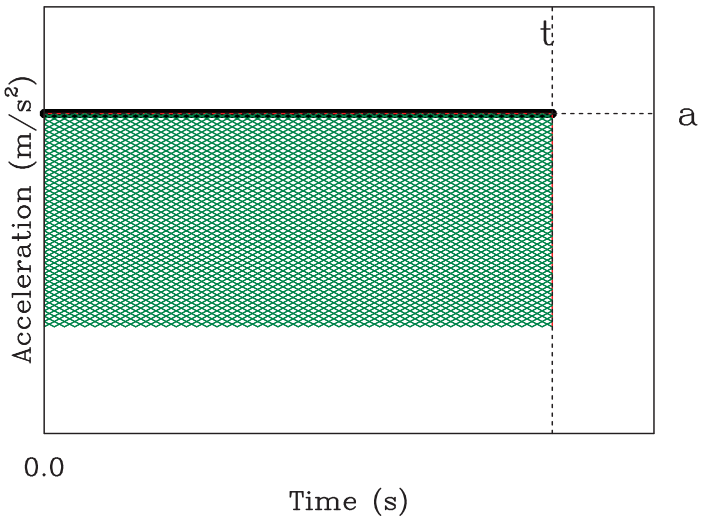
- Figure out the area under the acceleration curve to get the velocity curve
- Figure out the area under the velocity curve to get the position curve

Remember the area under the curve of (velocity, acceleration) just gives the *change in* (position, velocity) – *i.e.* initial minus final.

We'll start by assuming x_0 and v_0 are zero – that is, we're dropping something from rest.

Constant acceleration



What's the area under the curve out to time t, which gives the change in the velocity – $\Delta v = v(t) - v_0$?

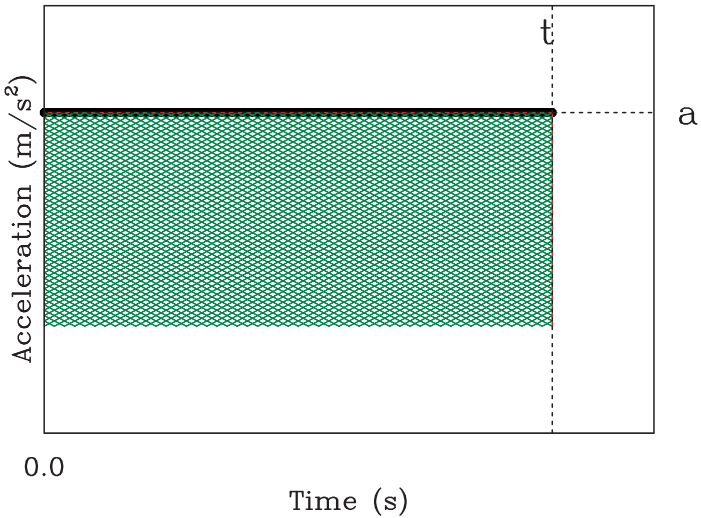
A: $\Delta v = at$

B: $\Delta v = at + v_0$

C: $\Delta v = \frac{1}{2}at^2$

D: $\Delta v = a$

Constant acceleration



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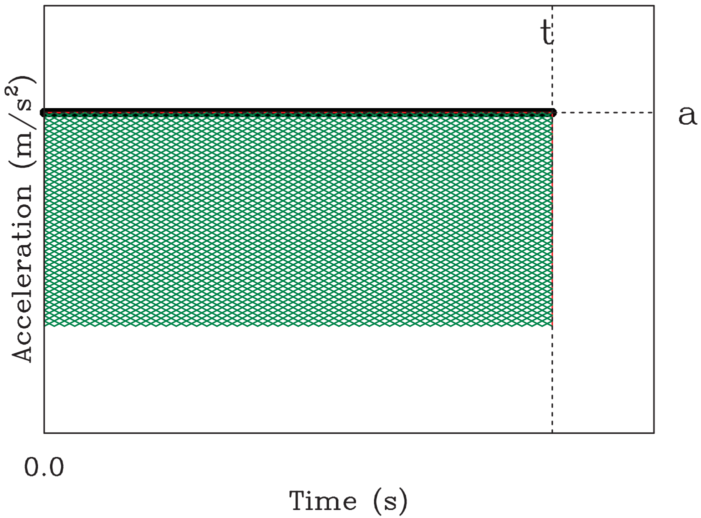
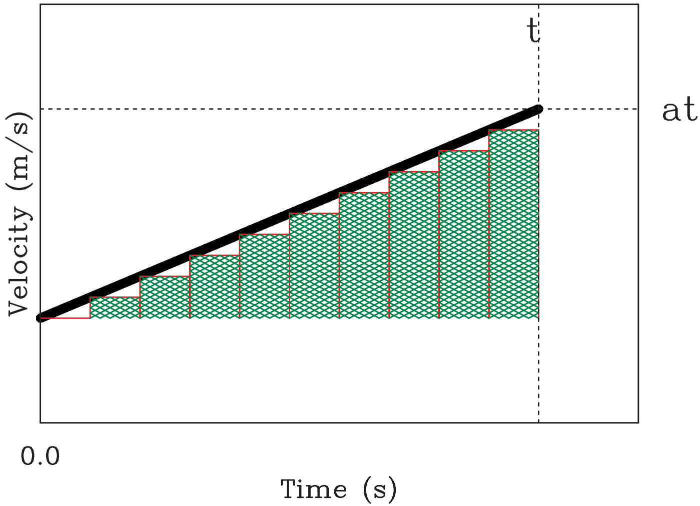


What's the area under the curve out to time t, which gives the change in the velocity – $\Delta v = v(t) - v_0$?

Δv, the change in velocity, is $v(t) - v_0 = at$, so $v(t) = at + v_0$

Same thing again to get position



Now the area under the velocity curve gives the change in position: $\Delta x = x(t) - x_0$. What is that?

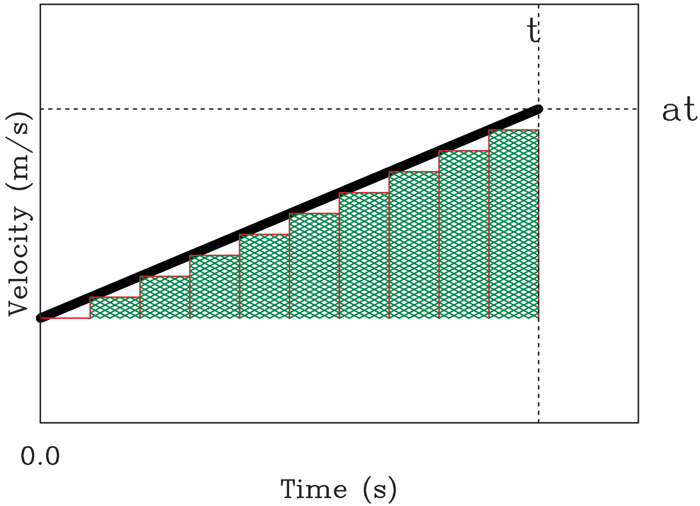
A: $\Delta x = at$

B: $\Delta x = vt$

C: $\Delta x = \frac{1}{2}at^2$

D: $\Delta x = v$

Same thing again to get position



Now the area under the velocity curve gives the change in position: $\Delta x = x(t) - x_0$. What is that?

A: $\Delta x = at$

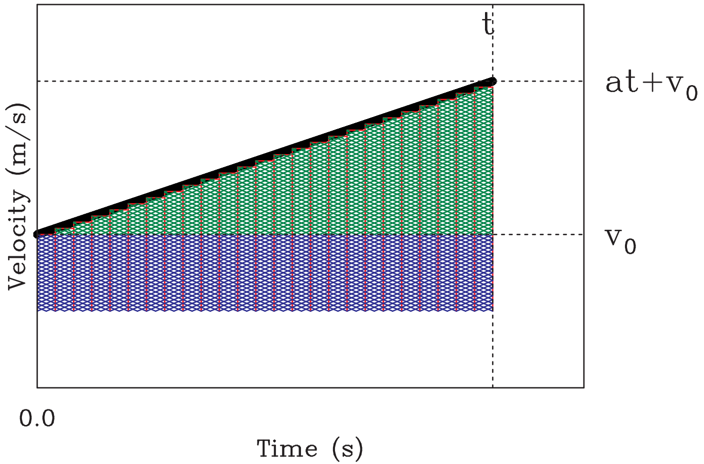
B: $\Delta x = vt$

C: $\Delta x = \frac{1}{2}at^2$

D: $\Delta x = v$

$$x(t) - x_0 = \frac{1}{2}at^2, \text{ thus } x(t) = \frac{1}{2}at + x_0$$

Now if v_0 is not zero...



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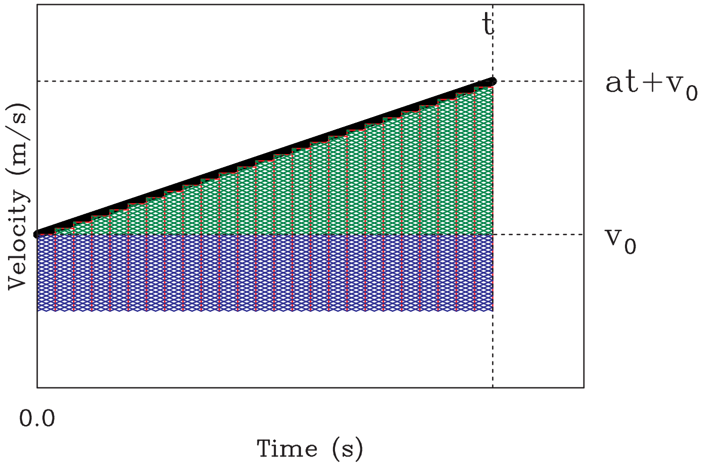


Area under blue part: $v_0 t$
Area under green part: $\frac{1}{2}at^2$
Total change in position: $x(t) - x_0 = \frac{1}{2}at^2 + v_0 t$

$$\text{Thus, } x(t) = \frac{1}{2}at^2 + v_0 t + x_0$$

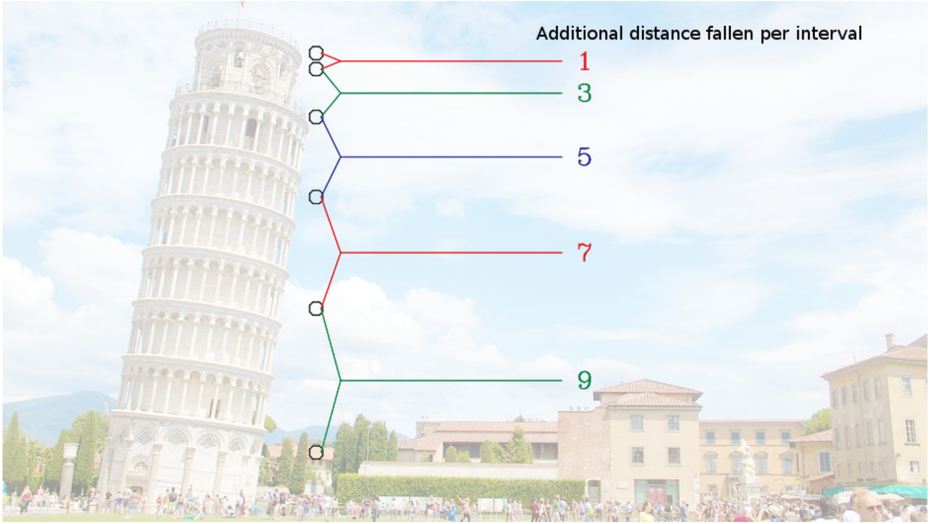
For those who are familiar with calculus:

$$
\begin{aligned}
a(t) &= \text{const.} \\
v(t) &= \int a\,dt &&= at + C_1 \\
x(t) &= \int v\,dt = \int (at + C_1)dt &&= \frac{1}{2}at^2 + C_1 t + C_2
\end{aligned}
$$

A little thought reveals that C_1 is the initial velocity v_0 and C_2 is the initial position x_0. This gives us the things we just derived, but much more easily:

$$
\begin{aligned}
v(t) &= at + v_0 \\
x(t) &= \frac{1}{2}at^2 + v_0 t + x_0
\end{aligned}
$$

Free fall revisited



Adding these numbers together gives us 1, 4, 9, 16, 25...
The calculus above explains this: distance is proportional to *time squared!*