Recitation Questions

15 February

Two very small cats, Fifi and Tali, are sitting on a smooth table when the table begins to tip. Fifi has a mass of m_f kg and Tali has a mass of m_t kg.[1]

The coefficients of friction between the kitties and the table are the following (Tali is slightly fuzzier):

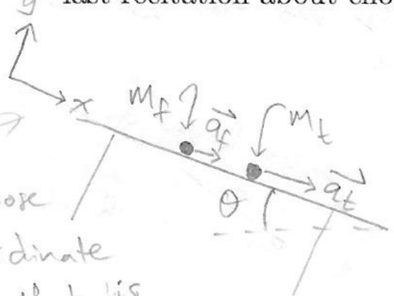
	Fifi	Tali
μ_k	0.4	0.3
μ_s	0.5	0.4
mass (kg)	3.4	3.6

As the angle θ between the table and the horizontal becomes larger and larger, eventually the cats will slide off the table.[2]

Remember two things about friction for this problem:

1. If two things are *already sliding* past one another, the force of kinetic friction between them is equal to $\mu_k F_N$ in whatever direction opposes that motion;
2. If two things are *not sliding*, the force of static friction is *however big it needs to be* in order to stop them from sliding, up to a *maximum* of $\mu_s F_N$.

a) Draw a cartoon of the problem, and choose a coordinate system. Recall what you learned last recitation about choosing coordinate systems that make your life easy.



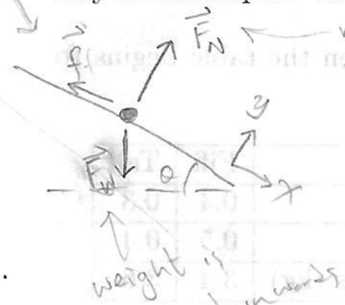
we choose a coordinate system that is aligned with the acceleration of the cats. That way, we have $a_y = 0$ for both cats (since they can only accelerate down the ramp, not off the ramp).

[1] This problem was inspired by the joke: "Q: Two kittens are sitting on a roof. Which one slides off first? A: The one with the smallest mew."

[2] They will land on their feet, since they are graceful cats. Their brother Pierre is a klutz and would land on his head, which is why we're not using him for this problem. But he's cute.

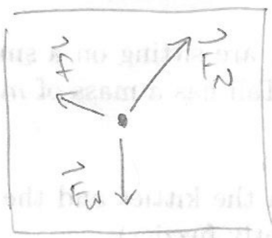
friction always opposes slipping

b) Draw a force diagram for the cat. Make it nice and large, since you'll need to do trigonometry to decompose the weight force into components.

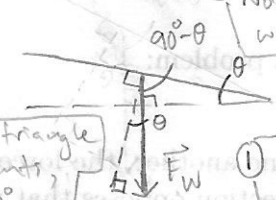


normal force perpendicular to the slope!

weight is always downwards



note that $\vec{F}_N$ is pointed in the y-direction and $\vec{f}$ is pointed in the negative x-direction, so only $\vec{F}_W$ needs to be split into components



(2) Notice the right triangle with angles θ, $90°$, and $90°-\theta$

As a check, if you draw your slope with θ small, other angles that are θ will also be small.

(3) In the triangle with the $\vec{F}_W$ components, see that $(90°-\theta)+\theta=90°$

(1) Draw the slope and $\vec{F}_W$ together to find θ in the components

c) Decompose the weight force into components. Do this as always: draw a right triangle with the weight force as its hypotenuse, and with its legs aligned with your coordinate system. Then, figure out which angle in the right triangle is the same as θ. (Do this on your diagram above.)

d) Write down Newton's second law $\sum F = ma$ in both $x-$ and $y-$directions.

You know your triangle is correct if your force vector forms the hypotenuse, and the components meet at a right angle

your force diagram tells you that $\sum F$ contains three different forces

Include the forces in your force diagram

$$\sum F_x = ma_x$$

$$f_x + F_{Wx} + F_{Nx} = ma_x$$

$$\boxed{-f + mg\sin\theta + 0 = ma_x}$$

Write the components in terms of the forces' magnitudes

$$\sum F_y = ma_y$$

$$f_y + F_{Wy} + F_{Ny} = 0$$

we know a_y is zero since the cats don't accelerate off the slope

$$\boxed{0 - mg\cos\theta + F_N = 0}$$

e) Right before the cat begins to slide off the table, what is true about the frictional force on them? Use this mathematical condition to solve for the angle θ at which each cat begins to slip off the table.

Since the cats aren't sliding down the table yet, we have static friction. Since they are close to sliding off, we know static friction is at its maximum:

$f = \mu_s F_N$. Also, a motionless cat has $a_x = 0$.

	Fifi	Tali
μ_s	0.5	0.4
θ	27°	22°

slipping angle

Now $-\mu_s F_N + mg\sin\theta = 0$, and $-mg\cos\theta + F_N = 0 \longrightarrow F_N = mg\cos\theta$.

Substituting, $-\mu_s(mg\cos\theta) + mg\sin\theta = 0$

when we divide by $\cos\theta$, we get $\frac{\sin\theta}{\cos\theta} = \tan\theta$

$-\mu_s\cos\theta + \sin\theta = 0$, $-\mu_s + \tan\theta = 0$, $\tan\theta = \mu_s$, so $\boxed{\theta = \tan^{-1}(\mu_s)}$

$m = 3.4$ kg, $\mu_k = 0.4$

$a_x \neq 0$ now, and sliding means kinetic friction is acting instead of static.

$m = 3.6$ kg, $\mu_k = 0.3$

f) Right after Fifi begins to slide, what will her acceleration be? What will Tali's be?

Our force diagram is the same the size of the friction force f is the only thing that changes. So we can reuse our 2nd law equations:

$$-f + mg\sin\theta = ma_x \qquad F_N = mg\cos\theta$$

Now we have $f = \mu_k F_N$ (always true for kinetic friction)

$$-\mu_k F_N + mg\sin\theta = ma_x,$$ so we can substitute in F_N in terms of knowns:

$$-\mu_k \cancel{m} g\cos\theta + \cancel{m} g\sin\theta = \cancel{m} a_x,$$

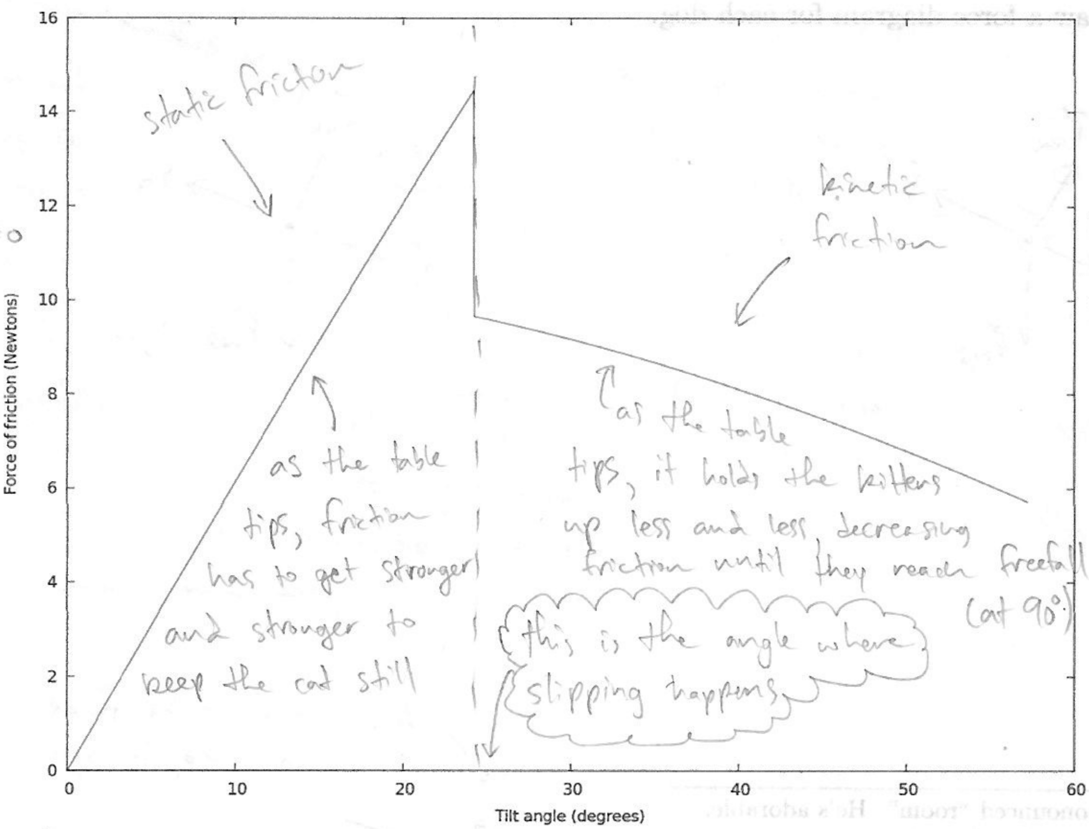
$$a_x = g(\sin\theta - \mu_k\cos\theta).$$

Using the angles from before,

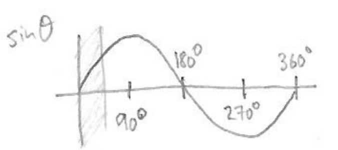
	Fifi	Tali
a	0.96 m/s²	0.95 m/s²

g) Here is a graph of the frictional force (whether static or kinetic) vs. tilt angle. Interpret as many of its features as you can; call your TA and/or coach over to join your conversation.

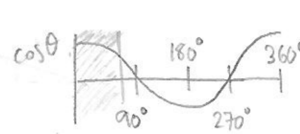
When $a_x = 0$ (the cat isn't moving), we have $-f + mg\sin\theta = 0$ (from $\Sigma F_x = ma_x$), or $f = mg\sin\theta$



When $a_x > 0$, (the cat is sliding), we have $f = \mu_k F_N$, or $f = \mu_k mg\cos\theta$



for small angles, $\sin\theta$ looks like a straight line



$\cos\theta$ starts flat and slopes downward for angles below 90°

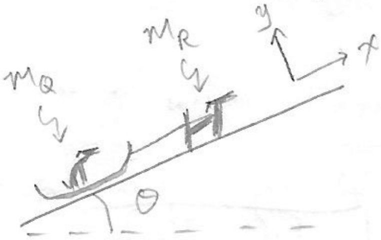
Rum[3], Ohana's larger dog, is harnessed to a sled carrying Quanta, her smaller dog.[4]

Rum has mass m_R and coefficient of static friction μ_s; Quanta and her sled have mass m_Q and coefficient of kinetic friction μ_k. (Remember that traction is just a special kind of static friction, and so the maximum traction force that Rum can exert is also equal to $\mu_s F_N$.)

Rum is trying to pull Quanta and her sled up a hill sloping up at an angle θ at a constant speed. In this problem, you'll solve for the steepest hill that they can climb.

a) Draw a cartoon of the problem, and choose a coordinate system.

if speed and direction are constant, acceleration is zero

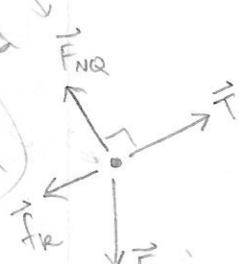


Again, if motion is only in the x-direction, we know $a_y = 0$ (since the dogs won't accelerate off of the slope)

b) Draw a force diagram for each dog.

the normal force is perpendicular to the ground

tension from the rope pulls Quanta up the hill

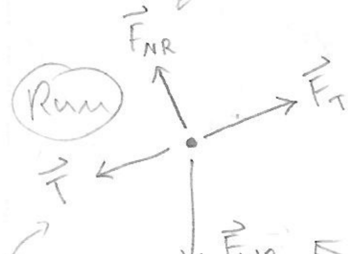


the weight force is always straight down

kinetic friction opposes sliding up the hill

Rum's normal force can be different from Quanta's

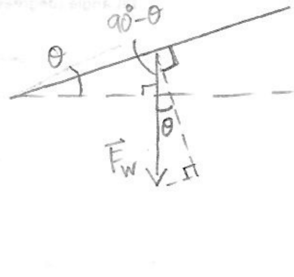
the traction force allows Rum to climb up the hill



Rum's weight is different from Quanta's

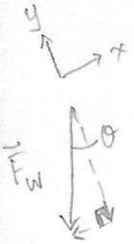
the tension on Rum is the same magnitude as the tension on Quanta's sled, since the same rope connects them

Again, the weight force is the only one we need to split into components



[3]Pronounced "room". He's adorable.

[4]She's also adorable.



Each force diagram gives its own copies of Newton's 2nd law

c) Write down Newton's second law in both directions – that is, $\sum F_x = ma_x$ and $\sum F_y = ma_y$ – in both directions.

Quanta

consider all four forces from Quanta's force diagram

$\sum F_x = ma_x$

$\downarrow$

$f_{kx} + F_{NQx} + F_{WQx} + T_x = m_Q a_x$

Rewrite

$\boxed{-f_k + 0 - M_Q g \sin\theta + T = M_Q a_x}$

$\sum F_y = ma_y$

$\downarrow$

$f_{ky} + F_{NQy} + F_{WQy} + T_y = 0.$

$\boxed{0 + F_{NQ} - M_Q g \cos\theta + 0 = 0}$

Rum

$\sum F_x = ma_x$

$\downarrow$

$T_x + F_{NRx} + F_{WRx} + F_{Tx} = Ma_x$

$\downarrow$

$\boxed{-T + 0 - M_R g \sin\theta + F_T = M_R a_x}$

$\sum F_y = ma_y$

$\downarrow$

$T_y + F_{NRy} + F_{WRy} + F_{Ty} = 0$

$\boxed{0 + F_{NR} - M_R g \cos\theta + 0 = 0}$

these components in terms of the magnitudes of the forces, including minus signs to indicate direction

a_x is the same for both dogs, since they are connected by a rope

d) This will result in four equations. Plug in things that you know. (What do you know about their accelerations?) This will result in four equations with four unknowns. What is true about the traction force on Rum when he's climbing the steepest hill that he can? Underneath each equation, identify the physical meaning of each term (i.e. "component of Quanta's weight parallel to the slope").

We know $f_k = \mu_k F_{NQ}$, since the kinetic friction on Quanta comes from the normal force on Quanta. We also know (since traction comes from static friction) that if he's climbing the steepest hill he can, $F_T = \mu_s F_{NR}$, the maximum of static friction.

Therefore, we have

$$\boxed{\begin{aligned} &-\mu_k F_{NQ} - M_Q g \sin\theta + T = 0, \quad F_{NQ} - M_Q g \cos\theta = 0, \\ &-T - M_R g \sin\theta + \mu_s F_{NR} = 0, \quad F_{NR} - M_R g \cos\theta = 0 \end{aligned}}$$

e) Discuss how you'd do the algebra to solve these equations; if you have time, work on doing so. We can see that $F_{NQ} = M_Q g \cos\theta$, and $F_{NR} = M_R g \cos\theta$.

Substituting, $-\mu_k M_Q g \cos\theta - M_Q g \sin\theta + T = 0$,

$-T - M_R g \sin\theta + \mu_s M_R g \cos\theta = 0$. If we add these equations the T's will cancel, leaving

$-\mu_k M_Q g \cos\theta - M_Q g \sin\theta - M_R g \sin\theta + \mu_s M_R g \cos\theta = 0$. We divide by $\cos\theta$, turning $\sin\theta$ into $\frac{\sin\theta}{\cos\theta} = \tan\theta$, so

$-\mu_k M_Q \cancel{g} - M_Q \cancel{g} \tan\theta - M_R \cancel{g} \tan\theta + \mu_s M_R \cancel{g} = 0$. Collecting, $\tan\theta (M_Q + M_R) = \mu_s M_R - \mu_k M_Q$

Finally, $\boxed{\theta = \tan^{-1}\left(\frac{\mu_s M_R - \mu_k M_Q}{M_Q + M_R}\right)}$