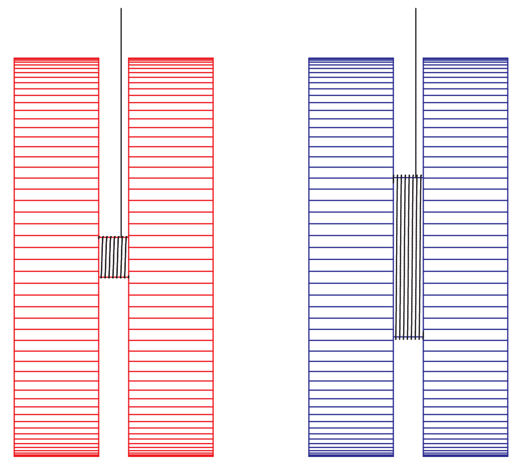
Recitation Exercises

27 April

Exercise 1: on combining rotation and translation

A Yo-Yo consists of a cylinder of radius R with a thin slit cut in it. Inside the slit is a smaller inner cylinder of radius r with a string attached to it and then wound around the cylinder. Note that the moment of inertia of a cylinder of radius R is $I = \frac{1}{2}mR^2$; since the slit in the Yo-Yo is so thin, you do not need to consider it in computing the moment of inertia. (Thus, both have the same moment of inertia: $I = \frac{1}{2}mR^2$.)
If a person holds the end of the string and drops the Yo-Yo, it will begin to spin as it falls, unwinding the string as it does.



a) Suppose that you have a red Yo-Yo with $r = 0.1R$ (that is, with a very small inner cylinder) and a blue Yo-Yo with $r = 0.4R$ (with a thicker inner cylinder). Predict which one will fall faster when it is dropped, and describe why it will do so. *(You shouldn't do any calculations here.)*

b) Now, you'll calculate the downward acceleration of the Yo-Yo. In this case, the Yo-Yo both *translates* and *rotates* as it does so

Start by drawing an extended force diagram for the Yo-Yo, showing all the forces acting on it *and where they act.*

c) Since it both translates and rotates, you will need both $\vec{F} = m\vec{a}$ to relate the forces on it to its translational acceleration and $\tau = I\alpha$ to relate the torques on it to its linear acceleration. Construct both of these equations, using the forces that appear on your force diagram. *(Hint: The tension in the string both applies a torque to the Yo-Yo and affects its translational acceleration.)*

d) In the above two equations, you will have three unknowns: the tension in the string, the translational acceleration, and the angular acceleration α. However, you can relate two of them to each other. What is that relation? *(Hint: Think carefully about minus signs here!)*

e) Now you should have enough information to solve for a in terms of g, r, and R. Once you have a value for your acceleration, call your GTA or coach over and have them check your work. Discuss with them whether the red or blue Yo-Yo in part (a) would fall faster.

Exercise 2

Consider the demonstration you saw in class yesterday. A person stands on top of a platform that is free to rotate.

a) Estimate the moment of inertia of the person around their center. You will need to figure out which of our simple shapes best approximates a person, then estimate the person's radius and mass. *(The table of moments of inertia is at the end.)*

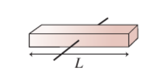
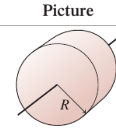
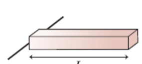
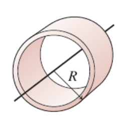
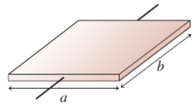
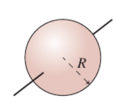
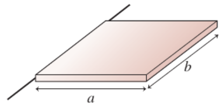
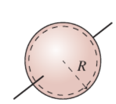
b) Someone else standing on the ground carries a bicycle wheel filled with concrete of mass $m = 5$ kg with radius 30 cm. They make the bicycle wheel spin at an angular velocity $\omega = 20$ radians/sec, turn it so that it is spinning clockwise when viewed from above, then hand it to the person standing on the platform.

When the person standing on the platform grabbed the wheel to stop it from spinning, they began to rotate slowly. Determine which direction and how fast they begin to rotate after they do this.

c) Imagine now that instead of grabbing the wheel, they turned it upside down, so it would be spinning counterclockwise rather than clockwise when seen from above. Without doing any mathematics, predict what should happen. Call your TA or coach over to discuss with your group.

d) Now, calculate which direction they will rotate and how fast when they rotate the wheel upside down. Does the result of your calculation agree with your prediction?

TABLE 12.2 Moments of inertia of objects with uniform density

Object and axis	Picture	I	Object and axis	Picture	I
Thin rod, about center		$\frac{1}{12}ML^2$	Cylinder or disk, about center		$\frac{1}{2}MR^2$
Thin rod, about end		$\frac{1}{3}ML^2$	Cylindrical hoop, about center		MR^2
Plane or slab, about center		$\frac{1}{12}Ma^2$	Solid sphere, about diameter		$\frac{2}{5}MR^2$
Plane or slab, about edge		$\frac{1}{3}Ma^2$	Spherical shell, about diameter		$\frac{2}{3}MR^2$

Recitation Exercises

29 April

In this last exercise of the semester, you will study *transmissions* – the assemblages of gears that transmit torque from a motor to the machine that it turns. For instance, the motor might be the engine of a car or the legs of a cyclist; the "machine" is the drive wheel applying traction to the ground. This works the same for all sorts of machines that use rotary motion to transmit power.

In all of these cases, the motor applies a torque to the driveshaft, which is connected to a machine that applies an equal and opposite torque. Thus, the motor delivers power to the machine. For this problem, the motor will always be spinning at a constant angular velocity, so $\sum \tau = 0$.

Motors have two limitations:

- They are limited in the torque that they can apply. For instance, for a human riding a bicycle, there is only so much force they can apply to the pedals.
- They are limited in their angular velocity. For instance, a person riding a bicycle can only spin their legs so fast.

We will see how we can partially overcome these limitations using gears. Let's think about this in an idealized case of an electric motor spinning a machine, and then apply it to a person riding a bike. Suppose that the motor can apply a maximum torque $\tau_{\text{max}} = 100\ \text{N} \cdot \text{m}$ to the driveshaft, but it has a maximum angular velocity $\omega_{\text{max}} = 50\ \text{rad/sec}$.

1. What is the maximum power that the motor can deliver to the machine? *(Hint: For translational motion, $P = \vec{F} \cdot \vec{v}$. What is the analogous formula for rotation?)*

2. However, in general, machines need to run at different speeds; for instance, cars can drive at many different speeds. Suppose that the operator of the machine wants to run it at low speed – say, at $25\ \text{s}^{-1}$. Can the motor still deliver its full power in this case? If not, how much power can it deliver?

The motor is simply *unable* to rotate any faster than 50 rad/sec. Since the machine operator may want to run the machine at any speed, and will likely want the full power from the motor at *any* speed, they construct a transmission out of gears.

In this figure, the motor is connected to the red gear with $r_{in} = 10$ cm; the machine is connected to the blue gear. We will first think about how this transmission works using a single blue gear with a radius of $R_{out} = 20$ cm in order to understand the principles at work here; then, we will think about the advantages of *shifting* gears, as in a bicycle or car transmission.

`Gears.png`

Here everything is rotating at a constant angular velocity. This means that the motor applies a clockwise torque to the red gear; the blue gear applies an equal and opposite counterclockwise torque to it.

3. Newton's third law applies to the forces that the two gears exert on each other: the two gears push on each other with equal and opposite forces. Given this, determine the relationship between τ_{motor} (the torque the motor applies to the red gear) and τ_{machine} (the torque the blue gear applies to the machine) in terms of r_{in} and R_{out}. Record this formula on the back page.

4. The velocities of the gears' teeth must also be the same as they turn. Given this, determine the relationship between ω_{motor} (the speed at which the motor and red gear turn) and ω_{machine} (the speed at which the blue gear and the machine turn). Again, you should have a result in terms of r_{in} and R_{out}. Record this formula on the back page; call your coach or TA over to check your result so far.

5. Suppose that the motor is running at its maximum angular velocity and torque, and the output (blue) gear has $R = 20$ cm. Calculate the angular velocity, torque supplied to the machine, and power supplied to the machine.

   - Angular velocity of machine (blue gear) = ____________________ rad/s
   - Torque applied to machine = ____________________ $\mathrm{N \cdot m}$
   - Power applied to machine = ____________________ W

6. How does this power compare to the maximum power that the motor could deliver to the machine at this angular velocity without the transmission?

7. Suppose that the engineer designing the machine wants to be able to run the machine at an angular velocity of $\omega = 200$ rad/s. This would be simply impossible without a transmission, since the motor can only turn at 50 rad/s. What radius should the output gear have so the machine can spin at $\omega = 200$ rad/s?

8. A bicycle uses a chain to connect the input and output gears, but the principle is the same: the rider's legs are limited in both their torque and their angular velocity. They want to be able to apply maximum power to the output gear (connected to the rear wheel) for a range of angular velocities – whether this is to climb a steep hill at low speed, or to go as fast as possible on flat ground.

   On a bicycle, the rider can change the radius of both input and output gears. Discuss how this allows the rider to deliver maximum power to the bicycle's wheel at any speed. What combination of gears does a rider want when they are climbing a steep hill at low speed? What combination do they want when they are going very fast on flat ground?

9. As you've seen, with appropriate choice of gear sizes, a transmission lets a motor (or a human) produce either extremely large torque or extremely high angular velocity. Is a transmission able to increase the amount of *power* that a motor (or cyclist) can produce? Why or why not?