

## WORK AND ENERGY

In this exercise, you will discover for yourself one of the most useful laws of mechanics – the *work-energy theorem*. I could derive it for you, but that would be boring; this is something you can do on your own!

Don't be intimidated by the fact that this is new material: everything that you will do follows logically from things that you have already learned.

In this class, usually you spend most of your practice time applying the laws of physics that you have learned already. This time, you will see how the work-energy theorem is just a *mathematical consequence* of what you know already.

In the process, you will learn some new mathematics, too. We will explore the meaning of those new mathematics together after you're done.

## Section 1: Work and Energy in One Dimension

Consider a constant force  $F$  acting on an object of mass  $m$  moving only in the  $x$ -direction as it moves; its change in position during some interval (from “before” to “after”) is  $\Delta x$ . (We call  $\Delta x$  – the change in the  $x$ -position during some motion – the  $x$ -displacement.)

Our starting point is the “third kinematics relation”

$$v_f^2 - v_i^2 = 2a\Delta x$$

which you worked out in recitation a while back.

1. Our goal here is to rewrite the “third kinematics equation” above so that the right-hand side references only the *forces* that act on the object, rather than its acceleration. You know that  $F = ma$ . Multiply both sides of that equation by some factor so that the right-hand side becomes only the force multiplied by the  $x$ -displacement – that is,  $F\Delta x$ .

Two hints:

- (a) Since  $F = ma$ , you can replace  $ma$  with  $F$  wherever it appears. How can you get  $ma$  to appear together on the right-hand side?
- (b) If you have any numerical factors that you want to get rid of, you can multiply both sides by their inverse – so, for instance, if the right-hand side is  $3F\Delta x$ , since you want just  $F\Delta x$ , you can multiply by  $1/3$ .

What do you get?

Multiply by  $\frac{1}{2}m$  to get

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = ma\Delta x$$

and substitute  $F = ma$  to get

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = F\Delta x.$$

2. This equation will become a good friend in the next few weeks, so we give the terms that appear in it special meanings. There are two terms on the left, representing the final and initial *kinetic energy* during the motion. (Kinetic energy is a type of energy associated with motion.)

What is the formula for the kinetic energy of something? (*Give only the general form – one term, either initial or final.*)

$$\frac{1}{2}mv^2.$$

3. The right-hand side of this equation – force times displacement – is also new thing we have not met before, called *work*.

In one dimension, we have two choices for our coordinate system: we can choose either right or left to be positive. Does the value of *work* depend on which coordinate system you choose? (*If you want,*

come up with a hypothetical situation: choose a direction for the force and the displacement, and then think about the result in each coordinate system.)

No it doesn't: if the force and displacement are in the same direction, then multiplying either two positives or two negatives gives you a positive.

4. Under what conditions is work positive? Under what conditions is it negative?

Positive if the force and displacement are in the same direction; negative if they are not.

5. Does the formula for the *kinetic energy* that you found above depend on your choice of coordinate system? Why or why not?

No it doesn't, because changing the coordinate system in one dimension just makes  $v$  either positive or negative, and it is being squared:  $(-v)^2 = v^2$ .

6. The result you got above is called the *work-energy theorem*. It says in words:

(change in kinetic energy due to a force) = (work done by that force).

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = F\Delta x.$$

Write the mathematical expressions corresponding to these ideas below the words.

The terms are: final kinetic energy, initial kinetic energy, work done.

## Section 2: Work and Energy in Two Dimensions

Like the rest of kinematics, the “third kinematics equation” applies separately for  $x$  and  $y$  directions. Since the work-energy theorem is based on the third kinematics equation, let’s see how we can modify the above result for motion in more than one dimension.

1. Begin by writing down the “third kinematics equation” in both  $x$  and  $y$ . (*You will need double subscripts, such as  $v_{f_x}$ ; that is okay. This will give you two equations.*)

$$v_{f_x}^2 - v_{i_x}^2 = 2a_x\Delta x$$

$$v_{f_y}^2 - v_{i_y}^2 = 2a_y\Delta y$$

2. You know that Newton’s second law applies separately in  $x$  and  $y$ ; that is  $\vec{F} = m\vec{a}$  implies both  $F_x = ma_x$  and  $F_y = ma_y$ . As you did before, multiply both sides of the equations by  $\frac{1}{2}m$ , then substitute the force in for the right-hand side.

$$\frac{1}{2}mv_{f_x}^2 - \frac{1}{2}mv_{i_x}^2 = F_x\Delta x$$

$$\frac{1}{2}mv_{f_y}^2 - \frac{1}{2}mv_{i_y}^2 = F_y\Delta y$$

3. In the past, we emphasized the fact that  $x$ – and  $y$ –directions were independent, and we could not combine them without great care. However, we’ll do that now. *Add the two equations* above together. (Yes, it is gross, but it won’t be gross for long. It will simplify greatly in two steps!)

*They will write the terms in some order or other, but typically:*

$$\frac{1}{2}mv_{f_x}^2 - \frac{1}{2}mv_{i_x}^2 + \frac{1}{2}mv_{f_y}^2 - \frac{1}{2}mv_{i_y}^2 = F_x\Delta x + F_y\Delta y$$

4. Group the terms on the left-hand side so that all the “initial” terms are together, and all the “final” terms are together. (In doing this, we are trying to figure out what the initial and final kinetic energy in two dimensions looks like.)

All they need to do here is flip the order of terms. Encourage them to put parentheses around them:

$$\left(\frac{1}{2}mv_{fx}^2 + \frac{1}{2}mv_{fy}^2\right) - \left(\frac{1}{2}mv_{ix}^2 + \frac{1}{2}mv_{iy}^2\right) = F_x\Delta x + F_y\Delta y$$

5. You may see right away how to simplify the left-hand side. If you don't, think about the Pythagorean theorem; if you are still stuck, ask Walter or the coach for help. (*You may use the simple symbol  $v$ , without an arrow, for “the magnitude of the velocity.”*) Write your simplified form of the work-energy theorem in two dimensions below.

This is the hardest step here probably. They need to see that

$$v_{fx}^2 + v_{fy}^2 = v_f^2.$$

Then the whole thing simplifies to

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = F_x\Delta x + F_y\Delta y$$

6. In two dimensions, what is an expression for the kinetic energy? Does it depend on which coordinate system you choose, or which direction the velocity vector is pointing?

It is just

$$(\text{kinetic energy}) = \frac{1}{2}mv^2$$

with no vector components; it doesn't depend on direction or coordinate system, only the magnitude of  $v$ .

7. Now, let's think about the right-hand side – the “work side”. You have found that (work in two dimensions) =  $F_x\Delta x + F_y\Delta y$ .

A moment ago, you found that the kinetic energy doesn't have a direction or depend on the coordinate system. If that is true for the left-hand side of an equation, it must also be true for the right-hand side! Here we will see why.

Imagine that you have a force  $\vec{F}$  acting 30 degrees south of east on an object that moves a distance  $d$  due east. Draw arrows representing the force vector and the displacement vector below.

8. Using the definition of work  $W = F_x\Delta x + F_y\Delta y$ , calculate the work that the force does on the object during the motion using two different coordinate systems:
- A coordinate system where the positive  $x$ –direction is pointing in the direction of the displacement
  - A coordinate system where the positive  $x$ –direction is pointing in the direction of the force

In both cases, you should get  $W = Fd\cos\theta$ , where  $\theta$  is the angle between the force and the displacement.

In the first case for the work, things simplify because  $\Delta y$  is zero:

$$\begin{aligned}W &= F_x\Delta x + F_y\Delta y \\W &= (F\cos\theta)(d) + (F\sin\theta)(0)\end{aligned}$$

and only the first term survives.

In the second case, things simplify because  $F_y$  is zero:

$$\begin{aligned}W &= F_x\Delta x + F_y\Delta y \\W &= (F)(d\cos\theta) + (0)(d\sin\theta)\end{aligned}$$

and only the first term survives, giving the same result.

9. Note that  $\Delta x$  and  $\Delta y$  are components of the displacement vector  $\Delta \vec{s} = (\Delta x, \Delta y)$ . Our form for the work  $W = F_x \Delta x + F_y \Delta y$  has the form  $A_x B_x + A_y B_y$ . This combination occurs quite a bit in physics, so we give it a special name: the **dot product**. Specifically,

$$A_x B_x + A_y B_y = \vec{A} \cdot \vec{B}.$$

Rewrite your version of the work-energy theorem so that the work is written in terms of a dot product between the force vector  $\vec{F}$  and the displacement vector  $\Delta \vec{s}$  (whose components are  $\Delta x$  and  $\Delta y$ ). Then label the three terms:

- work
- initial kinetic energy
- final kinetic energy

Note that *none* of them depend on coordinate system!

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = \vec{F} \cdot \Delta \vec{s}$$

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## Summary

In the previous sections, you found that

(change in kinetic energy during a motion) = (work done by all forces during that motion)

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \sum \vec{F} \cdot \Delta \vec{s}$$

Here  $\vec{F} \cdot \Delta \vec{s}$  means whichever of the following is easiest to figure out:

- The size of the force, multiplied by the distance moved in the direction of that force
- The distance moved, multiplied by the size of the force in the direction of motion

- The distance moved, multiplied by the size of the force, multiplied by the cosine of the angle between them

You may calculate the work done by different forces using different methods if you choose.