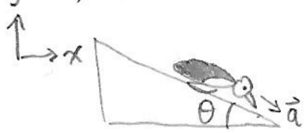
Recitation Questions

13 February

A penguin slides down a frictionless icy hill; the hill is inclined at an angle θ. In this problem, you will calculate the penguin's acceleration. However, I want you to do it two different ways, using two different coordinate systems.

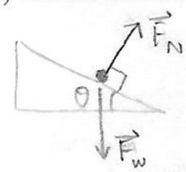
First, solve the problem using the conventional coordinate system, where x is horizontal and y is vertical. As usual, take the following steps:

a) Draw a cartoon of the problem, and label your coordinate system.



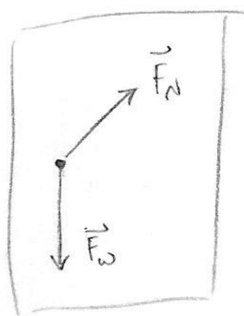
What are the forces on the penguin?

b) Draw a force diagram for the penguin.



← the ground holds the penguin up with a normal force. The normal force is always perpendicular to the surface

← the force of weight always points straight down



Since there is no friction and nothing else is touching the penguin, we have identified all the forces.

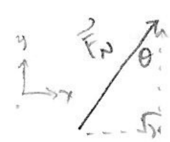
c) Write down Newton's second law in both directions – that is, $\sum F_x = ma_x$ and $\sum F_y = ma_y$. If you have any forces that don't lie along the x or y directions, use trigonometry to break them into components.

From $\Sigma \vec{F} = m\vec{a}$, we can break this vector equation into components

$\Sigma F_x = ma_x$ $\qquad$ $\Sigma F_y = ma_y$

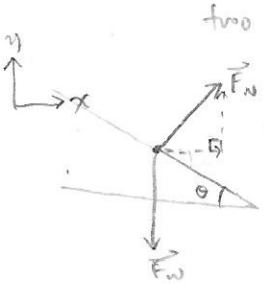
From our force diagram, we include the two forces

$F_{Nx} + F_{wx} = ma_x$ $\qquad$ $F_{Ny} + F_{wy} = ma_y$



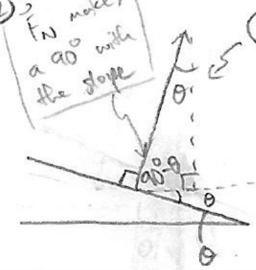
Now we know $F_{Nx} = F_N \sin\theta$, $F_{Ny} = F_N \cos\theta$.

$\vec{F}_w$ is straight down so $F_{wx} = 0$, $F_{wy} = -mg$



we want the components of our forces along the axes

Start identifying unknown angles one-by-one from the known angle θ



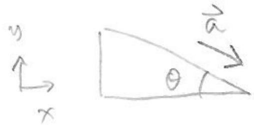
② F_N makes a 90° with the slope

③ Now we know the top angle is θ, since the angles in a triangle add up to 180°

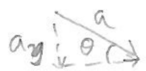
these lines ① are parallel, so both angles are θ

$$\boxed{\begin{aligned} F_N \sin\theta &= ma_x \\ F_N \cos\theta - mg &= ma_y \end{aligned}}$$

d) This will result in two equations with three unknowns: a_x, a_y, and F_N. However, in this problem, a_x and a_y are related. What is their relation? This should reduce you to two equations and two unknowns; write them below.



a_x and a_y are related because they are components of $\vec{a}$, so $a_x = a\cos\theta$, $a_y = -a\sin\theta$. a_y points in the negative y-direction



From the $\Sigma\vec{F} = m\vec{a}$ equations in part c), we now have

$$\boxed{\begin{aligned} F_N \sin\theta &= ma\cos\theta \\ F_N\cos\theta - mg &= -ma\sin\theta \end{aligned}}$$

Now F_N and a are the only unknowns, since θ is given.

e) Solve those equations to find the acceleration of the penguin. Use trigonometry to find the magnitude of $\vec{a}$.

the magnitude of $\vec{a}$

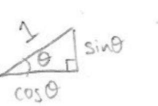
We can solve for a by eliminating F_N.

Let's solve the first equation for F_N

$F_N = ma\dfrac{\cos\theta}{\sin\theta}$, and plug it into the second equation:

$\left(\cancel{m}a\dfrac{\cos\theta}{\sin\theta}\right)\cos\theta - \cancel{m}g = -\cancel{m}a\sin\theta$. Now we can collect on a:

$a\left(\dfrac{\cos^2\theta}{\sin\theta} + \sin\theta\right) = g$. Multiplying both sides by $\sin\theta$ gets rid of the denominator:



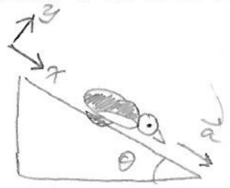
$a(\cos^2\theta + \sin^2\theta) = g\sin\theta$

Since $\cos\theta$, $\sin\theta$, and 1 form a right triangle, $\cos^2\theta + \sin^2\theta = 1$ for any θ.

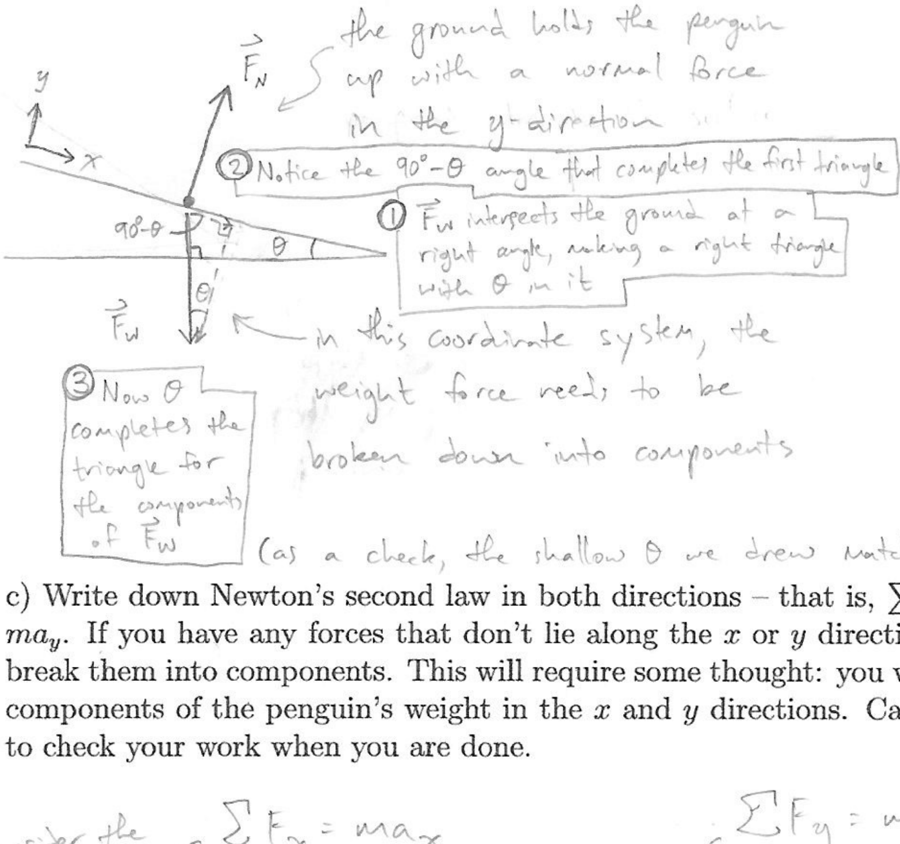
$$\boxed{a = g\sin\theta}$$

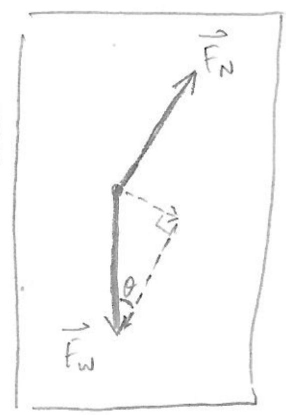
Now you will solve the problem again using a rotated coordinate system, where x is the direction parallel to the hill and y is the direction perpendicular to it. Again:

a) Draw a cartoon of the problem, and label your coordinate system.



b) Draw a force diagram for the penguin. (Draw this one large, since you will need to construct a right triangle with one of the forces as its hypotenuse to break it into components.)





(as a check, the shallow θ we drew matches the shallow angle in the $\vec{F}_W$ triangle).

c) Write down Newton's second law in both directions – that is, $\sum F_x = ma_x$ and $\sum F_y = ma_y$. If you have any forces that don't lie along the x or y directions, use trigonometry to break them into components. This will require some thought: you will need to figure out the components of the penguin's weight in the x and y directions. Call over your TA or coach to check your work when you are done.

Consider the two forces from your force diagram

$$\sum F_x = ma_x$$

$$F_{Nx} + F_{Wx} = ma_x$$

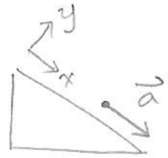
$$\boxed{0 + mg\sin\theta = ma_x}$$

Write the components in terms of the forces' magnitudes and the known angle θ

$$\sum F_y = ma_y$$

$$F_{Ny} + F_{Wy} = ma_y$$

$$\boxed{F_N - mg\cos\theta = ma_y}$$



d) This will result in two equations with three unknowns: a_x, a_y, and F_N. However, a little thought will tell you what one of these is. What is it? This should reduce you to two equations and two unknowns; write them below.

Since the penguin accelerates down the slope (only in the x-direction), $a_x = a$, and $a_y = 0$.

the magnitude of the acceleration

Therefore, $\boxed{\begin{array}{l} mg\sin\theta = ma, \\ F_N - mg\cos\theta = 0 \end{array}}$

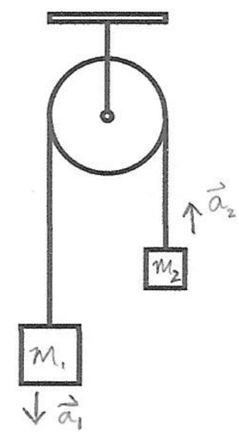
e) Solve those equations to find the acceleration of the penguin.

Notice that the acceleration comes just from $\Sigma F_x = ma_x$, and $\Sigma F_y = ma_y$ gives us the normal force with this coordinate system.

$\boxed{a = g\sin\theta}$, just like we found with the other coordinate system

f) Discuss the difference in the two approaches. In one, you aligned your coordinate system with gravity, and in the other, you aligned your coordinate system with the direction that you knew the penguin would accelerate in. Which was easier? Which should you adopt for future problems? Invite your TA or coach over to join your conversation.

Two weights of mass m_1 and m_2 are attached to either end of a string. This string is passed over a light frictionless pulley, as shown in the image. Clearly the heavier mass will go down and the lighter one will go up, but at what rate? In this problem, you will calculate their acceleration.

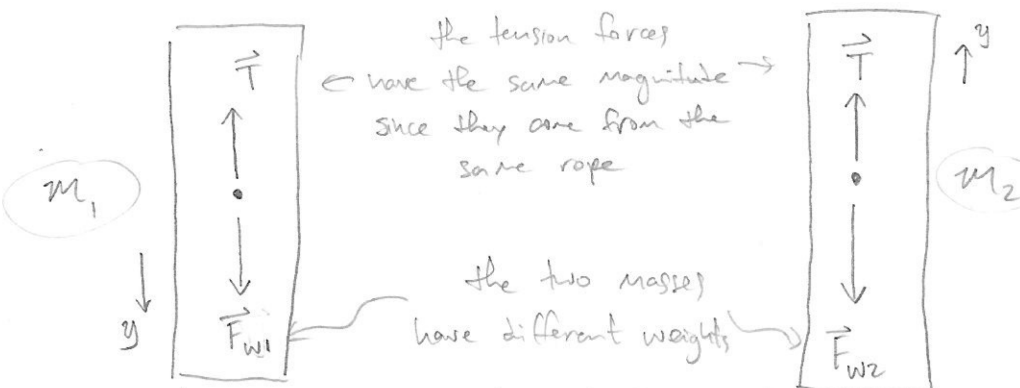


a) What do you expect the system to do if one of the masses is much heavier than the other? What do you expect if the two masses are equal?

If one mass is very heavy, it will outweigh the other mass and fall almost with acceleration g.

If the two masses are equal, the masses should balance motionless.

b) Draw force diagrams for both objects. Label your choice of coordinate system separately for each object – you don't have to choose the same coordinate system for each!



We can choose the y-direction to point in the direction of each mass' acceleration, so a_1 and a_2 are both positive.

c) State Newton's law for both objects. Note that their accelerations aren't necessarily the same, depending on your choice of coordinate system, so you should introduce separate variables a_1 and a_2 for both. The tension forces *are* the same.

The forces and accelerations are along one direction, so we only need one copy of the 2^{nd} law for each mass:

m_1 $\quad \sum F_y = ma_y$

We can read these components off our force diagram, checking which y-direction we chose

$$\boxed{-T + m_1 g = m_1 a_1}$$

m_2 $\quad \sum F_y = ma_y$

$$\boxed{T - m_2 g = m_2 a_2}$$

d) Since you have two objects, you have two copies of Newton's law. However, you have three unknowns: T, a_1, and a_2. What other statement can you make about the accelerations that lets you solve the system?

Since the masses are connected by a rope, the magnitudes of their accelerations must be the same: $\boxed{a_1 = a_2}$

We can therefore just use the symbol "a" for both accelerations

e) Actually solve the system, giving values of a_1 and a_2 in terms of m_1, m_2, and g. Then, translate your expressions for a_1 and a_2 into words. (Your TA and coaches can help with this.) Does your result make sense? Does it agree with your predictions in part (a)?

$-T + m_1 g = m_1 a$ $\quad\quad$ $T - m_2 g = m_2 a$

Since we don't need T, we can eliminate it by adding the equations together (for example).

$$-T + m_1 g = m_1 a$$
$$T - m_2 g = m_2 a$$
$$\downarrow$$

$m_1 g - m_2 g = m_1 a + m_2 a$. Collecting on a and g,

$(m_1 - m_2) g = (m_1 + m_2) a$, so

$$\boxed{a = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) g}$$

Acceleration is proportional to the differences in the masses.

Notice: if m_1 is much greater than m_2, $m_1 - m_2 \approx m_1$ and $m_1 + m_2 \approx m_1$. In that case, $a \approx \frac{m_1}{m_1} g = g$, and acceleration is near freefall, like predicted.

Also, if $m_1 = m_2$, $m_1 - m_2 = 0$, and $a = 0$, so the masses are motionless, like predicted.