Work and potential energy

Physics 211
Syracuse University, Physics 211 Spring 2023
Walter Freeman

March 28, 2023

Announcements

Homework 7 is due Friday and is quite long; you'll have all the material you need after today's class.

My help hours this week in the Physics Clinic:

- Tuesday (today), 2-4 PM
- Wednesday, 3-6 PM

Group Exam 3 is this Thursday/Friday in recitation.

Exam 3 is next Tuesday, covering energy and momentum. (I'll give full details Thursday.)

Weekend exam review: **Sunday, 1:30-4:30 PM.**

A new force: elasticity

Many things in nature stretch and squish.

- Springs are very stretchy
- Some of them are squishy too
- Everything will stretch a bit if you pull it hard enough
- Guitar strings, rubber bands, steel cables...

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An experiment with a spring

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- **The amount it is being stretched**

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How does this spring's elastic force depend on the amount we stretch it?

A simple model of elasticity

We saw that the elastic force is nearly proportional to the amount of stretch:

- F_e: elastic force (newtons)
- L: stretched length (meters)
- L_0: unstretched length (meters)
- k: proportionality constant (newtons per meter)

A reasonable model:

$$F_\mathrm{elastic} = -k(L - L_0) \equiv -k\Delta x$$

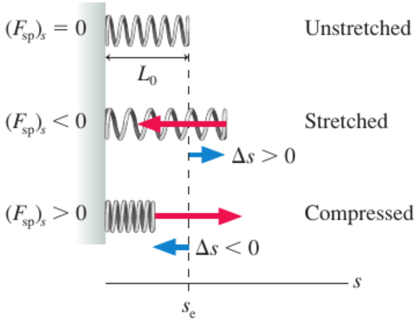
The minus sign means that the force is in the opposite direction as the stretch/squish.

This is called "Hooke's law" after Robert Hooke, an English physicist and contemporary of Newton.

A new force: elasticity and Hooke's law

Our model of elasticity:

- Springs have a particular length that they like to be: "equilibrium length" L_0
- A spring stretched to be longer than this pulls inward to shorten itself
- A spring compressed to be shorter than this pushes outward to lengthen itself
- Flexible things like strings and ropes only pull; they kink instead of compressing
- The force is proportional to the deviation from the optimum length



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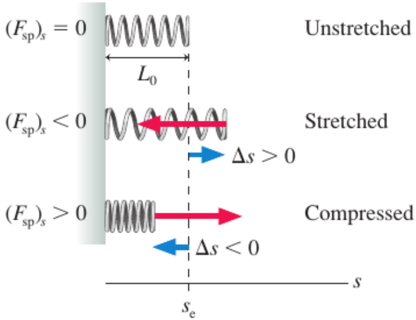

$$F_{\text{elastic}} = -k(L - L_0) \equiv -k\Delta x$$

k is called the "spring constant":

- Measures the stiffness of the spring/rope
- Units of newtons per meter: "restoring force of k newtons per meter of stretch"

The work done by a spring

An object of mass m is placed against a spring of spring constant k compressed by an amount Δx_1.

When the spring is released, it will propel the object forward.

When the spring reaches a compression amount Δx_2, how fast will the object be moving?

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How much work will the spring do on the object as it moves from position 1 to position 2?

- A: $W_{\text{elastic}} = k\Delta x_1$
- B: $W_{\text{elastic}} = k\Delta x_2$
- C: $W_{\text{elastic}} = k\Delta(x_1 - x_2)$
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- E: Something else

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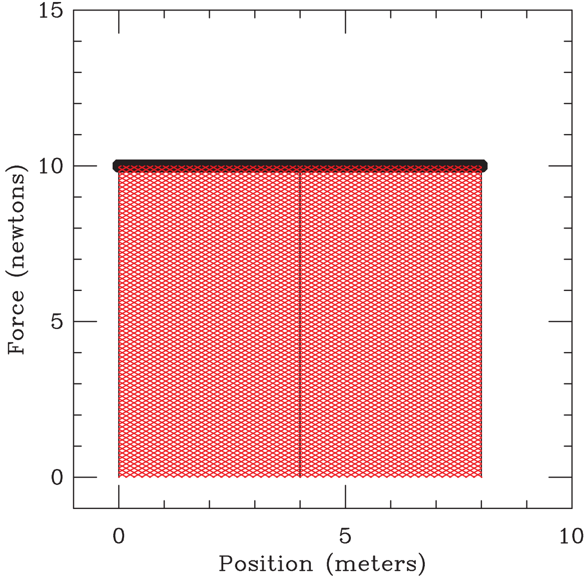
E: The force from the spring is changing as the object moves!
This means we can't do "work = force times distance moved"

How much work is being done here?



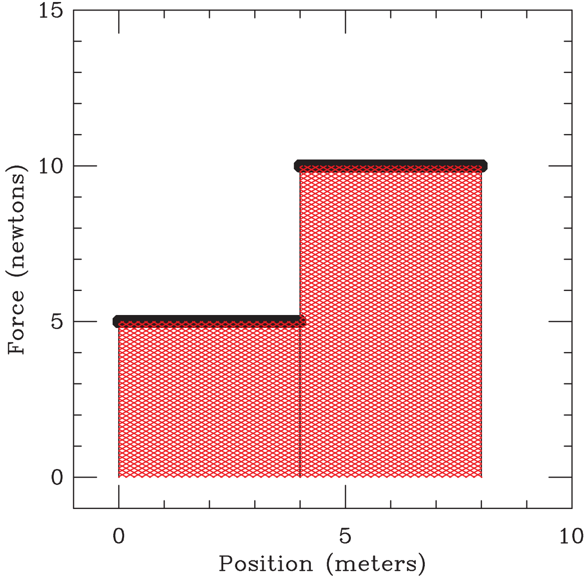
A: 8 joules

B: 18 joules

C: 10 joules

D: 80 joules

How much work is being done here?



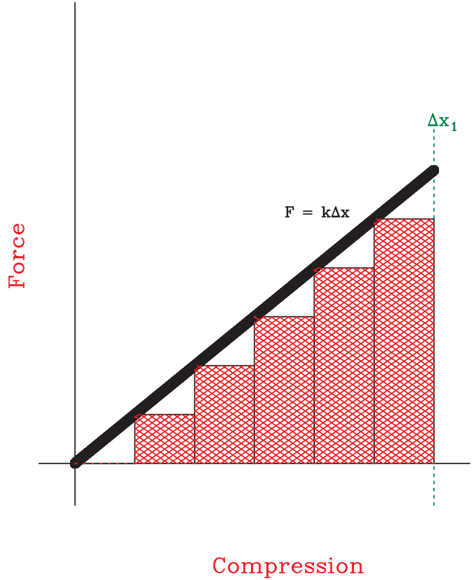
A: 15 joules

B: 60 joules

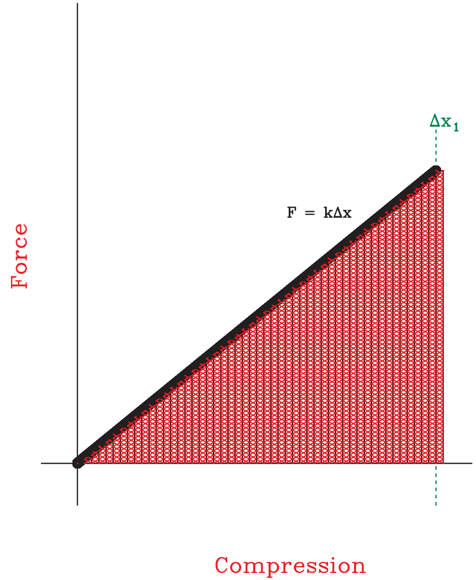
C: 40 joules

D: 140 joules

How do we compute the work done by the spring here?



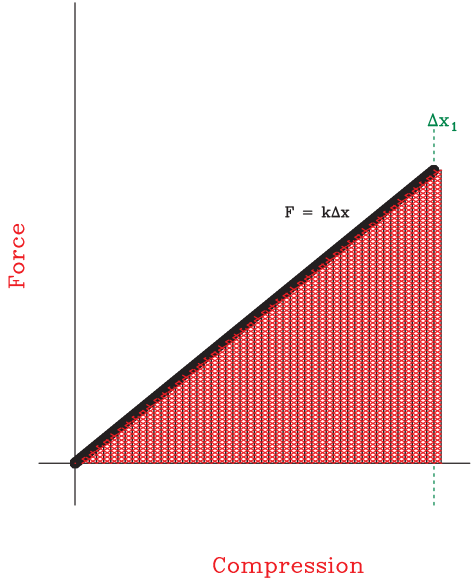
The work done is the area under the force vs. distance curve...



... just like the change in position is the area under the velocity vs. time curve:

$$W = \int F dx$$

If the spring relaxes all the way, how much work does it do?



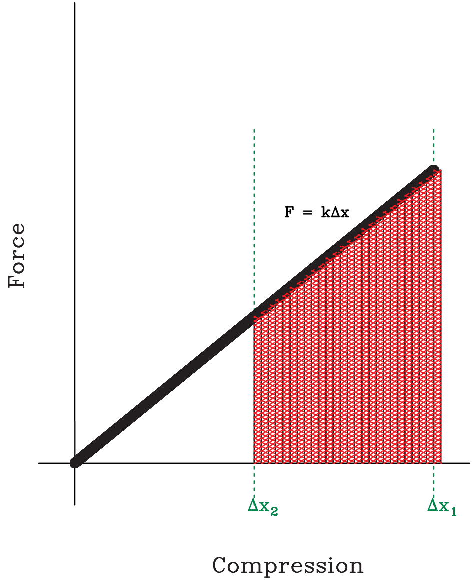
A: $W = k\Delta x_1$

B: $W = \frac{1}{2}k\Delta x_1^2$

C: $W = k\Delta x_1^2$

D: $W = \frac{1}{2}k\Delta x_1$

What if it doesn't relax all the way, and instead goes from Δx_1 to Δx_2?



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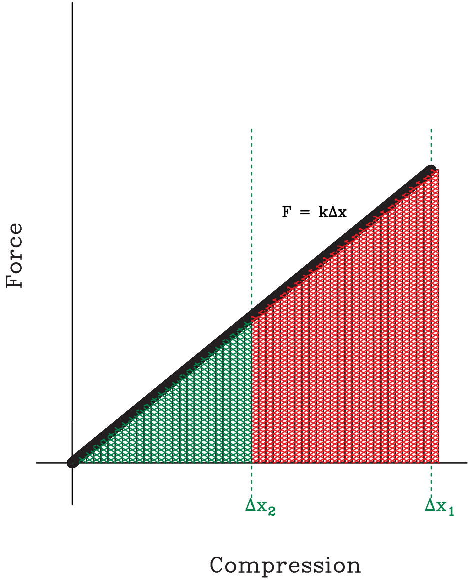

The red area is the area of the whole triangle, minus the area of the small green triangle:

$$W_e = \frac{1}{2}k\Delta x_1^2 - \frac{1}{2}k\Delta x_2^2$$

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We can collect all the "before" terms on the left and the "after" terms on the right, and identify $\frac{1}{2}k\Delta x^2$ as the *elastic potential energy* stored in a spring:

$$\begin{aligned} \frac{1}{2}k\Delta x_1^2 - \frac{1}{2}k\Delta x_2^2 &= \frac{1}{2}mv_2^2 \\ \frac{1}{2}k\Delta x_1^2 &= \frac{1}{2}mv_2^2 + \frac{1}{2}k\Delta x_2^2 \end{aligned}$$

The potential energy of a spring

Recall that the gravitational potential energy associated with an object at height y is

$$\text{gravitational potential energy} = mgy$$

Now we see that the elastic potential energy associated with an object of spring constant k stretched or compressed a distance Δx is

$$\text{elastic potential energy} = \frac{1}{2}k(\Delta x)^2.$$

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We can just remember this and never need to do this integral again (whew!)

A simple spring problem: done with potential energy

A person of mass $m = 100$ kg falls from a height of $h = 3$m onto a trampoline. If the person makes an impression $d = 40$ cm deep on the trampoline when he lands, what is the spring constant?

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- We have no "other forces": we're accounting for gravity and elasticity using potential energy
- The person begins and ends at rest, so we know the initial and final kinetic energy is zero
- Put $y = 0$ at the surface of the trampoline

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- $U_{\text{grav},0} = mgh$
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- $U_{\text{grav,f}} = -mgd$ (the person falls below $y = 0$; PE can be negative!)
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- $k = \frac{mg(h+d)}{2d^2}$

That spring problem: a recap

We don't care about time $\rightarrow$ energy methods

Work-energy theorem

- Initial KE + all work done = final KE
- Need to compute work done by gravity: easy
- Need to compute work done by spring: harder
  (need to integrate Hooke's law)

Potential energy treatment

- Initial KE + initial PE + other work = final KE + final PE
- No "other work" in this problem; all forces have a PE associated
- Need to know the expressions for PE:
  - $U_{\mathrm{grav}} = mgy$
  - $U_{\mathrm{elas}} = \frac{1}{2}kx^2$ (x is the distance from the equilibrium point)
- No integrals required (they're baked into the above)

Potential energy with other forces

What about associating a potential energy with other forces?

- Friction is a no-go: the work done by friction depends on the path, not just where you start and stop
- "Ephemeral" forces like tension and normal force are easiest to deal with by computing work directly

Problem-solving guide for problems involving energy

- Identify the various parts of the motion and what you need to know about them
  - Motion where you care only about begin/end states and not time: use work/energy methods
  - "Where does it land" projectile motion problems: can not use energy methods
- Draw a series of snapshots, showing what your "before" and "after" pictures look like (you may have more than two in some problems)
- Use force diagrams to calculate any forces you need to know
- Application of the work-energy theorem:

$$KE_{\text{initial}} + W_{\text{all}} = KE_{\text{final}}$$

OR

$$KE_{\text{initial}} + PE_{\text{initial}} + W_{\text{other}} = KE_{\text{final}} + PE_{\text{final}}$$

Sample problems

A mass m is hung from a spring of spring constant k and released. Which equation would let me find the distance d that it falls before it comes back up?

- A: $mgd - \frac{1}{2}kd^2 = 0$
- B: $\frac{1}{2}kx^2 = mgd + \frac{1}{2}mv^2$
- C: $0 = -mgd + \frac{1}{2}kd^2$
- D: $mgd + \frac{1}{2}kd^2 = 0$

Sample problems

A spring is used to launch a block up a ramp of total length L. The spring has spring constant k, the block has mass m, the ramp is inclined an angle θ, and it has a coefficient of kinetic friction μ_k. I compress the spring a distance x and let it go. How fast will the block be traveling when it reaches the top of the ramp?

Which equation would let me solve for this?

- A: $\frac{1}{2}kx^2 + \mu mgL\cos\theta = \frac{mgL}{\sin\theta} - \frac{1}{2}mv_f^2$
- B: $-\frac{1}{2}kx^2 + \mu mgL\sin\theta = \frac{mgL}{\sin\theta} + \frac{1}{2}mv_f^2$
- C: $\frac{1}{2}kx^2 - \mu mgL\sin\theta = mgL\cos\theta + \frac{1}{2}mv_f^2$
- D: $\frac{1}{2}kx^2 - \mu mgL\cos\theta = mgL\sin\theta + \frac{1}{2}mv_f^2$

Power: rate of doing work

A bit of mathematics that will be useful to you:

"An object moves at a constant speed $\vec{v}$, subject to some force $\vec{F}$; at what rate does that force do work on the object?"

An example: an airplane flies at v=1000 m/s, and its engines exert F=300 kN of thrust. What is the rate at which the engines do work (power)?

$$\text{Work} = \text{force} \cdot \text{distance}$$

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$$\text{Power} = \text{force} \cdot \text{velocity}$$

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\text{Power} = \text{force} \cdot (\text{distance} \ / \ \text{time}) \\
\text{Power} = \text{force} \cdot \text{velocity} \\
P = \vec{F} \cdot \vec{v} = 300 \text{ MW}
\end{gathered}$$

- The engines output 300 MW of power: this is around 10 liters per second of fuel even at 100% efficiency!
- Some of that 300 MW of energy dissipated by drag heats up the airplane... (real numbers for a SR-71 Blackbird, the fastest airplane in history)

Sample problems

A truck pulling a heavy load with mass $m = 4000$ kg wants to drive up a hill at a 30° grade.

If the truck's engine can produce 100 kW of power (134 hp), how fast can the truck go? (Neglect drag.)

57. ‖ The spring shown in **FIGURE P11.57** is compressed 50 cm and used to launch a 100 kg physics student. The track is frictionless until it starts up the incline. The student's coefficient of kinetic friction on the 30° incline is 0.15.

a. What is the student's speed just after losing contact with the spring?

b. How far up the incline does the student go?

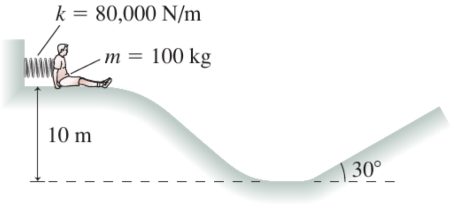

FIGURE P11.57

Sample problems

49. || Truck brakes can fail if they get too hot. In some mountainous areas, ramps of loose gravel are constructed to stop runaway trucks that have lost their brakes. The combination of a slight upward slope and a large coefficient of rolling resistance as the truck tires sink into the gravel brings the truck safely to a halt. Suppose a gravel ramp slopes upward at 6.0° and the coefficient of rolling friction is 0.40. Use work and energy to find the length of a ramp that will stop a 15,000 kg truck that enters the ramp at 35 m/s (≈75 mph).

Summary

- Potential energy is two things:
  - An accounting device that makes it easier to keep track of work done
  - Part of *conservation of total energy*, a powerful statement about nature
- Gravitational potential energy (on Earth): $U_g = mgy$

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- Potential energy is two things:
  - An accounting device that makes it easier to keep track of work done
  - Part of *conservation of total energy*, a powerful statement about nature
- Gravitational potential energy (on Earth): $U_g = mgy$
- We learned about a new force: **elasticity**
  - Restoring force in a stretched or compressed spring, or a stretched string:
    $F = -k(x - x_0)$ (x_0 is the equilibrium length)
  - k is the spring constant, measured in force per distance, that gauges stiffness
  - Elastic potential energy: $U_{\text{elas}} = \frac{1}{2}k(x - x_0)^2$

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  - An accounting device that makes it easier to keep track of work done
  - Part of *conservation of total energy*, a powerful statement about nature
- Gravitational potential energy (on Earth): $U_g = mgy$
- We learned about a new force: **elasticity**
  - Restoring force in a stretched or compressed spring, or a stretched string:
    $$F = -k(x - x_0) \text{ } (x_0 \text{ is the equilibrium length})$$
  - k is the spring constant, measured in force per distance, that gauges stiffness
  - Elastic potential energy: $U_{\text{elas}} = \frac{1}{2}k(x - x_0)^2$