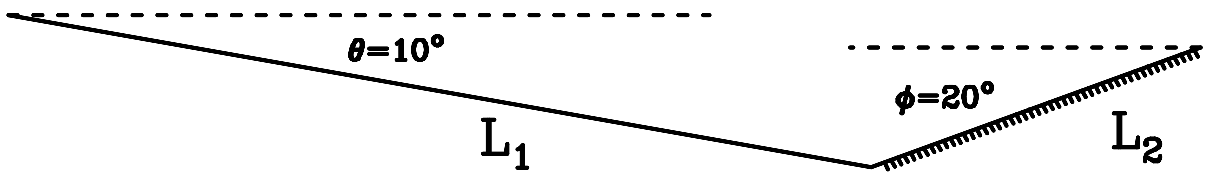
Physics 211 Exam 3, Form A
Answer Key

Question 1

Heavy trucks driving down steep mountains must continually apply their brakes to maintain a safe speed. If their brakes fail, these roads are equipped with "runaway truck ramps", which are short uphill pathways (made of sand or gravel) with a large coefficient of rolling friction. A truck whose brakes fail can steer into the ramp and come safely to a stop. Suppose that a truck of mass m is driving down the hill at a speed v_0 when its brakes fail. It is a distance L_1 away from the ramp, traveling at a speed v_0. When it reaches the ramp, it exits the highway and heads up the ramp, traveling a distance L_2 before coming to rest. In this problem, you will calculate the distance L_2 in terms of μ_r, g, m, L_1, v_0, θ, and ϕ.



a) Write an expression for the total work done by gravity during the entire motion in terms of g, m, L_1, L_2, θ, and ϕ. (5 points)

The work done by gravity is most easily found as

$$\vec{F}_g \cdot \Delta s = mg(y_i - y_f) = mg(L_1 \sin\theta - L_2 \sin\phi)$$

.

b) Write an expression for the total work done by friction during the entire motion in terms of μ_r, g, m, L_2, and ϕ. (5 points)

There is only friction on the ascent. The normal force is $F_N = mg\cos\phi$, giving a force of friction of $F_f = \mu mg\cos\phi$.

Since the force of friction is antiparallel to the motion,

$$W_f ric = -L_2 \mu mg \cos\phi$$

.

c) Write a statement of the work-energy theorem/conservation of energy in terms of μ_r, g, m, L_1, L_2, v_0, θ, and ϕ that you could solve for L_2. (You do not need to solve it.) (10 points)

The total work done is the work done by friction plus the work done by gravity. Using the pure work-energy theorem, we have

$$\frac{1}{2}mv_0^2 + mg(L_1 \sin\theta - L_2 \sin\phi) - L_2 \mu mg \cos\phi = 0$$

since the final kinetic energy is zero. Alternatively, we could treat gravity as associated with a potential energy:

$$\frac{1}{2}mv_0^2 + mg(L_1 \sin\theta) - L_2 \mu mg \cos\phi = mgL_2 \sin\phi.$$

d) Now, consider a truck whose brakes are working. It has a mass of $m = 10^4$ kg (10 tons) and is driving down a hill with a grade of $\theta = 10°$. If the driver wants to maintain a speed of $v = 15$ m/s, what is the power that the brakes must dissipate? (5 points)

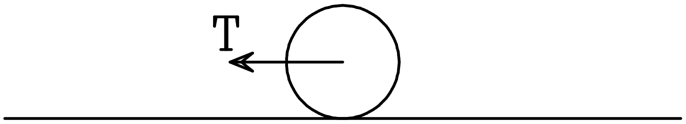
The power applied by the brakes must be equal and opposite to the power applied by gravity, so that the net work done is zero and the truck does not speed up or slow down.

$$P_g = \vec{F}_g \cdot \vec{v} = (mg\sin\theta)(v) = 260\,\mathrm{kW}.$$

The power applied by the brakes is equal and opposite this (the brakes do negative work on the truck).

Question 2

A wheel of mass m and radius r, with moment of inertia $I = \frac{1}{2}mr^2$, rests on the ground with coefficient of static friction μ_s and coefficient of kinetic friction μ_k. There is no rolling friction. Ropes are tied to the axle, and a total force T is applied directly toward the left.



Suppose first that the tension force is very small, so that the wheel rolls without slipping.

a) Is the force of friction on the wheel equal to $\mu_k mg$, $\mu_s mg$, zero, or some other value? Explain briefly. (4 points)

It is some other value – some small value between zero and $\mu_s mg$. If the tension force is small, then the force of static friction has whatever value is necessary to provide an α sufficient to ensure $\alpha = a/r$.

b) Draw a force diagram for the wheel, showing both the forces that act on it, and the positions where they act. Write an expression for the torque applied by each of these forces. (5 points)

- Gravity acts at the middle and points down: $\tau = 0$ since $r = 0$
- The normal force acts at the bottom and points up: $\tau = 0$ since $F_\perp = 0$
- The tension acts at the middle and points to the left: $\tau = 0$ since $r = 0$
- Friction acts at the bottom and points to the right. (To find this, imagine what would happen if there were no friction: the wheel would slip to the left. Friction opposes slipping, so it points to the right.) Its torque is $\tau = F_f r$.

c) Is the magnitude of the acceleration equal to T/m, less than this, or more than this? Explain briefly. (4 points)

It is less than this. If there were only tension Newton's second law gives $a = T/m$, but there is an opposing frictional force, so it is less.

d) Is the magnitude of the angular acceleration equal to a/r, less than this, or more than this? Explain briefly. (4 points) *It is equal. The rolling-without-slipping constraint is $v = \pm\omega r$; differentiating both sides gives $a = \pm\alpha r$.*

Now, suppose that the tension force is very large, so that the wheel slips on the ground.

e) Is the force of friction on the wheel equal to $\mu_k mg$, $\mu_s mg$, zero, or some other value? Explain briefly. (4 points)

Now it slips and feels kinetic friction. Kinetic friction is always given by $F_f = \mu_k F_N = \mu_k mg$.

f) Is the magnitude of the angular acceleration equal to a/r, less than this, or more than this? Explain briefly. (4 points)

It is less. The acceleration a may be very large if T is large, but the angular acceleration is limited by the torque provided by kinetic friction $\tau = \mu_k mgr$.

g) Calculate the maximum value of the tension T that can be applied so that the wheel rolls without slipping. (5 points extra credit)

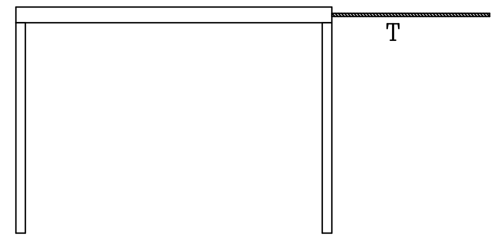
Now $F_f = \mu_s mg$ (maximum force of static friction). Treating left as positive, we have

$$F = ma \to T - \mu_s mg = ma$$
$$\tau = I\alpha \to \mu_s mgr = \frac{1}{2}mr^2\alpha \to \mu_s mg = \frac{1}{2}ma$$

Substituting and solving gives $T = 3\mu mg$.

Question 3

A table has a mass m, and its height is 2/3 of its width. The legs of the table are very light; all of the mass is in the top. The legs of the table are located at the ends, and a rope is tied to one side; a tension T is applied to the rope. Assume that the coefficient of friction between the legs and the ground is very large, so that the table does not slide.



In this problem, you will calculate the required tension T to tip the table.

a) Draw a force diagram for the table. Indicate your choice of pivot. (10 points)

There are, in principle, six forces. However, as the table begins to tip, the normal force on the left leg and the frictional force associated with it are zero. Thus, we choose our pivot at the bottom of the right leg.

- The normal force on the right leg, acting at its bottom, pointing up. This applies no torque.
- The frictional force on the right leg, acting at its bottom, pointing left. This applies no torque either.
- The tension on the table, acting at the top of the right side, pointing right. This applies a torque $-\frac{2}{3}WT$.
- The weight of the table, acting at the center of the top, pointing down. This applies a torque $\frac{1}{2}Wmg$.

b) What tension force T is required to tip the table (so that the back legs come off the ground)? (15 points)

When the table is just about to tip, $\alpha = \sum \tau = 0$, but the normal force on the back leg is zero (as discussed previously). Thus we set the net torque to zero:

$$\frac{1}{2}Wmg - \frac{2}{3}WT \rightarrow T = \frac{3}{4}mg$$

.

c) What coefficient of static friction between the legs and the floor is required so that the table tips, rather than sliding? (5 points extra credit)

The normal force on the front leg is just mg when the table is about to tip. For it to tip rather than sliding, F_f must be equal and opposite to the tension. Thus, $\mu_s mg > \frac{3}{4}mg$, giving a minimum μ_s of 3/4.

Question 4

The top end of a spring is attached to a stationary point; a $m = 2$ kg mass hangs from the bottom end. It has spring constant $k = 200$ N/m; the mass is at rest. You may give your answers either numerically or in terms of m, k, and g.

a) How far below the spring's equilibrium point does the mass hang? (5 points)

If the mass is hanging at rest, its acceleration is zero. The two forces on it are its own weight mg and the elastic force of magnitude $k\Delta x$. Newton's second law says

$$k\Delta x - mg = 0 \to \Delta x = mg/k = y_1 = 10\,\text{cm}.$$

Note that I am treating down as positive here.

Then, a second $2kg$ mass is added to the spring and released. When the mass is added, the extra weight stretches the spring out further, falling down an additional distance before the elastic force pulls it back up.

b) How far below the spring's equilibrium point do the masses fall in total? (15 points)

Here we treat the spring's equilibrium point as the zero for computing gravitational potential energy, allowing us to use only one variable in our equations for conservation of energy. If the masses are released at rest at height y_1 and come to rest again at height y_2, we have

$$\frac{1}{2}ky_1^2 - 2mgy_1 = \frac{1}{2}ky_2^2 - 2mgy_2$$

where the gravitational potential energy terms are negative since we are treating down as positive here.

We divide through by k, and then substitute $y_1 = mg/k$. This gives

$$\frac{1}{2}y_1^2 - 2y_1^2 = \frac{1}{2}y_2^2 - 2y_1y_2.$$

Rearranging this and dividing by $\frac{1}{2}y_1^2$ gives:

$$\left(\frac{y_2}{y_1}\right)^2 - 4\left(\frac{y_2}{y_1}\right) + 3 = 0.$$

Factoring this gives $(y_2/y_1) = (1, 3)$. The first is just the solution we started with (we'll eventually bounce back there!); the second is the one we want, so $y_2 = 3y_1 = 3mg/k$. (Note that there are many other ways to do this algebra.)

c) The masses bounce up and down for a while before eventually coming to rest (due to air drag, friction, and the like). Once they come to rest, how far below the equilibrium point will they be located? (5 points)

Using the same logic as a), but plugging in twice the mass, the ball will hang at a point $y_3 = 2mg/k$ below the spring's equilibrium point.