

# Work and potential energy

Physics 211  
Syracuse University, Physics 211 Spring 2023  
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March 23, 2023

- Your next homework assignment will be posted tonight or tomorrow and due next Friday.
  - The first 6-7 problems will be things you can do after today's class
  - The last few will require what we do next Tuesday
  - Please come prepared to ask questions next Tuesday!

# Where we've been and where we're going

- Last time: kinetic energy and the work-energy theorem
- This time: the idea of potential energy and conservation of energy
  - Potential energy: “the most meaningful bookkeeping trick in physics”
  - Lets us understand many phenomena without difficult mathematics
  - Conservation of energy: there's always the same amount of energy, and it just changes forms

# Review: kinetic energy

We will see that things are often simpler when we look at something called “energy”

- Basic idea: don't treat  $\vec{a}$  and  $\vec{v}$  as the most interesting things any more
- Treat  $v^2$  as fundamental:  $\frac{1}{2}mv^2$  called “kinetic energy”

## Previous methods:

- Velocity is fundamental
- Force: causes velocities to change over time
- Intimately concerned with vector quantities

## Energy methods:

- $v^2$  (related to kinetic energy) is fundamental
- Force: causes KE to change over distance
- Energy is a *scalar*

Energy methods: useful when you don't know and don't care about time

# Energy: measurements and units

We didn't talk about how we *measure* energy last time:

$$\text{Kinetic energy} = \frac{1}{2}mv^2$$

- Energy has units  $\text{kg m}^2/\text{s}^2$
- This unit is called a *joule*
- 1 joule = the energy required to lift an apple one meter

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- Work = force  $\times$  distance
- Unit of work: newton  $\times$  meter

These are the same thing:

$$\begin{aligned} 1 \text{ newton} \times \text{meter} \\ 1 \text{ kg m/s}^2 \times \text{m} \\ 1 \text{ kg m}^2/\text{s}^2 &\equiv 1 \text{ joule} \end{aligned}$$



## Another new quantity: power, the rate of doing work

Often we are concerned with the *rate* at which something can do work.

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Often we are concerned with the *rate* at which something can do work.

This rate is called **power**.

It is measured in joules per second or **watts**.

Example: my car's engine can do work at a rate of 75 kJ per second = 75 kW.

# The work-energy theorem in 1D

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Some new terminology:

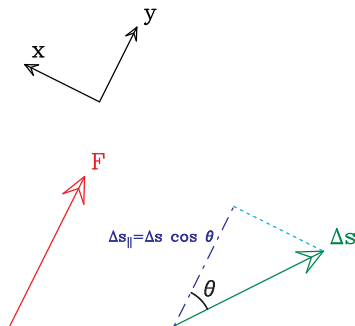
- $\frac{1}{2}mv^2$  called the “kinetic energy” (positive only!)
- $\vec{F} \cdot \Delta\vec{s}$  called the “work” (negative or positive!)
- “Work is the change in kinetic energy”

# Dot products: calculating work

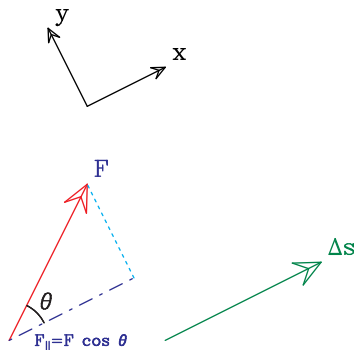
The work done by a force  $\vec{F}$  on an object as it moves through a displacement  $\Delta\vec{s}$  is

$$W = \vec{F} \cdot \Delta\vec{s}.$$

What does this “dot product” mean?



- $\vec{F} \cdot \Delta\vec{s} = (F)(\Delta s_{\parallel}) = (F)(\Delta s \cos \theta)$
- “The component of the displacement parallel to the force, times the force



- $\vec{F} \cdot \Delta\vec{s} = (F_{\parallel})(\Delta s) = (F \cos \theta)(\Delta s)$
- “The component of the force parallel to the motion, times the displacement

Different cases where each form is useful, but it's the same trig either way

# Pendulum demos

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- The kinetic energy can't go below zero
- The height at each end of the swing must be the same!
- ... and the return height can't be greater than the initial height...

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(If physics stops working and I go splat, have a nice summer!

# Potential energy: an accounting trick

- Notice that the work done by gravity depends *only* on the change in height.
- Some other forces are like this as well: the work done depends only on initial and final position
  - These are called *conservative forces*
  - Soon we'll see that the elastic force is like this too
- Separate out gravity and all other forces in the work-energy theorem:

$$KE_f - KE_i = W_{\text{grav}} + W_{\text{other}}$$
$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = mg(y_0 - y_f) + W_{\text{other}}$$

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- Collect all the “initial” things on the left and the “final” things on the right:

$$\begin{array}{ccc} \frac{1}{2}mv_0^2 + mgy_0 & + W_{\text{other}} = & \frac{1}{2}mv_f^2 + mgy_f \\ KE_0 + GPE_0 & + W_{\text{other}} = & KE_f + GPE_f \end{array}$$

- Identify  $mgy$  as “gravitational potential energy”: how much work will gravity do if something falls?

Potential energy lets us easily calculate the work done by conservative forces

# Potential energy: more than accounting!

- Another way to look at the roller coaster: **gravitational potential energy being converted to kinetic energy.**
- This perspective is universal: **all** forces just convert energy from one sort into another
- Some of these types are beyond the scope of this class, but we should be aware of them!

A short history of energy conversion:

- Hydrogen in the sun fuses into helium
- Hot gas emits light
- Light shines on the ocean, heating it
- Seawater evaporates and rises, then falls as rain
- Rivers run downhill
- Falling water turns a turbine
- Turbine turns coils of wire in generator
- Electric current ionizes gas
- Recombination of gas ions emits light
- Nuclear energy  $\rightarrow$  thermal energy
- Thermal energy  $\rightarrow$  light
- Light  $\rightarrow$  thermal energy
- Thermal energy  $\rightarrow$  gravitational potential energy
- Gravitational PE  $\rightarrow$  kinetic energy and sound
- Kinetic energy in water  $\rightarrow$  kinetic energy in turbine
- Kinetic energy  $\rightarrow$  electric energy
- Electric energy  $\rightarrow$  chemical potential energy
- Chemical PE  $\rightarrow$  light

# Potential energy: more than accounting!

- This class is just the study of motion: we can't treat light or nuclear energy here.
- ... but in physics as a whole, the *conservation of energy* – that processes just change energy from one form to another – is universal!
- Conservation of energy is one of the most tested, ironclad ideas in science
- Nuclear and chemical potential energy: nuclear forces do mechanical work on particles, much like gravity
- Light, and others: kinetic energy of little particles called “photons”
- Heat: kinetic energy of atoms in random motion
- Sound: kinetic energy of atoms in coordinated motion
- Food: Just chemical potential energy...
- ... so all of these things aren't as far removed from mechanics after all!
- Einstein: “Mass is just another form of energy”

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- ... so all of these things aren't as far removed from mechanics after all!
- Einstein: “Mass is just another form of energy”
- Maybe it's all, ultimately, just kinetic energy! (I believe it is; others will argue!)



A laptop battery has a potential energy of 100 watt-hours. This means that it can deliver energy at a rate of 100 watts for an hour (or 10 watts for 10 hours) before being exhausted.

How many joules is this?

- A: 100 J
- B: 6000 J
- C: 3600 J
- D: 360,000 J
- E: 60 J

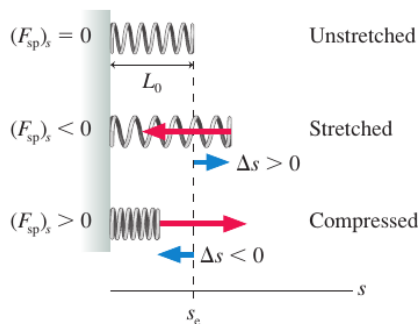
Suppose this 100 watt-hour battery was used to power a climbing robot (climbing an extremely tall mountain) with a mass of 5 kg. How high could the robot get before its battery ran out?

- A: 360 km
- B: 360 m
- C: 72 km
- D: 7200 m
- E: 36 km

# A new force: elasticity and Hooke's law

To best see how potential energy can be useful, let's introduce a new force: elasticity.

- Springs have a particular length that they like to be: “equilibrium length”  $L_0$
- A spring stretched to be longer than this pulls inward to shorten itself
- A spring compressed to be shorter than this pushes outward to lengthen itself
- Flexible things like strings and ropes only pull; they kink instead of compressing
- The force is proportional to the deviation from the optimum length

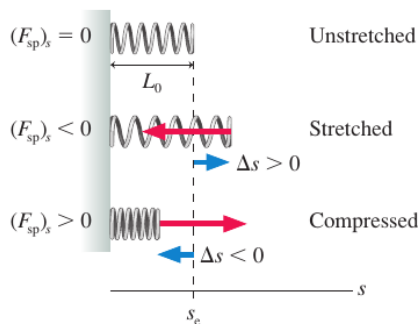


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$$F_{\text{elastic}} = -k(L - L_0) = -k\Delta x \text{ (Hooke's law)}$$

$k$  is called the “spring constant”:

- Measures the stiffness of the spring/rope
- Units of newtons per meter: “restoring force of  $k$  newtons per meter of stretch”

## A simple spring problem: done with the work-energy theorem

A person of mass  $m = 100\text{kg}$  falls from a height of  $h = 3\text{m}$  onto a trampoline. If the person makes an impression  $d = 40\text{ cm}$  deep on the trampoline when he lands, what is the spring constant?

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- Initial kinetic energy + work done by spring + work done by gravity = final kinetic energy
  - Need to use the integral form of the work-energy theorem since the force isn't constant
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- $KE_0 + W_{\text{grav}} + W_{\text{elas}} = KE_f$
- $0 + (mg)(h + d) - \frac{1}{2}kd^2 = 0$
- $k = \frac{mg(h+d)}{2d^2}$

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- A natural choice is  $\Delta x = 0$ , the equilibrium position of the spring.

“How much work is done by a spring as it goes from  $\Delta x = a$  to  $\Delta x = 0$ ?

$$U_{\text{elastic}} = W_{a \rightarrow 0} = \int_a^0 -kx \, dx = \int_0^a kx \, dx = \frac{1}{2}ka^2$$

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Now that we have this, we never have to do this integral again!

$$U_{\text{elastic}} = \frac{1}{2}kx^2, \text{ where } x \text{ is the distance from equilibrium}$$

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- $U_{\text{grav},0} = mgh$
- $U_{\text{elas},0} = 0$  (trampoline starts at equilibrium)
- $U_{\text{grav},f} = -mgd$  (the person falls below  $y = 0$ ; PE can be negative!)
- $U_{\text{elas},f} = \frac{1}{2}kd^2$  (see last slide)

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- $k = \frac{mg(h+d)}{d^2}$

# That spring problem: a recap

We don't care about time  $\rightarrow$  energy methods

## Work-energy theorem

- Initial KE + all work done = final KE
- Need to compute work done by gravity: easy
- Need to compute work done by spring: harder  
(need to integrate Hooke's law)

## Potential energy treatment

- Initial KE + initial PE + other work = final KE + final PE
- No “other work” in this problem; all forces have a PE associated
- Need to know the expressions for PE:
  - $U_{\text{grav}} = mgy$
  - $U_{\text{elas}} = \frac{1}{2}kx^2$  (x is the distance from the equilibrium point)
- No integrals required (they're baked into the above)

# Potential energy with other forces

What about associating a potential energy with other forces?

- Friction is a no-go: the work done by friction depends on the path, not just where you start and stop
- “Ephemeral” forces like tension and normal force are easiest to deal with by computing work directly

- Potential energy is two things:
  - An accounting device that makes it easier to keep track of work done
  - Part of *conservation of total energy*, a powerful statement about nature
- Gravitational potential energy (on Earth):  $U_g = mgy$

- Potential energy is two things:
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- Gravitational potential energy (on Earth):  $U_g = mgy$
- We learned about a new force: **elasticity**
  - Restoring force in a stretched or compressed spring, or a stretched string:
$$F = -k(x - x_0) \text{ (} x_0 \text{ is the equilibrium length)}$$
  - $k$  is the spring constant, measured in force per distance, that gauges stiffness
  - Elastic potential energy:  $U_{\text{elas}} = \frac{1}{2}k(x - x_0)^2$