ASEN 5090
Introduction to GNSS

Lecture 14: GPS Satellite Orbit

Jade Morton

Questions For This Lecture

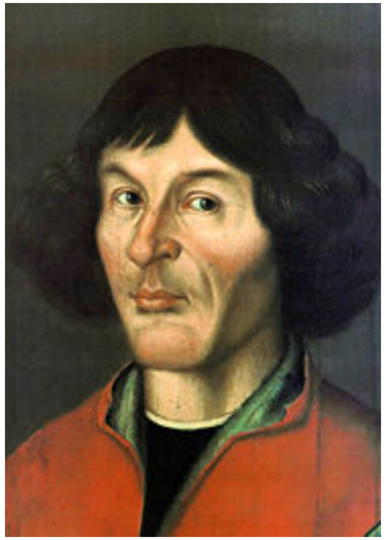
- Is there a model that we can use to describe (represent) GPS satellite orbits?
- What is the model based on?
- How accurate is the model?
- How do we implement the model?

Lecture 14 Outline

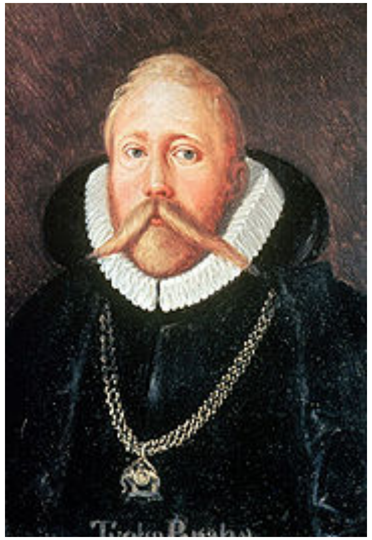
- Kepler’s law
- Ideal Elliptical Orbits
- Satelltie Position and Velocity
- Perturbed Keplerian Orbits
- GPS Orbit Parameters
- GPS Navigation Data Message
- GPS Satellite Constellation and Visibility Display

Five Giants

"There is a force in the Earth which causes the moon to move"

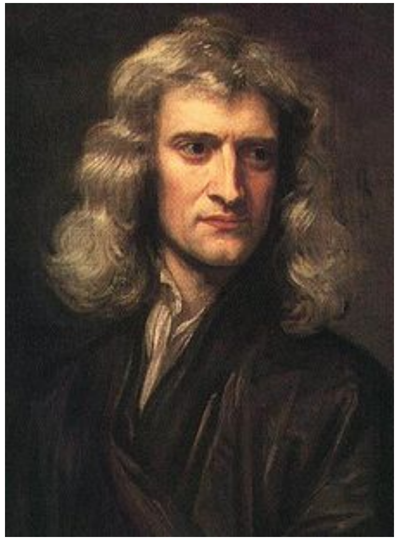


Copernicus
1473-1543
Polish

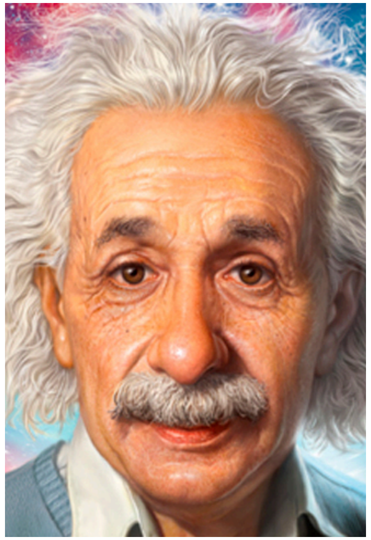


Tycho Brahe
1546-1601
Danish

Johannes Kepler
1571-1630
German



Issac Newton
1643-1727
English



Albert Einstein
1879-1955
US

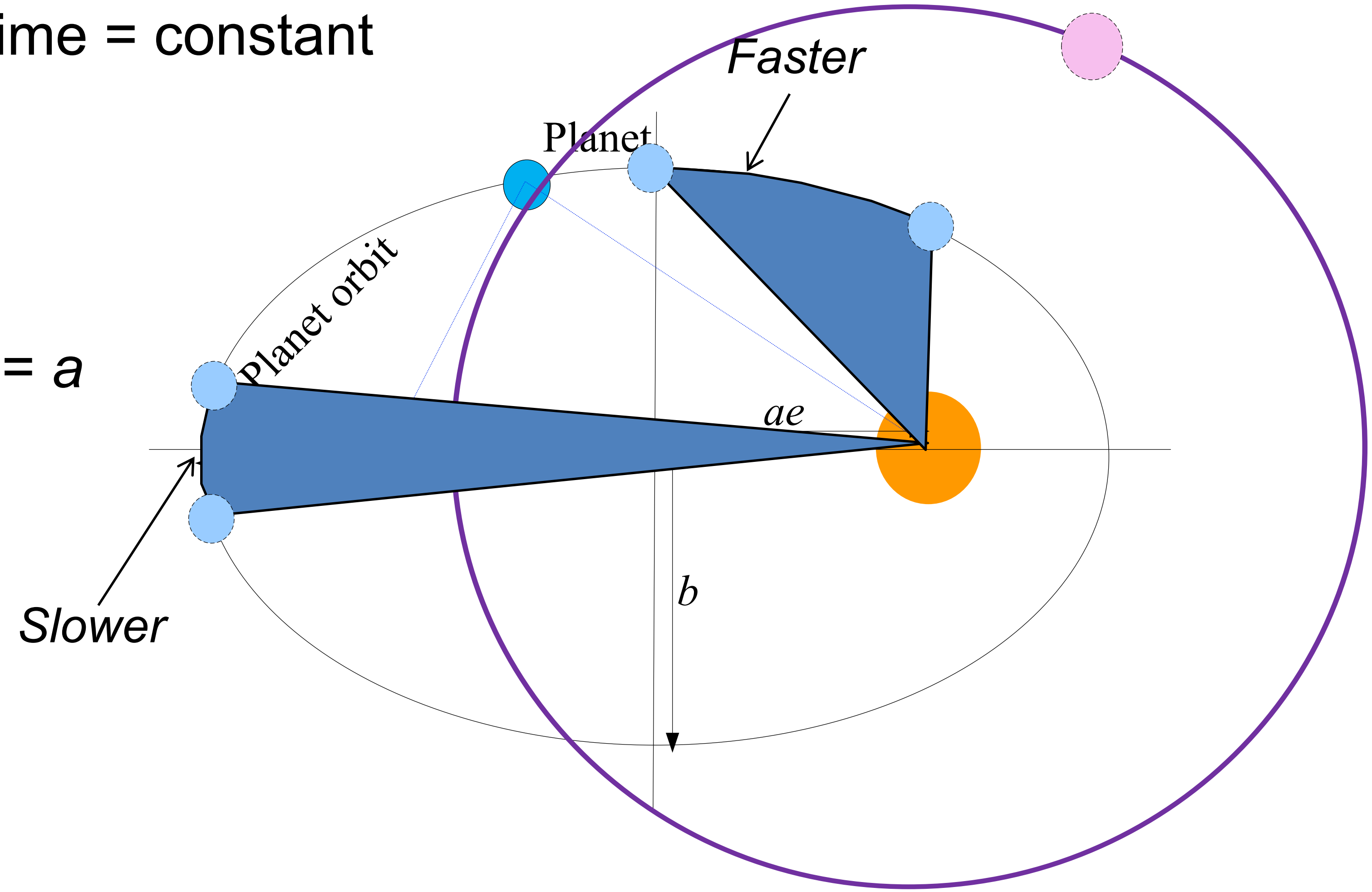
"If I have seen further it is by standing on the shoulders of Giants"

Kepler's Three Laws

1. The orbit of each planet is an ellipse with the sun at one of the foci
2. Orbital sweep areas/unit time = constant
3. $$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{\langle r_1\rangle}{\langle r_2\rangle}\right)^3$$

   T: Orbit period

   $\langle r \rangle$: Mean distance to foci = a



Newton's 2nd Law of Motion

$$\vec{F} = m\vec{a}$$

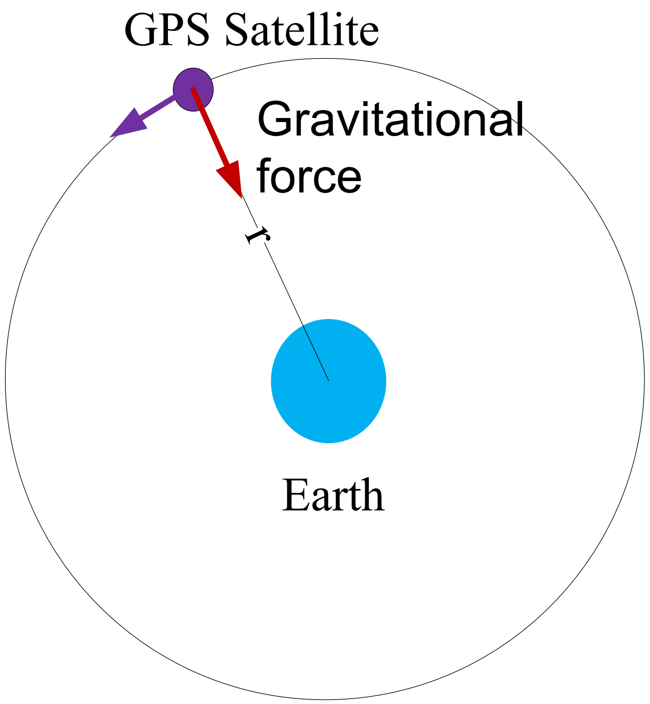
$$\vec{F} = g\frac{mM}{r^2}\hat{r} \qquad \vec{a} = \frac{d^2\vec{r}}{dt^2}$$

Equation that governs the satellite orbit:

$$\boxed{g\frac{M}{r^2}\hat{r} = \frac{d^2\vec{r}}{dt^2}}$$

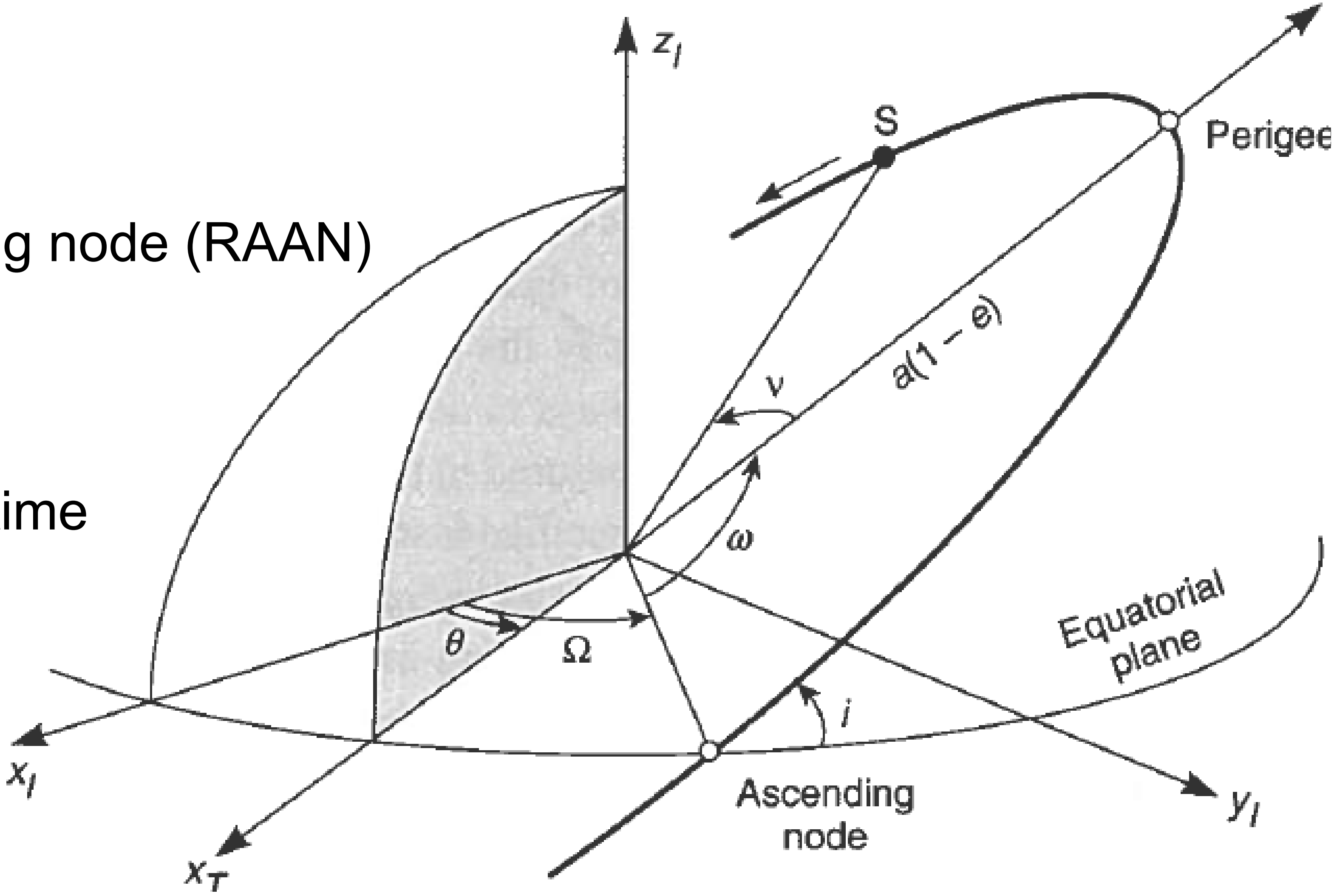
Initial condition that specifies a satellite orbit:

$$\vec{r}_0 = (x_0, y_0, z_0) \qquad \left.\frac{d\vec{r}}{dt}\right|_{t=0} = \left(\left.\frac{dx}{dt}\right|_{t=0}, \left.\frac{dy}{dt}\right|_{t=0}, \left.\frac{dz}{dt}\right|_{t=0}\right)$$



Ideal Elliptical Orbits: Keplerian Elements

a: semi-major axis
e: eccentricity
i: orbit inclination angle
Ω: right ascension of ascending node (RAAN)
ω: argument of perigee
ν: true anomaly



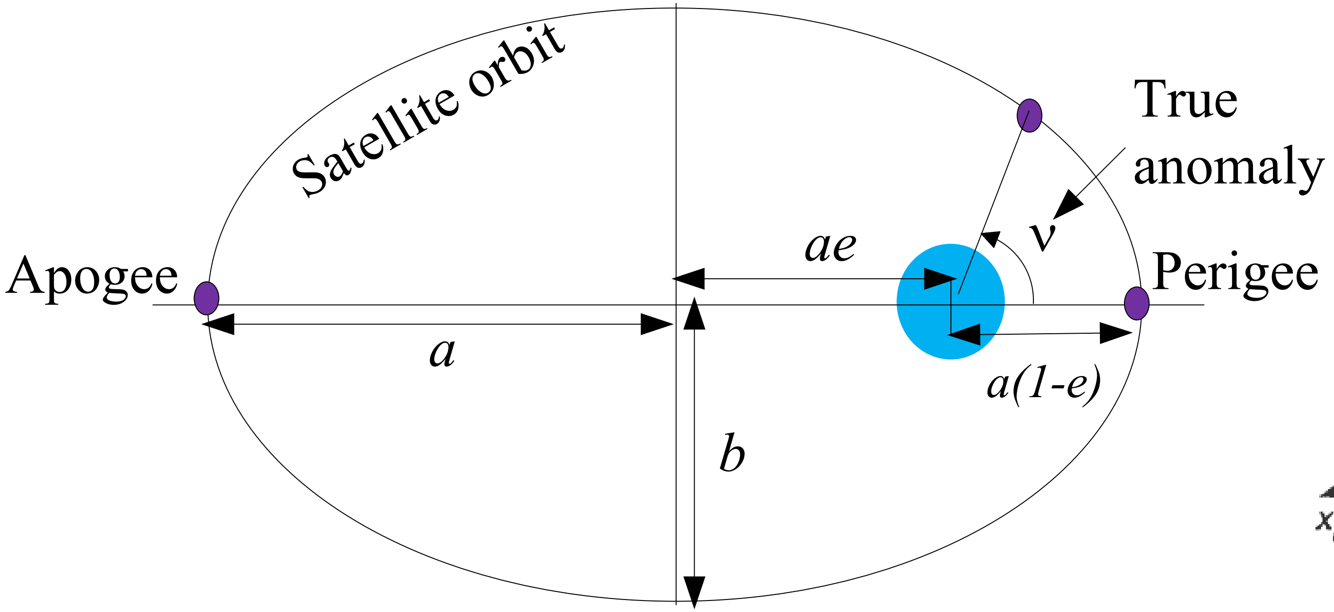
Satellite position at a specific time

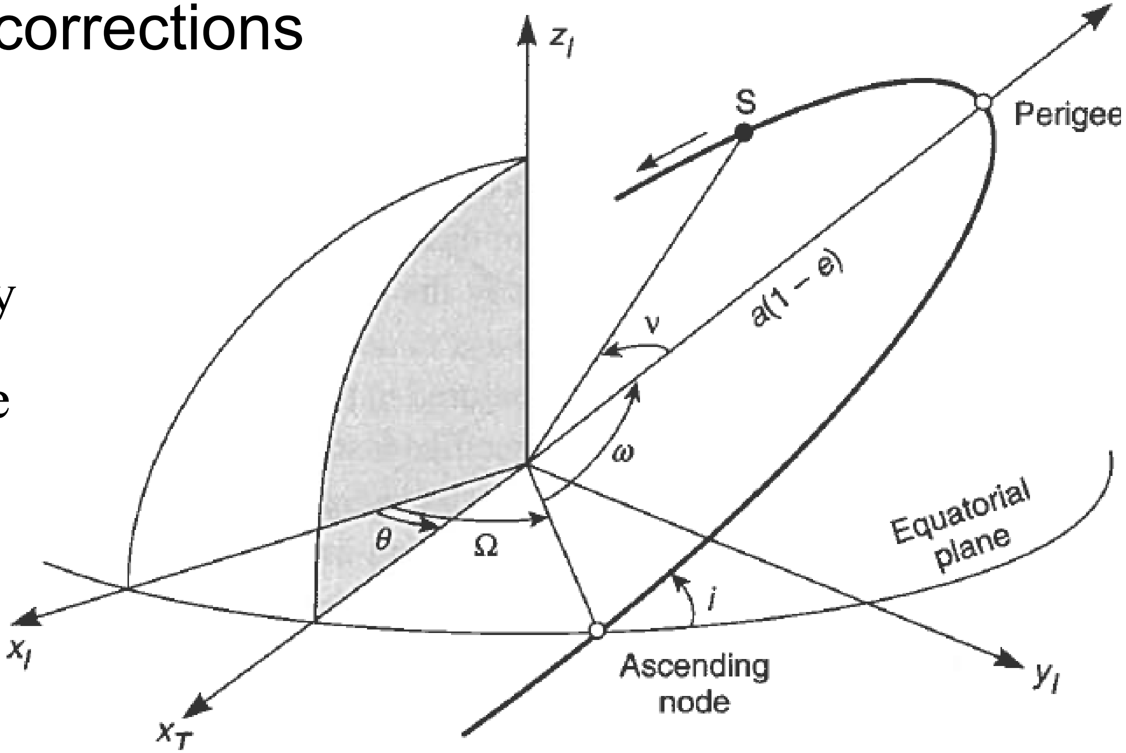
$$\{a, e, i, \Omega, \omega, \nu\}$$

Satellite orbit size/shape/ orientation in space

Keplerian Elements Definitions

Ascending node: the intersection of satellite orbit and earth equatorial plane
Perigee: the point at which the satellite is closest to the center of earth
Apogee: the point at which the satellite is furthest away from the center of earth
True anomaly: satellite position at a specified time
Argument of latitude: $\Phi=\omega+\nu$ used for orbit corrections





Expanded Quasi-Keplerian Parameters

(16-parameter set)

Parameters	Description
t_{0e}	Ephemeris reference time
A	Square root of semi-major axis
e	Orbit eccentricity
i_0	Inclination angle at t_{0e}
Ω_0	Longitude of the ascending node at the beginning of the GPS week
ω	Argument of perigee
M_0	Mean anomaly at t_{oe}
Δn	Correction to the computed mean motion
i-dot	Rate of change of inclination with time
Ω-dot	Rate of change of RAAN with time
C_{uc}, C_{us}	Correction terms for argument of latitude
C_{rc},C_{rs}	Correction terms for orbit radius
C_{ic}, C_{is}	Correction terms for inclination angle

Orbit shape/ size/ orientation

ν

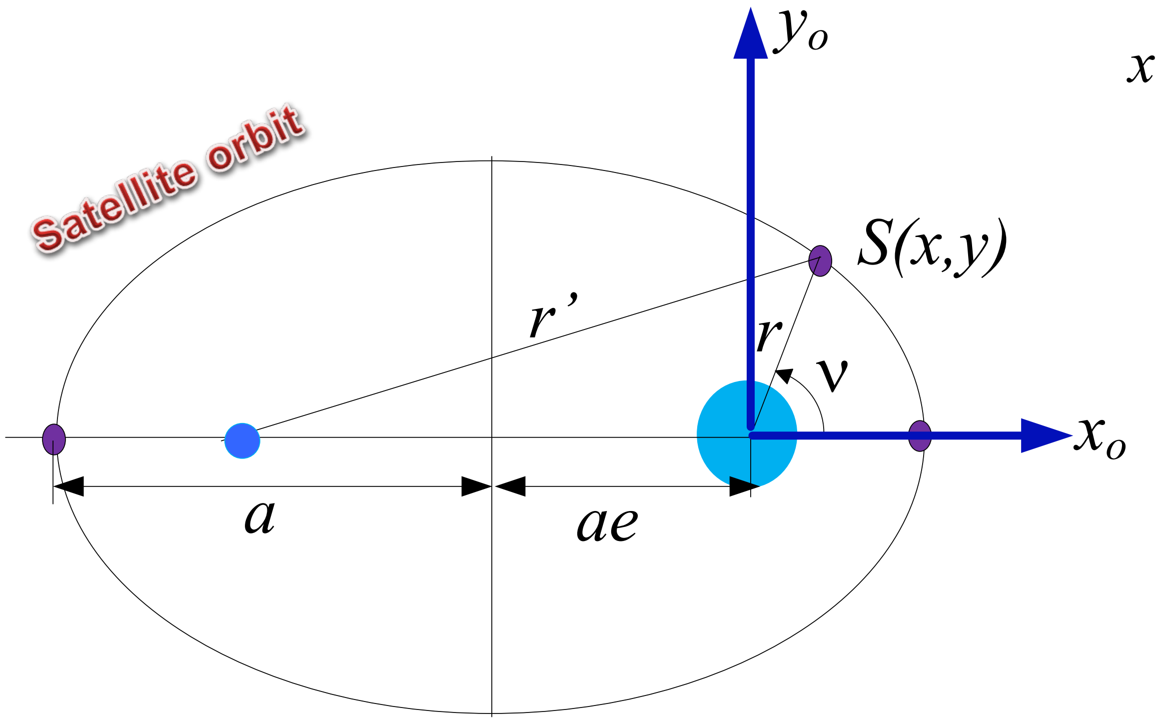
Perturbations

$$\Phi_k = \nu_k + \omega$$

$$\delta u_k = c_{us}\sin 2\Phi_k + c_{uc}\cos 2\Phi_k$$
$$\delta r_k = c_{rs}\sin 2\Phi_k + c_{rc}\cos 2\Phi_k$$
$$\delta i_k = c_{is}\sin 2\Phi_k + c_{ic}\cos 2\Phi_k$$

Orbital Coordinate System

SV position within its orbital coordinate system is determined by *a*, *e*, ν.



$$x = r\cos\nu \qquad y = r\sin\nu$$

Orbit radius: $$r = \frac{a\left(1-e^2\right)}{1+e\cos\nu}$$

By the way, how did we get this?

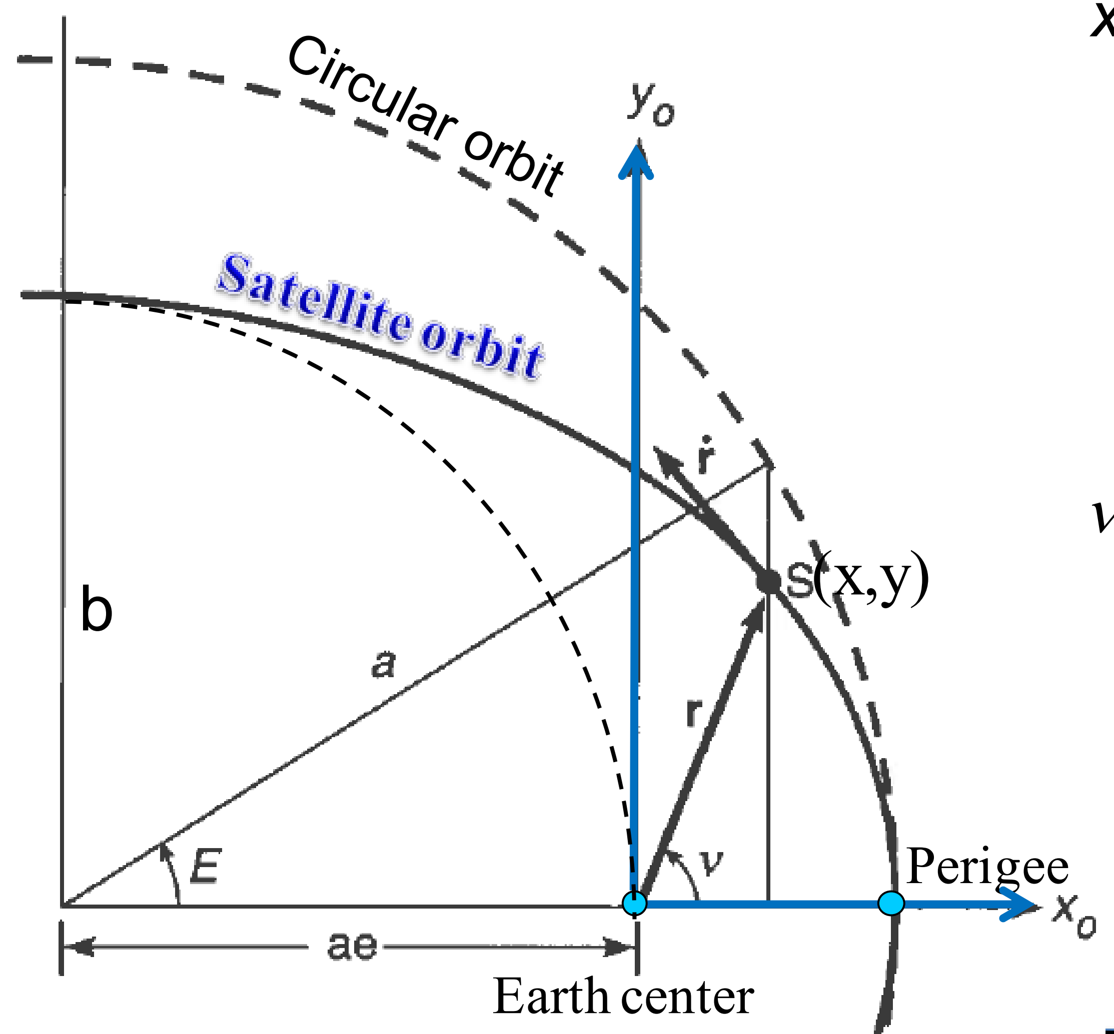
$$r + r' = 2a$$

$$r'^2 = r^2 + (2ae)^2 + 4aer\cos\nu$$

Our objective is to compute **x and y for a given time**

How do we **relate ν to time?**

Eccentric Anomaly: E (A convenient intermediate quantity)



x, y, and r are related to E:

$$x = a\cos E - ae$$

$$y = b\sin E = a\sqrt{1-e^2}\,\sin E$$

$$r = \sqrt{x^2 + y^2} = a\left(1 - e\cos E\right)$$

ν can be computed from E:

$$\tan\nu = \frac{\sin\nu}{\cos\nu} = \frac{\sqrt{1-e^2}\,\sin E}{\cos E - e}$$

How is E related to time?

By **relating *E* to Mean Anomaly *M***

Mean Motion= Mean SV Angular Velocity: n

But, we have to first define **Mean Motion**:

$$m_{SV} n^2 a = g \frac{m_{SV} m_E}{a^2}$$

$$n = \sqrt{\frac{g m_E}{a^3}} = \sqrt{\frac{\mu}{a^3}}$$

$$n = \frac{2\pi}{T}$$

μ: *Earth gravitational constant =3.986005x10^{14} m^3/s^2*
a: Fictitious circular orbit radius = orbit semi-major axis
T: SV orbit period

Mean Anomaly: *M*

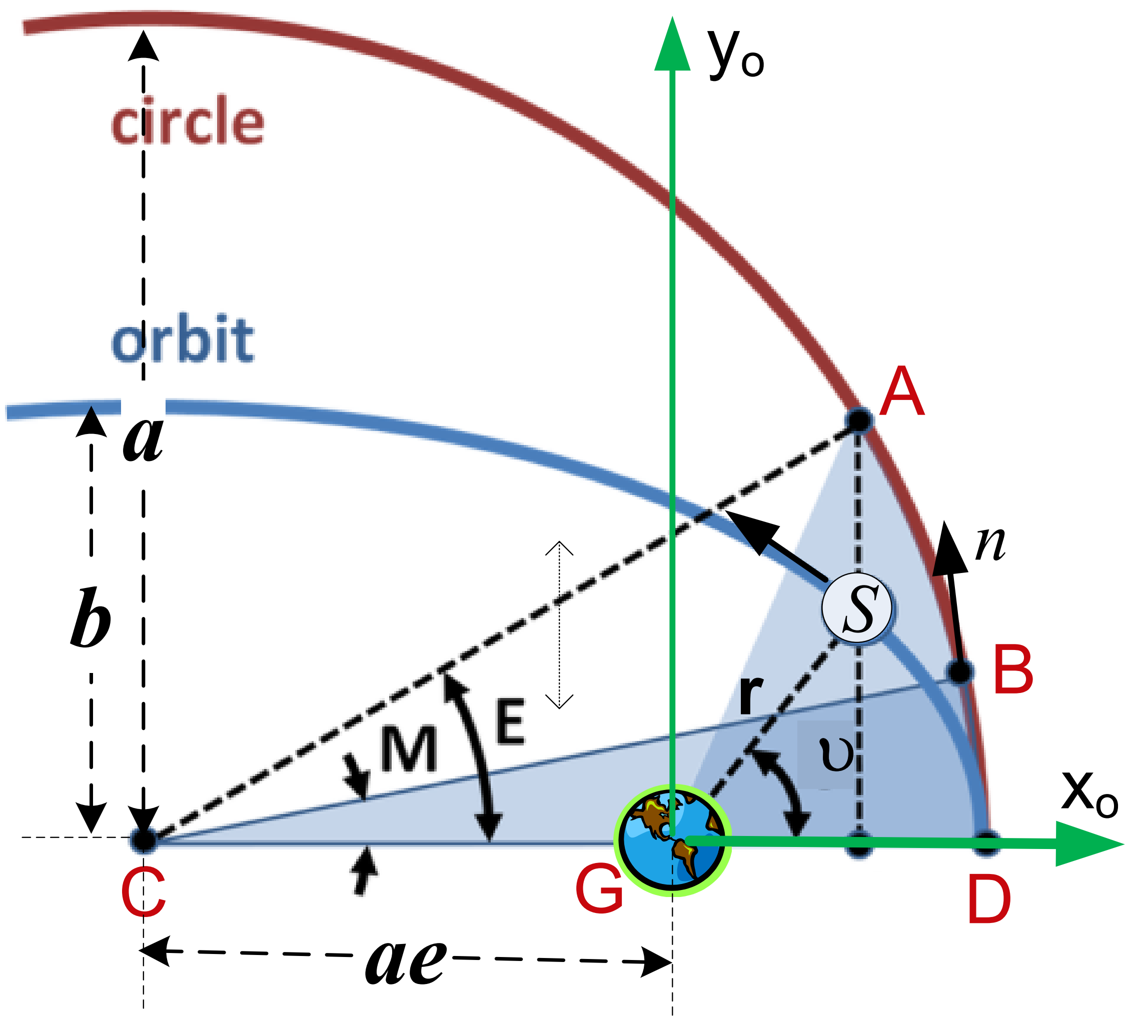
$$\frac{dM}{dt} = n$$

$$M = M_0 + n(t - t_{oe})$$

Satellite ephemeris gives M_0

Kepler's Equation: $M = E - e\sin E$

Solve for E iteratively: $E = M + e\sin E$



SV Velocity

$$x = a\cos E - ae \longrightarrow \frac{dx}{dt} = -a\sin E\frac{dE}{dt}$$

$$y = a\sqrt{1-e^2}\sin E \longrightarrow \frac{dy}{dt} = a\sqrt{1-e^2}\cos E\frac{dE}{dt}$$

$$M = E - e\sin E \longrightarrow \frac{dM}{dt} = \frac{dE}{dt} - e\cos E\frac{dE}{dt} \longrightarrow \frac{dE}{dt} = \frac{n}{1-e\cos E}$$

$$\boxed{\frac{dx}{dt} = -\frac{na\sin E}{1-e\cos E} \qquad \frac{dy}{dt} = -\frac{na\sqrt{1-e^2}\cos E}{1-e\cos E}}$$

Summarize Steps to Obtain SV Position/Velocity in Satellite Orbital Coordinate System

$$\Phi_k = \nu_k + \omega$$

$$\delta u_k = c_{us}\sin 2\Phi_k + c_{uc}\cos 2\Phi_k$$
$$\delta r_k = c_{rs}\sin 2\Phi_k + c_{rc}\cos 2\Phi_k$$
$$\delta i_k = c_{is}\sin 2\Phi_k + c_{ic}\cos 2\Phi_k$$

Compute mean motion n & mean anomaly M:

$$n = \sqrt{\frac{\mu}{a^3}} \qquad M = M_0 + n(t - t_{oe})$$

↓

Compute eccentricity anomaly E (iteratively)

$$E = M + e\sin E$$

↓

Compute orbit radius r and true anomaly *v:*

$$r = a(1 - e\cos E)$$

$$\tan\nu = \arctan\left(\frac{\sqrt{1-e^2}\,\sin E}{\cos E - e}\right)$$

↓

Compute SV coordinate (x,y) in Orbit System

$$x = r\cos\nu \qquad y = r\sin\nu$$

↓

Compute SV velocity (x’,y’) in Orbit System

$$\frac{dx}{dt} = -\frac{na\sin E}{1 - e\cos E} \qquad \frac{dy}{dt} = -\frac{na\sqrt{1-e^2}\cos E}{1 - e\cos E}$$

Transformation from SV Orbital to ECEF Coordinate

Let us do it backward: from ECEF to SV orbital

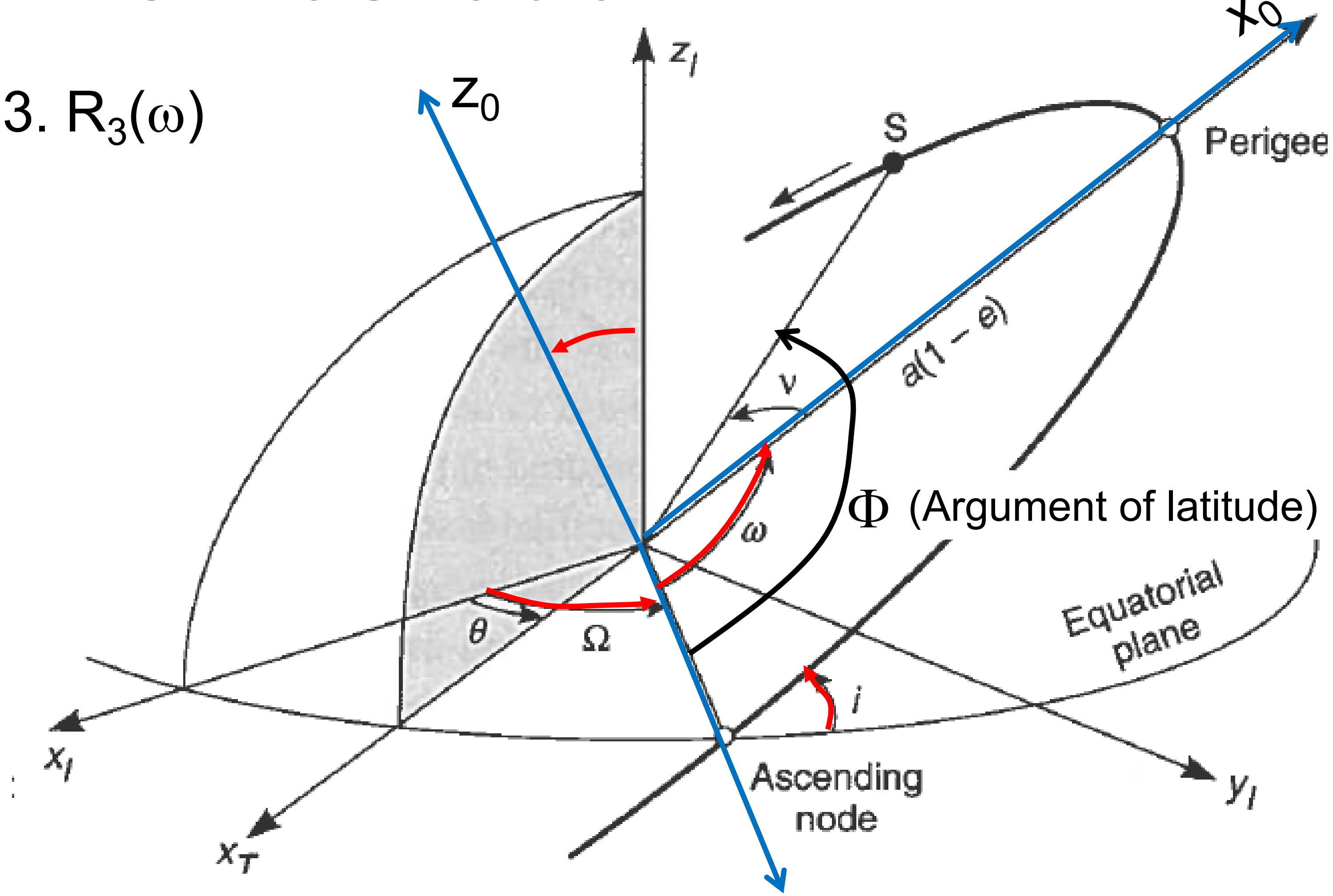
1. $R_3(\Omega)$ 2. $R_1(i)$ 3. $R_3(\omega)$

$$\vec{\mathbf{x}} = R_3(\omega) R_1(i) R_3(\Omega) \vec{\mathbf{x}}_I$$

$$\vec{\mathbf{x}}_I = R_3(-\Omega) R_1(-i) R_3(-\omega) \vec{\mathbf{x}}$$

$$\vec{\mathbf{x}}_T = R_3(\theta) \vec{\mathbf{x}}_I$$

$$\vec{\mathbf{x}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} r\cos\upsilon \\ r\sin\upsilon \\ 0 \end{bmatrix}$$



Simplifications in Transformation from Orbit to ECEF

$$R_3(-\omega)\vec{\mathbf{x}} = \begin{bmatrix} \cos\omega & -\sin\omega & 0 \\ \sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos\nu \\ r\sin\nu \\ 0 \end{bmatrix} = \begin{bmatrix} r\cos(\omega+\nu) \\ r\sin(\omega+\nu) \\ 0 \end{bmatrix} = \vec{\mathbf{x}}_{\mathbf{p}}$$

Instead of: $\boxed{\begin{aligned} x &= r\cos\nu \\ y &= r\sin\nu \end{aligned}}$ Compute: $\boxed{\begin{aligned} x_p &= r\cos(\omega+\nu) \\ y_p &= r\sin(\omega+\nu) \end{aligned}}$

$$\vec{\mathbf{x}}_{\mathbf{T}} = R_3(\theta)R_3(-\Omega)R_1(-i)R_3(-\omega)\vec{\mathbf{x}} = R_3(-(\Omega-\theta))R_1(-i)\vec{\mathbf{x}}_{\mathbf{p}}$$

$\theta = \Omega_e t$ Ω_e: Earth rotation rate=7.2921151467x10^{-5} rad/s

Ω has perturbation correction term: $\dot{\Omega} \Longrightarrow \Omega = \Omega_0 + \dot{\Omega}(t - t_{oe})$

$$\Omega_p = \Omega - \theta = \Omega_0 + \dot{\Omega}(t - t_{oe}) - \Omega_e t \qquad \vec{\mathbf{x}}_{\mathbf{T}} = R_3(-\Omega_p)R_1(-i)\vec{\mathbf{x}}_{\mathbf{p}}$$

SV Coordinate in ECEF

$$R_3(-\Omega) = \begin{bmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad R_1(-i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{bmatrix}$$

$$R_3(-\Omega_p)R_1(-i) = \begin{bmatrix} \cos\Omega_p & -\sin\Omega_p\cos i & 0 \\ \sin\Omega_p & \cos\Omega_p\cos i & 0 \\ 0 & \sin i & \cos i \end{bmatrix}$$

$$\vec{\mathbf{x}}_{\mathbf{T}} = R_3(-\Omega_p)R_1(-i)\vec{\mathbf{x}}_{\mathbf{p}} = \begin{bmatrix} x_p\cos\Omega_p - y_p\sin\Omega_p\cos i \\ x_p\sin\Omega_p + y_p\cos\Omega_p\sin i \\ y_p\sin i \end{bmatrix}$$

Steps to Obtain SV Position in ECEF System

Compute mean motion n & mean anomaly M:

$$n = \sqrt{\frac{\mu}{a^3}} \qquad M = M_0 + n(t - t_{oe})$$

Compute eccentricity anomaly E (iteratively)

$$\boldsymbol{E = M + e\sin E}$$

Compute orbit radius r and true anomaly *v:*

$$r = a(1 - e\cos E)$$

$$\tan\nu = \arctan\left(\frac{\sqrt{1-e^2}\sin E}{\cos E - e}\right)$$

Compute SV coordinate in rotated Orbit (by ω)

$$x_p = r\cos(\omega + \nu) \qquad y_p = r\sin(\omega + \nu)$$

Convert SV position to ECEF coordinate:

$$\Omega_p = \Omega_0 + \dot{\Omega}(t - t_{oe}) - \Omega_e t$$

$$x_T = x_p \cos\Omega_p - y_p \sin\Omega_p \cos i$$

$$y_T = x_p \sin\Omega_p + y_p \cos\Omega_p \sin i$$

$$z_T = y_p \sin i$$

IS-GPS-200, p94

Table 20-II. Ephemeris Data Definitions

M_0	Mean Anomaly at Reference Time
Δn	Mean Motion Difference From Computed Value
e	Eccentricity
$\sqrt{A}$	Square Root of the Semi-Major Axis
Ω_0	Longitude of Ascending Node of Orbit Plane at Weekly Epoch
i_0	Inclination Angle at Reference Time
ω	Argument of Perigee
$\dot{\Omega}$	Rate of Right Ascension
IDOT	Rate of Inclination Angle
C_{uc}	Amplitude of the Cosine Harmonic Correction Term to the Argument of Latitude
C_{us}	Amplitude of the Sine Harmonic Correction Term to the Argument of Latitude
C_{rc}	Amplitude of the Cosine Harmonic Correction Term to the Orbit Radius
C_{rs}	Amplitude of the Sine Harmonic Correction Term to the Orbit Radius
C_{ic}	Amplitude of the Cosine Harmonic Correction Term to the Angle of Inclination
C_{is}	Amplitude of the Sine Harmonic Correction Term to the Angle of Inclination
t_{oe}	Reference Time Ephemeris (reference paragraph 20.3.4.5)
IODE	Issue of Data (Ephemeris)

IS-GPS-200, p93. Steps to compute SV position (page 1)

$\mu = 3.986005 \times 10^{14}$ meters^3/sec^2 ← μ=gM	WGS 84 value of the earth's gravitational constant for GPS user
π=3.1415926535898, speed of light c = 299792458 m/s	
$\dot{\Omega}_e = 7.2921151467 \times 10^{-5}$ rad/sec	WGS 84 value of the earth's rotation rate
$A = \left(\sqrt{A}\right)^2$	Semi-major axis
$n_0 = \sqrt{\frac{\mu}{A^3}}$	Computed mean motion (rad/sec)
$t_k = t - t_{oe}$*	Time from ephemeris reference epoch
$n = n_0 + \Delta n$	Corrected mean motion
$M_k = M_0 + nt_k$	Mean anomaly
$M_k = E_k - e \sin E_k$	Kepler's Equation for Eccentric Anomaly (may be solved by iteration) (radians)
$\nu_k = \tan^{-1}\left\{\frac{\sin \nu_k}{\cos \nu_k}\right\}$	True Anomaly
$= \tan^{-1}\left\{\frac{\sqrt{1-e^2}\sin E_k / (1 - e\cos E_k)}{(\cos E_k - e)/(1 - e\cos E_k)}\right\}$	

IS-GPS-200, p91. Steps to compute SV position (page 2)

$$E_k = \cos^{-1}\left\{\frac{e + \cos\nu_k}{1 + e\cos\nu_k}\right\}$$ Eccentric Anomaly

$$\Phi_k = \nu_k + \omega$$ Argument of Latitude

$$\left.\begin{aligned} \delta u_k &= c_{us}\sin 2\Phi_k + c_{uc}\cos 2\Phi_k && \text{Argument of Latitude Correction} \\ \delta r_k &= c_{rs}\sin 2\Phi_k + c_{rc}\cos 2\Phi_k && \text{Radius Correction} \\ \delta i_k &= c_{is}\sin 2\Phi_k + c_{ic}\cos 2\Phi_k && \text{Inclination Correction} \end{aligned}\right\}$$ Second Harmonic Perturbations

$$u_k = \Phi_k + \delta u_k$$ Corrected Argument of Latitude

$$r_k = A(1 - e\cos E_k) + \delta r_k$$ Corrected Radius

$$\Omega_p = \Omega_0 + \dot{\Omega}(t - t_{oe}) - \dot{\Omega}_e t$$

Equivalent formula

$$i_k = i_0 + \delta i_k + (\mathrm{IDOT})\, t_k$$ Corrected Inclination

$$\left.\begin{aligned} x_k' &= r_k\cos u_k \\ y_k' &= r_k\sin u_k \end{aligned}\right\}$$ Positions in orbital plane.

$$\Omega_k = \Omega_0 + (\dot{\Omega} - \dot{\Omega}_e)\, t_k - \dot{\Omega}_e\, t_{oe} \qquad t_k = t - t_{oe}$$ Corrected longitude of ascending node.

$$\left.\begin{aligned} x_k &= x_k'\cos\Omega_k - y_k'\cos i_k\sin\Omega_k \\ y_k &= x_k'\sin\Omega_k + y_k'\cos i_k\cos\Omega_k \\ z_k &= y_k'\sin i_k \end{aligned}\right\}$$ Earth-fixed coordinates.

GPS Broadcast Orbit Parameter: Ephemeris

- Ephemeris: satellite orbital parameters shown in previous slides
- Derived from SV positions and velocity data over 4 hours intervals using least square curve fit.
- Control station upload the parameters once per day to SVs. Each update can potentially last 2 weeks. Accuracy degrades over time.
- SV broadcasts a subset of the parameters every two hours.
- Pseudorange rms error due to orbit ~2m

Precise Ephemeris

- Post-processed ephemeris based on GPS monitor & ref. stations measurements.
- International GNSS Services (IGS) products

Products	Parameters	Accuracy	Latency	Update rates	Sample interval
Broadcast	Orbits	~160 cm	Real time		Daily
	Sat. clocks	~7 ns			
Ultra-rapid (predicted half)	Orbits	~10 cm	Real time	4 times/day	15 min
	Sat. clocks	~5 ns			
Ultra-rapid (observed half)	Orbits	<5 cm	3 hours	4 times/day	15 min
	Sat. clocks	~0.2 ns			
Rapid	Orbits	<5 cm	17 hours	Once/day	15 min
	Sat. clocks	0.1 ns			5 min
Final	Orbits	<5 cm	~13 days	Once/week	15 min
	Sat. clocks	<0.1 ns			5 min

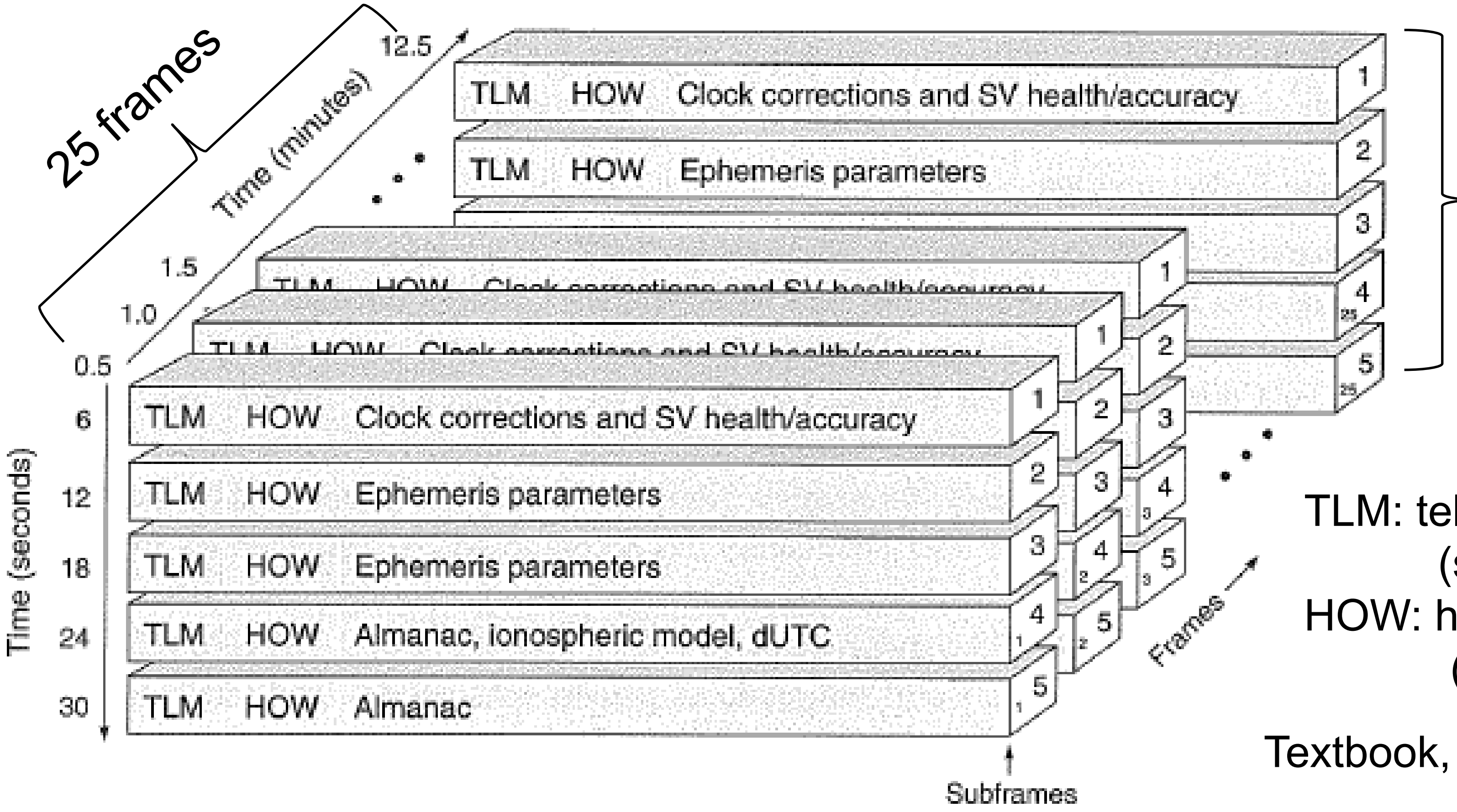
Almanac

- Coarse version of all satellites ephemeris
- A subset of ephemeris with reduced precision
- Allow RX to determine which SV's are in direct view
- Updated about once/week
- Orbit accuracy: 1~2km
- Clock correction: linear model
- Only the basic Keplerian parameters + Ω-dot at a common reference time are included

Example Almanac Data

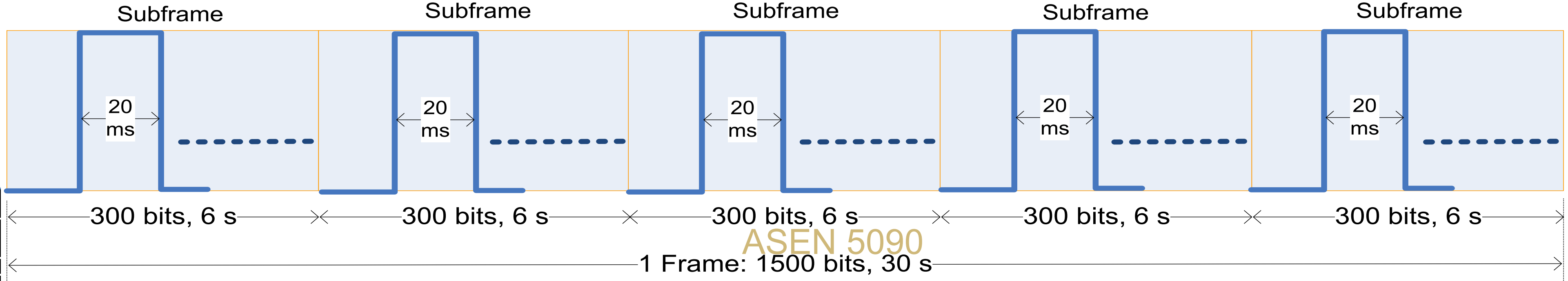
******** Week 250 almanac for PRN-01 ********	
ID:	01
Health:	000
Eccentricity:	0.5140781403E-002
Time of Applicability(s):	589824.0000
Orbital Inclination(rad):	0.9797727040
Rate of Right Ascen(r/s):	-0.7588887536E-008
SQRT(A) (m 1/2):	5153.507812
Right Ascen at Week(rad):	0.2765689089E+001
Argument of Perigee(rad):	-1.687860857
Mean Anom(rad):	0.6236332231E+000
Af0(s):	0.3480911255E-003
Af1(s/s):	0.0000000000E+000
week:	250

GPS Navigation Data Message



TLM: telemetry word.
(synch pattern)
HOW: hand-over word.
(SV time)

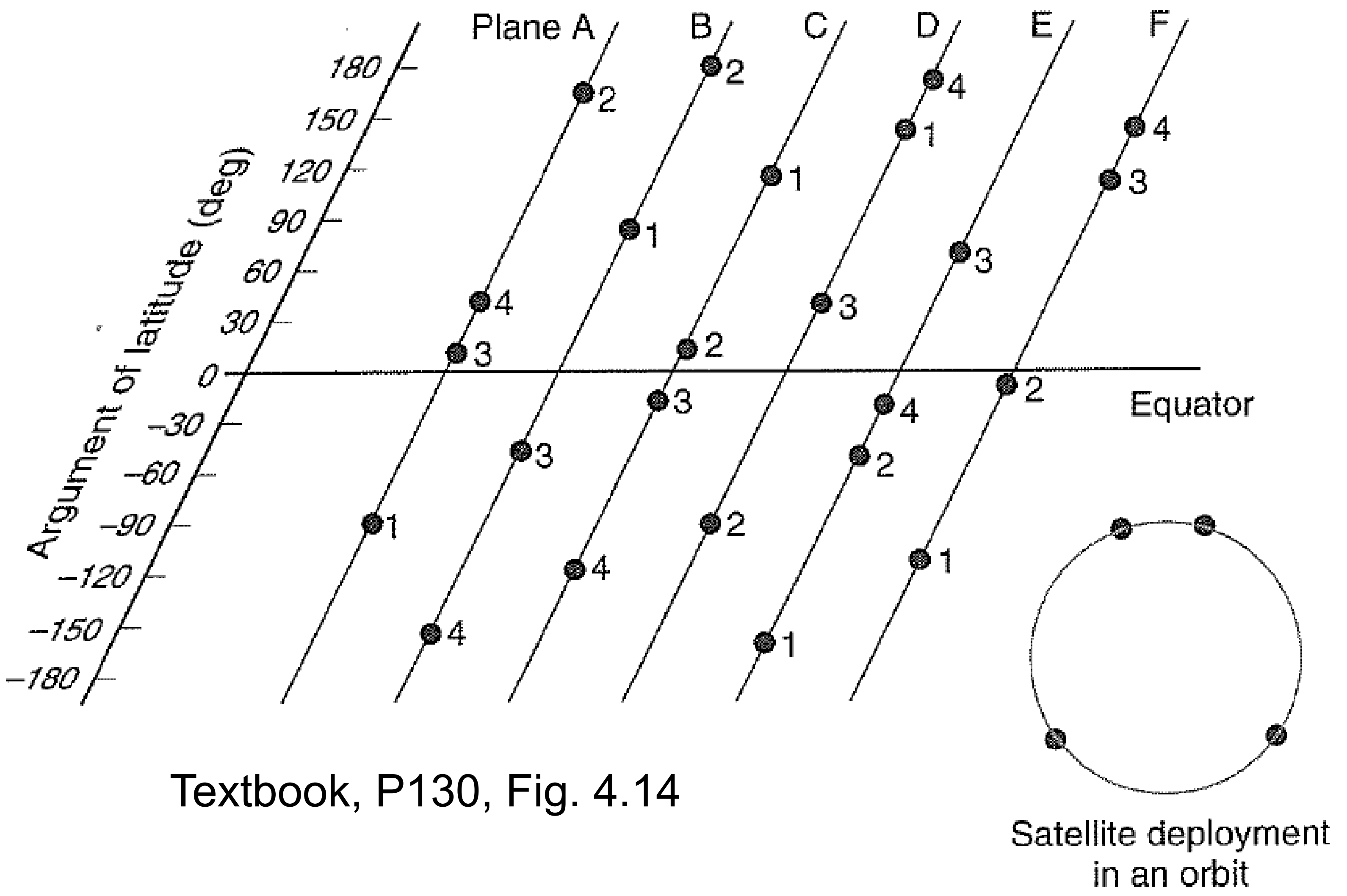
Textbook, P128, Fig. 4.13



GPS Satellite Constellation

- Baseline 24-satellite GPS constellation
- a=26,560km
- T=11 hours 58 min
- e < 0.02
- 6 orbital plane (A through F), all with $i = 55^o$. RAAN are separated by 60^o between orbital planes.
- 4 SVs per plane distributed unevenly
  - 2 SVs separated by 30^o~32.1^o.
  - 3 SVs are separated by $92.38^o \sim 130.98^o$.

Baseline GPS Constellation Configuration



Textbook, P130, Fig. 4.14

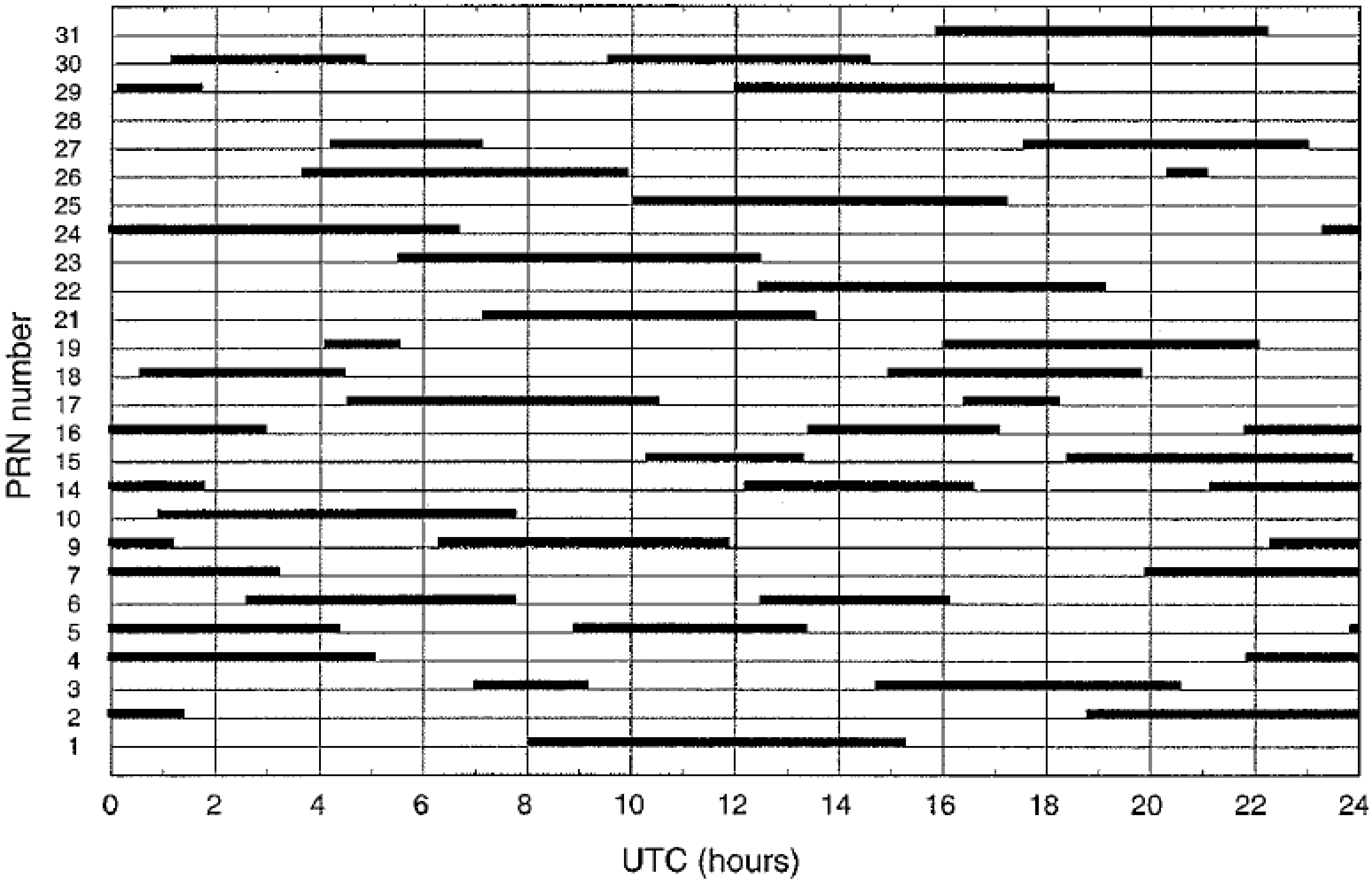
Orbital Parameters for Baseline 24-Satellite Constellation

Semi-major axis: 26,559.8 km; eccentricity: 0; inclination: 55°; argument of perigee: 0 [SPS (2001)]

Slot ID	Right ascension (deg)	Mean anomaly (deg)	Slot ID	Right ascension (deg)	Mean anomaly (deg)
A3	272.85	11.68	D1	92.85	135.23
A4	272.85	41.81	D4	92.85	167.36
A2	272.85	161.79	D2	92.85	265.45
A1	272.85	268.13	D3	92.85	35.16
B1	332.85	80.96	E1	152.85	197.05
B2	332.85	173.34	E2	152.85	302.60
B4	332.85	204.38	E4	152.85	333.69
B3	332.85	309.98	E3	152.85	66.07
C1	32.85	111.88	F1	212.85	238.89
C4	32.85	241.56	F2	212.85	345.23
C3	32.85	339.67	F3	212.85	105.21
C2	32.85	11.80	F4	212.85	135.35

Textbook, P130, Table 4.4

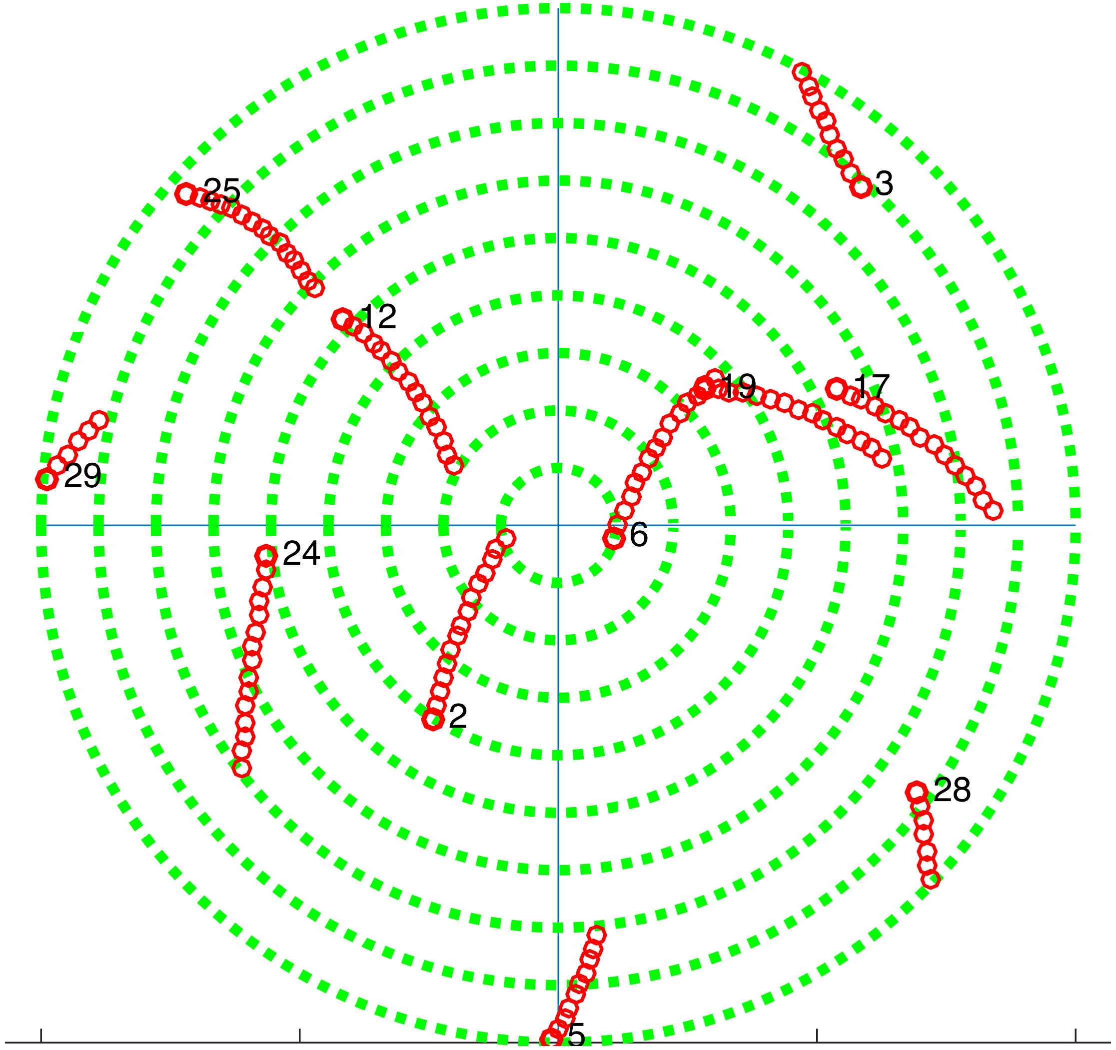
Example Satellite Visibility



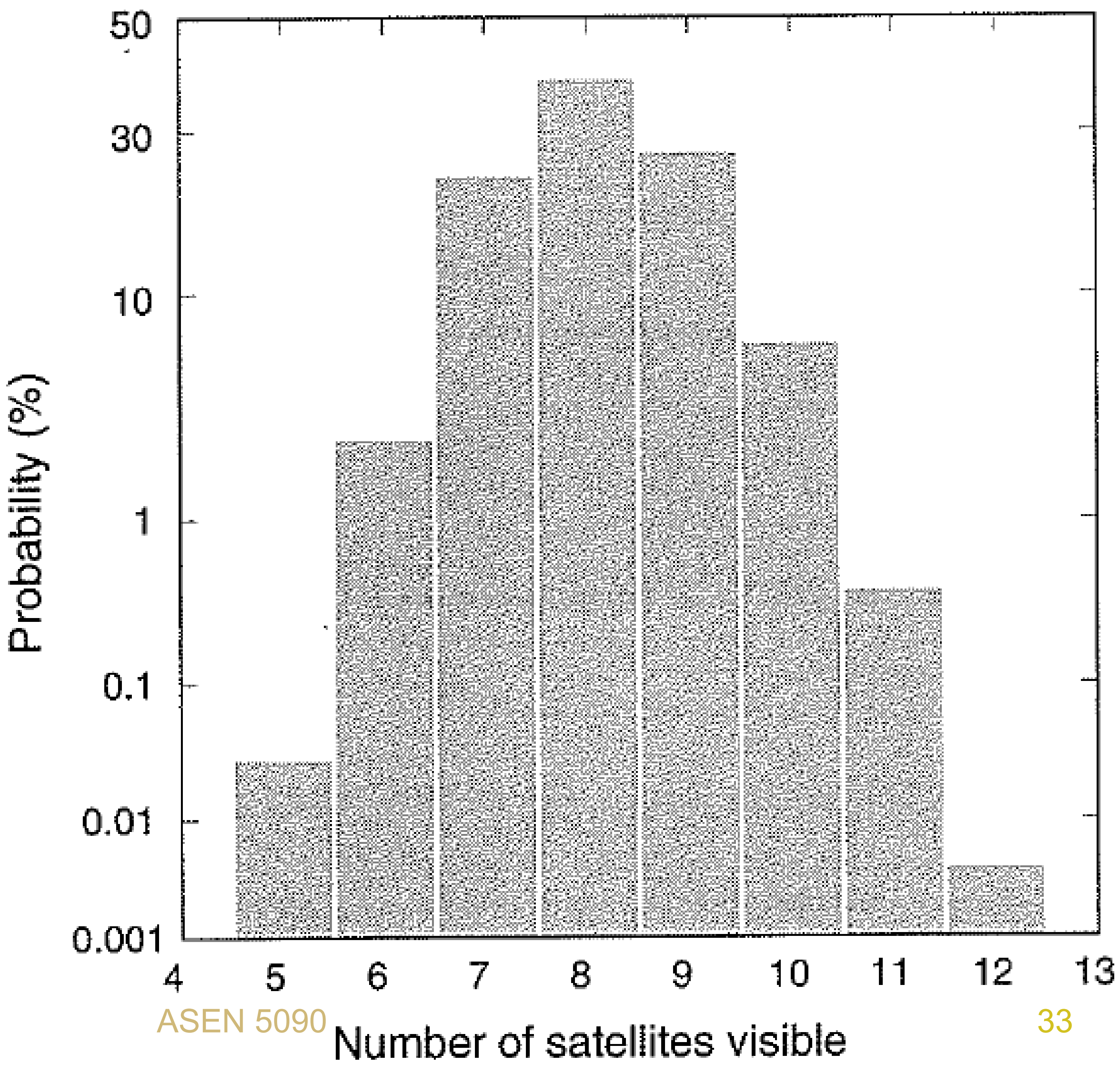
Textbook, P131, Fig. 4.15

Sky View of Satellite Tracks

9/4/2017,
9:30-10:45AM
Boulder, CO



Statistical Distributions of Visible SVs



Textbook, P132
Fig. 4.17

Block Diagram for Satellite Position Calculation

User Inputs:
Observation time t
Observation location: λ,ϕ,h

Data Inputs:
Read Yuma file to obtain almanac

Convert observer's geodetic coordinate to ECEF coordinate:
$\lambda,\phi,h \rightarrow x_{oT}, y_{oT}, z_{oT}$

Compute SV ECEF coordinate:
x_T, y_T, z_T

Convert SV ECEF coordinate to ENU coorinate:
$x_T, y_T, z_T \rightarrow E, N, U$

Compute SV elevation and azimuth, only keeping ones with elevation>masking angle
$E, N, U \rightarrow ele, azi$

Display routines…….

Assignments

- Lecture 14 coverage: P115-133. IS-GPS-ICD200 (handouts p91-94)
- Lecture 15 coverage: P147-157
- Project 4: Satellite orbit calculation using Almanac. Please see detailed instructions in D2L Project folder. Due date: 11/1.