MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013
ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS

ANSWERS

A) ______________________________

B) (________ , ________ , ________)

C) ______________________________

A) Let $f(x)=\sqrt{3x-4}$. Compute: $f^{-1}\left(\sqrt{5}\right)+f\left(\frac{8}{3}\right)$.

B) Given: $\begin{cases} f(x)=3x-2 \\ g(x)=(x-2)(x+3)+A, \text{ where } A<0 \end{cases}$

For several integer values of A, the composite function $h(x)=g\circ f(x)=g\left(f(x)\right)$ has two distinct rational zeros, r_1 and r_2, where $r_1<r_2$.

For the <u>largest</u> possible value of A, compute the ordered triple $\left(A,r_1,r_2\right)$

C) $f(x)$ is a 4^{th} degree polynomial with a leading coefficient of 1.

Curiously, $f(1)=f(2)=f(3)=f(4)=6$.

Determine the <u>sum</u> of the zeros of this function.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013
ROUND 2 ARITHMETIC / NUMBER THEORY

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

A) How many <u>two-digit</u> primes leave a remainder of 1 when divided by 4?
Assume prime refers to positive integers only.

B) The positive integers N and $(N + 1)$ have 3 and 6 positive factors respectively.
Compute the <u>smallest</u> possible value of N.

C) Find the <u>smallest</u> prime factor of $2^{30} - 2^{16} + 1$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013
ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS

ANSWERS

A) ______________________________

B) ______________________________

C) $y =$ __________________________

A) Given: $Arc\sin\left(\frac{5}{7}\right) = Arc\tan(k)$

Determine the value of k in terms of a simplified radical.
If necessary, rationalize the denominator.

B) Let $A =$ $Arc\tan\left(-4\sqrt{3}\right)$. Compute $\sin(\pi + A)$.

C) Given: $\begin{cases} x = 9\cos^4\theta \\ y = 9\sin^4\theta \end{cases}$, where $0 \le \theta < 2\pi$

Clearly, $0 \le x \le 9$ and $0 \le y \le 9$
Express y strictly in terms of x, where $0 \le x \le 9$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013
ROUND 4 ALG 1: WORD PROBLEMS

ANSWERS

A) ________________________________

B) ________________________________

C) ________________________________

A) A square of side N is cut from one corner of a 3 x 5 index card, leaving 85% of the index card (by area). Compute N.

B) A jar contains 504 jelly beans, either red (R) or green (G).
The initial ratio is $R : G = 5 : 3$. The red jelly beans are more tasty and disappear twice as fast as the green. How many jelly beans are left in the jar when $R : G = 3 : 5$?

C) A simple majority wins the election for senior class president.
Boris and Miles were the final two candidates for senior class president.
Boris won the election; however, if exactly 8% of those who voted for Boris switched their vote to Miles and those who voted for Miles originally voted for him again, Miles would have won the election by one vote. More than 100 votes were cast in the election.
Compute the <u>minimum</u> possible number of votes that were cast.

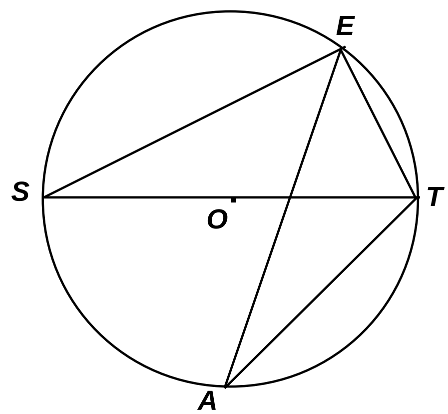
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013
ROUND 5 PLANE GEOMETRY: CIRCLES

ANSWERS

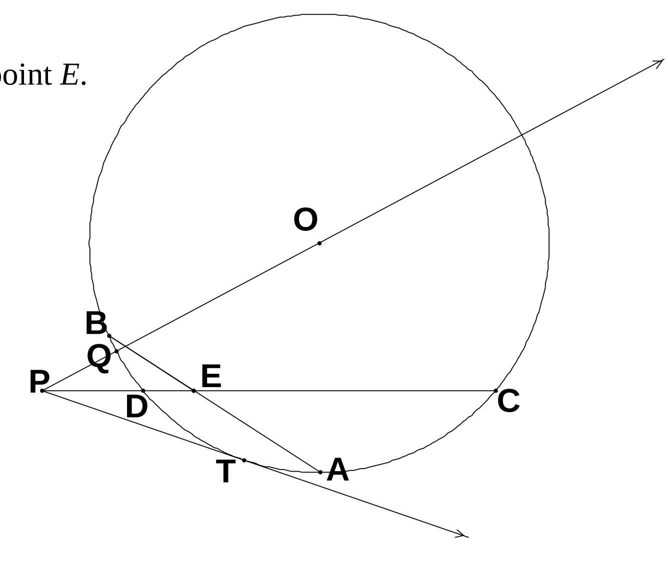
A) ______________________________ °

B) ______________________________

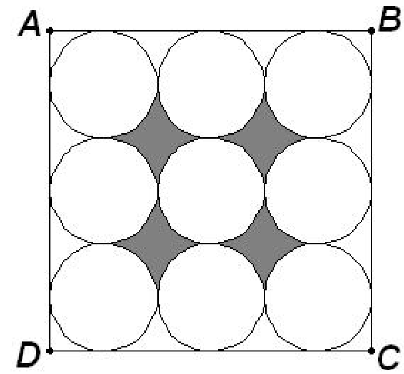
C) ______________________ sq. inches



A) Given: $\overline{ST}$ is a diameter of circle O.
$m\angle ETS = 53°$, $m\angle SEA = m\angle TEA$
Compute the degree-measure of minor arc $\overset{\frown}{ETA}$.



B) In circle O, chords $\overline{AB}$ and $\overline{CD}$ intersect at point E.
Ray $\overrightarrow{PT}$ is tangent to circle O at point T.
Ray $\overrightarrow{PQ}$ pass through point O.
$CE = 6$, $DE = 1$, $AE = 3$, $PQ = 1.5$, $BE = PD$
Compute the radius of circle O.



C) Nine circles are inscribed inside a square whose area is 162 square inches. Compute the area of the shaded region.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013
ROUND 6 ALG 2: SEQUENCES AND SERIES

ANSWERS

A) ______________________________

B) ______________ : ______________

C) ______________________________

A) Let S denote the sum of the 10^{th} and 13^{th} terms of the geometric progression $3, \frac{3}{i}, -3, 3i, \ldots$. Compute S^2.

B) $3,\ y,\ x$ are the first three terms in a geometric progression (GP) and $y \neq 0$.
$3,\ y+2,\ x+1$ are the first three terms in an arithmetic progression (AP).
Compute the ratio of the 12^{th} term of the AP to the 5^{th} term of the GP.

C) We know that the arithmetic mean is always greater than or equal to the harmonic mean. Suppose for some positive value(s) of *A*, the arithmetic mean and harmonic mean of 2 and *A* differ by 0.1. Compute <u>all</u> possible positive values of the geometric mean of 2 and *A*.

FYI: The harmonic mean of *x* and *y* is defined to be the reciprocal of the arithmetic mean of $\frac{1}{x}$ and $\frac{1}{y}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013
ROUND 7 TEAM QUESTIONS

ANSWERS

A) ____________________________ D) __________________________ ¢

B) ____________________________ E) (________________ , ________)

C) ____________________________ F) ____________________________

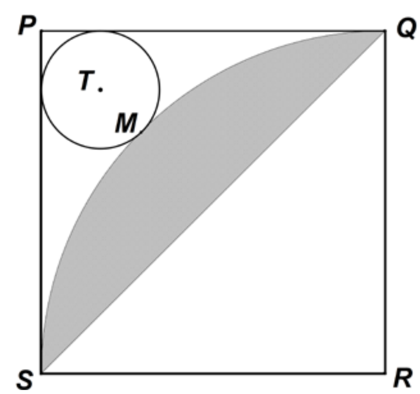
A) Given: $f(x)=\dfrac{Ax^3+Bx^2-6x+3}{4x^2-1}$, $f(5)=27$, $f(-1)=-1$
It has a linear asymptote $y=mx+b$ as $x\to\pm\infty$.
Compute the ordered pair (m, b).

B) Let N be a 4-digit integer consisting exclusively of prime base 10 digits, but not necessarily distinct. How many of these integers are divisible by 11?

C) Compute x such that $Arc\cos\left(\dfrac{25}{32}\right)+Arc\cos\left(x\right)=Arc\cos\left(\dfrac{1}{20}\right)$.

D) Suppose a sheet of first-class FOREVER stamps costs \$6.72 in 2021.
Suppose that due to a 4¢ increase in the rate, a sheet of FOREVER stamps with 8 more stamps cost \$12.00 in 2022. No sheet ever contains more than 50 stamps.
If a FOREVER stamp costs 46¢ in 2013, <u>*how much more*</u> will a FOREVER stamp cost in 2022?

E) $PQRS$ is a square, $PQ = 6$.
$\overset{\frown}{QMS}$ is an arc of a circle with center at R and radius RQ.
Circle T is tangent to 2 sides of the square and to the arc at point M. The ratio of the area of circle T to the area of the segment on $\overline{SQ}$ (i.e. the shaded region) may be expressed as $\dfrac{A\pi}{\pi-B}$, where B is an integer. Compute the ordered pair (A, B).



F) Suppose a sequence is defined by the recursive relation
$a_n-2a_{n+1}=1$.
If the first five terms are all <u>positive integers</u>, compute the <u>minimum sum</u> of these five terms.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 ANSWERS

Round 1 Alg 2: Algebraic Functions

A) 5 | B) $\left(-6,-\frac{2}{3},\frac{5}{3}\right)$ | C) 10

Round 2 Arithmetic/ Number Theory

A) 10 | B) 49 | C) 7

Round 3 Trig Identities and/or Inverse Functions

A) $\frac{5\sqrt{6}}{12}$ | B) $\frac{4\sqrt{3}}{7}$ | C) $y=\left(3-x^{\frac{1}{2}}\right)^2$ or $9-2\sqrt{x}+x$

Round 4 Alg 1: Word Problems

A) 1.5 | B) 72 | C) 139

Round 5 Geometry: Circles

A) 164 | B) 5.25 | C) $72-18\pi$ or $18(4-\pi)$

Round 6 Alg 2: Sequences and Series

A) $-18i$ | B) 29 : 24 | C) $\sqrt{6},\ \frac{2\sqrt{15}}{5}$

Team Round

A) (5, 2)

B) 32

C) $\frac{53}{80}$

D) 14¢

E) $\left(68-48\sqrt{2},\ 2\right)$ or $\left(4\left(17-12\sqrt{2}\right),\ 2\right)$

F) 57

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 1

A) $f\left(\frac{8}{3}\right)=\sqrt{3\left(\frac{8}{3}\right)-4}=\sqrt{4}=2$.

Let $a=f^{-1}\left(\sqrt{5}\right)$. Then: $f(a)=\sqrt{5}\Leftrightarrow\sqrt{3a-4}=\sqrt{5}\Leftrightarrow 3a-4=5\Leftrightarrow a=3$.

Thus, $f^{-1}\left(\sqrt{5}\right)+f\left(\frac{8}{3}\right)=3+2=$ **<u>5</u>**.

B) $\begin{cases} f(x)=3x-2 \\ g(x)=(x-2)(x+3)+A, \text{ where } A<0 \end{cases} \Rightarrow g\circ f(x)=\left(3x-4\right)\left(3x+1\right)+A=9x^2-9x+(A-4)$

To find the zeros, we use the quadratic formula to solve $9x^2-9x+(A-4)=0$:

$x=\dfrac{9\pm\sqrt{81-36(A-4)}}{18}=\dfrac{9\pm\sqrt{225-36A}}{18}$

Examining the discriminant, we want the largest possible negative integer for which $225-36A$ is a perfect square. For $A=-1,\ldots-5$, we get non-perfect squares 261, 297, 333, 369, and 405; but $A=-6$ gives us $441=21^2$

Therefore, $x=\dfrac{9\pm 21}{18}\Rightarrow -\dfrac{12}{18},-\dfrac{30}{18}\Rightarrow -\dfrac{2}{3},\dfrac{5}{3}$

Since $r_1<r_2$, we have $\left(A,r_1,r_2\right)=$ <u>$\left(\mathbf{-6,-\frac{2}{3},\frac{5}{3}}\right)$</u>

The question alluded to several other values for which the composite function had rational zeros. Some other values of A are: –14, –24, –36, –50, –66, …
Do you see a pattern? Investigate. Can you prove a conjecture?

C) $\begin{cases} f(x)=x^4+ax^3+bx^2+cx+d \\ f(1)=f(2)=f(3)=f(4)=6 \end{cases}\Rightarrow$

We do <u>not</u> have to determine the actual zeros, since the coefficient a is the opposite of the sum of the zeros.

Subtracting, $\begin{cases} 175+37a+7b+c=0 \\ 65+19a+5b+c=0 \\ 15+7a+3b+c=0 \end{cases}\Rightarrow\begin{cases} 110+18a+2b=0 \\ 50+12a+2b=0 \end{cases}\Rightarrow a=-10$

Thus, the sum of the roots is **<u>10</u>**.

<u>FYI</u>:
Backtracking with a = –60 in the remaining equations,
$f(x)=x^4-10x^3+35x^2-50x+30$.

Alternative: Let $f(x)=(x-1)(x-2)(x-3)(x-4)+6$ and the given requirements are satisfied and the sum of the zeros is the opposite of the coefficient of x^3, i.e. **<u>+10</u>**.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 2

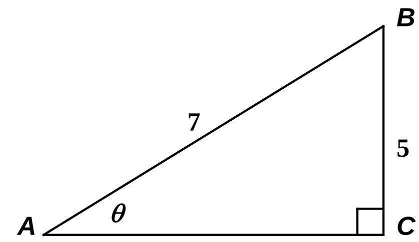
A) Positive integers leaving a remainder of 1 when divided by 4 are of the form $N = 4k + 1$, for $k = 3, 4, 5, \ldots$ Substituting $\Rightarrow (k, N) = (3, 13), (4, 17), (7, 29), (9, 37), (10, 41), (14, 53), (15, 61), (18, 73), (22, 89)$ and $(24, 97) \Rightarrow$ **<u>10</u>** two-digit primes

B) The only numbers with an odd number of divisors are perfect squares.
$2^2 = 4$ (4, 5) fails, since 5 is prime
$3^2 = 9$ (9, 10) fails, since 10 has 4 divisors
$5^2 = 25$ (25, 26) fails, since 26 has 4 divisors
$7^2 = 49$, $50 = 2 \cdot 5^2 \Rightarrow$ 6 factors
Thus, $N =$ **<u>49</u>**.

C) $2^{30} - 2^{16} + 1 = 1 \cdot 2^{30} - 2 \cdot 2^{15} + 1 = \left(2^{15} - 1\right)^2$
$2^{15} = 2^{10} \cdot 2^5 = 1024(32) = 32768$
We require the smallest prime factor of 32767.
Obviously 2, 3 and 5 fail, but 32767 = 7(4681).
4681 may not be prime, but it does not have a smaller factor than **<u>7</u>**.
FYI: In fact, 4681 is composite, since (31)(151) = 4681.
Also, you may have noticed that $2^{15} - 1 = \left(2^3 - 1\right)\left(2^{12} + 2^9 + 2^6 + 2^3 + 1\right)$ and the same result follows. Verify that $A^5 - 1 = (A - 1)\left(A^4 + A^3 + A^2 + A + 1\right)$ and substitute $A = 2^3$.
[In the world of computers, it's very useful to remember (<u>memorize</u>) the fact that **1 K (kilobyte) = 2^{10} = 1024 bytes**. K means thousand and 2^{10} is the closest power of 2.]
1 megabyte (Mb) = 2^{20} (1,048,576 bytes). M means million and 2^{20} is the closest power of 2.
1 gigabyte (Gb) = 2^{30}(1,073,741,824 bytes). G means billion and 2^{30} is the closest power of 2.
1 terabyte (Tb) = 2^{40} (1,099,511,627,776 bytes). T means trillion and 2^{40} is the closest power of 2.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 3

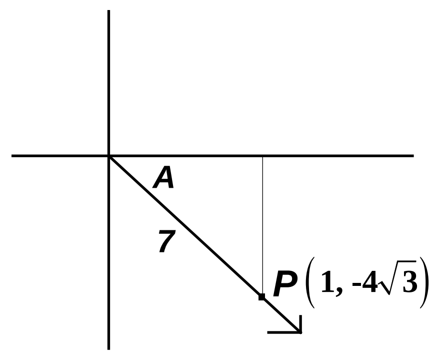


A) Let $\theta = Arc\sin\left(\frac{5}{7}\right)$. This is equivalent to $\sin\theta = \frac{5}{7}$ <u>and</u>

$0 < \theta < \frac{\pi}{2}$ (quadrant 1), since the argument was positive.

$AC = \sqrt{49-25} = 2\sqrt{6}$

Thus, $\tan\theta = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12}$ or, equivalently, $\theta = Arc\tan\left(\frac{5\sqrt{6}}{12}\right)$ and $k = \underline{\mathbf{\frac{5\sqrt{6}}{12}}}$.



B) The range of $y = Arc\tan(x)$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Since A is in quadrant 4 $\left(\tan A = -4\sqrt{3}\right)$, $\pi + A$ must be in quadrant 2.

The sine in quadrant 2 is positive $\Rightarrow$ $\underline{+\mathbf{\frac{4\sqrt{3}}{7}}}$.

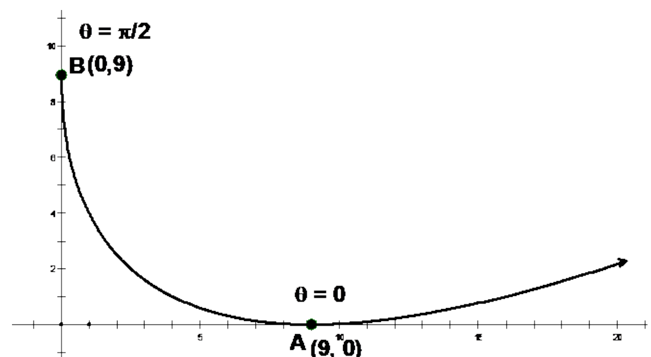
C) Since $\begin{cases} x = 9\cos^4\theta \\ y = 9\sin^4\theta \end{cases}$, it is clear that both x and y must be nonnegative.

For real numbers x and y, $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt{y} = y^{\frac{1}{2}}$ denote nonnegative numbers.
Taking the square root and summing, we have

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = 3\cos^2\theta + 3\sin^2\theta = 3\left(\cos^2\theta + \sin^2\theta\right) = 3\cdot 1 = 3$$

$$\Rightarrow y^{\frac{1}{2}} = 3 - x^{\frac{1}{2}} \Rightarrow y = \underline{\left(\mathbf{3 - x^{\frac{1}{2}}}\right)^{\mathbf{2}}} = \underline{\mathbf{9 - 2\sqrt{x} + x}}$$

What does the graph of this function look like?
Compare your graph with the graph is at the end of this solution key.
Carefully consider the accompanying commentary.



The tail to the right of point A is extraneous for our pair of parametric equations, since the value of x could not be greater than 9.

$\begin{cases} x = 9\cos^4\theta \\ y = 9\sin^4\theta \end{cases}$, where $0 \le \theta < 2\pi$ and $y = \left(3 - x^{\frac{1}{2}}\right)^2$

are equivalent only over $0 \le x \le 9$.

For a real-valued function $y = \left(3 - x^{\frac{1}{2}}\right)^2$,

the only restriction on the domain is $x \ge 0$ and the graph would continue to the right.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 4

A) If 85% is left, then 15% was removed. Thus, $N^2 = 0.15(15) = \frac{15^2}{100} \Rightarrow N = \underline{\mathbf{1.5}}$.

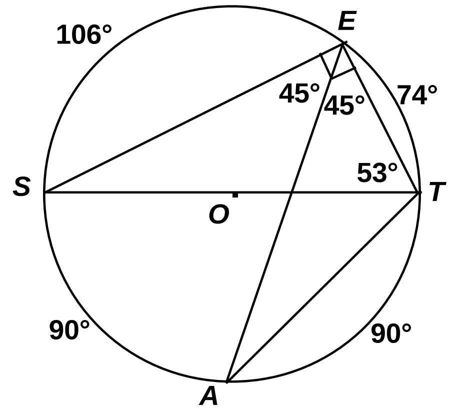
B) Let (R, G) = (5x, 3x) initially. Then: 8x = 504 ⇒ x = 63 and there are 315 red jelly beans and 189 green jelly beans at the outset. We require that
$\frac{315-2x}{189-x} = \frac{3}{5} \Rightarrow 1575-10x = 567-3x \Rightarrow 7x = 1008 \Rightarrow x = 144$
Thus, there are 315 – 288 = 27 red jelly beans and 189 – 144 = 45 green jelly beans, for a total of **<u>72</u>** jelly beans.

C) Let B and M denote votes for Boris and Miles respectively.
$M-1$ must be divisible by 21 $\Rightarrow (M, B) = \cancel{(22, 25)}, \cancel{(43, 50)}, (64, 75) \Rightarrow \underline{\mathbf{139}}$ votes cast.

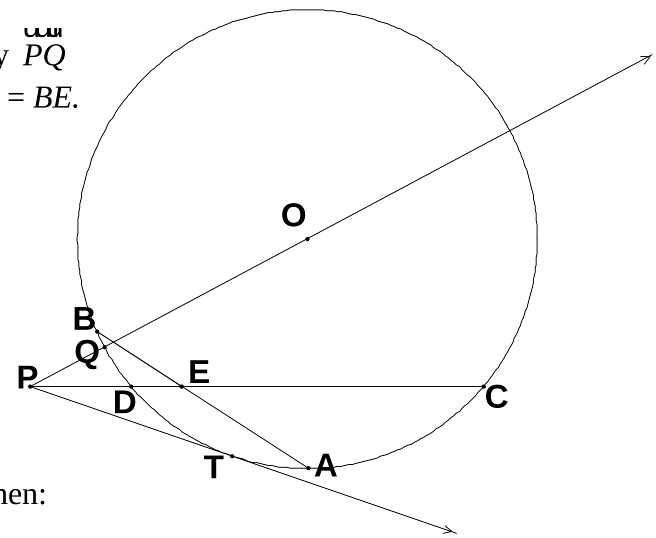
Check: This is the first total that exceeds 100. 8% of 75 votes is 6 votes. If Boris loses 6 votes then, instead of a 75 to 64 victory for Boris, the results become a 70 to 69 victory for Miles, a one vote differential as required.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 5



A) Since $\angle SET$ is inscribed in a semi-circle, it's a right angle. Thus, $m\angle SEA = m\angle TEA = 45°$. As intercepted arcs of $\angle SEA$ and $\angle ETS$, $m\overset{\frown}{SA} = 90°, m\overset{\frown}{SE} = 106° \Rightarrow m\overset{\frown}{ET} = 74°$
$\Rightarrow \overset{\frown}{ETA} = 90 + 74 = \underline{\mathbf{164}}$.



B) Let R denote the second point of intersection of ray $\overrightarrow{PQ}$ and circle O and recall that we were given that $PD = BE$.
Applying the product chord theorem,
$AE \cdot BE = CE \cdot DE \Rightarrow 3AE = 1 \cdot 6 \Rightarrow BE = PD = 2$
Applying the tangent-secant rule,
$PD \cdot PC = PT^2 \Rightarrow PT^2 = 2 \cdot (2+7) = 18$ and
$PQ \cdot PR = 18 \Rightarrow 1.5(1.5 + 2r) = 18$
$\Rightarrow 2r + 1.5 = 12 \Rightarrow r = \underline{\mathbf{5.25}}$

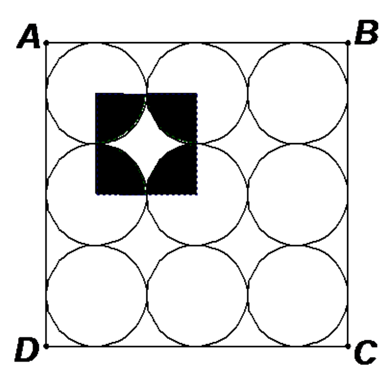
C) Let D denote the diameter of one of the circles. Then:

$$3D = \sqrt{162} \Rightarrow 6R = 9\sqrt{2} \Rightarrow R = \frac{3\sqrt{2}}{2}$$

From the diagram at the right, we see that the "star-shaped" region between the circles has an area equal to a square minus a full circle,

i.e. $\left(3\sqrt{2}\right)^2 - \pi\left(\dfrac{3\sqrt{2}}{2}\right)^2 = 18 - \dfrac{9\pi}{2}$

Multiplying by 4, the required area is $\underline{\mathbf{72 - 18\pi}}$ or $\underline{\mathbf{18(4-\pi)}}$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 – FEBRUARY 2013 SOLUTION KEY

Round 6

A) The second term of the geometric progression can be written as $-3i$.

The common ratio r for the geometric progression is $\frac{1}{i} = -i$.

Therefore, the progression is $3, -3i, -3, 3i, \boxed{3}, \ldots$ consists of a repetition of four terms.

If n is 1 more than a multiple of 4, i.e. n = 1, 5, 9, …, $t_n = 3$.

Since 10 is 2 more than a multiple of 4, $t_{10} = \frac{3}{i} = -3i$.

Since 13 is 1 more than a multiple of 4, $t_{13} = 3$

$\Rightarrow (-3i+3)^2 = -9-18i+9 = \underline{\boldsymbol{-18i}}$ (Also accept $0-18i$.) Do **not** accept $\frac{18}{i}$.

B) (1) $y^2 = 3x$

(2) $x + 1 - y - 2 = y + 2 - 3 \Rightarrow x = 2y$

Thus, $y^2 = 6y \Leftrightarrow y(y-6) = 0$. Since $y \neq 0$, $(x, y) = (12, 6)$

$\Rightarrow$ in the arithmetic progression: $t_{12} = 3 + 11(5) = 58$

$\Rightarrow$ in the geometric progression: $t_5 = 3(2^4) = 48$

The required ratio is **29: 24**

C) $\text{AM} = \text{HM} + 0.1 \Rightarrow \frac{A+2}{2} = \frac{4A}{A+2} + \frac{1}{10}$ Multiplying through by 10(A + 2), we have

$5(A+2)^2 = 40A + (A+2) \Rightarrow 5A^2 - 21A + 18 = (5A-6)(A-3) = 0$

Therefore, $A = 3, \frac{6}{5}$.

$A = 3 \Rightarrow GM = \underline{\boldsymbol{\sqrt{6}}}$

$A = 1.2 \Rightarrow GM = \sqrt{2 \cdot \frac{6}{5}} = \sqrt{\frac{3 \cdot 4 \cdot 5}{25}} = \underline{\boldsymbol{\frac{2\sqrt{15}}{5}}}$ (or $\frac{2}{5}\sqrt{15}$)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013 SOLUTION KEY

Team Round

A) $$\begin{cases} f(-1) = -1 \Rightarrow \dfrac{-A+B+9}{3} = -1 \Rightarrow A - B = 12 \\ f(5) = 27 \;\; \Rightarrow \dfrac{125A + 25B - 27}{99} = 27 \Rightarrow 125A + 25B = 99(27) + 27 = 27(99+1) = 2700 \\ \qquad\qquad\quad \Rightarrow 5A + B = 108 \end{cases}$$

Thus, (A, B) = (20, 8). By long division, $\dfrac{20x^3 + 8x^2 - 6x + 3}{4x^2 - 1} = 5x + 2 + \dfrac{-x+5}{4x^2 - 1}$.
As $x \to \infty$, the value of the fractional third term approaches zero, since the degree of the denominator is 1 more than the degree of the numerator implying the denominator grows much faster than the numerator.
Therefore, the functional values become closer and closer to $y = 5x + 2$ and (m, b) = **<u>(5, 2)</u>**.
Is the graph of the function above or below the line as $x \to \infty$? For other values of x?
Think about it and then look at the graph at the end of this solution key.

B) The test: A 4-digit integer <u>*a*</u> <u>*b*</u> <u>*c*</u> <u>*d*</u> is divisible by 11 *<u>if and only if</u>* $(a + c) - (b + d)$ is divisible by 11, that is, equal to either 0 or 11. Sum digits in even positions, sum digits in odd positions and then subtract.

The prime digits are 2, 3, 5 and 7.
Case 1: (all digits the same)
All <u>4</u> possibilities have $(a + c) - (b + d) = 0$ and are divisible by 11.

Case 2: (3 digits same) - 4·3 = 12 digit selections ⇒ x 4 arrangements ⇒ 48 possible N-values
For <u>any</u> integer of this form, for example, <u>*x*</u> <u>*x*</u> <u>*x*</u> <u>*y*</u>, $(a + c) - (b + d) = |x - y|$.
The minimum and maximum differences are 1 and 5 respectively.
<u>None</u> of these N-values are divisible by 11.

Case 3: (2 digits same, 2 different) - 6 digit selections ⇒ x 12 ⇒72 possible N-values
For 2235, 2237, 2257 min = 2, 4, 2 (sum of largest and smallest minus sum of other two)
max = 4, 6, 8 (sum largest two – sum smallest two) ⇒ <u>None</u>
For 3325, 3327, 3357 min = 1, 1, 2 max = 3, 5, 6 ⇒ <u>None</u>
For 5523, 5527, 5537 min = 1, 1,**<u>0</u>** max = 5, 5, 4 ⇒ <u>4</u> (7535, 3575, 5357, 5753)
For 7723, 7725, 7735 min = 1, 3, 2 max = 9, 7, 6 ⇒ <u>None</u>

Case 4: (2 pairs of digits the same) - 6 digit selections ⇒ x 6 ⇒ 36 possible N-values
Consider one of the 6 selections, e.g., <u>2</u> <u>2</u> <u>3</u> <u>3</u>. The required difference is either 6 – 4 = 2 or 5 – 5 = 0. There are 4 arrangements giving a difference of 0, namely <u>2</u> <u>3</u> <u>3</u> <u>2</u>, <u>3</u> <u>2</u> <u>2</u> <u>3</u>, <u>2</u> <u>2</u> <u>3</u> <u>3</u>, <u>3</u> <u>3</u> <u>2</u> <u>2</u>. This is true for each of the 6 selections. Thus, there are <u>24</u> possible N-values.

Case 5: (all digits different) - 4! = 24 possibilities
Using all distinct digits 2, 3, 5 and 7, the minimum difference 9 – 8 = 1 and the maximum difference is 12 – 5 = 7. Therefore, <u>none</u> of these are possible N-values.
Thus, the total is 0 + 4 + 4 + 24 + 0 = **<u>32</u>**.

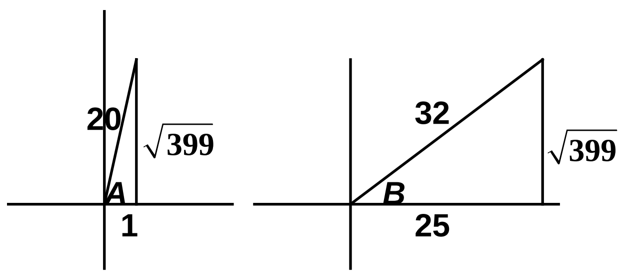
MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 5 - FEBRUARY 2013 SOLUTION KEY

Team Round - continued

C) Rewrite $Arc\cos\left(\frac{25}{32}\right) + Arc\cos(x) = Arc\cos\left(\frac{1}{20}\right)$ as

$Arc\cos(x) = Arc\cos\left(\frac{1}{20}\right) - Arc\cos\left(\frac{25}{32}\right)$ and let $A =$ $Arc\cos\left(\frac{1}{20}\right)$ and $B =$ $Arc\cos\left(\frac{25}{32}\right)$. Then, taking the cosine of both sides,

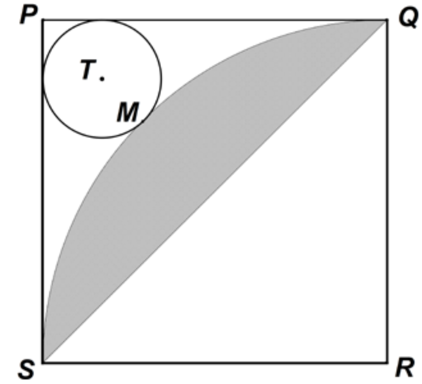
$x = \cos(A - B) =$ $\cos A \cos B + \sin A \sin B = \frac{1}{20} \cdot \frac{25}{32} + \frac{\sqrt{399}}{20} \cdot \frac{\sqrt{399}}{32} = \frac{424}{20 \cdot 32} =$ $\underline{\mathbf{\frac{53}{80}}}$



D) Suppose in 2021 there are n stamps on the sheet costing c cents each.
In 2022, the sheet consists of $(n + 8)$ stamps costing $(c + 4)$ cents each.
Then: $\begin{cases} nc = 672 \\ (n+8)(c+4) = 1200 \end{cases} \Rightarrow 4n + 8c = 1200 - 32 - nc = 1200 - 704 = 496$ or $n = 124 - 2c$

$(124 - 2c)c = 672 \Rightarrow 2c^2 - 124c + 672 = 2\left(c^2 - 62c + 336\right) = 2(c - 6)(c - 56) = 0 \Rightarrow c = \cancel{6}, 56$

Thus, in 2021, there were 12 stamps on the sheet, costing 56¢ each.
(112 stamps at 6¢ is rejected, since 112 > 50.)
Thus, the cost of a FOREVER stamp in 2022 is 60¢, <u>**14**</u>¢ more than in 2013.



Team Round - continued

E) The shaded region plus the region bounded by ΔSQR is a quarter circle with area $\frac{\pi(6)^2}{4} = 9\pi$. Thus, the area of the shaded region is $9\pi - \frac{1}{2}\cdot 6 \cdot 6 = 9(\pi - 2)$.

Draw diagonal $\overline{PR}$, intersecting $\overline{SQ}$ at point O.

$PR = 6\sqrt{2} \Rightarrow PO = 3\sqrt{2}$.

Since points T and M lie on $\overline{PR}$, $PT + TM + MO = 3\sqrt{2}$ (***).

If r denotes the radius of circle T, $PT = r\sqrt{2}$ (since P, T and the points of tangency on the square form a small square with side r). Therefore, (***) is equivalent to:

$r\sqrt{2} + r + \left(6 - 3\sqrt{2}\right) = 3\sqrt{2}$ $\quad (MO = RM - RO)$

$\Leftrightarrow r\left(\sqrt{2}+1\right) = 6\left(\sqrt{2}-1\right)$

$\Leftrightarrow r = \dfrac{6\left(\sqrt{2}-1\right)}{\sqrt{2}+1} = 6\left(\sqrt{2}-1\right)^2 = 6\left(3-2\sqrt{2}\right)$

Finally, the required ratio is $\dfrac{\pi r^2}{9(\pi-2)} = \dfrac{\pi \cdot 36 \cdot \left(3-2\sqrt{2}\right)^2}{9(\pi-2)} = \dfrac{\pi\left(68-48\sqrt{2}\right)}{\pi - 2}$ and

(A, B) = $\underline{\mathbf{\left(68 - 48\sqrt{2},\ 2\right)}}$ or $\underline{\mathbf{\left(4\left(17-12\sqrt{2}\right),\ 2\right)}}$

$4\left(\sqrt{2}-1\right)^4$ is <u>**not**</u> an acceptable as the computed value of A.

Team Round - continued

F) $a_n - 2a_{n+1} = 1 \Rightarrow a_{n+1} = \dfrac{a_n - 1}{2}$

Using this form of the recursive definition, we can find expressions for the first five terms, all in terms of a_1.

$$\begin{cases} a_2 = \dfrac{a_1 - 1}{2} \\ a_3 = \dfrac{a_2 - 1}{2} = \dfrac{\frac{a_1 - 1}{2} - 1}{2} = \dfrac{a_1 - 3}{4} \\ a_4 = \dfrac{a_1 - 7}{8} \\ a_5 = \dfrac{a_1 - 15}{16} \end{cases}$$

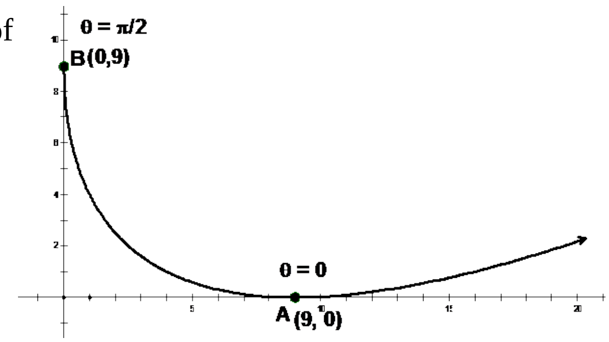
All of these expressions must generate integers. The smallest value of a_1 for which a_5 is an integer is 31, producing $a_5 = 1$

Substituting back up the chain, $(a_5,\ a_4,\ a_3,\ a_2,\ a_1) = (1,\ 3,\ 7,\ 15,\ 31)$ and the sum is **57**.

FYI:

The graph of 3C) connects points *A* and *B*.

The tail to the right of point *A* is extraneous for our pair of parametric equations, since the value of *x* could not be greater than 9.



$\begin{cases} x = 9\cos^4\theta \\ y = 9\sin^4\theta \end{cases}$, where $0 \le \theta < 2\pi$ and $y = \left(3 - x^{\frac{1}{2}}\right)^2$ are equivalent only over $0 \le x \le 9$

For a real-valued function $y = \left(3 - x^{\frac{1}{2}}\right)^2$, the only restriction on the domain is $x \ge 0$ and the graph would continue to the right.

Here's the graph of Team A)

The graph always lies above $y = 5x + 2$ for $x < -\frac{1}{2}$ or $x > \frac{1}{2}$ and below for *x*-values in-between these two critical values.. As $x \to +\infty$ (to the right) or $x \to -\infty$ (to the left), the graph is almost indistinguishable from the straight line. In fact, this is true for $x < -3$ and for $x > +2$. $x = \pm\frac{1}{2}$ and $y = 5x + 2$ are called asymptotes, lines which the function gets arbitrarily close to, but never actually makes contact!

Over $-\frac{1}{2} < x < \frac{1}{2}$, the graph reaches a maximum value over the interval $\left(0, \frac{1}{2}\right)$ and opens downward. Using calculus or a graphing calculator, over this interval, the maximum value of approximately –2.57 occurs at approximately $x = 0.135$.