

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 – JANUARY 2012
ROUND 1 ANALYTIC GEOMETRY: ANYTHING**

ANSWERS

A) _____

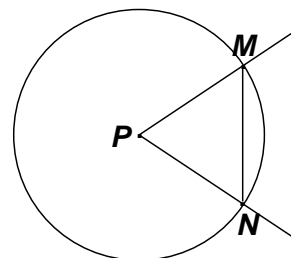
B) (_____ , _____)

C) (_____ , _____)

- A) The area of an ellipse is given by the formula πab , where a and b are the lengths of the semi-major and semi-minor axes, respectively.
Compute the absolute value of the difference between the areas of following conic sections:

$$\begin{aligned}x^2 + y^2 &= 36 \\ 4x^2 + 9y^2 &= 36\end{aligned}$$

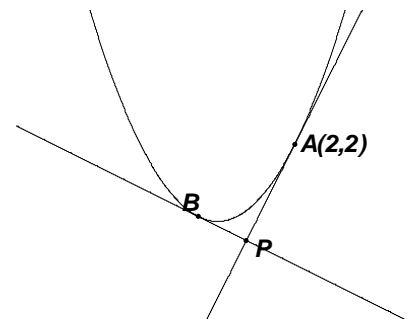
- B) Given: $C_1 = \{(x, y) \mid (x+1)^2 + (y-2)^2 = 64\}$
 $m\angle P = 60^\circ$, $M(a, b)$ and $N(a, c)$ lie on C_1 ,
where $a > 0$ and $b > c$.



Determine the ordered pair (a, b) .

- C) Lines are drawn tangent to the parabola $y = \frac{1}{2}x^2$. If $x = a$, the tangent line has equation

$ax - y = \frac{a^2}{2}$. The tangent line through $A(2, 2)$ and the tangent line at B are perpendicular and intersect at point P . Find the coordinates of point P .



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012
ROUND 2 ALG 1: FACTORING AND/OR EQUATIONS INVOLVING FACTORING

ANSWERS

A) _____

B) _____

C) _____

A) The product of two positive integers a and b is 210. Determine the minimum value of $a + b$.

B) If $x^3y - xy^3 = 91$ and $x^3y^2 - x^2y^3 = 105$, compute the numerical value of $\frac{1}{x} + \frac{1}{y}$.

C) The quadratic trinomial $4x^2 - 17x + A$ factors as $(ax - b)(cx - d)$, where A is a positive integer and a, b, c and d are also positive integers. Determine all possible values of A .

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012
ROUND 3 TRIG: EQUATIONS WITH A REASONABLE NUMBER OF SOLUTIONS**

ANSWERS

A) _____

B) _____

C) _____

***** All answers should be expressed in degrees over the specified intervals. *****

A) Solve for x over $-90^\circ < x < 0^\circ$. $\sin(3x) = -\frac{1}{2}$

B) Solve for x , where $0^\circ \leq x < 360^\circ$. $3(\cot x + \csc x) = 2 \sin x$

C) Solve for x over $0^\circ < x < 180^\circ$. $\cos(2x) - (1 - \sin^2 x) < -0.5$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012
ROUND 4 ALG 2: QUADRATIC EQUATIONS

ANSWERS

A) _____

B) _____

C) _____

A) A is a positive root of $x^2 - 3Ax + B = 0$. Compute A , if $B = 72$.

B) Solve for x .

$$\frac{x+2}{x-2} - \frac{x-3}{x+3} = \frac{7}{5}$$

C) Find the sum of all integral values of m for which the equation $9 + 2mx = 4x - x^2$ will have no real roots.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012
ROUND 5 GEOMETRY: SIMILARITY OF POLYGONS

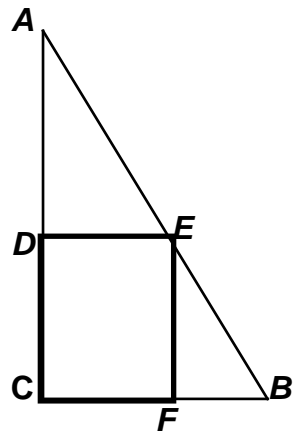
ANSWERS

A) _____

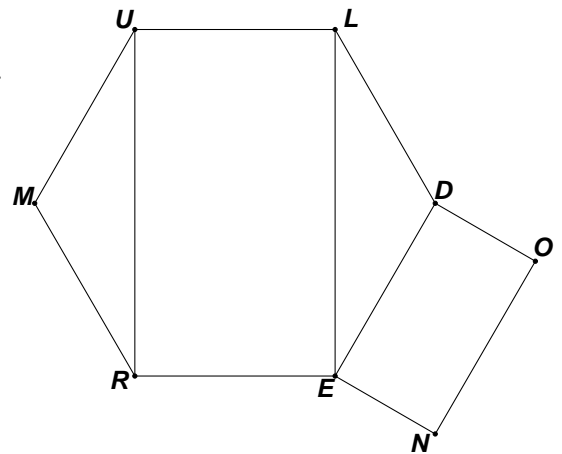
B) _____

C) _____

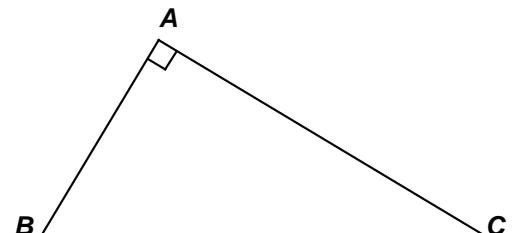
- A) $DEFC$ is a rectangle with vertices on the sides of right triangle ABC as indicated in the diagram at the right. If $(DE, DC, DA) = (2, 3, k)$, find a simplified expression for FB in terms of k .



- B) Given: Regular hexagon $MULDER$ with $MR = 2$.
 $DONE$ is a rectangle, where $DO < DE$.
 $DONE$ is similar to, but not congruent to, $RULE$.
 Compute DO .



- C) In $\triangle ABC$, $AB = 9$, $BC = 15$, $\overline{BA} \perp \overline{CA}$, D is the midpoint of \overline{AB} and E lies on \overline{AC} . The triangle with vertices A , D and E is similar to $\triangle ABC$, in some order. The length of \overline{AE} can be expressed as a simplified ratio of positive integers, $\frac{a}{b}$.
 Compute all possible sums $a + b$.



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012
ROUND 6 ALG 1: ANYTHING

ANSWERS

A) _____

B) (_____ , _____)

C) _____ seconds

A) 10W30 oil protects a car engine for air temperatures T which satisfy the inequality

$$|T - 55| \leq 50.$$

If T_{low} and T_{high} denote the low and high air temperatures for safe range of operation, compute $T_{\text{low}} + T_{\text{high}}$.

B) Carol starts at 1000 and counts down by 7s. Carol's list starts out 1000, 993, 986,...

Tarah starts at 100 and counts up by 13s. Tarah's list starts out 100, 113, 126, ...

Let C and T denote numbers in Carol's and Tarah's list respectively between 600 and 700.

Compute the ordered pair (C, T) , where $|C - T|$ is a minimum.

C) Two walkers, Chris and Christine, initially 12 miles apart, walk towards each other and meet in 1 hour and 20 minutes. Chris and Christine walk at constant, but different rates.

The faster walker walks twice as fast as the slower walker. Suppose they walk at these rates in the same direction, starting from the same point at the same time.

Compute the time (in seconds) when they are exactly $\frac{1}{4}$ mile apart.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012
ROUND 7 TEAM QUESTIONS**

ANSWERS

A) _____ D) _____

B) _____ E) _____

C) _____ F) \$ _____ . ____

A) Compute the slopes of both tangent lines to $x^2 + y^2 = 25$ from $P(7, -1)$.

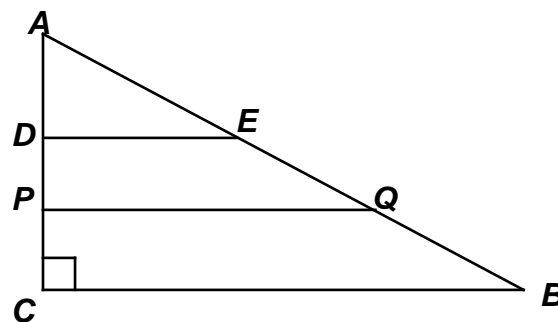
B) $\frac{x^4 + 3x^2 + k}{x^2 + 3}$ is an integer for exactly five positive integer values of x .
Determine the smallest possible positive integer value of k .

C) Solve for x over $0 \leq x < 360^\circ$: $\sin 4x - \cos 2x = 4 \sin x \cos x - 1$

D) For positive integer constants k, A and B , $x^2 - (k - 4)x + 6k - 2 = (x - A)^2 + B^2$ is an identity in x .
Find all possible ordered triples (k, A, B) .

E) Given: $\overline{AC} \perp \overline{BC}$, \overline{PQ} is a median of trapezoid $BCDE$
 $AC = 5$, $BC = 12$ and $AD = k$

Determine the range of values of k for which
the perimeter of $PQBC$ is less than or equal to 25.



F) In Maine, at one time, a customer could buy a Frisbee for the selling price of $\$X$, plus a 5% sales tax. However, due to inflation, the vendor raised his selling price by 25% and, adding insult to injury, Maine raised the sales tax to 8%. As a result, the cost to the customer for a Frisbee, including the tax, was an additional 48¢. Compute X (to the nearest cent).

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 ANSWERS**

Round 1 Analytic Geometry: Anything

- A) 30π B) $(4\sqrt{3}-1, 6)$ C) $\left(\frac{3}{4}, -\frac{1}{2}\right)$

Round 2 Alg1: Factoring

- A) 29 B) $\frac{13}{15}$ C) 4, 13, 15 and 18
(in any order)

Round 3 Trig: Equations

- A) $-10^\circ, -50^\circ$ B) $120^\circ, 240^\circ$ C) $45^\circ < x < 135^\circ$

Round 4 Alg 2: Quadratic Equations

- A) 6 B) $-\frac{6}{7}, 7$ C) 10

Round 5 Geometry: Similarity

- A) $\frac{6}{k}$ B) $\frac{2\sqrt{3}}{3}$ C) 7 and 35

Round 6 Alg 1: Anything

- A) 110 B) (685, 685) C) 300 seconds

Team Round

- A) $-\frac{4}{3}, \frac{3}{4}$ D) (30, 13, 3), (10, 3, 7), (22, 9, 7)
- B) 84 E) $\frac{10}{3} \leq k < 5$
- C) $0^\circ, 15^\circ, 75^\circ, 180^\circ, 195^\circ, 255^\circ$ F) \$1.60

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Round 1

- A) The circle has radius 6 and area 36π .

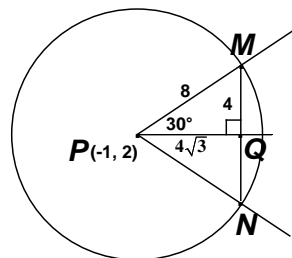
The ellipse $\left(\frac{x^2}{9} + \frac{y^2}{4} = 1\right)$ has $a = 3$ and $b = 2 \Rightarrow \text{area} = 6\pi$

Thus, the difference in areas is 30π .

- B) C_1 has its center at $(-1, 2)$ and a radius of 8.

Since \overline{MN} is vertical, a horizontal line through P will be the perpendicular bisector of \overline{MN} and the bisector of $\angle P$.

Thus, we have a 30 – 60 – 90 right triangles and $MQ = 4$,
 $PQ = 4\sqrt{3}$ and the coordinates of M must be $(4\sqrt{3} - 1, 6)$.



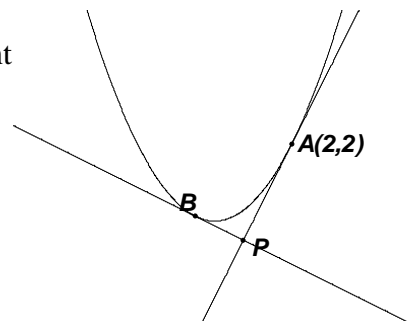
- C) The tangent through $(2, 2)$ has the equation $2x - y = 2$.

If the coordinates of B are $\left(b, \frac{1}{2}b^2\right)$, then the equation of the tangent

through B is $bx - y = \frac{b^2}{2}$. This line has slope b and is

perpendicular to $2x - y = 2$, so $b = -1/2$.

The coordinates of B are then $\left(-\frac{1}{2}, \frac{1}{8}\right)$.



This second line then has equation $-\frac{1}{2}x - y = \frac{\left(-\frac{1}{2}\right)^2}{2}$ or $x + 2y = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

and both equations simplify to $4x + 8y = -1$.

Solving $\begin{cases} 2x - y = 2 \\ 4x + 8y = -1 \end{cases}$, $x = \frac{3}{4}$, $y = -\frac{1}{2}$ and $P\left(\frac{3}{4}, -\frac{1}{2}\right)$.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Round 2

- A) Note: $210 = 210(1)$ (sum: 211), $105(2)$ (sum: 107), $3(70)$ (sum: 73) The value of the sum $a + b$ decreases as the difference between the factors decreases. Thus, we are looking for the pair of factors of 210 with the minimum difference. The prime factorization of 210 is $2 \cdot 3 \cdot 5 \cdot 7$. Pairing the outer and the inner factors, we have 14 and 15. Clearly, 1 is the minimum possible difference (unless 210 were a perfect square which it is not). Thus, the minimum value of $a + b = \underline{29}$.

$$\text{B) } \frac{x^3y - xy^3}{x^3y^2 - x^2y^3} = \frac{xy(x^2 - y^2)}{x^2y^2(x - y)} = \frac{xy(x + y)\cancel{(x - y)}}{x^2y^2\cancel{(x - y)}} = \frac{x + y}{xy} = \frac{1}{y} + \frac{1}{x} = \frac{91}{105} = \frac{\underline{13}}{\underline{15}}$$

- C) Certainly, $(x - 4)(4x - 1)$ gives the proper lead coefficient and coefficient of the middle term, so $A = 4$ works. But there are many possible factorizations to examine. How do we approach the search systematically? How can we limit the search? First, notice that the lead coefficient either factors as $4 \cdot 1$ or $2 \cdot 2$. Since filling in the blanks in $(2x + \underline{\hspace{1cm}})(2x - \underline{\hspace{1cm}})$ with integers would always produce an even coefficient in the middle term, our search is limited to products of the form $(4x - \underline{\hspace{1cm}})(x - \underline{\hspace{1cm}})$. One of the solutions will always be a positive integer. Our first factorization has established an upper limit for integer solutions, since $(x - 5)$ requires a second factor of $(4x + 3)$ and $d = -3$ violates the condition of positive constants.
- Thus, we try products $(x - 1)(4x - \underline{\hspace{1cm}})$, $(x - 2)(4x - \underline{\hspace{1cm}})$ and $(x - 3)(4x - \underline{\hspace{1cm}})$. FOILING out these products, each can produce a middle coefficient of 17, since the outer product simply makes up the difference. The fill-ins are 13, 9 and 5 respectively. Therefore, there are 4 possibilities for the product A , namely $A = \underline{13}, \underline{18}$ and $\underline{15}$, plus $A = \underline{4}$ that we had found originally (Answers allowed in any order).

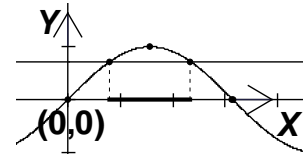
**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Round 3

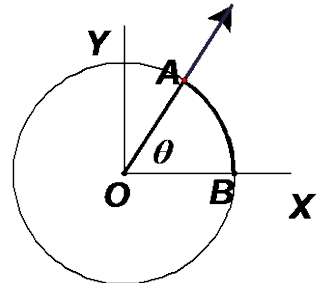
A) $3x = 330^\circ + 360n$ or $210^\circ + 360n \Rightarrow x = 110^\circ + 120n$ or $70^\circ + 120n$
 $n = -1 \Rightarrow x = \underline{-10^\circ, -50^\circ}$

B) $3(\cot x + \csc x) = 3\left(\frac{\cos x + 1}{\sin x}\right) = 2\sin x \quad (x \neq 0^\circ + 180n)$
 $\Rightarrow 3\cos x + 3 = 2\sin^2 x = 2 - 2\cos^2 x$
 $\Rightarrow 2\cos^2 x + 3\cos x + 1 = (2\cos x + 1)(\cos x + 1) = 0$
 $\Rightarrow \cos x = -\frac{1}{2}, -1 \Rightarrow \underline{120^\circ, 240^\circ}$, (180° is extraneous.)

C) $(\cos^2 x - \sin^2 x) - 1 + \sin^2 x < -0.5 \Rightarrow \cos^2 x - 1 < -0.5 \Rightarrow \sin^2 x > 0.5$
 Since $\sin x$ is always positive over the specified interval,
 we have $\sin x > \frac{\sqrt{2}}{2}$ and we know that $\sin 45^\circ = \frac{\sqrt{2}}{2}$.
 Appealing to the graph of $y = \sin x$, we have $45^\circ < x < 135^\circ$.



[In the graph above, x is measured in radians, so the first point of intersection occurs between 0 and 1 $\left(45^\circ = \frac{\pi}{4}^{rad} \approx 0.785 \text{ radians}\right)$, the second point of intersection occurs between 2 and 3 $\left(135^\circ = \frac{3\pi}{4}^{rad} \approx 2.356 \text{ radians}\right)$ and the function intersects the x -axis at $(180^\circ = \pi^{rad} \approx 3.142 \text{ radians})$.]



For those unfamiliar with radian measure, you may want to discuss the following with your coach or a teammate.

$m\angle AOB = \theta = 1$ radian if and only if the radius of the circle (\overline{OA}) has the same length as the intercepted (minor) arc (\widehat{AB}). Imagine a series of concentric circles, with center at O . In all cases, the intercepted arc has the same length as the radius and the central angle measures 1 radian! So to keep things simple, consider the unit circle (radius 1).

The circumference of the unit circle is 2π .

Thus, 2π copies of central angle AOB complete 1 revolution or 360° .

Dividing by 2, we have the equivalence $\boxed{180^\circ = \pi^{rad}}$ or $\boxed{1^{rad} = \frac{180}{\pi}^\circ}$ ($\approx 57.296^\circ \approx 57^\circ 17' 44.8''$).

Radian measure of angles allows us to use the same scale on both axes in the graph of $y = \sin x$ above!

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Round 4

A) Given: $x^2 - 3Ax + B = 0$ and A is a positive root.

Since the coefficient of x is $-3A$, the roots must be A and $2A$, to make a sum of $3A$.

Since B must be the product of the roots and A is positive, we have $2A^2 = 72 \Rightarrow A = \underline{6}$.

Alternate Solution: (Brute Force)

Over integers, 72 factors as either $8 \cdot 9$, $6 \cdot 12$, $3 \cdot 24$, $2 \cdot 36$, or $1 \cdot 72$

Considering the possible factorizations and the coefficients of the middle term, we have

$3A = 17, 18, 27, 38$ and 73 . Only $3A = 18$ produces a value of A which is also a root.

B) $5[(x+2)(x+3) - (x-2)(x-3)] = 7(x-2)(x+3)$

$$5\left[(x^2 + 5x + 6) - (x^2 - 5x + 6)\right] = 7(x^2 + x - 6)$$

$$5(10x) = 7x^2 + 7x - 42$$

$$7x^2 - 43x - 42 = (7x + 6)(x - 7) = 0 \Rightarrow x = \underline{-\frac{6}{7}, 7}$$

C) $9 + 2mx = 4x - x^2 \Leftrightarrow x^2 + (2m - 4)x + 9 = 0$

Using the quadratic formula, no real roots \Leftrightarrow a negative discriminant

Therefore,

$$(2m - 4)^2 - 36 < 0 \Leftrightarrow [2(m - 2)]^2 - 36 < 0 \Leftrightarrow (m - 2)^2 - 9 < 0 \Leftrightarrow m^2 - 4m - 5 < 0 \Leftrightarrow (m - 5)(m + 1) < 0$$

The critical values are $+5$ and -1 and the product is negative in between.

Thus, the integer values are $0, 1, 2, 3$ and 4 , resulting in a sum of **10**.

Note: $m = -1 \Rightarrow x^2 - 6x + 9 = (x - 3)^2 = 0$ which has one real root of $+3$.

$m = 5 \Rightarrow x^2 + 6x + 9 = (x + 3)^2 = 0$ which has one real root of -3 .

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Round 5

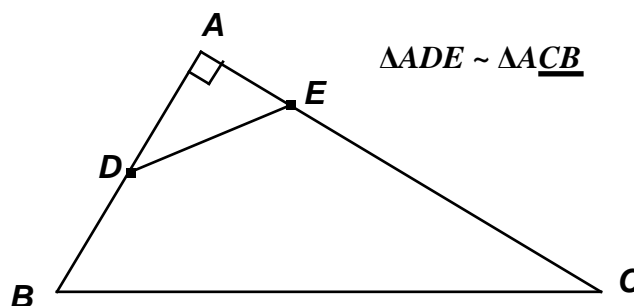
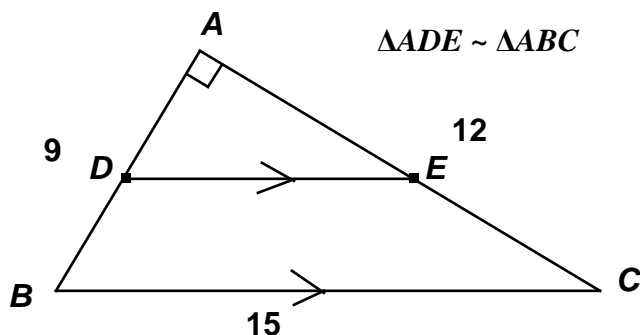
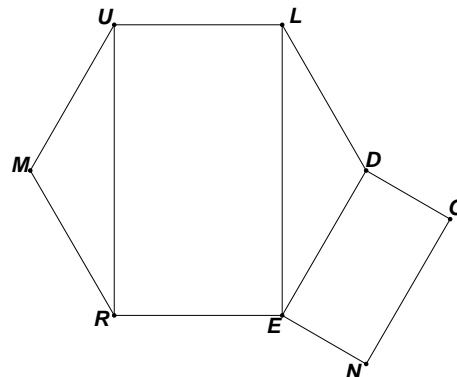
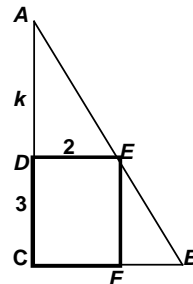
A) $\triangle ADE \sim \triangle EFB \Rightarrow \frac{AD}{DE} = \frac{EF}{FB} \Rightarrow \frac{k}{2} = \frac{3}{FB} \Rightarrow FB = \frac{6}{k}$

B) Drop a perpendicular from M to \overline{UR} , intersecting \overline{UR} in point Q , forming two 30 – 60 – 90 right triangles.

$MR = 2 \Rightarrow QU = \sqrt{3}, UR = 2\sqrt{3}$

Rectangles whose lengths and widths are in the same ratio are similar.

Thus, $\frac{\text{short}}{\text{long}} = \frac{UL}{UR} = \frac{DO}{DE} \Rightarrow \frac{2}{2\sqrt{3}} = \frac{DO}{2} \Rightarrow DO = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$



C) Right angle at A and $(AB, BC) = (9, 15) \Rightarrow AC = 12$.

If $\overline{DE} \parallel \overline{BC}$ ($\triangle ADE \sim \triangle ABC$), then E is also a midpoint and $AE = 6 = \frac{6}{1} \Rightarrow a + b = \underline{7}$.

If $\triangle ADE \sim \triangle ACB$ (Note: D now corresponds to C , and E to B), then $\frac{AD}{AC} = \frac{AE}{AB} \Rightarrow$

$$\frac{4.5}{12} = \frac{AE}{9} \Rightarrow \frac{9}{24} = \frac{3}{8} = \frac{AE}{9} \Rightarrow AE = \frac{27}{8}$$

$\Rightarrow a + b = \underline{35}$.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Round 6

A) $|T - 55| \leq 50 \Rightarrow -50 \leq T - 55 \leq +50 \Rightarrow 5 \leq T \leq 105 \Rightarrow (T_{\text{low}}, T_{\text{high}}) = (5, 105) \Rightarrow \underline{\underline{110}}$

B) The numbers in Carol's list are generated by $1000 - 7k$.

The numbers in Tarah's list are generated by $100 + 13k$.

Let's see if the lists share any common numbers. For the same value of k , does

$$1000 - 7k = 100 + 13k ?$$

Since $900 = 20k$, for $k = 45$, the lists do share a common number.

Multiplying by 7 is easier than by 13, so we substitute in $100 - 7k$.

The common number is $1000 - 7(45) = 1000 - 315 = 685$.

Thus $(C, T) = (\underline{\underline{685}}, \underline{\underline{685}})$.

Note: Since the least common multiple of 7 and 13 is 91, other numbers common to both lists will be of the form $685 + 91n$, for any integer n . For example, $n = \pm 1$ results in 594 and 776.

$$594 = 100 + 13(38) = 1000 - 7(58) \text{ and } 776 = 100 + 13(52) = 1000 - 7(32)$$

The 39th number in Carol's list and the 59th number in Tarah's list is 594

The 53rd number in Carol's list and the 33rd number in Tarah's list is 776

Between 600 and 700, only 685 is common to both lists and $|C - T| = 0$, the minimum value for the absolute value of a difference.

C) Suppose the walkers rates are x and $2x$.

You may argue later whether the faster walker is Chris or Christine.

When travelling in the same direction, they are separating at $(2x - x) = x$ mph.

$$RT = D \Rightarrow x \text{ (in mph)} \cdot T \text{ (in hrs)} = \frac{1}{4} \text{ mi. or } T \text{ (in hrs)} = \frac{1}{4x}$$

$$x \cdot \frac{4}{3} + 2x \cdot \frac{4}{3} = 12 \Leftrightarrow 12x = 36 \Leftrightarrow x = 3. \text{ Thus, } T = \frac{1}{12} \text{ hr.} = 5 \text{ minutes} = \underline{\underline{300}} \text{ seconds.}$$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Team Round

A) Let m denote the slope. Then: $mx - y = 7m - (-1) = 7m + 1$ or $mx + (-1)y - (7m + 1) = 0$

Applying the point to distance formula $\frac{|Ah + Bk + C|}{\sqrt{A^2 + B^2}}$ (from $P(h, k)$ to $Ax + By + C = 0$), we

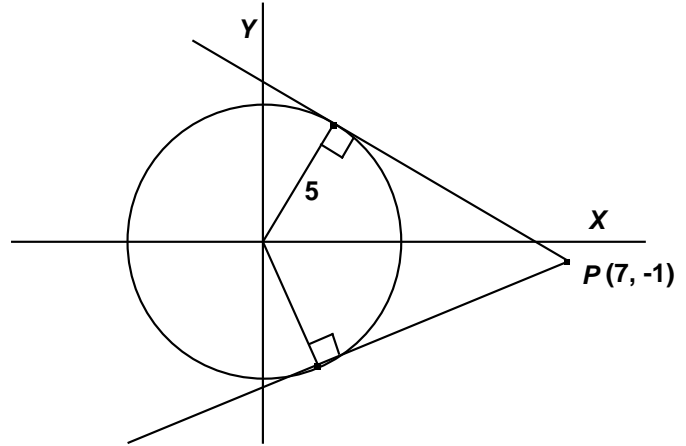
$$\text{have } \frac{|m(0) + (-1)0 - (7m + 1)|}{\sqrt{m^2 + 1}} = 5$$

$$(7m + 1)^2 = 25(m^2 + 1)$$

$$49m^2 + 14m + 1 = 25m^2 + 25$$

$$24m^2 + 14m - 24 = 2(3m + 4)(4m - 3) = 0$$

$$m = -\frac{4}{3}, \frac{3}{4}$$



B) Since $\frac{x^4 + 3x^2 + k}{x^2 + 3} = x^2 + \frac{k}{x^2 + 3}$, we see that $x^2 + 3$ must be a factor of k , if the original quotient is to be an integer.

Plan: List the values of $x^2 + 3$ for consecutive positive integer values of x until we find the smallest value in the list that is divisible by exactly four values preceding it.

$x = 1, 2, 3, \dots \Rightarrow$ divisors: 4, 7, 12, 19, 28, 39, 52, 67, 84, ...

Since $k = 84$ is divisible by each of the underlined values, $k = 84$ is the smallest value for which the given quotient has integral values for exactly five positive integer values of x , namely, 1, 2, 3, 5 and 9.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Team Round – continued

$$\begin{aligned} \text{C) } \sin 4x - \cos 2x &= 4\sin x \cos x - 1 \Leftrightarrow 2\sin 2x \cos 2x - \cos 2x = 2\sin 2x - 1 \\ &\Leftrightarrow \cos 2x(2\sin 2x - 1) = 2\sin 2x - 1 \\ &\Leftrightarrow (\cos 2x - 1)(2\sin 2x - 1) = 0 \\ &\Leftrightarrow \cos 2x = 1 \text{ or } \sin 2x = 1/2 \\ &\Leftrightarrow 2x = 0^\circ + n \cdot 360^\circ \text{ or } 2x = \begin{cases} 30^\circ + n \cdot 360^\circ \\ 150^\circ + n \cdot 360^\circ \end{cases} \end{aligned}$$

Dividing by 2 and letting $n = 0, 1 \Rightarrow x = \underline{0^\circ, 180^\circ, 15^\circ, 195^\circ, 75^\circ, 255^\circ}$.
(Answers in any order.)

$$\text{D) } \begin{cases} (1) \ k - 4 = 2A \\ (2) \ 6k - 2 = A^2 + B^2 \end{cases} \quad \text{Substituting } 2A + 4 \text{ for } k \text{ in (2), we have } 12A + 22 = A^2 + B^2.$$

$$\text{Completing the square, } (A^2 - 12A + \underline{36}) = (A - 6)^2 = 22 - B^2 + \underline{36} = 58 - B^2$$

For positive integer values of B ($1 \leq B \leq 7$), $58 - B^2$ must evaluate to a perfect square.

When $B = 3$, we have $(A - 6)^2 = 49 \Rightarrow A = 6 \pm 7 \Rightarrow A = 13$ (-1 is rejected).

When $B = 7$, we have $(A - 6)^2 = 9 \Rightarrow A = 6 \pm 3 \Rightarrow A = 3, 9$

For all other values of B , $58 - B^2$ is not a perfect square.

Thus, there are three solutions (30, 13, 3), (10, 3, 7) and (22, 9, 7).

(The ordered triples may be listed in any order.)

E) Since \overline{PQ} is a median of trapezoid $BCDE$, $x > 0$ and, consequently $k < 5$.

$$\text{Since } AC = 5, \ x = \frac{5-k}{2}$$

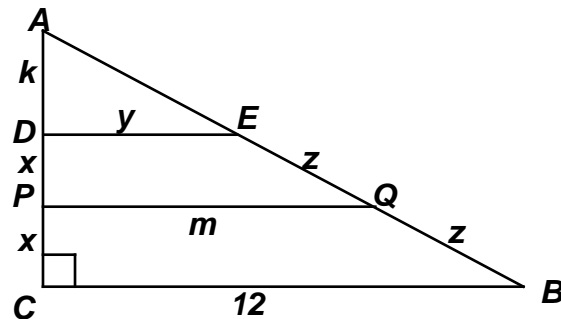
$$\triangle ADE \sim \triangle ACB \Rightarrow \frac{k}{5} = \frac{y}{12} = \frac{AE}{13} \Rightarrow \begin{cases} y = \frac{12k}{5} \\ AE = \frac{13k}{5} \end{cases}$$

$$\text{Therefore, } m = \frac{y+12}{2} = \frac{\frac{12k}{5}+12}{2} = \frac{12(k+5)}{10} \text{ and}$$

$$z = \frac{13 - \frac{13k}{5}}{2} = \frac{13(5-k)}{10}. \text{ The perimeter of } PQBC = x + m + z + 12 \leq 25 \Rightarrow$$

$$\frac{5-k}{2} + \frac{12(k+5)}{10} + \frac{13(5-k)}{10} \leq 13 \Rightarrow 25 - 5k + 12k + 60 + 65 - 13k \leq 130 \Rightarrow 20 \leq 6k \Rightarrow k \geq \frac{10}{3}$$

$$\text{Combining the restrictions, } \underline{\frac{10}{3} \leq k < 5}.$$



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 4 - JANUARY 2012 SOLUTION KEY**

Team Round – continued

F) Suppose the price on a Frisbee is \$X.

Adding a 5% sales tax, the original cost to the customer was $X + 0.05X = 1.05X$ (dollars).

The current cost is $1.08 \cdot 1.25X$ (dollars) – an 8% sales tax added to the increased selling price.

The difference is \$0.48.

$$1.08(1.25X) - 1.05X = 0.48$$

$$\Leftrightarrow X(108(1.25) - 105) = 48$$

$$\Leftrightarrow X\left(108\left(\frac{5}{4}\right) - 105\right) = 48$$

$$\Leftrightarrow X(135 - 105) = 48$$

Thus, $X = \frac{48}{30} = \frac{8}{5} = 1.6$ and the original selling price of the Frisbee was exactly **\$1.60**.