

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 – DECEMBER 2011
ROUND 1 TRIG: RIGHT ANGLE PROBLEMS, LAWS OF SINES AND COSINES

ANSWERS

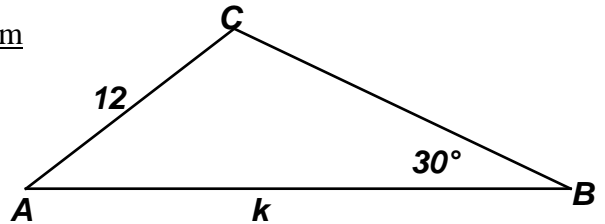
A) _____

B) _____

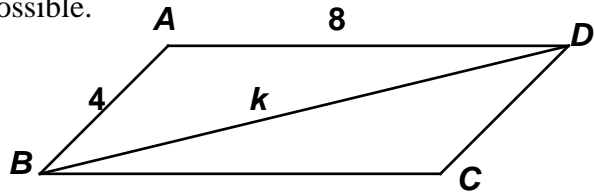
C) _____

- A) Let $\triangle ABC$ be a right triangle with legs of length 5 and 12.
If A is the right angle, compute $\cos B + \cos C$.

- B) In $\triangle ABC$, $\sin C$ is a rational number with a terminating decimal representation. Compute the sum of all possible positive integer values of k .



- C) In parallelogram $ABCD$, $BD = k$, where k is a positive integer.
Compute the positive difference between the value of $\cos A$, when k is as large as possible,
and the value of $\cos A$, when k is as small as possible.



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011
ROUND 2 ARITHMETIC/NUMBER THEORY**

ANSWERS

A) _____

B) _____

C) _____ base (-2)

A) The difference between two primes is 45. Compute the sum of these two primes.

B) The Gregorian calendar is now used virtually everywhere in the secular world.

A non-century year is a leap year (366 days) if and only if it is divisible by 4.

A century year is a leap year if and only if it is divisible by 400.

To the nearest integer, how many weeks (7 days) were there in the 400-year cycle from 1600 through 1999 inclusive, using the Gregorian calendar?

C) Consider integers written in “base (-2) ”, instead of the customary base (10) .

In base (-2) , suppose the allowable digits are only 0 and 1.

For integers, the place values are $(-2)^0$, $(-2)^1$, $(-2)^2$, etc., instead of $(10)^0$, $(10)^1$, $(10)^2$, etc.

Here are some examples:

In base (-2) , $5_{(10)}$ is expressed as $101_{(-2)}$ and $3_{(10)}$ is expressed as $11_{(-2)}$.

All base (-2) representations of base (10) integers are unique.

Express $10101_{(-2)} + 11010_{(-2)}$ in base (-2) .

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011
ROUND 3 COORDINATE GEOMETRY OF LINES AND CIRCLES**

ANSWERS

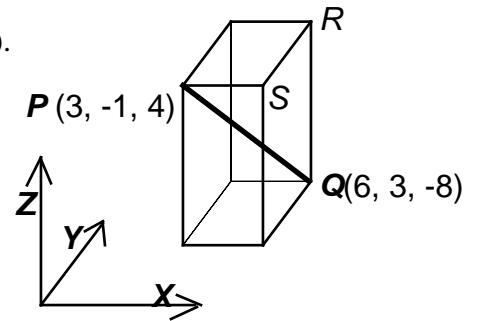
A) _____

B) _____

C) X: (_____ , _____) Y: (_____ , _____)

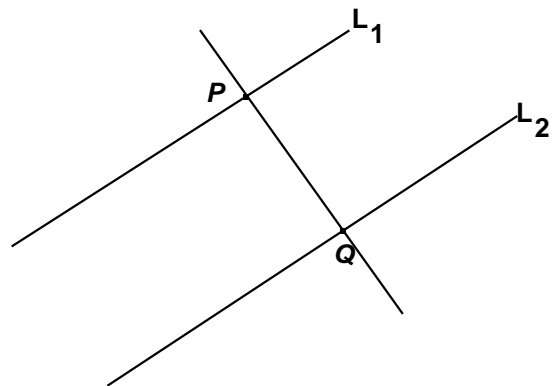
- A) A line segment in space has endpoints $P(3, -1, 4)$ and $Q(6, 3, -8)$.
Compute the length of the line segment.

Hint: Consider triangles PSR and PRQ in the box illustrated in the diagram at the right.



- B) Compute the length of a tangent from point $P(4, -3)$ to the circle $C_1: (x + 1)^2 + y^2 = 6$

- C) Given: $L_1 \parallel L_2$, $\overline{PQ} \perp L_1$, $P(7, 11)$, $PQ = 10$ and Q is located to the right of point P .
If $L_1 = \{(x, y) \mid 3x - 4y + 23 = 0\}$, compute the coordinates of the x - and y -intercepts of L_2 .



MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011
ROUND 4 ALG 2: LOG & EXPONENTIAL FUNCTIONS

ANSWERS

A) (_____ , _____)

B) _____

C) _____

A) Let $f(x) = 2^{3x+1}$ and $g(x) = 4^x$. These exponential functions intersect at $P(a, b)$.
Compute the ordered pair (a, b) .

B) Let P denote a point represented by the coordinates of an x -intercept of
 $y = f(x) = \log_2(8(2x-1)^2) - 5$.
Let Q denote a point represented by the coordinates of a y -intercept of this function.
Compute all possible distances PQ .

C) Given: $A = \log 2 + \log \frac{3}{2} + \log \frac{4}{3} + \dots + \log \frac{2000}{1999}$ and $\log_{1024} 10 = k$
Express A as a simplified ratio in terms of k .

Note: \log denotes common logs, i.e. \log_{10} .

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011
ROUND 5 ALG 1: RATIO, PROPORTION OR VARIATION

ANSWERS

A) (_____ , _____)

B) _____ gal

C) _____

A) Determine the ordered pair (x, y) on $4x - 3y = 12$ for which the x and y values are integers and in a ratio of either $2 : 1$ or $1 : 2$.

B) A full tank holds about 4.0 gallons of liquid propane (LP).
Suppose the percent of LP used is directly proportional to the grilling time. If I start with a full tank and use 5% of the LP in 24 minutes of continuous grilling, to the nearest tenth, how many gallons of LP remain in the tank after 3 hours of grilling?

C) If $xy = 2010$ and $\frac{x+z}{x-z} = -2011$, compute yz .

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011
ROUND 6 PLANE GEOMETRY: POLYGONS (no areas)

ANSWERS

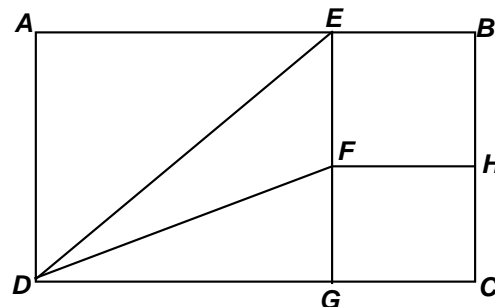
A) _____

B) _____^o

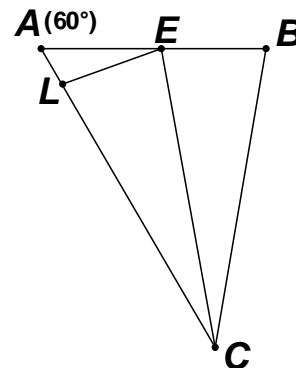
C) _____^o

- A) The interior and exterior angles of a regular polygon have measures in a 44 : 1 ratio.
 How many diagonals may be drawn from a single vertex?

- B) $ABCD$, $AEGD$, $EBCG$ and $FHCG$ are rectangles.
 $EBHF$ is a square. $BE = 4$, $DF = 6$
 The ratio of the perimeter of $EBHF$ to the perimeter
 of $FHCG$ is 8 : 7. Compute the measure of $\angle DFE$.



- C) In $\triangle ABC$, $m\angle A = 60^\circ$. L and E are located on \overline{AC} and \overline{AB}
 respectively such that $CE = CL = CB$ and $BE = EL$.
 Compute $m\angle BEL$.



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011
ROUND 7 TEAM QUESTIONS**

ANSWERS

A) _____ D) _____

B) _____ E) _____

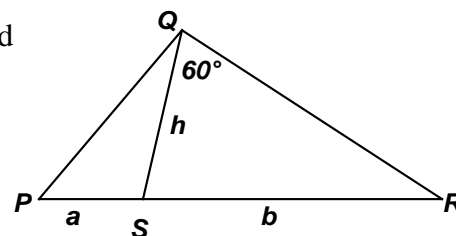
C) _____ F) _____

- A) $\triangle PQR$ is a right triangle with right angle at Q .

$PS = a$, $RS = b$ and QS divides $\angle Q$ into 30° and 60° as indicated in the diagram at the right.

For positive integers L , M and N , $h^2 = \frac{La^2b^2}{Ma^2 + Nb^2}$.

Compute the ordered triple (L, M, N) for which the sum $L + M + N$ is a minimum.



- B) Consider two groups of Pythagorean triples:

Group A: row 1: 3, 4, 5 row 2: 5, 12, 13 row 3: 7, 24, 25 row 4: 9, 40, 41

Group B: row 1: 8, 15, 17 row 2: 12, 35, 37 row 3: 16, 63, 65 row 4: 20, 99, 101

Compute the quotient of the third term in the 13th row of group A to the second term in the 36th row in group B.

- C) Compute the coordinates of all points $P(x, y)$, where x and y are positive integers and P is twice as far from $A(5, 2)$ as it is from $B(11, 8)$

- D) Compute all possible real values of x for which $3^{\left(\log_3 x^4 - \frac{1}{\log_x 3}\right)} + 2^{2\log_2 x} + 7^{5\log_7 x} = 3x^4$.

- E) In the spring, the ratio of girls to boys was 7 : 11 in the junior class at a local high school. Over the summer two boys joined this class and three girls in this class moved out of the district, making the new ratio of girls to boys 5 : 8. All these students advanced to the next grade. If there are no additional changes, in the fall, how many seniors will be attending this local high school?

- F) P and Q are regular polygons. Q has five times as many sides as P . The ratio of the measure of an interior angle of Q to the measure of an interior angle of P is 7 to 5. Regular polygon R has more sides than polygon Q and interior angles whose measures are an integral number of degrees. Compute the minimum number of degrees in one of the interior angles of R .

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 ANSWERS**

Round 1 Trig: Right Triangles, Laws of Sine and Cosine

- A) $\frac{17}{13}$ B) 108 C) $\frac{3}{2}$
[3, 6, 9, 12, 15, 18, 21 and 24]

Round 2 Arithmetic/Elementary Number Theory

- A) 49 B) 20,871 C) 1101111

Round 3 Coordinate Geometry of Lines and Circles

- A) 13 B) $2\sqrt{7}$ C) X(9, 0) Y $\left(0, -\frac{27}{4}\right)$

Round 4 Alg 2: Log and Exponential Functions

- A) $\left(-1, \frac{1}{4}\right)$ B) $2.5, \frac{1}{2}\sqrt{17}$ C) $\frac{30k+1}{10k}$
or equivalent

Round 5 Alg 1: Ratio, Proportion or Variation

- A) (-6, -12) B) 2.5 gal C) 2012

Round 6 Plane Geometry: Polygons (no areas)

- A) 87 B) 120 C) 160

Team Round

- A) (4, 3, 1) D) $1+\sqrt{2}$
B) 1 : 15 E) 611
C) (17, 14), (17, 6), (9, 14), (9, 6) F) 170

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Round 1

A) Either knowing common Pythagorean Triples or using the Pythagorean Theorem, the hypotenuse has length 13. Therefore, $\cos B + \cos C = \frac{5}{13} + \frac{12}{13} = \frac{17}{13}$

B) By the law of Sines, $\frac{\sin 30^\circ}{12} = \frac{1}{24} = \frac{\sin C}{k}$

$$\Rightarrow \sin C = \frac{k}{24} = \frac{k}{2^3 \cdot 3}$$

Only fractions with denominations containing exclusively powers of 2, powers of 5 or products thereof have terminating decimal representations.

Therefore, k must be a positive multiple of 3 and less than or equal to 24.

$\Rightarrow 3, 6, 9, 12, 15, 18, 21, 24 \Rightarrow \underline{108}$.

C) Using the Law of Cosines, $k^2 = 64 + 16 - 2 \cdot 4 \cdot 8 \cos A = 80 - 64 \cos A$.

$$\text{Solving for } \cos A, \cos A = \frac{80 - k^2}{64}.$$

Applying the Triangle Inequality to $\triangle BAD$, the minimum value of k is 5 and the maximum

$$\text{value is 11. Substituting for } k, \cos A = \frac{80 - 25}{64} = \frac{55}{64}, \frac{80 - 121}{64} = \frac{-41}{64}$$

$$\text{Thus, the positive difference is } \frac{96}{64} = \frac{3}{2}.$$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Round 2

- A) Since primes are odd (except for 2), the difference between two primes is odd if and only if one of the two primes is 2. Therefore, the two primes must be 2 and 47, resulting in a sum of **49**.
- B) There are $400(365)$ days plus 100 leap days minus leap days in 1700, 1800 and 1900.
 $[400(365) + 100 - 3] = 146097$ days or, dividing by 7, **20,871** weeks

FYI: The Julian calendar (aka Julius Caesar) was instituted in 45BC, specifying a year to be exactly 365.25 days long. A pretty amazing result for the time! The currently accepted year length is 365.2425 days, i.e. 365 days 5 hours 49 minutes and 12 seconds – a discrepancy of 10 minutes and 48 seconds per year. Not much of an error, but over the centuries it added up. The Gregorian calendar we use today was adopted in most countries in 1582. According to a Papal Bull issued by Pope Gregory XIII on 2/24/1582, the day after Thursday 10/4/1582 was to be Friday 10/15/1582. The accumulated error had grown to 10 days. Imagine the reaction from the general population!! People who accept as truth:

“There are three types of people: those who do math and those who don’t.”

The last country to make the transition was Greece on Thursday, March 1st **1923**. The day before had been Wednesday, February 15th (on the old Julian calendar). The gap had grown to 13 days. Great stuff! Check it out for yourself.

- C) $10101_{(-2)} = 1(16) + 1(4) + 1 = 21$
 $11010_{(-2)} = 1(16) + 1(-8) + 1(-2) = 6$
 Thus, the sum in base 10 is 27.
 Converting to base (-2) , the place values are 1, -2 , 4, -8 , 16, -32 , 64, ...
 Since 27 is odd and all place values are even except 1, we must have a 1 in the unit’s position.
 $(64)1 + (-32)1 = 32$ too large
 $32 + (-8)1 = 24$ too small
 $24 + (4)1 = 28$ too large
 $28 + (-2)(1) = 26$ too small
 $26 + (1)1 = 27$
 \Rightarrow **1101111** _{(-2)}

Alternate solution: (do the arithmetic in the given base!!)

10101

11010 What is $1 + 1$ in base (-2) ? $2_{(10)} = 110_{(-2)}$ Why?

?1111

$$[1(-2)^2 + 1(-2)^1 + 0(-2)^0 = 1(4) + 1(-2) + 0(1) = 4 - 2 + 0 = 2]$$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Round 3

- A) Applying the Pythagorean Theorem, in right $\triangle PSR$ ($PS = 3$, $SR = 4$ $\Rightarrow PR = 5$) and in right $\triangle PRQ$ ($PR = 5$, $RQ = 12 \Rightarrow PQ = 13$).

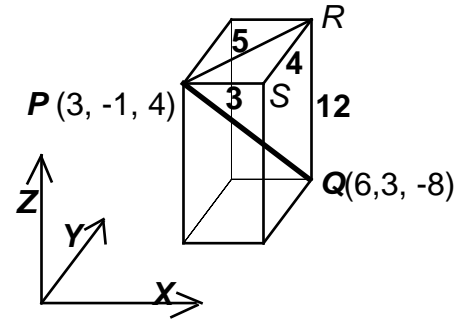
Alternate Solution:

The Pythagorean Theorem extends to 3D as follows:

The distance between points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

$$\text{Thus, } PQ = \sqrt{(6-3)^2 + (3-(-1))^2 + (-8-4)^2} = \sqrt{9+16+144} = \sqrt{169} = \underline{13}$$



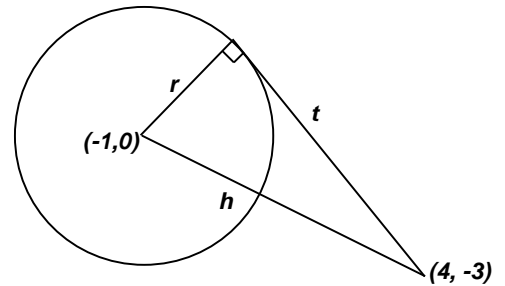
- B) $(x+1)^2 + y^2 = 6 \Rightarrow$ Center $(-1, 0)$ and $r = \sqrt{6}$

$$h = \sqrt{(4+1)^2 + (-3-0)^2} = \sqrt{34} \Rightarrow t^2 = 34 - 6 = 28$$

$$\Rightarrow t = \underline{2\sqrt{7}}$$

Since translations do not alter distances, translating the circle and given point one unit to the right (so that the circle is centered at the origin) gives the same result.

$$[x^2 + y^2 = 6 \text{ and } (5, -3)]$$



- C) The slope of L_2 is $\frac{3}{4}$ and the slope of \overleftrightarrow{PQ} is $-\frac{4}{3}$.

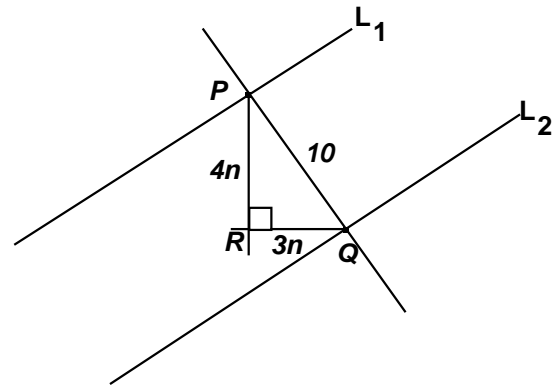
$$\text{Therefore, } \frac{PR}{RQ} = \frac{4}{3} \Rightarrow 25n^2 = 100 \Rightarrow n = 2.$$

$$Q(7+6, 11-8) = (13, 3)$$

The equation of L_2 is

$$(y-3) = \frac{3}{4}(x-13) \Leftrightarrow 3x-4y = 27$$

$$\text{Letting } x, y = 0, \text{ we have } X(\underline{9, 0}) \text{ and } Y(\underline{0, -\frac{27}{4}}).$$



Alternate Solution (Norm Swanson)

Since $L_1 \parallel L_2$, its equation has the form $3x - 4y + k = 0$.

Any point on L_2 is 10 units from L_1 . Applying the point to line distance formula, the

$$\text{distance from } (7, 11) \text{ on } L_1 \text{ to } L_2 \text{ is given by } \frac{|3(7) - 4(11) + k|}{\sqrt{3^2 + 4^2}} = 10$$

$$\Leftrightarrow |21 - 44 + k| = 50 \Leftrightarrow |k - 23| = 50 \Leftrightarrow k = -27, 53$$

$3x - 4y + 53 = 0$ is a line "above" L_1 and is rejected (Q would be to the left of P),

so the equation of L_2 is $3x - 4y - 27 = 0$ and we proceed as above.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Round 4

A) $2^{3x+1} = 4^x = 2^{2x} \Leftrightarrow 3x+1 = 2x \Leftrightarrow x = -1$. Thus, $(a, b) = \underline{\underline{\left(-1, \frac{1}{4}\right)}}$.

B) $x = 0 \Rightarrow y = 3 - 5 = -2$

$$y = 0 \Rightarrow \log_2(8(2x-1)^2) = 5 \Rightarrow 8(2x-1)^2 = 2^5 = 32 \Rightarrow 2x-1 = \pm 2 \Rightarrow x = \frac{3}{2}, -\frac{1}{2}$$

$$(0, -2) \text{ to } \left(\frac{3}{2}, 0\right) \Rightarrow \underline{\underline{2.5}} \quad (0, -2) \text{ to } \left(-\frac{1}{2}, 0\right) \Rightarrow \underline{\underline{\frac{1}{2}\sqrt{17}}}$$

C) Using the formula $\log a + \log b = \log ab$ for $a, b > 0$, we get

$$A = \log 2 + \log \frac{3}{2} + \log \frac{4}{3} + \dots + \log \frac{2000}{1999} = \log \left(\frac{2 \cdot 3 \cdot 4 \cdot \dots \cdot 2000}{2 \cdot 3 \cdot 4 \cdot \dots \cdot 1999} \right) = \log 2000 = 3 + \log 2$$

$$\log_{1024} 10 = k \Leftrightarrow \frac{1}{\log(2^{10})} = k \Leftrightarrow \frac{1}{10 \log 2} = k \Leftrightarrow \log 2 = \frac{1}{10k}$$

$$\text{Thus, } A = 3 + \frac{1}{10k} = \underline{\underline{\frac{30k+1}{10k}}}.$$

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Round 5

A) Given: $4x - 3y = 12$

If $x = 2y$, then $8y - 3y = 12$ and y is not an integer.

If $y = 2x$, then $4x - 6x = -2x = 12 \Rightarrow (x, y) = \underline{(-6, -12)}$.

$$\text{B) } \frac{5}{24} = \frac{x}{180} \Rightarrow x = \frac{5(180)}{24} = \frac{5(15)}{2} = 37.5\% \text{ or } \frac{3}{8} \text{ used} \Rightarrow \frac{5}{8} \text{ remains in tank}$$

$$\frac{5}{8}(4.0) = \underline{2.5}$$

FYI: LP turns into a gas and it's the gas that burns when we are grilling. As the gas is burned, the bottom of the tank sweats and the sweat line gives us a way of determining how full the tank is after we have been using it.

$$\text{C) } \frac{x+z}{x-z} = -2011 \Leftrightarrow \frac{(x+z)y}{(x-z)y} = \frac{xy+yz}{xy-yz} = -2011$$

$$\Rightarrow \frac{2010+yz}{2010-yz} = -2011 \Leftrightarrow -2011(2010-yz) = 2010+yz$$

$$\Rightarrow 2010yz = 2011(2010) + 2010 = 2010(2011+1) = 2010(2012) \Rightarrow yz = \underline{2012}.$$

Generalization: You should verify that if $\begin{cases} xy = k \\ \frac{z+x}{z-x} = k+1 \end{cases}$, then $yz = k+2$.

**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Round 6

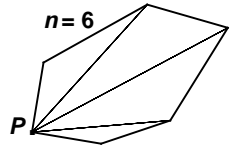
- A) Since the interior and exterior angles are supplementary, $44k + 1k = 180 \Rightarrow k = 4$.
An exterior angle of $4^\circ \Rightarrow$ there must be $360/4 = 90$ sides.

In an n -sided polygon the number of diagonals from each vertex is $(n - 3) \Rightarrow \underline{87}$ diagonals.

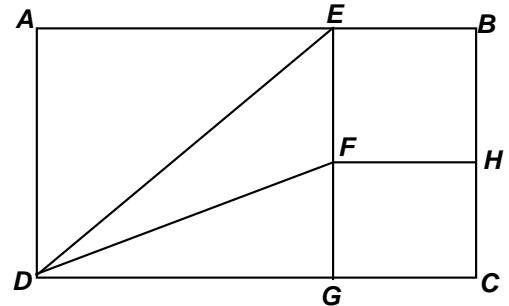
Recall: In the formula for the number of diagonals in a polygon with n sides,

namely $d = \frac{n(n-3)}{2}$, n denoted the number of vertices from which a diagonal

could start, $n - 3$ denoted the number of vertices to which a diagonal could be drawn and division by 2 was necessary to avoid counting each diagonal twice.



- B) Let $FG = HC = x$. Then $16 : (8 + 2x) = 8 : 7 \Rightarrow x = 3$
 $DF = 6$ and $FG = 3 \Rightarrow m\angle FDG = 30$, $m\angle DFE = \underline{120^\circ}$



- C) $CL = CE = CB \Rightarrow$ both $\triangle CLE$ and $\triangle CEB$ are isosceles.

$BE = EL \Rightarrow \triangle CLE \cong \triangle CEB$ (SSS)

Let $m\angle ECL = m\angle ECB = x$.

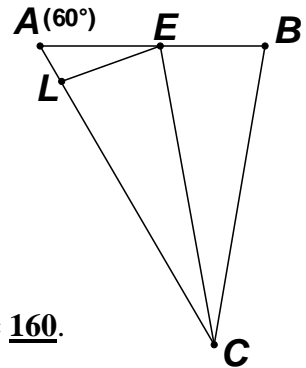
Let $m\angle CLE = m\angle CEL = m\angle CLB = m\angle LCB = y$.

Then: $4y + 2x = 360 \Rightarrow x + 2y = 180$

In $\triangle ACE$, $m\angle ACE = 180 - (60 + x) = 120 - x$

$\Rightarrow (120 - x) + y = 180$ or $y = 60 + x$.

Substituting, $x + 2(60 + x) = 180 \Rightarrow 3x = 60 \Rightarrow x = 20$, $y = 80$, $m\angle BEL = \underline{160}$.



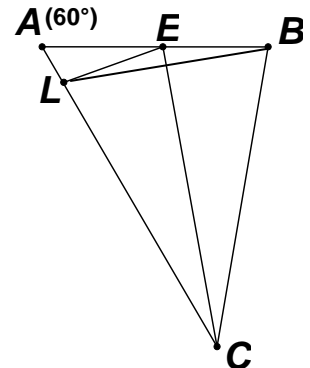
Alternative Solution (Norm Swanson)

Draw \overline{BL} . $BELC$ is a kite. Let $m\angle CLB = m\angle CBL = x^\circ$ and

$m\angle ELB = m\angle EBL = y^\circ$. Since $\triangle CLE$ is isosceles, $m\angle CEL = (x + y)^\circ$.

As an exterior angle of $\triangle ALE$, $m\angle BEL = 60 + (180 - (x + y))$.

Thus, $2(x + y) = 240 - (x + y) \Rightarrow x + y = 80 \Rightarrow m\angle BEL = \underline{160^\circ}$.



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Team Round

- A) Using the Law of Sines on $\triangle PQS$ and $\triangle RQS$,

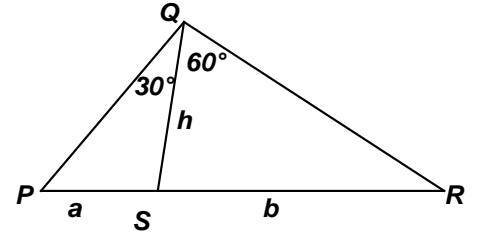
$$\frac{\sin 30}{a} = \frac{\sin P}{h} \text{ and } \frac{\sin 60}{b} = \frac{\sin R}{h}.$$

Squaring each equation and adding,

$$\frac{\sin^2 30}{a^2} + \frac{\sin^2 60}{b^2} = \frac{\sin^2 P}{h^2} + \frac{\sin^2 R}{h^2} = \frac{\sin^2 P + \cos^2 P}{h^2} = \frac{1}{h^2}$$

(since P and R are complementary)

$$\frac{1}{4a^2} + \frac{3}{4b^2} = \frac{b^2 + 3a^2}{4a^2b^2} = \frac{1}{h^2} \Rightarrow (L, M, N) = \underline{(4, 3, 1)}.$$



- B) In group A, the first number in each triple is $2k + 1$.

We want the 3rd term in the 13th row, i.e. we need the triple $(27, x, x + 1)$

$$27^2 + x^2 = (x + 1)^2 \Rightarrow 729 = 2x + 1 \Rightarrow x = 364 \Rightarrow 3^{\text{rd}} \text{ term} = 365 = 5(73)$$

In group B, the first number in each triple is $4(k + 1)$.

We want the 2nd term in the 36th row, i.e. we need the triple $(148, x, x + 2)$

Avoid as much computation as possible.

$$148^2 + x^2 = (x + 2)^2 \Rightarrow 148^2 = 4x + 4 \Rightarrow 4x = 148^2 - 2^2 = (148 + 2)(148 - 2) = 150(146)$$

$$\Rightarrow x = 75(73) = 2^{\text{nd}} \text{ term}$$

$$\text{The required ratio is } \frac{5(73)}{75(73)} = \underline{\frac{1}{15}}.$$

C) $\sqrt{(x-5)^2 + (y-2)^2} = 2\sqrt{(x-11)^2 + (y-8)^2}$

$$\Leftrightarrow (x-5)^2 + (y-2)^2 = 4((x-11)^2 + (y-8)^2)$$

$$\Leftrightarrow x^2 - 10x + 25 + y^2 - 4y + 4 = 4(x^2 - 22x + 121 + y^2 - 16y + 64) = 4x^2 - 88x + 484 + 4y^2 - 64y + 256$$

$$\Leftrightarrow 3x^2 - 78x + 3y^2 - 60y = -711$$

$$\Leftrightarrow 3(x^2 - 26x + \underline{169}) + 3(y^2 - 20y + \underline{100}) = -711 + \underline{507} + \underline{300} = 96$$

$$\Leftrightarrow (x-13)^2 + (y-10)^2 = 32 \text{ (A circle with center at (13, 10) and radius } 4\sqrt{2}.)$$

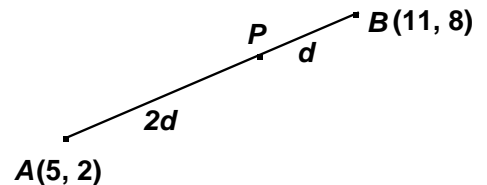
We need to examine perfect squares which sum to 32.

The list of candidates is 1, 4, 9, 16, 25. Only $16 + 16 = 32$!

$$(x-13)^2 = 16 \Rightarrow x-13 = \pm 4 \Rightarrow x = 17, 9$$

$$(y-10)^2 = 16 \Rightarrow y-10 = \pm 4 \Rightarrow y = 14, 6$$

Thus, the possible points are **(17, 14), (17, 6), (9, 14), (9, 6).**



**MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 3 - DECEMBER 2011 SOLUTION KEY**

Team Round - continued

- D) Simplifying $3^{\left(\log_3 x^4 - \frac{1}{\log_x 3}\right)} + 2^{2\log_2 x} + 7^{5\log_7 x}$, the 2nd and 3rd terms are clearly, x^2 and x^5 , but let's look carefully at the 1st term. Since x occurs as a base of the logarithm, $x \neq 1$.

$$\log_3 x^4 - \frac{1}{\log_x 3} = \log_3 x^4 - \log_3 x = \log_3 \left(\frac{x^4}{x} \right) = \log_3 x^3 \quad \text{Thus, the 1st term is, in fact, } x^3.$$

$$\text{Thus, we have } x^3 + x^2 + x^5 = 3x^4 \Rightarrow x^2(x^3 - 3x^2 + x + 1) = 0$$

Since x is the argument of the log function, $x > 0$ and the only roots come from the cubic factor.

$$\text{By synthetic substitution, } \begin{array}{r|rrrr} 1 & 1 & -3 & 1 & 1 \\ & & 1 & -2 & -1 \\ \hline & 1 & -2 & -1 & 0 \end{array} \Rightarrow (x-1)(x^2 - 2x - 1) = 0 \Rightarrow \cancel{x=1} \text{ or } x = \frac{2 \pm 2\sqrt{2}}{2}$$

$$\Rightarrow x = \underline{1 + \sqrt{2}} \quad (1 - \sqrt{2} < 0 \text{ and is, therefore, also extraneous.})$$

E) First condition: $\frac{G}{B} = \frac{7}{11} \Leftrightarrow G = \frac{7}{11}B$

$$\text{Second condition: } \frac{G-3}{B+2} = \frac{5}{8}$$

$$\text{Cross multiplying, } 8G - 24 = 5B + 10$$

$$\Leftrightarrow 8\left(\frac{7}{11}B\right) - 24 = 5B + 10 = \frac{55}{11}B + 10 \Leftrightarrow \frac{B}{11} = 34 \Leftrightarrow B = 374$$

$$\text{According to the first condition, } G = \frac{7}{11}(34 \cdot 11) = 238$$

Thus, in the fall, the total number of students is $(374 + 2) + (238 - 3) = \underline{611}$.

- F) Since Q has five times as many sides as P , the exterior angle of $P\left(\frac{360}{n}\right)$ is five times as large as

the exterior angle of $Q\left(\frac{360}{5n}\right)$. The relationship between the interior and the exterior angles of

these two polygons is summarized in the following diagram:

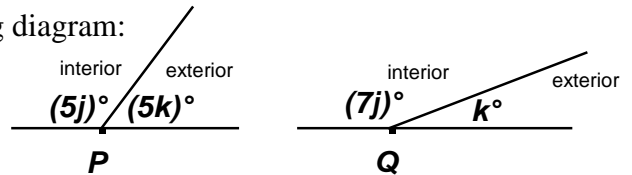
$$\text{Thus, } 5j + 5k = 180 = 7j + k \Rightarrow j = 2k$$

$$\text{Substituting, } 7(2k) + k = 180 \Rightarrow k = 12.$$

Since an exterior angle of Q measures 12° ,

Q must have 30 sides. As the number of sides increases, the measure of the interior angle increases.

This, if R is an N -gon, N must be the smallest factor of 360 larger than 30, i.e. 36° , producing an exterior angle of 10° and an interior angle of 170 $^\circ$.



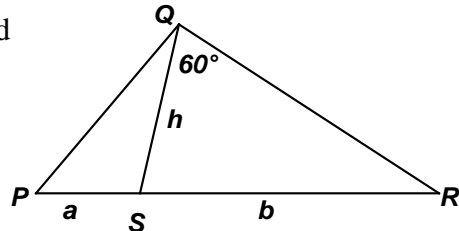
Addendum #1: Team A Question

$\triangle PQR$ is a right triangle with right angle at Q .

$PS = a$, $RS = b$ and QS divides $\angle Q$ into 30° and 60° as indicated in the diagram at the right.

For positive integers L , M and N , $h^2 = \frac{La^2b^2}{Ma^2 + Nb^2}$.

Compute the ordered triple (L, M, N) .

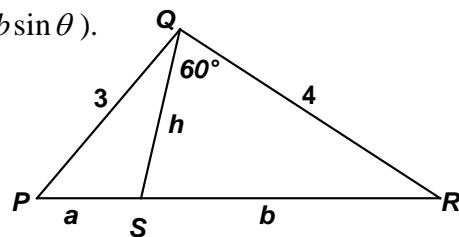


The original team A question specified a right triangle with $PQ = 3$ and $QR = 4$ and requested an expression for $1/h^2$ exclusively in terms of a and b .

In this case, we could have actually solved for a and b (using $\frac{1}{2}ab \sin \theta$).

$$3h \sin 30^\circ + 4h \sin 60^\circ = 3 \cdot 4$$

$$\Rightarrow h = \frac{12}{\frac{3}{2} + 2\sqrt{3}} = \frac{8}{13}(4\sqrt{3} - 3).$$



Using the Law of Sines in triangles QPS and QRS , $h = \frac{8}{5}a$ and $h = \frac{6b}{5\sqrt{3}}$.

Equating, $a = \frac{\sqrt{3}}{4}b$, so $a = \frac{5}{13}(4\sqrt{3} - 3)$ and $b = \frac{20}{13}(4 - \sqrt{3})$.

But none of these values needed to come into play when solving for $\frac{1}{h^2}$ in terms of a and b .

Depending on how you proceeded, different-looking 'formulas' would have been possible besides the official solution.

Using Stewart's Theorem, we get

$$3^2 \cdot b + 4^2 \cdot a = h^2 \cdot 5 + 5ab \Rightarrow h^2 = \frac{9b + 16a - 5ab}{5} \Rightarrow h^2 = \frac{9b + 16a - 5ab}{5} \Rightarrow \frac{1}{h^2} = \frac{5}{9b + 16a - 5ab}.$$

Using the Law of Cosines (twice) we have:

$$\begin{cases} h^2 = 9 + a^2 - 6a \cos P = 9 + a^2 - 6a \left(\frac{3}{5}\right) \\ h^2 = 16 + b^2 - 8b \cos R = 16 + b^2 - 8b \left(\frac{4}{5}\right) \end{cases} \quad \text{Adding, } 2h^2 = 25 + a^2 + b^2 - 2\left(\frac{9a + 16b}{5}\right) \text{ and we}$$

have a much different-looking result for $\frac{1}{h^2}$.

You could even find the numerical value of $\frac{1}{h^2}$, namely $\frac{13}{64}\left(\frac{4}{\sqrt{3}} + 1\right)$, and then determined

coefficients to write this value as a linear combination of $a = \frac{5}{13}(4\sqrt{3} - 3)$ and $b = \frac{20}{13}(4 - \sqrt{3})$.

It's quite challenging to show that all these variants are equivalent for the known values of (h, a, b) .

Addendum #2: Team A Question

The actual question used did not add the restriction the sum $L + M + N$ is a minimum. Strangely, no appeal was made and there should have been!

Here is the case made by Norm Swanson – Hamilton Wenham

Since no lengths are given, we can choose any convenient lengths. Furthermore, there are many possible locations for point Q for which $m\angle PQR$ is a right angle, since any point on a semi-circle with center O and diameter \overline{PR} will suffice. By choosing a specific point Q_2 , we can let \overline{PS} (with length h) be an altitude to \overline{PR} , dividing $\angle PQR$ into 30° and 60° angles.

Convenient lengths:

$$PQ_2 = 2 \Rightarrow Q_2R = 2\sqrt{3}, PR = 4, a = PS_2 = 1, b = S_2R = 3, h = Q_2S_2 = \sqrt{3}$$

The given formula $h^2 = \frac{La^2b^2}{Ma^2 + Nb^2}$ becomes $3 = \frac{9L}{M + 9N}$ or $1 = \frac{3L}{M + 9N}$ or $L = \frac{M}{3} + 3N$.

Since all three constants are positive integers, M must be a multiple of 3 and we get the minimum sum if we let $M = 3$, resulting in $L = 1 + 3N$.

Thus, the minimum sum occurs when $N = 1$ and $(L, M, N) = \underline{(4, 3, 1)}$.

However, (L, M, N) is not unique.

In fact, any triples of the form $(k + 3j, 3k, j)$, where j and k are positive integers, are solutions.

