MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011
ROUND 1 COMPLEX NUMBERS (No Trig)

ANSWERS

A) ______________________________

B) I II III IV

C) ______________________________

****** NO CALCULATORS IN THIS ROUND ******

A) Evaluate $\sqrt{-18} \cdot \sqrt{-8}$.

B) Let $z = x + yi$ be represented by the point $P(x, y)$.
In what quadrant is P, if $z^{-1} = 3 - 4i$?

C) Evaluate $\sum_{k=0}^{k=3} \left(1 - \sqrt{3}i\right)^k$.

Refuse to be intimidated by the mathematical symbolism.
"Σ" is just a summation operator.
For example, $\sum_{k=1}^{k=4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2011
ROUND 2 ALGEBRA 1: ANYTHING

ANSWERS

A) ________________________ inches

B) ______________________________

C) ______________________________

****** NO CALCULATORS IN THIS ROUND ******

A) A rectangular piece of cardboard is 4" by 6".
Squares of equal size are cut from each of the 4 corners.
If the total area of the cutouts is 37.5% of the area of the original piece of cardboard, compute the length (in inches) of the side of the squares cut from each corner.

B) When a two-digit number is divided by the sum of its digits, the quotient is 7 and the remainder is 0. When the same number is multiplied by the sum of its digits, the product is 567. Find this number.

~~**Joe Ford – 2009**~~
~~When a two-digit number is divided by the sum of its digits, the quotient is 7.~~
~~When the same number is multiplied by the sum of its digits, the product is 567.~~
~~Find this number.~~

~~Ans: 63~~

$$\begin{cases} (1)\ \ 10t+u=7(t+u) \\ (2)\ \ (10t+u)(t+u)=567 \end{cases}$$

~~(1) → $t=2u$~~

~~Substituting for $10t+u$ in (2), $7(t+u)^2=567$ → $(t+u)^2=81$ → $t+u=3u=9$~~

~~→ $u=3, t=6$ → **63**~~

C) Suppose I drive at a certain speed $S > 0$ (in mph) for a certain time $T > 0$ (in hours). Increasing my driving speed by 20 mph and decreasing my driving time by 1 hour, I cover the same distance. If the speed limit is 65 mph and I never break the speed limit (a governor has been installed in my vehicle), compute the number of possible ordered pairs (S, T), where S and T are both integers.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2011
ROUND 3 PLANE GEOMETRY: AREAS OF RECTILINEAR FIGURES

ANSWERS

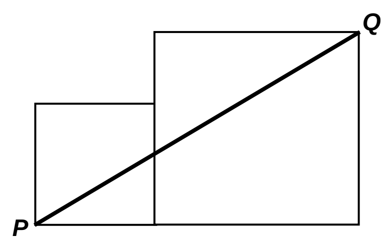
A) ______________________________ in^2

B) ______________________________ units

C) ______________________________ units

****** NO CALCULATORS IN THIS ROUND ******

A) Rectangle *PARK* is 3 inches by 5 inches.
T lies on $\overline{KR}$ such that ΔPTK is isosceles.
Compute the area of *TRAP*.



B) The ratio of the lengths of the sides of the two squares at the right is 3 : 2. The sum of their areas is 3328 $units^2$.
Compute the length of $\overline{PQ}$.

D) If the side of a square is increased by 25%, its area equals that of a rectangle whose sides have lengths in a 2 : 5 ratio and whose perimeter is 28 units.
Compute the <u>original</u> length of a side of the square.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2011
ROUND 4 ALG 1: FACTORING AND ITS APPLICATIONS

ANSWERS

A) ______________________________

B) ______________________________

C) ______________________________

****** NO CALCULATORS IN THIS ROUND ******

A) How many positive integer divisors does $N = 2^4 \cdot 3^2 \cdot 6$ have?

B) $4x^4 + 1 - 5x^2$ is to be completely factored over the integers, where each factor is of the form $ax + b$, a and b are integers and $a > 0$. Express the sum of the factors in terms of x.

C) If $x + y = 6$ and $x^3 + y^3 = 58.5$, compute the numerical value of xy.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2011
ROUND 5 TRIG: FUNCTIONS OF SPECIAL ANGLES

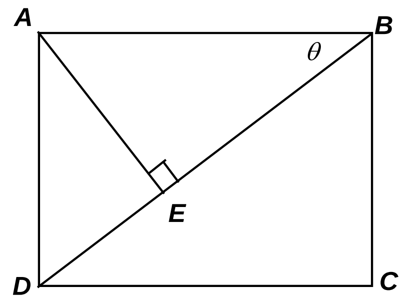
ANSWERS

A) ______________________________

B) (________, ________ , _________)

C) ______________________________

****** NO CALCULATORS IN THIS ROUND ******



A) $ABCD$ is a rectangle, $\overline{AE} \perp \overline{BD}$
$AD = 5$ and $AE = 4$
Compute $\cos\theta$.

B) In simplest form, $(\sin 45° + \tan 135°)^4 = \dfrac{A - B\sqrt{2}}{C}$, where A, B and C are positive integers.
Determine the ordered triple (A, B, C).

C) Compute the <u>sum</u> of the values of x (in degrees) that satisfy $\cos(270° + x) = \sin(-600°)$ and lie between 1500° and 1900° <u>inclusive</u>.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2011
ROUND 6 PLANE GEOMETRY: ANGLES, TRIANGLES AND PARALLELS

ANSWERS

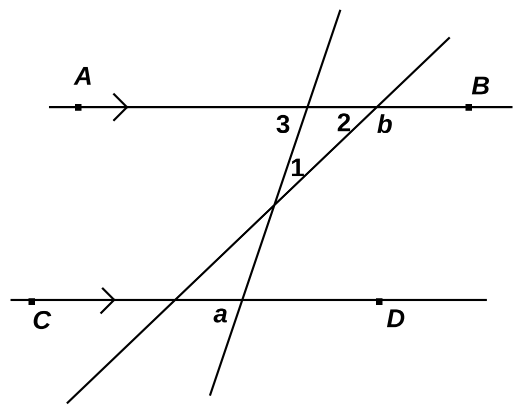
A) ______________________________ °

B) ______________________________ °

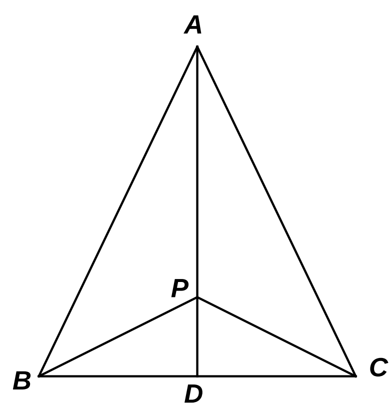
C) ______________________________ °

****** NO CALCULATORS IN THIS ROUND ******

A)
$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle 1 = m\angle 2 + 15°$, $m\angle 3 = 73°$
Compute $b - a$.



B) ΔABC is isosceles with base $\overline{BC}$.
$\overrightarrow{BP}$ bisects $\angle ABC$ and $\overrightarrow{CP}$ bisects $\angle ACB$.
If $AB > BC$ and $m\angle BAC = (2k)°$,
compute $m\angle BPA - m\angle BPC$ in terms of k.



C)
Two angles in an isosceles triangle measure $(x + 5)°$ and $(2x - 30)°$.
Compute the sum of the measures of all possible vertex angles.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2011
ROUND 7 TEAM QUESTIONS
ANSWERS

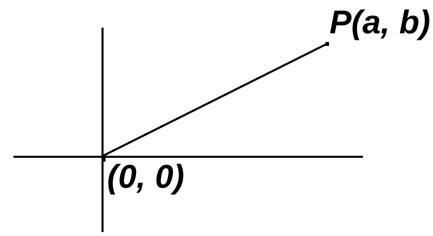
A) ____________________________ D) ____________________________

B) ___A B C D E___ E) ____________________________

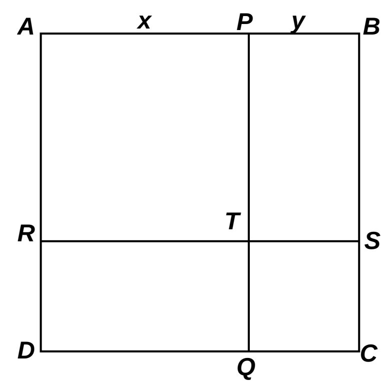
C) ____________________________ F) (____________, ____________)

****** NO CALCULATORS IN THIS ROUND ******

A) Given: $z = a + bi$ and $|z| = 7$
The graph of z in the complex plane is represented by the point P.
Compute all possible values of a for which the distance between $z+(i-2)$ and $\overline{z}+(1-i)$ is 5.



B) Given: $0 < 2x - A < 10$, for integer values of A.
Which of the following statements about the number of integer values of x satisfying the inequality are true? Circle the appropriate letters on the answer blank.
- A) As A increases, the number of integer values of x increases.
- B) As A decreases, the number of integer values of x decreases.
- C) Regardless of the value of A, there are 4 integer values of x.
- D) If A is even, there are 4 integer values of x.
- E) If A is odd, there are 5 integer values of x.

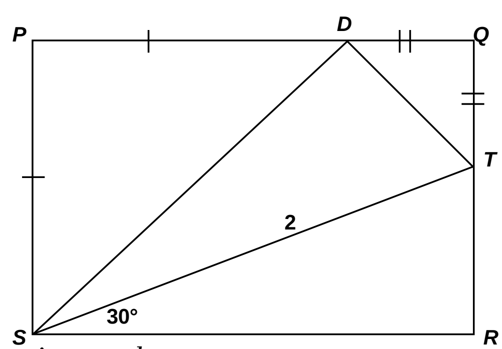


C) Given: a square $ABCD$, subdivided into two squares and two rectangles by $\overline{PQ}$ and $\overline{RS}$ drawn parallel to the sides of the square.
$AP = PT = x$ and $PB = y$, where $x > y > 0$.
The sum of the areas of the two squares is k times the sum of the areas of the two rectangles.

Determine a simplified expression for $\frac{x}{y}$ in terms of k.

D) There are several positive integer values of k for which $\sqrt{k^2 - 96}$ is an integer. Find all of them.

E) Given: Rectangle $PQRS$ with $ST = 2$
$m\angle TSR = 30°$, $PS = PD$ and $QD = QT$
Compute: $\cot(\mathrm{R}STD)$.



F) Given: $m\mathrm{R}A$ is an integer, $m\mathrm{R}B = m\mathrm{R}A + k°$, for some positive integer k.
For each value of k, $m\mathrm{R}A$ is as large as possible.
Each angle in a convex polygon P is congruent to either A or B.
There are 10 of the larger angle B and $0 < n < 10$ of the smaller angle A.
Let m and M denote the minimum and maximum possible values of k. Compute (m, M).

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 - NOVEMBER 2011 ANSWERS

Round 1 Algebra 2: Complex Numbers (No Trig)

A) -12 B) 1 C) $-8-3\sqrt{3}i$

Round 2 Algebra 1: Anything

A) 1.5 B) 63 C) 2

Round 3 Plane Geometry: Area of Rectilinear Figures

A) 10.5 B) $16\sqrt{34}$ C) $\frac{8}{5}\sqrt{10}$

Round 4 Algebra 1: Factoring and its Applications

A) 24 B) $6x$ C) $\frac{35}{4}$ ($8\frac{3}{4}$ or 8.75)

Round 5 Trig: Functions of Special Angles

A) $\frac{4}{5}$ (or 0.8) B) (17, 12, 4) C) 4920 [1500, 1560, 1860]

Round 6 Plane Geometry: Angles, Triangles and Parallels

A) 78° B) $45-\frac{3k}{2}$ or equivalent C) 222°

Team Round

A) $\pm 4\sqrt{3}, \pm 2\sqrt{10}$

B) D and E only

C) $k+\sqrt{k^2-1}$

D) k = 10, 11, 14, 25 (in any order)

E) $2-\sqrt{3}$

F) (2, 15)

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011 SOLUTION KEY

Round 1

A) $\sqrt{-18}\cdot\sqrt{-8}=i\sqrt{18}\cdot i\sqrt{8}=i^2\sqrt{144}=\underline{\mathbf{-12}}$

B) If $z^{-1}=3-4i$, then $z=\dfrac{1}{3-4i}\cdot\dfrac{3+4i}{3+4i}=\dfrac{3+4i}{9+16}=\dfrac{3}{25}+\dfrac{4}{25}i$

Since $x>0$ and $y>0$, P is located in quadrant $\underline{\mathbf{1}}$.

C) $\sum_{k=0}^{k=3}\left(1-\sqrt{3}i\right)^k=\left(1-\sqrt{3}i\right)^0+\left(1-\sqrt{3}i\right)^1+\left(1-\sqrt{3}i\right)^2+\left(1-\sqrt{3}i\right)^3$

$\left(1-\sqrt{3}i\right)^2=1-2\sqrt{3}i+3i^2=-2-2\sqrt{3}i$

The first three terms evaluate easily as $1+\left(1-\sqrt{3}i\right)+\left(-2-2\sqrt{3}i\right)=-3\sqrt{3}i$.

The last term evaluates as follows:

$\left(1-\sqrt{3}i\right)^3=\left(1-\sqrt{3}i\right)^2\cdot\left(1-\sqrt{3}i\right)^1=\left(-2-2\sqrt{3}i\right)\left(1-\sqrt{3}i\right)=-2+2\sqrt{3}-2\sqrt{3}-6=-8$

If you recalled that the three cubes roots of –8 are –2, $-1+\sqrt{3}$ and $1-\sqrt{3}$, this expansion was unnecessary. Thus, the required sum is $\underline{\mathbf{-8-3\sqrt{3}i}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011 SOLUTION KEY

Round 2

A) Let x denote the side of each square cutout.

Since 37.5% = 3/8, we have $\frac{3}{8}(4\cdot 6)=4x^2 \Rightarrow x^2=\frac{9}{4} \Rightarrow x=$ **1.5** (or $\underline{\frac{3}{2}}$).

B) $\begin{cases}(1)\ \ 10t+u=7(t+u)\\(2)\ \ (10t+u)(t+u)=567\end{cases}$

(1) $\Rightarrow t=2u$

Substituting for $10t+u$ in (2), $7(t+u)^2=567 \Rightarrow (t+u)^2=81 \Rightarrow t+u=3u=9$

$\Rightarrow u=3, t=6 \Rightarrow$ **63**

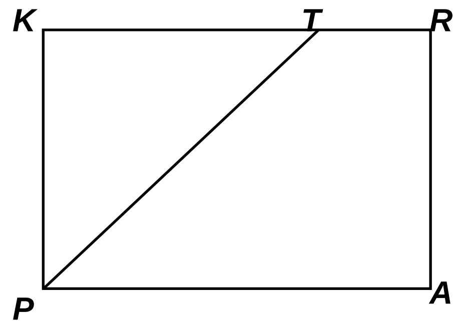
~~B)~~C) $ST=(S+20)(T-1)=ST-S+20T-20 \Rightarrow S=20(T-1)$

$S>0 \Rightarrow T\geq 2$, $S\leq 65 \Rightarrow T\leq 4$.

Thus, there are 3 possible ordered pairs (S, T), namely (20, 2), (40, 3) and (60, 4).
However, the last ordered pair fails, since to travel 240 miles (60 mph for 4 hours), I would have to travel 80 mph for 3 hours, breaking the speed limit.
Thus, there are only **2** ordered pairs satisfying the never-break-the-speed-limit condition.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011 SOLUTION KEY

Round 3



A) Since K is a right angle, isosceles triangle PTK must be 45 – 45 – 90 and $KT = KP$.
Thus, $\overline{KR}$ must be the longer side.
$KP = RA = 3$, $KR = PA = 5 \Rightarrow TR = 2$
Use the area formula for a trapezoid directly

Area = $\frac{1}{2}3(2+5) = \frac{21}{2} = \underline{\mathbf{10.5}}$

or determine the area of the rectangle $PARK$ and subtract the area of the triangle PKT.

Area = $15 - \frac{1}{2}\cdot 3\cdot(5-3) = 15 - 4.5 = \underline{\mathbf{10.5}}$

B) $(3x)^2 + (2x)^2 = 3328 \Rightarrow x^2 = \frac{3328}{13} = 256 \Rightarrow x = 16$
Thus, the sides of the squares are 32 and 48 units.
$\overline{PQ}$ is the hypotenuse of a right triangle with a horizontal base of 80 and a vertical height of 48.
$(48, 80, PQ)$ is a Pythagorean triple. $(48, 80, PQ) = 16(3, 5, x) \Rightarrow x = \sqrt{34} \Rightarrow PQ = \underline{\mathbf{16\sqrt{34}}}$.

C) Let x denote the side of the square. The side of the larger square is $x + \frac{1}{4}x = \frac{5}{4}x$
The perimeter of the rectangle is $2(2k + 5k) = 28 \Rightarrow k = 2$.
The rectangle is 4 x 10, resulting in an area of 40.
Thus, $\left(\frac{5}{4}x\right)^2 = 40 \Rightarrow x^2 = \frac{40\cdot 4^2}{5^2} = \frac{2^2\cdot 4^2\cdot 10}{5^2} \Rightarrow x = \underline{\mathbf{\frac{8}{5}\sqrt{10}}}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011 SOLUTION KEY

Round 4

A) $N = 2^5 \cdot 3^3$
Any divisor (other than 1) will have factors of 2 or 3, but no other prime.
Thus, the exponents of 2 may be any integer from 0 to 5 inclusive – 6 possibilities.
The exponents of 3 may be any integer from 0 to 3 inclusive – 4 possibilities.
Choosing both exponents to be 0 gives us 1.
Thus, there are 4(6) = **<u>24</u>** possible positive divisors.
In general, determine the unique prime factorization of N, add 1 to each exponent and multiply.

B) $4x^4 + 1 - 5x^2 = (4x^2 - 1)(x^2 - 1) = (2x + 1)(2x - 1)(x + 1)(x - 1)$.
The sum of the factors is **<u>$6x$</u>**.

C) $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + y^3 + 3xy(x + y)$
$\Rightarrow 6^3 = 58.5 + 3xy(6)$
$\Rightarrow 216 - 58.5 = 18xy \Rightarrow xy = \dfrac{157.5}{18} = \dfrac{315}{36} = \underline{\boldsymbol{\dfrac{35}{4}}} = \underline{\mathbf{8.75}}$
The alternate solution below uses this assertion (fact): ***if a + b + c = 0, then $a^3 + b^3 + c^3 = 3abc$***
In other words, if the sum of 3 numbers is 0, then the sum of the cubes of the 3 numbers will always be 3 times the product of the 3 numbers.

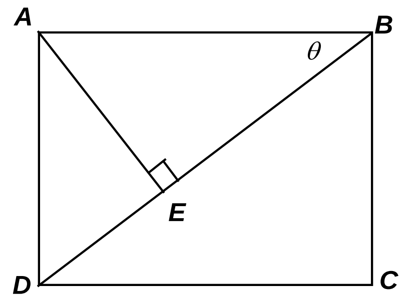
$x + y = 6 \Rightarrow x + y - 6 = 0 \Rightarrow x^3 + y^3 + (-6)^3 = 3xy(-6) \Rightarrow 58.5 - 216 = -18xy$
which is the same result as above.

Proof of the assertion
$(x - a)(x - b)(x - c) = 0$ has solutions $x = a, b, c$
Expanding, and regrouping, $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0$

Substituting for x: $\begin{cases} a^3 - (a+b+c)a^2 + (ab+ac+bc)a - abc = 0 \\ b^3 - (a+b+c)b^2 + (ab+ac+bc)b - abc = 0 \\ c^3 - (a+b+c)c^2 + (ab+ac+bc)c - abc = 0 \end{cases}$

Adding these three equations,
$a^3 + b^3 + c^3 - (a+b+c)(a^2 + b^2 + c^2) + (ab+ac+bc)(a+b+c) - 3abc = 0$
Finally, if $(a + b + c) = 0$, $a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow \boxed{a^3 + b^3 + c^3 = 3abc}$.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011 SOLUTION KEY



Round 5

A) In ΔDBA, $\cos\theta = \frac{AB}{BD}$

$AD = 5$ and $AE = 4 \Rightarrow DE = 3$

$\Delta ABE \cong \Delta DBA$ (by AA)

$\Rightarrow \frac{AB}{DB} = \frac{AE}{DA} = \underline{\mathbf{\frac{4}{5}}}$

[Using the dimensions of right ΔABE is a distraction, since computing $BE = \frac{16}{3}$ and $AB = \frac{20}{3}$ is unnecessary.]

B) $\left(\frac{\sqrt{2}}{2} - 1\right)^4 = \left(\frac{\sqrt{2}}{2}\right)^4 - 4\left(\frac{\sqrt{2}}{2}\right)^3 + 6\left(\frac{\sqrt{2}}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right) + 1 = \frac{1}{4} - \sqrt{2} + 3 - 2\sqrt{2} + 1 = \frac{17}{4} - 3\sqrt{2}$

$= \frac{17 - 12\sqrt{2}}{4} \Rightarrow (A, B, C) =$ **<u>(17, 12, 4)</u>**

C) $\cos(270° + x) = \sin(-600°) \Leftrightarrow \sin x = \sin(120°) \Leftrightarrow x = \begin{cases} 120° + 360n \\ 60° + 360n \end{cases}$

$1500 \le 120 + 360n \le 1900 \Leftrightarrow \frac{138}{36} \le n \le \frac{178}{36} \Rightarrow n = 4\,(\text{only}) \Rightarrow 1560°$

$1500 \le 60 + 360n \le 1900 \Leftrightarrow \frac{144}{36} \le n \le \frac{184}{36} \Rightarrow n = 4, 5 \Rightarrow 1500°, 1860°$

Adding, **<u>4920°</u>**.

Alternate Solution:

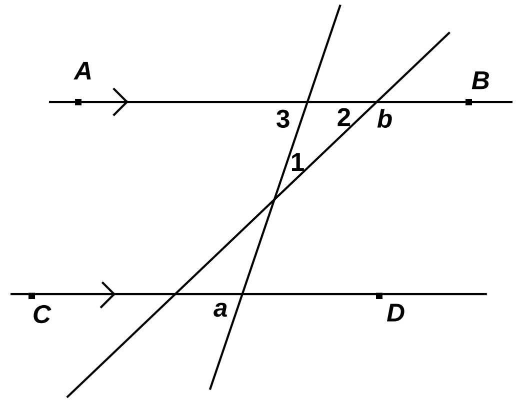
$\cos(270° + x) = \sin(-600°) \Leftrightarrow$

$\cos(270° + x) = \sin(-600° + 720°) = \sin(120°) = \sin 60° = \cos(\pm 30°)$

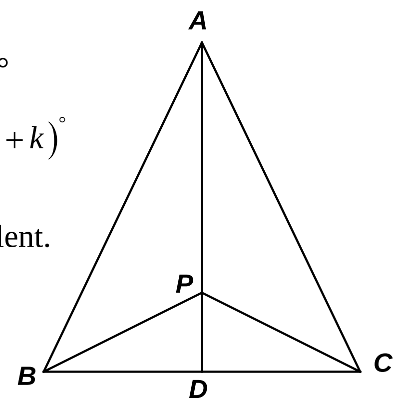
$270 + x = \pm 30 + 360n \Rightarrow x = \begin{cases} -240 + 360n \to 120, 480, 840, 1200, \underline{1560}, 1920 \\ -300 + 360n \to 60, 420, 780, 1140, \underline{1500, 1860} \end{cases} \Rightarrow \underline{\mathbf{4920}}.$

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 2 – NOVEMBER 2011 SOLUTION KEY

Round 6



A) Let m∠2 = x° and m∠1 = (x + 15)°
m∠3 = 73° ⇒ (x + 15) + x + 107 = 180 ⇒ x = 29
⇒ m∠1 = 44°, m∠2 = 29° ⇒ b = 151
Since a is vertical to the alternate interior angle of ∠3,
a = 73. Thus, b – a = 151 – 73 = **<u>78°</u>**.



B) In isosceles ΔABC, a vertex angle of (2k)° leaves base angles of (90 – k)°

Thus, $m\angle ABP = \left(45 - \frac{k}{2}\right)^{\circ} \Rightarrow m\angle BPA = \left(135 - \frac{k}{2}\right)^{\circ}$ and $m\angle BPC = \left(90 + k\right)^{\circ}$

The required difference is $\left(135 - \frac{k}{2}\right)^{\circ} - \left(90 + k\right)^{\circ} = \underline{\left(\mathbf{45 - \frac{3k}{2}}\right)^{\circ}}$ or equivalent.

C) Case 1 - Given angles are the base angles:
x + 5 = 2x – 30 ⇒ x = 35
⇒ base angles 40° , vertex angle 100°

Case 2 - Both base angles measure (x + 5)°:
2(x + 5) + 2x – 30 = 180 ⇒ 4x = 200 ⇒ x = 50
⇒ vertex angle = 2(50) – 30 = 70°

Case 3 – Both base angles measure (2x – 30)°:
2(2x – 30) + (x + 5) = 180 ⇒ 5x = 235 ⇒ x = 47
⇒ vertex angle = 47 + 5 = 52°
Thus, the required sum is **<u>222</u>**°.

MASSACHUSETTS MATHEMATICS LEAGUE
CONTEST 1 - NOVEMBER 2011 SOLUTION KEY

Team Round

A) Let $P(x_1, y_1)$ denote $z + (i-2) = (a-2) + (b+1)i$. Then: $x_1 = a-2,\ y_1 = b+1$

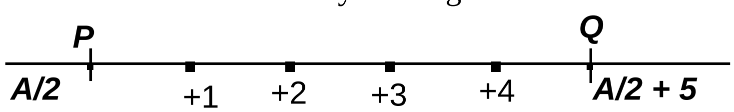
Let $Q(x_2, y_2)$ denote $\bar{z} + (1-i) = (a+1) + (-b-1)i$. Then: $x_2 = a+1,\ y_2 = -b-1$

$PQ = 5 \Rightarrow ((a-2)-(a+1))^2 + ((b+1)-(-b-1))^2 = 25 \Rightarrow (-3)^2 + (2b+2)^2 = 25$

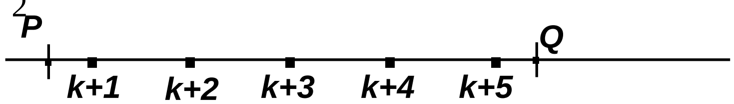
$\Rightarrow 4(b+1)^2 = 16 \Rightarrow b = -1 \pm 2 = 1, -3$

$|z| = \Rightarrow\ a^2 + b^2 = 49 \Rightarrow a^2 = 49 - 1$ or $49 - 9 \Rightarrow a =$ $\underline{\boldsymbol{\pm 4\sqrt{3},\ \pm 2\sqrt{10}}}$

B) If $0 < 2x - A < 10$, then the solutions in x lie strictly between $A/2$ and $A/2 + 5$.
If A is even, then so are the coordinates of the endpoints P and Q. Since this a strict inequality, the endpoints are excluded and there are always 4 integer solutions.

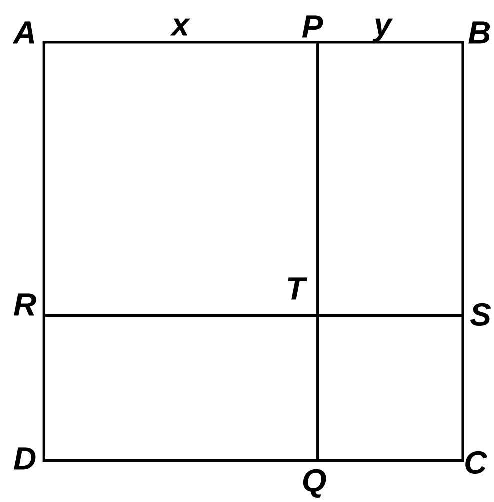


If A is odd (say $2k + 1$ for some integer k), then the coordinates of the endpoints P and Q are $k + \frac{1}{2}$ and $k + 5\frac{1}{2}$. There are 5 integer solutions, namely $(k + 1) \ldots (k + 5)$



Thus, only D and E are true statements.

C) $x^2 + y^2 = k(2xy)$ Moving all terms to the left side and dividing both sides by y^2, we get a quadratic equation in $\frac{x}{y}$:



$$\left(\frac{x}{y}\right)^2 - 2k\left(\frac{x}{y}\right) + 1 = 0$$

Using the quadratic formula,

$$\frac{x}{y} = \frac{2k \pm \sqrt{(2k)^2 - 4}}{2} = k \pm \sqrt{k^2 - 1}.$$

Since $x > y$, the required ratio is greater than 1 and

$$\frac{x}{y} = \underline{\boldsymbol{k + \sqrt{k^2 - 1}}}$$

The same result is obtained if $x^2 - (2ky)x + y^2 = 0$ is treated as a quadratic in x and solved for x in terms of k and y. Verify that you get $x = ky \pm y\sqrt{k^2 - 1}$ and the same result follows.

Team Round - continued

D) $k^2 - 96$ must be positive, so $k \geq 10$ for starters.
$k^2 - 96$ must also be a perfect square, i.e. there is an integer N such that $k^2 - 96 = N^2$ or
$k^2 - N^2 = 96$
Let's examine a list of consecutive perfect squares and look for gaps of 96.

1 4 9 16 25 36 49 64 81 100 121 144 169 196 ….

The gap between consecutive entries grows uniformly by 2, so eventually the gap can be any odd integer. Consecutive perfect squares always differ by an odd amount, so we are looking for non-consecutive perfect squares.
We are looking for perfect squares which sandwich an odd number of consecutive perfect squares.
(Perfect squares sandwiching an even number of perfect squares would differ by an odd number.)
The minimum difference if there are 9 intermediate numbers is 121-1 = 120, so 7 is the maximum number of intermediate perfect squares. There are 4 possible answers.
100 – 4 = 96 $\Rightarrow$ k = **10**

5 intermediate numbers $\Rightarrow$ 6 gaps $\Rightarrow$ a, $a + 2$, … $a + 10$ $\Rightarrow$ $6a + 30 = 96$ $\Rightarrow$ $a = 11$
25 (36 49 64 81 100) 121 $\Rightarrow$ $k^2 = 121$ $\Rightarrow$ k = **11**

3 intermediate numbers $\Rightarrow$ 4 gaps $\Rightarrow$ a, $a + 2$, … $a + 6$ $\Rightarrow$ $4a + 12 = 96$ $\Rightarrow$ $a = 21$
100 (121 144 169) 196 $\Rightarrow$ $k^2 = 196$ $\Rightarrow$ k = **14**

1 intermediate number $\Rightarrow$ $a + (a + 2) = 96$ $\Rightarrow$ $a = 47$ which occurs between ….. hmm?
Continuing the list of perfect squares …, 400, 441, 484, 529, (576), 625 $\Rightarrow$ $k^2 = 625$ $\Rightarrow$ k = **25**

Alternate solution: (**Aaargh!** - Why didn't I go down this road first!!!)

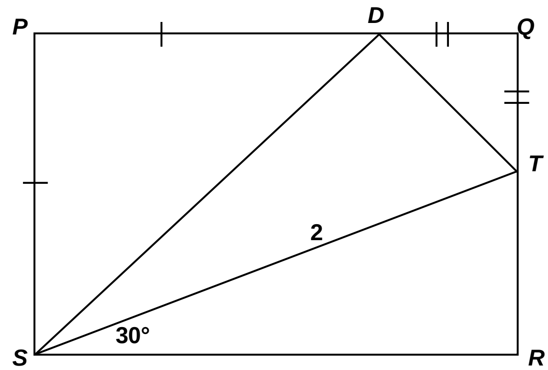
$k^2 - 96 = N^2 \Leftrightarrow k^2 - N^2 = 96 \Leftrightarrow (k + N)(k - N) = 96$
So just look at factor pairs of 96, namely (1, 92), (2, 48), (3, 32), (4, 24), (6, 16), (8, 12)
Equating $k - N$ with the smaller factor and $k + N$ with the larger, we have:

$$\begin{cases} k - N = 1 \quad 2 \quad 3 \quad 4 \quad 6 \quad 8 \\ k + N = 92 \; 48 \;\; 32 \;\; 24 \;\; 16 \;\; 12 \end{cases}$$

Adding and ignoring odd sums, we have $2k$ = 50, 28, 22, 20 $\Rightarrow$ k = **25, 14, 11** and **10** (in any order).

Team Round - continued

E) $ST = 2$ and m$\angle TSR = 30° \Rightarrow TR = 1$ and $SR = \sqrt{3}$.
Isosceles triangles PSD and $DQT \Rightarrow$ m$\angle SDT = 90°$
Let $DQ = QT = x \Rightarrow PS = PD = x + 1 \Rightarrow PQ = 2x + 1$



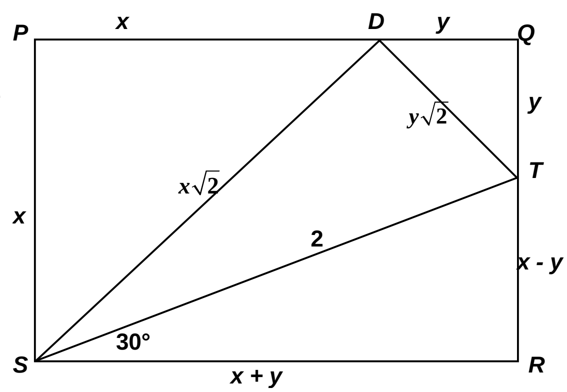
$$\cot(\mathrm{R}STD) = \frac{DT}{DS} = \frac{QT\sqrt{2}}{PD\sqrt{2}} = \frac{x}{x+1}$$

$PQ = SR \Rightarrow 2x + 1 = \sqrt{3} \Rightarrow x = \dfrac{\sqrt{3}-1}{2}$

$\Rightarrow (x + 1) = \dfrac{\sqrt{3}+1}{2}$

Substituting, $$\cot(\mathrm{R}STD) = \frac{\left(\frac{\sqrt{3}-1}{2}\right)}{\left(\frac{\sqrt{3}+1}{2}\right)} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\left(\sqrt{3}-1\right)^2}{2} = \underline{2-\sqrt{\mathbf{3}}}$$

Alternate Solution:
Let $PD = x$ and $QD = y$.
The remaining lengths are as indicated in the diagram. Since $ST = 2$ and m$\angle TSR = 30°$, we have the following system of equations: $\begin{cases} x - y = 1 \\ x + y = \sqrt{3} \end{cases}$



$\Rightarrow (x, y) = \left(\dfrac{1+\sqrt{3}}{2}, \dfrac{-1+\sqrt{3}}{2}\right)$ and

$$\cot(\angle STD) = \frac{DT}{DS} = \frac{y\sqrt{2}}{x\sqrt{2}} = \frac{y}{x} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{4-2\sqrt{3}}{3-1} = \underline{\mathbf{2-\sqrt{3}}}$$

Note: m$\angle STD = 180° – (45° + 60°) = 75°$, but is was not necessary to invoke the expansions for $\sin(A + B)$ and $\cos(A + B)$ to evaluate $\cot(75°)$ as $\dfrac{\cos(30°+45°)}{\sin(30°+45°)}$ to get an exact answer.
This makes the problem solvable by anyone with minimal knowledge of right triangle trig, i.e. SOHCAHTOA, reciprocal relationships and special angles (30°, 60°, 45°).

Team Round - continued

F) Let $x = m\mathrm{R}A$.
P has (10 + n) sides, i.e. at least 11 sides and at most 19 sides.
Thus, $180((10+n)-2) = 10(x+k) + nx = x(10+n) + 10k$
$\Rightarrow (180 - x)(10 + n) = 360 + 10k \Rightarrow x = 180 - \dfrac{360+10k}{10+n}$
As n increases from 1 to 9, the denominator of the fraction increases from 11 to 19.
If n = 1, then the numerator must be the smallest possible multiple of 11.
This occurs for k = 8 and we have $x = A = 180 - 440/11 = 140$.
k = 19 also insures divisibility by 11, but $x = 180 - 550/11 = 130$ and it's not as large as possible.

$n = 2$: $\dfrac{360+10k}{12} = 30 + \dfrac{5k}{6} \Rightarrow k = 6, x = 180 - 35$, $(A, B) = (145,151)$
$n = 3$: $\dfrac{360+10k}{13} = \dfrac{351}{13} + \dfrac{10k+9}{13} \Rightarrow k = 3, x = 180 - 30$, $(A, B) = (150,153)$
$n = 4$: $\dfrac{360+10k}{14} = \dfrac{2\cdot 5(36+k)}{2\cdot 7} \Rightarrow k = 6, x = 180 - 30$, $(A, B) = (150,156)$
$n = 5$: $\dfrac{360+10k}{15} = 24 + \dfrac{2k}{3} \Rightarrow k = 3, x = 180 - 26$, $(A, B) = (154,157)$
$n = 6$: $\dfrac{360+10k}{16} = \dfrac{2\cdot 5(36+k)}{2\cdot 8} \Rightarrow k = 4, x = 180 - 25$, $(A, B) = (155,159)$
$n = 7$: $\dfrac{360+10k}{17} = \dfrac{357}{17} + \dfrac{10k+3}{17} \Rightarrow k = 15, x = 180 - 30$, $(A, B) = (150,165)$
$n = 8$: $\dfrac{360+10k}{18} = 20 + \dfrac{5k}{9} \Rightarrow k = 9, x = 180 - 25$, $(A, B) = (155,164)$
$n = 9$: $\dfrac{360+10k}{19} \Rightarrow k = 2, x = 180 - 20$, $(A, B) = (160,162)$
Thus, (m, M) = **<u>(2, 15)</u>**.

Checks that $10(m\angle B) + n(m\angle A) = 180(n-2)$
(not necessary, but included just to be on the safe side!)
n = 2: 1510 + 2(145) = (12 – 2)180 = 1800
n = 3: 1530 + 3(150) = (13 – 2)180 = 1980
n = 4: 1560 + 4(150) = (14 – 2)180 = 2160
n = 5: 1570 + 5(154) = (15 – 2)180 = 2340
n = 6: 1590 + 6(155) = (16 – 2)180 = 2520
n = 7: 1650 + 7(150) = (17 – 2)180 = 2700
n = 8: 1640 + 8(155) = (18 – 2)180 = 2880
n = 9: 1620 + 9(160) = (19 – 2)180 = 3060