**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2012**
**ROUND 1 ALGEBRA 2: ALGEBRAIC FUNCTIONS**

**ANSWERS**

A) ________________________________

B) ________________________________

C) ( ________ , ________ , ________ )

******* NO CALCULATORS ON THIS ROUND *******

A) Let $f(x)=\begin{cases} 4x+5 & \text{for } x>2 \\ 20 & \text{for } -2<x\le 2 \\ 6x-8 & \text{for } -10\le x\le -2 \\ 12 & \text{for } x<-10 \end{cases}$. Compute $f\left(f(2)\right)+f\left(f(-2)\right)+f(0)$.

B) Functions *f* and *g* are defined as follows:

$f : f(x) = 4 - 2x$
$g = \{(1, -2), (2, -3), (-2, 4), (3, 20), (0, 12)\}$

Compute $f^{-1}\left(g(3)\right)+g^{-1}\left(f(3)\right)$.

C) Given $f(x) = \dfrac{3x+1}{x-1}$. When the input value *x* is doubled, the new output can be expressed in terms of the original output. Specifically, $f\left(2x\right)=\dfrac{Af(x)+B}{f(x)+C}$ for integer constants *A*, *B* and *C*. Compute the ordered triple (*A*, *B*, *C*).

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 - FEBRUARY 2012**
**ROUND 2 ARITHMETIC / NUMBER THEORY**

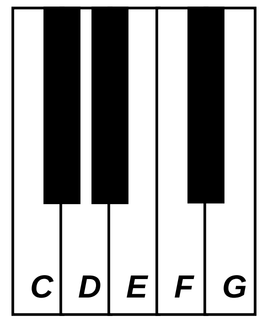
**ANSWERS**

A) C D E F G

B) ______________________________

C) ( ____________ , ____________ )

******* NO CALCULATORS ON THIS ROUND *******




A) Engelbert practices proper finger position on the piano using only his right hand. Starting with his thumb he plays note #1, middle C, followed immediately by DEFGFED, notes #2 - #8, using the other fingers of his right hand. Starting again with middle C, note #9, he continues to repeat this exercise (ad nauseum). He gets to stop for lunch after he plays note #2012. On the answer blank above, circle note #2012.

B) Determine all primes between 300 and 500 ending in 7 whose digit sum is a multiple of 11.

C) A spinner has 13 equally spaced positions, numbered 1 through 13 clockwise.
Pointer $A$ is initially pointing at 3 and moves clockwise 7 positions every second.
Pointer $B$ is initially pointing at 3 and moves counterclockwise 5 positions every second.
Pointer $C$ is initially pointing at 3 and moves clockwise 2 positions every second.
Let $(a, b, c)$ denote the numbers being referenced by the pointers $A$, $B$ and $C$ at one second intervals and $S(n) = a + b + c$, after $n$ seconds have elapsed.
For example, at $n = 1$, $(a, b, c) = (10, 11, 5)$ and $S(1) = 26$.
Compute $(n, m)$, where $m$ = the maximum value of S($n$) and $n$ is as small as possible.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 - FEBRUARY 2012**
**ROUND 3 TRIG: IDENTITIES AND/OR INVERSE FUNCTIONS**

**ANSWERS**

A) ______________________________

B) ______________________________

C) ( ________ , ________ , ________ )

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $Arc\tan(0.75) = Arc\sin(N)$
Compute $N$.

B) Compute: $\cos\left(Cos^{-1}\left(-\frac{3}{5}\right) - Tan^{-1}\left(-\frac{40}{9}\right)\right)$

C) $\left(2\sin 7\frac{1}{2}^\circ \cos 7\frac{1}{2}^\circ\right)\left(1 - 2\sin^2 37\frac{1}{2}^\circ\right) = \frac{A-\sqrt{B}}{C}$ Compute the ordered triple ($A$, $B$, $C$).

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 - FEBRUARY 2012**
**ROUND 4 ALG 1: WORD PROBLEMS**

**ANSWERS**

A) ______________________ mph

B) ______________________

C) ______________________

******* NO CALCULATORS ON THIS ROUND *******

A) A train traveled 300 miles in $t$ hours. When the speed of the train was increased by 5 mph, it covered 20 more miles in the same amount of time. Find the faster rate of the train.

B) There are three types of coins in the fountain - pennies, dimes and quarters worth \$6.66. If there are 156 pennies, what is the maximum number of coins in the fountain?

C) From past experience, an assembly line of $A$ robots all programmed to work at the same rate could have completed a job in $B$ days. However, before the job even got started, $C$ robots are removed from the original crew for maintenance. How many more days will be needed to complete the original job? Give your answer as a simplified expression in terms of $A$, $B$ and $C$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 - FEBRUARY 2012**
**ROUND 5 PLANE GEOMETRY: CIRCLES**

**ANSWERS**

A) ________________________________

B) ________________________________

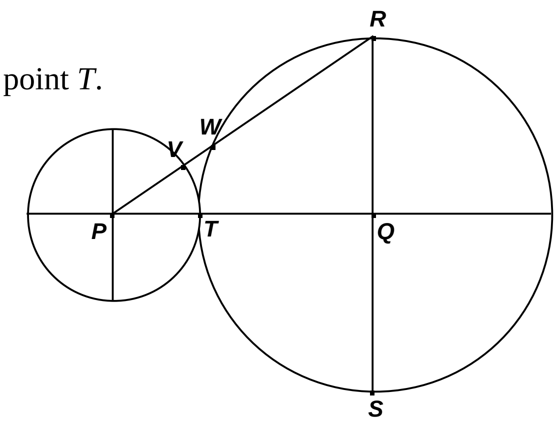
C) ________________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Four congruent circles are inscribed in a square. If the circumference of each is $12\pi$, compute the area of the circle circumscribed about this square.

B) Point $P$ is exterior to (but in the same plane as) circle $O$.
$Q$ is the point on circle $O$ closest to $P$.
$R$ is the point on circle $O$ farthest from $P$.
If $PQ = 4$ and $PR = 20$, compute the length of a tangent to circle $O$ from point $P$.

C) Circle $P$ with radius 1 is tangent to circle $Q$ with radius 3 at point $T$.
Diameter $\overline{RS}$ is perpendicular to the line of centers
at point $Q$. $\overrightarrow{PR}$ intersects the circles at points $V$ and $W$.
Compute $VW$.



**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 - FEBRUARY 2012**
**ROUND 6 ALG 2: SEQUENCES AND SERIES**

**ANSWERS**

A) ______________________________

B) ______________ : ______________

C) ______________________________

******* NO CALCULATORS ON THIS ROUND *******

A) $4x+1,\ 7x,\ 8x+3$ are the first three terms of an arithmetic progression.
Find the <u>sum</u> of the first 30 terms of this progression.

B) $(x + 2)$, $(4x + 2)$ and $(12x + 6)$ are the first, second and third terms of a geometric sequence. When 12 is added to the middle term, this sequence of three terms becomes the first three terms of an arithmetic sequence. Compute the ratio of the <u>sixth</u> term of the arithmetic sequence to the <u>fifth</u> term of the geometric sequence

C) Given: $T = \sum_{n=1}^{\infty} A(-0.7)^{n-1}$ and $B_n = \sum_{k=1}^{n} b_k$, where $\begin{cases} b_{k+1} = 2b_k + k \\ b_1 = 8 \end{cases}$.

Compute $A$, if $\frac{1}{T} = B_4$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 - FEBRUARY 2012**
**ROUND 7 TEAM QUESTIONS**
**ANSWERS**

A) ____________________________ D) ____________________________

B) ____________________________ E) ___________________________ °

C) ( ____________ , ____________ ) F) ____________________________

******* NO CALCULATORS ON THIS ROUND *******

A) Given: $f(x)=2x^4+x^3$, $f(h(x))=32x^4-56x^3+36x^2-10x+1$
If $h(x)=Ax+B$, where $A$ and $B$ are integer constants, compute $h^{-1}(3)$.

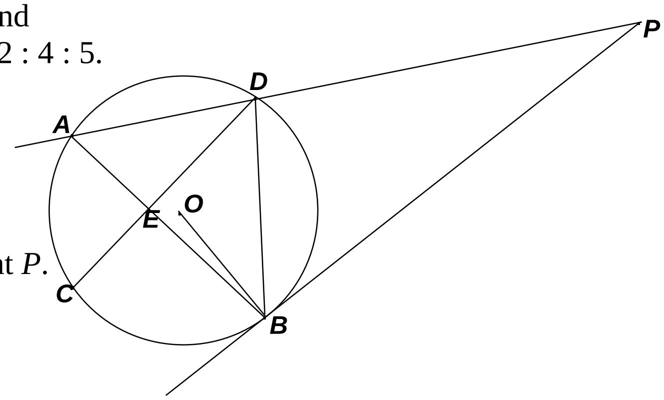
B) Let $M$(b) be the base 10 representation of the minimum natural number in base $b$ that has a digit sum greater than 10. For example, $M(10)=29$ and $M(3)=122222_{(3)}=845_{(10)}$.
Compute $\sum_{b=4}^{b=9} M(b)$. Recall: $\sum$ is the summation symbol. (Ex: $\sum_{x=1}^{x=4} x^2=1+4+9+16=30$)

C) Compute the ordered pair ($A$, $B$) for which the following equation is an identity, for all values of $x$ for which both sides of the equation are defined.

$$\frac{2\tan x\left(1-\tan^2 x\right)}{\left(1+\tan^2 x\right)^2}=A\sin\left(Bx\right)$$

D) In a special summer session of Hogwarts School of Witchcraft and Wizardry, three courses were offered to new students: Charms ($C$), Potions ($P$) and Flying ($F$).
Every student chose to take at least one course and some chose to take multiple courses.
Let $XY$ denote taking both course $X$ and course $Y$.
Let $X + Y$ denote taking either course $X$ or course $Y$ (or both).
Let $n(X)$ denote the number of students signed up for course $X$.

Given: $n(C) = 30$, $n(C + P + F) = 116$, $n(CPF) = 6$ and
$n(CP) : n(CF) : n(PF) = n(C) : n(P) : n(F) = 2 : 4 : 5$.
Compute the <u>largest</u> possible number of students who could have signed up <u>just for flying</u>.




E) $\overline{AB}$ and $\overline{CD}$ are chords in circle $O$ that intersect at point $E$. A secant line through points $A$ and $D$ and a line tangent to the circle at point $B$ intersect at point $P$.
If m$\angle DBA$ = m$\angle ADC$ + 10°, m$\angle P$ = 5° and m$\angle AED$ : m$\angle BED$ = 4 : 5, compute m$\angle EBO$.

F) The sum of an infinite geometric progression with first term $a$ and common multiplier $r$, is one more than the sum of its first two terms. If $2 \le a \le 6$, compute <u>all</u> possible values of $r$.

# MASSACHUSETTS MATHEMATICS LEAGUE
# CONTEST 5 – FEBRUARY 2012 ANSWERS

**Round 1 Alg 2: Algebraic Functions**

A) 117 B) $-7$ C) (7, 3, 5)

**Round 2 Arithmetic/ Number Theory**

A) F B) 317 C) (5, 29)

**Round 3 Trig Identities and/or Inverse Functions**

A) 0.6 (or $\frac{3}{5}$) B) $-\frac{187}{205}$ C) (2, 3, 4)

**Round 4 Alg 1: Word Problems**

A) 80 mph B) 204 C) $\frac{BC}{A-C}$

**Round 5 Geometry: Circles**

A) $288\pi$ B) $4\sqrt{5}$ C) 0.4 (or $\frac{2}{5}$)

**Round 6 Alg 2: Sequences and Series**

A) 2445 B) 7 : 27 C) $\frac{1}{80}$

**Team Round**

A) 2 D) 40

B) 476 E) 20°

C) $\left(\frac{1}{2},4\right)$ F) $-1<r\le-\frac{1}{2}$ or $\frac{1}{3}\le r\le\frac{1}{2}$

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2012 SOLUTION KEY**

## Round 1

A) Given: $f(x)=\begin{cases} 4x+5 & \text{for } x>2 \\ 20 & \text{for } -2<x\le 2 \\ 6x-8 & \text{for } -10\le x\le -2 \\ 12 & \text{for } x<-10 \end{cases}$

$f\left(f(2)\right)+f\left(f(-2)\right)+f(0)=f(20)+f(-20)+20$ = 85 + 12 + 20 = **<u>117</u>**.

B) Given $f(x) = 4 - 2x$. Since $f$ multiplies $x$ (the input value) by –2 and then adds 4, the inverse function $f^{-1}$ will perform the "opposite" operations in the opposite order, namely subtract 4 (from $x$) and then divide by –2.

Thus, $f^{-1}(x)=\dfrac{x-4}{-2}=\dfrac{4-x}{2}$.

$f^{-1}\left(g(3)\right)=f^{-1}(20)=(4-20)/2=-8$

$g^{-1}\left(f(3)\right)=g^{-1}(-2)$

The function $g^{-1}$ is obtained by transposing the coordinates of the ordered pairs in $g$.

Thus, $g^{-1}(-2)=1$ and the required sum is **<u>–7</u>**.

C) Replace $f(x)$ with $y$, i.e. let $y=\dfrac{3x+1}{x-1}$ and solve for $x$ in terms of $y$.

$\Rightarrow xy - y = 3x + 1$. Solving for $x \Rightarrow xy - 3x = y + 1 \Rightarrow x = \dfrac{y+1}{y-3}$

Thus, $f(2x)=\dfrac{6x+1}{2x-1}=\dfrac{6\dfrac{y+1}{y-3}+1}{2\dfrac{y+1}{y-3}-1}=\dfrac{6(y+1)+(y-3)}{2(y+1)-(y-3)}=\dfrac{7y+3}{y+5}=\dfrac{7f(x)+3}{f(x)+5}$

and $(A, B, C)$ = **<u>(7, 3, 5)</u>**.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2012 SOLUTION KEY**

**Round 2**

A) The notes are being played in repeating blocks of 8 in the sequence CDEFGFED.
$\frac{2012}{8} = 251, \quad r = 4$.
Thus, Engelbert completed 251 blocks and stops on the $4^{th}$ note in the $252^{nd}$ block
$\Rightarrow$ Note #2012 = **F**.

B) The 3-digit number must be of the form 3*x*7 or 4*x*7 $\Rightarrow$ digit sum is 10 + *x* or 11 + *x*.
For integers in the 300s, we must test only 317.
For integers in the 400s, we must test only 407
Since 407 = 11(37), we need only rigorously test 317.
We must test for divisibility only by primes smaller than $\sqrt{317}$. Since $19^2 = 361 > 317$, the divisors to be tested are: 2, 3, 5, 7, 11, 13 and 17.
The first three divisors are easily eliminated.
$7 \Rightarrow r = 2$ $\quad 11 \Rightarrow r = 9$ $\quad 13 \Rightarrow r = 5$ $\quad 17 \Rightarrow r = 11$
Thus, **317** is prime.

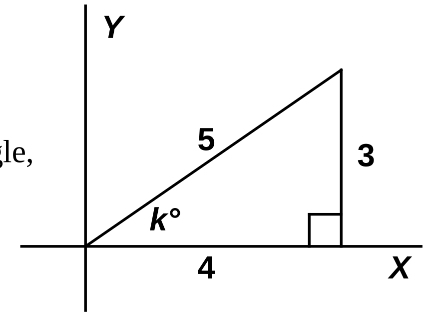
C) Each pointer cycles through the 13 numbered positions, before returning to 3.
For pointer *A*, spin clockwise (7 spaces)
3 + 7 = 10, 10 + 7 = 17 (–13) = 4 etc. Whenever the sum exceeds 13, subtract 13.
For pointer *B*, spin counterclockwise (5 spaces)
5 spaces counterclockwise is equivalent to 8 spaces clockwise, since 5 + 8 = 13!
It's easier to add 8 and deal with positive numbers than to subtract 5 and deal with negative numbers. As above, when the total exceeds 13, subtract 13.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Pointer *A*: | 3 | 10 | 4 | 11 | 5 | 12 | 6 | 13 | 7 | 1 | 8 | 2 | 9 | 3 |
| Pointer *B*: | 3 | 11 | 6 | 1 | 9 | 4 | 12 | 7 | 2 | 10 | 5 | 13 | 8 | 3 |
| Pointer *C*: | 3 | 5 | 7 | 9 | 11 | 13 | 2 | 4 | 6 | 8 | 10 | 12 | 1 | 3 |
| Sums: | | 26 | 17 | 21 | 25 | $\boxed{29}$ | 20 | 24 | 15 | 19 | 23 | 27 | 18 | |

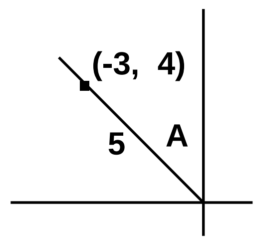
Thus, the maximum sum is 29 and it occurs first for *n* = 5 $\Rightarrow$ **(5, 29)**.

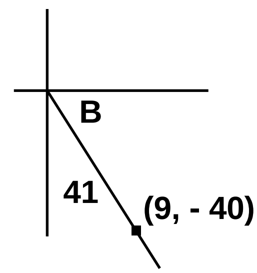
**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2012 SOLUTION KEY**

**Round 3**




A) Think 3 – 4 – 5 right triangle!
$Arc\tan\left(0.75\right)$ is the measure of the smaller acute angle,
namely $k°$, since $\tan(k°) = \dfrac{y}{x} = \dfrac{OPP.}{ADJ.} = 0.75 = \dfrac{3}{4}$.
$\sin(k°) = \underline{\mathbf{\dfrac{3}{5}}}$ (or **<u>0.6</u>**) Recall: SOH-CAH-TOA




B) Let $A$ denote $Cos^{-1}\left(-\dfrac{3}{5}\right)$. Let $B$ denote $Tan^{-1}\left(-\dfrac{40}{9}\right)$.
Think Pythagorean triples (3,4,5) and (9, 40, 41).
The <u>principle</u> inverse cosine ( $Arc\cos(\theta)$ or $Cos^{-1}(\theta)$ )
denotes an angle in the first or second quadrant,
specifically $0 \le \theta \le \pi$ . The principle inverse tangent ( $Arc\tan(\theta)$
or $Tan^{-1}(\theta)$ ) denotes an angle in the first or fourth quadrant, specifically $-\dfrac{\pi}{2} < \theta < \dfrac{\pi}{2}$.
Since $A < 0$, $A$ is located in quadrant 2 as indicated in the diagram.
Since $B < 0$, $B$ is located in quadrant 4 as indicated in the diagram.
$\cos(A - B) = \cos A\cos B + \sin A\sin B =$

$$\left(-\frac{3}{5}\right)\left(\frac{9}{41}\right)+\left(\frac{4}{5}\right)\left(-\frac{40}{41}\right)=\frac{-27-160}{205} = \underline{\mathbf{-\frac{187}{205}}}$$

C) $$\left(2\sin 7\frac{1}{2}°\cos 7\frac{1}{2}°\right)\left(1-2\sin^2 37\frac{1}{2}°\right) = \sin 15°\cdot\cos 75° = \sin^2 15° = \left(\sin(45-30)\right)^2$$

$$=\left(\sin 45\cos 30-\sin 30\cos 45\right)^2 = \left(\frac{\sqrt{6}-\sqrt{2}}{4}\right)^2$$

$$= \frac{8-2\sqrt{12}}{16} = \frac{8-4\sqrt{3}}{16} = \frac{2-\sqrt{3}}{4} \Rightarrow (A, B, C) = \underline{\mathbf{(2, 3, 4)}}.$$

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2012 SOLUTION KEY**

**Round 4**

A) $\frac{300}{t}+5=\frac{320}{t} \Rightarrow 300 + 5t = 320 \Rightarrow t = 4$.

Thus, the faster rate is $\frac{320}{4}=$ **80** mph

Alternately, since (Rate)(Time) = (Distance), examine possible ordered pairs $(R, T)$ for which $RT = 300$.

$(R, T) = (300, 1), (150, 2), (100, 3), (75, 4), (60, 5), (50, 6), \ldots$..

If the rate is increased by 5 mph, the distance traveled in the same time is 320 miles.

This happens for $R = 75$, since $(75 + 5)4 = 320$ and we have the same result.

B) Let $D$ and $Q$ denote the number of dimes and quarters respectively. Then:

$10D + 25Q = 666 - 156 = 510 \Rightarrow 2D + 5Q = 102$

$(D, Q) = (1, 20), (6, 18), \ldots, (46, 2), (51, 0)$

Since there are 3 types of coins in the fountain (51, 0) is rejected and the maximum number of coins in the fountain is 156 + 46 + 2 = **204**.

C) Let $X$ denote the number of days it would take for $(A - C)$ men to complete the job. The time it takes to complete a job is inversely proportional to the size of the work force. The larger the workforce, the less time it takes to complete the job.

Thus, $X > B$ and we have the proportion $\frac{A}{A-C}=\frac{X}{B} \Rightarrow X = \frac{AB}{A-C}$.

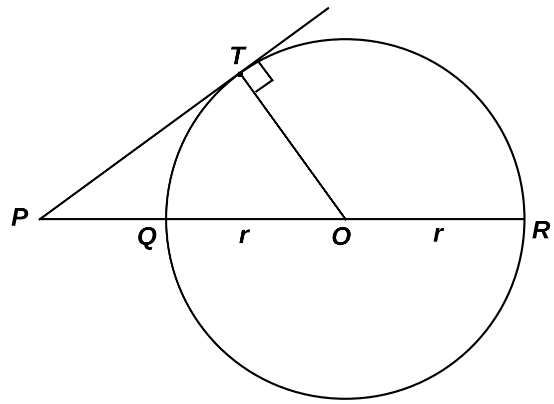
The additional days needed is $x - B = \frac{AB}{A-C}-B=\frac{AB-B(A-C)}{A-C}=\boldsymbol{\frac{BC}{A-C}}$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2012 SOLUTION KEY**

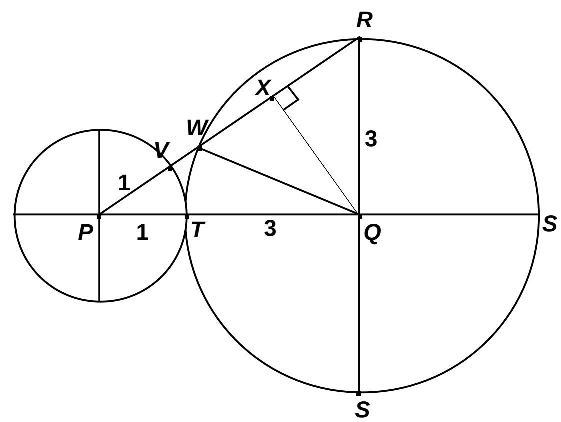
## Round 5

A) $C = 12\pi \Rightarrow$ radius $r = 6 \Rightarrow$ diameter $d = 12 \Rightarrow$ side of square $s = 24$
$\Rightarrow$ diagonal of the square = $24\sqrt{2}$
$\Rightarrow$ radius of the circumscribed circle = $12\sqrt{2}$ $\Rightarrow$ area = $\underline{\mathbf{288\pi}}$

B) Draw $\overleftrightarrow{PO}$. This line intersects the circle twice.
The point between $P$ and $O$ is the closet point, $Q$.
The other point of intersection is point $R$.
$PR = PQ + QR \Rightarrow 20 = 4 + QR$
Since $QR$ is a diameter, the radius of circle $O$ is 8.
Using the Pythagorean Theorem on $\Delta TPO$,
$PT^2 + 8^2 = 12^2 \Rightarrow PT = \sqrt{80} = \underline{\mathbf{4\sqrt{5}}}$




C) $\Delta RQP$ is a 3 – 4 – 5 right triangle (with area 6).
Thus, $\frac{1}{2}(QX)(PR) = 6 \Rightarrow \frac{1}{2} \cdot QX \cdot 5 = 6 \Rightarrow QX = \frac{12}{5}$.
Note since $\Delta RXQ \sim \Delta RQP$, $\Delta RXQ$ is a scaled version of a 3 – 4 – 5 triangle.
$\left(\frac{12}{5}, \_\_, 3\right) = \frac{1}{5}(12, \_\_, 15) = \frac{3}{5}(4, \_\_, 5)$
Rather than grinding out the Pythagorean Theorem for $\Delta RXQ$, we see that $XR = \frac{3}{5}(3) = \frac{9}{5} = 1.8$
Therefore, $WR = 3.6$ and $VW = 5 – 1 – 3.6 = \underline{\mathbf{0.4}}$




Alternately, consider two secants from point $P$ to the larger circle $Q$.
$PR = 5$. Let $VW = x$. We know that, for any secant lines drawn to a circle from the same external point, the product of the length of the external segment and the sum of the external and internal segments is constant. In the diagram above, this means $PW(PR) = PT(PS)$
This gives us $(1 + x)(5) = (1)(7)$ and we quickly have $x = \frac{7}{5} - 1 = \underline{\mathbf{0.4}}$.

**MASSACHUSETTS MATHEMATICS LEAGUE**
**CONTEST 5 – FEBRUARY 2012 SOLUTION KEY**

**Round 6**

A) $8x+3-7x=7x-(4x+1) \Rightarrow 2x=4 \Rightarrow x=2$
AP: 9, 14, 19, …
$a=9$, $d=5$ and $n=30 \Rightarrow S_{30}=\frac{30}{2}\left(2(9)+(30-1)5\right)=15(18+29\cdot 5)=15(163)=$ **2445**

B) The first three terms of the AP are $(x+2)$, $(4x+14)$ and $(12x+6)$.
The common difference $d=(4x+14)-(x+2)=(12x+6)-(4x+14) \Rightarrow 3x+12=8x-8$
$\Rightarrow x=4$ and $d=24$. The AP is 6, 30, 54, … and the GP is 6, 18, 54, … .
The 6th term in the AP is $(54+3\cdot 24)=126=6\cdot 21=2\cdot 3^2\cdot 7$.
The 5th term in the GP is $54\cdot 3^2=2\cdot 3^5$.
The required ratio is **7 : 27**.

C) $T$ is the sum of an infinite geometric sequence whose first term is $A$ and $r$, the ratio between successive terms, is $-\frac{7}{10}$. Since $|r|<1$, the infinite geometric series converges to

$$T=\frac{a}{1-r}=\frac{A}{1-\left(-\frac{7}{10}\right)}=\frac{10A}{17}.$$

$b_1=8 \Rightarrow b_2=2(8)+1=17 \Rightarrow b_3=2(17)+2=36 \Rightarrow b_4=2(36)+3=75$

$B_4=8+17+36+75=136$.

Equating, $\frac{17}{10A}=136 \Rightarrow A=\frac{\cancel{17}^{1}}{\cancel{136}^{8}(10)}=\mathbf{\frac{1}{80}}$.

## MASSACHUSETTS MATHEMATICS LEAGUE
## CONTEST 5 - FEBRUARY 2012 SOLUTION KEY

### Team Round

A) $f(x) = 2x^4 + x^3 = x^3(2x+1)$

Thus, $f(h(x)) = \left(h(x)\right)^3\left(2h(x)+1\right) = \left(Ax+B\right)^3(2Ax+(2B+1))$.

Expanding $\left(Ax+B\right)^3(2Ax+(2B+1))$, the lead coefficient would be $2A^4$ and the constant term would be $B^3(2B+1)$. Therefore, $2A^4 = 32 \Rightarrow A = \pm 2$ and

$B^3(2B+1) = 1 \Leftrightarrow 2B^4 + B^3 - 1 = 0 \Rightarrow B = -1$

Consequently, $h(x) = 2x - 1$ or $-2x - 1$.

However, checking the other coefficients of $f(h(x))$, only $A = 2$ produces the correct coefficients for $x^3$, $x^2$ and $x$ and $h(x) = 2x - 1$ only.

Thus, $h^{-1}(x) = \dfrac{x+1}{2}$ and $h^{-1}(3) = \dfrac{4}{2} = \underline{\mathbf{2}}$.

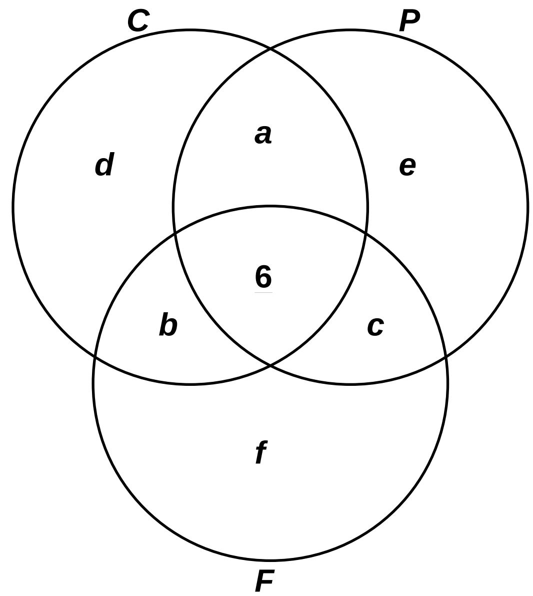
B) Base 4: $2333_{(4)} = 2(4^3) + (4^3 - 1) = 3(64) - 1 = 191$
Base 5: $344_{(5)} = 3(5^2) + (5^2 - 1) = 99$
Base 6: $155_{(6)} = 36 + 30 + 5 = 71$
Base 7: $56_{(7)} = 35 + 6 = 41$
Base 8: $47_{(8)} = 32 + 7 = 39$
Base 9: $38_{(9)} = 27 + 8 = 35$
Total : **<u>476</u>**

C) $$\frac{2\tan x\left(1-\tan^2 x\right)}{\left(1+\tan^2 x\right)^2} = \frac{2\dfrac{\sin x}{\cos x}\left(\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x}\right)}{\sec^4 x} = 2\sin x\left(\frac{\cos^2 x - \sin^2 x}{\cos^3 x}\right)\cos^4 x$$

$$= 2\sin x\cos x\left(\cos^2 x - \sin^2 x\right) = \sin 2x\cos 2x = \frac{1}{2}\left(2\sin 2x\cos 2x\right) = \frac{1}{2}\sin 4x$$

Thus, $(A, B) = \underline{\left(\mathbf{\frac{1}{2}, 4}\right)}$.

**Team Round - continued**




D) (A classic Venn Diagram problem)

$\frac{a+6}{b+6}=\frac{2}{4}=\frac{1}{2} \Leftrightarrow 2a+12=b+6 \Leftrightarrow b=2a+6$

$\frac{a+6}{c+6}=\frac{2}{5} \Leftrightarrow 5a+30=2c+12 \Leftrightarrow c=\frac{5a+18}{2}$

To insure that $c$ is an integer, $a$ must be even.
Since the total number of students involved is 116, to maximize $f$, we must minimize $a$.
Let $n(C) = 2N$, $n(P) = 4N$ and $n(F) = 5N$.
If $a = 2$, $(b, c) = (10, 14)$

$\begin{cases} d+18=2N \\ e+22=4N \text{ and} \\ 30+f=5N \end{cases}$

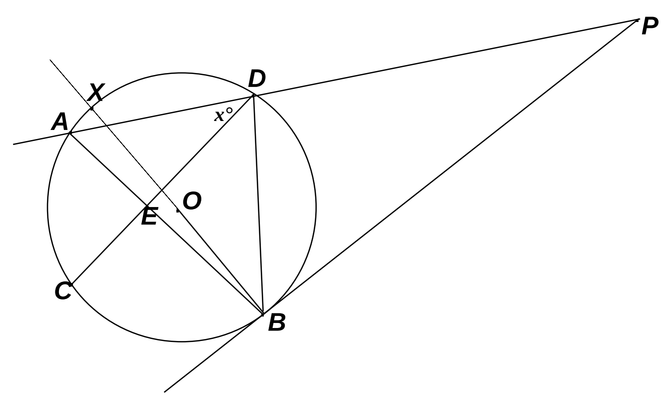
$(2+10+14)+d+e+f+6=116 \Leftrightarrow d+e+f=84$

Consequently,

$\Rightarrow (2N-18)+(4N-22)+(5N-30)=84 \Leftrightarrow 11N=154 \Leftrightarrow N=14 \Rightarrow f=70-30=\underline{\mathbf{40}}$.

E) Since angles $AED$ and $BED$ are supplementary, a 4 : 5 ratio implies m$\angle AED$ = 80° and m$\angle BED$ = 100°.

Let m$\angle ADC$ = $x$° and m$\angle DBA$ = $(x + 10)$° and $m\left(\overset{\frown}{BD}\right)=y°$. All arcs named with two letters refer to minor arcs. As arcs subtended by inscribed angles, $m\left(\overset{\frown}{AC}\right)=2x°$ and $m\left(\overset{\frown}{AD}\right)=(2x+20)°$.




As a leftover arc, $m\left(\overset{\frown}{BC}\right)=(340-4x-y)°$

As an angle formed by intersecting chords,

$m\angle BED=\frac{1}{2}(2x+y)=100 \Leftrightarrow 2x+y=200$

As an angle formed by a tangent and a secant line

$m\angle P=\frac{1}{2}\left((340-2x-y)-y\right)=5$

$\Leftrightarrow 340-2x-2y=10$

$\Leftrightarrow x+y=165$

Thus, $x = 35$, $y = 130 \Rightarrow m\left(\overset{\frown}{AD}\right)=90° \Rightarrow m\left(\overset{\frown}{ADB}\right)=220°$ (a major arc)

If $\overrightarrow{BO}$ intersects the circle in point $X$, then $m\left(\overset{\frown}{AX}\right)=220°-180°=40°$.

As an inscribed angle, m$\angle EBO$ = $\underline{\mathbf{20}}$°.

**Team Round - continued**

F) An infinite geometric series converges to a sum $\left(\dfrac{a}{1-r}\right)$ if and only if $|r| < 1$.

Thus, $\dfrac{a}{1-r} = a + ar + 1 = a(1+r) + 1 \Leftrightarrow a = a\left(1-r^2\right) + \left(1-r\right) \Leftrightarrow ar^2 + r - 1 = 0 \Leftrightarrow a = \dfrac{1-r}{r^2}$.

This means $2 \le \dfrac{1-r}{r^2} \le 6 \Leftrightarrow 2r^2 \le 1-r$ and $1-r \le 6r^2$ or $\begin{cases} 2r^2 + r - 1 \le 0 \\ 6r^2 + r - 1 \ge 0 \end{cases}$.

We must take the intersection of these two conditions.

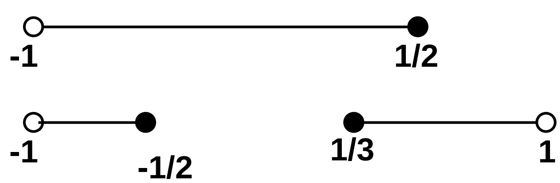
$$\begin{cases} 2r^2 + r - 1 \le 0 \\ 6r^2 + r - 1 \ge 0 \end{cases} \Leftrightarrow \begin{cases} (2r-1)(r+1) \le 0 \\ (2r+1)(3r-1) \ge 0 \end{cases}$$

The first condition requires that $-1 \le x \le \dfrac{1}{2}$, but convergence requires $r \neq -1$; hence,

$-1 < x \le \dfrac{1}{2}$.

The second condition requires $r \le -\dfrac{1}{2}$ or $r \ge \dfrac{1}{3}$, but convergence requires $-1 < r < 1$; hence,

$-1 < r \le -\dfrac{1}{2}$ or $\dfrac{1}{3} \le r < 1$




Carefully taking the intersection, we have $\underline{\boldsymbol{-1 < r \le -\frac{1}{2}} \text{ or } \boldsymbol{\frac{1}{3} \le r \le \frac{1}{2}}}$.

Extra Problems Contest #5 – return to database
Round 1
$y = f(x)$ is a polynomial function of minimum degree with integer coefficients and zeros of $-1$, $\sqrt{6}$ and $i$. If the lead coefficient is positive and the greatest common factor of the coefficients is 1, compute the ratio $\dfrac{2f(-2)}{3f(3)}$.

Ans: 1 : 18

Since $y = f(x)$ has roots of $-1$, $\sqrt{6}$ and $i$ and has integral coefficients, it must also have roots of $-\sqrt{6}$ and $-i$. Thus, $f(x) = (x+1)\left(x-\sqrt{6}\right)\left(x+\sqrt{6}\right)(x-i)(x+i) = (x+1)\left(x^2-6\right)\left(x^2+1\right)$ and $\dfrac{2f(-2)}{3f(3)} = \dfrac{2\left(-1\cdot -2\cdot 5\right)}{3\left(4\cdot 3\cdot 10\right)} = \dfrac{20}{360} = \underline{\boldsymbol{\dfrac{1}{18}}}$.