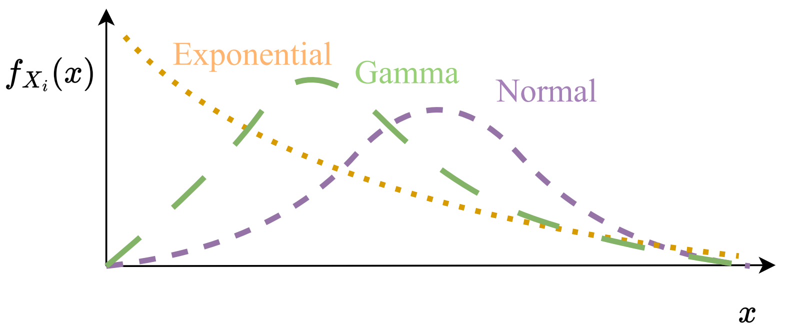
Compound Poisson Process

$(t;\lambda,f_{X_i})$

Parameters are continuous, states are discrete



The magnitude of i^{th} event $X_i\in(0,+\infty)$ could follow any specified distribution, e.g. continuous dist.: normal, gamma (exponential is a special case of gamma distribution); or discrete dist.

If X_i is gamma distribution with shape parameter a and rate parameter b

$$X_i\in(0,+\infty)\sim\Gamma(a,b),a>0,b>0$$

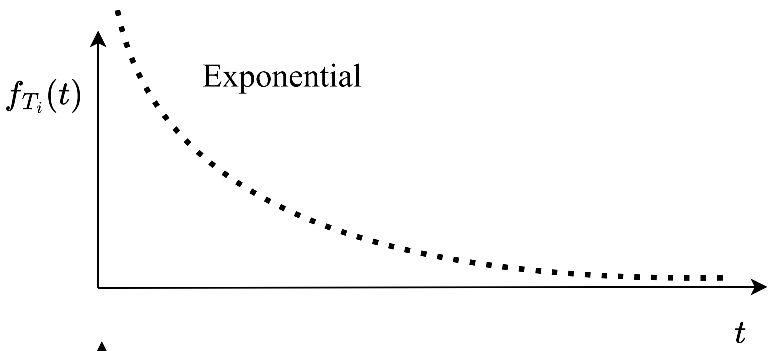
PDF: $f_X(x)=\frac{x^{a-1}e^{-x/b}}{b^a\Gamma(a)}$

CDF: regularized incomplete gamma function $F_X(x)=\frac{\gamma(a,x/b)}{\Gamma(a)}$

lower incomplete gamma function $\gamma(a,x/b)=\int_0^{x/b}t^{a-1}e^{-t}dt$

gamma function $\Gamma(a)=\int_0^{\infty}t^{a-1}e^{-t}dt$

$\mu_X=ab,\sigma_X=\sqrt{a}b$



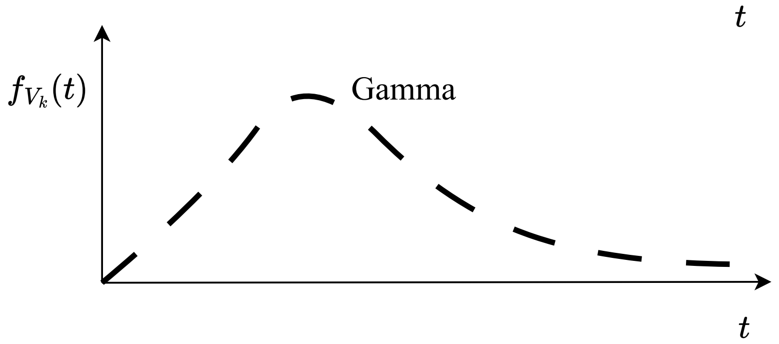
The time or distance between $(i-1)^{th}$ and i^{th} success:

$$T_i\in(0,+\infty)\sim exp(\lambda),\lambda>0$$

PDF $f_{T_i}(t)=\lambda e^{-\lambda t}$

CDF $F_{T_i}(t)=1-e^{-\lambda t}$

$\mu_{T_i}=\frac{1}{\lambda},\sigma_{T_i}=\frac{1}{\lambda}$



The time until obtaining the k^{th} success:Erlang dist.

$$V_k\in(0,+\infty)\sim\Gamma(k,\lambda),\lambda>0,k=1,2,\cdots$$

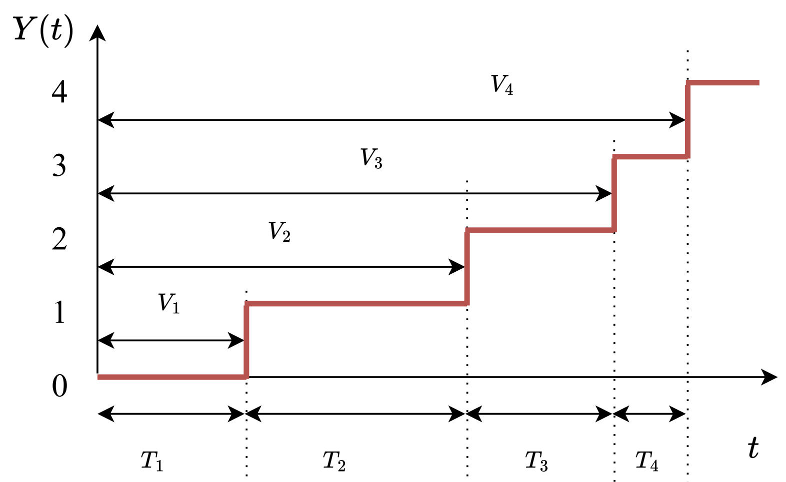
In numpy

V_k = np.random.gamma(shape = k,scale = 1/lambda)

PDF $f_{V_k}(t)=\frac{\lambda^k t^{k-1}e^{-\lambda t}}{\Gamma(k)}=\frac{\lambda^k t^{k-1}e^{-\lambda t}}{(k-1)!}$

CDF $F_{V_k}(t)=\frac{\gamma(k,\lambda t)}{\Gamma(k)}=\frac{\int_0^{\lambda t}\tau^{k-1}e^{-\tau}d\tau}{(k-1)!}$

$\mu_{V_k}=\frac{k}{\lambda},\sigma_{V_k}=\frac{\sqrt{k}}{\lambda}$



The total number of successes until time t

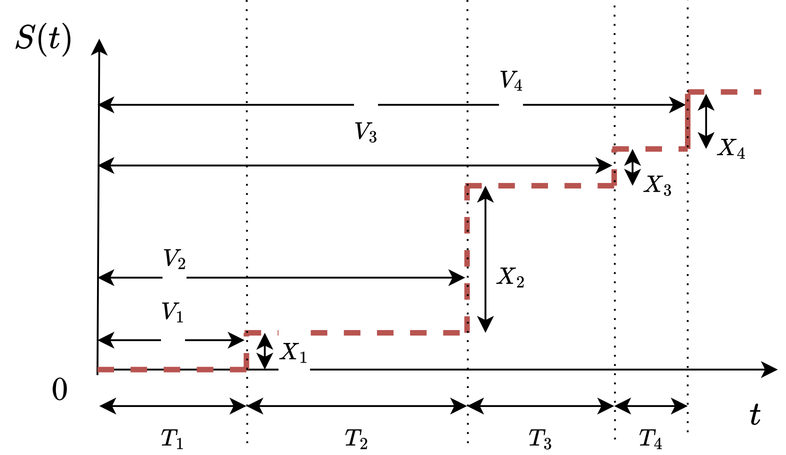
$$Y(t)\in\{0,1,2,\cdots\}\sim poisson(\nu),\nu=\lambda t,\lambda>0,t>0$$

$$\lim_{y\to\infty,p\to 0,yp\to\lambda t} binomial(y,p)=poisson(\lambda t)$$

PMF $p_Y(y)=\frac{\nu^y}{y!}e^{-\nu}=\frac{(\lambda t)^y}{y!}e^{-\lambda t}$

CDF $F_Y(y)=1-p(1-p)^y$

$\mu_Y=\nu=\lambda t,\sigma_Y=\sqrt{\nu}=\sqrt{\lambda t}$



The accumulated total amount until time t

$$S(t)=\sum_{i=1}^{Y(t)}X_i\in(0,+\infty)\sim CPP(t;\nu,F_X(x)),\nu=\lambda t,\lambda>0,t>0$$

$$\mathbb{E}[S(t)]=\mathbb{E}[\mathbb{E}[S(t)|Y(t)]]=\mathbb{E}[Y(t)\mathbb{E}[X(t)]]=\lambda t\mathbb{E}[X_i]$$

$$\mathbb{D}[S(t)]=\mathbb{E}[\mathbb{D}[S(t)|Y(t)]]+\mathbb{D}[\mathbb{E}[S(t)|Y(t)]]=\mathbb{E}[\sigma_X^2Y(t)]+\mathbb{D}[\mu_XY(t)]=\sigma_X^2\mu_Y+\mu_X^2\sigma_Y^2=\lambda t(\mu_X^2+\sigma$$

$$S|_{Y(t)=y}=\sum_{i=1}^{y}X_i\sim\Gamma(ya,b)\qquad f_{S|_{Y=y}}(x)=(f_X*\cdots f_X)(x)=\frac{b^{ya}x^{ya-1}e^{-bx}}{\Gamma(ya)}$$

$$f_S(x)=\sum_{y=0}^{\infty}p_Y(y)f_{S|_{Y=y}}(x)=e^{-\lambda t}+\sum_{y=1}^{\infty}\frac{(\lambda t)^y}{y!}e^{-\lambda t}\frac{b^{ya}x^{ya-1}e^{-bx}}{\Gamma(ya)}=e^{-\lambda t}+\sum_{y=1}^{\infty}\frac{(\lambda t)^y b^{ya}x^{ya-1}}{y!\Gamma(ya)}e^{-\lambda t-bx}$$

$$W(x,z)=\sum_{y=1}^{\infty}\frac{x^y}{y!\Gamma(zy)}\qquad f_S(x)=e^{-\lambda t}+\frac{e^{-\lambda t-bx}}{x}\sum_{y=1}^{\infty}\frac{(\lambda t(bx)^a)^y}{y!\Gamma(ya)}=e^{-\lambda t}+\frac{e^{-\lambda t-bx}}{x}W(\lambda t(bx)^a,a)$$

$$F_S(s)=\int_0^s f_S(x)dx=e^{-\lambda t}+\int_0^s\frac{e^{-\lambda t-bx}}{x}\sum_{y=1}^{\infty}\frac{(\lambda t(bx)^a)^y}{y!\Gamma(ya)}dx$$