Numerical Analysis HW1

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1 Problem1

引理：Intermediate Value Theorem:

If f∈C[a,b] and K is any number between f (a) and f (b), then there exists a number c in (a, b) for which f (c) = K.

1.1 a.

不妨假设 $f(x_1) \leq f(x_2)$，那么一定有 $f(x_1) \leq \frac{f(x_1)+f(x_2)}{2} \leq f(x_2)$，根据介质定理，一定 $\exists \xi \in [a,b]$，使得 $\frac{f(x_1)+f(x_2)}{2} = f(\xi)$。

1.2 b.

不妨假设 $f(x_1) \leq f(x_2)$, 设 $K = \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}$，则

$$\begin{cases} f(x_1) - K = \frac{c_2(f(x_1)-f(x_2)}{c_1+c_2} \leq 0 \\ f(x_2) - K = \frac{c_1(f(x_2)-f(x_1)}{c_1+c_2} \geq 0 \end{cases}$$

得到 $f(x_1) \leq K \leq f(x_2)$, 根据介质定理，一定 $\exists \xi \in [a,b]$，使得 $f(\xi) = \frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2}$。

1.3 c.

取 $f(x) = x^2$，令 $x_1 = 1, x_2 = 2$ $c_1 = 2, c_2 = -1$, 则 $\frac{c_1 f(x_1)+c_2 f(x_2)}{c_1+c_2} = -2 \notin [f(x_1), f(x_2)]$

2 Problem2

2.1 a.

absolute error:$|f(x_0) - f(\tilde{x_0})| = |f^{'}(\xi)\epsilon| = |f^{'}(x_0 + \theta\epsilon)\epsilon|, 0 < \theta < 1$, 当 ϵ 足够小时，原式 =$|f^{'}(x_0)\epsilon|$

relative errors:$|f(x_0) - f(\tilde{x_0})|/|f(x_0)| = |f^{'}(\xi)\epsilon| = |f^{'}(x_0 + \theta\epsilon)\epsilon|/|f(x_0)|, 0 < \theta < 1$, 当 ϵ 足够小时，原式 =ϵ

2.2 b.

$|f(x_0) - f(\tilde{x_0})| = |f^{'}(\xi)\epsilon| = |f^{'}(x_0 + \theta\epsilon)\epsilon|, 0 < \theta < 1$

i)$f(x) = e^x$

absolute errors:$(1.359140914229522 \times 10^{-5}, 1.359147709951083 \times 10^{-5})$

relative errors:$(5.000000000000000 \times 10^{-6}, 5.000025000062500 \times 10^{-6})$

ii)$f(x) = sin(x)$

absolute errors:$(2.701490492532310 \times 10^{-6}, 2.701511529340699 \times 10^{-6})$

relative errors:$(3.210438079631523 \times 10^{-6}, 3.210463079671654 \times 10^{-6})$

2.3 c.

i)$f(x) = e^x$

absolute errors:$(0.110132328974034, 0.110132879637055)$

relative errors:$(5.000000000000000 \times 10^{-6}, 5.000025000062499 \times 10^{-6})$

ii)$f(x) = sin(x)$

absolute errors:$(4.195344044802048 \times 10^{-6}, 4.195357645382262 \times 10^{-6})$

relative errors:$(7.711730226688204 \times 10^{-6}, 7.711755226784599 \times 10^{-6})$

3 Problem3

3.1 a.

i)exactly:$\frac{17}{15}$

ii)three-digit chopping:1.133=0.113$\times 10^1$

relative errors:2.94$\times 10^{-4}$

iii)three-digit rounding:1.133=0.113$\times 10^1$

relative errors:2.94$\times 10^{-4}$

3.2 b.

i)exactly:$\frac{301}{660}$

ii)three-digit chopping:0.452

relative errors:8.90$\times 10^{-3}$

iii)three-digit rounding:0.453

relative errors:6.71$\times 10^{-3}$

4 Problem4

4.1 a.

令 $f(x) = F(x) - c_1L_1 - C_2L_2 = O(x^\alpha) + O(x^\beta)$, 只需证明 $\lim_{x\to 0} \frac{f(x)}{x^\gamma} = A(A \neq 0)$，而

$$\lim_{x\to 0} \frac{f(x)}{x^\gamma} = \begin{cases} c_1(\alpha \leq \beta) \\ c_2(\alpha > \beta) \end{cases} \neq 0$$

证毕

4.2 b.

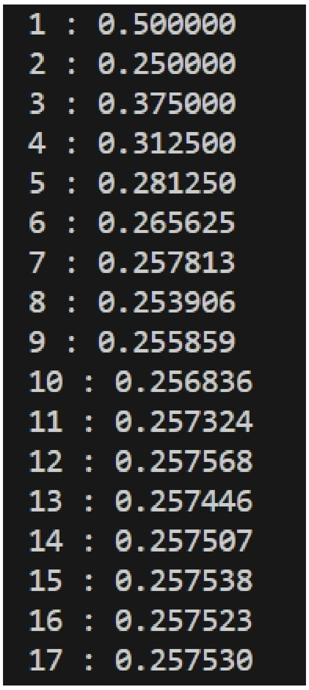
令 $g(x)=G(x)-L_1-L_2=O(c_1^{\alpha}x^{\alpha})+O(c_2^{\beta}x^{\beta})$, 只需证明 $\lim_{x\to0}\frac{g(x)}{x^{\gamma}}=A(A\neq0)$，而

$$\lim_{x\to0}\frac{g(x)}{x^{\gamma}}=\lim_{x\to0}*(O(x^{\alpha})+O(x^{\beta}))\neq0$$

证毕

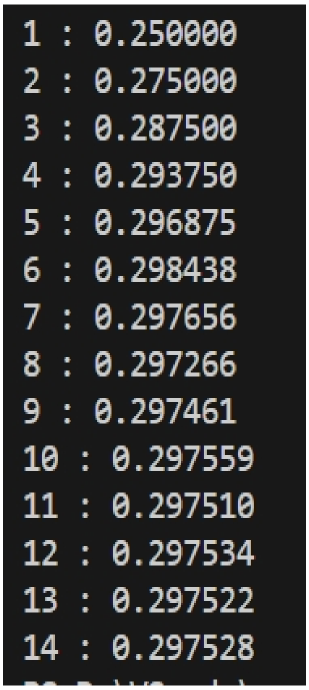
5 Problem5

result:



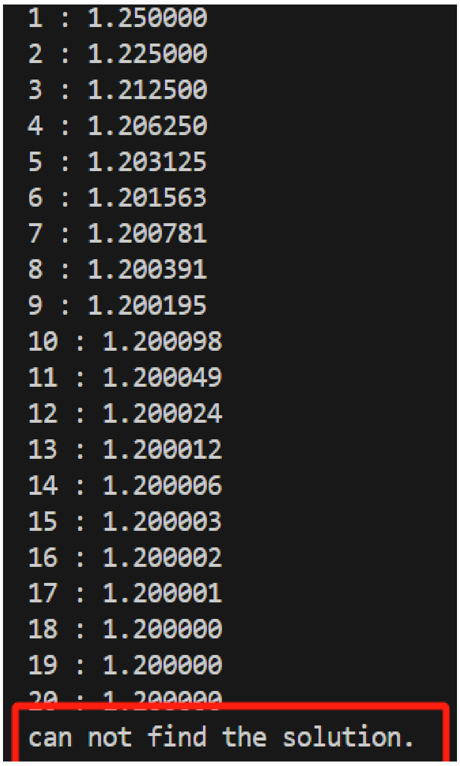
```
1 : 0.500000
2 : 0.250000
3 : 0.375000
4 : 0.312500
5 : 0.281250
6 : 0.265625
7 : 0.257813
8 : 0.253906
9 : 0.255859
10 : 0.256836
11 : 0.257324
12 : 0.257568
13 : 0.257446
14 : 0.257507
15 : 0.257538
16 : 0.257523
17 : 0.257530
```

图 1: 5a



```
1 : 0.250000
2 : 0.275000
3 : 0.287500
4 : 0.293750
5 : 0.296875
6 : 0.298438
7 : 0.297656
8 : 0.297266
9 : 0.297461
10 : 0.297559
11 : 0.297510
12 : 0.297534
13 : 0.297522
14 : 0.297528
```

图 2: 5b1



```
1 : 1.250000
2 : 1.225000
3 : 1.212500
4 : 1.206250
5 : 1.203125
6 : 1.201563
7 : 1.200781
8 : 1.200391
9 : 1.200195
10 : 1.200098
11 : 1.200049
12 : 1.200024
13 : 1.200012
14 : 1.200006
15 : 1.200003
16 : 1.200002
17 : 1.200001
18 : 1.200000
19 : 1.200000
20 : 1.200000
can not find the solution.
```

图 3: 5b2

6 Problem6

6.1 a.

选取 g(x)=$(2sin(\pi x)+4x)/3$, 选取原因如下：

i) 寻找根的大致范围，发现在 [1,2] 的范围内存在两个根，第一个根的范围在 [1,1,3], 在寻找 g(x) 的时候先考虑这一个区间

ii) 加入参数 t, 使得 $x=\frac{2sin(\pi*x)+tx}{t-1}=g(x)$ 则 $g^{'}(x)=\frac{2\pi cos(\pi x)+t}{t-1}$，令 $|g^{'}(x)|<1$, 可以化简为 $-2\leq\frac{2\pi cos(\pi x)+1}{t-1}\leq0$, 取 t=4, 可以满足要求

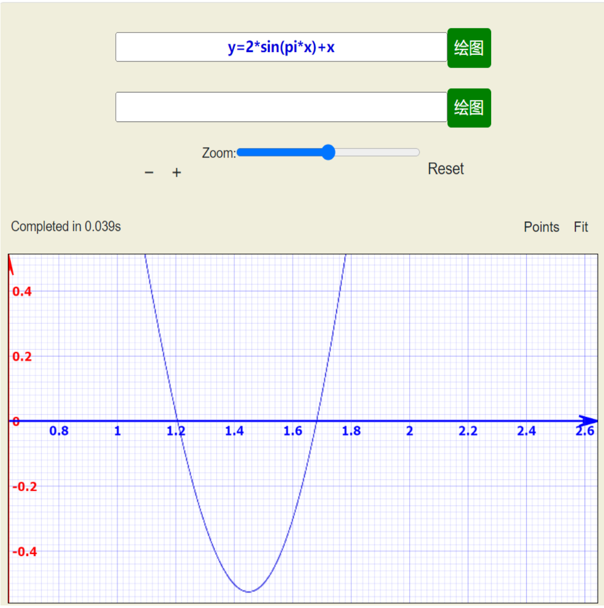


图 4: 判断根范围

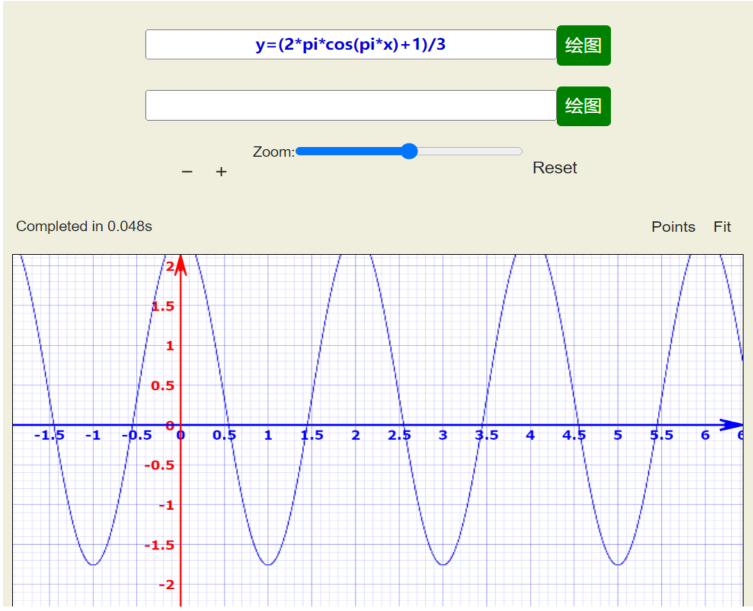


图 5: 待定系数 t

结果如图

```
error     p         p0
0.427413  1.427413  1.000000
0.130131  1.297282  1.427413
0.022122  1.275160  1.297282
0.003846  1.279006  1.275160
solution is 1.279006  end.
```

图 6: 6a 结果

6.2 b.

选取 $g(x) = \frac{e^x}{3x}$, 取 p0=0.5（这只是一个解，事实上应该有三个解）

```
error     p         p0
0.599148  1.099148  0.500000
0.188864  0.910283  1.099148
0.000300  0.909983  0.910283
solution is 0.909983  end.
```

图 7: 6b 结果

7 Problem7

$|p_1 - p| \geq |g(p_0) - p| = |g(p_0) - g(p)| = |(p_0 - p)g^{'}(p_0 + \epsilon(p_0 - p)| = |(p_0 - p)||g^{'}(p_0 + \epsilon(p_0 - p)|,$

且 $|g^{'}(p_0 + \epsilon(p_0 - p)| > 1,$

故 $|p_1 - p| > |(p_0 - p)|$