x^y Problems

Dylan Yu

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Remark 1. These are problems of the form x^y, i.e. variables to the power of variables.

Problem 1. Find all positive integer solutions $x^y + y^x = (x-y)^{x+y}$.

Problem 2 (Estonia TST 2005/3). Find all pairs (x, y) of positive integers satisfying the equation $(x+y)^x = x^y$.

Problem 3 (APMC Team 1999/1). Find all pairs (x, y) of positive integers such that $x^{x+y} = y^{y-x}$.

Problem 4 (Delta Polish Magazine). Find all rational numbers $x, y > 1$ satisfying

$$x^y = xy.$$

Problem 5 (M.M. Circles). Prove that if $x > 1, y > 1$, and $x^y + y^x = x^x + y^y$, then $x = y$.

Problem 6 (Kazakhstan MO Grade 11 2000/4). Find all triples of natural numbers (x, y, z) that satisfy the condition $(x+1)^{y+1} + 1 = (x+2)^{z+1}$.

Problem 7. Let $0 < x, y < 1$. Prove $x^y + y^x > 1$.

Problem 8 (IMAC Arhimede 2009/5). Find all natural numbers x and y such that $x^y - y^x = 1$.

Problem 9 (Romania MO Grade 10 2009/1).

(a) Show that two real numbers $x, y > 1$ chosen so that $x^y = y^x$, are equal or there exists a positive real number $m \neq 1$ such that $x = m^{\frac{1}{m-1}}$ and $y = m^{\frac{m}{m-1}}$.

(b) Solve the following equation in $(1, \infty)^2$: $x^y + x^{x^{y-1}} = y^x + y^{y^{x-1}}$.

Problem 10 (USAMTS 5/1/32). Find all pairs of rational numbers (a, b) such that $0 < a < b$ and $a^a = b^b$.