ASE 2020-21 Advanced Notes

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Last Updated September 4, 2020

Contents

§ 1 Exponents, Logarithms, and Radicals

§ 1.1 Definitions

Definition 1 (Exponent). An **exponent** is a symbol written above and to the right of a mathematical expression to indicate the operation of raising to a power.

In other words, in a^b, b is the exponent. An example of raising a number to an exponent or power is 3^2. All this means is that you are multiplying 3 by itself 2 times, so it's the same thing as $3 \cdot 3$. You can also have 3^{100} which is the same thing as $3 \cdot 3 \cdot 3 \cdot 3 \ldots$, but multiplying 3 by itself a total of 100 times. We can also do the same thing with variables. For example, x^3 is the same thing as $x \cdot x \cdot x$.

*The ASE playlist can be found here.

Example 1. Compute 2^3.

Solution. This means 2 multiplied with itself 3 times, so $2 \times 2 \times 2 = \boxed{8}$. $\square$

Definition 2 (Logarithm). A **logarithm** indicates how many of a number do we need to multiply to get that number?

In other words, if $c = a^b$, then $\log_a c = b$, where b is the logarithm.

Example 2. Compute $\log_2 8$.

Solution. If we let $c = \log_2 8$, then $2^c = 8$, so $c = \boxed{3}$. $\square$

Definition 3 (Radical). If $c = a^b$, then $\sqrt[b]{c} = a$ is the **radical**.

Example 3. Compute $\sqrt[3]{8}$.

Solution. If we let $c = \sqrt[3]{8}$, then $c^3 = 8$, so $c = \boxed{2}$. $\square$

Notice how in PEMDAS, exponents are right after parenthesis. Logarithms and radicals have the same position as exponents, but *PERLMDAS* doesn't sound as good.

§ 1.2 Exponent Rules

1. (Multiplication) $x^m \cdot x^n = x^{m+n}$
2. (Division) $\frac{x^m}{x^n} = x^{m-n}$
3. (Power Rule) $(x^m)^n = x^{m \cdot n}$
4. $\dfrac{1}{x^a} = x^{-a}$
5. $x^0 = 1$ when $x \neq 0$
6. $1^a = 1$
7. $x^{\frac{a}{b}} = x^{a \cdot \frac{1}{b}} = \sqrt[b]{x^a}$

Example 4. Compute $(2^3)^2$.

Solution. Using the power rule, we get $2^6 = \boxed{64}$. $\square$

Example 5. Compute $\frac{(4x^2y^4)^3}{(8xy^3)^2}$.

Solution. The top is $64x^6y^{12}$ and the bottom is $64x^2y^6$, so the answer is $\boxed{x^4y^6}$. □

§ 1.3 Logarithm Rules

1. (Identity) $\log_a a = 1$
2. $\log_a 1 = 0$
3. (Addition) $\log_a b + \log_a c = \log_a(bc)$
4. (Subtraction) $\log_a b - \log_a c = \log_a\left(\frac{b}{c}\right)$
5. (Power Rule) $\log_a(b^c) = c\log_a b$
6. (Change of Base) $\log_a b = \frac{\log_c b}{\log_c a}$

Example 6. Compute $\log_{100} 2 + \log_{100} 50$.

Solution. Using the addition rule, we get $\log_{100}(2 \cdot 50) = \log_{100} 100 = \boxed{1}$. □

Example 7. Given $m = \log_3 2$ and $n = \log_3 5$, find $\log_2 5$ in terms of m and n.

Solution. Using the change of base rule, we get $\log_2 5 = \frac{\log_3 5}{\log_3 2} = \boxed{\dfrac{n}{m}}$. □

§ 1.4 Radical Rules

1. (Radical to Exponent) If $\sqrt[n]{x} = r$, then $x = r^n$
2. $a\sqrt{b} + c\sqrt{b} = (a+c)\sqrt{b}$
3. $a\sqrt{b} - c\sqrt{b} = (a-c)\sqrt{b}$
4. $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$
5. $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$
6. (Simplifying Radicals) $\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}$ and $\frac{1}{\sqrt{a}+\sqrt{b}} = \frac{\sqrt{a}-\sqrt{b}}{a-b}$

Example 8. Simplify $3\sqrt{6} + 2\sqrt{2} \cdot 3\sqrt{3}$.

Solution. Note that $2\sqrt{2} \cdot 3\sqrt{3} = 6\sqrt{6}$, so $3\sqrt{6} + 6\sqrt{6} + \boxed{9\sqrt{6}}$. □

Example 9. Simplify $\frac{4}{\sqrt{6}-\sqrt{2}}$.

Solution. Note that $\frac{4}{\sqrt{6}-\sqrt{2}} = 4 \cdot \frac{1}{\sqrt{6}-\sqrt{2}} = 4 \cdot \frac{\sqrt{6}+\sqrt{2}}{6-2} = \boxed{\sqrt{6}+\sqrt{2}}$. □

§ 1.5 Two Important Exponents

Both 10 and e are very important. It is easy to understand why 10 is: we use it a lot in decimals and counting place values! For example, if $n = \overline{a_1 a_2 \dots a_k}$, then $n = a_1 \cdot 10^{k-1} + a_2 \cdot 10^{k-2} + \dots + a_k \cdot 10^0$ (e.g. $103 = 1 \cdot 10^2 + 0 \cdot 10^1 + 3 \cdot 10^0$). But what is e, and why is it important?

Definition 4 (e). The constant e is approximately 2.71828 and is the **natural base**.

This constant e is important for **exponential growth** as we will see later. When we write $\log x$, we are referring to $\log_{10} x$ by default. When we write $\ln x$, we are referring to $\log_e x$.

§ 1.6 Exponential Growth & Decay

Theorem 1 (Continuous Exponential Growth/Decay). If A_0 is the initial value, A is the final value, k is the continuous growth rate (a.k.a. the **constant of proportionality**, $k > 0$ means growing, $k < 0$ means decaying), and t is the time elapsed, $A = A_0 e^{kt}$ when the value changes continuously and exponentially.

Example 10. A strain of bacteria growing on your desktop doubles every 5 minutes. Assuming that you start with only one bacterium, how many bacteria could be present at the end of 96 minutes?

Solution. If there is a doubling every 5 minutes, this means that $2 = 1e^{k \cdot 5}$, so $k = \frac{\ln 2}{5}$. Thus, in 96 minutes, we have $A = 1e^{\frac{\ln 2}{5} \cdot 96} \approx \boxed{602248}$. Notice how I **rounded down**, since the extra part left is not a full bacteria yet. □

Theorem 2 (Exponential Growth Formula). If a population of a grows by a **growth rate** of r every 1 unit of time, after t time the population is $a(1+r)^t$.

Theorem 3 (Exponential Decay Formula). If a population of a decays by a **decay rate** of r every 1 unit of time, after t time the population is $a(1-r)^t$.

Definition 5 (Growth Factor). If the growth rate is r, the **growth factor** is $1+r$.

Definition 6 (Decay Factor). If the decay rate is r, the **decay factor** is $1-r$.

Example 11. The population of Sugar Land in 2016 was estimated to be 35,000 people with an annual rate of increase of 2.4%.

(a) What is the growth factor of Sugar Land?

(b) Write an equation to model future growth.

(c) Find the population in 2020 to the nearest hundred people.

Solution. Let's use our growth formula:

(a) There is a growth rate of $2.4\% = 0.024$, so the growth factor is $\boxed{1.024}$.

(b) We can use our formula to get $\boxed{35000(1.024)^t}$, where t is in years.

(c) 4 years have passed, so we get $35000(1.024)^4 \approx \boxed{38500}$.

□

Here is another example of growth:

Theorem 4 (Compound Interest Formula). If A is the final amount, P is the principle balance (i.e. the initial amount), r is the interest rate, n is the number of times interest is applied per time period, and t is the number of time periods, $A = P\left(1+\frac{r}{n}\right)^{nt}$.

§ 1.7 Exponents in Number Theory

Usually, you will see huge exponents and you'll be terrified of that. Here is how to deal with them:

Theorem 5 (Large Exponents Method). If there is something in the form a^b, where b is large, and we are looking for the units digit, for example, try to find a pattern.

Example 12. Find the last two digits of $5^{20202020}$.

Solution. Through testing, we see that $5^2, 5^3, \ldots$ all end in 25, so the answer is $\boxed{25}$ (probably). □

We will put this to the test in the problems below, and we will refine this method when we cover number theory.

§ 1.8 Further Reading

More examples and rules can be found below:

1. Properties of logarithmic functions, Monterey Institute

2. Algebra review, Unknown

§ 1.9 Problems

Problem 1 (MathLeague Sprint 11123/16). Express five hundred thousand squared in scientific notation.

Problem 2. Compute $\log_7 0$.

Problem 3. Compute $\log_7 1$.

Problem 4. Compute $\log_7 343$.

Problem 5. If there are 2197 unit cubes in a cube of side length n, what is n?

Problem 6. Solve for x in

$$\sqrt{x-5}+2=9.$$

Problem 7. Find $\frac{x}{y}$ if $2^{2^x}=2^{2^y}$.

Problem 8. If $2^x=\left(\frac{1}{4}\right)^8$, find x.

Problem 9. If $2^{3x+5}=8^{-23}\cdot 4^x$, find x.

Problem 10. If $3^{9x}=\frac{9^{24}}{81^14}$.

Problem 11. Find all x that satisfy $4^x-5\cdot 2^x+4=0$.

Problem 12 (MathLeague Sprint 11323/27). How many ordered pairs of real numbers (a,b) satisfy the equations $|ab|=\sqrt{3}$ and $\frac{a}{b}=\frac{b}{a}$?

Problem 13. A bacteria colony doubles its population every 10 seconds. If it has a population of 200 right now, how many seconds ago did it have a population of 100?

Problem 14. A bacteria colony doubles its population every 10 seconds. After x seconds, the population increases from 1 bacterium to 2^{20} bacteria. What is x?

Problem 15. If an amount of \$5,000 is deposited into a savings account at an annual interest rate of 5%, compounded monthly, what is the value of the investment after 10 years?

Problem 16 (MathLeague Target 11322/2). Find the units digit of 147^{148}.

Problem 17 ($*$). Find the last two digits of 6^{2020}.

Problem 18 ($*$). Compute

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{99}+\sqrt{100}}.$$