ISL 2019/A5

Dylan Yu

April 15, 2021

Problem 1. Let $x_1, \dots, x_n$ be different real numbers. Prove that

$$\sum_{1 \le i \le n} \prod_{\substack{j \neq i \\ 1 \le j \le n}} \frac{1 - x_i x_j}{x_i - x_j} = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Solution. We will use Lagrange interpolation:

Theorem 1 (Lagrange Interpolation)
Let $x_1, x_2, \dots, x_n$ be different real numbers and let $y_1, y_2, \dots, y_n$ also be different real numbers. There is a unique polynomial P of degree at most d such that $P(x_i) = y_i$. It is given explicitly by

$$P(X) = \sum_{1 \le i \le n} y_i \prod_{\substack{j \neq i \\ 1 \le j \le n}} \frac{X - x_j}{x_i - x_j}.$$

Consider a polynomial P such that $P(x_i) = x_i^k$ for some positive integer k. Then by Lagrange interpolation,

$$P(X) = \sum_{1 \le i \le n} \left(x_i^k \cdot \prod_{\substack{j \neq i \\ 1 \le j \le n}} \frac{X - x_j}{x_i - x_j} \right),$$

and this is obviously equivalent to X^k. The coefficient of X^{n-k-1} on the LHS is

$$\sum_{1 \le i \le n} \left((-1)^k \cdot \sigma_k(i) \cdot x_i^k \cdot \prod_{\substack{j \neq i \\ 1 \le j \le n}} \frac{1}{x_i - x_j} \right),$$

where $\sigma_k(i)$ is the symmetric sum of $x_1, \dots, x_n$ excluding x_i taken k at a time, and the coefficient of X^{n-k-1} on the RHS is 1 if $n - k - 1 = k \implies n = 2k + 1$ and 0 otherwise (i.e. $n \neq 2k + 1$). It is easy to show that

$$\prod_{\substack{j \neq i \\ 1 \le j \le n}} (1 - x_i x_j) = \sum_{0 \le k \le n-1} (-1)^k \cdot \sigma_k(i) \cdot x_i^k$$

by simply computing the product (there's a lot of symmetry to be exploited), implying

$$\sum_{1 \le i \le n} \prod_{\substack{j \ne i \\ 1 \le j \le n}} \frac{1 - x_i x_j}{x_i - x_j} = \sum_{k=0}^{n-1} \left(\sum_{1 \le i \le n} \left((-1)^k \cdot \sigma_k(i) \cdot x_i^k \cdot \prod_{\substack{j \ne i \\ 1 \le j \le n}} \frac{1}{x_i - x_j} \right) \right).$$

Thus,

$$\sum_{1 \le i \le n} \prod_{\substack{j \ne i \\ 1 \le j \le n}} \frac{1 - x_i x_j}{x_i - x_j}$$

must be 1 when $n = 2k + 1$ and 0 otherwise, i.e. when n is even and when n is odd, respectively. $\square$

Remark 2. This problem was solved with the help of Srinivas Arun.