NiCε

# Navigable Inaugural Computational Exam

Spring 2021

## Contents

# 1 Results

Prize winners will be emailed by the NICE committee by February 6th. Please check your email in the coming days if you won a prize.

## 1.1 Top 10 Scorers

Congratulations to our top scorers!

- The top 3 win a $25 AoPS Gift Certificate and one year of Wolfram|Alpha Notebook Edition,
- places 4 through 6 win one year of Wolfram|Alpha Notebook Edition, and
- places 7 through 10 win one year of MapleSoft Student.

Ties are broken by considering the highest problem solved by each contestant that the other did not.

Here are the top 10 scorers for the NICE Spring 2021 Contest:

1. Rishabh Das (24 points)
2. Raminh Nugit (23 points)
3. Alex Xu (21 points)
4. Jason Mao (20 points)
5. Elliott Liu (20 points)
6. Tovi Wen (19 points)
7. Jakub Kadziolka (19 points)
8. Miku Hatsune (18 points)
9. Perryn Chang (18 points)
10. Vivian Loh (18 points)

## 1.2 Special Prizes

We also have some special prizes for other contestants!

- The top 3 female contestants overall win one year of Wolfram|Alpha Notebook Edition,
- and the top 3 middle school contestants (excluding other prize winners) win one year of Wolfram|Alpha Notebook Edition as well.

Here are our top 3 female contestants:

1. Miku Hatsune (18 points)
2. Vivian Loh (18 points)
3. Emily Liu (13 points)

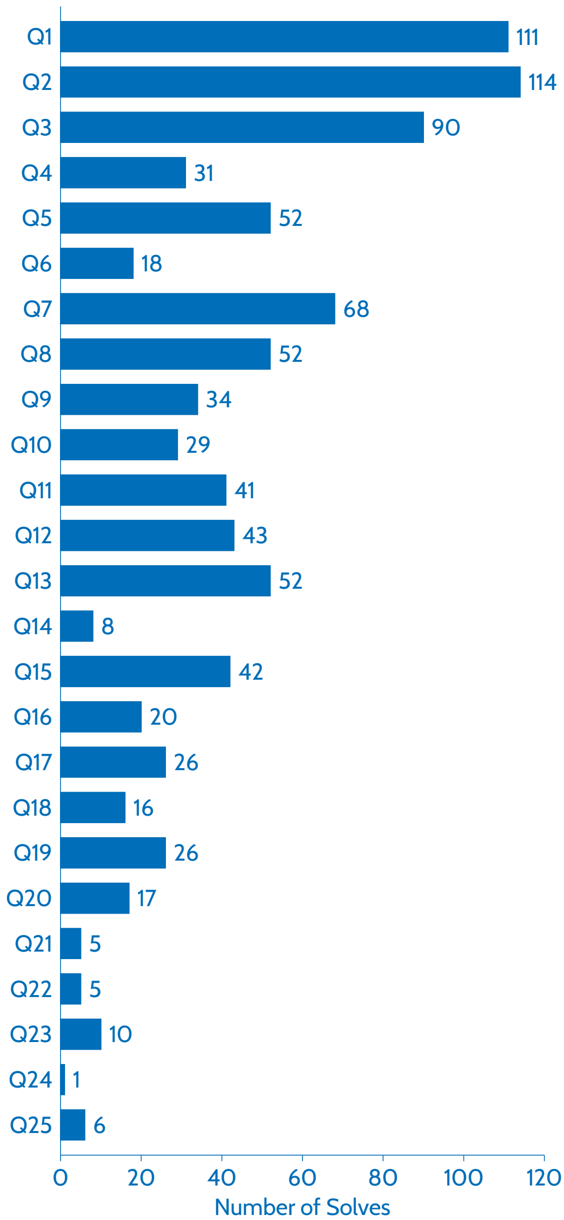
Here are our middle school prize winners:

1. Lerchen Zhong (17 points)
2. Athithan Elamaran (15 points)
3. Steven Wei (12 points)

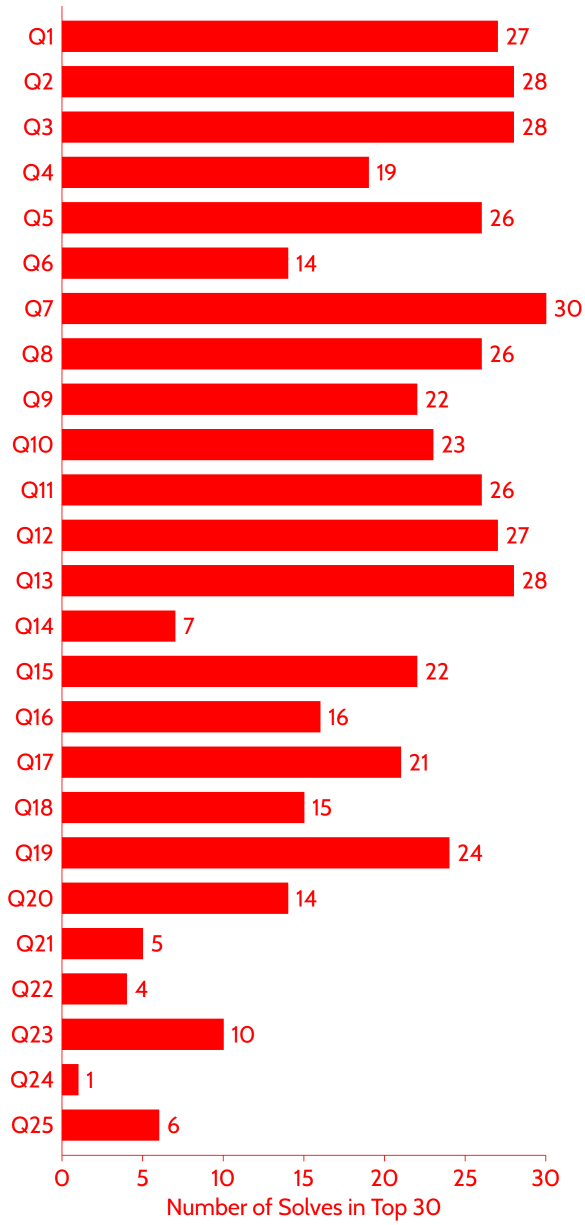
In addition to these special prizes, there was a raffle for a $25 AoPS Gift Certificate and a year of MapleSoft Student. Congratulations to Kanishk Sharma for winning the gift certificate and to David Dong for winning the MapleSoft Student license! You will receive an email in the coming days to collect your prize.

# 2 Statistics

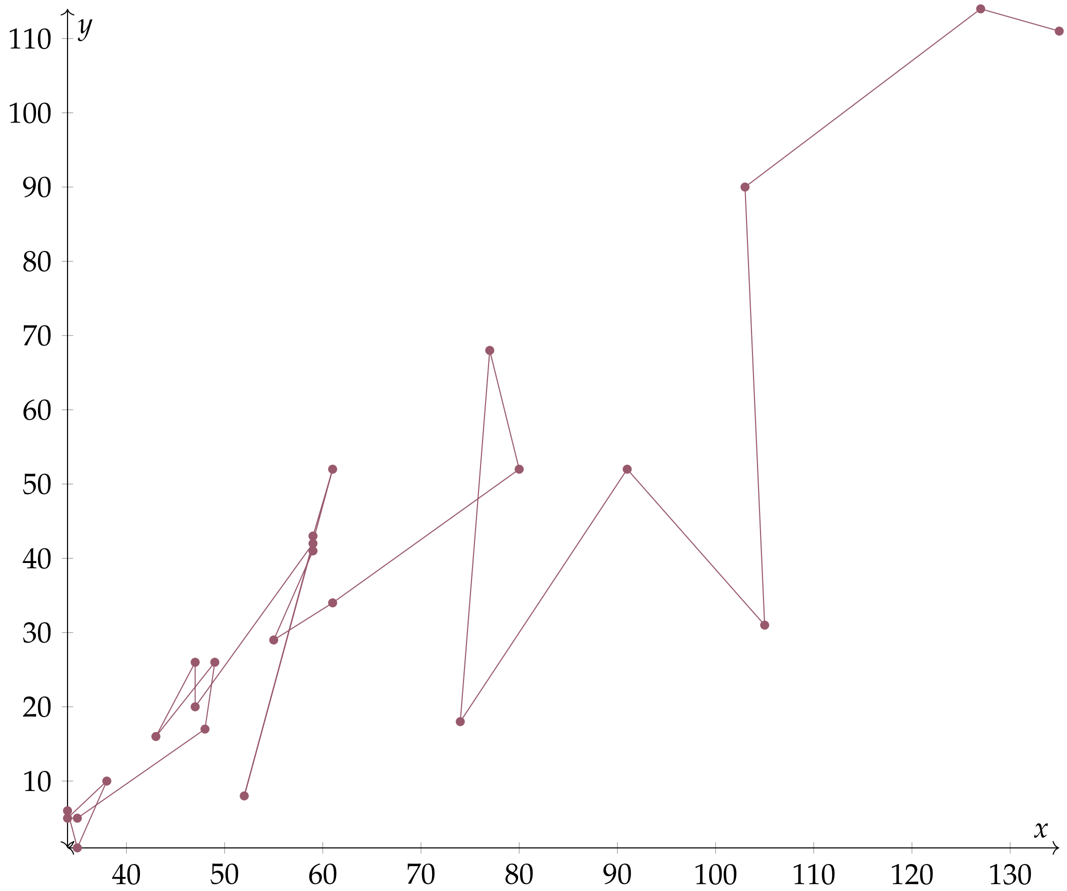
## 2.1 Number of Solves per Problem




## 2.2 Number of Solves per Problem in Top 30



## 2.3 Number of Solves v.s. Number of Attempts



# 3 Solutions

## 3.1 Problem 1

Sally's teacher wrote the expression

$$\left(\frac{1}{3}\right)\sqrt[3]{a}$$

on the board. Unfortunately, Sally misinterpreted this as

$$\left(\frac{1}{3}\right)^3\sqrt{a}.$$

However, for some positive real value of $a$, these two expressions are both equal to the same number $b$. Determine $b$.

*Proposed by Linus Tang*

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### Solution 1

The given equality implies

$$\left(\frac{1}{3}\right)^2 = a^{\frac{1}{3}-\frac{1}{2}} = a^{-\frac{1}{6}},$$

so $a = 3^{12}$. Hence the common value of $b$ is $\frac{1}{3}\cdot 3^4 = \boxed{27}$.

## 3.2 Problem 2

In a hat, there are ten slips, each containing a different integer from 1 to 10, inclusive. David reaches into the hat and keeps for himself two numbers that differ by seven. Ankan then reaches into the hat and picks two numbers that differ by five. What is the largest possible sum of the four picked numbers?

*Proposed by David Altizio*

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### Solution 2

Let $m$ and $m-7$ be the two integers that David picks, and let $n$ and $n-5$ be the two integers that Ankan picks. The sum of the four picked numbers is then

$$m + (m-7) + n + (n-5) = 2(m+n) - 12.$$

But David and Ankan cannot both pick the number 10, so $m + n \leq 19$. Hence the sum can be at most $2 \cdot 19 - 12 = \boxed{26}$. This is achieved if David picks 3 and 10 while Ankan picks 4 and 9.

## 3.3 Problem 3

In parallelogram $ABCD$, let $P$ be a point on $\overline{AC}$. Suppose that the sum of the perimeters of $\triangle PAB$ and $\triangle PAD$ is 13, the sum of the perimeters of $\triangle PCB$ and $\triangle PCD$ is 7, and $AC = 9$. Find $AP$.

*Proposed by Arul Kolla*

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### Solution 3

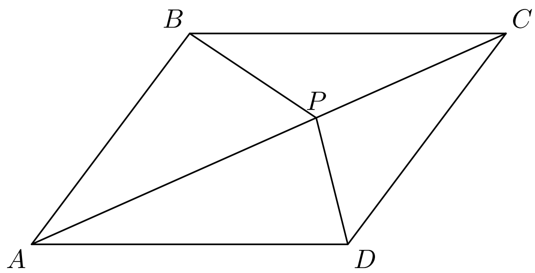
Note that the sum of the perimeters of $\triangle PAB$ and $\triangle PAD$ is

$$(PB + BA + AP) + (PD + DA + AP) = (AB + AD + BP + PD) + 2 \cdot AP.$$

Analogously, the sum of the perimeters of $\triangle PBC$ and $\triangle PCD$ is

$$(BC + CD + BP + DP) + 2 \cdot PC$$

Subtracting these two conditions induces rampant cancellation and yields $PA - PC = 3$. But additionally $PA + PC = AC = 9$; solving yields $AP = \boxed{6}$.



## 3.4 Problem 4

What is the smallest possible area of a rhombus $\mathcal{R}$ whose sides have length 5 and whose vertices are all points in the plane with integer coordinates?

*Proposed by David Altizio*

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### Solution 4

Let $ABCD$ be the rhombus in question; we may assume that $A$ is at the origin of the coordinate plane and $\angle DAB$ is acute. Note that the area of $\mathcal{R}$ is twice the area of $\triangle DAB$, which in turn is minimized when $\angle DAB$ is minimized. This is equivalent to minimizing the length of $\overline{BD}$.

From here, note that the only lattice points on the circle $x^2 + y^2 = 25$ are $(0, \pm 5)$, $(\pm 5, 0)$, $(\pm 3, \pm 4)$, and $(\pm 4, \pm 3)$. The minimum length of $\overline{BD}$ is therefore $\sqrt{2}$, achieved when, for example, $B = (4, 3)$ and $D = (3, 4)$. In this case, the area of $\triangle BAD$ is $\frac{7}{2}$, so the desired minimum area of $\mathcal{R}$ is $\boxed{7}$.

## 3.5 Problem 5

Aeren needs to memorize a table about a new binary operation $\heartsuit$. He is given the table below by his teacher and is also told that

$$A-(A\heartsuit B)=(B\heartsuit A)-B$$

for all positive integers $A$ and $B$ between 1 and 6 inclusive. At most how many additional entries in the table can he fill out (without guessing)?

| $\heartsuit$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | | 1 | | | | |
| 2 | | | | | | |
| 3 | | 1 | | | 8 | |
| 4 | 3 | | | | 7 | |
| 5 | | | | | | |
| 6 | 4 | | | 9 | | |

*Proposed by Arul Kolla*

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### Solution 5

Rewrite the given equation as $(B\heartsuit A)=A+B-(A\heartsuit B)$. Now, given $A,B,(A\heartsuit B)$, we can find the value of $(B\heartsuit A)$ uniquely. In addition if $A=B$ we have $(A\heartsuit A)=A+A-(A\heartsuit A)$ or $(A\heartsuit A)=A$, meaning that we can uniquely determine $(A\heartsuit A)$ for each $A$. Therefore, in the table, we can determine 7 entries using the first relation and 6 entries using the second relation, for a total of $6+7=\boxed{13}$ entries.

## 3.6 Problem 6

Archimedes is playing *jackblack* with a deck of 5 cards, labeled with the integers from 1 to 5. Archimedes chooses up to 3 cards at random, with replacement, and wins if the cards in his hand sum to 9 at any point. (The game immediately stops once he wins.) However, Archimedes has an advantage: he secretly slipped an extra card labelled 2 into his pocket. He will only use the extra 2 if the cards in his hand sum to exactly 7 after two turns, leading to a guaranteed win.
The probability that Archimedes wins is $\frac{m}{n}$, where $m$ and $n$ are positive relatively prime integers. Find $1000m + n$.

*Proposed by Neil Shah*

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### Solution 6

Let $k$ be the sum of the first two cards in Archimedes' hand. There are three cases to consider:

- If $k$ equals 4, 5, 6, or 8, then there is exactly one card in his hand that sums to exactly 9, so the probability of success is $\frac{1}{5}$.
- If $k = 9$, then Archimedes automatically wins. Similarly, if $k = 7$, then Archimedes uses his extra card to automatically win.
- Otherwise, Archimedes has no chance of winning.

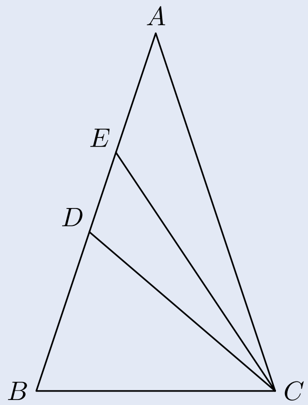
This means the desired probability is

$$\frac{1}{5} \cdot \frac{3+4+5+3}{25} + 1 \cdot \frac{4+2}{25} = \frac{9}{25}$$

for a requested answer of $\boxed{9025}$.

## 3.7 Problem 7

Let $\triangle ABC$ be an isosceles triangle with $AB = AC > BC$. Points $D$ and $E$ lie on $\overline{AB}$ such that $CB = CD$ and $DB = 2DE$, as shown in the diagram. If $\angle ECD = \angle ECA$, then the value of $\frac{AB}{BC}$ can be written in the form $\frac{m}{n}$, where $m$ and $n$ are positive relatively prime integers. What is $1000m + n$?




*Proposed by Linus Tang*

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### Solution 7

Without loss of generality let $BC = 1$, and set $AB = AC = x$. Since $CB = CD$, $\triangle ABC \sim \triangle CDB$, so $BC = \frac{1}{x}$. Then

$$DE = \frac{1}{2x} \quad \text{and} \quad AE = x - \frac{3}{2x}.$$

By the Angle Bisector Theorem, therefore,

$$x = \frac{AC}{DC} = \frac{AE}{DE} = \frac{x - \frac{3}{2x}}{\frac{1}{2x}} = 2x^2 - 3.$$

Solving yields $x = \frac{3}{2}$, so the requested answer is $\boxed{3002}$.

## 3.8 Problem 8

How many positive integers $N \leq 1000$ are there for which $N^2$ and $(N+1)^2$ both have at least five positive divisors? It is known there are 168 primes between 1 and 1000.

*Proposed by David Altizio*

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### Solution 8

Observe that $N^2$ has at least five positive divisors if and only if $N$ is composite. Indeed, if $N$ is prime then the only factors of $N^2$ are 1, $p$, and $p^2$. Conversely, if $N$ is composite, $N^2$ always has at least five positive divisors:

- If $N$ is divisible by the square of a prime $p^2$, then 1, $p$, $p^2$, $p^3$, and $p^4$ are all divisors of $N^2$;
- If $N$ is divisible by two distinct primes $p$ and $q$, then 1, $p$, $q$, $p^2$, and $q^2$ are all divisors of $N^2$.

Thus, it suffices to find the number of positive integers $N \leq 1000$ such that both $N$ and $N+1$ are composite.

To compute this, note that each such pair of composite numbers contains exactly one even and one odd number. For $N \geq 4$, this even number is always composite; therefore, the condition is satisfied if and only if either $N$ or $N+1$ is a composite odd number. As there are 168 primes between 1 and 1000, there are $499 - 167 = 332$ composite odd numbers between 1 and 1000, leading to $2 \cdot 332 = 664$ possible values of $N$. Adding the lone $N = 1000$ case ($1001 = 7 \cdot 11 \cdot 13$ is composite!) yields the desired answer of $\boxed{665}$.

## 3.9 Problem 9

Ethan draws a $2 \times 6$ grid of points on a piece of paper such that each pair of adjacent points is distance 1 apart. How many ways can Ethan divide the 12 points into six pairs such that the distance between the points in each pair is in the interval $[2, 4)$?

*Proposed by David Altizio*

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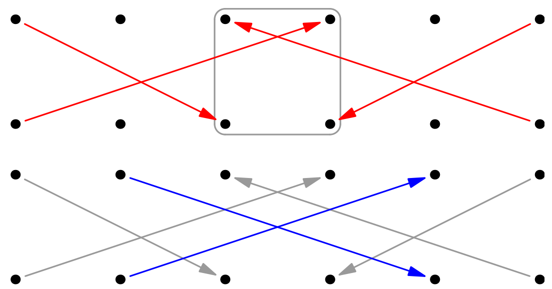
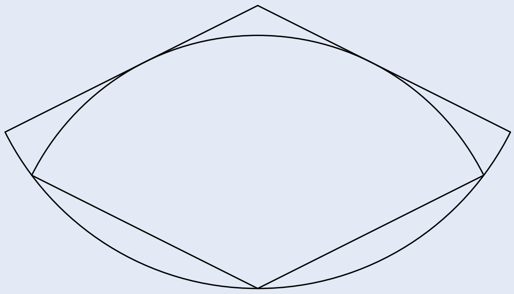
### Solution 9

Observe that the allowable distances between points in each pair are 2, $\sqrt{5}$, 3, and $\sqrt{10}$. This means each of the corner points must be paired with any of the four points in the middle $2 \times 2$ square of the grid, as shown below. The only way this can happen is if each point in this square is the partner of some point in the corner; there are $4! = 24$ ways to arrange the four partners to these four points. Now there are two ways to pair the remaining four points in the grid for a total of $24 \cdot 2 = \boxed{48}$ partitions.



## 3.10 Problem 10

A sector of angle $\alpha < 180^\circ$ is inscribed inside another sector of the same angle, as shown below. If $\cos(\alpha) = -\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers, what is $1000m + n$?



*Proposed by Linus Tang*

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### Solution 10

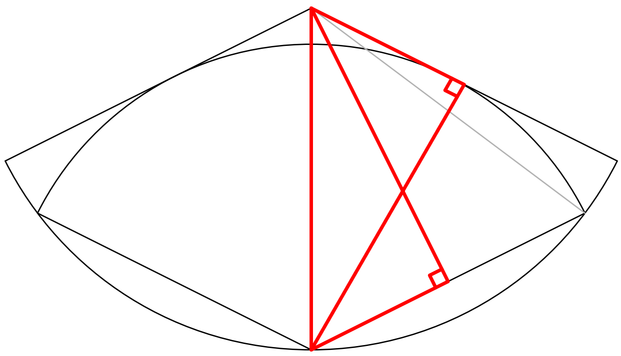
Let $r < R$ be the radii of the two given sectors. The two highlighted triangles are congruent and have sides $r$, $\frac{r}{2}$, and $R$. By the Pythagorean Theorem,

$$r^2 + \left(\frac{r}{2}\right)^2 = \frac{5}{4}r^2 = R^2,$$

so $\sin\frac{\alpha}{2} = \frac{r}{R} = \frac{2}{\sqrt{5}}$. Thus

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = -\frac{3}{5}$$

and the requested answer is $\boxed{3005}$.



## 3.11 Problem 11

Fifty rooms of a castle are lined in a row. The first room contains 100 knights, while the remaining 49 rooms contain one knight each. These knights wish to escape the castle by breaking the barriers between consecutive rooms, ending with the barrier from room 50 to the outside.
At the stroke of midnight, each knight in the $i^{\text{th}}$ room begins breaking the barrier between the $i^{\text{th}}$ and $(i+1)^{\text{st}}$ rooms, where we count the $51^{\text{st}}$ room as the exterior. Each person works at a constant rate and is able to break down a barrier in 1 hour, and once a group of knights breaks down the $i^{\text{th}}$ barrier, they immediately join the knight breaking down the $(i+1)^{\text{st}}$ barrier.
The number of hours it takes for the knights to escape the castle is $\frac{m}{n}$, where $m$ and $n$ are positive relatively prime integers. Compute the product $mn$.

*Proposed by Ethan Liu*

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### Solution 11-A

We know that the $50^{\text{th}}$ wall is broken last. Thus, all 149 knights exert equal work in breaking the 50 walls. Since each knight individually can break a wall in exactly 1 hour, it follows that the number of hours it takes for all walls to break is $\frac{50}{149}$. The requested answer is $149 \cdot 50 = \boxed{7450}$.

### Solution 11-B

Let $t_n$ denote the time in hours until $n^{\text{th}}$ barrier to be broken, so that $t_0 = 0$ and $t_1 = \frac{1}{100}$.
We claim that

$$t_{n+1} = \max\left\{1, t_n + \frac{1-t_n}{100+n}\right\}.$$

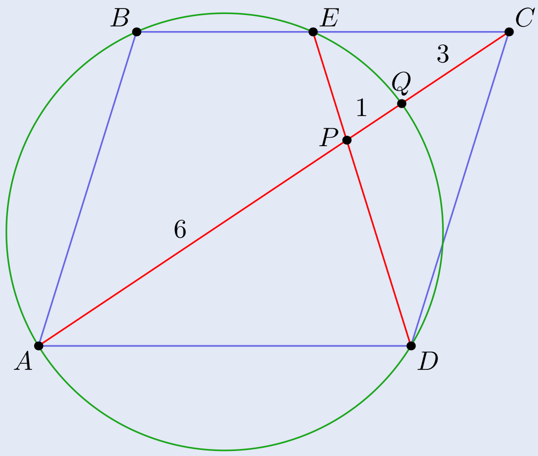
Indeed, the breaking of the $(n+1)^{\text{st}}$ barrier proceeds in two steps:

- In the beginning, the knight in the $n^{\text{th}}$ room works alone and breaks $t_n$ fraction of the barrier.
- If $t_n > 1$, then this is all that happens. Otherwise, the previous $99+n$ knights now assist in breaking down the remaining $1-t_n$ fraction of the barrier; this occurs in time $\frac{1-t_n}{100+n}$ hours.

Combining both bullets proves the claim.
From this, we obtain $t_n = \frac{n}{99+n}$ for all $n$, yielding $t_{50} = \frac{50}{149}$ as before.

## 3.12 Problem 12

Let $ABCD$ be a parallelogram with $\angle BAD < 90^\circ$. The circumcircle of $\triangle ABD$ intersects $\overline{BC}$ and $\overline{AC}$ again at $E$ and $Q$, respectively. Let $P$ be the intersection point of $\overline{ED}$ with $\overline{AC}$. If $AP = 6$, $PQ = 1$, and $QC = 3$, find the area of parallelogram $ABCD$.




*Proposed by David Altizio*

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### Solution 12

First observe that triangles $APD$ and $CPE$ are similar with ratio of similarity $\frac{3}{2}$. This means there exists some positive real number $x$ such that $EC = 2x$ and $BC = AD = 3x$. By Power of a Point at point $C$,

$$30 = CQ \cdot CA = CE \cdot CB = 6x^2.$$

This means $x = \sqrt{5}$, so $CE = 2\sqrt{5}$ and $BC = 3\sqrt{5}$.

Similarly, there exists some positive real number $y$ such that $EP = 2y$ and $PD = 3y$. By Power of a Point at point $P$,

$$6 = AP \cdot PQ = EP \cdot PD = 6y^2,$$

so $y = 1$. This means $EP = 2$ and $PD = 3$.

Finally, observe that, since $ABED$ is a quadrilateral inscribed in a circle and $BE \parallel AD$, this quadrilateral is actually an isosceles trapezoid, i.e. $ED = AB = CD = 5$. In particular, $\triangle CDE$ is an isosceles triangle with leg length 5 and base length $2\sqrt{5}$. The height from $D$ to the side $\overline{CE}$ of this triangle has length $2\sqrt{5}$. This means the area of parallelogram $ABCD$ is $(3\sqrt{5})(2\sqrt{5}) = \boxed{30}$.

## 3.13 Problem 13

Suppose $x$ and $y$ are nonzero real numbers satisfying the system of equations

$$\begin{aligned} 3x^2 + y^2 &= 13x, \\ x^2 + 3y^2 &= 14y. \end{aligned}$$

Find $x + y$.

*Proposed by David Altizio*

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### Solution 13

The system of equations is equivalent to the system

$$\begin{aligned} 3x^2y + y^3 &= 13xy \\ x^3 + 3xy^2 &= 14xy. \end{aligned}$$

Adding and subtracting the two equations yields

$$\begin{aligned} (x+y)^3 &= 27xy \\ (x-y)^3 &= xy. \end{aligned}$$

Thus $x + y = 3(x - y)$, or $x = 2y$. Making that substitution we get $12y^2 + y^2 = 26y$, or $y = 2$ and $x = 4$. So $x + y = \boxed{6}$.

## 3.14 Problem 14

A nonempty string $X$ of letters is called a *strict subsequence* of another string $Y$ if a nonzero number of letters can be removed from $Y$ to result in $X$. For example, *AD*, *WAR*, and *WARD* are strict subsequences of *WIZARD*, but *RAID* is not. Blitz writes a nonempty list of distinct strict subsequences of *AAAABBBBBB* such that, for any two strings in Blitz's list, neither is a (strict) subsequence of the other. How many such lists are possible? Note that order does not matter, so the list *AAB*, *ABBB* is the same as *ABBB*, *AAB*.

*Proposed by Linus Tang*

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### Solution 14

Let $P(4,-1)$ and $Q(-1,6)$ be points in the coordinate plane. We will show a correspondence between paths from $P$ to $Q$ that consist of movements 1 unit up and 1 unit to the left. (Almost) every such path corresponds to a list of substrings: the list containing $x$ $A$'s and $y$ $B$'s for every $(x,y)$ such that $(x,y)$, $(x-1,y)$, and $(x,y-1)$ all lie on the path. There are 3 exceptions: one path corresponds to an empty list of substrings, one path corresponds to the list containing only an empty string, an one path corresponds to the list containing the original string, *AAAABBBBBB* (and none of these three are allowed). Since there are $\binom{7+5}{5} = 792$ paths total, $\boxed{789}$ of them correspond to valid lists of substrings.

## 3.15 Problem 15

Let $r_1$, $r_2$, and $r_3$ be the roots of the polynomial $x^3 - x + 1$. Then

$$\frac{1}{r_1^2 + r_1 + 1} + \frac{1}{r_2^2 + r_2 + 1} + \frac{1}{r_3^2 + r_3 + 1} = \frac{m}{n},$$

where $m$ and $n$ are positive relatively prime integers. Find $1000m + n$.

*Proposed by David Altizio*

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### Solution 15

For each $i = 1, 2, 3$, we know that $r_i^3 = r_i - 1$, so

$$\frac{1}{r_i^2 + r_i + 1} = \frac{r_i - 1}{r_i^3 - 1} = \frac{r_i - 1}{r_i - 2}.$$

Hence the sum in question becomes

$$\frac{r_1 - 1}{r_1 - 2} + \frac{r_2 - 1}{r_2 - 2} + \frac{r_3 - 1}{r_3 - 2} = 3 + \left( \frac{1}{r_1 - 2} + \frac{1}{r_2 - 2} + \frac{1}{r_3 - 2} \right).$$

Now observe that the polynomial with roots $r_1 - 2$, $r_2 - 2$, and $r_3 - 2$ is

$$(x+2)^3 - (x+2) + 1 = x^3 + 6x^2 + 11x + 7,$$

so by Vieta the sum of the roots of the polynomial on the right hand side equals $-\frac{11}{7}$. It follows that the expression equals $3 - \frac{11}{7} = \frac{10}{7}$, and so the requested answer is $\boxed{10007}$.

## 3.16 Problem 16

An arrow points from $(0,0)$ to $(1,0)$. Define a *flop* as an operation where the arrow either moves one unit in the direction it is pointing, or the arrow switches between pointing right and pointing up, each with equal probability. Let $(x,y)$ be the average position of the arrow after all possible sequences of 100 flops. If $xy$ can be expressed in the form $\frac{a}{b}$, where $a,b$ are relatively prime, find $a+b$.

*Proposed by Ethan Liu*

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### Solution 16

Let $\vec{v}_i$ denote the random variable equal to the vector that the arrow moves on the $i^{\text{th}}$ term. By Linearity of Expectation,

$$\mathbb{E}[\vec{v}_1 + \cdots + \vec{v}_{100}] = \mathbb{E}[\vec{v}_1] + \cdots + \mathbb{E}[\vec{v}_{100}].$$

Observe that $\mathbb{E}[\vec{v}_1] = \frac{1}{2}\langle 1,0\rangle + \frac{1}{2}\langle 0,0\rangle = \langle \frac{1}{2},0\rangle$. Furthermore, for all other $2 \le i \le 100$, a symmetry argument (toggle the first move) shows that $\vec{v}_i$ equals $\langle 0,0\rangle$ with probability $\frac{1}{2}$ and it equals the two vectors $\langle 1,0\rangle$ and $\langle 0,1\rangle$ with probability $\frac{1}{4}$ each; hence $\mathbb{E}[\vec{v}_i] = \langle \frac{1}{4},\frac{1}{4}\rangle$. All in all,

$$\mathbb{E}[\vec{v}_1 + \cdots + \vec{v}_{100}] = \left\langle \frac{1}{2}, 0 \right\rangle + 99\left\langle \frac{1}{4}, \frac{1}{4} \right\rangle = \left\langle \frac{101}{4}, \frac{99}{4} \right\rangle.$$

Thus $xy = \frac{9999}{16}$ and the requested sum is $\boxed{10015}$.

## 3.17 Problem 17

What is the sum of all two-digit primes $p$ for which there exist positive integers $x$, $y$, and $D$ such that

$$x^2 - Dy^2 = (x+1)^2 - D(y+1)^2 = p?$$

*Proposed by David Altizio*

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### Solution 17

The possible primes are $p = 13$, $p = 37$, and $p = 73$, for a grand sum of $\boxed{123}$. Indeed, notice that

$$\begin{aligned} 4^2 - 3 \cdot 1^2 &= 5^2 - 3 \cdot 2^2 = 13, \\ 7^2 - 3 \cdot 2^2 &= 8^2 - 3 \cdot 3^2 = 37, \text{ and} \\ 10^2 - 3 \cdot 3^2 &= 11^2 - 3 \cdot 4^2 = 73. \end{aligned}$$

It remains to show that these are the only primes that work.

To prove this, first observe that $x^2 - Dy^2 = (x+1)^2 - D(y+1)^2$ implies $2x + 1 = D(2y+1)$. From this, we see that $D$ is odd. Now plugging in $x = \frac{D(2y+1)-1}{2}$ into the equation $x^2 - Dy^2 = p$ yields

$$\begin{aligned} p &= \left(\frac{D(2y+1)-1}{2}\right)^2 - Dy^2 \\ &= (D^2 - D)y^2 + (D^2 - D)y + \frac{(D-1)^2}{4}. \end{aligned} \tag{1}$$

Multiplying by 4 yields

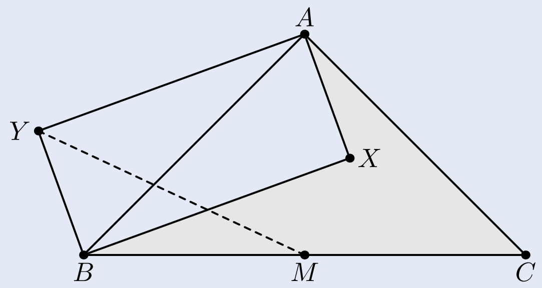
$$4p = 4D(D-1)y^2 + 4D(D-1)y + (D-1)^2,$$

so $D - 1$ divides $4p$. Now observe that $D = 0$ fails - since this yields $4p = 1$ - so $4D(D-1)y + (D-1)^2 \geq 0$. In particular, $D(D-1) \leq p$. Hence $D$ cannot be at least $p+1$. As a result, $D$ must be either 3 or 5, but plugging in $D = 5$ gives $p = 20y^2 + 20y + 4$, which is a contradiction since then $p$ is divisible by 4. It follows that $D = 3$.

Finally, plugging $D = 3$ back into (1) yields $p = 6y^2 + 6y + 1$, so $y^2 + y = \frac{p-1}{6}$. Since $p \leq 100$, we have $y^2 + y \leq 16$, so $y$ must be either 1, 2, or 3. Plugging these values back in to solve for $p$ yields that $p$ equals 13, 37, or 73, which proves the claim.

## 3.18 Problem 18

Let $ABC$ be an isosceles right triangle with $AB = AC$, and denote by $M$ the midpoint of $\overline{BC}$. Construct rectangle $AXBY$, with $X$ in the interior of $\triangle ABC$, such that $YM = 8$ and $AY^3 + BY^3 = 10^3$. The area of quadrilateral $AXBC$ can be expressed in the form $\frac{m\sqrt{n}}{p}$, where $m$, $n$, and $p$ are positive integers such that $m$ and $p$ are relatively prime and $n$ is not divisible by the square of any prime. Find $1000000m + 1000n + p$.




*Proposed by David Altizio and Arul Kolla*

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### Solution 18

Observe that, since

$$\angle AYB = \angle AXB = \angle AMB = 90^\circ,$$

pentagon $AXMBY$ is cyclic with $AM = MB$. Let $AX = x$ and $AY = y$. By Ptolemy on cyclic quarilateral $AYBM$,

$$AM(x+y) = YM \cdot AB = 8 \cdot (AM\sqrt{2}),$$

whence $x + y = 8\sqrt{2}$. In turn, the area of quadrilateral $AXBC$ is

$$\begin{aligned}[AXBC] = [ABC] - [AXB] &= \frac{1}{2} \cdot AB^2 - \frac{1}{2} \cdot xy \\ &= \frac{x^2+y^2}{2} - \frac{xy}{2} = \frac{x^2 - xy + y^2}{2} = \frac{x^3+y^3}{2(x+y)}.\end{aligned}$$

But $x^3 + y^3 = 10^3$ and $x + y = 8\sqrt{2}$, so this expression simplifies to $\frac{10^3}{2(8\sqrt{2})} = \frac{125\sqrt{2}}{4}$. The requested answer is $\boxed{125002004}$.

## 3.19 Problem 19

Suppose $a$ and $b$ are positive integers with $a < b$ such that the four intersection points of the lines $y = x + a$ and $y = x + b$ with the parabola $y = x^2$ are the vertices of a quadrilateral with area 720. Find $1000a + b$.

*Proposed by David Altizio*

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### Solution 19

In general, let $a > 0$, and suppose $(c, c^2)$ and $(d, d^2)$ are the two intersection points of the line $y = x + a$ and the parabola $y = x^2$, where $c < d$. Note that $c$ and $d$ are the roots of the polynomial $x^2 - x - a = 0$, so $c + d = 1$ and $cd = -a$. Hence

$$(d - c)^2 = (d + c)^2 - 4cd = 1 + 4a,$$

and so $d - c = \sqrt{1 + 4a}$. In turn, the distance between these two points is $\sqrt{2(1 + 4a)}$.

Observe that the quadrilateral in question has bases of lengths $\sqrt{2(1 + 4a)}$ and $\sqrt{2(1 + 4b)}$ and has height $\frac{1}{\sqrt{2}}(b - a)$. Thus, the condition rewrites to

$$\begin{aligned} 720 &= \frac{1}{2} \cdot \frac{1}{\sqrt{2}}(b - a) \cdot \left(\sqrt{2(1 + 4b)} + \sqrt{2(1 + 4a)}\right) \\ &= \frac{1}{2}(b - a)\left(\sqrt{1 + 4b} + \sqrt{1 + 4a}\right). \end{aligned}$$

It suffices to find the unique solution to this Diophantine equation.

Notice that $\sqrt{1 + 4b} + \sqrt{1 + 4a}$ is rational, hence an integer; in turn, $1 + 4b$ and $1 + 4a$ are perfect squares. Let $1 + 4b = (2m + 1)^2$ and $1 + 4a = (2n + 1)^2$, so that $b = m^2 + m$ and $a = n^2 + n$. Then

$$720 = \frac{1}{2}(m^2 + m - n^2 - n)(2m + 1 + 2n + 1) = (m - n)(m + n + 1)^2.$$

Now $720 = 2^4 \cdot 3^2 \cdot 5$, so $m + n + 1$ has to divide $2^2 \cdot 3 = 12$. Testing all divisors of 12 – and remembering to exclude cases with $m - n \geq m + n$ – yields only one solution in positive integers $(m, n)$, namely $(8, 3)$. Hence $b = 8 \cdot 9 = 72$ and $a = 3 \cdot 4 = 12$. The requested answer is $1000a + b = \boxed{12072}$.

## 3.20 Problem 20

Determine the least integer $k \geq 2$ for which

$$\frac{1}{2^{1000}} \prod_{i=2}^{k} (i^{100} - 1) := \frac{(2^{100} - 1)(3^{100} - 1) \cdots (k^{100} - 1)}{2^{1000}}$$

is an integer.

*Proposed by Andrew Wen*

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### Solution 20

We will invoke the following two important results in this solution. Their proofs are omitted from this packet but can easily be found online.

**Lemma 1 (Lifting the Exponent, $p = 2$ Case)**
For all odd integers $n$ and even positive integers $k$,

$$\nu_2(n^k - 1) = \nu_2(n - 1) + \nu_2(n + 1) + \nu_2(k) - 1.$$

**Lemma 2 (De Polignac, Special Case)**
Let $s(n)$ denote the sum of the digits of a positive integer $n$ when written in binary. Then

$$\nu_2(n!) = n - s(n).$$

Note that the minimum value of $k$ must be odd since all even terms contribute nothing to the 2-adic valuation of the numerator. Let $k = 2k_0 + 1$. From Lemma 1, we know that

$$\nu_2(i^{100} - 1) = \nu_2(i - 1) + \nu_2(i + 1) + 1$$

for all odd integers $i$, and furthermore

$$\nu_2(i^{100} - 1) = 0 = \nu_2(i - 1) + \nu_2(i + 1)$$

for all even integers $i$. Therefore, we have

$$\begin{aligned}
\nu_2\left(\prod_{i=2}^{2k_0+1} (i^{100} - 1)\right) &= \sum_{i=2}^{2k_0+1} \nu_2(i^{100} - 1) \\
&= \sum_{i=2}^{2k_0+1} [\nu_2(i - 1) + \nu_2(i + 1)] + k_0 \\
&= \nu_2((2k_0)!) + \nu_2(\tfrac{1}{2}(2k_0 + 2)!) + k_0 \\
&= \nu_2((2k_0)!) + \nu_2((2k_0 + 2)!) + k_0 - 1.
\end{aligned}$$

But Lemma 2 implies the expression above can be further simplified to

$$\begin{aligned}
&\big[2k_0 - s(2k_0)\big] + \big[(2k_0 + 2) - s(2k_0 + 2)\big] + k_0 - 1 \\
&= 5k_0 + 1 - s_2(k_0) - s_2(k_0 + 1).
\end{aligned}$$

It suffices to find the smallest value of $k_0$ for which this expression is at least 1000.

Let $f(k_0)$ denote the expression above. Notice that $f(k_0) \leq 5k_0$, so $k_0 \geq 200$. We may compute $f(200) = 994$, $f(201) = 998$, and $f(202) = 1002$. Thus the minimum value of $k_0$ is 202, and so the minimum value of $k$ is $\boxed{405}$.

## 3.21 Problem 21

Let $n \geq 4$ be an integer. A lamplighter has $n$ lamps, placed at the vertices of a regular $n$-gon. Initially, all the lamps are off. In a move, the lamplighter may toggle four lamps placed at the vertices of an isosceles trapezoid or rectangle.
Let $a_n$ denote the number of configurations of lamps the lamplighter can obtain after applying some number of moves. Compute the remainder when $a_4 + a_5 + \cdots + a_{100}$ is divided by 1000.

*Proposed by Ankan Bhattacharya*

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### Solution 21

We split the solution into two parts.

**Part 1: Determining $a_n$.** We claim that

$$a_n = \begin{cases} 2 & \text{if } n = 4, \\ 2^{n-1} & \text{if } n \geq 5 \text{ is odd,} \\ 2^{n-2} & \text{if } n \geq 6 \text{ is even.} \end{cases}$$

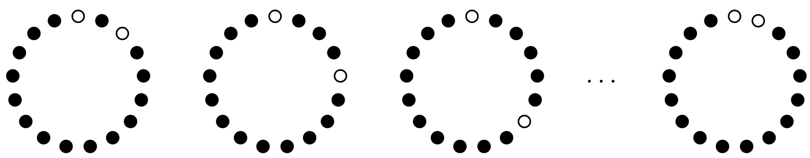
To prove this, ignore $n = 4$. Enumerate the lamps from 0 to $n-1$. First note that, since we always change the status of an even number of lamps, the number of lit lamps must always be even (due to parity restrictions). This is enough for the case where $n$ is odd. In the case where $n$ is even, letting $a$, $b$, $c$, and $d$ be the labels on the lamps in clockwise order, observe that the condition implies

$$a + c \equiv b + d \pmod{n}.$$

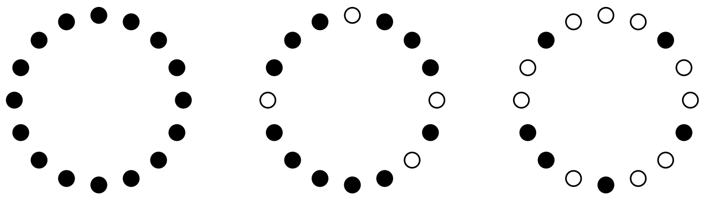
Since $n$ is even, this implies $a + c \equiv b + d \pmod{2}$. That is, the number of lit lamps in the sub-polygon $\mathcal{P}_1 := \{0, 2, \ldots, n-2\}$ (i.e. the number of lamps with even labels) must be even, as must the number of lit lamps in the sub-polygon $\mathcal{P}_2 := \{1, 3, \ldots, n-1\}$.

It remains to establish that these restrictions are sufficient. Call such a configuration of lamps *admissible*, where the exact definition depends on the parity of $n$ as above. If $n \geq 5$, then for any $a$, toggling $\{a, a+1, a+3, a+4\}$ and $\{a+1, a+2, a+3, a+4\}$ has the net effect of toggling $\{a, a+2\}$. This is enough to generate any admissible configuration.

- If $n$ is odd, then 2 is invertible modulo $n$, so by iterating this process several times we may toggle any two given lamps $i$ and $j$:



- If $n$ is even, then iterating this process has the effect of toggling any two given lamps in $\mathcal{P}_1$ or any two given lamps in $\mathcal{P}_2$, so we can build the any admissible configuration in $\mathcal{P}_1$ before building it in $\mathcal{P}_2$:



This completes the classification of $a_n$.

**Part 2: Answer Extraction.** It remains to compute

$$2 + 2(2^4 + 2^6 + \cdots + 2^{98}) \pmod{1000}.$$

To do this, use the Chinese Remainder Theorem modulo 8 and 125. Clearly, the sum is congruent to 2 modulo 8 since all terms beyond the first are divisible by 8. Furthermore, remark that

$$\begin{aligned} 2^4 + 2^6 + \cdots + 2^{98} &= 2^4(1 + 4 + \cdots + 4^{47}) = 16 \cdot \frac{4^{48} - 1}{3} \\ &\equiv \frac{2^{100} - 16}{2} \equiv \frac{1 - 16}{3} \equiv -5 \pmod{125}. \end{aligned}$$

Therefore $a_4 + \cdots + a_{100} \equiv 2 + 2 \cdot (-5) \equiv -8 \pmod{125}$. That is, letting $X$ be the sum, we have

$$X \equiv 2 \pmod 8 \quad \text{and} \quad X \equiv -8 \pmod{125}.$$

The second condition implies $X = 125k - 8$ for some positive integer $k$. Taking modulo 8 yields $2 \equiv 5k \pmod 8$, so $k \equiv 2 \pmod 8$. Therefore

$$X = 125(8\ell + 2) - 8 = 1000\ell + 242$$

for some nonnegative integer $\ell$. All in all, $X \equiv \boxed{242} \pmod{1000}$.

## 3.22 Problem 22

Determine the number of ordered pairs $(b, c)$ of integers with $0 \leq b \leq 2016$ and $0 \leq c \leq 2016$ such that 2017 divides $x^3 - bx^2 + c$ for exactly one integer $x$ with $0 \leq x \leq 2016$.

*Proposed by David Altizio*

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### Solution 22

Replace 2017 with an arbitrary odd prime $p$. First suppose $x^3 - bx^2 + c$ has exactly one solution modulo $p$. By unique factorization of polynomials modulo $p$, there are two possible cases.

- In the first case,
$$x^3 - bx^2 + c \equiv (x-q)^3 \pmod{p}$$
for some $q \in \mathbb{F}_p$. Expanding and equating coefficients yields $q = 0$, so the only polynomial in this case is $x^3$, which works.

- In the second case, the given polynomial is not a perfect cube, which implies a more general factorization
$$x^3 - bx^2 + c \equiv (x-q)(x^2 + rx + s) \pmod{p}.$$
Since $q$ is the only root of the cubic polynomial, $x^2 + rx + s$ must be irreducible. Furthermore, if $r = 0$, then
$$(x-q)(x^2+s) \equiv x^3 - qx^2 + sx - sq,$$
so $s = 0$ as well and the polynomial $x^2 + rx + s$ is no longer irreducible.

  Conversely, suppose $x^2 + rx + s$ is an irreducible quadratic in $\mathbb{F}_p$ with $r \neq 0$. Since $p \nmid r$, the multiplicative inverse $r^{-1}$ exists modulo $p$. Now we may take $q \equiv sr^{-1}$, so that
$$(x^2 + rx + s)(x - sr^{-1}) \equiv x^3 - (sr^{-1} - r)x^2 - s^2r^{-1} \pmod{p},$$
which satisfies the given conditions.

Therefore, excluding the trivial case $b = c = 0$, the set of cubic polynomials satisfying the given condition bijects to the set of irreducible quadratics $x^2 + rx + s$ with $r \neq 0$.

We now proceed to count this quantity in two steps; each step exploits a slightly technical but important fact in number theory.

- First, we count the number of irreducible quadratics modulo $p$. To do this, we complementary count. Every reducible quadratic factors uniquely into two linear factors. These factors can either be the same (for perfect square trinomials) or different, leading to a total of
$$p + \binom{p}{2} = \frac{p^2 + p}{2}$$

reducible quadratics. As there are $p^2$ total quadratics modulo $p$, we conclude there are

$$p^2 - \frac{p^2+p}{2} = \frac{p^2-p}{2}$$

irreducible quadratics modulo $p$.

- Now we add in the constraint that $r \neq 0$. Again, we complementary count. If $r = 0$, then the polynomial $x^2 + s$ must be irreducible modulo $p$. This holds if and only if $s$ is a quadratic non-residue modulo $p$. There are precisely $\frac{p-1}{2}$ such quadratic non-residues, ergo $\frac{p-1}{2}$ irreducible quadratics with $x$-coefficient equal to zero. In turn, there are

$$\frac{p^2-p}{2} - \frac{p-1}{2} = \frac{(p-1)^2}{2}$$

irreducible quadratics modulo $p$ with nonzero $x$-coefficient.

Plugging in $p = 2017$ and remembering to add the lone 1 yields a final answer of $\boxed{2032129}$.

*Remark 3.* In fact, one can compute the number of irreducible polynomials of any degree modulo $p$. A fact of abstract algebra says that all irreducible polynomials with degree dividing $d$ can be realized as irreducible factors of the polynomial $x^{p^d} - x$, and vice versa. Thus, letting $A_n$ be the number of irreducible polynomials of degree exactly $n$, a degree count of both sides yields

$$p^n = \sum_{d|n} dA_d.$$

Möbius inversion thus yields

$$A_n = \frac{1}{n}\sum_{d|n} p^d \mu\left(\frac{n}{d}\right).$$

## 3.23 Problem 23

Let $ABC$ be a triangle with $\angle ABC = 90^\circ$ and incircle $\omega$ of radius 4, which is tangent to $\overline{BC}$ and $\overline{AC}$ at $E$ and $F$ respectively. Suppose that $\overline{BF}$ is tangent to the circumcircle of $\triangle CEF$. Then $BC$ can be written in the form $p + \sqrt{q}$ for positive integers $p$ and $q$. Determine $1000p + q$.

*Proposed by Andrew Wen*

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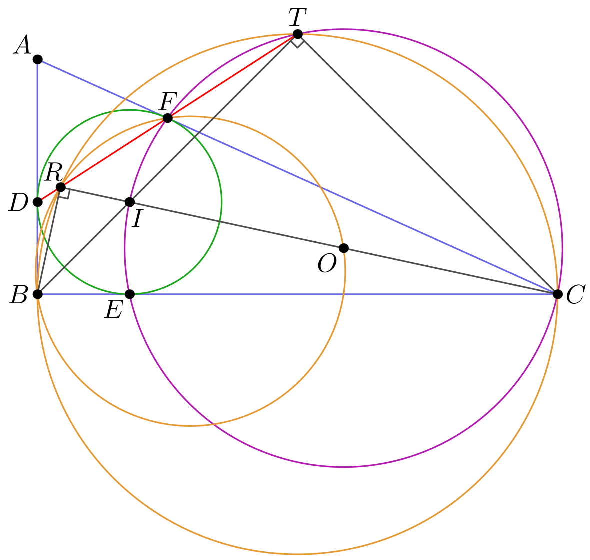
### Solution 23

Let $I$ be the incenter of $\triangle ABC$ and $O$ be the midpoint of $\overline{CI}$, which is also the circumcenter of $\odot(CEF)$. The condition that $BF$ is tangent to $\odot(CEF)$ implies $\angle BFO = 90^\circ$.




Let $R$ be the projection of $B$ onto $CI$ and let $T$ be the projection of $C$ onto $BI$. It is known (via, for example, the Iran Lemma or Sharygin 2009) that the points $D$, $R$, $F$, and $T$ are collinear. Now $\angle BRO = \angle BFO = 90^\circ$ implies that $BRFO$ is a cyclic quadrilateral, while $\angle IFC = \angle ITC = 90^\circ$ implies that $IFTC$ is a cyclic quadrilateral. It follows via spiral similarity that

$$\triangle BFO \sim \triangle BTC \sim \triangle BEI.$$

Now let $CE = 2d$. Note that $CO = EO = FO = \sqrt{d^2+4}$ and $BO = \sqrt{(d+4)^2+4}$. Then the above similarity yields

$$\frac{BO}{FO} = \frac{\sqrt{(d+4)^2+4}}{d^2+4} = \frac{BI}{FI} = \sqrt{2}.$$

Solving for $d$ and discarding the negative solution yields $d = 4 + 2\sqrt{7}$. Thus

$$BC = BE + 2d = 12 + \sqrt{112}$$

and the requested answer is $\boxed{12112}$.

## 3.24 Problem 24

Seven teams play in a round robin tournament (each pair of teams plays exactly one game against each other). In each game, one of the two participating teams wins and the other loses; there are no ties. 21 games are played in total, so there are $2^{21}$ possible results of the tournament. For how many of these results is it true that for every pair $(A, B)$ of teams, there exists a third team, $C$, that won against both of them?

*Proposed by Linus Tang*

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### Solution 24

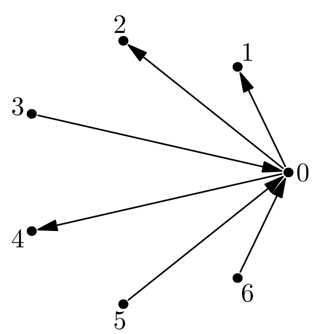
We first claim that every team in the tournament lost and won exactly three games (so that the tournament is *3-regular*). To prove this, note the following facts:

- If a team $P$ won against all other teams, then the condition is violated because we can choose another arbitrary team, $Q$, and no team beat both $P$ and $Q$ (because no team beat $P$).
- If a team $P$ won against all other teams except for $Q$, then the condition is again violated because no team beat $P$ and $Q$.
- If a team $P$ won against all other teams except for $Q$ and $R$, then it isn't as easy to see why this is impossible. The only team that could have beaten both $P$ and $Q$ is $R$. Therefore, $R$ beat $Q$. But the only team that could have beaten both $P$ and $R$ is $Q$, implying that $Q$ beat $R$. This is a contradiction.

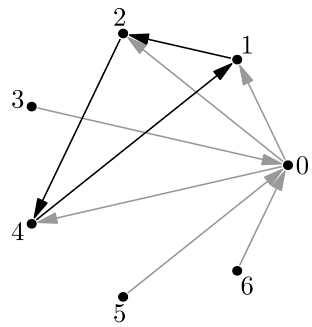
Therefore each team lost at least 3 games, for a total of at least 21 losses. But the tournament must have exactly 21 losses, so equality holds. This implies each team lost exactly three games.

The rest of this solution is dedicated to showing the surprising fact that there is exactly one tournament up to isomorphism that satisfies the conditions: the so-called *quadratic residue tournament*, defined by labeling the vertices as integers from 0 to 6 and saying that team $x$ beat team $y$ iff $x - y \in \{1, 2, 4\} \pmod 7$.

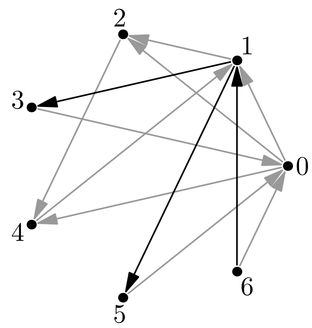
We start with a graph of seven dots, labeled from 0 to 6, and fill in directed edges between them. An arrow from $P$ to $Q$ indicates that team $P$ won against team $Q$. To begin, we know that team 0 won against exactly 3 teams, so we can assume WLOG (without loss of generality) that those three teams were 1, 2, and 4.



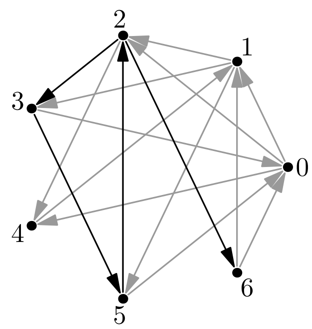
Since team 0 beat all three of the teams 1, 2, and 4, each of these teams must have been beaten by another of them. (In general, if teams $P$, $Q$, and $R$ were beaten by the same team, then $P$, $Q$, and $R$ form a cycle.) Without loss assume the cycle goes $1 \to 2 \to 4 \to 1$.




Since team 1 won against exactly 3 teams, it must have lost against exactly one of 3, 5, and 6. Without loss, assume it lost against team 6 and won against the other two.



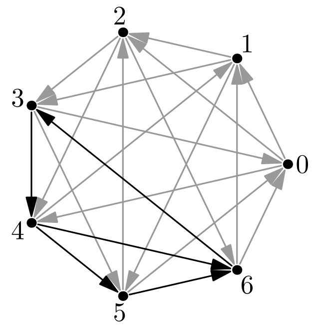

Team 2 has already lost to teams 0 and 1, so it must win against either team 3 or 5 (or both). Since teams 3 and 5 are symmetrical with respect to each other, we can assume WLOG that team 2 beats 3. Then teams 2, 3, and 5 were all beaten by team 1, so 2, 3, and 5 must form a cycle. This allows us to completely determine the games that team 2 won and lost.




Team 2 beat teams 3, 4, and 6, so 3, 4, and 6 must form a cycle. If the cycle goes $3 \to 6 \to 4 \to 3$, then team 6 beats teams 0, 1, and 4, yet these three teams themselves do not form a cycle. Therefore, the cycle must go $3 \to 4 \to 6 \to 3$. At this point, we know enough information about the tournament to completely deduce the rest of the graph.

So, indeed, all solutions are symmetrical to the quadratic residue tournament. It remains to count the number of symmetries.

There are $7! = 5040$ ways to assign seven numbers to seven teams, but this is an overcount for two reasons. First, we need to divide by seven because the assignments 0123456,




1234560, 2345601, $\cdots$, and 6012345 lead to the same graph (that is, the tournament is invariant under rotation). We also need to divide by 3 because 0123456, 0246135, and 0415263 lead to the same graph (that is, the tournament is invariant under cyclic permutation of the three cycles of length 7.) Therefore, the answer is $\frac{5040}{7 \cdot 3} = \boxed{240}$.

## 3.25 Problem 25

David attempts to draw a regular hexagon $ABCDEF$ inscribed in a circle $\omega$ of radius 1. However, while the points $A$ through $E$ are accurately placed, his point $F$ is slightly off, and so the diagonals $AD$, $BE$, and $CF$ do not concur. (Fortunately for him, the location of point $F$ on $\omega$ is the only inaccurate part of the diagram.) Undeterred by his artistic deficiencies, David pretends that the diagram is accurate by shading in the circumcircle of the triangle bounded by these three line segments, which has radius $\frac{1}{24}$. He then continues to draw the rest of the diagram.
The area of the hexagon $ABCDEF$ that David actually drew can be expressed in the form $a - \frac{b}{c}\sqrt{d}$, where $a$, $b$, $c$, and $d$ are positive integers such that $b$ and $c$ are relatively prime and $d$ is not divisible by the square of any prime. Find $a + b + c + d$.

*Proposed by David Altizio*

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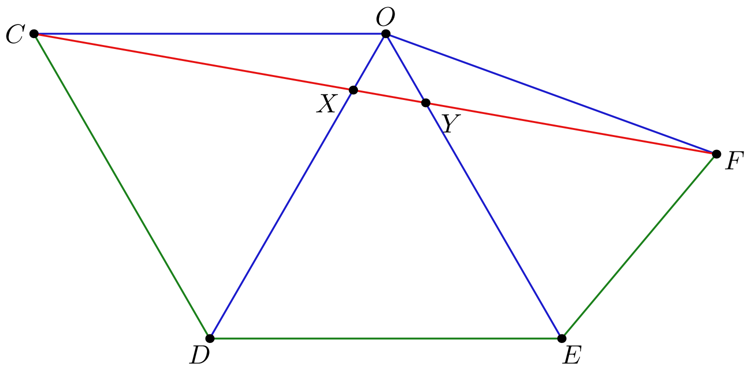
### Solution 25

Let $O$ be the center of $\omega$. Without loss of generality assume that $F$ lies on the same side of $OC$ as $D$ and $E$ do; then $\overline{CF}$ intersects $\overline{OD}$ and $\overline{OE}$ at points $X$ and $Y$, respectively. The given condition is equivalent to the circumradius of $\triangle OXY$ equaling $\frac{1}{24}$.




Let $\alpha := \angle OCF = \angle OFC$. Then

$$\angle OXY = 60^\circ + \alpha \quad \text{and} \quad \angle OYX = 60^\circ - \alpha.$$

The Extended Law of Sines applied to $\triangle OXY$ yields $OY = \frac{1}{12}\sin(60^\circ + \alpha)$. Then the Law of Sines applied to $\triangle OCY$ gives

$$\frac{1}{\sin(60^\circ - \alpha)} = \frac{OY}{\sin\alpha} = \frac{\frac{1}{12}\sin(60^\circ + \alpha)}{\sin\alpha},$$

or

$$12\sin\alpha = \sin(60^\circ + \alpha)\sin(60^\circ - \alpha) = \frac{3}{4} - \sin^2\alpha.$$

Solving the quadratic in $\sin\alpha$ and disregarding the extraneous solution yields $\sin\alpha = \frac{7\sqrt{3}}{2} - 6$. (This means $\alpha \approx 3.565^\circ$.)

Finally, a bit more angle chasing yields $\angle FOE = 60^\circ - 2\alpha$ and $\angle FOA = 60^\circ + 2\alpha$, so

$$\begin{aligned}[OAFE] &= [AOF] + [EOF] = \tfrac{1}{2}(\sin(60^\circ + 2\alpha) + \sin(60^\circ - 2\alpha)) \\ &= \sin(60^\circ)\cos(2\alpha) = \tfrac{\sqrt{3}}{2}\left(1 - 2\left(\tfrac{7\sqrt{3}}{2} - 6\right)^2\right) = 126 - \tfrac{289}{4}\sqrt{3}.\end{aligned}$$

Therefore the area of the entire hexagon $ABCDEF$ is $4 \cdot \frac{\sqrt{3}}{4} + [OAFE] = 126 - \frac{285}{4}\sqrt{3}$. The requested sum is $126 + 285 + 4 + 3 = \boxed{418}$.