Coloring Polygons

Dylan Yu

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Problem 1 (USAMO 2003/6). At the vertices of a regular hexagon are written 6 nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers at the neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all 6 vertices.

Problem 2 (USAMO 2011/2). An integer is assigned to each vertex of a regular pentagon so that the sum of the five integers is 2011. A turn of a solitaire game consists of subtracting an integer m from each of the integers at two neighboring vertices and adding 2m to the opposite vertex, which is not adjacent to either of the first two vertices. (The amount m and the vertices chosen can vary from turn to turn.) The game is won at a certain vertex if, after some number of turns, that vertex has the number 2011 and the other four vertices have the number 0. Prove that for any choice of the initial integers, there is exactly one vertex at which the game can be won.

Problem 3 (APMC 2000). For which integers $n \geq 5$ is it possible to color the vertices of a regular n-gon using at most 6 colors in such a way that any 5 consecutive vertices have different colors?

Problem 4 (BWM 2007/R1/1). The vertices and the midpoints of the sides of a given regular 2007-gon are to be numbered with numbers $1, 2, \ldots, 4014$ so that the sum of the three numbers at every side is the same. Show that such a numbering is possible.

Problem 5 (ToT Spring 2000/4). Can one place positive integers at all vertices of a cube in such a way that for every pair of numbers connected by an edge, one will be divisible by the other, and there are no other pairs of numbers with this property?

Problem 6 (IRMO 2015/P1/2). A regular polygon with $n \geq 3$ sides is given. Each vertex is coloured either red, green or blue, and no two adjacent vertices of the polygon are the same colour. There is at least one vertex of each colour. Prove that it is possible to draw certain diagonals of the polygon in such a way that they intersect only at the vertices of the polygon and they divide the polygon into triangles so that each such triangle has vertices of three different colours.

Problem 7 (German TST 2011/6). Vertices and edges of a regular n-gon are numbered $1, 2, \ldots, n$ clockwise such that edge i lies between vertices $i, i+1 \pmod n$. Now non-negative integers $(e_1, e_2, \ldots, e_n)$ are assigned to corresponding edges and non-negative integers $(k_1, k_2, \ldots, k_n)$ are assigned to corresponding vertices such that:

- $(e_1, e_2, \ldots, e_n)$ is a permutation of $(k_1, k_2, \ldots, k_n)$.
- $k_i = |e_{i+1} - e_i|$ with indices in mod n.

(a) Prove that for all $n \geq 3$, such non-zero n-tuples exist.

(b) Determine for each m the smallest positive integer n such that there is an n-tuples stisfying the above conditions and also $\{e_1, e_2, \ldots, e_n\}$ contains all $0, 1, 2, \ldots m$.

Problem 8 (Saudi Arabia Pre-TST 2016). Ten vertices of a regular 20-gon $A_1A_2....A_{20}$ are painted black and the other ten vertices are painted blue. Consider the set consisting of diagonal A_1A_4 and all other diagonals of the same length.

(a) Prove that in this set, the number of diagonals with two black endpoints is equal to the number of diagonals with two blue endpoints.

(b) Find all possible numbers of the diagonals with two black endpoints.

Problem 9 (Path to Combinatorics). Each vertex of a regular polygon is colored with one of a finite number of colors so that the points of the same color are the vertices of some new regular polygon. Prove that at least two of the polygons obtained are congruent.

Problem 10 (ToT Spring 1983). k vertices of a regular n-gon P are coloured. A colouring is called almost uniform if for every positive integer m the following condition is satisfied: if M_1 is a set of m consecutive vertices of P and M_2 is another such set then the number of coloured vertices of M_1 differs from the number of coloured vertices of M_2 at most by 1. Prove that for all positive integers k and n ($k \leq n$) an almost uniform colouring exists and that it is unique within a rotation.

Problem 11 (Caucasus MO 2020). All vertices of a regular 100-gon are colored in 10 colors. Prove that there exist 4 vertices of the given 100-gon which are the vertices of a rectangle and which are colored in at most 2 colors.

Problem 12 (ToT Fall 2011). The vertices of a regular 45-gon are painted into three colors so that the number of vertices of each color is the same. Prove that three vertices of each color can be selected so that three triangles formed by the chosen vertices of the same color are all equal.

Problem 13 (Polish 1994/R2/4). Each vertex of a cube is assigned 1 or -1. Each face is assigned the product of the four numbers at its vertices. Determine all possible values that can be obtained as the sum of all the 14 assigned numbers.

Problem 14 (239 2010/8). Consider the graph G with 100 vertices, and the minimum odd cycle goes through 13 vertices. Prove that the vertices of the graph can be colored in 6 colors in a way that no two adjacent vertices have the same color.

Problem 15 (ToT Spring 1988). The vertices of a regular n-gon are painted in turn black and white. Two people play the following game. Each in turn draws a diagonal connecting two vertices of the same color. These diagonals must not intersect. The winner is the player who is able to make the last move. Who will win if both players adopt the best strategy, and

(a) $n = 10$?

(b) $n = 12$?

Problem 16 (USEMO 2019/5). Let $\mathcal{P}$ be a regular polygon, and let $\mathcal{V}$ be its set of vertices. Each point in $\mathcal{V}$ is colored red, white, or blue. A subset of $\mathcal{V}$ is *patriotic* if it contains an equal number of points of each color, and a side of $\mathcal{P}$ is *dazzling* if its endpoints are of different colors.

Suppose that $\mathcal{V}$ is patriotic and the number of dazzling edges of $\mathcal{P}$ is even. Prove that there exists a line, not passing through any point in $\mathcal{V}$, dividing $\mathcal{V}$ into two nonempty patriotic subsets.

Problem 17 (ISL 2005/C8). In a certain n-gon, some $(n-3)$ diagonals are colored black and some other $(n-3)$ diagonals are colored red, so that no two diagonals of the same color intersect strictly inside the polygon, although they can share a vertex. (Note: a side is not a diagonal.) Find the maximum number of intersection points between diagonals colored differently strictly inside the polygon, in terms of n.