# ELMO 2020/3

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**Problem 1.** Janabel has a device that, when given two distinct points $U$ and $V$ in the plane, draws the perpendicular bisector of $UV$. Show that if three lines forming a triangle are drawn, Janabel can mark the orthocenter of the triangle using this device, a pencil, and no other tools.

*Solution.* Let the triangle be $ABC$. Then clearly we can find the circumcenter $O$ of $\triangle ABC$ by taking the perpendicular bisectors of any two of the sides of the triangle. Now take the perpendicular bisector of $AO$ and let it intersect $AB$ and $AC$ at $D$ and $E$, respectively. Call the midpoint of $AO$ as $M$. We can find the circumcenter of triangle $ADE$ through the same method as before; call that $P$. Then clearly

$$\angle MAE = 90^\circ - \angle AEM$$

so

$$\angle APD = 180^\circ - 2\angle AEM,$$

implying $\angle DAP = \angle AEM$. Thus, line $AP$ must contain $H$, the orthocenter of $\triangle ABC$ (using the fact that $O$ and $H$ are isogonal conjugates).

> **Claim** — Given points $X, Y$ respectively on parallel lines $x, y$ such that $XY$ is not perpendicular to $x, y$, we can construct the line $XY$.

*Proof.* Let the perpendicular bisector of $XY$ meet lines $x, y$ at $A, B$, respectively. Taking the perpendicular bisector of $AB$ gives us the line $XY$ as desired (sketch: prove $AXBY$ is a parallelogram). $\square$

Thus, because $AO \parallel PM$, and $AO$ and $PM$ are both lines we can "use" (because they are both perpendicular bisectors), we must be able to draw line $AP$. We can do the same for $B$ and intersect the two lines to get $H$ as desired. $\square$