Christmas Problems

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Problem 1 (239 2009 J8). Each of the 11 girls wants to mail each of the other a gift for Christmas. The packages contain no more than two gifts. If they have enough time, what is the smallest possible number of packages that they have to send?

Problem 2 (ToT 2014/6). During Christmas party Santa handed out to the children 47 chocolates and 74 marmalades. Each girl got 1 more chocolate than each boy but each boy got 1 more marmalade than each girl. What was the number of the children?

Problem 3 (Slovenia TST 2018/1). Let n be a positive integer. On the table, we have n^2 ornaments in n different colours, not necessarily n of each colour. Prove that we can hang the ornaments on n Christmas trees in such a way that there are exactly n ornaments on each tree and the ornaments on every tree are of at most 2 different colours.

Problem 4 (Chistmas Assessment Exercise). The elves in Santa's factory have received, between them, exactly 2^n Christmas cards. After Christmas they seek to dispose of them at the Lapland recycling centre. Santa is a harsh taskmaster, and will only allow one elf to take time off work to visit the dump. The elves can pass bundles of cards between themselves as follows: if elf X has x cards and elf Y has y cards, and $x \geq y$, then X may pass y cards to Y, so that X has $x - y$ cards and Y has $2y$ cards. Can the cunning elves arrange their exchanges so that they can recycle all the cards?

> *Remark 0.1.* More of these problems can be found here.

Problem 5 (Estonia TST 2020/2). The city has 2020 inhabitants. Everyone is happy before the New Year; but if before on holiday, the resident will not receive a greeting card from another resident, he will be sad. Unfortunately, there is only one postal company in the city, which provides only one service: each a resident can name two other citizens before the New Year, and during the holiday the postal company will send a postcard on behalf of the sender one of them of your choice. The postal company is known to choose recipients in such a way that as many citizens as possible become sad. Find the smallest possible number of townspeople who can become sad.

Problem 6 (Italian MO 2004/1). Observing the temperatures recorded in Cesenatico during the December and January, Stefano noticed an interesting coincidence: in each day of this period, the low temperature is equal to the sum of the low temperatures the preceeding day and the succeeding day. Given that the low temperatures in December 3 and January 31 were 5°C and 2°C respectively, find the low temperature in December 25.

Problem 7 (Geoff Smith). The Elves have a huge supply of wrapped presents in identical cubical boxes, with plenty wrapped in red paper, and plenty more wrapped in blue paper. To prepare for the big night, Santa orders that 12 boxes be placed on the ground in a row. Then 11 boxes are placed on top of these, so that the rightmost bottom box has no box on top of it. Then 10 boxes are to be placed on top of these, so that the last box on the right in the second row has no box on top of it. This continues until the last row has just one box in it, and it is on the far left. Thus 78 boxes are stacked in a triangular shape.

Santa is very superstitious, and insists that the stacking obeys *Lapland Rules*.

(a) In each of the 12 columns, the colours of the top box and the bottom box must be the same.

(b) If a box has a blue box underneath and a blue box to the left, then it must be a red box.

(c) If a box has a red box underneath and a red box to the left, then it must be a red box.

(d) If a box has different coloured boxes underneath and to the left, then it must be a blue box.

How many different stackings are there which obey the Lapland rules?

Problem 8 (Geoff Smith). **Follow up**: How many such completed stackings are possible, given that Santa insists that each column should have the same colour top present as its bottom present?

Problem 9 (ToT 2015/7). Santa Clause has k candies from each of n types of candies. He puts them randomly in k bags with and equal number of candies in each bag and gives the bags to k children, one to each. The children organise trade exchanges as follows: If one of them doesn't have candies of type i but has some of type j and the other has some of type i but none of type j then they may exchange one candy of these types among themselves. Prove that they can organise a sequence of exchanges which will eventually lead each of them having candies of all types.

Problem 10 (expii). The Christmas Tree at Rockefeller Center in New York City is an annual tradition. The 2016 tree is 94 feet high and 56 feet wide. Conveniently, tree weight does not scale with the volume of the cone determined by the tree, because the internal branching pattern makes the tree sparser inside than at the fringes. Evergreen trees grow this way so they have a large surface area to capture sunlight while the interior can support the tree structurally.

To see how quickly a branching structure can expand, consider a growth process which starts with a single branch, forking into two sub-branches after a foot of growth, and where every sub-branch forks into two more sub-branches after a foot of growth. After how many feet of growth would there be 1 million sub-branches?