USAMO 2021/1

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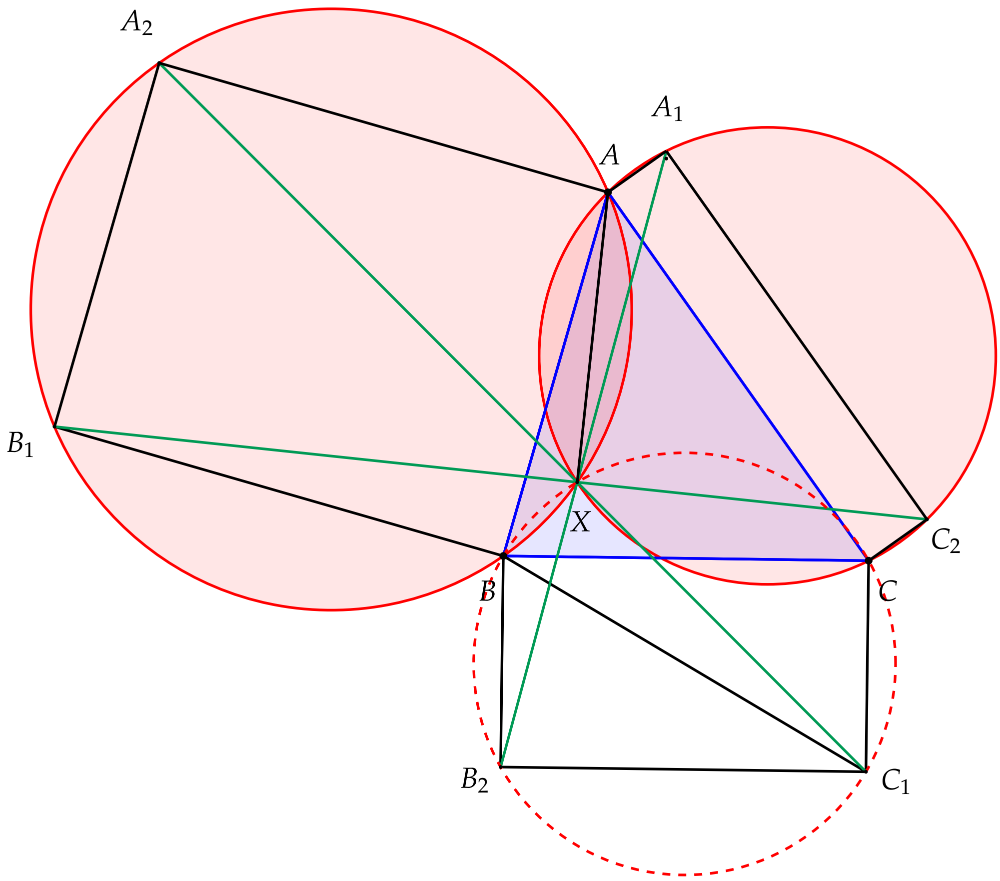
Problem 1 (USAMO 2021/1). Rectangles BCC_1B_2, CAA_1C_2, and $ABB_1A_@$ are erected outside an acute triangle ABC. Suppose that

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ.$$

Prove that lines B_1C_2, C_1A_2, and A_1B_2 are concurrent.

Solution. Let X be the foot of A to B_1C_2.



Claim — The circumcircles of BCC_1B_2, CAA_1C_2, and AB_1A_2 are concurrent on BC_2.

Proof. Because $\angle AXB_1 = 90^\circ$, we must have that AXB_1A_2 is cyclic. Furthermore, ABB_1A_2 is also cyclic because it is a rectangle, so X lies on it. By similar reasoning, X lies on CAA_1C_2.

Let $\angle AB_1B = \alpha$ and $\angle CA_1A = \beta$. Then

$$\angle BXB_1 = \angle BAB_1 = 90^\circ - \alpha,$$

and similarly $\angle CXC_2 = 90^\circ - \beta$. Thus, $\angle BXC_1 = \alpha + \beta$. But by definition,

$$\angle BC_1C + \angle CA_1A + \angle AB_1B = 180^\circ,$$

so

$$\angle BC_1C = 180^\circ - (\alpha + \beta),$$

implying

$$\angle BXC + \angle BC_1C = 180^\circ,$$

and thus $BXCC_1$ is cyclic. Again, BCC_1B_2 is a rectangle, so X lies on it. □

Similarly, the three circles are concurrent on C_1A_2 and A_1B_2. Thus, X lies on B_1C_2, C_1A_2, and A_1B_2, implying the desired. □