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Problem 1. Let ABC be a fixed acute triangle inscribed in a circle ω with center O. A variable point X is chosen on minor arc AB of ω, and segments CX and AB meet at D. Denote by O_1 and O_2 the circumcenters of triangles ADX and BDX, respectively. Determine all points X for which the area of triangle OO_1O_2 is minimized.

Solution. We prove that $[OO_1O_2]$ is minimized when X is the intersection of the height from C to AB and ω. Without loss of generality, let $\angle ADX \leq \angle BDX$ (alternatively, we can use directed angles, but this isn't too large of an issue).

Claim — $\triangle ABC \sim \triangle O_2OO_1$.

Proof. Note that $\angle BAC = \angle BXC$. Since $BX \perp OO_2$ and $CX \perp O_1O_2$ (because the perpendicular bisectors of AX, BX are used in determining O, O_1, O_2), we can easily deduce $\angle BXC = \angle O_1O_2O$. Similarly, we can find the other angles to get that these two triangles are indeed similar. □

Claim — O, O_1, O_2, X are concyclic.

Proof. It is well known that $\angle AO_1X = 2\angle ADX$ and so $\angle AXO_1 = 90^\circ - \angle ADX$. Similarly, $\angle BXO_2 = \angle BDX - 90^\circ = 90^\circ 0 \angle ADX$. Thus, $\angle AXB = \angle O_1XO_2$, and since $\angle ACB = \angle O_2OO_1$, we must have that

$$\angle O_1OO_2 + \angle O_2XO_1 = 180^\circ$$

as desired. □

Claim — Let Y be a point on the minor arc AB of ω such that $\angle BCX = \angle ACY$. Then quadrilateral $AYBC$ is similar to quadrilateral O_2XO_1O.

Proof. Note that

$$\angle BCX = \angle BAX = \frac{1}{2}\angle BO_1D = \angle XO_1O_2 = \angle XOO_2.$$

Equivalently, $\angle ACY = \angle XOO_2$, and by inspection, all the angles in (O_2XO_1O) match up with those in $(AYBC)$ (this isn't exactly rigorous, but one can easily show all angles are equal) as desired. □

We want to minimize the area of $\triangle OO_1O_2$, which implies we want to minimize the ratio of two corresponding parts in O_2XO_1O to $AYBC$. Let's focus on OX to CY in particular. Note that OX is the radius, which is fixed, implying we simply need to maximize CY, which is largest when CY is the diameter. It is well known then that CX is perpendicular to AB (using the fact that the circumcenter and orthocenter of a triangle are isogonal conjugates), thus concluding our proof. $\square$