ASE 2020-21 Advanced Notes

Lecture Notes by Dylan Yu

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Contents

§ 1 Sunday, 08/09/20

§ 1.1 Introductory Problems

These problems are to see what level you all are at.

Problem 1. How many numbers less than or equal to 100 are divisible by 2 and 3 but not 4?

Problem 2. There are n people at a party. Each person shakes hands with every person (besides themselves) exactly once. How many handshakes occur?

Problem 3. How many different squares of any size are there on a 3×3 board?

Problem 4. Jamie has 2 dimes, 4 nickels and 8 pennies. In how many different ways can she make 26 cents?

§ 1.2 Casework

Sometimes, we cannot simply multiply things out and get the right answer. This means we have to consider **cases**, and this is known as **casework**. Let's try an example to get started:

Example 1. The Dylan alphabet has only 5 letters, and every word in the Dylan language has no more than 3 letters in it. How many words are possible. Note that a word must have at least one letter.

Solution. Let's do casework on the number of letters:

1. There are 5 letters, so each letter could be a word. Thus, in this case there are **5** possible words with one letter.

2. There are two letters now, and the first letter can be any of the 5 letters, as well as the second letter. Thus, $5 \cdot 5 = \mathbf{25}$ possible words with two letters.

3. Following the same pattern, we have $5 \cdot 5 \cdot 5 = \mathbf{125}$ possible words with 3 letters.

Thus, there are $5 + 25 + 125 = \boxed{155}$ words in the Dylan language. □

Example 2. How many pairs of positive integers (a, b) satisfy $a^2 + b < 37$?

Solution. Notice that if $a = 7$ or higher, then there is no way the equation will work. So we only have to try $a = 1$ to 6:

1. If $a = 1$, then $b < 36$, and there are **35** possibilities for b.

2. If $a = 2$, then $b < 33$, and there are **33** possibilities for b.

3. If $a = 3$, then $b < 28$, and there are **27** possibilities for b.

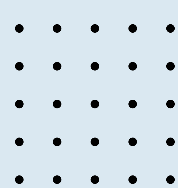
4. If $a = 4$, then $b < 21$, and there are **20** possibilities for b.

5. If $a = 5$, then $b < 12$, and there are **11** possibilities for b.

6. If $a = 6$, then $b < 1$, and there are **0** possibilities for b.

Thus, there are $35 + 33 + 27 + 20 + 11 + 0 = \boxed{126}$ pairs of positive integers. □

Exercise 1. How many squares of any size can be formed by connecting dots in the grid shown below?

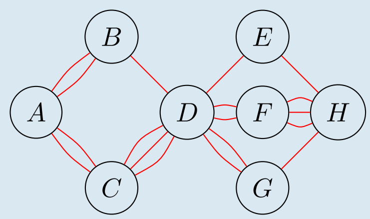


Exercise 2. How many words with 3 letters or less are there with only one A? Note that a *word* doesn't have to be a real word, for example ABC is a word.

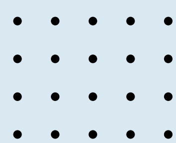
Exercise 3. How many 5-letter words can we make from the letters $A, B, C,$ and D, if we must use the letter A at least once?

Exercise 4. I have two hats. In one hat are balls numbered 1 through 15. In the other hat are balls numbered 16 through 25. I first choose a hat, then from that hat, I choose 3 balls, without replacing the balls between selections. How many different ordered selections of 3 balls are possible?

Exercise 5. How many paths are there from A to H in the diagram shown below, if we can only travel left to right? Note that we can only go on the red lines, and if there is a path between two letters, then we can only travel from the letter that comes first in the alphabet to the letter that comes after it (for example, we can go from A to B, A to C, C to D, F to H, but **not** H to G).

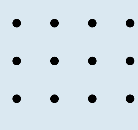


Exercise 6. How many different rectangles can be formed by connecting four of the dots in the following grid, with the sides parallel to the sides of the grid?



This next one is **very hard**:

Exercise 7. How many different isosceles triangles can be formed by connecting three of the dots in a 4×4 square array of dots as shown below?



§ 1.3 Complementary Counting

When we do **complementary counting**, we are counting the part we do **not** want and subtract that from the total. Again, let's try an example:

Example 3. How many three-digit numbers are **not** multiples of 8?

Solution. Let's count the number of three-digit multiples of 8. The smallest three-digit multiple of 8 is 104, and the largest is 992. That means they range from $104, 112, \ldots, 992$, and if we divide by 8, we get $13, 14, \ldots, 124$, so there must be $124 - 13 + 1 = 112$ three-digit multiples of 8. In total, there are $999 - 100 + 1 = 900$ three-digit numbers, so there are $900 - 112 = \boxed{788}$ three-digit numbers that are not multiples of 8. ☐

Remark 1. *Whenever you see the word not, it usually means we will do some sort of complementary counting.*

Fact 1. You should use complementary counting whenever it is easier to count the opposite.

In this example, it is a lot easier to count the multiples of 8 rather than the non-multiples, so that's what we did.

Example 4. The Smith family has 4 sons and 3 daughters. In how many ways can they be seated in a row of 7 chairs such that at least 2 boys are next to each other?

Solution. The opposite of *at least 2 boys next to each other* is *no boys next to each other*. This means there is at least one girl between every boy, but the only way this is possible is if we have $BGBGBGB$. There are $4! = 24$ ways to arrange the boys, and $3! = 6$ ways to arrange the girls, so together there are $24 \cdot 6 = 144$ ways to arrange the boys and girls in the pattern $BGBGBGB$. In total, there are $7! = 5040$ ways to arrange the kids in any order, so the answer we want is $5040 - 144 = \boxed{4896}$. ☐

Exercise 8. How many 4-letter words with at least one vowel can be constructed from the letters A, B, C, D, and E?

Exercise 9. How many 5-digit numbers have at least one two?

Exercise 10. How many 5-letter strings (a 5-letter string is just 5 letters combined together) have at least two consecuive integers which are the same?