MAO Invariants

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November 23, 2020

1 Introduction

1.1 Definitions

Invariant

An ***invariant*** is a property or quantity that does not change under certain operations

Monovariant

A ***semi-invariant*** or ***monovariant*** is a quantity that always increases or always decreases after the corresponding operation.

1.2 More Exposition

Classical examples of invariants are parity or algebraic expressions such as sums or products. Finding an invariant is a common idea in problems asking to prove that something cannot be achieved. Monovariants are also very efficient in showing that the corresponding process must stop after finitely many moves.

2 Classics

I'll skip over the "find the invariant and win" questions, since those just involve ***algebraic manipulation***. In other words, I'm skipping over to the main course.[1] In these problems we do one of three things (or a combination of them):

1. use algorithms, or
2. use modular arithmetic, or
3. use AM-GM.

Note AM-GM is for bounding.

Example 2.1 (ISL 1989)

A natural number is written in each square of an $m \times n$ chessboard. The allowed move is to add an integer k to each of two adjacent numbers in such a way that nonnegative numbers are obtained (two squares are adjacent if they share a common side). Find a

[1] Try the problems in the problem set if you would like to see examples of these.

necessary and sufficient condition for it to be possible for all the numbers to be zero after finitely many operations.

The following solution is by Pranav Sriram:

Solution. Note that in each move, we are adding the same number to 2 squares, one of which is white and one of which is black (if the chessboard is colored alternately black and white). If S_b and S_w denote the sum of numbers on black and white squares respectively, then $S_b - S_w$ is an invariant. Thus if all numbers are 0 at the end, $S_b - S_w = 0$ at the end and hence $S_b - S_w = 0$ in the beginning as well. Thus, this condition is necessary; now we prove that it is sufficient.

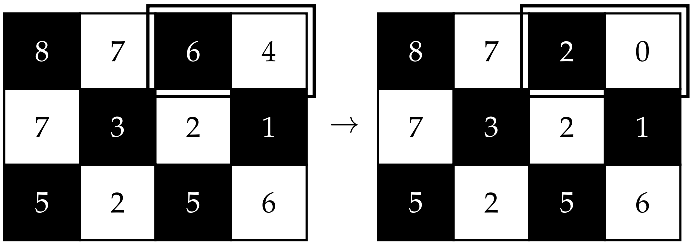

Figure 1: A move on the $m \times n$ board

Suppose a, b, c are numbers in cells A, B, C respectively, where A, B, C are cells such that A and C are both adjacent to B. If $a \leq b$, we can add $(-a)$ to both a and b, turning a to 0. If $a \geq b$, then add $a - b$ to b and c. Then b becomes a, and now we can add $-a$ to both of them, making them 0. Thus we have an algorithm for reducing a positive integer to 0. Apply this in each row, making all but the last 2 entries 0. Now all columns have only zeroes except the last two. Now apply the algorithm starting from the top of these columns, until only two adjacent nonzero numbers remain. These two numbers must be equal since $S_b = S_w$. Thus we can reduce them to 0 as well. □

This implies an important heuristic: **use an invariant to show a condition is necessary, and use an algorithm to show it's sufficient**.

Example 2.2 (ELMO 1999)
Jimmy moves around on the lattice point. From points (x, y) he may move to any of the points $(y, x), (3x, -2y), (-2x, 3y), (x + 1, y + 4)$ and $(x - 1, y - 4)$ show that if he starts at $(0, 1)$ he can never get to $(0, 0)$.

Solution. Let us take mod 5 of $x + y$. Note that since

$$3x - 2y \equiv 3(x + y) \pmod 5,$$

$$-2x + 3y \equiv 3(x + y) \pmod 5,$$

$$x + 1 + y + 4 \equiv x + y \pmod 5,$$

$$x - 1 + y - 4 \equiv x + y \pmod 5,$$

the sum of the two coordinates is either constant or multiplied by 3. Thus, $(0, 0)$ cannot be achieved. □

Always try invariants like:

1. sums
2. products
3. AM/GM/HM

This helps motivate what modulo is necessary.

Example 2.3 (IMO Shortlist 2014)
The number 1 is written on each of 2^n sheets of paper. Each minute we are allowed to choose two distinct sheets, erase the two numbers a and b appearing on them and writing the number $a+b$ instead on both sheets. Prove that after $n2^{n-1}$ minutes the sum of the numbers on all sheets is at least 4^n.

Solution. Consider the product P of the number on the sheets. Say we choose a, b and replace them by $a+b, a+b$. The quotient between the product of all numbers after the operation and the one before the operation is $\frac{(a+b)^2}{ab}$, which by AM-GM is greater than or equal to 4. Thus, the product is at least $4^{n \cdot 2^{n-1}}$, and by AM-GM again, we get that the sum S is

$$\left(\frac{S}{2^n}\right)^{2^n} \geq P,$$

implying the desired result $S \geq 4^n$. □

Example 2.4
The numbers $1, 2, \ldots, 2008$ are written on a blackboard. Every second, Jimmy erases four numbers of the form $a, b, c, a+b+c$, and replaces them with the numbers $a+b, b+c, c+a$. Prove that this can continue for at most 10 minutes.

Solution. Note that $a+b+c+(a+b+c) = (a+b)+(b+c)+(c+a)$. Thus, for every operation he does, the sum is constant, but the number of numbers decreases by 1. Even more important,

$$a^2+b^2+c^2+(a+b+c)^2 = (a+b)^2+(b+c)^2+(c+a)^2,$$

implying the sum of squares is also invariant. Let $x_1, x_2, \ldots, x_n$ be on the blackboard. Then by Cauchy-Schwarz,

$$n(x_1^2+x_2^2 \ldots + x_n^2) \geq (x_1+x_2+\ldots+x_n)^2.$$

Taking into account our two invariants, we obtain

$$n \geq \frac{(1+2+\ldots+2008)^2}{1^2+2^2+\ldots+2008^2} = 1506 + \frac{502}{1339},$$

implying this can take place at most $2008 - 1506 = 502$ times, which is less than 600 seconds, or 10 minutes. □

Example 2.5 (Saint Petersburg 2013)
There are 100 numbers from the interval $(0, 1)$ on the board. Every minute we can replace two numbers a, b on the board with the roots of $x^2 - ax + b = 0$ (if it has two real roots). Prove that this process must stop at some moment.

Solution. Suppose this process is endless. There exist real number $N < 1$ such that all of 100 initial numbers are smaller than N. Since if $a, b < N$, we get

$$\frac{a + \sqrt{a^2 - 4b}}{2} < N.$$

So, all numbers on the board will always smaller than N. Denote by S and P the sum and product respectively of all numbers on the board. Let S_0 and P_0 be that of the initial 100 numbers. Each move gives us $S \to S - b$ and $P \to \frac{P}{a} > \frac{P}{N}$. After M moves, we get that $S < S_0$ and $P > \frac{P_0}{N^M}$. By AM-GM, we get $S \geq 100 \sqrt[100]{P}$. This gives contradiction for sufficiently large M, implying the desired result. □

Fact 2.6. When sums and products are used in invariant/monovariant questions, it is a good idea to use AM-GM.

Now for a monovariant:

Example 2.7 (IMO Shortlist 2012)
Several positive integers are written in a row. Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

Solution. Clearly the maximum number on the board does not change, say it was M initially. Let $a_1, a_2, \ldots, a_n$ be the numbers on the board. I claim the quantity

$$S = a_1 + 2a_2 + \ldots + ka_k + \ldots + na_n$$

always increases. Say that at some moment $x > y$ with x to the left of y and x in position i. The difference between the new value of S and the old one is either

$$i(y + 1) + (i + 1)x - (ix + (i + 1)y) = x - y + i \geq 1$$

or

$$i(x - 1) + (i + 1)x - (ix + (i + 1)y) = -i + (i + 1)(x - y) \geq -i + i + 1 = 1,$$

proving the claim and the desired result. □

3 Brutal Examples

Example 3.1 (Tournament of Towns 2016)
On a blackboard several polynomials of degree 37 are written, each of them having leading coefficient equal to 1 and all coefficients nonnegative.
It is allowed to erase any pair of polynomials f, g and replace it by another pair of polynomials f_1, g_1 of degree 37 with leading coefficients to 1 such that either $f_1 + g_1 = f + g$ or $f_1 g_1 = fg$.
Can we reach a blackboard on which all polynomials have 37 distinct positive roots?

Solution. Let's suppose that

$$(X^{37}+a_1X^{36}+\ldots+a-37)(X^{37}+b_1X^{36}+\ldots+b_{37})$$
$$=(X^{37}+c_1X^{36}+\ldots+c-37)(X^{37}+d_1X^{36}+\ldots+d_{37}).$$

Looking at the coefficient of X^{36+37} we obtain

$$a_1+b_1=c_1+d_1.$$

This also holds if

$$(X^{37}+a_1X^{36}+\ldots+a-37)+(X^{37}+b_1X^{36}+\ldots+b_{37})$$
$$=(X^{37}+c_1X^{36}+\ldots+c-37)+(X^{37}+d_1X^{36}+\ldots+d_{37}).$$

Thus, the sum of coefficients of X^{36} is invariant. Since initially all coefficients are nonnegative, at each step the sum of coefficients of X^{36} stays nonnegative. This implies that at least one polynomial, say $P(X)=X^{37}+a_1X^{36}+\ldots+a-37$, has $a_1\geq 0$ at every step. If $x_1,x_2,\ldots,x_{37}$ are the complex roots of P, then

$$x_1+x_2+\ldots+x_{37}=-a_1\leq 0.$$

Thus, the polynomial cannot have 37 positive roots, which implies the desired result. □

Fact 3.2. If we have more than one option as to what to turn the objects into (e.g. Tournament of Towns 2016), it is often good to find a way to find an invariant that works for all options.

The following is not necessarily hard, but realizing the weighting is nontrivial.

Example 3.3

The first quadrant is divided into unit squares. We are allowed to perform the following move: if the square (x,y) has a token, while $(x,y+1),(x+1,y)$ are empty, then we take the token on (x,y) and put a token on each of the other two squares. Initially, we have tokens on $(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)$. Can we clear these six squares by a sequence of moves?

Solution. Consider the sum of $\frac{1}{2^{x+y}}$ over all pairs (x,y) for which there is a token at (x,y). Thus,

$$\frac{1}{2^{x+(y+1)}}+\frac{1}{2^{(x+1)+y}}=\frac{1}{2^{x+y}},$$

implying the sum is invariant. Initially, the sum is

$$\frac{1}{2^2}+\frac{2}{2^3}+\frac{3}{2^4}=\frac{11}{16}.$$

Suppose that at some moment the six initial squares have no token. Then the sum is at most

$$\sum_{x,y\geq 1}\frac{1}{2^{x+y}}-\frac{11}{16}=1-\frac{11}{16}=\frac{5}{16},$$

which is a contradiction, proving the desired result. □

Fact 3.4. Sometimes ***weighting*** is important for invariant questions, especially grid problems.

In general, when there's symmetry, attempt to weight the objects.

Example 3.5 (Russia 2014)

The polynomials $X^3 - 3X^2 + 5$ and $X^2 - 4X$ are written on the blackboard. If the polynomials $f(X)$ and $g(x)$ are written on the blackboard, we are allowed to write down the polynomials $f(X) \pm g(X), f(X) \cdot g(X), f(g(X))$ and $c \cdot f(x)$, where c is an arbitrary real constant. Can we write a nonzero polynomial of form $X^n - 1$ after a finite number of steps?

Solution. Let $f(X) = a_0 + a_1 X + \ldots + a_n X^n$, then its derivative is

$$f'(x) = a_1 + 2a_2 X + \ldots + na_n X^{n-1}.$$

This satisfies all conditions, because

$$\begin{aligned} (f \pm g)' &= f' \pm g', \\ (cf)' &= cf', \\ (f \cdot g)' &= f' \cdot g + g' \cdot f, \\ (f \circ g)' &= (f' \circ g) \cdot g'. \end{aligned}$$

Thus, if f' and g' have a common root z, then z is a common root of $(f \pm g)', (cf)', (f \cdot g)'$ and $(f \circ g)'$. The derivatives of the initial polynomials are $3X^2 - 6X$ and $2X - 4$, and 2 is a common root. However, $(X^n - 1)' = nX^{n-1}$ does not have the root $X = 2$, implying we can never get $X^n - 1$. $\square$

Example 3.6 (RMM Shortlist 2016)

Start with any finite list of distinct positive integers. We may replace any pair $n, n+1$ (not necessarily adjacent in the list) by the single integer $n-2$, now allowing negatives and repeats in the list. We may also replace any pair $n, n+4$ by $n-1$. We may repeat these operations as many times as we wish. What is the most negative integer which can appear in a list?

Solution. Let's look for an invariant of the form $\sum_{n \in \mathbb{L}} x^n$, where $\mathbb{L}$ is a subset of $\mathbb{Z}$. To have an invariant, we want

$$\begin{aligned} x^n + x^{n+1} &= x^{n-2}, \\ x^n + x^{n+4} &= x^{n-1}, \end{aligned}$$

for all n. This reduces to

$$\begin{aligned} x^2 + x^3 &= 1, \\ x^5 + x &= 1, \end{aligned}$$

which is easily solvable since they are secretly the same equation, because

$$x^5 + x - 1 = (x^3 + x^2 - 1)(x^2 - x + 1).$$

Thus, we choose x such that $x^3 + x^2 = 1$ and get $\sum_{n \in \mathbb{L}} x^n$ is constan.. Thus,

$$\sum_{n \in \mathbb{L}} x^n \le \sum_{n \ge 1} x^n = \frac{x}{1-x} = x^{-4}.$$

This must be true for all steps, and since $0 < x < 1$, we know that $n > -4$. Working backwards from -3, we get eventually get $1, 2, 3, 4, 5$ implying $\boxed{-3}$ works. $\square$

> *Remark* 3.7. The invariant in this problem is similar to the one in Conway's soldiers. The motivation behind this is ***recursion***, then transfer it to a ***characteristic polynomial***. Note that this is again a weighting problem.

Fact 3.8. For most invariant/monovariant questions, it is pretty easy to identify if the answer is yes or no (otherwise it wouldn't be a invariant/monovariant question!). The hard part is **proving** why your claim is true.

4 Problems

Let's eat a three course meal.

4.1 Appetizer

Problem 1. The cells of a 7×7 board are chess-painted (alternating colors) so that the corners are black. One is allowed to repaint any two adjacent cells to the opposite color. Is it possible to repaint the entire board white using such operations?

Problem 2. The numbers $1, 2, \ldots, 20$ are written on the board. One is allowed to erase any two numbers a and b and instead write the number $a + b - 1$. What number can remain on the board after 19 such operations?

Problem 3. Given a 1000-digit number with no zeroes, prove that from this number you can delete several (or none) last digits so that the resulting number is not a natural power less than 500 (a^1 is not considered a power).

Problem 4. The numbers 1 through 1000 are written on the board. One is allowed to erase any two numbers and and write the numbers ab and $a^2 + b^2$ instead. Is it possible with such operations to ensure that among the numbers written on the board, there are 700 at least that are the same?

Problem 5. Initially we have the numbers $\frac{49}{1}, \frac{49}{2}, \ldots, \frac{49}{97}$ on a board. A move consists in replacing two numbers, say a and b, with $2ab - a - b + 1$. After a series of moves, there is only one number left on the board. Find it!

4.2 Entree

Problem 6 (Russia 2008). A natural number is written on the blackboard. Whenever a number x is written, one can write either the number $2x + 1$ or $\frac{x}{x+2}$. At some point the number 2008 appears on the blackboard. Show that it was there from the beginning.

Problem 7 (Saint Petersburg 2020). The points $(1, 1)$, $(2, 3)$, $(4, 5)$ and $(999, 111)$ are marked in the coordinate system. If points (a, b) are marked then (b, a) and $(a - b, a + b)$ can be marked. If points (a, b) and (c, d) are marked then so can be $(ad + bc, 4ac - 4bd)$.

Can we, after some finite number of these steps, mark a point belonging to the line $y = 2x$?

Problem 8 (Tuymaada Junior 2018). The numbers $1, 2, 3, \ldots, 1024$ are written on a blackboard. They are divided into pairs. Then each pair is wiped off the board and non-negative difference of its numbers is written on the board instead. 512 numbers obtained in this way are divided into pairs and so on. One number remains on the blackboard after ten such operations. Determine all its possible values.

4.3 Dessert

Full yet?

Problem 9. The numbers $1, 2, \ldots, n$ are written on a blackboard. Each minute, a student goes up to the board, chooses two numbers x and y, erases them, and writes the number $2x + 2y$ on the board. This continues until only one number remains. Prove that this number is at least $\frac{4}{9}n^3$.

Problem 10. Let n be a fixed positive integer. Initially, n 1's are written on a blackboard. Every minute, David picks two numbers x and y written on the blackboard, erases them, and writes the number $(x + y)^4$ on the blackboard. Show that after $n - 1$ minutes, the number written on the blackboard is at least $2^{\frac{4n^2-4}{3}}$.

Problem 11 (USAMO 2019/5). Two rational numbers $\frac{m}{n}$ and $\frac{n}{m}$ are written on a blackboard, where m and n are relatively prime positive integers. At any point, Evan may pick two of the numbers x and y written on the board and write either their arithmetic mean $\frac{x+y}{2}$ or their harmonic mean $\frac{2xy}{x+y}$ on the board as well. Find all pairs (m, n) such that Evan can write 1 on the board in finitely many steps.

Problem 12 (Russia 2017). Initially a positive integer n is on the blackboard. Every minute we are allowed to take a number a on the blackboard, erase it and write instead all divisors of a except for a. After some time there are n^2 numbers on the blackboard. For which n is this possible?

Problem 13 (Iran RMM TST 2020). A 9×9 table is filled with zeroes. At every step we can either take a row, add 1 to every cell and shift it one unit to the right (the rightmost number in that row ends up in the leftmost position of the row) or take a column, subtract 1 from every number on that column and shift it one cell down (with the same convention as for rows). Can the table with the top right -1 and bottom left $+1$ and all other cells zero be reached?