Math Level 2 Week 8

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Contents

1 Some Formulas

Theorem 1.1
The number of ways to go from the bottom left corner to the top right corner of a $m \times n$ grid if you can only go right or up is $\binom{m+n}{m}$.

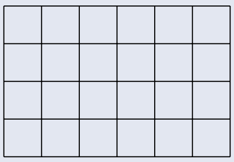


Figure 1: A 6×4 grid.

Proof. We move to the right m times and go up n times. The number of ways to order these moves is $\frac{(m+n)!}{m!n!} = \binom{m+n}{m}$. □

Theorem 1.2
Let A and B the bottom left corner and top right corner of a $n \times n$ grid, respectively. The number of paths from A to B that don't go above $\overline{AB}$ is

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

Proof. Note C_n is equivalent to the number of ways to go from $(0,0)$ to (n,n) without going above $y = x$ (you can go on it, but not above it). There are $\binom{n+n}{n} = \binom{2n}{n}$ ways to go from $(0,0)$ to (n,n). In all the illegal (meaning going above $y = x$) ways to go from

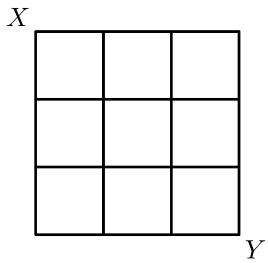
$(0,0)$ to (n,n), let the first illegal point be where we first go above $y = x$. This can only happen if we are on the line, and go one above it. This means all illegal points lie on the line that is 1 above $y = x$, i.e. $y = x + 1$. Furthermore, let us reflect the portion of the path after reaching the illegal path across $y = x + 1$, implying (n,n) will be reflected to $(n-1, n+1)$. Thus, all illegal paths are the same as going from $(n-1, n+1)$, and there are $\binom{n-1+n+1}{n-1} = \binom{2n}{n-1}$ ways to do this. Thus,

$$\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}.$$

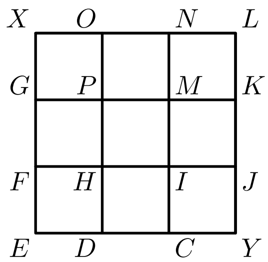
□

2 An Example

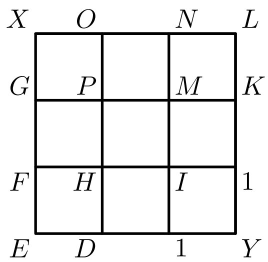
If we were to find how many paths could be made in a 5 by 5 grid if you could only go up and to the right, there would be $\binom{10}{5}$ ways because if we determine which steps will be to the right, then we can decide for the number of up ways. We choose from 10 because there will always be 10 steps.



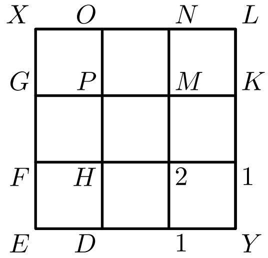
We are going to move either right or down. We start off with some easier means of this problem from A to B by labeling closer distances. Let's label these points with letters.



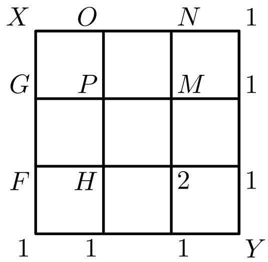
We first find that $\overline{YJ}$ and $\overline{YC}$ which is 1 for both of them "for $\overline{YJ}$ we have to move down, and for $\overline{YC}$ we move right 1".



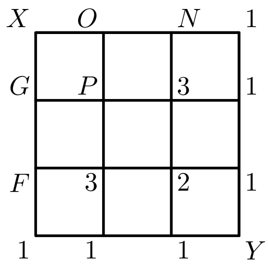
For YI she is going to have to either go right or down to start off with when we go right, we already know how many ways we have, which would be 1. For down, we again know how many ways we have, which is 1. Therefore for YI we have 2 total ways.



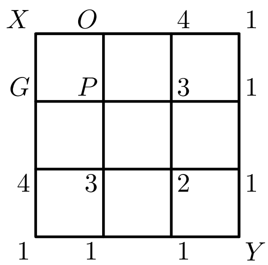
We also know for KY, LY, DY, EY we are going to have to go 2 or 3 down/right in both cases, which is a total of 1.



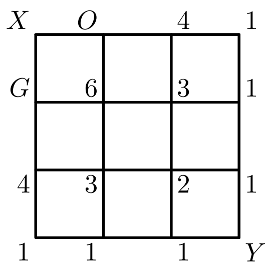
Using this same process, we find for MY and HY we are going to have to go down or right. From there, we already have the numbers filled in. For MY it is $1 + 2$, and for HY is it $2 + 1$ which in both cases is 3.



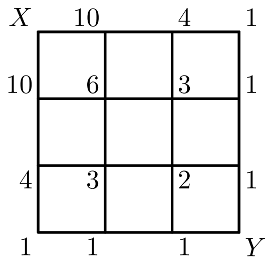
For NY and FY we have to move down or right, and find we have $3 + 1 = 4$ total ways once we move down/right.



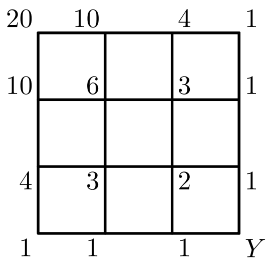
YP is going to have the two cases of going down/right again, which is going to give us $3 + 3 = 6$



For GY and OY we are going to have 6+4=10 total ways (since we start going down and right, and the cases have already been done for the points from that location).



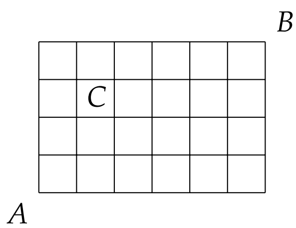
Now we evaluate XY. For XY, we have two cases. We can go down, or right. When we go right, we get O which has 10 possible ways from there. When we go down, we get G which has 10 possible ways. Therefore XY has $\boxed{20}$ possibles ways when only going right or down.



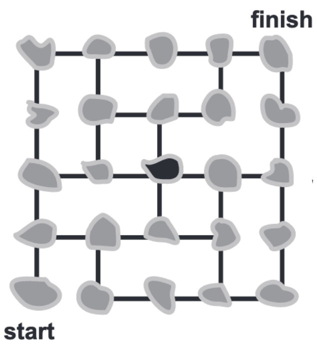
3 Problems

Problem 1. If we go from $(0,0)$ to $(3,3)$ on a coordinate grid, how many paths are there of length 6?

Problem 2. How many paths are there from A to B not passing through C?



Problem 3 (Math Kangaroo Grades 5/6 2020/30, edited). On the map you see some islands that are connected by bridges. A postman has to visit all the islands exactly once, starting at “start” and finishing at "finish". In how many ways can he do this?



> *Remark 3.1.* The original problem can be found here.

This next one is challenging:

Problem 4 (Summer Mock AIME 2019/6). A tortoise starts at the bottom left corner of a 4×4 grid of points. Each point is colored red or blue, and there are exactly 8 of each color. It can only move up or right, and can only move to blue points. If the bottom left corner is always blue, find the number of ways we can color the grid such that the tortoise can reach the top right corner.

Problem 5 (Mandelbrot Regionals 2013/5.6). How many paths are there from A to B through the network shown if you may only move up, down, right, and up-right? A path also may not traverse any portion of the network more than once. A sample path is highlighted.