1 Lecture Notes

1.1 Introduction

Probability

Take an event E. Suppose that in A possible outcomes, E will happen, and in B possible outcomes, E will not happen. Then, the ***probability*** that E happens is

$$\frac{A}{A+B}.$$

Thus, the maximum a probability can be is 1, or 100%. Additionally, the sum of all possible probabilities in a sample space (or all available probabilities) is 1, or 100%. The least value a probability can attain is 0.

Example 1.2

Suppose we flip a penny, a nickel, a dime, and a quarter. Find the following probabilities:

1. All coins are heads?
2. Only the penny and the nickel come up heads?
3. The penny and the nickel have the same side facing up?
4. At least 15 cents are showing face up?

Solution.

1. There are $2^4 = 16$ possibilities and only 1 has all heads. Thus, the answer is $\frac{1}{16}$.
2. This implies the dime and quarter are tails, so the probability remains the same: $\frac{1}{16}$.
3. It can be heads or tails, so we simply multiply by 2: $\frac{1}{8}$.
4. Either the quarter is heads, and so the probability of that occurring is $\frac{1}{2}$. Or we have a dime and a nickel and possibly a penny, in which the probability is $\frac{2}{16}$. Thus, the answer is $\frac{5}{8}$.

$\square$

1.2 Independent and Dependent Events

Independent events are events that do not effect the outcome of one another. For example, flipping a coin and choosing a card from a deck are independent. The probabilities of these events are often multiplied together. In dependent events, since the probability of one event is a consequence on the one before, we must take any changes into account before multiplying.

1.3 Geometric Probability

In geometric probability, we do not count objects but rather lengths and areas.

Theorem 1.3 (Geometric Probability)

$$P(\text{good}) = \frac{\text{size of good region}}{\text{size of total region}}.$$

1.4 Your list of tricks

- **Complementary probability**: instead of finding the probability $P(A)$ something happens, find the probability $P(\overline{A})$ it **doesn't** happen then calculate $1 - P(\overline{A})$.
- **Casework**: Consider each case, find the probability of each occurring, then sum the parts up.
- **Conditional probability**: Given two events A and B, the probability that A happens given that B happens can be written has $P(A \mid B)$, and we have

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

2 Problems

Problem 1. If two vertices of an octagon are chosen at random, what is the probability that they are adjacent?

Problem 2. How many ways are there to choose 3 people from 10 boys and 10 girls, given that not all boys and not all girls are chosen?

Problem 3. What is the probability you choose all boys or all girls when trying to choose 3 people from 10 boys and 10 girls?

Problem 4. Point P is chosen at random atop a 5 foot by 5 foot table. A circular disk with radius 1 is placed on the table with its center directly on Point P. What is the probability that the entire disk is on top of the table?

Problem 5. My friend and I are hoping to meet for lunch. We each will arrive at the restaurant at a random time between 12 and 1, stay for 10 min, and then leave. What is the probability that we meet each other?

Problem 6. There are 5 red balls and 8 green balls in a bin. You randomly draw 2 balls from the bin. What is the probability that you draw one red and one green ball?

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Problem 8 (AMC 8 2007/24). A bag contains four pieces of paper, each labeled with one of the digits 1, 2, 3 or 4, with no repeats. Three of these pieces are drawn, one at a time without replacement, to construct a three-digit number. What is the probability that the three-digit number is a multiple of 3? **(A)** $\frac{1}{4}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{3}{4}$

Problem 9 (AMC 8 2014/12). A magazine printed photos of three celebrities along with three photos of the celebrities as babies. The baby pictures did not identify the celebrities. Readers were asked to match each celebrity with the correct baby pictures. What is the probability that a reader guessing at random will match all three correctly?

Problem 10 (AMC 8 2013/8). A fair coin is tossed 3 times. What is the probability of at least two consecutive heads?

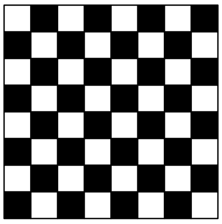
Problem 11 (AMC 8 2001/18). Two dice are thrown. What is the probability that the product of the two numbers is a multiple of 5?

Problem 12 (AMC 8 2003/12). When a fair six-sided dice is tossed on a table top, the bottom face cannot be seen. What is the probability that the product of the faces that can be seen is divisible by 6?

Problem 13 (AMC 8 2011/12). Angie, Bridget, Carlos, and Diego are seated at random around a square table, one person to a side. What is the probability that Angie and Carlos are seated opposite each other?

Problem 14 (AMC 8 2011/18). A fair 6 sided die is rolled twice. What is the probability that the first number that comes up is greater than or equal to the second number?

Problem 15 (AMC 8 2009/10). On a checkerboard composed of 64 unit squares, what is the probability that a randomly chosen unit square does not touch the outer edge of the board?



Problem 16 (AMC 8 2008/24). Ten tiles numbered 1 through 10 are turned face down. One tile is turned up at random, and a die is rolled. What is the probability that the product of the numbers on the tile and the die will be a square?

Problem 17 (AMC 8 2007/21). Two cards are dealt from a deck of four red cards labeled A, B, C, D and four green cards labeled A, B, C, D. A winning pair is two of the same color or two of the same letter. What is the probability of drawing a winning pair?

Problem 18 (AMC 8 2002/21). Harold tosses a coin four times. What is the probability that he gets at least as many heads as tails?

Problem 19 (AMC 8 2000/21). Keiko tosses one penny and Ephraim tosses two pennies. What is the probability that Ephraim gets the same number of heads that Keiko gets?

Problem 20 (AMC 10B 2013/12). Let S be the set of sides and diagonals of a regular pentagon. A pair of elements of S are selected at random without replacement. What is the probability that the two chosen segments have the same length?