ARML Relay 2021 2E-3

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Problem 1. *Let T be the number you will receive* (in this case $T = 120$). A set $\mathcal{U}$ has $\frac{T}{10}$ elements. Compute the sum of $|S|$ over all subsets $S \subseteq \mathcal{U}$.

Solution. We present two solutions:

Official solution via combinatorial sums Note that there are $\binom{12}{k}$ subsets S such that $|S| = k$. Thus, our sum is equivalent to

$$X = \sum_{k=0}^{12} k \cdot \binom{12}{k},$$

which we can easily identify is equal to

$$\sum_{k=0}^{12} k \cdot \binom{12}{12-k} = \sum_{k=0}^{12} (12-k) \cdot \binom{12}{k},$$

implying

$$2X = \sum_{k=0}^{12} k \cdot \binom{12}{k} + \sum_{k=0}^{12} (12-k) \cdot \binom{12}{k} = 12 \sum_{k=0}^{12} \binom{12}{k} = 12 \cdot 2^{12},$$

so $X = \boxed{24576}$.

Solution via linearity of expectation (Rowechen Zhong) The expected value of $|S|$ is 6, and so over all subsets we have an expected sum of

$$6 \cdot 2^{12} = \boxed{24576}.$$

□