Dr. Namli Summer Session Solutions (2019)

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Language Used: $\LaTeX$

- List of Symbols: http://tug.ctan.org/info/symbols/comprehensive/symbols-letter.pdf

Rendering Software: Code Cogs, TeXeR, and OverLeaf
(**Note**: As a result, the formatting of the problems may appear different.)

Formatting

Problem #. TYPE THE PROBLEM HERE
(**Possible**) **Motivation for Problem #.** WHAT IS THE MOTIVATION FOR THE SOLUTION?
Solution # by PERSON. TYPE THE SOLUTION IN LaTeX

Notes

- Box in your answers!
- Bold the first part and then put a PERIOD (.) after it. Do not use COLON (:) or a DASH (-) - that way when I go over the solutions it will be easier to read.

Example of Formatting

Problem 1000. Prove the first prime number is 2.
Motivation for Problem 1000. Proof by Definition.
Solution 1000 by Dr. Namli. Assume 2 is the first prime number. Thus, by definition, the first prime number is 2.

Notes

- This is a terrible proof.
- However, it highlights the formatting, which is what is important.
- A good example is **Problem 12**, which is a good solution, even if it doesn't match all the other people's formatting.

Problems Solved and To Be Solved

Problem #	**Solved?**	**Problem #**	**Solved?**	**Problem #**	**Solved?**
1	✔	**21**	✔	**41**	
2	✔	**22**	((a) Left)	**42**	✔
3	✔	**23**		**43**	✔
4	✔	**24**	✔	**44**	✔
5	✔	**25**	✔	**45**	✔
6	✔	**26**	✔	**46**	
7	✔	**27**	((a) Left)	**47**	✔
8	✔	**28**	✔	**48**	✔
9	✔	**29**	✔	**49**	
10	✔	**30**	✔	**50**	
11	✔	**31**		**51**	✔
12	✔	**32**		**52**	
13	✔	**33**	✔	**53**	Rich
14	✔	**34**	✔	**54**	
15	✔	**35**	✔	**55**	✔
16	✔	**36**	✔	**56**	✔
17		**37**	((b),(c) Left)	**57**	✔
18	✔	**38**		**58**	
19	✔	**39**		**59**	
20	✔	**40**	✔	**60**	

Problem #	**Solved?**	**Problem #**	**Solved?**	**Problem #**	**Solved?**
61		**81**	Eric		
62		**82**			
63		**83**			
64		**84**			
65		**85**			
66		**86**			
67		**87**			
68		**88**			
69		**89**			
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71		**91**			
72		**92**			
73		**93**			
74		**94**			
75		**95**			
76		**96**			
77		**97**			
78		**98**			
79		**99**			
80		**100**			

Problems and Solutions

Day 1

Problem 1. In a group of males, there are exactly 2 father-son relationships, 2 uncle-nephew relationships, a grandfather-grandson relationship, and one elder and younger brother. What is the least number of people in this group? For example: If there were exactly 1 father and 1 son, it would be 2, because we would just need a father and a son.

Motivation: Let's find a reasonably small solution first then look for improvements.

Solution 1 by Gavin Wang. First we start with the two brothers. We note that in order to have the two uncle-nephew pairs, each brother must have a son. We realize that only the grandfather-grandson relationship is missing from this group of people.

However, by adding a grandfather, we must make sure that an extra father-son relationship is not added. Thus the grandfather must not be the father of any of the two uncles. Therefore, the grandfather must be the father of the wife of one of the uncles, thus making him the grandfather of exactly one other person in the group.

This group of $\boxed{5}$ people satisfies the problem's constraints, and we have proved it is the minimum.

Problem 2.

$$\sqrt{\frac{x^2-1}{x}} = \sqrt{\frac{x-1}{x}} + \frac{x-1}{x},$$

Find all $x \in \mathbb{R}$ that satisfy this equation.

Solution 2 by Karthik Vedula.

Let

$$a = \sqrt{x - \frac{1}{x}} + \sqrt{1 - \frac{1}{x}}$$

Since $\sqrt{x - \frac{1}{x}} - \sqrt{1 - \frac{1}{x}} = \frac{x-1}{x}$, multiplying these together gives

$$\frac{a(x-1)}{x} = x-1 \implies x = 1 \text{ or } a = x$$

Keeping in mind that $x = 1$ is a valid solution, adding the original equations gives

$$x + \frac{x-1}{x} = \frac{x^2+x-1}{x} = 2\sqrt{x - \frac{1}{x}} \implies \frac{(x^2+x-1)^2}{x^2} = \frac{4x^2-4}{x}$$

$$x^4 + 2x^3 - x^2 - 2x + 1 = 4x^3 - 4x \implies x^4 - 2x^3 - x^2 + 2x + 1 = 0$$

$$\implies (x^2-x-1)^2 = 0 \implies x^2 - x - 1 = 0 \implies x = \frac{1 \pm \sqrt{5}}{2}$$

Checking for extraneous solutions gives

$$x = \boxed{1, \frac{1+\sqrt{5}}{2}}$$

Problem 3. (Vietnam)

$$\sqrt{3x}(1 + \frac{1}{x+y}) = 2,$$

$$\sqrt{7x}(1 - \frac{1}{x+y}) = 4\sqrt{2},$$

Find all $(x, y) \in \mathbb{R}^2$ that satisfy this system.

Motivation for Solution. We notice that the denominator resembles a sum of squares, which inspires us to think of magnitudes of complex numbers.

Solution 3 by Karthik Vedula. Simplifying gives

$$\sqrt{x} + \frac{\sqrt{x}}{x+y} = \frac{2}{\sqrt{3}}$$

$$\sqrt{y} - \frac{\sqrt{y}}{x+y} = \frac{4\sqrt{2}}{\sqrt{7}}$$

Adding the first equation to $i = \sqrt{-1}$ times the second equation gives

$$\sqrt{x} + i\sqrt{y} + \frac{\sqrt{x} - i\sqrt{y}}{x+y} = \frac{2}{\sqrt{3}} + \frac{4i\sqrt{2}}{\sqrt{7}}$$

Now we can make the substitution $z = \sqrt{x} + i\sqrt{y}$ to get

$$z + \frac{\overline{z}}{|z|} = z + \frac{1}{z} = \frac{2}{\sqrt{3}} + \frac{4i\sqrt{2}}{\sqrt{7}}$$

$$\implies z^2 - \left(\frac{2}{\sqrt{3}} + \frac{4i\sqrt{2}}{\sqrt{7}}\right) z + 1 = 0$$

Note that

$$\pm\sqrt{\Delta} = \pm\sqrt{\left(\frac{2}{\sqrt{3}} + \frac{4i\sqrt{2}}{\sqrt{7}}\right)^2 - 4} = \pm\sqrt{\frac{-152 + 16i\sqrt{42}}{21}}$$

$$= \pm\frac{4\sqrt{21} + 42i\sqrt{2}}{21}$$

This means

$$z = \frac{\left(\frac{2}{\sqrt{3}} + \frac{4i\sqrt{2}}{\sqrt{7}}\right) \pm \frac{4\sqrt{21} + 42i\sqrt{2}}{21}}{2}$$

$$= \left(\frac{\sqrt{7} \pm 2}{\sqrt{21}}\right) + \left(\frac{2\sqrt{2} \pm \sqrt{14}}{\sqrt{7}}\right) i$$

However, since $z = \sqrt{x} + i\sqrt{y}$, the real and imaginary parts of z must be positive. Equating and squaring gives

$$x = \left(\frac{\sqrt{7} \pm 2}{\sqrt{21}}\right)^2 = \frac{11 \pm 4\sqrt{7}}{21}$$

$$y = \left(\frac{2\sqrt{2} + \sqrt{14}}{\sqrt{7}}\right)^2 = \frac{22 + 8\sqrt{7}}{7}$$

Checking for extraneous solutions gives

$$\boxed{(x, y) = \left(\frac{11 + 4\sqrt{7}}{21}, \frac{22 + 8\sqrt{7}}{7}\right)}$$

Problem 4.

$$z^3 + 2z^2 + 2z + m = 0,$$

Does there exist solutions to this cubic in the rationals when m is an integer? What about when m is rational?

Solution 4 by Karthik Vedula. We claim that the answer to both is $\boxed{\text{no}}$.

Suppose otherwise. If m is an integer, the Rational Root Theorem guarantees that all rational solutions will be integers. If the roots are a, b, and c, by Vieta's we know that

$$a+b+c=2, ab+bc+ac=-2$$
$$\implies a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ac)=8$$

If the squares of three integers add to 8, clearly one of them is 0 and the other two are ± 2. However, there is no possible arrangement such that $a+b+c=2$.

If $m \in \mathbb{Q}$, let the roots be $\frac{a}{d}$, $\frac{b}{d}$, and $\frac{c}{d}$, where $a, b, c, d \in \mathbb{Z}$ and

$$\gcd(a,d)=\gcd(b,d)=\gcd(c,d)=1$$

Note that the sum of the squares of the roots is still 8, so

$$\left(\frac{a}{d}\right)^2+\left(\frac{b}{d}\right)^2+\left(\frac{c}{d}\right)^2=8 \implies a^2+b^2+c^2=8d^2$$

Since all squares are congruent to 0 or 1 modulo 4, all of the squares on the left-hand-side must be congruent to 0 modulo 4. Thus, a, b, and c are even, so we can rewrite $a=2a_1$ and similarly to get

$$a_1^2+b_1^2+c_1^2=2d^2$$

Since a, b, and c are even, d must be odd, so the right-hand-side must be congruent to 2 modulo 4. This means two of the squares on the left-hand-side are 1 modulo 4, and the other one is 0 modulo 4. This means one of a_1, b_1, and c_1 are even, and the other two are odd.

WLOG let $a_1=2a_2$, $b_1=2b_2+1$, and $c_1=2c_2+1$. However, remember that

$$2=\frac{a+b+c}{d} \implies a_1+b_1+c_1=d=2a_2+2b_2+2c_2+2$$

Since we assumed d is odd, this is a contradiction, and we are done.

Problem 5. If

$$x^2+y^2+z^2+t^2=50,$$
$$y^2+t^2-x^2-z^2=24,$$
$$xz=yt,$$
$$x+z+t=y,$$

Find x, y, z, t where $x, y, z, t \in \mathbb{R}$.

Solution 5 by Karthik Vedula. Adding and subtracting the first two equations results in $x^2+z^2=13$ and $y^2+t^2=37$. Substituting for y from the fourth equation into the first equation gives

$$x^2+z^2+t^2+(x^2+z^2+t^2+2xz+2xt+2zt)=50$$

$$\implies x^2+z^2+t^2+xz+zt+xt = 13+t^2+xz+zt+xt = 25$$
$$\implies t^2+xz+t(x+z) = 12 \implies t^2+xz+t(y-t) = xz+yt = 12$$

Since $xz = yt$, we know that they are both equal to 6. We know that $x^2+z^2 = 13$ and $xz = 6$, and $y^2+t^2 = 37$ and $yt = 6$.

Note that $y = \pm 1, \pm 6$.

- If $(y,t) = (1,6)$, then $x+z = -5$, so $(x,y,z,t) = (-2,1,-3,6), (-3,1,-2,6)$.
- If $(y,t) = (6,1)$, then $x+z = 5$, so $(x,y,z,t) = (2,6,3,1), (3,6,2,1)$.
- If $(y,t) = (-1,-6)$, then $x+z = 5$, so $(x,y,z,t) = (2,-1,3,-6), (3,-1,2,-6)$.
- If $(y,t) = (-6,-1)$, then $x+z = -5$, so $(x,y,z,t) = (-2,-6,-3,-1), (-3,-6,-2,-1)$.

To conclude, the set of solutions are

$$\boxed{(x,y,z,t) = (\pm 2, \pm 6, \pm 3, \pm 1), (\pm 3, \pm 6, \pm 2, \pm 1), (\pm 2, \mp 1, \pm 3, \mp 6), (\pm 3, \mp 1, \pm 2, \mp 6)}$$

Problem 6.

$$\sqrt{1+\sqrt{1-x^2}} \cdot \left(\sqrt{(1+x)^3} - \sqrt{(1-x)^3}\right) = 2+\sqrt{1-x^2},$$

Solve the equation where $x \in \mathbb{R}$.

Motivation for Problem 6. The first term seems suspiciously nested.

Solution 6 by Dylan Yu. We see that the first term looks like a nested radical - thus, we can rewrite it as

$$\sqrt{1+\sqrt{1-x^2}} = \frac{\sqrt{2+2\sqrt{1-x^2}}}{\sqrt{2}} = \frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{2}}.$$

The second part of the left hand side can be factored as a difference of cubes:

$$\sqrt{(1+x)^3} - \sqrt{(1-x)^3} = (\sqrt{1+x}-\sqrt{1-x})((\sqrt{1+x})^2 + \sqrt{1-x^2}$$
$$+(\sqrt{1-x})^2) = (\sqrt{1+x}-\sqrt{1-x})(2+\sqrt{1-x^2}),$$

So multiplying them all together, and cancelling the positive $2+\sqrt{1-x^2}$ term, gives:

$$\frac{(\sqrt{1+x}+\sqrt{1-x})(\sqrt{1+x}-\sqrt{1-x})(2+\sqrt{1-x^2})}{\sqrt{2}} = 2+\sqrt{1-x^2},$$

$$(1+x)-(1-x) = \sqrt{2},$$

$$2x = \sqrt{2},$$
$$x = \boxed{\frac{\sqrt{2}}{2}}.$$

Problem 7. A certain quartic, shown below

$$16x^4 - mx^3 + (2m+17)x^2 - mx + 16 = 0,$$

has the property that the roots are in a real, geometric sequence. If $x, m \in \mathbb{R}$, find m.

Different Motivation for 7. Try to use Vieta's formulas, and seeing that the roots multiply to 1 make us think to express the roots as $\frac{a}{r^3}, \frac{a}{r}, ar,$ and ar^3. In addition, the substitution $k = r + \frac{1}{r}$ aids us heavily throughout the problem.

Solution 7 by Rich Wang. We claim that the only answer is $m = 170$.

Now to prove that this is the only solution:

By Vieta's, we obtain the fact that the roots multiply to 1, which allows us to express our roots in the form of $\frac{a}{r^3}, \frac{a}{r}, ar,$ and ar^3.

Using Vietas gives us:

$$\frac{a}{r^3} + \frac{a}{r} + ar + ar^3 = \frac{m}{16}.$$
$$\frac{a^2}{r^4} + \frac{a^2}{r^2} + a^2 + a^2 + a^2r^2 + a^2r^4 = \frac{2m+17}{16}.$$
$$\frac{a^3}{r^3} + \frac{a^3}{r} + ar + ar^3 = \frac{m}{16}.$$

Factoring out the $a's$ gives us:

$$a\left(\frac{1}{r^3} + \frac{1}{r} + r + r^3\right) = \frac{m}{16}.$$
$$a^2\left(\frac{1}{r^4} + \frac{1}{r^2} + 1 + 1 + r^2 + r^4\right) = \frac{2m+17}{16}.$$

$$a^3(\frac{1}{r^3}+\frac{1}{r}+r+r^3)=\frac{m}{16}.$$

Dividing the third equation by the first equation gives us $a^2=1$, so we have $a=1,-1$.

Note we only need to consider the case when $a=1$, as choosing $a=-1$ simply changes the parity of each of the terms, which can already be done when $a=1$ by flipping the sign of r, therefore not affecting the value of m.

From the first equation we get $2(\frac{1}{r^3}+\frac{1}{r}+r+r^3)=\frac{2m}{16}$. Substituting this into the second equation gives us

$$\frac{1}{r^4}+\frac{1}{r^2}+1+1+r^2+r^4=2(\frac{1}{r^3}+\frac{1}{r}+r+r^3)+\frac{17}{16}.$$

We now make the substitution $k=r+\frac{1}{r}$, which implies that. We get:

$$k^2-2=r^2+\frac{1}{r^2},k^3-3k=r^3+\frac{1}{r^3},k^4-4k^2+2=r^4+\frac{1}{r^4}$$

$$k^4-4k^2+2+2+k^2-2=2(k+k^3-3k)+\frac{17}{16}.$$

This simplifies to

$$k^4-2k^3-3k^2+4k+\frac{15}{16}=0.$$

Using Rational Root Theorem, we can guess roots $-\frac{3}{2},\frac{5}{2}$. Using the quadratic equation on the remaining quadratic gets us roots of $\frac{1-\sqrt{2}}{2},\frac{1+\sqrt{2}}{2}$.

However, note that we set $r+\frac{1}{r}=m$. Solving for r here with quadratic formula gives us

$$r=\frac{m+\sqrt{m^2-4}}{2}$$

implying that we must have $|m|\geq 2$. The only solution here with that property is $m=\frac{5}{2}$.

Now, solving for r with quadratic equation gets us solutions of $r=2,\frac{1}{2}$. However, these give identical roots of $\frac{1}{8},\frac{1}{2},2,$ and 8. From here, we can easily solve for m by using the fact that:

$$(\frac{1}{r^3}+\frac{1}{r}+r+r^3)=\frac{m}{16}$$

to get that $m = \boxed{170}$.

Now, we check to make sure this works. Plugging in $m = 170$ into

$$16x^4 - mx^3 + (2m+17) - mx + 16 = 0$$

gets us the quartic:

$$16x^4 - 170x^3 + 357x^2 - 170x + 16 = 0$$

from which we can easily check that roots of $\frac{1}{8}, \frac{1}{2}, 2,$ and 8 work.

Day 2

Problem 8. Solve the following system of equations, where $x, y, z \in \mathbb{R}$:

$$x^3 + x(y-z)^2 = 2 \qquad (1)$$
$$y^3 + y(z-x)^2 = 30 \qquad (2)$$
$$z^3 + z(x-y)^2 = 16 \qquad (3).$$

Solution 8 by Chris Qiu: Notice that the right-hand sides of the three equations form an arithmetic sequence. This gives us the motive to subtract some of the equations, in order to get rid of pesky constant terms.

Take $(2) - (3)$:

$$(y-z)(x^2+y^2+z^2) = 14.$$

Take $(3) - (1)$:

$$(z-x)(x^2+y^2+z^2) = 14.$$

Subtract these two resulting equations to get $(2z-x-y)(x^2+y^2+z^2) = 0$.
It is obvious that x, y, z cannot all equal 0, as the equations are not satisfied.
So, $x^2+y^2+z^2 > 0$.

Therefore, $2z - x - y = 0$, so $z = \dfrac{x+y}{2}$.

Now, we can plug this back into (1) and (2) to get:

$$x(5x^2+y^2-2xy) = 8 \qquad (1)'$$
$$y(x^2+5y^2-2xy) = 120 \qquad (2)'.$$

Again, we would like to get rid of constant terms. Take $(1)' - 15 \cdot (2)'$:

$$75x^3 - 31x^2y + 17xy^2 - 5y^3 = 0.$$

Let $a = \frac{x}{y}$. If we divide the equation above by y^3, we get

$$75\left(\frac{x}{y}\right)^3 - 31\left(\frac{x}{y}\right)^2 + 17\left(\frac{x}{y}\right) - 5 = 0$$

or just $75a^3 - 31a^2 + 17a - 5 = 0$. The only real root of this equation is $a = \frac{x}{y} = \frac{1}{3}$.

Therefore, we can plug $y = 3x$ back into equation $(1)'$ to get $8x^3 = 8$, so $x = 1$.

In conclusion, $y = 3$ and $z = 2$. So, the solution to this system of equations is

$$\boxed{(x, y, z) = (1, 3, 2)}.$$

Problem 8 Variation:

Solve the following system of equations, where $x, y, z \in \mathbb{R}$:

$$x^3 + x(y - z)^2 = A,$$
$$y^3 + y(z - x)^2 = B,$$
$$z^3 + z(x - y)^2 = C.$$

Solution 8 (Variation) by Chris Qiu.

Solution:

$$\begin{aligned} x^3 + x(y-z)^2 &= A \qquad (1)\\ y^3 + y(z-x)^2 &= B \qquad (2)\\ z^3 + z(x-y)^2 &= C \qquad (3) \end{aligned}$$

For now, assume that $A \neq B, B \neq C, C \neq A$ so that x, y, and z are distinct.

We want to get rid of the constant terms in the equation, so we perform the following operations:

$B \cdot (1) - A \cdot (2)$:

$$(Bx - Ay)(x^2 + y^2 + z^2) + 2(A - B)xyz = 0$$

$C \cdot (2) - B \cdot (3)$:

$$(Cy - Bz)(x^2 + y^2 + z^2) + 2(B - C)xyz = 0$$

$C \cdot (1) - A \cdot (3)$:

$$(Cx - Az)(x^2 + y^2 + z^2) + 2(A - C)xyz = 0$$

Now, by rearranging each of the equations, we get the following result:

$$-\frac{-2xyz}{x^2 + y^2 + z^2} = \frac{Bx - Ay}{A - B} = \frac{Cy - Bz}{B - C} = \frac{Cx - Az}{A - C}$$

However, if $\frac{a}{b} = \frac{c}{d} = k$, then $\frac{a+c}{b+d} = k$.
Therefore, combining the second and third terms gets us

$$\frac{(Bx - Ay) + (Cy - Bz)}{(A - B) + (B - C)} = \frac{B(x - z) - Ay + Cy}{A - C} = \frac{Cx - Az}{A - C}$$

which can be simplified to

$$z = \frac{(x-y)(C-B)}{A-B} + y$$

.

Plugging this back into equations (1) and (2) gives us the following equations:

$$x^3 + \frac{x(x-y)^2(C-B)^2}{(A-B)^2} = A$$

$$y^3 + \frac{y(x-y)^2(C-B)^2}{(A-B)^2} = B$$

. Now, we let $D = \frac{(C-B)^2}{(A-B)^2}$.
The equations above become:

$$x^3 + Dx(x-y)^2 = A$$

$$y^3 + Dy(x-y)^2 = B$$

. Now, again, we want to cancel out the constant terms. Multiplying the first equation by B and the second equation by A and subtracting the former by the latter gives the equation:

$$(B-A)(D-1)x^3-(2BD+AD)x^2y+(BD+2AD)xy^2-(B-A)(D-1)y^3 = 0$$

Let $a = \frac{x}{y}$. Now, we can rewrite the equation above.

$$(B-A)(D-1)a^3-(2BD+AD)a^2+(BD+2AD)a-(B-A)(D-1) = 0$$

From this point, one can solve for a in terms of A, B, D, substitute into the first equation, solve for x and y in terms of A, B, C, and plug the results into z to get a complete solution that is unattainable by hand.

Problem 9. Given

$$x^3 - 3x^2 - 8x + 40 = 8\sqrt[4]{4x+4},$$

Find x, given it is real.

Motivation for Problem 9: If one checks a solution it is clear that $x = 3$ works. Since this is a math olympiad problem chances are that it is the only one. (No math professor of the right mind will ask 8-9th graders to solve an equation of 12th degree)

Also we don't like the $4x + 4$ on the right side. Can we somehow make it nicer?

Solution 9 by Leo Lei: Since the solution is in reals, we know that $4x+4$ has to be greater than or equal to 0. Thus x has to be greater than or equal to -1, thus let's define $y=x+1$, and this now gives us the condition that y is a non-negative number. This is great, because inequalities such as AM-GM rely on variables being non-negative.

Now plugging in $x=y-1$ into the given equation, we obtain $y^3-6y^2+y+44=8\sqrt[4]{4y}$.

Now $y=4$ is a solution, and a technique is to factor that out as a square,and the remaining factor is a $y-2$ (so the coefficient of y^2 can still remain a -6) and we have $(y-4)^2(y+1)+y+12=8\sqrt[4]{4y}$. Now clearly the first term on the left hand side will always be positive except when y is equal to 4, and thus we only have to look at the $y+12$ term. Now let's cleverly split it up into 8 terms using AM-GM (for the factor of 8 on the RHS of the equation). So by AM-GM,

$$(y/2+y/2+2+2+2+2+2)/8 \geq \sqrt[8]{y/2*y/2*2^6}.$$

Simplifying yields the desired $y+12\geq 8\sqrt[4]{4y}$., with equality at $\frac{y}{2}=2$, or $y=4$.

Thus the only solution is $y=4$ and hence $x=\boxed{3}$.

Problem 10. Show that the equation $x^3-3xy^2+y^3=2891$ has no solutions when x and y are integers.

Motivation: Mod 3 looks helpful here...

Solution 10 by Chris Qiu: Notice that the left-hand side of the equation $x^3-3xy^2+y^3=n$ looks a lot like $(x+y)^3$. We can write this equation as

$$(x+y)^3-3x^2y-6xy^2=2891\equiv 2 \pmod 3.$$

Since the second and third terms are already divisible by 3, we can conclude that $x+y\equiv 2 \pmod 3$.

Now, for the sake of contradiction, assume that $x\equiv 0 \pmod 3$ or $y\equiv 0 \pmod 3$.

If this is true, then $(x+y)^3=3x^2y+6xy^2+2891\equiv 2 \pmod 9$. But, cubes can only be $-1,0,1 \pmod 9$. Contradiction!

Therefore, $x\equiv y\equiv 1 \pmod 3$. Let $x=3a+1$ and $y=3b+1$.

Plug this into $(x+y)^3-3x^2y-6xy^2=2891$ to get:
$(3a+1)^3-3(3a+1)(3b+1)^2+(3b+1)^3=2891$
$27a^3+27a^2-31ab^2+27a^2-54ab-9b=2892\not\equiv 0 \pmod 9$.
Again, contradiction. Therefore, no solutions exist.

Variation of Problem 10. Show that if a solution to $x^3-3xy^2+y^3=n$ exists, there must be at least three solutions, where n is a positive integer not congruent to $2 \pmod 3$.

Motivation: If we can find 3 solutions in terms of variables, we are done, because that means that the values of the answer will change with the values of the variables.

Solution by Chris Qiu. We can simply observe that if we replace solution (a,b) with $(-b,a-b)$, and the equation is still satisfied.
We can then get solutions $(a,b)\to(-b,a-b)\to(b-a,a)\to(a,b)$.
Now, there are at least three solutions, unless they equal - and they only equal at $(0,0)$, which gives you $n=0$, which is impossible. Thus, there are at least **three** solutions.

Problem 11. Given

$$x^3+3xy^2=-49,$$
$$x^2-8xy+y^2-8y+17x=0,$$

Find all real x and y.

Solution 11.1 by Eric Peng.

$x^2 - 8xy + y^2 - 8y + 17x = 0$

Multiply by 3

$3(x^2 - 8xy + y^2 - 8y + 17x) = 0$

Expand

$3x^2 - 24xy + 3y^2 - 24y + 51x = 0$

Add the two equations

$x^3 + 3xy^2 + 3x^2 - 24xy + 3y^2 - 24y + 51x = -49$

Take out $(x+1)^3$

$(x+1)^3 + 3xy^2 - 24xy + 3y^2 - 24y + 48x + 48 = 0$

Factor $(x+1)$ **out**

$(x+1)^2 + (3y^2 - 24y + 48) = 0$

Don't forget that at the end, we might need to check $x = -1$

Realize that $3y^2 - 24y + 48 = 3(y-4)^2$

$(x+1)^2 + 3(y-4)^2 = 0$

Apply trivial inequality

$\boxed{x = -1, y = 4}$

Checking $x = -1$**for other solutions yields**

$\boxed{x = -1, y = -4}$

Solution 11.2 by Karthik Vedula.

Solution 1: Note that solving for y in the second equation gives

$$y^2 - (8x+8)y + (x^2 + 17x) = 0 \implies y = \frac{(8x+8) \pm \sqrt{60x^2 + 60x + 64}}{2} \implies$$

$$y = (4x+4) \pm \sqrt{15x^2 + 15x + 16}$$

Substituting this into the first equation gives

$$x^3 + 3xy^2 + 49 = x^3 + 3x\left((4x+4) \pm \sqrt{15x^2+15x+16}\right)^2 + 49$$

$$(94x^3 + 141x^2 + 96x + 49) \pm 24x(x+1)\sqrt{15x^2+15x+16} = 0$$

$$\implies (94x^3 + 141x^2 + 96x + 49)^2 = 576x^2(x+1)^2(15x^2+15x+16)$$

$$\implies (x+1)^2\left((94x^2 + 47x + 49)^2 - 576x^2(15x^2+15x+16)\right) = 0$$

$$\implies (x+1)^2(196x^4 + 196x^3 + 2205x^2 + 4606x + 2401) = 0$$

$$\implies 49(x+1)^4(4x^2 - 4x + 49) = 49(x+1)^4\left((2x-1)^2 + 48\right) = 0$$

This means that $x = -1$. Therefore, substituting back in either equation gives

$$\boxed{(x, y) = (-1, \pm 4)}$$

$\square$

Problem 12. Given that the sum of n positive integers is 1976, where n is a positive integer greater than or equal to 1, determine the maximum possible product of these n integers.

Solution 12 by Eric Peng.

We write this as $\sum_{i=1}^{n} a_i = 1976$ where n is the number of positive integers we sum.

We realize that any $a_i = 1$ is **utterly useless**, and can be added to another term to actually increase the product.

For any $a_i \geq 5$, we can preserve the sum of 1976, but split the term into two terms which are 3, and $(a_k - 3)$. We see that if we do this, we increase the product becaue $3(a_k - 3) = 3a_k - 9$, which is greater than a_k because we restricted $a_i \geq 5$, and $3a_k - 9 \geq a_k$ simplifies to $a_k \geq \frac{9}{2}$.

Now, we know that all of our a_i satisfy $1 < a_i < 5$. However, we notice that if $a_i = 4$, we can break that down into two terms of 2. This preserves both the sum and product. We now have only values of 2 and 3 for each a_k.

Finally, we can realize that

$$2 + 2 + 2 = 3 + 3 \text{ and } 2 * 2 * 2 < 3 * 3$$

. This means that we can swap out any 3 twos with 2 threes to increase our product. This statement is equivalent to maximizing the number of threes in our sum, and filling in at the end with 2 because we can't fit a 3. This maximization leads to 658 threes and 1 two.

Answer: $\boxed{2 \cdot 3^{658}}$

Problem 13. Given the general cubic

$$ax^3 + bx^2 + cx + d = 0,$$

Find a method to find all x.

Motivation for Problem 13. Can we make the cubic a quadratic?

Solution 13 by Leo Lei .

First, divide the equation by a so the coefficient of x^3 is 1. Assume the new cubic is now

$$x^3 + px^2 + qx + r = 0.$$

Let's first start off with the common technique of substituting a new variable $x = y - \frac{p}{3}$ into the equation. Now we have a new equation

$$y^3 + y(q - p^3/3) + r - pq/3 + 2p^3/27 = 0,$$

But the key difference now is that the x^2 term is gone!

Let's rewrite as

$$y^3 + By = T \text{ ,}$$

For some constants B and T. Now look at this amazing identity - $(s+t)^3 - 3st(s+t) = s^3 + t^3$. Using wishful thinking to map this back to our current problem, this means that we have $y = s + t$, $B = -3st$, and $T = s^3 + t^3$. Solving for t gives us $t = -B/3s$, and plugging this into the last equation gives us $T = s^3 - B^3/27s^3$. Clearing denominators gives us $s^6 - Ts^3 - B^3/27$. This is a quadratic in s^3. We can then find s using the quadratic formula, then t from $T = s^3 + t^3$, then $y = s + t$, then finally x. The ending result is

$$\begin{aligned} x &= \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} \\ &+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) - \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}. \end{aligned}$$

Problem 14. Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r. Find all pairs (a, b) such that $q^2 + r = 1977$.

Solution 14 by Rich Wang:

We claim that the answer is $\boxed{(50,7),(50,37),(7,50),(37,50)}$**.**

Now we prove that these are the only solutions:

LEMMA 1: q **must equal** 44**.**
PROOF:

We know that $(a^2 + b^2) = (a + b)q + r$. We realize that $q < (a + b)$, as if otherwise $(a + b)^2 + r > (a^2 + b^2)$, meaning equality is impossible.

Noting that by trivial inequality, $a^2 + b^2 \geq 2ab$, meaning $\frac{a+b}{2} \leq \frac{a^2+b^2}{a+b}$, if we divide both sides of $(a^2 + b^2) = (a + b)q + r$ by $(a + b)$ we get $\frac{a+b}{2} \leq \frac{a^2+b^2}{a+b} = q + \frac{r}{a+b}$.

From here, noting that $r < a+b$ and $1 - \frac{2r}{a+b} < 1$ we can get that
$r < a+b - \frac{2r}{a+b} + 1 \leq 2q+1$.

Now, we have $q^2 < q^2 + r = 1977 < q^2 + 2q + 1 = (q+1)^2$. As a result, $q = 44, r = 41$ must hold, (1977 only between one pair of consecutive squares: $1936, 2025$).
(End Lemma)

Now, we use the quadratic formula on $a^2 + b^2 = (a+b)44 + 41$, or
$a^2 - 44a + (b^2 - 44b - 41)$ to get solutions of:
$22 \pm \sqrt{-(b^2 - 44b - 525)}$.

Now, we need $-(b^2 - 44b - 525) = k^2$, where k is a non-negative integer. This is equivalent to needing $k^2 + (b-22)^2 = 1009$. We can bash out the solutions to be
$(28, 7), (28, 37), (15, 50)$. This means b must equal either $7, 37,$ or 50.

Plugging this into our solutions for a gives us (when $b = 7, 37, 50$ respectively) $a = 50, 50,$ and (2 sols for last case) $a = 7, 37$. Now, to plug in to check if all of these cases work:

$50^2 + 7^2 = 2549 = (44)57 + 41$ is correct.
$50^2 + 37^2 = 3869 = (44)87 + 41$ is correct.

And note that by symmetry the other two solutions work as well, as they just switch around a and b.

Therefore, our only solutions are $\boxed{(50,7),(50,37),(7,50),(37,50)}$**.**

Q.E.D

Problem 15. Find the factorization of

$$(a+b)^2 - (a^2 + b^2),$$

$$(a+b)^3 - (a^3 + b^3),$$

$$(a+b)^5-(a^5+b^5),$$
$$(a+b)^7-(a^7+b^7),$$
$$(a+b)^{11}-(a^{11}+b^{11}),$$

And if possible, note the pattern.

Motivation for Problem 15. The pattern is the motivation.

Solution 15 by Dylan Yu. We see that after factoring, we get

$$(a+b)^2-(a^2+b^2)=2ab,$$
$$(a+b)^3-(a^3+b^3)=3ab(a+b),$$
$$(a+b)^5-(a^5+b^5)=5ab(a+b)(a^2+ab+b^2),$$
$$(a+b)^7-(a^7+b^7)=7ab(a+b)(a^2+ab+b^2)^2,$$
$$(a+b)^{11}-(a^{11}+b^{11})=11ab(a+b)(a^2+ab+b^2)(\ldots),$$

Which shows that all the factorizations (note the exponents are prime numbers) are divisible by their exponent, and ab. 3 and up are divisible by $a+b$, and 5 and up are divisible by a^2+ab+b^2.

Problem 16. Given

$$3n^2+3n+7=x^3,$$

Find all integers n, x that suffice.

Solution 16 by Dylan Yu. Let us take mod 3. From here, we get

$$x^3\equiv 1(\text{mod } 3),$$

So

$$x\equiv 1(\text{mod } 3).$$

This implies

$$x=3y+1,$$

So

$$3n^2+3n+7=27y^3+27k^2+9k+1,$$

Which leads to

$$n^2+n+2=9y^3+9y^2+3y.$$

If we take mod 3, we get

$$n^2+n+2\equiv 0(\text{mod } 3).$$

If we plug in

$n \equiv 0, 1, 2 \pmod 3$,

None of them lead to $0 \pmod 3$, so there are $\boxed{\text{no solutions}}$.

Problem 17. Given

$2^x = 3^y + 5$,

Find all positive integers x and y that suffice.

Solution 17. We claim that $(3, 1), (5, 3)$ are the only solutions.

Problem 18. (Chinese TST) Given

$2x^4 + 1 = y^2$

Find all integers x and y that suffice.

Solution 18 by Ray Tang.

Obviously y is odd so let $y = 2a + 1$ plugging this in we get that:

$2x^4 + 1 = 4a^2 + 4a + 1$

So $x^4 = 2(a + 1)(a)$

So x is even and we let $x = 2b$ plugging this in we get

$8b^4 = (a + 1)(a)$

Since $gcd(a, a + 1) = 1$ we can either have that $a = c^4$ and $a + 1 = 8d^4$ where $gcd(c, 2d) = 1$ and $cd = b$

Or we can have that $a = 8c^4$ and $a + 1 = d^4$

For the first case we have that $c^4 + 1 = 8d^4$

We know a 4th power is $0, 1$, or 3 (mod 8) and by taking modulo 8 we get:

$c^4 = 7$ (mod 8)

Which is impossible

For our next case, $8c^4 + 1 = d^4$, we know that d is odd, so we can substitute $d = 2e + 1$.

Plugging this in we get that $8c^4 + 1 = 16e^4 + 32e^3 + 24e^2 + 8e + 1$.

Simplifying, we get that

$c^4 = 2e^4 + 4e^3 + 3e^2 + e = e(e+1)(2e^2 + 2e + 1) = e(e+1)((e^2) + (e+1)^2)$.

Now we notice that all the pairwise greatest common divisors of the 3 terms are relatively prime.

So we need each of them to be a fourth power. Let $e = f^4$ and $e + 1 = g^4$ and $e^2 + (e + 1)^2 = h^4$ where $fgh = c$, and the pairwise gcd of f, g, h is 1.

Since $e = f^4$ and $e+1 = g^4$, we get that $f^4 + 1 = g^4$, which means the only solutions in Z to this equation are $f = 0,\ g = +-1$. Plugging this back in we get that:
$e = 0, d = 1, c = 0, b = 0, a = 0 or - 1, x = 0, y = +-1.$

$\boxed{\text{Our roots are (0,1) and (0,-1)}}$

Problem 19. (IMO 1996) The positive integers a, b are such that $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

Solution 19 by Eric Peng.

Write as

$$15a + 16b = u^2 \text{ and } 16a - 15b = v^2$$

We can square both equations and add them, giving $481(a^2 + b^2) = u^4 + v^4$.

Since $481 = 13 * 37$ **We take mod 13 and 37 giving** $u^4 + v^4 \equiv 0 \pmod{13}$ **and** $u^4 + v^4 \equiv 0 \pmod{37}$

Case 1: Both u and v are divisible by 13.

We know that $u^{12} + v^{12} \equiv 0 \pmod{13}$ **because if u, v are divisible by 13,** $u + v$ **is divisible by 13, and that is a factor of** $u^{12} + v^{12}$ **so that is also divisible by 13. We apply the same logic to mod 37.**

Case 2: Both u and v are not divisible by 13.

By Fermat's Little Theorem $u^{12}, v^{12} \equiv 1, 1 \pmod{13}$

which means that their sum is equivalent to 2 mod 13, a contradiction.

We can apply the same logic to mod 37.

Since Case 2 doesn't work, u and v must both be divisible by

both 13 and 37 which implies they are both divisible by 481, meaning 481 is the minimum.

We can check the original equations to see if $u = v = 481$ **has a solution, and it has the solution**

$a = 481 \cdot 31, b = 481$

Therefore, the least possible value of either square is $481^2 = \boxed{231361}$

Day 3

Problem 19. Find non trivial infinite families for the equations ($x, y, z, n \in \mathbb{Z}^+$)

a) $x^{2n+1} + y^{2n} = z^{2n+1}$

b) $x^{2n} + y^{2n+1} = z^{2n+2}$

c) $x^A + y^B = z^B$ where $A, B, C \in \mathbb{Z}^+$ and $\gcd(A, B, C) = 1$

Solution 19a.

Solution 19b.

Solution 19c.

Problem 20. Solve the following system of equations, where $x, y \in \mathbb{R}^+$:

$$\sqrt{x}\left(1 - \frac{12}{3x+y}\right) = 2$$
$$\sqrt{y}\left(1 + \frac{12}{3x+y}\right) = 6$$

Motivation 20. This looks quite identical to #3…

Solution 20 by Karthik Vedula. Adding $\sqrt{3}$ times the first equation to $i = \sqrt{-1}$ times the second equation gives

$$\sqrt{3x} + i\sqrt{y} - 12\left(\frac{\sqrt{3x} - i\sqrt{y}}{3x+y}\right) = 2\sqrt{3} + 6i$$

Again, we make the substitution $z = \sqrt{3x} + i\sqrt{y}$ to get

$$z - \frac{12}{z} = 2\sqrt{3} + 6i \implies z^2 - \left(2\sqrt{3} + 6i\right) z - 12 = 0$$

Using the quadratic formula, we get

$$z = \frac{\left(2\sqrt{3} + 6i\right) \pm \sqrt{\left(2\sqrt{3} + 6i\right)^2 + 48}}{2} = \frac{\left(2\sqrt{3} + 6i\right) \pm \sqrt{24 + 24i\sqrt{3}}}{2}$$
$$= \sqrt{3} + 3i \pm \sqrt{6 + 6i\sqrt{3}} = \sqrt{3} + 3i \pm \sqrt{3}(\sqrt{3} + i)$$

Since the real and imaginary parts of z are positive, we must have

$$\sqrt{3x} = 3 + \sqrt{3} \implies x = 4 + 2\sqrt{3}$$
$$\sqrt{y} = 3 + \sqrt{3} \implies y = 12 + 6\sqrt{3}$$

This means that the unique solutions for this system is

$$\boxed{(x, y) = (4 + 2\sqrt{3}, 12 + 6\sqrt{3})}$$

Problem 21. Find all solutions to $x^3 - y^3 = 2xy + 8$ where $x, y \in \mathbb{Z}$.

Solution 21 by Ray Tang

$$\boxed{\text{We claim that there are no solutions.}}$$

We plug in $x = y + z$.
We get $(3z-2)y^2 + (3z^2-3z)y + z^3 - 8 = 0$.
$3z-2|3z^2-2z, 3z-2|3z-2$ so $3z-2|z^3-8$.

Since $\gcd(3z-2,27)=1$, we can multiply z^3-8 by 27 and get the same remainder.
$z^3 - 8 \equiv 27z^3 - 216 \equiv -208(mod 3z-2)$.
So $z = 1, 4, 16, 13, 52, 208, -1, -4, -16, -13, -52, -208$.

Now we look at the discriminant of the equation and we get:
$-3z^4 + 8z^3 - 8z^2 + 96z - 64$, which we know is nonnegative.

We solve for the real roots and they are between $(0,1)$ and $(3,4)$.
Thus our only possible root for z is 1.
Plugging this into the original equation we get that $y^2 + y - 7 = 0$ which has no roots in Z.

$\boxed{\text{So x,y have no roots in Z}}$.

Problem 22. Consider the equation $x^2 + 19y^2 = 198 \cdot 10^{1989}$ where $x, y \in \mathbb{Z}^+$.

- **a)** Show that if there exists a solution, then there exists another solution.
- **b)** Find a solution to this equation, as well as its twin solution.

Solution 22a by Chris Ho

By taking both sides modulo 3, we find that x^2 + y^2 = 0 (mod 3). Because a square cannot be 2 (mod 3), we find that both x and y are divisible by 3. We then make the substitution x=3a and y=3b, changing the equation to a^2 + 19b^2 = 22*10^1989. By taking both sides modulo 19, we have that x^2 = 11(mod 19). Therefore, x=7 or 12 mod 19.

Solution 22b by Chris Ho.

By dividing both sides of the given equation by 10^1988, we get that x^2/10^1988+19y^2/10^1988=198*10, or that (x/10^994)^2+19(y/10^994)^2 = 1980. Let a = x/10^994 and b = y/10^994. The equation then becomes a^2+19b^2=1980. By taking both sides modulo 19, we find that a^2 = 4(mod 19), and thus a=2,17(mod 19). After testing the values 2,17,21, and 36 for a (we only need to test values such that a^2<1980), we find that a=36,b = 6 and a=21, b = 9 are valid solutions. Therefore, one solution to the original equation is (36*10^994, 6*10^1994) and its twin solution is (21*10^994, 9*10^1994).

Problem 23. Find all the solutions to the equation where $x, y_1, y_2, y_3, \cdots y_n \in \mathbb{Z}$ and $n > 1$:
$(x+1)^2 + y_1^2 = (x+2)^2 + y_2^2 = \cdots = (x+k)^2 + y_k^2 = \cdots = (x+n)^2 + y_n^2$

Problem 24. Find all solutions to $(x+y)(1+xy) = 2^z$ where $x, y, z \in \mathbb{Z}^+$.

Solution 24 by Rich Wang.

$(2^m + 1 + d + \frac{4^m - 1}{d})^2(2^m) = 2^z$. In order for us to have solutions,
$(2^m + 1 + d + \frac{4^m - 1}{d})$ must equal some 2^a, where a is a positive integer. .Should x or y equal 1, (assume x in this case, as the equation is symmetric), we end up getting $(x+1)^2 = 2^z$. In order for there to be solutions, we must have $x = 2^n - 1, z = 2n$, where n is a positive integer. So this gives solutions of $(2^n - 1, 1, 2n), (1, 2^n - 1, 2n)$.

Now, if we have $x, y \geq 2$, we know that $1 + xy > x + y$. However, we also know that each of the factors must be a power of two, so we get the equation $(x+y) * 2^m = 1 + xy$, where m is a positive integer. This can be expressed as

$4^m - 1 = (x - 2^m)(y - 2^m)$. As a result, we know (x, y) must be in the form of $(2^m + d, 2^m + \frac{4^m - 1}{d})$, where d is a positive integer factor of $4^m - 1$.

However, we might have some extraneous solutions. We plug this into the original equation to get:

$(2^{m+1} + d + \frac{4^m - 1}{d})(1 + 2^{2m} + 4^m - 1 + 2^m(\frac{4^m - 1}{d})) = 2^z$, or

$(2^{m+1}+d+\frac{4^m-1}{d})^2(2^m)=2^z$. In order for us to have solutions, $(2^{m+1}+d+\frac{4^m-1}{d})$ must equal some 2^a, where a is a positive integer. .

Then wat.... Seems kinda impossible to find all sols from here.

Problem 25. Consider the $tetration\ operation$ which can be defined as $f_n(x)$, when $n \in \mathbb{Z}+$, where it satisfies the following:

- **a)** If $n > 1$, then $f_n(x) = x^{f_{n-1}(x)}$.
- **b)** If $n = 1$, then $f_n(x) = x$.

For example, $f_3(3) = 3^{3^3} = 3^{27}$. What is the least value for n such that for any x the following holds: $f_{n+1}(x) - f_n(x) \equiv 0 \pmod{1989}$?

Solution 25 by Rich Wang.

We claim that the answer is $\boxed{\text{n=4}}$.

We first prove that $n = 1, 2, 3$ fail:

For $n = 1$, 2^2 is not congruent to 2 mod 1989.

For $n = 2$, $2^{2^2} = 16$ is not congruent to 2^2 mod 1989

For $n = 3$, $2^{2^{2^2}} = 65536 \equiv 1888$ mod 1989 is not congruent to $2^{2^2} = 16$ mod 1989.

Now we prove that $n = 4$ holds for all x:

The prime factorization of 1989 is $3^2 * 13 * 17$, and we break this up into mods $9, 13, 17$.

By Chinese Remainder Theorem, if we prove $x^{x^{x^{x^x}}} \equiv x^{x^{x^x}}$ (mods $9, 13, 17$), we are done.

Starting with mod 9:

If x is divisible by 3, then we just have $0 \equiv 0$ (mod 9), so we assume x is relatively prime to 9.

LEMMA 1: Proving $f_n(x) \equiv f_{n-1}(x)$ (mod $\phi(a)$), where a is a positive integer relatively prime to x, is equivalent to proving $f_{n+1}(x) \equiv f_n(x)$ (mod a)

PROOF:

We know $f_n(x) \equiv f_{n-1}(x)$ (mod $\phi(a)$), and $x^{\phi(a)} \equiv 1$ (mod a).

Combining these two results together proves $x^{f_n(x)} \equiv x^{f_{n-1}(x)}$, or $f_{n+1}(x) \equiv f_n(x)$ (mod a).

(End of Lemma)

It is easy to see that (Just plug in) $x^{x^x} \equiv x^x$ (mod 2). So, by LEMMA 1: (when x is not divisible by 3):

$x^{x^{x^x}} \equiv x^{x}$ (mod 3), and since once again it is easy to see that (Just plug in) $x^{x^{x^x}} \equiv x^{x^x}$ (mod 2). By Chinese Remainder Theorem, we have:

$x^{x^{x^x}} \equiv x^{x^x}$ (mod 6) For all x not divisible by 3. Now, by LEMMA 1,

$x^{x^{x^{x^x}}} \equiv x^{x^{x^x}}$ (mod 9)

This finishes mod 9.

Next, the mod 13 case:

Above, we got that $x^{x^{x^x}} \equiv x^{x^x}$ (mod 2). When x does not divide 2, we get by **LEMMA 1** that:

$x^{x^{x^x}} \equiv x^{x^x}$ (mod 4). However, we quickly realize that this also holds for x dividing 2, as we just get 0 on both sides, so this also holds for all x.

Now we combine this with the fact that we got above: for all x not divisible by 3, $x^{x^{x^x}} \equiv x^{x^x}$ (mod 3). Note if $x = 3$ we just get 0 on both sides, meaning that this holds for all x. Now, by Chinese Remainder theorem we get that:

$x^{x^{x^x}} \equiv x^{x^x}$ (mod 12). Since this holds for all x relatively prime to 13, we can use **LEMMA 1** to get:

$x^{x^{x^{x^x}}} \equiv x^{x^{x^x}}$ (mod 13), when x is not divisible by 13 . However, we quickly realize if x divides 13 this still holds, as we just get 0 on both sides.

Finally, for the mod 17 case:

We know that $x^x \equiv x$ (mod 4) for all x not divisible by 2, as if $x \equiv 1$ (mod 4) we just have $1^x \equiv 1$ (mod 4), and if $x \equiv 3$ (mod 4) we just have $(-1)^x \equiv (-1)$ (mod 4), which always holds because x is always odd. Now, by **LEMMA 1**, we know:

$x^{x^x} \equiv x^x \pmod 8$ for all x not divisible by 2. Once again, by **LEMMA 1**, we have $x^{x^{x^x}} \equiv x^{x^x} \pmod{16}$ for all x not divisible by 2. However, we realize that this equivalence is still true for all x divisible by 2, as we just get 0 on both sides. Since it holds for all x, it must hold for all x not divisible by 17. Thus, we use **LEMMA 1** one last time to get $x^{x^{x^{x^x}}} \equiv x^{x^{x^x}} \pmod{17}$ for all x not divisible by 17. However, we realize that this equivalence is true for all x divisible by 17, as we just get 0 on both sides again.

Thus, we are finished.

Problem 26 (Russian MO). Solve the following equation in $\mathbb{Z}$: $x^3 - y^3 = xy + 61$.

Solution 26 by Rich Wang.

We claim that the answers are $\boxed{\text{(x,y)=(-5,-6),(x,y)=(6,5)}}$**.**

Now we prove that this works:
It is obvious that
$(-5)^3 - (-6)^3 = (-5)(-6) + 31$ and
$(6)^3 - (5)^3 = (5)(6) + 31$ are true.

Now we prove that these are the only solutions.

We substitute $x = y + z$. We get
$(3z-1)y^2 + (3z^2 - z)y + z^3 - 61 = 0$. Since the y^2 and y terms are divisible by $3z-1$, $z^3 - 61$ must be as well. We must have $3z-1$ divides $27z^3 - 1647$. Note we are allowed to multiply by 27 here, as $3z-1$ cannot be divisible by 7.

We end up getting the quotient is equal to $9z^2 + 3z + 1 - \dfrac{1646}{3z-1}$. Since $3z-1$ must divide 1646, which prime factorizes as $2 * 823$, we know $3z-1$ must equal $-1646, -823, -2, -1, 1, 2, 823, 1646$. The only integer values of z we get from here are $z = -274, 0, 1, 549$.

Yet the discriminant of this quadratic in y is equal to $\sqrt{(-1)(3z-1)(z^3+z^2-244)}$. It is obvious we get the square root of a negative number when $z = -274, 0, 549$, which means that the only possible value is $z = 1$. Plugging this into our equation gives $2y^2 + 2y - 60$, which has solutions $y = -6, 5$. Since $z = 1$ this means our only possible solutions are:

$\boxed{\text{(x,y)=(-5,-6),(x,y)=(6,5)}}$.

Q.E.D

Problem 27. Find all pairs of positive integers (a, b) that satisfy

a) $2^a + 1 \equiv 0 \pmod{2^b - 1}$ and $b > 1$

b) $2^a - 1 \equiv 0 \pmod{2^b + 1}$

Solution 27a.

Solution 27b by Dylan Yu. WLOG let $a > b$. Let us do some mod addition:

$$2^a - 1 + 2^b + 1 \equiv 2^a + 2^b \equiv 2^b(2^{a-b} + 1) \equiv -(2^{a-b} + 1) \equiv 2^{a-b} + 1 \pmod{2^b + 1}.$$

Thus, $a \geq 2b$ in order to obtain pairs. We also see that only when $a - b$ divides b that we can

Problem 28. Given

$$x^3 + y^4 = 19^{19},$$

Find all positive integer x and y.

We take the equation and convert into mod 13, giving $x^3 + y^4 \equiv 7 \pmod{13}$.

We check the first 12 whole numbers taking them to the 3rd and 4th power to find out the possible values for both in mod 13. The possible values for x^3 turn out to be $0, 1, 5, 8, 12$, and the possible values for y^4 turn out to be $0, 1, 3, 9$. We then see that there is no way to add one of $0, 1, 5, 8,$ or 12 and $0, 1, 3,$ or 9 to get 7 in mod 13. Therefore, there are $\boxed{\text{no solutions}}$.

Solution 28 by Eric.

Problem 29. Given

$$x^5 = y^2 + 4,$$

Find all positive integer x and y.

Solution 29 by Leo Lei, Gavin Wang, and Eric Peng.

We take the equation and convert into mod 1, giving $x^5 \equiv y^2 + 4 \pmod{11}$.

We check the first 10 whole numbers taking them to the 2nd and 5th power to find out the possible values for both in mod 11. The possible values for x^5 turn out to be $0, 1, 10$, and the possible values for y^2 turn out to be $0, 1, 3, 4, 5, 9$. We then see that there is no way to take one number out of $0, 1,$ or 10 and $0, 1, 3, 4, 5,$ or 9 to satisfy the equivalence. Therefore, there are $\boxed{\text{no solutions}}$.

Problem 30. Given

$$7^x = 3^y + 4,$$

Find all positive integer x and y.

Solution 30. By Gavin Wang

First we notice that the ordered pair (1,1) works. Now to prove that this is the only solution, we let a be x-1 and b be y-1.

Then, 7(7^a) = 3(3^b)+4.
By reordering, we get 7(7^a-1)=3(3^b-1).
Assume a,b > 0.
Because 7 must divide 3^b-1, then 3^b = 1(mod 7).
From this, we get that y = 0(mod 6) because the modular residues of 3^b are 3,2,6,4,5,1 with an order of 6.
Furthermore, since b is a factor of 6, then (3^6-1) must divide (3^b-1).
3^6-1 = 728 = 2^3*7*13.
Thus, 8 and 13 must divide (7^a-1).
Since 13 is a prime, notice that 7^a = 1(mod 13) because of fermat's little theorem.
Thus, 12 = 3*4 divides a.
From this we see that (7^3 - 1) divides (7^a-1).
7^3-1 = 342 = 2*3^2*13.
From this we get that 9 divides 3(3^b-1).
Thus 3 must divide 3^b-1, which is a contradiction for b>0.
Therefore, there are no solutions for a,b>0.

Problem 31. Given

$$2^x + 3 = 11^y,$$

Find all positive integer x and y.

Solution 31 by Ray Tang.

Plug in x=1 and we get 5=11^y which has no integer roots

Plug in x=2 and we get 7=11^y which has no integer roots
So we know x>=3
Lets take mod 8 and we get (3)^y==3(mod 8) so y==1(mod 3)

Problem 32. If the variables $x, y \in \mathbb{R}$ satisfy $x^3 + y^4 \leq x^2 + y^3$, what is the minimum and maximum of $x^3 + y^3$?

Solution 32.

Problem 33. A number x is a positive integer. It is concatenated with itself to get a new number y. For example, if $x = 534324$, then $y = 534324534324$. What is the least value of x such that y is a perfect square?

Solution 33 by Karthik Vedula. If the number x has d digits (or $10^{d-1} \leq x < 10^d$), then we know that y can be represented as $y = x(10^d + 1)$. Now we have two situations: $10^d + 1$ is squarefree or it is not. If it is squarefree, then x must be $x = k^2(10^d + 1)$ where k is a positive integer. However, this violates the condition that x must have d digits. Therefore, $10^d + 1$ cannot be squarefree. Searching for the smallest value of d gives $10^d + 1$ cannot be squarefree. Searching for the smallest value of d gives

$$10^1 + 1 = 11$$
$$10^2 + 1 = 101$$
$$10^3 + 1 = 1001 = 7 \cdot 11 \cdot 13$$
$$10^4 + 1 = 10001 = 73 \cdot 137$$
$$10^5 + 1 = 100001 = 11 \cdot 9091$$
$$10^6 + 1 = 1000001 = 101 \cdot 9901$$
$$10^7 + 1 = 10000001 = 11 \cdot 909091$$
$$10^8 + 1 = 100000001 = 17 \cdot 5882353$$
$$10^9 + 1 = 1000000001 = 7 \cdot 11 \cdot 13 \cdot 19 \cdot 52579$$
$$10^{10} + 1 = 10000000001 = 101 \cdot 3541 \cdot 27961$$

These all fail. However, for $d = 11$ we succeed because of LTE, Lifting the Exponent Lemma, which gives

$$v_{11}(10^{11} + 1) = v_{11}(10 + 1) + v_{11}(10 - 1) + v_{11}(11) = 2$$

So $11^2 | 10^{11} + 1$. Therefore, we can say $d = 11$. Now we let

$$10^{10} \leq x = \frac{m^2(10^{11} + 1)}{121} < 10^{11}$$

This means that $12.1 < m^2 < 120$. The least value of m which satisfies this is $m = 4$. This means our least solution is

$$\boxed{x = \frac{16(10^{11}+1)}{121} = 13223140496}$$

Problem 34. Consider the *tetration operation* which can be defined as $f_n(x)$, when $n \in \mathbb{Z}+$, where it satisfies the following:

c) If $n > 1$, then $f_n(x) = x^{f_{n-1}(x)}$.

d) If $n = 1$, then $f_n(x) = x$.

For example, $f_3(3) = 3^{3^3} = 3^{27}$. For all positive integers k, c, and p, show that there exists an x such that

$$\begin{aligned}
&x \equiv c \pmod{p},\\
&x^x \equiv c \pmod{p},\\
&\qquad \vdots\\
&f_k(x) \equiv c \pmod{p}.
\end{aligned}$$

Solution 34 by Rich Wang.

Note that the problem is trivial if $p = 2$, so we will assume p is an odd prime. Note that the problem is also trivial if $c \equiv 0$ (mod p), so we will assume that this is not the case.

By Fermat's Little Theorem, we know that $c^{p-1} \equiv 1$ (mod p). This means that all $c^n \equiv c$ (mod p) for all $n \equiv 1$ (mod $p - 1$), where n is a positive integer.

In addition, let us choose x to be a number congruent to c (mod p) and congruent to $p - 2$ (mod $p - 1$), taken to the nth power, where n is even, and congruent to 1 (mod p). This is possible by Chinese Remainder Theorem. Because of what we did above, we know that this satisfies the first $x \equiv c$ (mod p).

Yet now, since x is equal to a number that is congruent to $p - 2$ (mod $p - 1$) and is raised to an even power, any $f_n(x)$ will be congruent to 1 (mod $p - 1$). Since for all $n > 1$, the teration operation is defined as:If $n > 1$, then $f_n(x) = x^{f_{n-1}(x)}$, then this means that all tetration operations will satisfy being congruent to c (mod p), by Fermat's Little Theorem above, so we are done.

Problem 35. (USA TST P4) If $x^7 + 7 = n^2$, find all possible integers n and x.
Solution 35 by Eric Peng with help from Rich Wang.

Lemma 1: If $n \equiv 3 \pmod 4$, and $a^2 + b^2 \equiv 0 \pmod n$, then a and b both must divide n.

Proof: We try to prove that there is a contradiction is n doesn't divide into a. Given that, b must also not be divisible by n, because otherwise $a^2 + b^2 \equiv 0 \pmod n$ can not be true. By Fermat's Little Theorem, a^{n-1} and b^{n-1} are congruent to 1 in mod n. This means their sum is congruent to 2 in mod n.
We now write $n = 4k + 3$, where k is an integer because $n \equiv 3 \pmod 4$. We replace n with $4k + 3$ in the following congruence $a^{n-1} + b^{n-1} \equiv 2 \pmod n$ to get $a^{4k+2} + b^{4k+2} \equiv 2 \pmod n$. Now we write $4k + 2as2(2k + 1)$ to get $a^2 \cdot a^{2k+1} + b^2 \cdot b^{2k+1} \equiv 2 \pmod n$, a contradiction since we have $a^2 + b^2 \equiv 0 \pmod n$, and $a^2 + b^2$ divides this. Therefore, a and b both must divide n.

Lemma 2: Given $a \equiv 3 \pmod 4$, then $b \equiv 3 \pmod 4$ iff $ab \equiv 1 \pmod 4$.

Proof: We check all the possibilities in mod 4 to see that only $b \equiv 3$ works.

We see this equation and immediately try mod 15, as $15 = 7 * 2 + 1$. However, that doesn't help, but mod 4 tells us some information. When we take a 7th power in mod 4, we see by plugging in 0 through 3 that $x^7 \equiv 2 \pmod 4$ is not possible, and when we take a square in mod 4, we see by plugging in 0 through 3 that only $n^2 \equiv 0 (\text{mod } 4)$ and $n^2 \equiv 1 \pmod 4$ are possible. We check the two cases $n^2 \equiv 0(mod4)$ and $n^2 \equiv 1 \pmod 4$ to see that $n^2 \equiv 1$ (mojd 4) has no solution, so $n^2 \equiv 0 \pmod 4$ and $x^7 \equiv 1 \pmod 4$. We can try using other mods to our advantage, but this ends up not working.

We then try to find a nice factorization. We notice that. Which inspires us to add 121 to both sides, giving $x^7 + 128 = n^2 + 121$, and rewriting it as $(x + 2)(x^6 - 2x^5 + 4x^4 - 8x^3 + 16x^2 - 32x + 64) = n^2 + 121$. We know that $x + 2$ is a factor of the left hand side, so it must be a factor of the right hand side. Since $x^7 \equiv 1 \pmod 4$, we can check for the possible values of x in mod 4 leading to us realizing that $x \equiv 1 \pmod 4$, which means that $x + 2 \equiv 3 \pmod 4$. Since $x + 2$ is a factor of the left side of the equation, it must also be a factor on the right side of the equation. However, by Lemma 1, this means that

$n \equiv 0$ (mod 11). By Lemma 2, $\dfrac{x^7+2^7}{x+2} \equiv 3$ (mod 4). However, we use Lemma 1, which contradicts because $\dfrac{x^7+2^7}{x+2}$ can't divide into 121. Therefore, our answer is $\boxed{\text{No solutions}}$.

Day 4

Problem 36. Given

$$n^{10}+n^5+1$$

is prime, find n.

Solution 36 by Ray Tang.
We know that you can factor
$n^{10}+n^5+1=(n^2+n+1)(n^8-n^7+n^5-n^4+n^3-n+1)$
In order for $n^{10}+n^5+1$ to be prime we need either $n^2+n+1=\pm 1$ or
$n^8-n^7+n^5-n^4+n^3-n+1=\pm 1$
For the first one we must have that $n=-1$ or 0 and for the second we reset roots to find that $n=1$ is a root. The other roots are either $-1,1$ or complex by rational root theorem and we don't need to calculate them. Now we plug $n=-1,0,1$ into the original equation and find out that $n=1$ gives 3 and is the only possible root so $\boxed{\text{n=1}}$.

Problem 37. Find the smallest integer that cannot be represented as

a) $\dfrac{2^a-2^b}{2^c-2^d}$, for positive integer a,b,c,d.

b) $n=|2^a-3^b|$, where n is prime, and a and b are whole numbers.

c) $n=\dfrac{a^3+b^3}{c^3+d^3}$, where a,b,c,d are positive integer

Solution 37a by Ray Tang and Dylan Yu. Through manipulation we see that

$$\frac{2^a-2^b}{2^c-2^d}=\frac{2^{a-d}-2^{b-d}}{2^{c-d}-1}=\frac{2^{a-b}-1}{2^{c-d}-1}\cdot(2^{b-d}).$$

Thus, we simply need to find a number that cannot be expressed as

$$\frac{2^x-1}{2^y-1}.$$

We can do this by testing out odd numbers. Because if an odd number works every even multiple must also work by setting b-d to be a specific value. - through trial and error we see

$$1,3,5,\ldots,9$$

Work. Therefore, we only have to prove 11 doesn't. We can do this using Fermat's Little Theorem:

$$2^{10}\equiv 1 \pmod{11}.$$

By testing 1 through 9 we see that 10 is actually the smallest power that creates $1 \pmod{11}$. Because

$$2^{10} - 1 = 11 \cdot 93,$$

We must somehow get rid of the 93 using the denominator. However, the smallest power of 2 that is $1 \pmod{93}$ is actually 11 - this means both the power x in the numerator and the power y in the denominator are divisible by 10. Because $x > y$, we can test the smallest values to see

$$\frac{2^{20} - 1}{2^{10} - 1} > 11,$$

And because the value will only increase, the answer is $\boxed{11}$.

Solution 37b.

Solution 37c.

Problem 38. Let $n \geq 2$, and let A_i be the set of all positive integers that does not contain digit i in their base n. For all $i < n$,

$$f_n(i) = \sum_{r \in A_i} \frac{1}{r}.$$

Calculate $f_n(i)$ explicitly, and find an estimate for it.

a)solve $f_{10}(3)$

Sol a) 1+½+¼+⅕+⅙+...+1/10<2.6 (2.596)

Motivation for Problem 38. Note that if $i = 10$, then we are basically subtracting the harmonic series and one tenth of the harmonic series. Thus, it makes sense that a result can be achieved.

Problem 39. Let

$$4A^3 - 2A^2 - 15A + 9 = 0,$$

$$12B^3 + 6B^2 - 7B + 1 = 0,$$

Where the roots A_1, A_2, A_3 and B_1, B_2, B_3 are real. Given that

$$A_1 > A_2 > A_3,$$

$$B_1 > B_2 > B_3,$$

Prove

$${A_1}^2 + 3{B_2}^2 = 4.$$

Solution 39.

Problem 40. Find all integers m such that

$$x^3 + 2x + m | x^{12} - x^{11} + 3x^{10} + 11x^3 - x^2 + 23x + 30.$$

Solution 40 by Rich Wang.

Motivation: Long dividing with m as a variable won't be very nice. However, we realize that if there is an m that works, then for every integer x, after dividing we will also get an integer. This motivates us to try plugging in convenient numbers for x so that we can limit the values for m.

We claim that the only value that works is $\boxed{\text{x=3}}$.

This can easily be verified by long division.

To prove that this is the only value that works:

Plugging in $x = 0, 1$ gives us that $m + 3$ must divide 66, and m must divide 30. Solving this gives us the five possible values of $x = -6, -5, -2, 0, 3$ as possible values. However, after plugging in $x = 2$, we realize that

$12 + m$ must divide $2^{12} - 2^{11} + 3 * 2^{10} + 11 * 2^3 - 2^2 + 23 * 2 + 30 = 5280$. This eliminates -5 as a possibility.

Now, we unfortunately must use long division to figure out which of the last 4 work.

After long dividing, we realize that the only value that works is $\boxed{\text{m=3}}$.

Problem 41. For all a, b that satisfy the below, find n:

(a) $a, b | n,$

(b) $(a, b) = 1,$

(c) $a + b - 1 | n.$

Solution 41.

Problem 42. If $1, 2, \ldots, \lfloor\sqrt{n}\rfloor$ all divide n, find n.

Solution 42 by Rich Wang.

We claim that the answers are $\boxed{1,\ 2,\ 3,\ 4,\ 6,\ 8,\ 12,\ 15,\ 24}$.

It is very easy to verify that these numbers all work.

Now we prove these are the only answers that can work:

Let us define $k =$ the floor of $\sqrt{n}$

We know that in order for the floor of $\sqrt{n}$ to stay at the number it is currently at, n must be between k^2 and $(k+1)^2 - 1$, inclusive. This means that in order for n to be divisible by k, it must either be k^2, k^2k, k^2k.

In order for k^2 to work, k must also be divisible by every positive integer below it. This only happens if $n = 1, 4$.

In order for $k(k+1)$ to work, we know that $k-1$ must be a factor of $k(k+1)$. However, $k, k-1$ are relatively prime, and the max gcd between $k-1, k+1$ is 2, so this can only work if $k1 = 0, 1, 2$. This means that values of n of this form that work are $2, 6, 12$.

Finally, in order for $k(k+2)$ to work, we know that $k-1$ must be a factor of $k(k+2)$. However, $k, k-1$ are relatively prime, and the max gcd between $k-1, k+1$ is 3, so this can only work if $k - 1 = 0, 1, 2, 3$. This means that values of n of this form that work are $3, 8, 15, 24$.

Thus, we are done.

Day 6

Problem 43. Prove that $\forall$ n $\in \mathbb{Z}^+$,n $\notin \mathbb{P}, \exists a, b, c \in \mathbb{Z}^+$ s.t. $ab + bc + ac + 1 = n$.

Variation of Problem 43. Find all primes n such that
$\exists a, b, c \in \mathbb{Z}^+$ s.t. $ab + bc + ac + 1 = n$.

Solution 43 by Eric Peng. Let's say that p is a prime factor of n. We claim that $\frac{n-p}{p}, p-1, 1$ works for all n. We see that this is valid because the equation simplifies to $n = n$, and we can check to make sure that $a, b, c \in \mathbb{Z}^+$ by seeing that this isn't true if **(1)** $n - p = 0$, which is not possible because n is not prime, so it must have another factor. **(2)** $p - 1 = 0$, not possible, or **(3)** $p = 0$, also not possible. Therefore, this satisfies.

Problem 44. (APMO 1998) Find all n such that

$$1, 2, 3, \ldots, \lfloor \sqrt[3]{n} \rfloor$$

Divides n.

Solution 44 by Rich Wang.

Solution: (Note that we repeatedly use the fact that the gcd of two numbers divides their difference in our cases)

We claim the solution set is:

$\boxed{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 10,\ 12,\ 14,\ 16,\ 18,\ 20,\ 22,\ 24,\ 26,\ 30,\ 36,}$

$\boxed{42,\ 48,\ 54,\ 60,\ 72,\ 84,\ 96,\ 108,\ 102,\ 180,\ 240,\ 300,\ 420}$.

We can easily verify that each of these work, and that all other $n < 420$ do not.

Now to prove that these are the only solutions:

Assume $n \geq 420$

Let us express $n = x^3 + k$, where k is a non-negative integer and $n < (x+1)^3$. Since $x, x-1$ relatively prime, we know $x(x-1) | x^3 + k < x^3 + 3x^2 + 3x + 1$. Now, since $x^3 + k \geq x^3$, and $x > 7$, we get that the only k which satisfy these conditions are $k = x^2 - 2x, 2x^2 - 3x, 3x^2 - 4x$. So now we just need to check that there are no n that work for these values of k.

The values of $x^3 + k$ can be factored as
$x(x-1)(x+2), x(x-1)(x+3), x(x-1)(x+4)$. Let us consider cases:

Case 1: $x(x-1)(x+2)$.
We know that $x-2$ must divide this value, yet is relatively prime to $x-1$, has a maximum gcd of 2 with x, and a maximum gcd of 4 with $x+2$. This means that we must either have $x-2=1,2,4,8$. However, since we assumed $x>7$, we only have to check $x=10$ here. It can easily be verified that 7 does not divide $x(x-1)(x+2)$ in that case, so no solutions here.

Case 2: $x(x-1)(x+3)$.
We know that $x-2$ must divide this value, yet is relatively prime to $x-1$, has a maximum gcd of 2 with x, and a maximum gcd of 5 with $x+3$. This means that we must either have $x-2=1,2,5,10$. However, since we assumed $x>7$, we only have to check $x=12$ here. It can easily be verified that 7 does not divide $x(x-1)(x+3)$ in that case, so no solutions here.

Case 3: $x(x-1)(x+4)$.
We know that $x-2$ must divide this value, yet is relatively prime to $x-1$, has a maximum gcd of 2 with x, and a maximum gcd of 6 with $x+2$. This means that we must either have $x-2=1,2,3,4,6,12$. However, since we assumed $x>7$, we only have to check $x=8,14$ here. It can easily be verified that 5 does not divide $x(x-1)(x+4)$ in either of these cases, so no solutions here.

Thus, there are no solutions above $n=420$.

Problem 45. Given

$$m^2+4n=x^2,$$
$$n^2+4m=y^2,$$

Find all integer m,n.

Solution 45.

Problem 46. Given

$$(m^2-n)|(m+n^2),$$
$$(n^2-m)|(n+m^2),$$

Find all integer m,n.

Solution 46.

Problem 47. (Putnam) Given

$$m^{n+1} - (m+1)^n = 2001,$$

Find all positive integer m, n.

Solution 47 by Eric Peng. Trying mod m gives us m is a factor of 2002. Trying mod 3 gives us

$$m \equiv 1 \pmod{3}.$$

Trying mod 4 gives us

$$m \equiv 1 \pmod{4}.$$

If we look through the factors of 2002, we find $\boxed{\text{m} = 13,\ \text{n}=2}$.

Problem 48. Given

$$x - \frac{y}{2} = \frac{2}{y^4} - \frac{2}{x^4},$$

$$x + \frac{y}{2} = (\frac{3}{x^2} + \frac{1}{y^2})(\frac{1}{x^2} + \frac{3}{y^2}),$$

Find all real x, y.

Motivation for Problem 48. After adding and subtracting the equations, numbers similar to that of binomial coefficients appear, and we can capitalize on that.

Solution 48 by Dylan Yu. Let us expand the second equation to get

$$x - \frac{y}{2} = \frac{3}{x^4} + \frac{10}{x^2y^2} + \frac{3}{y^4}.$$

From here, we can add the two equations to get

$$2x = \frac{1}{x^4} + \frac{10}{x^2y^2} + \frac{5}{y^4},$$

And subtract to get

$$y = \frac{5}{x^4} + \frac{10}{x^2y^2} + \frac{1}{y^4}.$$

By multiplying each equation by x^4y^4, we get

$$2x^5y^4 = 5x^4 + 10x^2y^2 + y^4,$$

$$x^4y^5 = x^4 + 10x^2y^2 + 5y^4,$$

And multiplying the first equation by y and the second by x and then adding and subtracting them, we get

$$3x^5y^5 = (x+y)^5,$$

$$x^5y^5 = (x-y)^5,$$

So

$$xy\sqrt[5]{3} = x + y,$$
$$xy = x - y,$$
$$xy(\sqrt[5]{3} + 1) = 2x,$$
$$y = \frac{2}{1 + \sqrt[5]{3}},$$
$$x = \frac{2}{-1 + \sqrt[5]{3}},$$

So the only solution is $\boxed{(\frac{2}{\sqrt[5]{3} - 1}, \frac{2}{\sqrt[5]{3} + 1})}$.

Problem 49. For a sequence of nonzero real numbers $a_1, a_2, \ldots, a_n,$, $\forall j \geq 0,$

$$\sum_{i=j+1}^{j+7} a_i > 0,$$
$$\sum_{i=j+1}^{j+11} a_i < 0,$$

What is the largest n possible?

Solution 49.

Wdym n=76

Problem 50. Find all n such that if $\gcd(a, b) = 1$ and $n = a^2 + b^2$, then for all prime numbers under $\sqrt{n}$,

$$p|ab.$$

Solution 50.

Problem 51. Find all N where there exists real $a, b, c, d \geq N$ such that

$$a^2 + b^2 + c^2 + d^2 = abc + abd + acd + bcd.$$

Solution 51 by Eric Peng. WLOG, we can say d is the smallest out of the 4. We write the equation above as a quadratic in terms of d giving $d^2 - (ab + bc + ac)d + \ldots = 0$. We can use Vieta's to see that the sum of the roots is $ab + ac + bc$, and since we know d is a root, we can find the other root to be $ab + bc + ac - d$. Since we said d is the smallest, this root is bigger than d, which means we can recursively apply this to the smallest value of the 4 to increase it, meaning all N work.

Problem 52. (IMO Shortlist) Find all pairs of m, n such that there exists k such that

$$\sigma(mk) = \sigma(nk),$$

Where $\sigma(n)$ is the sum of divisors of n.

Solution 52.

Variation of Problem 52. Find all pairs of m, n such that there exists k such that

$$d(mk) = d(nk),$$

Where $d(n)$ is the number of divisors of n.

Solution for the Variation of Problem 52.

Problem 53. Find all integer x, y, z, t such that

$$xy - zt = x + y = z + t.$$

Solution 53 by Rich Wang.

We claim that there are:

Now to prove these are the only ones that work:

We quickly start by eliminating the case if one of the variables is 0. Assume $x=0$.

We get that $x=-zt$, so $1=(z+1)(t+1)$, giving us solutions of $(x,y,z,t)=(-4, 0, -2, -2)$ and $(0,0,0,0)$. Note x,y, z,t are interchangeable, so we also get a solution of $(0, -4, -2, -2)$

Now, if we assume $z=0$, we similarly get that $(x-1)(y-1)=1$, which gives us solutions of $(x,y,z,t)=(2,2,0,4)$ and $(0,0,0,0)$. Due to interchangeability again, we also have $(2,2,4,0)$.

From here, assume x,y,z,t do not equal 0, and let us call $z=k+x, t=k+y$.

We get that since $xy-(x+k)(y-k)=x+y$, we know that $k^2+xk-yk=x+y$. As a result, since both sides are integers, we know that $x+y|k$.

Now, we solve for x,y. We get:
$x=\frac{yk+y-k^2}{k-1}$, $y= {xk-x+k^2}{k+1}$, or
$x=-k+(y-1)+\frac{2y-1}{k-1}, y=k+(x-1)-\frac{2k-1}{k+1}$.
This implies $k+1|2x-1, k-1|2y-1$.

Case 1: x,y are both positive or both negative.

Since we have $x+y|k, k+1|2x-1, k-1|2y-1$, If x,y positive we know that:

$x+y<2x-1, x+y-1 \le 2y-1$, or $y<x-1, x \le y$, implying there are no solutions here.

If they are both negative, we know:
$x+y>2x-1, x+y-1 \ge 2y-1$, or $y>x-1, x \ge y$, implying $y=x-1$ as the only possible solution.

Plugging this into $xy-zt=x+y$ gives us the solution set:
$(x,y,z,t)=(\frac{k^2+k+1}{2},\frac{k^2+k-1}{2},\frac{k^2+3k+1}{2},\frac{k^2-k-1}{2})$.
However, this is unusable, as we can never have both k and x,y,z,t as integers. (Note: $z=x+t$), so if x is an integer, t needs to be an integer to garuntee that z is an integer.

Thus, there are no solutions here.

I'm sorry this problem has sucked away my will to do anything else....

How in the world do you do the x,y one neg one pos case...

Day 7

Problem 54. (2010 IMO Shortlist) Given

$$(1-\frac{1}{s_1})(1-\frac{1}{s_2})\ldots(1-\frac{1}{s_n})=\frac{51}{2010},$$

Find the smallest n for which $s_1, s_2, \ldots, s_n$ are positive integers.

Solution 54a by Ray Tang. First notice how 51/2010 can be simplified into 17/670. We can create this product with (1-1/67*16)(1-1/9)(1-1/7)(1-⅓)(1-⅕)(1-½)(1-½)(1-½)(1-½)=17/670
This uses 9 fractions. We need to prove that we cannot use 8 or less fractions to create such a product.
As s_i grows 1-1/s_i grows and approaches 1. Since we want to use the least amount of fractions and we need a value of 67 on the denominator, it makes sense that we should have 66/67 (we can try a ton of multiples of 67 and all of them possess a high amount of primes in the numerator or denominator and thus are not efficient) now we need to get rid of the 66- we need an 11, and we need a 17 our first possible case is if we have two fractions for these factors 66/67*17a/17a+1*11b-1/11b*... however if the … were replaced with all 1/2s we would need at least 6 of them to make the fraction close to 66/67 so this will require at least 9 fractions
Our second case is if the two factors are contained in the same fraction. This would require a fraction like 153/154 (any multiple of 154 will give meaningless primes and would make the factor closer to 1 so we don't want that) since 153=9*17 and 154=11*14 we need to get rid of the 9 and the 14 therefore we will need another fraction that is very close to 1 for this (as having 8/9 and 13/14 is impossible because of the ½ argument) therefore we will need a fraction like 224/225. Then we can have the rest as ½ so that we can see if we can reach 17/670 with less than 9 fractions but this is not possible.
Thus we must use at least 9 fractions to create the product. Since we have shown that n=9 works, the smallest such n is 9.

Solution 54b by Ray Tang. We can construct such a value with the fractions ½ *⅔*¾*...*32/33*34/35*35/36*36/37*...*39/40*66/67=17/670
This product uses 39 fractions.

Problem 55. (2010 IMO Shortlist) Given

$$m^2+2\cdot 3^n=m(2^{n+1}-1),$$

Find all positive integer m, n.

Solution 55.

Problem 56. (2010 IMO Shortlist) Given

$$x^2 + 7 = p_1(x)^2 + p_2(x)^2 + \ldots + p_n(x)^2,$$

Where p_i are polynomials. Find the smallest n that satisfies if p_i have integer, rational, or real coefficients.

Solution 56 by Dylan Yu. If we have real coefficients, the answer is just $\boxed{2}$:

$$p_1(x) = x,$$
$$p_2(x) = \sqrt{7}.$$

If we have integer coefficients, one of p_i has to be x, and the rest must be integers. Thus, we have

$$p_1(x) = x,$$
$$p_2(x) = 2,$$
$$p_3(x) = p_4(x) = p_5(x) = 1.$$

Thus, for integers, the answer is 5. If we have rational coefficients, we see that each function is linear or constant (if it were bigger, then $x^2 + 7$ couldn't be quadratic). Thus,

$$p_i(x) = a_i x + b_i.$$

Therefore,

$$a_1^2 + \ldots + a_n^2 = 1,$$
$$b_1^2 + \ldots + b_n^2 = 7.$$

Substituting

$$x_i = \frac{a_i + b_i}{2},$$
$$y_i = \frac{a_i - b_i}{2},$$

We see that

$$x_1^2 + \ldots + x_n^2 = y_1^2 + \ldots + y_n^2 = 8,$$
$$x_1 y_1 + \ldots + x_n y_n = -6.$$

We see that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ must have a common denominator m, so

$$c_1^2 + \ldots + c_n^2 = d_1^2 + \ldots + d_n^2 = 8m^2,$$
$$c_1 d_1 + \ldots + c_n d_n = -6m^2,$$

Where $c_i = \frac{x_i}{m}$, and $d_i = \frac{y_i}{m}$. When we check $n = 4$, there are no solutions, but when $n = 5$, we get the same solutions we got from the integers. Thus, this answer is also $\boxed{5}$.

Problem 57. (2010 IMO Shortlist) Given

$$a^n + pb = b^n + pc = c^n + pa,$$

And p is prime, find all positive integer a, b, c that satisfy.

Solution 57 by Eric Peng. We can rearrange the equation to get

$$p = \frac{a^n - b^n}{c - b} = \frac{b^n - c^n}{a - c} = \frac{c^n - a^n}{b - a}.$$

Multiply the three to get $p^3 = \frac{a^n - b^n}{a - b} = \frac{b^n - c^n}{b - c} = \frac{c^n - a^n}{a - c}$. This is equivalent to $-p^3 = \frac{a^n - b^n}{a - b} = \frac{b^n - c^n}{b - c} = \frac{c^n - a^n}{c - a}$. Since we know that $a, b, c \in \mathbb{Z}^+$, none of the terms can be negative, meaning that there is $\boxed{\text{No Solutions}}$.

Problem 58. Given

$$x^3(y^3 + z^3) = 2012(xy + 2),$$

Find all positive integer x, y, z.

Motivation for Problem 58. Maybe Legendre's Symbol will give us some insight…

Solution 58.

Day 8

Problem 59. $1 + 2^x + 2^{2x+1} = y^2$, $x, y \in \mathbb{Z}^+$.

Solution 59 by Eric Peng and Ray Tang.

This equation is equivalent to $2^x(1 + 2^{x+1}) = (y-1)(y+1)$. We see that this means that y is odd, so we replace it with $2z + 1$ to get $2^{x-2}(1 + 2^{x+1}) = z(z+1)$.

We need to check $x = 1, 0$ for when $x = 1 y = \sqrt{11}$ which is irrational and when $x = 0 y = 2$.

Both factors in both equations are coprime so we have three cases which are $z = 2^{x-2}$ and $z + 1 = 2^{x-2}$ and $z = 1$.

$z = 1$ has no solution.

If $z + 1 = 2^{x-2}$

Then we have that $z = 2^(x+1) + 1$. However, since x is in positive integers, $z > z + 1$ which is impossible

If $z = 2^(x-2)$

Then we have that $z+1 = 2^(x+1)+1$

$2^(x-2)+1 = 2^(x+1)+1$

$2^(x-2) = 2^(x+1)$

This will never happen because x is in positive integers.

We see that both cases don't work so $\boxed{x=0, y=2}$.

Problem 60. Given

$$ab - c = 2^m,$$
$$ac - b = 2^n,$$
$$bc - a = 2^k,$$

Find all positive integer a, b, c, m, n, k that suffice.

Solution 60.

Variation of Problem 60. What if 2 was replaced with a random prime p?

Solution for the Variation of Problem 60.

Problem 61. Given

$$\frac{(p+q)^{p+q}(p-q)^{p-q}-1}{(p+q)^{p-q}(p-q)^{p+q}-1}$$

Is an integer, find all prime p, q that suffice.

Solution 61.

Problem 62. Let $p(n)$ be the largest prime factor of n. Find all positive integers n such that

$$p(n^4+n^2+1) = p((n+1)^4+(n+1)^2+1).$$

Solution 62.

Problem 63. Find all n for which there exists a k such that

$$1 + \frac{2^k-1}{n} = (1+\frac{1}{m_1})\dots(1+\frac{1}{m_k}),$$

Where n, k and m_i for all $i = 1, 2, \dots, k$ are positive integers.

Solution 63.

Problem 64. Given

$$7^k - 3^n | k^4 + n^2,$$

Find all positive integer n, k.

Solution 64.

Problem 65. Find n for which there exists a m such that

$$2^m + m \equiv 0 \pmod{n}.$$

Solution 65.

Problem 66. Given

$$\frac{x^7 - 1}{x - 1} = y^5 - 1,$$

Find all positive integer x, y that suffice.

Solution 66 by Karthik Vedula. Let p be a prime divisor of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$ (which is equal to the RHS). If $p|x - 1$, then

$$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \equiv 1 + 1 + 1 + 1 + 1 + 1 + 1 \equiv 0 \pmod{7}$$

or the order of x modulo p is 7 and $p \equiv 1 \pmod{7}$. Therefore all divisors of x are 0 or 1 modulo 7.

Now we will proceed by contradiction. Suppose there exists a solution (a, b). Since $y - 1$ and $y^4 + y^3 + y^2 + y + 1$ are divisors of $x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$, we know that they are both 0 or 1 modulo 7. However, testing both possibilities, we see that this cannot happen, so we have proved that there are no solutions.

Problem 67. Let each vertex of a cube have either 1 or -1 on it. Every number on each face is determined by taking the product of the numbers on the vertex that make up that face (i.e. the vertices of the square face). What are the possibilities for the sum of the numbers on the faces with the numbers on the vertices?

Solution 67 by Karthik Vedula. We try and just list them.

- If all vertices are the same, two sums are possible, 14 and -2 are clearly the only possibilities.
- If five vertices are the same, then only two sums are possible, ± 6.
- If four vertices are the same and two are different, then orientation matters. The two vertices could be on the ends of an edge, a face diagonal, or a space diagonal. These orientations and signs give the values of ± 2, ± 6, and 10.
- Having three of each will be even harder. Fix the position of a 1.
    - Subcase 3.1: The ones are not adjacent to each other at all.
        - There is only one possible orientation here with sums of
        - **Can someone finish this?**

Problem 68. (1992 IMO Shortlist)

Prove that $\frac{5^{5^3}-1}{5^{5^2}-1}$ is a composite number.

Solution 68.

Letting $x = 5^{25}$, we have that

$$\frac{x^5-1}{x-1} = x^4 + x^3 + x^2 + x + 1 = (x^2+3x+1)^2 - 5x(x+1)^2$$

which factors into

$$(x^2+3x+1+5^{13}(x+1))(x^2+3x+1-5^{13}(x+1)) = (5^{50}+3*5^{25}+1+5^{13}(5^{25}+1))(5^{50}+3*5^{25}+1-5^{13}(5^{25}+1)).$$

We know the latter factor is greater than 1 because $5^{50} >> 5^{39} > 5^{13}(5^{25}+1)$.

Day 9

Problem 69. $p^2 - p = 37q^2 - q$

Solution 69 by Eric Peng.

We see that q is a factor of $p-1$ so we write $p-1 = qk$. We put this back in the equation to get $q = \dfrac{k+1}{37-k^2}$. We only need to check 6 and below for k. 6 gives $p = 43, q = 7$. However, at $k = 5$, $q < 1$. Therefore, our answer is $\boxed{p = 43, q = 7}$.

Variation of Problem 69. $p^2 - p = dq^2 - q$

Solution to the Variation of Problem 69 by Eric Peng.

We get $q = \dfrac{k+1}{d-k^2}$ the same way as before. Checking k to be $\sqrt{d}$ will be a possible solution. However, we use $q > 1$ to get $\dfrac{k+1}{d-k^2} > 1$, simplifying to $d < k^2 + k + 1$. If we check the value of k 1 less than before, we get $k^2 - k + 1 > d$. However, since the denominator of q must be greater than 0, we have $d > k^2$, a contradiction. Therefore, we have 1 value of k we can use to find p and q.

Problem 70. Given $n = 7^{7^7} + 1$,

(a) Find the number of digits of n
(b) Prove that the number of prime factors of n is at least 17
(c) Prove that there are at least 5 **distinct** prime factors

Solution 70a by Gavin Wang and Leo Lei. If you know your logs, log 10 (8) is 0.845. Then, you can calculate that log 10 (7^7) is 5.915. 10^5.915 will be around 820000, and multiplying thay by .845 you get an answer of 692900, which is only 3000 digits off from the actual solution of 695975.

Solution 70b.

Problem 71. Given

$$n | p - 1,$$
$$p | n^3 - 1,$$

Find all prime p and positive integer n that suffice.

Solution 71.

Problem 72. Given

$$2^2 + 3^2 + \ldots + n^2 = p^k,$$

Find all prime p and positive integer n, k that suffice.

Solution 72.

Problem 73. Given

$$x^2 + 1 = m(2^n - 1),$$
Find all positive integers x, m, n that suffice.

Solution 73.

Problem 74. Given
$$pq|(5^p - 2^p)(5^q - 2^q),$$
Find all primes p, q that suffice.

Solution 74.

Problem 75. Given
$$p|q^r + 1,$$
$$q|r^p + 1,$$
$$r|p^q + 1,$$
Find all prime p, q, r that suffice.

Solution 75 by Dylan Yu. WLOG, let
$$p \leq q \leq r.$$
We know that if
$$q^r + 1 \equiv 0 \pmod{p},$$
Then
$$q^{2r} \equiv 1 \pmod{p}.$$
Because
$$q^{p-1} \equiv 1 \pmod{p},$$
We know that either $2r$ divides $p-1$ or $p-1$ divides $2r$. But because $p \leq r,$, the first case is impossible, so $p-1$ is either $1, 2, r,$ or $2r$. If p is 2 or 3 the cases are trivial, but if $p = r$, we get
$$p^q + 1 \equiv 1 \equiv 0 \pmod{p},$$
Which is impossible. If $p = 2r$,
$$(2r)^q + 1 \equiv 1 \equiv 0 \pmod{r},$$
Which is also impossible. Thus, we only need to solve $p = 2$ and 3. Because $q \leq r,$ we can say the same for q: it is either 2 or 3. If they are both 2, using the first equation we get a contradiction:
$$2^r + 1 \equiv 1 \equiv 0 \pmod{2}.$$
If $p = 2$ and $q = 3$, using the third equation we get

$$8 + 1 \equiv 9 \equiv 0 \pmod{r},$$

So $r = 3$. If $p = 3$ and $q = 2$,

$$9 + 1 \equiv 10 \equiv 0 \pmod{r},$$

So $r = 2, 5$. Finally, if $p = 3$, $q = 3$, we again get a contradiction from the first equation.

Plugging these in, we see only $\boxed{(3, 2, 5)}$ works.

Problem 76. Given

$$pq|5^p + 5^q,$$

Find all prime p, q that suffice.

Solution 76.

Problem 77. Given

$$x^{p-1}|(p-1)^x + 1,$$

And $x \leq 2p$, find all positive integer x and prime p that suffice.

Solution 77.

Problem 78. Given

$$\frac{2^n + 1}{n^2}$$

Is an integer, find all positive integer n that suffice.

Solution 78.

Problem 79. Given

$$x^2 - y! = 2001,$$

Find all positive integer x, y that suffice.

Solution 79.

Problem 80. Given

$$a^3 + 2b^3 + 4c^3 = 6abc + 1,$$

Find all positive integer a, b, c that suffice.

Solution 80 by Dylan Yu. Applying AM-GM, we see that

$$\frac{a^3 + 2b^3 + 4c^3}{3} \geq \sqrt[3]{8a^3b^3c^3},$$
$$a^3 + 2b^3 + 4c^3 = 6abc + 1 \geq 6abc,$$

Which is always true - therefore, the answer is all reals.

Variation of Problem 80. Given
$$2a^2 + 3b^3 + 4c^4 - 6abc = 1,$$
Find all positive integer a, b, c that suffice.

Solution of the Variation of Problem 80.

Day 10

Problem 81. Find all $a, b, c \in \mathbb{Z}^+$ such that $\sqrt{a^2 + b + c}, \sqrt{b^2 + c + a}, \sqrt{c^2 + a + b} \in \mathbb{Z}^+$

Solution 81 by Eric Peng. Let $a =$ max(a, b, c). Then, $a^2 < a^2 + b + c < (a + 1)^2$, meaning the first square root is not an integer. $\boxed{\text{No solutions.}}$

Problem 82. Given
$$ab(a + b) \not\equiv 0 \pmod{2037},$$
$$(a + b)^7 - a^7 - b^7 \equiv 0 \pmod{2037},$$
Find infinitely many positive integers a, b that suffice.

Motivation for Problem 82. Problem 15 solves the problem.

Solution 82 by Dylan Yu. Note that
$$(a + b)^7 - (a^7 + b^7) = 7ab(a + b)(a^2 + ab + b^2)^2,$$
So as long as $a^2 + ab + b^2$ is divisible by $97 \cdot 3$, we are good. Thus, we are done.

Problem 83. Given
$$n^2 = \frac{m^5 - 1}{m - 1},$$
Find all positive integer m, n that suffice.

Motivation for Problem 83. Squeeze Theorem seems nice here - we could use cyclotomic polynomials, but just finding a way to show no solutions is easier.

Solution 83 by Eric Peng and Dylan Yu.
This becomes $n^2 = m^4 + m^3 + m^2 + m + 1$. We try to bind the RHS by squeezing it between 2 squares. We can do this by seeing $(m^2 + \frac{1}{2}m)^2 = m^4 + m^3 + \frac{1}{4}m^2$, while $(m^2 + \frac{1}{2}m + 1)^2 = m^4 + m^3 + \frac{9}{4}m^2 + m + 1$, so we have that $m^2 + \frac{1}{2}m < n < m^2 + \frac{1}{2}m + 1$. This means that when m is even, there are no solutions, but when m is odd,

$$n = m^2 + \frac{1}{2}m + \frac{1}{2},$$
$$n^2 = m^4 + m^3 + \frac{5}{4}m^2 + \frac{1}{2}m + \frac{1}{4} = m^4 + m^3 + m^2 + m + 1,$$
$$\frac{1}{4}m^2 - \frac{1}{2}m - \frac{3}{4} = 0,$$
$$m^2 - 2m - 3 = 0,$$
$$(m-3)(m+1) = 0,$$
$$m = -3, 1.$$

m cannot be -3 because it is positive, and $m = 1$ is impossible due to the denominator being $m - 1$. Thus, $\boxed{\text{No Solutions}}$.

Problem 84. Let $a_1, a_2, \ldots, a_k$ be a positive integer sequence. Find all ks for which there exists n such that

$$(a_i + n, a_j + n) = 1,$$

For all i, j where $i < j$.

Solution 84.

Problem 85. Given

$$p(p+1) + q(q+1) = r(r+1),$$

Find all prime p, q, r that suffice.

Solution 85.

Problem 86. Prove that for all n, there exists infinitely many m where

$$n = \pm 1^2 \pm 2^2 \pm \ldots \pm m^2.$$

Solution 86.

Problem 87. Find all n, k where

$$n | 2^n + k.$$

(Solve for only certain cases - some are unsolved problems.)

Solution 87.

Problem 88. Given

$$(m^2 - n^2)^2 = 1 + 16n,$$

Find all positive integer m, n that suffice.

Solution 88.

Problem 89. Find 2019 digit positive integers A, B, C where

$$A + B = C.$$

Motivation for Problem 89. 2019 is not useful here. Simply replace it with n.
Solution 89.

Problem 90. Find an infinitely long, positive integer, increasing sequence $a_1, a_2, \cdots$ where $a_1 = 2019$, and

$$a_1^2 + a_2^2 + \ldots \equiv 0(a_1 + a_2 + \ldots).$$

Dr. Namli Winter Session Solutions (2019-2020)

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Day 1

Property 1.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$$

Property 2.

$$\sum_{n=0}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

Property 3.

$$(x^n)' = nx^{n-1} \forall n$$

Derivative = polynomial killer

Property 4.

$$(P_n)^{(n)} = 0,$$

Where P_n is a polynomial of degree n, and $(f(x))^{(n)}$ is the nth derivative.

Definition 1.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Product Rule of Derivatives.

$$(f \cdot g)' = f' \cdot g + g' \cdot f$$

Division Rule of Derivatives.

$$(\frac{f}{g})' = \frac{f' \cdot g - g' \cdot f}{g^2}$$

Property 5.

$$\int x^n dx = \frac{x^{n+1}}{n}, n \neq -1$$

$$\int \frac{1}{x} dx = \ln x$$

Integral = polynomial reviver

Problem 1. Compute

$$\sum_{n=0}^{\infty} nx^n.$$

Motivation for Problem 1. We could do an arithmetic-geometric sequence summation, or just take the derivative and multiply by x.

Problem 2. Compute

$$\frac{\sum_{n=0}^{90} f(x)}{90},$$

Where $f(x)$ is

(a) $\sin n$
(b) $\cos n$
(c) $\tan n$
(d) $\sec n$
(e) $\csc n$
(f) $\cot n$

Motivation for Problem 2. It might be nice to multiply by $\sin 1$, or whatever function it is.

Problem 3. Given $\sin a + \cos a$ is rational, prove $\sin^n a + \cos^n a$ is rational.

Solution 3 by Dylan Yu. If we square the first equation, we get

$$\sin^2 a + \cos^2 a + 2 \sin a \cos a = \text{rational},$$

$$\sin a \cos a = \text{rational}.$$

Thus, we can proceed by (strong) induction. The base case $n = 1$ is correct by definition. By multiplying $\sin^n a + \cos^n a$ by $\sin a + \cos a$, we get

$$\sin^{n+1} a + \cos^{n+1} a + \sin a \cos a(\sin^{n-1} a + \cos^{n-1} a) = \text{rational},$$

And because $\sin^{n-1} a + \cos^{n-1} a$ and $\sin^n a + \cos^n a$ are rational, $\sin^{n+1} a + \cos^{n+1} a$ must be rational.

Problem 4. Given infinite triominoes, prove that we can fill in an $2^k \times 2^k$ grid with one missing square.

Motivation for Problem 4. Induction seems good here.

Problem 5. Prove

$$a^3 + b^3 + c^3 \geq 3abc$$

For positive real a, b, c.

Problem 6. Compute

$$\sqrt{1+\sqrt{2+\sqrt{3+\ldots}}}.$$

Problem 7. Compute

$$\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}.$$

Solution 7 by Dylan Yu. Let the sum be S. Thus,

$$S^2 - 2 = S,$$
$$S^2 - S - 2 = 0,$$
$$S = 2, -1.$$

The sum cannot be a negative number, because it is under a square root; thus, the answer is $\boxed{2}$.

Problem 8. Compute

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}.$$

Solution 8 by Albert Zhu.

If we let $a = \sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$, then we must have $\sqrt{2}^a = a$, from which we arrive at $a = 2, 4$. I claim that $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdot^{\cdot^{\cdot}}}}}$ cannot be more than 2 for a finite number of $\sqrt{2}$s, which I will prove using induction on the number of $\sqrt{2}$s in the expression. Our base case follows from the observation that $1 < 2 < 4 \implies 1 < \sqrt{2} < 2$ so let x be a tower of n $(n \in \mathbb{Z}^+)$ $\sqrt{2}$s such that its value is less than 2. Note that $\sqrt{2}^x < \sqrt{2}^2 = 2$, so our inductive step is complete and thus $\boxed{2}$ is the only solution.

Problem 9 (Ramanujan). Compute

$$\sqrt{1+2\sqrt{1+3\sqrt{\ldots}}}.$$

Problem 10. For what value of x is the following maximum?

$$x^{x^{x^{\cdots}}}$$

Solution 10 by Eric Peng.

This is equivalent to finding a if $x^a = a$. Taking the natural log of both sides gives $\ln(x) = \dfrac{\ln(a)}{a}$. We are trying to maximize so we look for when the derivative is 0. By the quotient rule the derivative is $\dfrac{1-\ln(a)}{a^2}$, which is 0 when $a = e$, leading to $x = \boxed{e^{\frac{1}{e}}}$.

Problem 11. Compute

$$1 + -1 + 1 + \ldots.$$

Problem 12. Compute

$$\frac{2}{3-\frac{2}{3-\frac{2}{\cdots}}}.$$

Note: Such a quantity is called Cesaro Summable: Cesàro summation

Problem 13. Find all positive integers x, y such that

$$615 + x^2 = 2^y.$$

Solution 13 by Rich Wang.

We claim that $\boxed{\text{(x,y)=(59,12)}}$ is the only solution.

To start our proof, We claim that y must be even. We prove this by taking the equation mod 3. We know that $615 \equiv 0 \pmod 3$, so we now just have that $x^2 \equiv 2^y \pmod 3$. However, squares can only be $0, 1 \pmod 3$, while 2^y is congruent to $2 \pmod 3$ when y is even and

$1 \pmod 3$ when y is odd. Thus, y must be odd in order for the left and right hand sides to be congruent mod 3.

From here, substitute $y = 2z$.

Rearranging the equation and using the difference of squares formula, we have that $615=(2^z-x)(2^z+x)$. Note both factors must be positive, as 2^z+x is always positive.

We factor $615=3*5*41$. Since $2^z+x>2^z-x$, there are only a few cases of the factors we have to check.

Doing the cases, we have that:

Case 1: $2^z-x=1, 2^z+x=615$- Gives $2*2^z=616$ by adding the two equations, no solution.

Case 2: $2^z-x=3, 2^z+x=205$- Gives $2*2^z=208$ by adding the two equations, no solution.

Case 3: $2^z-x=5, 2^z+x=123$- Gives $2*2^z=128$ by adding the two equations. This gives us that $x=59, z=6$, or a solution set of $(x,y)=(59,12)$. We can easily plug this back into the equation and verify that it works.

Case 4: $2^z-x=15, 2^z+x=41$- Gives $2*2^z=56$ by adding the two equations, no solution.

Thus our only solution is $\fbox{(x,y)=(59,12)}$

Problem 14. Given

$$x(x+1) | y(y+1),$$

And

$$x \nmid y,$$
$$x \nmid y+1,$$
$$x+1 \nmid y,$$
$$x+1 \nmid y+1,$$

(a) Find one such possible positive integer (x, y)
(b) Prove that there are infinitely many positive integer solutions (x, y), and provide one such set of infinite solutions.
(c) Find all positive integer solutions. (Dr. Namli couldn't even solve this one)

Solution 14 by Rich Wang.
Why did I get this one… we haven't even solved it :(

(a) We claim that the set $(x, y) = (14, 35)$ is valid. It is easy to verify that this works.
(b) We will prove that, given one solution (provided above), we can construct infinitely many solutions out of it.

Let $x = a \cdot b$, $x + 1 = c \cdot d$, $y = a \cdot d \cdot m$, and $y + 1 = b \cdot c \cdot n$, where a, b, c, d are all relatively prime to m, n, and all (a, b, c, d, m, n) are positive integers greater than 1. Also, a, b, c, d, and m, n are coprime. We will prove we can construct another set (x, y) with these properties given an initial set (x, y) that satisfies these properties. Note $(14, 35)$ satisfies these constraints when $a = 7, b = 2, c = 3, d = 5, m = 1, n = 6$.

Now, since $\gcd(x, x + 1) = \gcd(y, y + 1) = 1$,

Problem 15. Given
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$$
Has exactly 2005 positive solutions for x, y, prove N is a perfect square.

Motivation for Problem 15. This looks suspiciously like SFFT.

Solution 15 by Dylan Yu. We can rearrange the equation to get
$$\frac{x+y}{xy} = \frac{1}{N},$$
$$xy - Nx - Ny = 0,$$
$$(x - N)(y - N) = N^2.$$
Because x, y are positive, there cannot be a negative solution, so if
$$N = p_1^{e_1} p_2^{e_2} \ldots p_n^{e_n},$$
The number of solutions, which is equal to the number of factors of N^2 is
$$(2e_1 + 1) \ldots (2e_n + 1) = 2005 = 5 \cdot 401.$$
Thus, either $N = p^{1002}$ or $N = p_1^2 p_2^{200}$. In either case, N is a perfect case, so we are done.

Problem 16. Find all positive integer n such that
$$\sqrt{\frac{1^2 + 2^2 + \ldots + n^2}{n}}$$

Is a positive integer.

Solution 16 by Ray Tang. The equation be equal to K in Z. Square the equation and simplifying, we get that $(n+1)(2n+1)=6k^2$. Now Expanding the LHS we know that $2n^2+3n+1=6k^2 so 48k^2=16n^2+24n+8$. Now we complete the square so we know that $(4n+3)^2=48k^2+1$. We do difference of square to get that (4n+3-krt48)(4n+3+krt48)=1. The first root to this is k=1 4n+3=7. We can take this to any power for all powers that get an integer n.

Problem 17. Find all positive integer n such that

$$\frac{98^n-68^n-31^n+1}{2010}$$

Is an integer.

Solution 17 by Jeffrey Chen. Since $2010=2\cdot 3\cdot 5\cdot 67$, we need to check $98^n-68^n-31^n+1$ in each mod. In $\mod 2$, we have $-1^n+1\equiv 0 \mod 2$. For $\mod 3$, we have $2^n-2^n-1^n+1\equiv 0 \mod 3$. For $\mod 5$, we have $3^n-3^n-1^n+1\equiv 0 \mod 5$. Finally, for $\mod 67$, we have $31^n-1^n-31^n+1^n\equiv 0 \mod 67$. Since this equation is congruent to $0 \mod 2,3,5,67$, it is also congruent to $0 \mod 2010$, therefore $\dfrac{98^n-68^n-31^n+1}{2010}$ is always an integer.

Problem 18. Given p is prime, and m,n are positive integers, find all m,n,p such that

$$(p-1)(p^n+1)=4m(m+1).$$

Solution 18 (i) by Dylan Yu. We expand to get $p^{n+1}-p^n+p-1=4m^2+4m$, so

$$p^{n+1}-p^n+p=(2m+1)^2,$$

Which means $p|2m+1$. Thus,

$$p^{n+1}-p^n+p=p(p^n-p^{n-1}+1)\equiv 0 \pmod p,$$

Which is obviously impossible since $p^n-p^{n-1}+1$ cannot be divisible by p. Thus, there are no solutions.

Solution 18 (ii) by Jeffrey Chen. We expand to get $p^{n+1}-p^n+p-1=4m^2+4m \Rightarrow p^{n+1}-p^n+p=4m^2+4m+1=(2m+1)^2$. Since there is a factor of p on the left hand side, we have $p|(2m+1)^2 \Rightarrow p|(2m+1)$. Then there is a factor of p^2 on the right hand side, so there must be a factor of p^2 in $p^{n+1} - p^n + p = p(p^n - p^{n-1} + 1)$. Therefore, $p|(p^n - p^{n-1} + 1)$, and since $n \geq 1$, $p|p^n$, so $p|(1 - p^{n-1})$. If $p > 1$, then there can't be a factor of p in $1 - p^{n-1}$, therefore $n = 1$, which satisfies $p|(p^n - p^{n-1} + 1)$. If $n = 1$, we have $p^2 = (2m+1)^2$, so $p = 2m + 1$, therefore our only solutions are $(p, m, n) = (p, \frac{p-1}{2}, 1)$ for $p \geq 3$.

Problem 19. Given a, b, m, n are positive integers, find all solutions to

$$a^m b^n = (a+b)^2 + 1.$$

Solution to Problem 19 by Ray Tang.

Problem 20. Given m, n, k are positive integers, find all solutions such that

$$k = \frac{m+1}{n} + \frac{n+1}{m}$$

Is an integer.

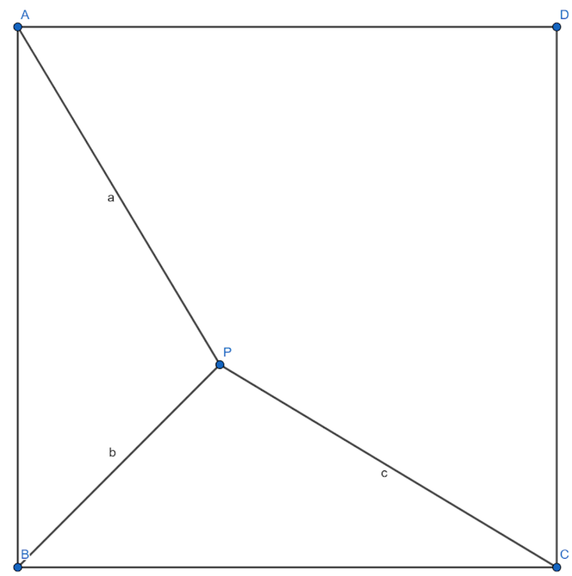
Solution 20 by Albert Zhu. We do casework on the value of m relative to n.

Case 1: $m = n$. In this case, we get $k = \frac{2(n+1)}{n}$, which is only an integer when n is $1, 2$ so \$(m,n,k)=(1,1,4),(2,2,3).$n, k) = (1, 1, 4), (2, 2, 3).$k)=(1,1,4),(2,2,3).$4), (2, 2, 3).$,(2,2,3)..\$

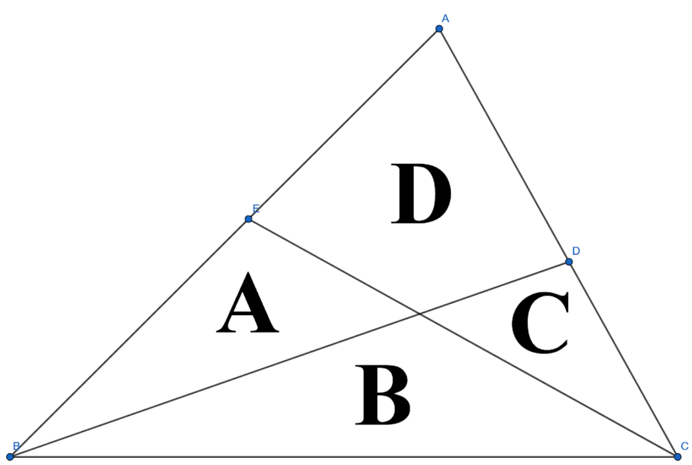
Case 2: $m = n + 1$. In this case, we also get $k = \frac{n+2}{n} + \frac{n+1}{n+1} = \frac{2(n+1)}{n} \implies (m, n, k) = (2, 1, 2), (3, 2, 3)$ as well.

Case 3: $m \geq n + 2$. In this case, we get $k = \frac{m^2 + m + n^2 + n}{mn}$. For this to be an integer, we require $m | n^2 + n = n(n+1)$, but since m is more than both factors and $\gcd(n, n+1) = 1$, it must be equal to $n(n+1)$. Thus, we require that $\frac{m^2 + m + n^2 + n}{mn} = \frac{m+2}{n} = \frac{n^2+n+2}{n}$ to be an integer. However, this clearly requires that $n = 2$, so the solution is $(6, 2, 4)$.

Problem 21. Which is bigger, e^{π}, or π^{e}?
Solution 21 by Jefreyll Chen
e^pi is bigger by duh theorem. Q.E.D.
Problem 22. Find the area of the following square in terms of a, b, c:



Problem 23. Find the relationship between areas A, B, C, D.



Solution 23 by Rich Wang. Set the intersection of AD and CE as point F. Now draw in line FA, and call $[FAE] = x, [FAD] = y$. From here we use area ratios.

We know $\dfrac{BF}{FD} = \dfrac{[BFC]}{[FCD]} = \dfrac{B}{C}$ and $\dfrac{CF}{FE} = \dfrac{[CFB]}{[BEF]} = \dfrac{B}{A}$.

Because of this, we know that:
$\dfrac{A+x}{y} = \dfrac{[BAF]}{[FAD]} = \dfrac{BF}{FD} = \dfrac{B}{C}$ and $\dfrac{C+y}{x} = \dfrac{[CAF]}{[CAE]} = \dfrac{CF}{FE} = \dfrac{B}{A}$.

From here we solve for x, y in terms of A, B, C, by using our equations of $\dfrac{A+x}{y} = \dfrac{B}{C}$ and $\dfrac{C+y}{x} = \dfrac{B}{A}$.

From the first equation we get that $y = \dfrac{(A+x)\cdot C}{B}$. Plugging this into our second equation gives us $\dfrac{C + \frac{(A+x)\cdot C}{B}}{x} = \dfrac{B}{A}$.
Getting a common denominator and eliminating fractions, we have

$$A \cdot B \cdot C + A \cdot C \cdot (A+x) = B^2 \cdot x.$$

We isolate x, to get that $x = \dfrac{ABC + A^2C}{B^2 - AC}$.

Now we just plug this back into our first equation and solve for \$y\$.

$$y = \frac{(A+x)\cdot C}{B}$$

$$y = \frac{(A + \frac{ABC+A^2C}{B^2-AC})\cdot C}{B}$$

$$y = \frac{(\frac{AB^2-A^2C+ABC+A^2C}{B^2-AC})\cdot C}{B}$$

$$y = \frac{(\frac{ABC+AC^2}{B^2-AC})\cdot C}{B}$$

$$y = \frac{ABC + AC^2}{B^2 - AC}.$$

Adding up x and y, which equals D, we have our answer,

$$\boxed{D = \frac{2ABC + AC^2 + A^2C}{B^2 - AC}}$$

Problem 24. In the following figure, find $\angle PBC$.

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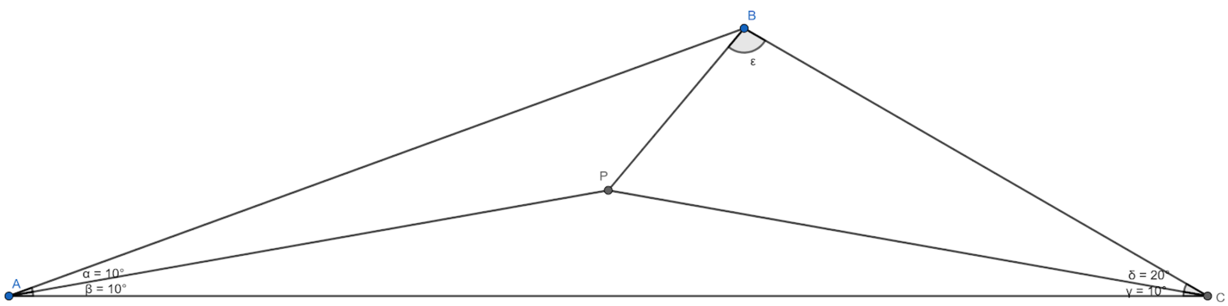
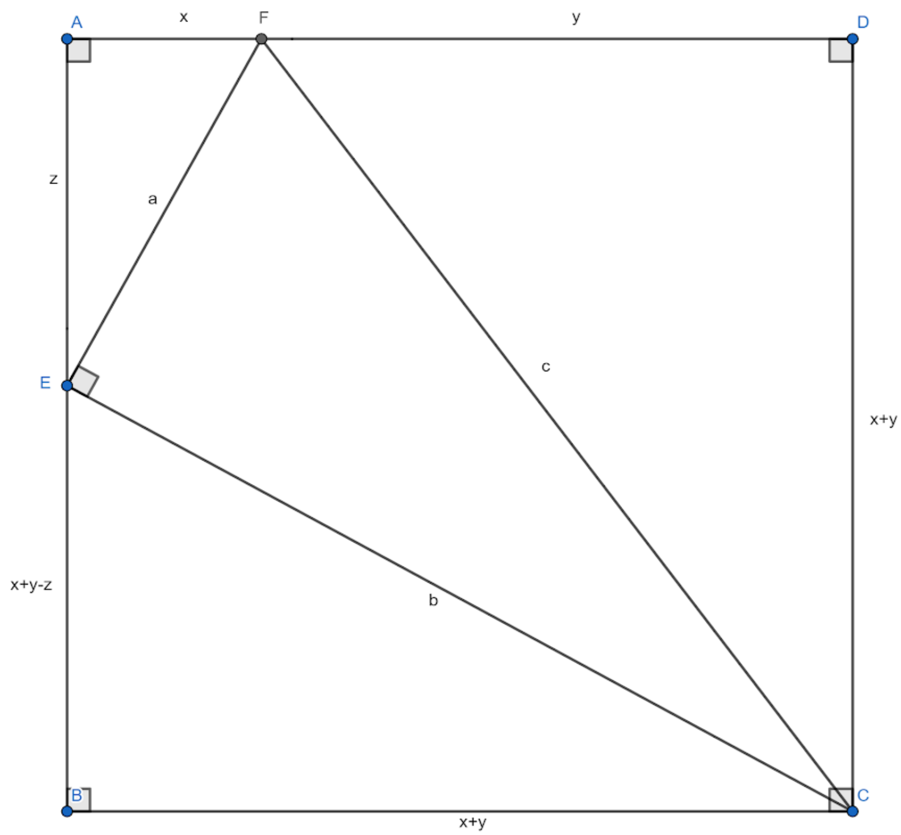


Solution 24 by Jeffrey Chen. By Trigonometric Ceva's, we have
$\frac{\sin 10}{\sin 10} \cdot \frac{\sin 10}{\sin 20} \cdot \frac{\sin \alpha}{\sin 130 - \alpha} = \frac{1}{\cos 10} \cdot \frac{\alpha}{\sin 130 - \alpha} = 1$. Plugging in $\alpha = 100$ works, therefore $\angle PBC = 100$

Problem 25. Find the relationship between a, b, c, d, e.

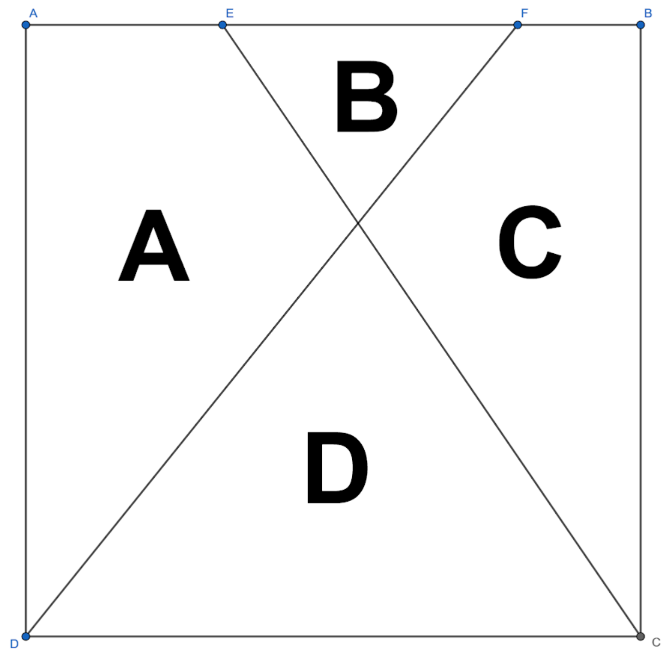
Solution 25 by Ray Tang. Extend EC to a point F such that BF perpendicular to EC. Since we have a rectangle, so we know that $BD = FE$.

Problem 26. Find x, y, z in terms of a, b, c.



Solution 26 by Ray Tang. By similarity, $\frac{x}{x+y-z} = \frac{z}{x+y} = \frac{a}{b}$. We also know by Pythagorean theorem that $x^2 + z^2 = a^2$. Three variables and three equations gives us our answer.

Problem 27. Find the relationship between A, B, C, D.



Day 2

Property 6. On a sphere, the shortest surface path is through the great circle passing through the two points of interest.

Problem 28. There are 3 points on a circle. Show that 2 of them lie on a semicircle of the circle.

Solution 28 by Ray Tang. Start with one point and construct a diameter, we know that at least one of the points is on one of the sides of the diameter. So we are done.

Problem 29. There are 5 points on a sphere. Show that 3 of them lie on a great hemisphere of the sphere.

Solution 28 by Ray Tang. Start with one point and construct a diameter, we know that at least one of the points is on one of the sides of the diameter. So we are done.

Problem 30. Find a figure such that each point connects to n other points, each with a distance of 1. Specifically, find $n = 2$ and $n = 3$.
Solution 30 by Jeffrey Chen. For n = 2, we can take an equilateral triangle with side length 1. For n =3, take a cube, and project it onto the plane. The result is two squares with side length 1, and 4 lines connecting vertice pairs with length 1.

Problem 31. A cone with radius r and height h has a point on the circumference of the base and a point on the cone not on the base. Find (a) the shortest distance to that point (i.e. the method to find it), and (b) given the point on the circumference of the base find the point that will minimize the distance to get to that point, given one must travel in a straight line, and find the distance in terms of r and h.

Problem 32. Given

$$\overline{a_1 a_2 \ldots a_n} = k \cdot \overline{a_n a_1 a_2 a_3 \ldots a_{n-1}},$$

Find all $k, \overline{a_1 a_2 \ldots a_n}$ that satisfy. **Unfinished!**

Casework for $k = 4$ to $k = 9$ by Eric Peng.

Let all of the digits except a_n be a, and a_n be b. We have $k \cdot (10^n \cdot b + a) = 10a + b$.

Case $k = 4$:

We see that if $b > 2$, then the number of digits increases after multiplication, meaning b must be 1 or 2. However, if $b = 1$, the two sides have different parities. Therefore, $b = 2$. This gives us $a = \dfrac{4 \cdot 10^{n-1} - 1}{3}$, which has n digits, which is too many for a.

Case $k = 5$, $k = 6$, and $k = 8$:

For this case and forward, we see that $b = 1$ because otherwise, the multiplication will result in us having an extra digit. We take the equation in mod 5 for $k = 5$, and mod 2 for $k = 6$ and $k = 8$, resulting in a contradiction for all 3 when the LHS does not equal the RHS.

Case $k = 7$ and $k = 9$:

Rearrange the equation. This results in $a = \dfrac{7 \cdot 10^{n-1} - b}{3}$, and $a = 9 \cdot 10^{n-1} - 1$, which have too many digits.

Problem 33. Given

$$\overline{a_1 a_2 a_3 \ldots a_n a_1 a_2 a_3 \ldots a_n} = x^2,$$

Find all $x, \overline{a_1 a_2 a_3 \ldots a_n}$ that satisfy.

Problem 34. Is there a k such that

$$2^n + k \notin P \forall n?$$

Solution 34 by Eric Peng.

If k is allowed to be even, it is easy to find a value for k, such as 8.

Problem 34b. Solve problem 34, except k is odd.

Problem 35. (17 or Bust Project) Is there a k such that

$$k \cdot 2^n + 1 \notin P \forall n?$$

Note: More information can be found at: Seventeen or Bust

Problem 36. Is there a k such that

$$k \cdot m^n + 1 \notin P \forall n?$$

Problem 37. Find n such that $n^4 + 4 \notin P \forall n$.

Problem 38. Given

$$1 = d_1 < d_2 < d_3 < d_4,$$

Where d_i are the smallest four divisors of n, and

$$n = d_1^2 + d_2^2 + d_3^2 + d_4^2,$$

Find n.

Solution 38 by Dylan Yu. Case bashing shows that $n = p^a$ for some prime p is impossible, so we try $n = p^a q^b x$, where x is some of the other divisors of n. We see that if $d_1 = 1, d_2 = p, d_3 = q, d_4 = pq$, then we have an easy solution of $p = 2, q = 5$. From here, we derive that $n = \boxed{130}$. Note that this does not guarantee all the solutions, but shows the motivation behind $n = 130$.

Problem 39. Given $d(n) = \sqrt{n}$, find n.

Problem 40. Given $d(n) = \sqrt[3]{n}$, find n.

Problem 41. If $1, 2, \ldots, \sqrt[3]{n}$ divides n, find n.

Problem 42. If $1, 2, \ldots, \frac{\sqrt{n}}{2}$ divide n, find n.

Problem 43. Given $d(n) = \frac{n}{12}$, find n.

Problem 44. Given

$$a_n = an + b_{n=1}^{\infty},$$

$$a_n \cap F_n = \emptyset,$$

Find the minimum value of a and b, where F_n is the Fibonacci sequence.

Day 3

Problem 45. Compute

$$\sum_{k=0}^{\infty} \arctan(\frac{2}{k^2}).$$

Problem 46. Find (a) one, then (b) infinitely many n such that it is the sum of two positive cubes but not the difference of two positive cubes.

Problem 47. Find the smallest prime p such that

$$p \neq |2^a - 3^b|,$$

Where a, b are nonnegative integers.

Problem 48. Given prime numbers $p, q < 2019,$ find all p, q such that

$$p|q^2 + 8, q|p^2 + 8.$$

Problem 49. For positive integers a, b, c, d,

$$a + b^2 + c^3 + d^4 = 2019.$$

What is the minimum value of $a + b + c + d$?

Problem 50. For a sequence $a_1, a_2, \cdots$, we are given $a_1 = 1, a_2 = 1, a_3 = 2.$ Given

$$a_{n+3} = \frac{a_n \cdot a_{n+1} + 7}{a_{n+2}},$$

Find a_n in terms of n.

Problem 51. Given

$$y^2 = x^3 + (x+4)^2,$$

For integers x, y, find all x, y.

Russian Guy Winter Session Solutions (2019)

Information about the Authors

Teacher and Problem Finder / Author: Russian Guy
Current Authors: Dylan Yu, Jeffrey Chen

Day 1

Problem 1. x, y, z, t are positive integers. Solve:
$$x^x + y^y = z^z + t^t.$$

Problem 2. A few natural numbers are written on the board. The sum of any two of them is a natural power of two. What is the largest possible number of distinct numbers among the problems on the board?

Problem 3. Prove that there is no tetrahedron in which every edge is a side of some obtuse angle of that tetrahedron.

Problem 4. Find all quadruples {a,b,c,d} of real numbers, in each of which any number is equal to the product of any two other numbers.

Problem 5. Prove that at least one of the bases of the perpendiculars dropped from the internal point of the convex polygon on its sides lies on the side itself, and not its extension.

Problem 6. There are n points on the plane. It is known that among any three of them there are two, the distance between which is not more than 1. Prove that two circles of radius 1 can be drawn on the plane, such that they cover all the points.

Problem 7. First-grader John knows only the number 1. Prove that he can write down a number that is divisible by 2019.

Problem 8. There are 6 points inside the 3×4 rectangle. Prove that among them there are two points such that the distance between them does not exceed $\sqrt{5}$.

Problem 9. There are 51 points in a square with side 1. Prove that some three of them can be covered with a circle of radius $\frac{1}{7}$.

Solution 9 by Jeffrey Chen. Divide the unit square into 25 squares each with side length $\frac{1}{5}$. By pigeonhole, there must be 3 points inside one of these squares, however, each square of side length $\frac{1}{5}$ can be covered with a circle of radius $\frac{1}{7}$, so we're done.

Problem 10. Given $n+1$ pairwise distinct natural numbers less than $2n$, where $n > 1$, prove that among them there are three such numbers that the sum of two of them is equal to the third.

Problem 11. There are $4n$ segments of length 1 inside a circle of radius n. Prove that you can draw a line parallel or perpendicular to a given line l and intersect at least two given segments.

Problem 12. Given endless checkered paper and a figure whose area is less than the area of one cell, prove that this figure can be put on the paper without covering any vertices of the cells.

Problem 13. Is there a convex 2020-gon in which all angles are expressed by an integer number of degrees?

Problem 14. One glass contains 5 spoons of milk, and the other glass contains 5 spoons of tea. A spoon of milk was taken from the second glass and put in the first, then mixed thoroughly. Next, a spoon of tea (with milk) was poured back into the second glass. Is there more milk in the first glass or more tea in the second glass? Will the answer change after 10 such transfusions?

Problem 15. Real numbers $a, b, c > 1$ are given. Prove that

$$a^a + b^b + c^c \geq a^b + b^c + c^a.$$

Problem 16. Each point A in the plane is assigned a real number $f(A)$. Given $f(M) = f(A) + f(B) + f(C)$, where M is the centroid of $\triangle ABC$, prove that $f(A) = 0$ for all points A.

Solution 16 by Dylan Yu. We can construct a figure where M is the centroid of $\triangle ABC$, D is the centroid of $\triangle BMC$, E is the centroid of $\triangle CMA$, and F is the centroid of $\triangle AMB$. From symmetry (or actual proof) we see that $\triangle DEF$ has centroid M. Thus,

$$f(A) + f(B) + f(C) = f(M) = f(D) + f(E) + f(F),$$

$$f(B)+f(C)+f(M)=f(D),$$
$$f(C)+f(A)+f(M)=f(E),$$
$$f(A)+f(B)+f(M)=f(F),$$

So plugging in the last 3 equations into the first, we get

$$f(M)=2(f(A)+f(B)+f(C))+3f(M)=5f(M),$$

So $f(M)=0$. Since any point can be a centroid $\boxed{f(A)=0}$ for all A.

Day 2

Problem 17. The cells of a 7×7 bored are chess-painted (alternating colors) so that the corners are black. One is allowed to repaint any two adjacent cells to the opposite color. Is it possible to repaint the entire board white using such operations?

Problem 18. The number -1 is written in the $b8$ cell of the chessboard, and the number 1 is written in all the other cells. One is allowed to change the sign of all the cells in a row or column. Prove that no matter how many times we do this, it is impossible to achieve all numbers in the table becoming positive.

Problem 19. The numbers $1,2,3,\ldots,19,20$ are written on the board. One is allowed to erase any two numbers a and b and instead write the number $a+b-1$. What number can remain on the board after 19 such operations?

Solution 19 by Dylan Yu. Note that the number of terms decrease by 1, as does the sum of the numbers. Thus, after 19 operations, there remains only one number, and the sum has decreased by 19. Thus, the answer is $1+2+\ldots+20-19=\boxed{191}$.

Problem 20. The circle is divided into 6 sectors, in which the numbers $1,0,1,0,0,0$ are put on the circle in that order clockwise. You can add 1 to any two numbers in neighboring sectors. Is it possible to make all numbers equal?

Problem 21. The set of all numbers a,b,c is replaced every second by $a+b-c,b+c-a,c+a-b$. At the beginning there is a set of numbers $2018,2019,2020$. Can a set of $2019,2020,2021$ be obtained?

Solution 21 by Dylan Yu. Note that the sum of all the numbers does not change. Thus, it is impossible for each number to increase by 1, because then the sum would increase by 3.

Problem 22. The numbers 1 through 20 are written on the board. One is allowed to erase any two numbers a and b and replace them with $ab + a + b$. What number can remain on the board after 19 such operations?

Solution 22 by Dylan Yu. We claim that adding one to all of the numbers left, multiplying all of them, and then subtracting one is invariant. This is true because
$(a+1)(b+1) = ab + a + b + 1$. Note the number of integers after 19 operations is 1. Thus, the answer is simply the answer to the method above - $2 \cdot 3 \cdot \ldots 21 - 1 = \boxed{21! - 1}$.

Problem 23. The KK42 device works like this: if you put four balls in it, then the second weighted ball will fall out into the first tray (i.e. a ball of weight b, if $a > b > c > d$), and the rest will fall out into the second tray. The device does not work with a different number of balls - only 4. There are 100 identical-looking (and identical-feeling) balls with different weights. They are numbered 1 through 100. Using the device, how can one find the heaviest ball, assuming you can only use the device 100 times?

Problem 24. Given a 1000-digit number with no zeroes, prove that from this number you can delete several (or none) last digits so that the resulting number is not a natural power less than 500 (a^1 is not considered a power).

Problem 25. The numbers 1 through 1000 are written on the board. One is allowed to erase any two numbers a and b and write the numbers ab and $a^2 + b^2$ instead. Is it possible with such operations to ensure that among the numbers written on the board, there are at least 700 that are the same?

Problem 26. Petya thinks of a natural number N, and Vasya wants to guess it. Petya tells Vasya the sum of the digits of the number $N+1$, then the sum of the digits of the number $N+2$, and so on. Is it true that sooner or later Vasya (who is completely rational) will find N?

Solution 26 by Jeffrey Chen. Observe that if Petya says 1, then that number must be a power of 10. The next time Petya says the number 1, it must be the next power of 10, therefore Vasya can count the amount of times Petya speaks between these two powers of 10, then count backwards to get N

Problem 27. Prove that if one chooses $n+1$ numbers from the numbers 1 through $2n$, one of the $n+1$ numbers must divide one of the other $n+1$ numbers.

Solution 27 by Jeffrey Chen. Divide the set of $2n$ numbers into subsets of $(1, 2, 4, 8, \cdots), (3, 6, 12, 24, \cdots), (5, 10, 20, \cdots), (7, 14, 28, \cdots)\cdots$. In other words, divide the set of $2n$ numbers into subsets where if you take a number and divide it by the highest power of 2 that divides that number, the resulting number is the same for all of the numbers in its subset. Because there are a total of n of these subsets, by pigeonhole, there must be 2 numbers in the same subset, therefore those two numbers divide each other, so we're done.

Problem 28. The positive integers are partitioned into finitely many subsets. Show that some subset S contains infinitely many multiples of n, where n is every positive integer.

Solution 28 by Jeffrey Chen. Take the set S $(1!, 2!, 3!, \cdots n!, (n+1)!, \cdots)$. Because there are finitely many subsets, and an infinite amount of elements in set S, there must be one subset that contains infinitely many elements of S. In that case, that set contains a bunch of really large factorials, which are all multiples of each n.

Problem 29. Let M be the set of the numbers 1 through 1000, and let X be a subset of S. Let a_X be the sum of the smallest and largest element. Find the arithmetic mean of all a_X, i.e. the sum of the largest and smallest elements for all X.
Solution 29 by Jeffrey Chen. For each element e in set S, take 1001 - e, and put that into a new set K. Next, notice that $a_S + a_K = 2002$, therefore our average is 1001.

Problem 30. Several positive numbers are placed on a circle, each less than 1. Prove that you can divide the circle into 3 arcs, such that the sum of the numbers on the adjacent arcs differ by at most 1.

Dr. Nal Winter Session Solutions (2019-2020)

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Day 1

Theorem 1. There are no odd cycles in a graph if and only if it is bipartite.

Day 5

Combinatorial Identities

Identity 1. $\binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = 2^n$.

Identity 2. (Pascal's Identity) $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

Identity 3. (Stirling Numbers of the 1st Kind) $\left\{ {n+1 \atop k} \right\} = \left\{ {n \atop k-1} \right\} + k\left\{ {n \atop k} \right\}$.

Identity 4. (Stirling Numbers of the 2nd Kind) $\left[{n+1 \atop k} \right] = \left[{n \atop k-1} \right] + n\left[{n \atop k} \right]$.

Problem 1. Prove that

$$\sum_{k=0}^{n} \binom{n}{k} 2^k \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = \binom{2n+1}{n}.$$

Graph Theory Lemmas (from Sriram's Olympiad Combinatorics)

Lemma 1. $\sum d_i = 2k$, where d_i is the degree of vertex i, and k is the total number of edges.

Lemma 2. $\sum d_i^2 \geq \frac{(\sum d_i)^2}{n} \implies \sum d_i^2 \geq \frac{4k^2}{n}$, where n is the number of vertices.

Proof of Lemma 2. Cauchy-Schwarz solves the problem.

Lemma 3. $\sum \binom{d_i}{2} \geq \frac{2k^2}{n} - k$.

Proof of Lemma 3. Using Lemma 1 and Lemma 2, we get

$$\sum \binom{d_i}{2} = \frac{1}{2}\sum d_i^2 - d_i = \frac{1}{2}\sum d_i^2 - \frac{1}{2}\sum d_i \geq \frac{1}{2}\left(\frac{4k^2}{n} - 2k\right) = \frac{2k^2}{n} - k$$

Lemma 4. $\sum_{v_i, v_j \in E} (d_i + d_j) = \sum_{i=1}^{n} d_i^2$, where E is the set of edges.

Proof of Lemma 4. Each vertex of degree d_i has i edges connecting out of it, so each degree d_i is counted a total of d_i times in the LHS, therefore giving us d_i^2 on the RHS.

Lemma 5. $\sum w_i = \sum l_i, \sum \binom{w_i}{2} = \sum \binom{l_i}{2} \implies \sum w_i^2 = \sum l_i^2$.

Problem 1. Show that a graph with n vertices and k edges has at least $\boxed{\frac{k(4k - n^2)}{3n}}$ triangles.

Problem 2. Let G be a graph with E edges, n vertices, and no cycles of 4. Show that $E \leq \frac{n}{4}(1 + \sqrt{4n - 3})$.

Dr. Nal Winter 701 Session Solutions (2020)

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Day 1

Problem 1. Prove that $d(n)$ is odd if and only if n is a perfect square, where $d(n)$ denotes the number of divisors of n .

Problem 2. (Wilson's Theorem) Prove that for all primes p ,

$$(p-1)! \equiv -1 \pmod{p}.$$

Problem 3. Solve $x^4 + x^3 = x^2 + x + 1 = 0$.

Problem 4. Prove that

$$(a+b)(b+c)(c+a) \geq 8abc$$

Is true for all positive numbers a, b, c with equality only if $a = b = c$.

Problem 5. Factor $a^3 + b^3 + c^3 - 3abc.$

Problem 6. If $x + y = xy = 3$, find $x^3 + y^3.$

Problem 7. Show that

$$\frac{b-c}{a} + \frac{c-a}{+}\frac{a-b}{c} = \frac{(a-b)(b-c)(c-a)}{abc}.$$

Problem 8. (Putnam) Let $ax^3 + bx^2 + cx + d = 0$ have integral coefficients and roots r, s, t . Find a polynomial equation with integer coefficients written in terms of a, b, c, d whose roots are $r^3, s^3, t^3.$

Problem 9. (USAMO) Factor the expression $(x+y)^7 - x^7 - y^7.$ (Note: this problem was mentioned before.)

Problem 10. (IMO 1995 / P2) Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Note for Problem 10. This reduces to something very similar to **Nesbitt's Inequality**:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}.$$

Problem 11. Show that the harmonic series diverges.

Problem 12. Show that the Riemann Zeta Function $\zeta(s)$ converges for all $s \geq 2.$

Problem 13. Let S be the set of positive integers which do not have a zero in their base-10 representation. Does the sum of the reciprocals of the elements in S converge or diverge? (Note: This problem is **Problem 38** from before)