MAΘ Diagram Perturbation

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1 Introduction

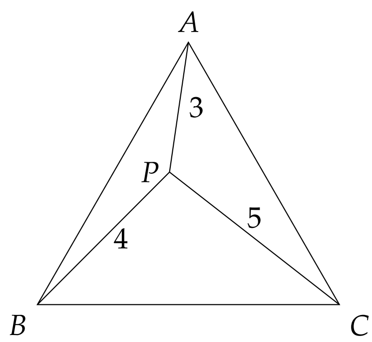
1.1 What is diagram perturbation?

The idea of ***diagram perturbation*** is to manipulate a diagram in a geometry problem in some way so that the result becomes easier to find. Let's try a few classic examples.

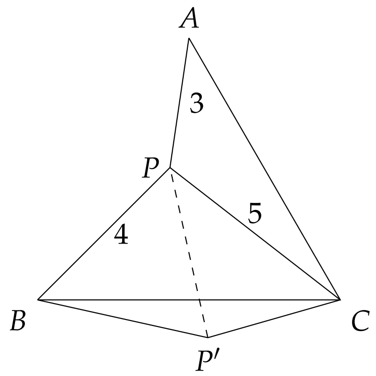
1.2 Rotations

Example 1.1
If a point P lies in an equilateral triangle ABC such that $AP = 3, BP = 4, CP = 5$, find the area of $\triangle ABC$.

Solution. Let's draw a figure:



Note that $3, 4, 5$ are special numbers, because they form the side lengths of a right triangle. Now we're going to rotate $\triangle APB$ around B such that A goes to C:



Let's angle chase. Note that $\angle ABP = \angle CBP'$ and $\angle APB + \angle CBP = 60°$. Thus, $\angle PBP' = 60°$, and combined with the fact $BP = BP' = 4$, we must have that $\triangle BPP'$ is an equilateral triangle. Furthermore, $CP' = 3$, implying $\triangle PP'C$ is a $3-4-5$ right triangle. Thus,

$$\angle APB = \angle CP'B = \angle BP'P + \angle CP'P = 60° + 90° = 150°,$$

and we can use Law of Cosines on $\triangle APB$ to get

$$AB^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cdot \cos 150° = 25 + 12\sqrt{3},$$

implying

$$[ABC] = \frac{AB^2\sqrt{3}}{4} = \boxed{\frac{25\sqrt{3}+36}{4}}.$$

□

We used a few strategies here:

- rotations (***cutting and pasting***),
- angle chasing, and
- trigonometry.

We will focus on rotations and similar ideas in this handout – in other words, rotations are an example of diagram perturbation.

> *Remark 1.2.* You will see a common pattern throughout problems of this type:
> - do something smart with the figure (i.e. diagram perturbation),
> - find some angles, and
> - apply length bashing techniques (e.g. trigonometry) to finish the problem.

Let's try one more example.

1.3 Reflections

I don't remember exactly how this was stated, but it's a problem any math enthusiast has heard of.

Example 1.3 (Folklore)
A man is at point A and wants to go to point B. However, he must first go to a river to get water, which is effectively a straight line. Note that points A and B are on the same side of the river. If he must go from A to a point on the river then to B, what is the path he should take?

Solution. Let's reflect B across the river line to get B'. Then if the point on the river he goes to is P, we are trying to maximize $AP + BP$, but $BP = B'P$ since it is a reflection, so

$$AP + BP = AP + B'P.$$

But the shortest distance from A to B' in general is just the line segment AB', and in this case, P would be the intersection of AB' and the river line. Thus, we reflect $B'P$ back across the line, and this is the path we should take. □

Convince yourself this is true. With rotations and reflections explained, let's move on to some harder ideas.

1.4 Isn't this just transformations?

You might be asking yourself, "I've known about rotations, reflections, translations, and dilations since 6th grade. What's different here?"

This is a good question. There actually is **no difference**. However, I've chosen to call this *diagram perturbation* as opposed to *transformations* because we aren't just reflecting the whole object. Finding what pieces to perturb and what auxillary lines to draw is wildly harder than moving all the pieces. We're going to focus on drawing extra lines here.

2 Parallelograms

Parallelograms are effectively just reflecting a triangle across one of its midpoints. Let's take advantage of the angles formed.

Example 2.1
Let M and N be the midpoints of $\overline{AB}$ and $\overline{AC}$ in triangle ABC. Prove $MN = \frac{1}{2}BC$ without using similar triangles.

Solution. Let L be the midpoint of BC. Then $MN \parallel LC$ and $NC \parallel ML$, implying $MNCL$ is a parallelogram. Thus, $MN = LC = LB$, and we're done. □

Example 2.2
Let M be the midpoint of $\overline{BC}$ in a triangle ABC. Given that $AM = 2$, $AB = 3$, $AC = 4$, find the area of ABC.

Solution. Reflect A across M to get A'. Then AA' and BC bisect each other, implying $ACA'B$ is a parallelogram. Furthermore, $AM = MA' = 2$, so $BA' = A'A = 4$, which means $\triangle AA'B$ is isosceles. We can easily find

$$[AA'B] = \frac{3\sqrt{55}}{4},$$

but we know that $[AA'B] = [AMB] + [MBA']$ and furthermore

$$[AMB] = [CMA] = [A'MC] = [BMA'],$$

implying

$$[ABC] = [AA'B] = \boxed{\frac{3\sqrt{55}}{4}}.$$

$\square$

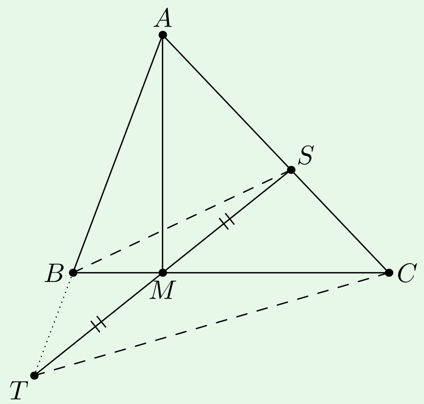
Example 2.3
A triangle ABC has medians of lengths m_a, m_b, m_c. Find the ratio of the area of the triangle formed by these medians to the area of triangle ABC.

Solution. Let l be the line through A parallel to BC, and let D, E, F be the midpoints of BC, CA, AB respectively. Furthermore, let A' be a point on l such that $AA' = EF$. We can easily prove through parallelograms that $\triangle A'CF$ is a triangle formed by the medians of $\triangle ABC$ (prove this yourself!). Thus, if we let S be one of the four equal areas formed by parallelogram $AA'EF$ and its diagonals, we have that $[AA'EF] = 4S$, and $[A'CF] = 6S$. Furthermore, since $[AEF] = 2S$, we have that $[ABC] = 8S$. Thus, the answer is

$$\frac{[AC'F]}{[ABC]} = \frac{6S}{8S} = \boxed{\frac{3}{4}}.$$

$\square$

Example 2.4 (NIMO 8.8)
The diagonals of convex quadrilateral $BSCT$ meet at the midpoint M of $\overline{ST}$. Lines BT and SC meet at A, and $AB = 91$, $BC = 98$, $CA = 105$. Given that $\overline{AM} \perp \overline{BC}$, find the positive difference between the areas of $\triangle SMC$ and $\triangle BMT$.



Solution. Let's get rid of B and C first. Set $\beta = \angle BAM$, $\gamma = \angle CAM$ and note that $\sin\beta = \frac{5}{13}$ and $\sin\gamma = \frac{3}{5}$. Compute $AM = 84$. Now, let A_1 be the reflection of A over M.

We can compute

$$\begin{aligned}
[AST] &= [AA_1T] \\
&= \frac{AM^2 \sin\beta \sin\gamma}{2\sin(\beta+\gamma)} \\
&= 7^2 \cdot 288 \cdot \frac{\frac{5}{13}\cdot\frac{3}{5}}{\frac{5}{13}\cdot\frac{4}{5}+\frac{12}{13}\cdot\frac{3}{5}} \\
&= 7^2 \cdot 288 \cdot \frac{15}{56} \\
&= 3780.
\end{aligned}$$

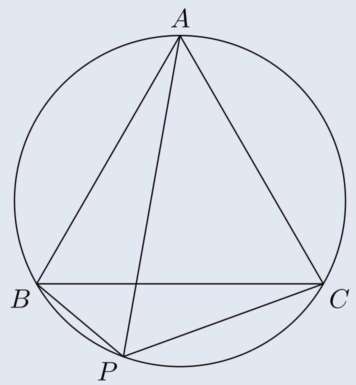
In that case, the desired quantity is $[ABC] - [AST] = 84 \cdot 7^2 - 3780 = \boxed{336}$. □

3 Equilateral Triangles

3.1 Same Point on the Same Side

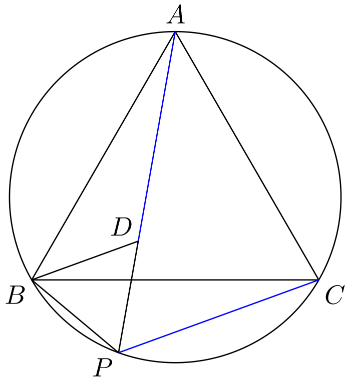
The idea is to split up a segment into two parts, or you can also think of it as adding two segments and seeing if that new segment can be found in the figure.

Theorem 3.1 (van Schooten's Theorem)
Let P be a point on the minor arc BC of equilateral triangle ABC. Then $PA = PB + PC$.



There is a quick solution using Ptolemy's theorem by applying it to quadrilateral $ABPC$. We'll try to prove this theorem without using Ptolemy's.

Proof. To prove that PA is the sum of PB and PC, let's try to split up PA into two segments. One will have length PB, and the other should have length PC. We pick the point D on segment PA such that $PD = PB$.



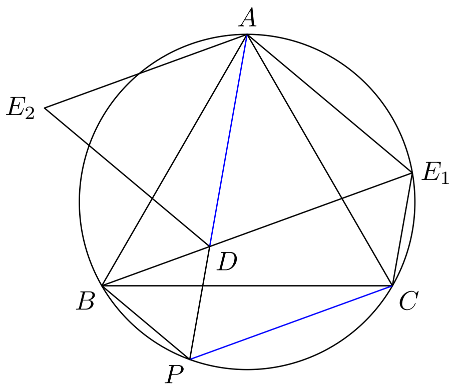
From construction, we have $PD = PB$, so to finish, we need to prove $AD = PC$. Let's look at what we can get from $PD = PB$. If we draw BD, it doesn't just look like triangle BDP is isosceles, but it looks like it's equilateral too.

Angle chasing, we get

$$\angle BPD = \angle BPA = \angle BCA = 60^\circ.$$

Since $\triangle BDP$ is isosceles, the remaining two angles are 60°, making it equilateral.

We can show that $AD = PC$ by constructing a similar equilateral triangle. Let E be the point such that $\triangle ADE$ is equilateral. But there's a problem: there are *two* possible choices of E on opposite sides of AD. Let's draw both and see what happens.



Surprisingly, it looks like E_1 lies on the circumcircle. It even looks like B, D, and E_1 are collinear! It also seems that this line is parallel to PC, which would make DE_1CP is a parallelogram. In fact, if it was a parallelogram, we'd be done.

We will try to prove these:

1. If DE_1CP is a parallelogram, then $AD = PC$.

2. DE_1CP is a parallelogram.

For the first one, if it was a parallelogram, then $PC = DE_1$ because they're opposite sides. But $DE_1 = AD$ because $\triangle ADE_1$ is equilateral, so that finishes the proof. The second one's a bit harder, and needs some more steps. Try to show them individually:

- *B, D and E_1 are collinear*: $\angle ADE_1 = 60^\circ = \angle BDP$.

- *E_1 lies on the circumcircle*: $\angle AE_1B = \angle AE_1D = 60^\circ = \angle ACB$.

- *DE_1 and PC are parallel*: $\angle ADE_1 = 60^\circ = \angle ABC = \angle APC$.

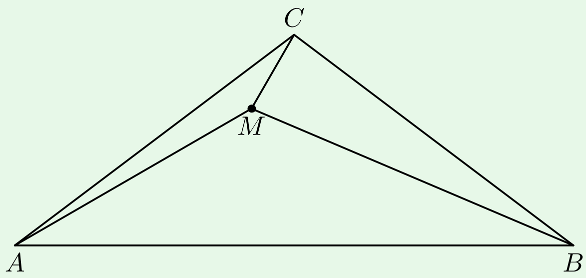
- *DP and E_1C are parallel*: $\angle BE_1C = \angle BAC = \angle PAE_1 = 180^\circ - \angle PCE_1$.

This gives us the desired result. □

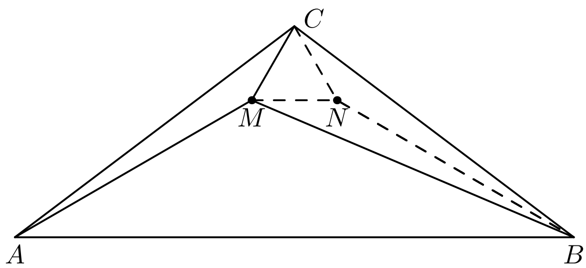
3.2 Reflections

Let's just dive into an example:

Example 3.2 (AIME I 2003/10)
Triangle ABC is isosceles with $AC = BC$ and $\angle ACB = 106^\circ$. Point M is in the interior of the triangle so that $\angle MAC = 7^\circ$ and $\angle MCA = 23^\circ$. Find the number of degrees in $\angle CMB$.



Solution. Let's reflect M across the perpendicular from C to AB to get N:



Then obviously $\angle CBN = 7^\circ$ and $\angle BCN = 23^\circ$. Thus,

$$\angle MCN = 106^\circ - 2 \cdot 23^\circ = 60^\circ.$$

Furthermore, since $\triangle AMC$ and $\triangle BNC$ are congruent by ASA, we must have that

$$CM = CN.$$

Hence $\triangle CMN$ is an equilateral triangle, so $\angle CNM = 60^\circ$. Thus

$$\angle MNB = 360^\circ - \angle CNM - \angle CNB = 360^\circ - 60^\circ - 150^\circ = 150^\circ.$$

We now see that $\triangle MNB$ and $\triangle CNB$ are congruent. Therefore, $CB = MB$, so $\angle CMB = \angle MCB = \boxed{83^\circ}$. □

4 Translations

We've seen how rotation and reflection are useful. What about translation? In this case, we're just going to move **one point** and see how everything else follows. Here's a problem I came up with to demonstrate this:

Example 4.1
An equilateral triangle ACK is located inside a regular decagon $ABCDEFGHIJ$. If the area of the decagon is 2020, find the area of $HIJAK$.

Solution. Let O be the center of this decagon. Note that O lies on the line BG, as does K. Note that $AH \parallel BG$, so O to AH is the same distance as K to AH. Thus, by same base-same height, we must have $[AHO] = [AHK]$. Thus,

$$[HIJAK] = [HIJA] + [AHK] = [HIJA] + [AHO] = [HIJAO],$$

which is equivalent to three-tenths of the decagon (because $[HIJAO] = [HIO] + [IJO] + [JAO]$, and each of these are equal isosceles triangles, and the total area of the decagon is 10 of these equal isosceles triangles). Thus, the answer is

$$\frac{3}{10} \cdot 2020 = \boxed{606}.$$

□

The idea was to take advatange of **same base-same height** (if the heights and bases are equal in length for two triangles, their areas are the same). This is the basis of moving a point, i.e. translation.

5 An IMO Shortlist Teaser

This is going to use the trick of creating a segment out of the sum of two segments.

⚠ We are **not** going to solve this problem! This is just a hint as to how to solve it. A solution is given here if you'd like to see the rest – we are just going to examine the construction. The other part, Ceva, is a part left to the reader to prove.

Example 5.1 (ISL 2000/G3)
ABC is an acute-angled triangle with orthocenter H and circumcenter O. Show that there are points D, E, F on BC, CA, AB respectively such that $OD + DH = OE + EH = OF + FH$ and AD, BE, CF are concurrent.

The constraint $OD + DH = OE + EH = OF + FH$ is extremely weird. It is well known that H and O share some nice connections. How could this help us?

In a triangle ABC, if we reflect the orthocenter H across BC, we get a point H_A that lies on the circumcircle. Thus, $HD = H_AD$. But wait a minute! What if we connect O to H_A? What if we let D be the intersection of OH_A and BC?

We realize that $R = OD + DH_A = OD + DH$! This tells us that our weird constraint is actually just saying that their sum is equal to R. From here we can apply the other constraint and use Ceva to solve the problem.

6 Strategies

- **Symmetry**: take advantage of this. In particular, you can create symmetry by applying transformations.

- **Angle chasing**: use cyclic quadrilaterals and similarity to get some angles. Transformations can also help.
- **Auxillary lines**: draw lines, because they help you find out what exactly you're missing.
  - **Parallograms**: construct them when dealing with midpoints or, more obviously, parallel lines.
  - **Equilateral triangles**: if there is one in the figure, refer back to the bullet point above about symmetry. If there isn't, try applying transformations to find a hidden one.
- **Same point on the same side**: make the sum of two segments into a segment. If $AX + AY$ appears on one side, construct a point Y' on ray AX such that $XY' = AY$. Then $AX + AY = AY'$, and AY' hopefully makes an isosceles triangle, parallelogram, isosceles trapezoid, or cyclic quadrilateral. Note that you can try constructing on ray AY instead. Try to construct in the opposite direction.
- **Same point on opposite sides**: make the difference of two segments into a segment.
- **Transformations**: obviously, these are useful. But how?
  - **Rotations**: cut and paste a bit of the figure and attach it elsewhere. Usually, you want to attach it so that two sides line up because they have the same length.
  - **Reflections**: reflect isosceles figures.
  - **Translations**: try moving one point and see what happens. We take advantage of same base-same height here.
- **Length bashing**: this is mostly just used for answer extraction. However, sometimes bashing out that two lengths are the same is a good indication something interesting is occurring.

7 Problems

7.1 Classics

These are examples of Langley's problems that might serve better as brainteasers. Here is a generalized way to solve them.

1. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20°$. Points D and E are on AC and AB respectively such that $\angle CBD = 40°$ and $\angle BCE = 50°$. Determine $\angle CED$.
2. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20°$. Points D and E are on AC and AB respectively such that $\angle CBD = 50°$ and $\angle BCE = 60°$. Determine $\angle CED$.
3. In isosceles triangle ABC, $AB = AC$ and $\angle BAC = 20°$. Points D and E are on AC and AB respectively such that $\angle CBD = 60°$ and $\angle BCE = 70°$. Determine $\angle CED$.
4. In convex quadrilateral $ABCD$, $\angle ABD = 12°$, $\angle ACD = 24°$, $\angle DBC = 36°$, and $\angle BCA = 48°$. Determine $\angle ADC$.

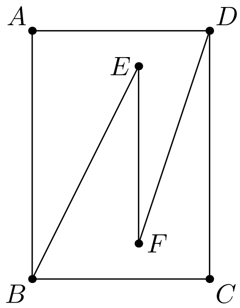
5. In convex quadrilateral $ABCD$, $\angle ABD = 38°$, $\angle ACD = 48°$, $\angle DBC = 46°$, and $\angle BCA = 22°$. Determine $\angle ADC$.

Have these shown up in contest? Yep!

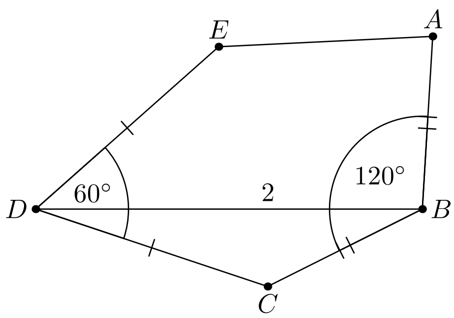
Problem 1 (AMC 10B 2008/24). In convex quadrilateral $ABCD$, $AB = BC = CD$, $\angle ABC = 70°$, and $\angle BCD = 170°$. Determine $\angle DAB$.

7.2 Parallelograms

Problem 2 (AIME 2011/2). In rectangle $ABCD$, $AB = 12$ and $BC = 10$. Points E and F lie inside rectangle $ABCD$ so that $BE = 9$,$DF = 8$,$\overline{BE} \parallel \overline{DF}$,$\overline{EF} \parallel \overline{AB}$, and line BE intersects segment $\overline{AD}$. The length EF can be expressed in the form $m\sqrt{n} - p$, where m,n, and p are positive integers and n is not divisible by the square of any prime. Find $m + n + p$.

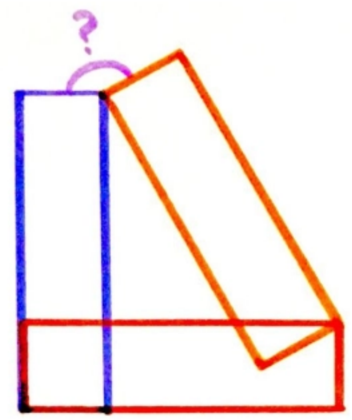


Problem 3. Let $ABCDE$ be a convex pentagon with $AB = BC$ and $CD = DE$. If $\angle ABC = 2\angle CDE = 120°$ and $BD = 2$, find the area of $ABCDE$.



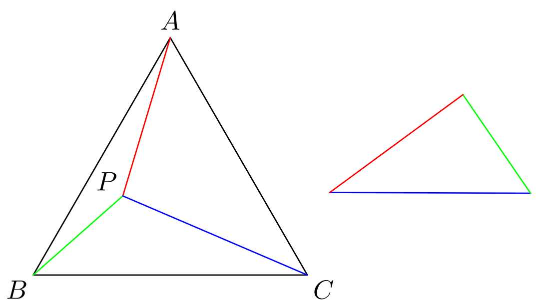
7.3 Equilateral Triangles

Problem 4 (Catriona Shearer). Find the angle labeled by the question mark.

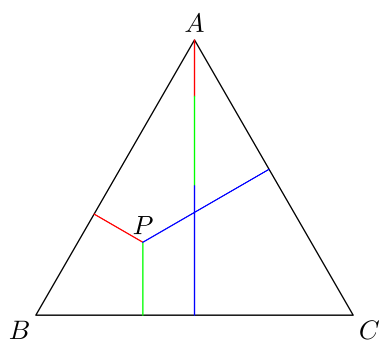


Remark 7.1. For the above problem, rotations would also work.

Problem 5 (Pompeiu's Theorem). Let P be a point *not* on the circumcircle of an equilateral triangle ABC. Then there exists a triangle with side lengths PA, PB, and PC.



Problem 6 (Viviani's Theorem). Let P be a point inside equilateral triangle ABC. Then the sum of the distances from P to the sides of the triangle is equal to the length of its altitude.

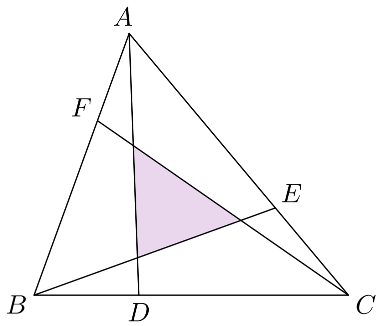


The following is not an equilateral triangle problem, but it is still has a similar idea nonetheless:

Problem 7 (One-Seventh Area Triangle). In triangle ABC, points D, E, and F lie on sides BC, CA, and AB respectively, such that

$$\frac{CD}{BD} = \frac{AE}{CE} = \frac{BF}{AF} = 2.$$

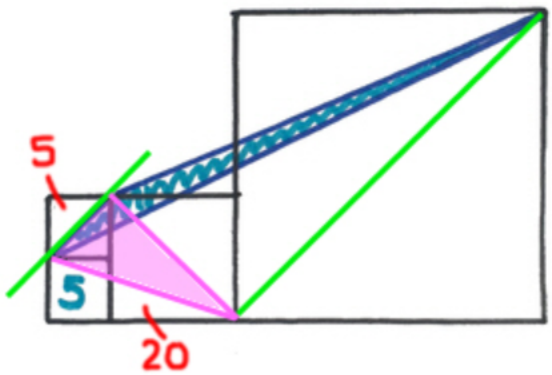
Then the area of the inner triangle formed by the lines AD, BE, and CF is one-seventh the area of ABC.



> *Remark 7.2.* A generalization of the above is Routh's theorem.

7.4 Translations

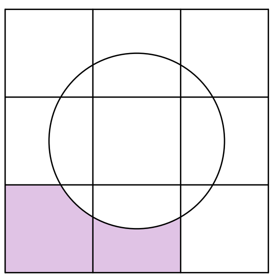
Problem 8 (Catriona Shearer). In this figure there are four squares. The area of the two little squares is 5 and the area of the middle square is 20. What is the area of the blue triangle? (Note that the figure hints at the answer.)



7.5 Miscellaneous

This is just a few problems that I thought were interesting. Have fun!

Problem 9. A circle with radius 1 is drawn centered on a 3×3 grid of unit squares. Find the area inside the lower-left and bottom squares but outside the circle.



Problem 10 (UKMT 2014). A circle with area 2500 is divided by two perpendicular chords into four regions. The two regions next to the region with the circle's center, shaded in the figure, have combined area 1000. The center of the circle and the intersection of the chords form opposite corners of a rectangle, whose sides are parallel to the chords. What is the area of this rectangle?