ASE 2020-21 Advanced Notes

Lecture Notes by Dylan Yu*

Last Updated August 29, 2020

Contents

§ 1 Coordinate Geometry

§ 1.1 Reading

1. Simple linear equations, Math Centre
2. Solving linear equations, Math Centre
3. Simultaneous equations, Math Centre

§ 1.2 Basics

A **point** in the coordinate plane is written as (x, y), where x is the x-coordinate and y is the y-coordinate. A few facts about **lines**:

Theorem 1 (Slope of a Line). The slope of a line going through (x_1, y_1) and (x_2, y_2) is $m = \frac{y_2-y_1}{x_2-x_1}$.

Example 1. Find the slope of a line going through $(1, 2)$ and $(2, 3)$.

Solution. Using our formula, we get $\frac{3-2}{2-1} = \boxed{1}$. □

*The ASE playlist can be found here.

Example 2. Write an equation of the line that contains $(6, -5)$ and has a slope of $\frac{3}{4}$.

Solution. Using point-slope form, we get $y + 5 = \frac{3}{4}(x - 6)$. □

Definition 1 (Intercept). The **x-intercept** is where a function intersects the x-axis, and the **y-intercept** is where a function intersects the y-axis.

Theorem 2 (Midpoint). The midpoint of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$.

Theorem 3 (Ratio Point Theorem). Let $A = (x_1, y_1)$ and $B = (x_2, y_2)$. The point

$$P = (rx_1 + (1-r)x_2, ry_1 + (1-r)y_2)$$

lies on the line between A and B, and splits the segment AB into $1 - r : r$ ratio, given $0 < r < 1$. If $r < 1 - r$, P is closer to B than A.

Theorem 4 (Distance between Two Points). The distance between two points (x_1, y_1) and (x_2, y_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Theorem 5 (Relation of Two Lines). For two linear functions $y = m_1x + b_1$ and $y = m_2x + b_2$:

1. (Inconsistent) If $m_1 = m_2$ and $b_1 \neq b_2$, then these two lines are **parallel**.
2. (Dependent) If $m_1 = m_2$ and $b_1 = b_2$, then these two lines **coincide**.
3. If $m_1 \cdot m_2 = -1$, then these two lines are **perpendicular**.
4. (Consistent) If $m_1 \neq m_2$, these two lines are **intersecting**.

The **standard form** of a linear function is $Ax + By + C = 0$, and so then

$$y = -\frac{A}{B}x - \frac{C}{B},$$

which is known as the **slope-intercept form**, since $m = -\frac{A}{B}$ is the slope, and $b = -\frac{C}{B}$ is the y-intercept (so we can also write the equation as $y = mx + b$). The **point-slope form** is $y - y_1 = m(x - x_1)$, where (x_1, y_1) is a point lying on the line and m is the slope. The slope of a horizontal line is 0, and its equation is $y = b$, where b is the y-intercept. Note that linear functions have at most one x-intercept and one y-intercept.
Unlike the slope-intercept form, the standard form is not very useful when you want to graph a linear

equation. Instead, it is used when you want to find the x and y intercepts or when you are solving systems of linear equations. The standard form of a linear graph is in the form $Ax + By = C$. If we solve this equation for y, we see that it gets us the slope-intercept form in terms of A, B, and C, which is $y = -\frac{A}{B}x + \frac{C}{B}$. This means that the slope of the graph in terms of A, B, and C is $-\frac{A}{B}$, and the y-intercept is $\frac{C}{B}$. We can use this conversion to quickly find the slope of a graph if we are given the standard form.

Theorem 6 (x/y-Intercept in Common Forms). In $Ax + By + C = 0$, the x-intercept is $-\frac{C}{A}$ and the y-intercept is $-\frac{C}{B}$. In $y = mx + b$, the x-intercept is $-\frac{b}{m}$ and the y-intercept is b.

Theorem 7 (2D Point to Point Distance). The distance d between the two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Theorem 8 (3D Point to Point Distance). The distance d from point $A(x_1, y_1, z_1)$ to point $B(x_2, y_2, z_2)$ in space:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

Using this theorem, we can now find the distance between the origin and a point:

Corollary 9 (Origin to Point Distance). The distance d from the origin to a point (a, b, c) is $d = \sqrt{a^2 + b^2 + c^2}$.

Example 3. What is the length of the diagonal of a rectangular solid with dimensions $3, 4$, and 12?

Solution. We can position the rectangular solid in the 3D plane such that its sides are parallel to the x, y, z-axes and put one vertex at the origin, so we get that the opposite vertex is at $(3, 4, 12)$. Thus, the diagonal has length $\sqrt{3^2 + 4^2 + 12^2} = \boxed{13}$. $\square$

Example 4. What is the distance between point $A(1, -1, 2)$ and point $B(3, 4, 1)$?

Solution. Using the formula we get $\sqrt{(3-1)^2 + (4-(-1))^2 + (1-2)^2} = \boxed{\sqrt{30}}$. $\square$

§ 1.3 Review

Problem 1. Solve the following systems of equations:

(a)
$$x + y = 1,$$
$$1.001x + y = 2.$$

(a)
$$2x + 3y = 42,$$
$$5x - y = 20.$$

(a)
$$x - y = 5,$$
$$3x + y = 1.$$

Problem 2. At what point do the lines $2x + 9y = 7$ and $x = 32 - 4.5y$ intersect?

Problem 3. Find the coordinates of the midpoint of AB if A has coordinates $(0, 2)$ and B has coordinates $(4, 6)$.

Problem 4. What is the distance between $(4, 3)$ and $(7, 7)$?

Problem 5. Find the x- and y-intercept of the line $y = 42x + 3$.

Problem 6. Find the x- and y-intercept of the line $3x + 6y = 12$.

Problem 7 (Cramer's Rule). Find the intersection of the lines $y = ax + b$ and $y = cx + d$ in terms of a, b, c, d, given that they are not parallel.