Math Level 2 Week 10

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Contents

1 Lecture Notes

1.1 Introduction

Example 1.1
Which of the integers from 2 to 10 have no positive divisors besides 1 and themselves?

Solution. By trying divisors, we get $\boxed{2, 3, 5, 7}$. □

Prime
A ***prime*** is a natural number $p > 1$ whose only positive divisors are 1 and p.

This means $2, 3, 5, 7$ are primes.

Composite
A ***composite number*** is a natural number c with some positive divisor besides 1 and c itself.

This means $4, 6, 8, 9, 1, 0$ are composite. Notice how we skipped 1. 1 is only divisible by itself, so it is neither prime nor composite.

Fact 1.4. There is only one even prime number: 2.

Fact 1.5. Multiples of composite numbers are still composite. Multiples of primes (besides the prime itself) are also composite.

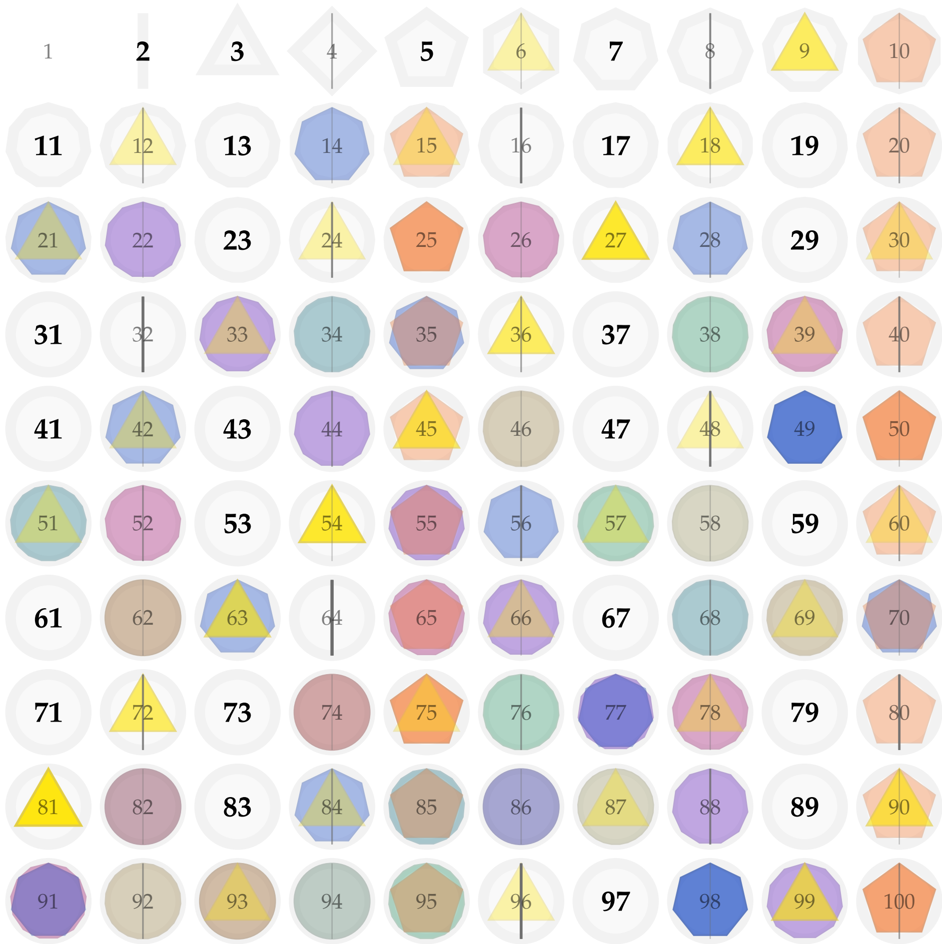
1.2 Prime Factorization

Theorem 1.6 (Fundamental Theorem of Arithmetic)
All positive integers n can be expressed as

$$n = 2^{e_1} \cdot 3^{e_2} \cdot 5^{e_3} \cdot \ldots,$$

where $e_1, e_2, e_3, \ldots$ are all nonnegative integers.

By now, you should know how to prime factorize. The ***Sieve of Eratosthenes*** can help you find the prime numbers, as shown below:



In this figure, if the number is divisible by 2, a line is drawn through it, if it is divisible by 3, a triangle is drawn on it, if it is divisible by 5, a pentagon is drawn on it, and so on. This lets us know which primes divide which numbers.

Remark 1.7. For a simpler Sieve of Eratosthenes, do not mark whether or not it is divisible by 2, 3, 5, etc. – just cross it out.

1.3 Divisors

Divisor

A ***divisor*** d of n is a number such that $\frac{n}{d}$ is an integer.

For example, 4 is a divisor of 12, and 1 is a divisor of every integer.

Example 1.9

Find all of the positive divisors of 20.

Solution. Through trial and error, we get $\boxed{1, 2, 4, 5, 10, 20}$. □

A ***common divisor*** of a group of integers is a divisor that divides every integer in that group.

1.4 Divisibility

Sometimes it is hard to tell if a number divides another number. So let's take a look at some divisibility tests here:

1. **2**: check if the last digit is even
2. **3**: check if the sum of the digits is divisible by 3 (repeat if necessary)
3. **4**: check if the last 2 digits is a 2-digit multiple of 4
4. **5**: check if the last digit is 0 or 5
5. **6**: use the rule for 2 and 3
6. **7**: take the last digit, multiply it by 2, and subtract it from the rest of the digits (repeat if necessary)
7. **8**: check if the last 3 digits is a 3-digit multiple of 8
8. **9**: check if the sum of the digits is divisible by 9 (repeat if necessary)
9. **10**: check if the last digit is 0
10. **11**: start from the last digit, then skip every other digit and up the ones you didn't skip; next, take all the digits you skipped and add those together; check if the difference between the first sum and the second sum is a multiple of 11

Example 1.10

Does 3 divide 11111199?

Solution. Summing the digits, we get $1 + 1 + 1 + 1 + 1 + 1 + 9 + 9 + 9 = 33$, which is divisible by 3, so the answer is yes. □

Example 1.11
Does 7 divide 7287?

Solution. Multiplying the last digit by 2 gives us 14, and subtracting it from the rest of the digits gives us $728 - 14 = 714$. Since it is still a little hard to tell if this is divisible by 7, we can apply it again: last digit is 4, so twice that is 8, and subtracting it we get $71 - 8 = 63$, which is 7×9, so the answer is yes. □

Example 1.12
Does 11 divide 9097?

Solution. If we sum the last digit, skip a digit, then sum the next digit, we get $7 + 0 = 7$. The skipped digits are 9 and 9, so $9 + 9 = 18$. $7 - 18 = -11$, which is a multiple of 11, so the answer is yes. □

1.5 GCD

You likely know the definitions for GCD and LCM:

GCD & LCM
The ***greatest common divisor***, or ***gcd*** of two numbers is the largest number dividing both numberes. The ***least common multiple***, or ***lcm*** of two numbers is the smallest number that can be divided by both numbers.

Theorem 1.14 (GCD & LCM Formula)
For two numbers

$$m = 2^{m_1} \cdot 3^{m_2} \cdot 5^{m_3} \cdot \ldots,$$
$$n = 2^{n_1} \cdot 3^{n_2} \cdot 5^{n_3} \cdot \ldots,$$

we have

$$\gcd(m, n) = 2^{\min(m_1,n_1)} \cdot 3^{\min(m_2,n_2)} \cdot 5^{\min(m_3,n_3)} \cdot \ldots,$$
$$\text{lcm}(m, n) = 2^{\max(m_1,n_1)} \cdot 3^{\max(m_2,n_2)} \cdot 5^{\max(m_3,n_3)} \cdot \ldots.$$

Theorem 1.15
For any two positive integers m, n,

$$\gcd(m, n) \cdot \text{lcm}(m, n) = mn.$$

There are a few methods to find the gcd and lcm, namely the ***Cake Method*** and simply applying the process above.

Example 1.16
Find the value of $\gcd(3, 6)$ and $\text{lcm}(3, 6)$.

Solution. By simply calculating, we get 3 and 6, respectively. □

Note that the value of $\gcd(0, n)$ is n for all positive integer n.

1.6 Euclidean Algorithm

Theorem 1.17 ("Dumb" Euclidean Algorithm)
For all positive integers $m > n$,

$$\gcd(m,n) = \gcd(m-n,n) = \gcd(n,m-n).$$

Example 1.18
Find the value of $\gcd(104,78)$.

Solution. Applying the algorithm above, we get

$$\gcd(104,78) = \gcd(26,78) = \gcd(26,52) = \gcd(26,26) = 26.$$

□

Theorem 1.19 (Euclidean Algorithm)
Let a, b be integers, with $b \neq 0$, and let q, r be the unique integers such that $a = qb + r$. Then

$$\gcd(a,b) = \gcd(b,r).$$

1.7 Coprime Integers

Coprime
Let a, b be integers. We say that a and b are ***coprime***, or ***relatively prime***, if a and b share no common factors. That is to say, a and b are coprime if $\gcd(a,b) = 1$.

Coprimality can be useful with thinking about common divisors. In particular, we have the following useful theorem:

Theorem 1.21 (Coprime)
Let a, b be nonzero integers, and let $d = \gcd(a,b)$. Then

- $\frac{a}{d}$ and $\frac{b}{d}$ are coprime.
- Write $a = dk$ for some integer k. Then for some integer y, if $a \mid (dy)$, then $k \mid y$.

2 Problems

Problem 1. Jon teaches a fourth grade class at an elementary school where class sizes are always at least 20 students and at most 28. One day Jon decides that he wants to arrange the students in their desks in a rectangular grid with no gaps. Unfortunately for Jon he discovers that doing so could only result in one straight line of desks. How many students does Jon have in his class?

Problem 2. Are 37 and 111 coprime?

Problem 3 (Mathcounts). The number 13 is prime. If you reverse the digits you also obtain a prime number, 31. What is the larger of the pair of primes that satisfies this condition and has a sum of 110?

Problem 4 (Mathcounts). A group of 25 pennies is arranged into three piles such that each pile contains a different prime number of pennies. What is the greatest number of pennies possible in any of the three piles?

Problem 5 (Mathcounts). A two-digit prime number is randomly selected. What is the probability that its digits sum to 9?

Problem 6 (Mathcounts). What is the first year in the twenty-first century that is a prime number?

Problem 7. Find the value of $\gcd(0, 4, 10)$.

Problem 8. Find the value of $\operatorname{lcm}(4, 6, 10)$.

Problem 9. What is the value of

$$\gcd(33, 121) \cdot \operatorname{lcm}(33, 121)?$$

Problem 10. Find the largest n such that $n \mid 8, 10, 12$.

Problem 11. Find $\gcd(819, 504)$.

Problem 12. Two different rectangles have the same width. All four sides of both rectangles have integer lengths. The areas of the rectangles are 1086 and 828. Find the largest possible value of the common width of the rectangles.

Problem 13. Mike has 24 red balls and 36 green balls that he places into several boxes. Each box contains the same number of balls and there are at least 2 balls in each box. If all the balls in any box are the same color, what are the possible numbers of balls in each box?

Problem 14. Find all of the positive common divisors of both 20 and 30.

Problem 15. Jon splits 42 blue marbles into piles containing n marbles each. Jon then divides 28 green marbles into n total piles with an equal

Problem 16. Find the greatest 3-digit number that leaves a remainder of 4 when divided by 11.