Math Level 2 Week 12

Dylan Yu

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1 Introduction

1.1 Definitions

Here we introduce some important notation and ideas that we will use throughout the handout.

Diophantine Equation
A ***diophantine equation*** is an equation that can be solved over the integers.

For example, $a + b = 32$, where a, b are integers, is a diophantine equation. A *linear example* would be $ax + by = c$, where a, b, c, x, y are integers.

$\mathbb{Z}$
If $a \in \mathbb{Z}$, then a is an integer.

Furthermore, $\mathbb{Z}^{-}$ is the set of negative integers, $\mathbb{Z}^{+}$ is the set of positive integers, $\mathbb{Z}^{0+}$ is the set of nonnegative integers, and $\mathbb{Z}^{0-}$ is the set of nonpositive integers.

1.2 Modular Arithmetic

Let us turn to one of the most important theorems for solving Diophantine equations:

Theorem 1.3 (Law of Diophantines)
Let $m > 1$ be a positive integer. If an equation has no solution modulo m, then it has no integer solutions.

1.3 Factoring

Simon's Favoring Factoring Trick, abbreviated SFFT, is useful here.

Theorem 1.4 (SFFT)
For all x, y, a, b (usually integers),

$$xy + ax + by + ab = (x + b)(y + a).$$

This isn't particularly special, but sometimes it is disguised.

Example 1.5
Find all integral solutions to $xy - x + y = 0$.

Solution. Note that this is equivalent to $x(y - 1) + y = 0$. If we subtract 1 from both sides, we get $x(y - 1) + y - 1 = -1$, so

$$(x + 1)(y - 1) = -1,$$

implying we have $x + 1 = 1$ and $y - 1 = -1$ or $x + 1 = -1$ or $y - 1 = 1$. Thus, the solutions for (x, y) are $\boxed{(0, 0)}$ or $\boxed{(-2, 2)}$. □

Here is an important theorem to keep in mind while solving:

Theorem 1.6 (Solutions to a common form of SFFT)
Let x, y be positive integers and let $n = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$ (in other words, its prime factorization). Then the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{n}$$

has

$$\tau(n^2) = (2e_1 + 1)(2e_2 + 1) \dots (2e_k + 1),$$

solutions, where $\tau(n)$ is the number of divisors of n.

Knowing key factorizations is important. For example,

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$

can help you solve problems of this nature quickly.

2 Examples

Theorem 2.1 (Linear Equation with Two Unknowns)
The equation $ax + by = c$ has integer solutions if and only if $\gcd(a, b) \mid c$. If $\gcd(a, b) = 1$, and x_0, y_0 is one integer solution for $ax + by = c$, then the general solution is $x = x_0 + bt, y = y_0 - at$, where t is an integer.

There is a lot of mapping solutions to solutions in Diophantine equations. In other words, if (x_0, y_0) is a solution, then $(f(x_o), g(y_0))$ is a solution, for some functions f, g.

Example 2.2
Find all positive integer solutions of

$$19x + 7y = 260.$$

Solution. A simple solution is $x = y = 10$. Thus, we can apply the method listed above and solve for all solutions. It turns out the only solutions are $(10, 10)$ and $(3, 29)$. □

Let's move on to higher degree equations:

Theorem 2.3 (Quadratic Equation with Three Unknowns)
For the equation

$$x^2 + y^2 = z^2,$$

all positive integer solutions (x, y, z) that satisfy $\gcd(x, y, z) = 1$ is

$$x = a^2 - b^2, y = 2ab, z = a^2 + b^2,$$

where integers $a > b > 0$, one even and one odd, and $\gcd(a, b) = 1$.

As you have probably realized, this is just the ***parameterization*** of the ***Pythagorean triples***. This underlies something important in solving Diophantine equations – parameterization.

Theorem 2.4 (Quadratic Equation with Four Unknowns)
For the equation

$$xy = zt,$$

all positive integer solutions can be found by letting

$$\frac{x}{z} = \frac{t}{y} = \frac{m}{n},$$

where $\gcd(m, n)$, then $x = pm, z = pn, t = qm, y = qn$, where $p = \gcd(x, z), q = \frac{y}{n}$.

Example 2.5
Let a, b, c, d be positive integers, and $ab = cd$. Prove that $a^4 + b^4 + c^4 + d^4$ is not prime.

Solution. From the theorem above, we have

$$\frac{a}{c} = \frac{d}{b} = \frac{m}{n},$$

so $a = pm, c = pn, d = qm, b = qn$. Now, plugging these numbers in, we get

$$a^4 + b^4 + c^4 + d^4 = p^4m^4 + q^4n^4 + p^4n^4 + q^4m^4 = (p^4 + q^4)(m^4 + n^4).$$

Because p, q, m, n are all positive integers, then

$$p^4 + q^4 > 1,$$

$$m^4 + n^4 > 1,$$

so the product cannot be prime. □

3 Problems

Problem 1 (AMC 12A 2004/3). For how many ordered pairs of positive integers (x, y) is $x + 2y = 100$?

Problem 2 (AMC 12B 2008/5). A class collects 50 dollars to buy flowers for a classmate who is in the hospital. Roses cost 3 dollars each, and carnations cost 2 dollars each. No other flowers are to be used. How many different bouquets could be purchased for exactly 50 dollars?

Problem 3 (AMC 12A 2006/14). Two farmers agree that pigs are worth 300 dollars and that goats are worth 210 dollars. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. (For example, a 390 dollar debt could be paid with two pigs, with one goat received in change.) What is the amount of the smallest positive debt that can be resolved in this way?

Problem 4 (AMC 12A 2005/8). Let A, M, and C be digits with

$$(100A + 10M + C)(A + M + C) = 2005$$

What is A?

Problem 5 (AHSME 1989/16). A lattice point is a point in the plane with integer coordinates. How many lattice points are on the line segment whose endpoints are $(3, 17)$ and $(48, 281)$? (Include both endpoints of the segment in your count.)

Problem 6 (AHSME 1967/24). What is the number of solution-pairs in the positive integers of the equation $3x + 5y = 501$?

Problem 7 (AHSME 1968/19). Let n be the number of ways 10 dollars can be changed into dimes and quarters, with at least one of each coin being used. Then what is n?

Problem 8 (AMC 12A 2006/9). Oscar buys 13 pencils and 3 erasers for \$1.00. A pencil costs more than an eraser, and both items cost a whole number of cents. What is the total cost, in cents, of one pencil and one eraser?

Problem 9 (AMC 12A 2014/19). There are exactly N distinct rational numbers k such that $|k| < 200$ and

$$5x^2 + kx + 12 = 0$$

has at least one integer solution for x. What is N?

Problem 10 (AIME 1997/1). How many of the integers between 1 and 1000, inclusive, can be expressed as the difference of the squares of two nonnegative integers?