USEMO 2020/1

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Problem 1. Which positive integers can be written in the form

$$\frac{\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)}{\operatorname{lcm}(x,z)}$$

for positive integers x, y, z?

Solution. I claim all even integers (and only even integers) suffice.

Let k be an integer, and let

$$f(x,y,z) = \frac{\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)}{\operatorname{lcm}(x,z)}.$$

Then

$$f(k,k^2,k) = \frac{\operatorname{lcm}(k,k^2)+\operatorname{lcm}(k^2,k)}{\operatorname{lcm}(k,k)} = \frac{k^2+k^2}{k} = 2k.$$

Thus if $k = 1, 2, \ldots$, then we can create all even integers as desired.

Now we will show all odds cannot be created. Two solutions (the first is recommended):

Solution via ν_2 Because $f(x,y,z)$ is an integer, we must show

$$\nu_2(\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)) > \nu_2(\operatorname{lcm}(x,z)) = \max(\nu_2(x),\nu_2(z)).$$

WLOG let $\nu_2(x) \geq \nu_2(z)$. Then again we have to show

$$\nu_2(\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)) > \nu_2(x).$$

Let us now do casework on the size of y with respect to x and z:

- **Case 1** ($\nu_2(y) \geq \nu_2(x) \geq \nu_2(z)$). Then $\nu_2(\operatorname{lcm}(x,y)) = \nu_2(\operatorname{lcm}(y,z)) = \nu_2(y)$, so

  $$\nu_2(\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)) > \min(\nu_2(\operatorname{lcm}(x,y)),\nu_2(\operatorname{lcm}(y,z))) = \nu_2(y) \geq \nu_2(x)$$

  as desired.

- **Case 2** ($\nu_2(x) > \nu_2(y) \geq \nu_2(z)$). Then $\nu_2(\operatorname{lcm}(x,y)) = \nu_2(x)$ and $\nu_2(\operatorname{lcm}(y,z)) = \nu_2(y)$, so

  $$\nu_2(\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)) \geq \min(\nu_2(\operatorname{lcm}(x,y)),\nu_2(\operatorname{lcm}(y,z))) = \nu_2(y) < \nu(x),$$

  implying $f(x,y,z)$ is not an integer, so this case is impossible.

- **Case 3** ($\nu_2(x) \geq \nu_2(z) > \nu_2(y)$). Also impossible, left as exercise to the reader.

Thus, $\nu_2(\operatorname{lcm}(x,y)+\operatorname{lcm}(y,z)) > \nu_2(x)$ as desired.

Solution via casework Let us consider all possible parities of x, y, z (assume $f(x, y, z)$ is an integer, of course):

$x \pmod 2$	$y \pmod 2$	$z \pmod 2$	$f(x,y,z) \pmod 2$
0	0	0	Case 1
0	0	1	Case 2
0	1	0	Case 3
0	1	1	Case 4
1	0	0	Case 5
1	0	1	Case 6
1	1	0	Case 7
1	1	1	Case 8

- **Case 1**. Let $x = 2a, y = 2b, z = c$. Then
$$f(x,y,z) = \frac{\operatorname{lcm}(2a,2b) + \operatorname{lcm}(2b,2c)}{\operatorname{lcm}(2a,2c)} = \frac{\operatorname{lcm}(a,b) + \operatorname{lcm}(b,c)}{\operatorname{lcm}(a,c)} = f(a,b,c).$$
Thus, we will eventually reach a case where not all x, y, z are divisible by 2, i.e. one of the cases 2 through 7.

- **Case 2**. Let $x = 2a, y = 2b, z = 2c + 1$. Then
$$f(x,y,z) = \frac{\operatorname{lcm}(2a,2b) + \operatorname{lcm}(2b,2c+1)}{\operatorname{lcm}(2a,2c+1)} = \frac{\operatorname{lcm}(a,b) + \operatorname{lcm}(b,2c+1)}{\operatorname{lcm}(a,2c+1)} = f(a,b,2c+1).$$
Thus, we will eventually reach another case.

- **Case 3**. Let $x = 2a, y = 2b + 1, z = 2c$. Then
$$f(x,y,z) = \frac{\operatorname{lcm}(2a,2b+1) + \operatorname{lcm}(2b+1,2c)}{\operatorname{lcm}(2a,2c)} = f(a,2b+1,c).$$
Thus, we will eventually reach another case.

- **Case 4**. Let $x = 2a, y = 2b + 1, z = 2c + 1$. Then
$$f(x,y,z) = \frac{\operatorname{lcm}(2a,2b+1) + \operatorname{lcm}(2b+1,2c+1)}{\operatorname{lcm}(2a,2c+1)}.$$
The numerator is odd whereas the bottom is even, so $f(x, y, z)$ cannot be an integer.

- **Case 5**. Identical to case 2.

- **Case 6**. Let $x = 2a + 1, y = 2b, z = 2c + 1$. Then
$$f(x,y,z) = \frac{\operatorname{lcm}(2a+1,2b) + \operatorname{lcm}(2b,2c+1)}{\operatorname{lcm}(2a+1,2c+1)} = 2f(2a+1,b,2c+1) = \text{even}.$$

- **Case 7**. Identical to case 4.

- **Case 8**. Let $x = 2a + 1, y = 2b + 1, z = 2c + 1$. Then
$$f(x,y,z) = \frac{\operatorname{lcm}(2a+1,2b+1) + \operatorname{lcm}(2b+1,2c+1)}{\operatorname{lcm}(2a+1,2c+1)} = \frac{\text{even}}{\text{odd}},$$
so if $f(x, y, z) \in \mathbb{Z}$, then it must be even.

Thus, $f(x, y, z)$ is always even as desired. $\square$