Work and Energy

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An important part of mechanics is work and energy. We will touch on the key principles of each and relate the two using the Work-Energy Theorem.

Contents

Vocabulary

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§1 Work, Energy, Power

Say ***force*** is the agent of change. Then:

Work
Work is a force applied over a certain displacement.

Energy
Energy is the ability to do work.

Note that work is also a way of transferring energy between systems. There are many forms of energy: gravitational energy (a meteor crashing into the Earth), elastic energy (a rubber band stretching), thermal energy (baking cookies in an oven), electrical energy (charging your phone), nuclear energy (power plants), and mass energy (this is where $E = mc^2$ comes from). Energy can flow into a system and flow out of it.

Remark 3 (Energy Consumption). Although we often hear people talking about ***energy consumption***, energy is never really destroyed. It is just transferred from one form to another, doing work in the process. Some forms of energy are less useful to us than others – for example, low level heat energy. It is better to talk about the consumption or extraction of energy resources, for example coal, oil, or wind, than consumption of energy itself.

This leads to a **very** important idea – ***Conservation of Energy***:

Theorem 4 (Conservation of Energy)
The total amount of energy in a process will stay constant; i.e., *conserved*.

Conservation of Energy is also sometimes known as the ***First Law of Thermodynamics***.

§1.1 Work

Lifting a book and pushing a box have two things in common – a change in position and an applied force.

Theorem 5 (Work Formula)
Let θ be the angle between the force F applied and the displacement Δd. Then the work applied is

$$W = F\Delta d \cos\theta.$$

The unit for work is ***joules***, which can be abbreviated as J. It is equivalent to $\mathrm{N} \cdot \mathrm{m}$.

Fact 6 (Work is a Scalar). Since work is the dot product of F and Δd, work is a **scalar**.

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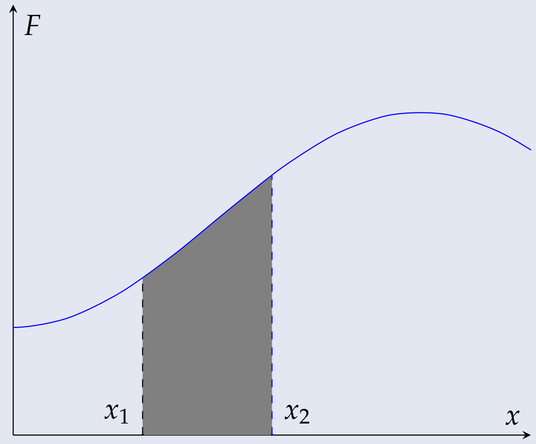
Theorem 7 (Area in a Force-Displacement Graph)
The area under a force-displacement graph is work.



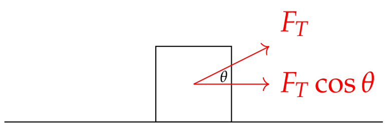
In this handout, we will be dealing with *easier areas* that don't involve calculus.

Example 8
You slowly lift a book of mass 2 kg at a constant velocity a distance of 3 m. How much work did you do on the book?

Solution. The force necessary to carry the book is $F = mg = (2\,\text{kg})(10\,\text{m/s}^2) = 20\,\text{N}$. Thus, $W = Fd = 20\,\text{N} \cdot 3\,\text{m} = \boxed{60\,\text{J}}$. □

Example 9
A 15 kg crate is moved along a horizontal floor by a warehouse worker who's pulling on it with a rope that makes a 30° angle with the horizontal. The tension in the rope is 200 N and the crate slides a distance of 10 m. How much work is done on the crate by the rope?

Solution. Let's draw a FBD:



Thus, $W = (F_T \cos\theta)d = (200\,\text{N} \cdot \cos 30°)(10\,\text{m}) = \boxed{1730\,\text{J}}$. □

§1.2 Kinetic Energy

Kinetic Energy
The ***kinetic energy*** is the energy obtained from its motion.

In particular,

Theorem 11 (Kinetic Energy Formula)
An object with mass m going at a speed v has kinetic energy $K = \frac{1}{2}mv^2$.

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§1.3 Potential Energy

Potential energy is independent of motion; it arises from the object's position. For example, a ball at the edge of a tabletop has energy that could be transformed into kinetic energy if it falls off. Work was done on the object to put it in the given configuration, and since work is the means of transferring energy, these things have *stored energy that can be retrieved* as kinetic energy. Potential energy is often denoted U. Because there are different types of forces, there are different types of potential energy. A ball on the edge of the table is an example of ***gravitational potential energy***, U_{grav}, which is the energy stored by virtue of an object's position in a gravitational field.

Theorem 12 (Gravitational Potential Energy Formula)
$U_g = mgh$, where U_g is the gravitational potential energy and h is the height above some reference.

We can see that regardless of the path the ball takes to get to a certain height, the GPE is the same. Thus, gravity is a ***conservative*** force.

Example 13
A stuntman of mass 60 kg scales a 40-meter-tall rock face. What is her gravitational potential energy (relative to the ground)?

Solution. Using the Gravitational Potential Energy Formula, we have $U_g = mgh = (60\,\text{kg})(9.81\,\text{m/s}^2)(40\,\text{m}) = \boxed{24\,000\,\text{J}}$. □

§2 Work-Energy Theorem

Kinetic energy is expressed in joules like work. As we have seen previously, if an object starts at rest, then $W = K$. What if the object has an non-zero initial velocity?

Theorem 14 (Work-Energy Theorem)
The total work done is equal to the change in kinetic energy, i.e. $W_{\text{total}} = \Delta K$.

Example 15
What is the kinetic energy of a ball with mass 10 kg moving with a speed of 30 m/s?

Solution. Using the Kinetic Energy Formula, we get $K = \frac{1}{2}mv^2 = \frac{1}{2}(10\,\text{kg})(30\,\text{m/s})^2 = \boxed{4500\,\text{J}}$. □

§3 Conservation of Mechanical Energy

The ***total mechanical energy*** E equals $K + U$. If there are no non-conservative forces (like friction) acting on the system, then ME (mechanical energy) is conserved. Thus:

Theorem 16 (Conservation of Mechanical Energy)
The initial and final mechanical energies are equal, implying $K_i + U_i = K_f + U_f$.

Note that because U is relative, so is E.

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Example 17
A ball of mass 2 kg is dropped from a height of 5 m above the floor. Find the speed of the ball as it strikes the floor.

Solution. Note that $K_i = U_f = 0$ if $h = 0$ is our reference level, so $K_i + U_i = K_f + U_f \implies mgh = \frac{1}{2}mv^2$. This means $v = \sqrt{2gh} = \boxed{10\,\text{m/s}}$. □

§4 Graphing Kinetic and Potential Energy

Fact 18. Given the graph of kinetic or potential energy, one can find the other.

This is assuming that $K + U$ is constant, but it is easy to see how to graph it from there.

Theorem 19 (Elastic Potential Energy Formula)
Let x be the displacement from the relaxed position of a spring. Then $U = \frac{1}{2}kx^2$.

§5 Power

Power
Power is the rate at which work is done, and is measured in units of watts.

This implies:

Theorem 21 (Power Equation)
For work W done in time t, the power is

$$P = \frac{W}{t}.$$

Alternatively, using the Work-Energy Theorem,

$$P = \vec{F} \cdot \vec{v}.$$

Example 22
1,000-kg car accelerates from 88 m/s to 100 m/s in 30 s. How much power does that require?

Solution. We know that

$$P = \frac{W}{t},$$

and since $W = \Delta KE = \frac{1}{2}m(v_2^2 - v_1^2)$, we can substitute in to get

$$W = \frac{1}{2t}m(v_2^2 - v_1^2) = \boxed{38\,000\,\text{W}}.$$

□

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§6 Problems

Problem 1. A 15 kg crate is moved along a horizontal floor by a warehouse worker who's pulling on it with a rope that makes a 30° angle with the horizontal. The tension in the rope is 200 N and the crate slides a distance of 10 m.

(a) How much work is done by the normal force?

(b) How much work is done by the friction force?

Problem 2. A box slides down an inclined plane with an angle of 37°. The mass of the block, m, is 35 kg, the coefficient of friction between the box and the ramp, μ_k, is 0.3, and the length of the ramp, d, is 8 m.

1. How much work is done by gravity?
2. How much work is done by the normal force?
3. How much work is done by friction?
4. What is the total work done?
5. If it starts from rest at the top of the ramp, with what speed does it reach the bottom?

Problem 3 (Hooke's Law). The force exerted by a spring when it's displaced by x from its natural length is given by $F(x) = -kx$, where k is a positive constant. What is the work done by a spring as it pushes out from $x = -x_2$ to $x = -x_1$, where $x_2 > x_1$?

Problem 4. A tennis ball with a mass of 0.06 kg is hit straight upward with an initial speed of 50 m/s. How high would it go if air resistance were negligible?

Problem 5. A pool cue striking a stationary billiard ball with mass 0.25 kg gives the ball a speed of 2 m/s. If the average force of the cue on the ball was 200 N, over what distance did this force act?

Problem 6. An object with mass 60 kg falls off a 40 meter cliff. What is the final speed right before the object hit the ground?

Problem 7. A box is projected up a long ramp (incline angle with the horizontal is 37°) with an initial speed of 10 m/s. If the surface of the ramp is very smooth, how high up the ramp will the box go? What distance along the ramp will it slide?

Problem 8. A skydiver jumps from a hovering helicopter that's 3000 m above the ground. If air resistance can be ignored, how fast will he be falling when his altitude is 2000 m?

Problem 9. A crash test dummy of mass 40 kg falls off a 50-meter-high cliff. On the way down, the force of air resistance has an average strength of 100 N

Problem 10. A skier starts from rest at the top of a 20° incline and skis in a straight line to the bottom of the slope, a distance d (measured along the slope) of 400 m. If the coefficient of kinetic friction between the skis and the snow is 0.2, calculate the skier's speed at the bottom of the run.

Problem 11. A mover pushes a large crate of mass 75 kg from the inside of the truck to the back end (a distance of 6 m), exerting a steady push of 300 N. If he moves the crate this distance in 20 s, what is his power output during this time?

Problem 12. What must be the power output of a car engine, which moves a 1000 kg car at a constant speed of 8 m/s?

Problem 13. Under the influence of a force, an object of mass 4 kg accelerates from 3 m/s to 6 m/s in 8 s. How much work was done on the object during this time?

Problem 14. A box of mass m slides down a frictionless inclined plane of length L and vertical height h. What is the change in its gravitational potential energy?

Problem 15. An astronaut drops a rock from the top of a crater on the Moon. When the rock is halfway down to the bottom of the crater, its speed is what fraction of its final impact speed?

Problem 16. A force of 200 N is required to keep an object sliding at a constant speed of 2 m/s across a rough floor. How much power is being expended to maintain this motion?

Problem 17. A 60.0-kg person is running and accelerates from 5.0 m/s to 7.0 m/s in 2.0 s. How much power does that require?

Problem 18. 120-kg linebacker accelerates from 5.0 m/s to 10.0 m/s in 1.0 s. How much power does that require?

Problem 19. You're driving a snowmobile that accelerates from 10 m/s to 20 m/s over a time interval of 10.0 s. If you and the snowmobile together have a mass of 500 kg, how much power is used?