NICE Computational Contest

Spring 2021

Rules

1. This is a 25 question test. All answers are integers between 0 and $2^{31} - 1$, inclusive.
2. Contestants are only allowed to use scratch paper, rulers, compasses, erasers, and four-function calculators. No other computing devices or resources such as Wolfram Alpha are allowed. No problems on the exam require the use of a calculator.
3. Submit your answers through the nicecontest.xyz web portal.
4. Do not discuss the contents of this exam with anyone until after the discussion period begins (which is an hour after the submission deadline). Failure to comply with this rule may result in disqualification from this and future NICE contests.
5. Enjoy the problems! ☺

Problems

1. Sally's teacher wrote the expression

$$\left(\frac{1}{3}\right)\sqrt[3]{a}$$

on the board. Unfortunately, Sally misinterpreted this as

$$\left(\frac{1}{3}\right)^3\sqrt{a}.$$

However, for some positive real value of a, these two expressions are both equal to the same number b. Determine b.

2. In a hat, there are ten slips, each containing a different integer from 1 to 10, inclusive. David reaches into the hat and keeps for himself two numbers that differ by seven. Ankan then reaches into the hat and picks two numbers that differ by five. What is the largest possible sum of the four picked numbers?

3. In parallelogram $ABCD$, let P be a point on $\overline{AC}$. Suppose that the sum of the perimeters of $\triangle PAB$ and $\triangle PAD$ is 13, the sum of the perimeters of $\triangle PCB$ and $\triangle PCD$ is 7, and $AC = 9$. Find AP.

4. What is the smallest possible area of a rhombus $\mathcal{R}$ whose sides have length 5 and whose vertices are all points in the plane with integer coordinates?

5. Aeren needs to memorize a table about a new binary operation $\heartsuit$. He is given the table below by his teacher and is also told that

$$A - (A\heartsuit B) = (B\heartsuit A) - B$$

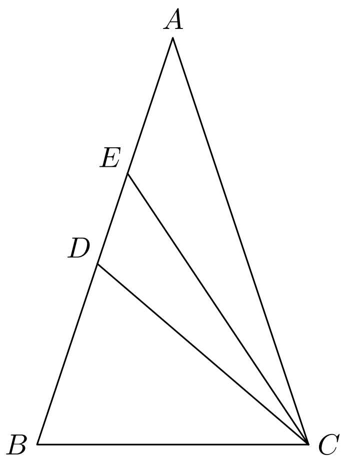
for all positive integers A and B between 1 and 6 inclusive. At most how many additional entries in the table can he fill out (without guessing)?

$\heartsuit$	1	2	3	4	5	6
1		1				
2						
3		1			8	
4	3				7	
5						
6	4			9		

6. Archimedes is playing *jackblack* with a deck of 5 cards, labeled with the integers from 1 to 5. Archimedes chooses up to 3 cards at random, with replacement, and wins if the cards in his hand sum to 9 at any point. (The game immediately stops once he wins.) However, Archimedes has an advantage: he secretly slipped an extra card labelled 2 into his pocket. He will only use the extra 2 if the cards in his hand sum to exactly 7 after two turns, leading to a guaranteed win.

The probability that Archimedes wins is $\frac{m}{n}$, where m and n are positive relatively prime integers. Find $1000m + n$.

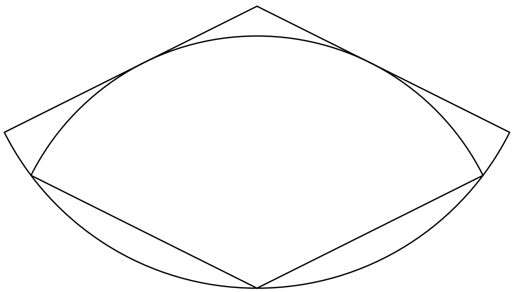
7. Let $\triangle ABC$ be isosceles with $AB = AC > BC$. Points D and E lie on $\overline{AB}$ such that $CB = CD$ and $DB = 2DE$, as shown in the diagram. If $\angle ECD = \angle ECA$, then the value of $\frac{AB}{BC}$ can be written in the form $\frac{m}{n}$, where m and n are positive relatively prime integers. What is $1000m + n$?



8. How many positive integers $N \leq 1000$ are there for which N^2 and $(N+1)^2$ both have at least five positive divisors? It is known there are 168 primes between 1 and 1000.

9. Ethan draws a 2×6 grid of points on a piece of paper such that each pair of adjacent points is distance 1 apart. How many ways can Ethan divide the 12 points into six pairs such that the distance between the points in each pair is in the interval $[2, 4)$?

10. A sector of angle $\alpha < 180°$ is inscribed inside another sector of the same angle, as shown below. If $\cos(\alpha) = -\frac{m}{n}$, where m and n are relatively prime positive integers, what is $1000m + n$?

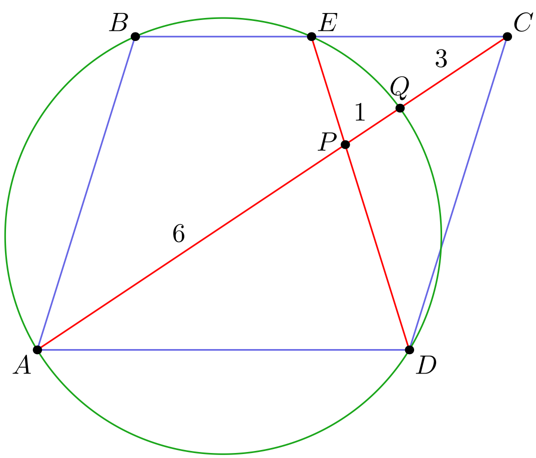


11. Fifty rooms of a castle are lined in a row. The first room contains 100 knights, while the remaining 49 rooms contain one knight each. These knights wish to escape the castle by breaking the barriers between consecutive rooms, ending with the barrier from room 50 to the outside.

At the stroke of midnight, each knight in the i^{th} room begins breaking the barrier between the i^{th} and $(i+1)^{\text{st}}$ rooms, where we count the 51^{st} room as the exterior. Each person works at a constant rate and is able to break down a barrier in 1 hour, and once a group of knights breaks down the i^{th} barrier, they immediately join the knight breaking down the $(i+1)^{\text{st}}$ barrier.

The number of hours it takes for the knights to escape the castle is $\frac{m}{n}$, where m and n are positive relatively prime integers. Compute the product mn.

12. Let $ABCD$ be a parallelogram with $\angle BAD < 90^\circ$. The circumcircle of $\triangle ABD$ intersects $\overline{BC}$ and $\overline{AC}$ again at E and Q, respectively. Let P be the intersection point of $\overline{ED}$ with $\overline{AC}$. If $AP = 6$, $PQ = 1$, and $QC = 3$, find the area of parallelogram $ABCD$.



13. Suppose x and y are nonzero real numbers satisfying the system of equations

$$\begin{aligned} 3x^2 + y^2 &= 13x, \\ x^2 + 3y^2 &= 14y. \end{aligned}$$

Find $x + y$.

14. A nonempty string X of letters is called a *subsequence* of another string Y if a nonzero number of letters can be removed from Y to result in X. For example, *AD*, *WAR*, and *WARD* are subsequences of *WIZARD*, but *RAID* is not. Blitz writes a nonempty list of subsequences of *AAAABBBBBB* such that, for any two strings in Blitz's list, neither is a subsequence of the other. How many such lists are possible? Note that order does not matter, so the list *AAB*, *ABBB* is the same as *ABBB*, *AAB*.

15. Let r_1, r_2, and r_3 be the roots of the polynomial $x^3 - x + 1$. Then

$$\frac{1}{r_1^2 + r_1 + 1} + \frac{1}{r_2^2 + r_2 + 1} + \frac{1}{r_3^2 + r_3 + 1} = \frac{m}{n},$$

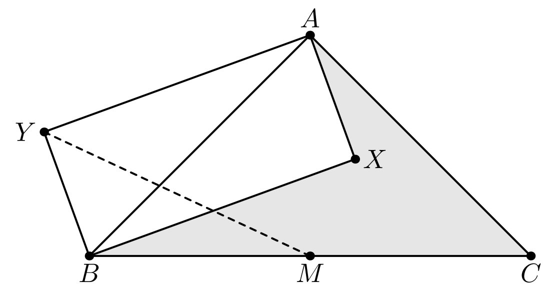
where m and n are positive relatively prime integers. Find $1000m + n$.

16. An arrow points from $(0, 0)$ to $(1, 0)$. Define a *flop* as an operation where the arrow either moves one unit in the direction it is pointing, or the arrow switches between pointing right and pointing up, each with equal probability. Let (x, y) be the average position of the arrow after all possible sequences of 100 flops. If xy can be expressed in the form $\frac{a}{b}$, where a, b are relatively prime, find $a + b$.

17. What is the sum of all two-digit primes p for which there exist positive integers x, y, and D such that

$$x^2 - Dy^2 = (x + 1)^2 - D(y + 1)^2 = p?$$

18. Let ABC be an isosceles right triangle with $AB = AC$, and denote by M the midpoint of $\overline{BC}$. Construct rectangle $AXBY$, with X in the interior of $\triangle ABC$, such that $YM = 8$ and $AY^3 + BY^3 = 10^3$. The area of quadrilateral $AXBC$ can be expressed in the form $\frac{m\sqrt{n}}{p}$, where m, n, and p are positive integers such that m and p are relatively prime and n is not divisible by the square of any prime. Find $1000000m + 1000n + p$.



19. Suppose a and b are positive integers with $a < b$ such that the four intersection points of the lines $y = x + a$ and $y = x + b$ with the parabola $y = x^2$ are the vertices of a quadrilateral with area 720. Find $1000a + b$.

20. Determine the least integer $k \geq 2$ for which

$$\frac{1}{2^{1000}} \prod_{i=2}^{k} (i^{100} - 1) := \frac{(2^{100} - 1)(3^{100} - 1) \cdots (k^{100} - 1)}{2^{1000}}$$

is an integer.

21. Let $n \geq 4$ be an integer. A lamplighter has n lamps, placed at the vertices of a regular n-gon. Initially, all the lamps are off. In a move, the lamplighter may toggle four lamps placed at the vertices of an isosceles trapezoid or rectangle.

Let a_n denote the number of configurations of lamps the lamplighter can obtain after applying some number of moves. Compute the remainder when $a_4 + a_5 + \cdots + a_{100}$ is divided by 1000.

22. Determine the number of ordered pairs (b, c) of integers with $0 \leq b \leq 2016$ and $0 \leq c \leq 2016$ such that 2017 divides $x^3 - bx^2 + c$ for exactly one integer solution x with $0 \leq x \leq 2016$.

23. Triangle ABC satisfies $AB < AC$ and has incircle ω. Suppose that ω is tangent to $\overline{BC}$ and $\overline{AC}$, respectively, at E and F. Suppose that $\overline{BF}$ is tangent to the circumcircle of $\triangle CEF$, circle ω has radius 4, and $\angle ABC = 90^\circ$. Then BC can be written in the form $p + \sqrt{q}$ for positive integers p and q. Determine $1000p + q$.

24. Seven teams play in a round robin tournament (each pair of teams plays exactly one game against each other). In each game, one of the two participating teams wins and the other loses; there are no ties. 21 games are played in total, so there are 2^{21} possible results of the tournament. For how many of these results is it true that for every pair (A, B) of teams, there exists a third team, C, that won against both of them?

25. David attempts to draw a regular hexagon $ABCDEF$ inscribed in a circle ω of radius 1. However, while the points A through E are accurately placed, his point F is slightly off, and so the diagonals AD, BE, and CF do not concur. (Fortunately for him, the location of point F on ω is the only inaccurate part of the diagram.) Undeterred by his artistic deficiencies, David pretends that the diagram is accurate by shading in the circumcircle of the triangle formed by these three line segments, which has radius $\frac{1}{24}$. He then continues to draw the rest of the diagram.

The area of the hexagon $ABCDEF$ that David actually drew can be expressed in the form $a - \frac{b}{c}\sqrt{d}$, where a, b, c, and d are positive integers such that b and c are relatively prime and d is not divisible by the square of any prime. Find $a + b + c + d$.