

# Generalized Dirichlet to Neumann Maps for Linear Dispersive Equations on Half-Line

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## Abstract

A large class of initial-boundary problems of linear evolution partial differential equations formulated on the half-line is analyzed via the unified transform method. In particular, explicit representations are presented for the generalized Dirichlet to Neumann maps. Namely, the determination of the unknown boundary values when an essential set of initial and boundary data is given.

## 1 Introduction

In this paper, we study a class of initial-boundary value problems of linear evolution equations formulated on the half line, by using so-called the *unified transform* method introduced in [1]. A major difficulty of solving certain initial-boundary value problems stems from the fact that the solution representation requires all boundary values, whereas only a subset of them is prescribed as boundary conditions. The determination of the unknown boundary values in terms of given data is often called the *Generalized Dirichlet to Neumann maps*.

In particular, we consider the P.D.E

$$(\partial_t + \omega(-i\partial_x))q(t, x) = 0, \quad 0 < t < T, \quad 0 < x < \infty \quad (1.1)$$

where  $\omega$  is defined by

$$\omega(\xi) = a_n \xi^n + a_{n-1} \xi^{n-1} + \cdots + a_1 \xi + a_0 \quad (1.2)$$

for which  $\{a_j\}_{j=0}^n$  are complex constant coefficients.

We assume that the initial condition

$$q_0(x) \doteq q(0, x) \quad (1. 3)$$

has sufficient decaying as  $x \rightarrow \infty$ , as discussed in [1]-[3]. Here, we analyze the generalized Dirichlet to Neumann maps for the following initial-boundary value problems:

**(a)** Let

$$\omega(\xi) = a_n \xi^n \quad (1. 4)$$

where a subset of  $n - N$  boundary values is prescribed as boundary conditions. The exact value of  $N$  will be specified later. We set

$$U \cup V = \{0, 1, \dots, n - 1\} \quad (1. 5)$$

where

$$U = \{u_1, u_2, \dots, u_{n-N}\}, \quad V = \{v_1, v_2, \dots, v_N\}. \quad (1. 6)$$

Let

$$g_l(t) \doteq \partial_x^{u_l} q(t, 0), \quad u_l \in U \quad (1. 7)$$

which are given smooth functions, and are compatible with  $q_0(x)$  at  $x = 0$ . Namely,

$$g_l(0) = q_0(0), \quad u_l \in U. \quad (1. 8)$$

We will present explicit formulae for the remaining unknown boundary values

$$\partial_x^{v_j} q(0, t), \quad v_j \in V \quad (1. 9)$$

in terms of  $\{g_l(t)\}_{l=1}^{n-N}$ . Some examples were computed in [3] for  $n = 2, 3$ .

**(b)** Let  $\omega$  in the general form (1. 2) for  $n \leq 5$  but assume the *canonical* boundary conditions. Namely,  $q(t, 0)$  and its first  $n - N - 1$  derivatives are prescribed as boundary conditions:

$$\partial_x^{u_l} q(t, 0) = g_l(t), \quad u_l \in \{0, 1, 2, \dots, n - N - 1\}. \quad (1. 10)$$

We will present explicit formulae for the unknown boundary values in terms of  $\{g_l(t)\}_{l=1}^{n-N}$ . These formulae involve the solution of an algebraic equation of order  $n - 1$ , which in general can be solved in terms of radicals only when  $n \leq 5$ .

The unified transform (or Fokas transform) was introduced in [1], [2] (see also the book [3] and the reviews [4], [5]). The implementation of the unified transform to evolution equations on the half-line and the finite interval is discussed in [6], [7] and [8]-[11] respectively. The case of periodic initial condition is discussed in [12]. The large  $t$  asymptotics of evolution PDEs on the half-line is analyzed in [13]-[15]. The numerical implementation of the unified transform to evolution PDEs is discussed in [16] and [17]. Rigorous results are presented in [18]. The implementation of the unified transform to evolution PDEs in the two dimensional space is discussed in [6], [20] and [21]. Implications of the unified transform in the area of spectral theory are discussed in [22]-[27]. The implementation of the unified transform to several problems of physical significance is discussed in [24]-[28]. Systems of evolution PDEs are considered in [25] and [26].

## 2 Formulation on the Main Results

In this section, we introduce our main results after certain preliminary settings, for which more background discussions can be found in the book [3].

Observe that equation (1. 1) admits the family of explicit solutions:

$$\exp\{ix\xi - \omega(\xi)t\}, \quad 0 < x < \infty, \quad 0 < t < T \quad (2. 1)$$

where  $\xi \in \mathbb{C}$ . The convergence of the solution is provided by  $\xi \in \mathbb{C}^+ \cup \mathbb{R}$  and  $\mathbf{Re}\omega(\xi) \geq 0$ . Taking into consideration that  $\omega(\xi)$  asymptotes to  $a_n\xi^n$  as  $|\xi| \rightarrow \infty$ , we thus assume that  $\mathbf{Re}a_n \geq 0$  if  $n$  is even, and  $\mathbf{Re}a_n = 0$  if  $n$  is odd. Without loss of generality, we let  $|a_n| = 1$ .

Let  $\mathbf{D}$  to be the *principal domain*:

$$\mathbf{D} = \{\xi \in \mathbb{C} : \mathbf{Re}(a_n\xi^n) < 0\} \quad (2. 2)$$

which is an union of  $n$  sectors in  $\mathbb{C}$ .

Throughout the rest of the paper, we fix the notations

$$a_n = \cos \varphi + i \sin \varphi \quad \text{and} \quad \xi = R(\cos \vartheta + i \sin \vartheta). \quad (2. 3)$$

Notice that  $\mathbf{Re}a_n\xi^n < 0$  implies  $\cos(\varphi + n\vartheta) < 0$ . The  $n$  sectors of  $\mathbf{D}$  are characterized by

$$\left( \vartheta + \frac{\varphi - 2m\pi}{n} \right) \in \left( \frac{\pi}{2n}, \frac{3\pi}{2n} \right) \quad m = 0, 1, \dots, n-1. \quad (2. 4)$$

Moreover,  $\mathbf{Re}a_n \geq 0$  if  $n$  is even and  $\mathbf{Re}a_n = 0$  if  $n$  is odd together imply that

$$\varphi \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]. \quad (2. 5)$$

A direct computation shows that there are exactly  $N$  sectors of  $\mathbf{D}$  lying in  $\mathbb{C}^-$ , whereas

$$N = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd } a_n = i, \\ \frac{n+1}{2} & \text{if } n \text{ is odd } a_n = -i. \end{cases} \quad (2. 6)$$

The solution  $q(t, x)$  of the initial boundary value problems (a) and (b) is well-posed provided that there are essentially  $N$  boundary values given. See [3] and [18] for references.

We write

$$\mathbf{D}^- = \mathbf{D} \cap \mathbb{C}^- = \bigcup_{j=1}^N \mathbf{D}_j^- \quad \text{and} \quad \mathbf{D}^+ = \mathbf{D} \cap \mathbb{C}^+ = \bigcup_{k=1}^{n-N} \mathbf{D}_k^+ \quad (2. 7)$$

where  $\mathbf{D}_k^+$  and  $\mathbf{D}_j^-$  represent the sectors as subdomains of  $\mathbf{D} \cap \mathbb{C}^+$  and  $\mathbf{D} \cap \mathbb{C}^-$  respectively. These sectors are pairwisely disjoint. We number them consecutively by starting with the one with the smallest degree.

We write the boundary of  $\mathbf{D}^+$  by  $\partial\mathbf{D}^+$ , whereas

$$\partial\mathbf{D}^+ = \bigcup_{k=1}^{n-N} \partial\mathbf{D}_k^+ \quad (2.8)$$

with a counterclockwise orientation. Each  $\partial\mathbf{D}_k^+$  consists of the rays  $\mathcal{R}_k^1$  and  $\mathcal{R}_k^2$  with  $\mathcal{R}_k^1$  emanating from the origin and  $\mathcal{R}_k^2$  directed towards the origin. Geometrically, we always have  $\mathcal{R}_k^1$  on the right of  $\mathcal{R}_k^2$  for every  $k = 1, 2, \dots, n - N$ .

By using the notations

$$\vartheta_k^1 = \arg \mathcal{R}_k^1 \quad \text{and} \quad \vartheta_k^2 = \arg \mathcal{R}_k^2, \quad (2.9)$$

Equations (2.4)-(2.5) imply respectively that

$$\vartheta_k^1 = \frac{\pi}{2n} - \frac{\varphi - 2(k-1)\pi}{n}, \quad \vartheta_k^2 = \frac{3\pi}{2n} - \frac{\varphi - 2(k-1)\pi}{n} \quad (2.10)$$

for  $k = 1, 2, \dots, n - N$ . In particular, we have

$$\vartheta_1^1 = \begin{cases} \frac{\pi - 2\varphi}{2n} & n \text{ is even,} \\ \frac{\pi}{n} & n \text{ is odd, } a_n = -\mathbf{i} \\ 0 & n \text{ is odd, } a_n = \mathbf{i} \end{cases} \quad \vartheta_{n-N}^2 = \begin{cases} \pi - \frac{\pi - 2\varphi}{2n} & n \text{ is even,} \\ \pi - \frac{\pi}{n} & n \text{ is odd, } a_n = -\mathbf{i} \\ \pi & n \text{ is odd, } a_n = \mathbf{i}. \end{cases} \quad (2.11)$$

Let

$$\rho = \exp\left(\frac{2\pi i}{n}\right). \quad (2.12)$$

Define the  $N \times N$  Vandermonde matrix

$$\mathbf{V}(\rho) = \begin{bmatrix} \rho^{n-v_1-1} & \rho^{n-v_2-1} & \cdots & \rho^{n-v_N-1} \\ \rho^{2(n-v_1-1)} & \rho^{2(n-v_2-1)} & \cdots & \rho^{2(n-v_N-1)} \\ \vdots & & \ddots & \\ \rho^{N(n-v_1-1)} & \rho^{N(n-v_2-1)} & \cdots & \rho^{N(n-v_N-1)} \end{bmatrix} \quad (2.13)$$

whose determinant is nonzero.

Let  $\mathbf{V}_{ji}$  to be the  $(i, j)$ -th principal minor of  $\mathbf{V}$ . When the matrix is a scalar, we take its  $(1, 1)$ -th principal minor to be 1. From (2.13) we have

$$\det \mathbf{V}(\rho) = \sum_{i=1}^N (-1)^{j+i} \det \mathbf{V}_{ji}(\rho) \rho^{i(n-v_j-1)}, \quad j = 1, 2, \dots, N. \quad (2.14)$$

Let  $\mathbf{V}^{jl}$  to be the matrix obtained from the matrix  $\mathbf{V}$ , after replacing its  $j$ -th column by the column vector  $(\rho^{i(n-u_l-1)})_{1 \leq i \leq N}$ .

Our first main result regarding problem (a) is given below.

**Theorem One:** Let  $q(t, x)$  satisfy the PDE in (1. 1) with  $\omega(\xi)$  defined in (1. 4). Given

$$q(0, x) = q_0(x), \quad 0 < x < \infty; \quad \partial_x^{u_l} q(t, 0) = g_l(t), \quad 0 < t < T, \quad u_l \in U \quad (2. 15)$$

where  $q_0(x)$  and  $g_l(t)$  are defined in (1. 3) and (1. 7), the unknown boundary values can be determined by the following formulae:

$$\begin{aligned} 2\pi \partial_x^{v_j} q(t, 0) &= -\frac{1}{n-N} \sum_{k=1}^{n-N} \int_{\partial D_k^+} \sum_{i=1}^N (-1)^{i+j} \left( \frac{\det \mathbf{V}_{ji}}{\det \mathbf{V}} \right) (\rho^{n-N+1-k}) n \widehat{q}_0(\rho^{n-N+i-k} \xi) (\mathbf{i} \xi)^{v_j} \\ &+ \frac{1}{n-N} \sum_{k=1}^{n-N} \int_{\partial D_k^+} \sum_{u_l < v_j} \left( \frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}} \right) (\rho^{n-N+1-k}) n q_0(0) (\mathbf{i} \xi)^{v_j - u_l - 1} e^{-\omega(\xi)t} d\xi \\ &+ \sum_{v_j < u_l} \mathbf{i}^{v_j - u_l + 1} \Lambda_{jl} \Gamma \left( \frac{v_j - u_l + n}{n} \right) \int_0^t \frac{g_l(\tau) d\tau}{(t - \tau)^{\frac{v_j - u_l + n}{n}}} - \sum_{u_l < v_j} \mathbf{i}^{v_j - u_l + 1} \Lambda_{jl} \Gamma \left( \frac{v_j - u_l}{n} \right) \int_0^t \frac{\dot{g}_l(\tau) d\tau}{(t - \tau)^{\frac{v_j - u_l}{n}}} \end{aligned} \quad (2. 16)$$

for every  $v_j \in V$ , where  $\Gamma$  is Gamma function and

$$\Lambda_{jl} = \left( \frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}} \right) (\rho^{n-N}) \exp \left( \mathbf{i}(v_j - u_l) \left( \vartheta_1^1 - \frac{\pi}{2n} \right) \right) - \left( \frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}} \right) (\rho) \exp \left( \mathbf{i}(v_j - u_l) \left( \vartheta_{n-N}^2 + \frac{\pi}{2n} \right) \right). \quad (2. 17)$$

Let  $\omega$  in the general form (1. 2) but assume  $n \leq 5$ . Consider the domain

$$D = \{ \xi \in \mathbb{C} : \operatorname{Re}(\omega(\xi)) < 0 \}. \quad (2. 18)$$

Since  $\omega(\xi) \approx a_n \xi^n$  as  $\xi$  is asymptotically large, the domain  $D$  coincides with the principal domain  $\mathbf{D}$  in (2. 2) at infinity. We define

$$D_L = \{ \xi \in D : |\xi| \geq L \} \quad D_L^- = D_L \cap \mathbb{C}^- = \bigcup_{j=1}^N D_{L,j}^- \quad D_L^+ = D_L \cap \mathbb{C}^+ = \bigcup_{k=1}^{n-N} D_{L,k}^+ \quad (2. 19)$$

for some  $L$  sufficiently large. Again, we number them consecutively by starting with the one with the smallest degree. Moreover, we write the boundary of  $D_L^+$  by  $\partial D_L^+$ , whereas

$$\partial D_L^+ = \bigcup_{k=1}^{n-N} \partial D_{L,k}^+ \quad (2. 20)$$

with a counterclockwise orientation, for  $L$  sufficiently large. By fundamental theorem of algebra, there are exactly  $n - 1$  functions  $z_i(\xi)$ ,  $i = 1, 2, \dots, n - 1$  other than  $\xi$  itself, determined implicitly by

$$\frac{\omega(z_i) - \omega(\xi)}{z_i - \xi} = 0 \quad (2. 21)$$

such that  $\omega(z_i(\xi)) = \omega(\xi)$  for every  $i = 1, 2, \dots, n - 1$ . Let  $\xi \in D_k^+$  for  $k = 1, 2, \dots, n - N$ , there are exactly  $N$  such functions

$$z_1^k(\xi), z_2^k(\xi), \dots, z_N^k(\xi) \quad (2. 22)$$

whose values lie inside  $\mathbb{C}^-$  for  $\xi$  asymptotically large. See [18] for more discussions.

Let  $\omega_n(\xi) = \omega(\xi)$  and define inductively

$$\omega_m(\xi) = \frac{1}{\xi} (\omega_{m+1}(\xi) - a_{n-m-1}), \quad m = 1, 2, \dots, n - 1. \quad (2. 23)$$

The  $N \times N$  alternant matrix

$$\mathbf{A}(k, \xi) \doteq \mathbf{A}(z_1^k(\xi), z_2^k(\xi), \dots, z_N^k(\xi)) = \begin{bmatrix} \omega_{n-v_1-1}(z_1^k(\xi)) & \omega_{n-v_2-1}(z_1^k(\xi)) & \cdots & \omega_{n-v_N-1}(z_1^k(\xi)) \\ \omega_{n-v_1-1}(z_2^k(\xi)) & \omega_{n-v_2-1}(z_2^k(\xi)) & \cdots & \omega_{n-v_N-1}(z_2^k(\xi)) \\ \vdots & \ddots & & \vdots \\ \omega_{n-v_1-1}(z_N^k(\xi)) & \omega_{n-v_2-1}(z_N^k(\xi)) & \cdots & \omega_{n-v_N-1}(z_N^k(\xi)) \end{bmatrix}. \quad (2. 24)$$

Let  $\mathbf{A}_{ji}$  to be the  $(i, j)$ -th principal minor of  $\mathbf{A}$ . From (2. 24) we have

$$\det \mathbf{A}(k, \xi) = \sum_{i=1}^N (-1)^{j+i} \det \mathbf{A}_{ji}(k, \xi) \omega_{n-v_j-1}(z_i^k(\xi)), \quad j = 1, 2, \dots, N. \quad (2. 25)$$

Let  $\mathbf{A}^{jl}$  to be the matrix obtained from the matrix  $\mathbf{A}$  after replacing its  $j$ -th column by the column vector  $(\omega_{n-u_l-1}(z_i^k(\xi)))_{1 \leq i \leq N}$ . Define

$$\lambda + \exp\left(\frac{\pi i}{n-N}\right). \quad (2. 26)$$

Our second main result regarding problem (b) is given below.

**Theorem Two:** Let  $q(t, x)$  satisfy the PDE in (1. 1) with  $\omega(\xi)$  defined in (1. 2) and  $n \leq 5$ . Given

$$\begin{aligned} q(0, x) &= q_o(x), \quad 0 < x < \infty; \\ \partial_x^{u_l} q(t, 0) &= g_l(t), \quad 0 < t < T, \quad u_l \in \{0, 1, \dots, n-N-1\} \end{aligned} \quad (2. 27)$$

where  $q_o(x)$  and  $g_l(t)$  are defined in (1. 3) and (1. 7), the unknown boundary values can be determined by the following formulae:

$$\begin{aligned} 2\pi(n-N)\partial_x^{v_j} q(t, 0) &= -i^{v_j} \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \sum_{i=1}^N (-1)^{i+j} \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \widehat{q}_o(z_i^k(\xi)) e^{-\omega(\xi)t} d\xi \\ &- \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \sum_{l=1}^{n-N} i^{v_j-u_l+1} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} q_o(0) e^{-\omega(\xi)t} d\xi \\ &+ \sum_{k,l=1}^{n-N} i^{v_j-u_l+1} \mathbf{p} \cdot \mathbf{v} \int_0^\infty \left[ \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \lambda^k \xi) - \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \lambda^{k-1} \xi) \right] \frac{\omega'(\xi)}{\omega(\xi)} \left( \int_0^t e^{\omega(\xi)(\tau-t)} \dot{g}_l(\tau) d\tau \right) d\xi \\ &+ \sum_{k,l=1}^{n-N} i^{v_j-u_l+1} (g_l(t) - g_l(0)) \left[ 2\pi i \sum_{\omega(\zeta)=0, \arg \zeta \in \Theta_k} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \zeta) + \pi i \sum_{\omega(\zeta)=0, \arg \zeta \in \Pi_k} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \zeta) \right] (|\zeta| > 0) \\ &+ \sum_{k,l=1}^{n-N} i^{v_j-u_l+1} (g_l(t) - g_l(0)) \left( \frac{\pi i}{n-N} \right) \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, 0) \quad (\omega(0) = 0) \end{aligned} \quad (2. 28)$$

where  $\Theta_k = \{\vartheta \in [0, 2\pi); \frac{(k-1)\pi}{n-N} < \vartheta < \frac{k\pi}{n-N}\}$  and  $\Pi_k = \{\vartheta \in [0, 2\pi); \vartheta = \frac{(k-1)\pi}{n-N} \text{ and } \vartheta = \frac{k\pi}{n-N}\}$ , for every  $v_j \in \{n-N, n-N+1, \dots, n-1\}$  and  $L$  sufficiently large.

### 3 Novel Integral Representations

In this section, we develop the main framework of unified transform method, known as the *novel integral representations*. It has been used for obtaining certain results in evolution P.D.E. See [1]-[2] and [6]. An introduction can be found in [3] and [5].

The PDE (1. 1) can be rewritten in the following divergence form:

$$\left( e^{-ix\xi+\omega(\xi)t} q(t, x) \right)_t - \left( e^{-ix\xi+\omega(\xi)t} \sum_{m=0}^{n-1} c_m(\xi) \partial_x^m q(t, x) \right)_x = 0, \quad (3. 1)$$

where  $c_m(\xi)$  can be explicitly computed by

$$\sum_{m=0}^{n-1} c_m(\xi) \partial_x^m = i \frac{\omega(\xi) - \omega(\eta)}{\xi - \eta} \Big|_{\eta=-i\partial_x}. \quad (3. 2)$$

From (3. 2) and (2. 23), we have

$$c_m(\xi) = i^{3m+1} \omega_{n-m-1}(\xi). \quad (3. 3)$$

Employing (3. 1) and using Green's theorem in the domain  $\{0 < x < \infty, 0 < s < t\}$ , we find the following *global relation*:

$$e^{\omega(\xi)t} \widehat{q}(t, \xi) = \widehat{q}_o(\xi) - \sum_{m=0}^{n-1} c_m(\xi) \int_0^t e^{\omega(\xi)\tau} \partial_x^m q(\tau, 0) d\tau \quad (3. 4)$$

which has an analytic continuation into the lower half plan for  $\text{Im}\xi \leq 0$ , where

$$\widehat{q}_o(\xi) = \int_0^\infty e^{-x\xi} q_o(x) dx \quad \text{and} \quad \widehat{q}(t, \xi) = \int_0^\infty e^{-x\xi} q(t, x) dx. \quad (3. 5)$$

We define the following  $t$ -transforms:

$$G_l(t, \xi) = \int_0^T e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau, \quad u_l \in U \quad \text{and} \quad Q_j(t, \xi) = \int_0^T e^{\omega(\xi)(\tau-t)} \partial_x^{v_j} q(0, \tau) d\tau, \quad v_j \in V. \quad (3. 6)$$

By using the notations  $\mathbf{a}_j = c_{v_j}$  and  $\mathbf{b}_l = c_{u_l}$ , equation (3. 3) implies

$$\mathbf{a}_j(\xi) = i^{3v_j+1} \omega_{n-v_j-1}(\xi), \quad \mathbf{b}_l(\xi) = i^{3u_l+1} \omega_{n-u_l-1}(\xi). \quad (3. 7)$$

Evaluating (3. 4) at  $t = T$  and multiplying the resulting equation by  $e^{-\omega(\xi)t}$ , we have

$$\sum_{j=1}^N \mathbf{a}_j(\xi) Q_j(T, \xi) = -e^{\omega(\xi)(T-t)} \widehat{q}(T, \xi) + e^{-\omega(\xi)t} \widehat{q}_o(\xi) - \sum_{l=1}^{n-N} \mathbf{b}_l(\xi) G_l(t, \xi). \quad (3. 8)$$

By replacing  $\xi$  in (3. 8) with  $z_i^k(\xi)$ , we obtain the following linear system of  $N$  equations:

$$\sum_{j=1}^N \mathbf{a}_j(z_i(\xi)) Q_j(t, \xi) = -e^{\omega(\xi)(T-t)} \widehat{q}(T, z_i^k(\xi)) + e^{-\omega(\xi)t} \widehat{q}_o(z_i^k(\xi)) - \sum_{l=1}^{n-N} \mathbf{b}_l(z_i^k(\xi)) G_l(t, \xi), \quad i = 1, 2, \dots, N. \quad (3. 9)$$

The inverse of the matrix  $\mathbf{A}$  defined in (2. 24) can be written as

$$\mathbf{A}^{-1}(k, \xi) = \left( \frac{\text{adj}\mathbf{A}}{\det \mathbf{A}} \right)(k, \xi) \quad (3. 10)$$

where  $\text{adj}\mathbf{A}$  is the *adjugate* of  $\mathbf{A}$  whose  $(i, j)$ -th entry equals the determinant of the  $(j, i)$ -th principal minor of  $\mathbf{A}$ , denoted by  $\mathbf{A}_{ji}$  multiplying  $(-1)^{j+i}$ . Recall from section 2 and (3. 7). The matrix  $\mathbf{A}^{jl}$  is constructed from  $\mathbf{A}$  by replacing its  $j$ -th column by column vector  $(\mathbf{b}_l(z_i(\xi)))_{1 \leq i \leq N}$ . By solving the system of the  $N$ -linear equations in (3. 9), we have

$$\begin{aligned} Q_j(t, \xi) &= -\mathbf{i}^{v_j-1} \sum_{i=1}^N (-1)^{j+i} \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right)(k, \xi) \left( e^{\omega(\xi)(T-t)} \widehat{q}(T, z_i^k(\xi)) - e^{-\omega(\xi)t} \widehat{q}_o(z_i^k(\xi)) \right) \\ &\quad - \mathbf{i}^{v_j-u_l} \sum_{l=1}^{n-N} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right)(k, \xi) G_l(t, \xi), \quad j = 1, 2, \dots, N \end{aligned} \quad (3. 11)$$

for every  $k = 1, 2, \dots, n - N$ . Suppose that we are given the canonical boundary values, i.e:  $\mathbf{U} = \{0, 1, \dots, n - N - 1\}$ . Then,

$$\det \mathbf{A}(k, \xi) = a_n^N \prod_{1 \leq i < j \leq N} (z_i^k(\xi) - z_j^k(\xi)) \quad (3. 12)$$

which is not vanishing for  $\xi$  asymptotically large, as was discussed in [18].

The expression in (3. 12) can be observed as follows. By (2. 24), if  $z_i = z_j$  for any pair of  $i \neq j$ , then  $\det \mathbf{A} = 0$ . The determinant is a polynomial of  $z_1, z_2, \dots, z_N$  in an order of  $N(N - 1)/2$ . Therefore, it must be devisable by the product in (3. 12). On the other hand, the number of the factors in the product equals to  $N(N - 1)/2$  which is a combinatorial fact.

Recall that  $\mathbf{A}^{jl}$  is the matrix obtained from  $\mathbf{A}$  after replacing its  $j$ -th column by  $\omega_{n-u_l-1}(z_i)$  for every  $i$ -th row. If  $z_i = z_j$  for any pair of  $i \neq j$ , then  $\det \mathbf{A}^{jl} = 0$ . This implies that the determinant of  $\mathbf{A}^{jl}$  is devisable by the product in (3. 12) as well. When  $n \leq 5$ , each  $z_i = z_i(\xi)$ ,  $i = 1, 2, \dots, N$  can be computed explicitly by radical formulae, which is analytic for  $\xi$  asymptotically large. From all above, the quotient  $\det \mathbf{A}^{jl}/\det \mathbf{A}$  for every  $j, l$  has possibly finitely many removable singularities, and is analytic for  $\xi \in D_L^+$  provided that  $L$  is sufficiently large.

By multiplying equation (3. 4) with  $-\mathbf{i}\omega'(\xi)$  and integrating along the contour  $\partial D_L^+$  defined in (2. 19), we have the following novel integral representation:

$$\begin{aligned} &\int_{\partial D_L^+} \left( \int_0^T -\mathbf{i}\omega'(\xi) e^{\omega(\xi)(\tau-t)} \partial_x^{v_j} q(\tau, 0) d\tau \right) d\xi = \\ &\sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \sum_{i=1}^N \mathbf{i}^{v_j} (-1)^{j+i} \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right)(k, \xi) \omega'(\xi) \left( e^{\omega(\xi)(T-t)} \widehat{q}(T, z_i(\xi)) - e^{-\omega(\xi)t} \widehat{q}_o(z_i(\xi)) \right) d\xi \\ &+ \sum_{l=1}^{n-N} \mathbf{i}^{v_j-u_l+1} \left\{ \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right)(k, \xi) \omega'(\xi) \left( \int_0^T e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi \right\} \end{aligned} \quad (3. 13)$$

for every  $j = 1, 2, \dots, N$ .

## 4 Proof of Theorem Two

Our estimation will be carried out in the general setting where the given boundary conditions prescribed in (1. 7) are not necessary in the canonical form. We assume that

$$\left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right)(k, \xi) \quad k, l = 1, 2, \dots, n - N, \quad j = 1, 2, \dots, N \quad (4. 1)$$

have only finitely many removable singularities, and are analytic for  $\xi$  asymptotically large. If (1. 7) is in the canonical form and  $n \leq 5$ , then this assumption can be removed. The proof will be accomplished within several steps.

1. Recall the definition of  $\mathbf{A}_{ji}$ ,  $\mathbf{A}^{jl}$  and (2. 23) in section 3. Since  $\omega(\xi) \approx a_n \xi^n$  for  $\xi$  is asymptotically large, we shall have

$$\left| \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right)(k, \xi) \right| \lesssim |\xi|^{v_j - n + 1}, \quad \left| \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right)(k, \xi) \right| \lesssim |\xi|^{v_j - u_l} \quad (4. 2)$$

as  $|\xi| \rightarrow \infty$ . From (2. 21)-(2. 22). We have  $|z_i(\xi)| \sim |\xi|$  with  $\text{Im} z_i^k(\xi) < 0$  as  $|\xi| \rightarrow \infty$  for  $i = 1, 2, \dots, N$  and  $k = 1, 2, \dots, n - N$ . Consider the Fourier transform

$$\widehat{q}(T, z_i^k(\xi)) = \int_0^\infty e^{-ixz_i^k(\xi)} q(T, x) dx \quad (4. 3)$$

where  $q(T, x)$  has sufficiently decay as  $x \rightarrow \infty$ . See [1], [3] and [18].

Integration by parts with respect to  $x$  inside (4. 3) gives

$$|\widehat{q}(T, z_i^k(\xi))| \lesssim |\xi|^{-1}, \quad \xi \in D_{L,k}^+. \quad (4. 4)$$

Recall the principal domain  $\mathbf{D}$  in (2. 2) and (2. 7). Define

$$\mathbf{M}(R) = \max_{|\xi|=R} \left\{ \left| \widehat{q}(T, z_i^k(\xi)) \right| : \xi \in \mathbf{D}_k^+, i = 1, 2, \dots, N; k = 1, 2, \dots, n - N \right\} \quad (4. 5)$$

which tends to zero as  $R \rightarrow \infty$ .

2. By writing

$$a_n = \cos \varphi + \mathbf{i} \sin \varphi \quad \text{and} \quad \xi = R(\cos \vartheta + \mathbf{i} \sin \vartheta),$$

we have  $\cos(\varphi + n\vartheta) < 0$  whereas  $\varphi + n\vartheta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$  from (2. 4), for  $\xi \in \mathbf{D}$ .

Consider the norm of

$$\mathbf{I}_j = \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbf{D}_k^+ : |\xi|=R\}} \sum_{i=1}^N (-1)^{j+i} \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right)(k, \xi) \omega'(\xi) e^{\omega(\xi)(T-t)} \widehat{q}(T, z_i^k(\xi)) d\xi. \quad (4. 6)$$

Let  $|\xi| = R$ . From the first inequality in (4. 2) we have

$$\mathbf{M}(R) \sum_{i=1}^N \left| \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right)(k, \xi) \omega'(\xi) \right| \lesssim \mathbf{M}(R) R^{v_j} \quad \text{as } R \rightarrow \infty. \quad (4. 7)$$

Recall that  $\cos(\varphi + n\vartheta) < 0$  is concave for  $\vartheta \in \arg \mathbf{D}^+$  as defined in (2. 4). For every  $t < T$ ,

$$\begin{aligned} |\mathbf{I}_j| &\lesssim \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbf{D}_k^+ : |\xi|=R\}} \sum_{i=1}^N \left| \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \widehat{q}(T, z_i^k(\xi)) \right| e^{\operatorname{Re}\omega(\xi)(T-t)} d\xi \\ &\lesssim \mathbf{M}(R) \int_{\arg \mathbf{D}^+} R^{v_j+1} e^{-R^n \cos(\varphi+n\vartheta)(T-t)} d\vartheta \\ &\lesssim \mathbf{M}(R) \int_{\arg \mathbf{D}^+} R^{v_j+1} e^{-R^n(T-t)\vartheta} d\vartheta \\ &\lesssim \mathbf{M}(R) \left( \frac{1}{R^{n-v_j-1}} \right) \left( \frac{1}{T-t} \right). \end{aligned} \quad (4. 8)$$

Since  $v_j$  is at most  $n - 1$  and  $\mathbf{M}(R)$  turns to zero as  $R$  approaching to infinity, we have  $|\mathbf{I}| \rightarrow 0$  as  $R \rightarrow \infty$ . Since  $D_L^+$  in (2. 18)-(2. 19) coincides with  $\mathbf{D}^+$  at infinity, by applying Cauchy's theorem on  $D_L^+$  we have

$$\sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \sum_{i=1}^N (-1)^{j+i} \left( \frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) e^{\omega(\xi)(T-t)} \widehat{q}(T, z_i^k(\xi)) d\xi = 0. \quad (4. 9)$$

**3.** Consider the left hand side of equation (3. 13). Since the integrant is analytic, we can deform back the contour from  $\partial D_L^+$  to  $\partial D^+$ . Recall that  $\partial D = \{\xi \in \mathbb{C} : \operatorname{Re}\omega(\xi) = 0\}$  and  $D^+$  coincides with  $\mathbf{D}^+$  at infinity which consist of  $n - N$  sectors. By changing variables  $\omega(\xi) = il$ , we have  $l \in \mathbb{R}$  and  $-i\omega'(\xi)d\xi = dl$ . Therefore,

$$\begin{aligned} \int_{\partial D^+} \left( \int_0^T -i\omega'(\xi) e^{\omega(\xi)(\tau-t)} \partial_x^{v_j} q(\tau, 0) d\tau \right) d\xi &= (n - N) \int_{-\infty}^{\infty} e^{il(\tau-t)} dl \left( \int_0^T \partial_x^{v_j} q(\tau, 0) d\tau \right) \\ &= 2\pi(n - N) \int_0^T \delta(\tau - t) \partial_x^{v_j} q(\tau, 0) d\tau \\ &= 2\pi(n - N) \partial_x^{v_j} q(t, 0) \end{aligned} \quad (4. 10)$$

where  $\delta$  is Dirac delta.

**4.** For the remaining terms in (3. 13), suppose  $v_j < u_l$  and consider

$$\begin{aligned} &\sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \left( \int_0^T e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi \\ &= \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \left( \int_0^t e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau + \int_t^T e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi. \end{aligned} \quad (4. 11)$$

Let

$$\mathbf{I}_{jl} = \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbf{D}_k^+ : |\xi|=R\}} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) e^{\omega(\xi)(\tau-t)} d\xi. \quad (4. 12)$$

From the second inequality in (4. 2), we have

$$\left| \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \right| \lesssim |\xi|^{v_j - u_l + (n-1)} \quad \text{as } |\xi| \rightarrow \infty. \quad (4. 13)$$

Recall that  $\cos(\varphi + n\vartheta) < 0$  is concave for  $\vartheta \in \arg \mathbf{D}^+$ . For every  $\tau > t$ ,

$$\begin{aligned} |\mathbf{I}_{jl}| &\lesssim \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbf{D}_k^+: |\xi|=R\}} \left| \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \right| e^{\operatorname{Re} \omega(\xi)(\tau-t)} d\xi \\ &\lesssim \int_{\arg \mathbf{D}^+} R^{v_j - u_l + n} e^{-R^n \cos(\varphi + n\vartheta)(\tau-t)} d\vartheta \\ &\lesssim \int_{\arg \mathbf{D}^+} R^{v_j - u_l + n} e^{-R^n(\tau-t)\vartheta} d\vartheta \\ &\lesssim R^{v_j - u_l} (\tau - t)^{-1}. \end{aligned} \quad (4. 14)$$

By applying Cauchy's theorem on  $D_L^+$ , we have

$$\sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \left( \int_t^T e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi = 0. \quad (4. 15)$$

5. From (2. 2)-(2. 4),  $\xi \in \mathbf{D}_k^+$  implies  $\frac{(k-1)\pi}{n-N} \leq \arg \xi \leq \frac{k\pi}{n-N}$ . Let

$$\mathbf{II}_{jl} = \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbb{C}^+ \setminus \mathbf{D}_k^+: |\xi|=R, \frac{(k-1)\pi}{n-N} \leq \arg \xi \leq \frac{k\pi}{n-N}\}} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) e^{\omega(\xi)(\tau-t)} d\xi. \quad (4. 16)$$

We have  $\cos(\varphi + n\vartheta) > 0$  is convex for  $\vartheta \in \arg(\mathbb{C}^+ \setminus \mathbf{D}^+)$ . For every  $\tau < t$ ,

$$\begin{aligned} |\mathbf{II}_{jl}| &\lesssim \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbb{C}^+ \setminus \mathbf{D}^+: |\xi|=R, \frac{(k-1)\pi}{n-N} \leq \arg \xi \leq \frac{k\pi}{n-N}\}} \left| \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \right| e^{\operatorname{Re} \omega(\xi)(\tau-t)} d\xi \\ &\lesssim \int_{\arg(\mathbb{C}^+ \setminus \mathbf{D}^+)} R^{v_j - u_l + n} e^{-R^n \cos(\varphi + n\vartheta)(t-\tau)} d\vartheta \\ &\lesssim \int_{\arg(\mathbb{C}^+ \setminus \mathbf{D}^+)} R^{v_j - u_l + n} e^{-R^n(t-\tau)\vartheta} d\vartheta \\ &\lesssim R^{v_j - u_l} (t - \tau)^{-1}. \end{aligned} \quad (4. 17)$$

By assumption,  $\det \mathbf{A}^{jl} / \det \mathbf{A}$  has only finitely many removable singularities. By applying Cauchy's theorem on  $\mathbb{C}^+ \setminus D_L^+$ , and Residue's theorem on  $\{\xi \in D^+: |\xi| \leq L\}$ , we have

$$\begin{aligned} &\sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \left( \int_0^t e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi \\ &= \sum_{k=1}^{n-N} \int_0^\infty \left[ \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \lambda^k \xi) - \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \lambda^{k-1} \xi) \right] \omega'(\xi) \left( \int_0^t e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi \end{aligned} \quad (4. 18)$$

where  $\lambda$  is defined in (2. 26).

6. Suppose  $v_j > u_l$  and consider

$$\begin{aligned} & \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \left( \int_0^T e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi \\ &= \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \omega'(\xi) \left( \int_t^T e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau + \int_0^t e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi \end{aligned} \quad (4. 19)$$

Integrating by parts with respect to  $\tau$  yields

$$\begin{aligned} & \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} \left( e^{\omega(\xi)(T-t)} g_l(T) - \int_t^T e^{\omega(\xi)(\tau-t)} \dot{g}_l(\tau) d\tau \right) d\xi \\ & - \sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} \left( e^{-\omega(\xi)t} g_l(0) + \int_0^t e^{\omega(\xi)(\tau-t)} \dot{g}_l(\tau) d\tau \right) d\xi. \end{aligned} \quad (4. 20)$$

Notice that  $\omega'(\xi)/\omega(\xi)$  has possibly simple poles which can occur only on the boundary  $\partial D$  whereas  $\operatorname{Re}\omega(\xi) = 0$ .

From the second inequality in (4. 2), we have

$$\left| \left( \frac{\det \mathbf{A}^{ji}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} \right| \lesssim |\xi|^{v_j - u_l - 1} \quad \text{as } |\xi| \rightarrow \infty. \quad (4. 21)$$

Let

$$\text{III}_{jl} = \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbf{D}^+: |\xi|=R\}} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} e^{\omega(\xi)(\tau-t)} d\xi. \quad (4. 22)$$

Recall that  $\cos(\varphi + n\vartheta) < 0$  is concave for  $\vartheta \in \arg \mathbf{D}^+$ . For every  $\tau > t$ ,

$$\begin{aligned} |\text{III}_{jl}| &\lesssim \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbf{D}^+: |\xi|=R\}} \left| \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} \right| e^{\operatorname{Re}\omega(\xi)(\tau-t)} d\xi \\ &\lesssim \int_{\arg \mathbf{D}^+} R^{v_j - u_l} e^{R^n \cos(\varphi + n\vartheta)(\tau-t)} d\vartheta \\ &\lesssim \int_{\arg \mathbf{D}^+} R^{v_j - u_l} e^{-R^n(\tau-t)\vartheta} d\vartheta \\ &\lesssim R^{v_j - u_l - n} (\tau - t)^{-1}. \end{aligned} \quad (4. 23)$$

The result holds for  $\tau - t$  replaced by  $T - t$ .

By applying Cauchy's theorem on  $D_L^+$ , we have

$$\sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} \left( e^{\omega(\xi)(T-t)} g_l(T) - \int_t^T e^{\omega(\xi)(\tau-t)} \dot{g}_l(\tau) d\tau \right) d\xi = 0. \quad (4. 24)$$

7. Let

$$\mathbf{IV}_{jl} = \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbb{C}^+ \setminus \mathbf{D}_k^+: |\xi|=R, \frac{(k-1)\pi}{n-N} \leq \arg \xi \leq \frac{k\pi}{n-N}\}} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} e^{\omega(\xi)(\tau-t)} d\xi. \quad (4.25)$$

Recall that  $\cos(\varphi + n\vartheta) > 0$  is convex for  $\vartheta \in (\arg \mathbb{C}^+ \setminus \mathbf{D}^+)$ . For every  $\tau < t$ ,

$$\begin{aligned} |\mathbf{IV}_{jl}| &\lesssim \sum_{k=1}^{n-N} \int_{\{\xi \in \mathbb{C}^+ \setminus \mathbf{D}^+: |\xi|=R, \frac{(k-1)\pi}{n-N} \leq \arg \xi \leq \frac{k\pi}{n-N}\}} \left| \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} \right| e^{\operatorname{Re} \omega(\xi)(\tau-t)} d\xi \\ &\lesssim \int_{\arg(\mathbb{C}^+ \setminus \mathbf{D}^+)} R^{v_j - u_l} e^{-R^n \cos(\varphi + n\vartheta)(t-\tau)} d\vartheta \\ &\lesssim \int_{\arg(\mathbb{C}^+ \setminus \mathbf{D}^+)} R^{v_j - u_l} e^{-R^n(t-\tau)\vartheta} d\vartheta \\ &\lesssim R^{v_j - u_l - n} (t - \tau)^{-1}. \end{aligned} \quad (4.26)$$

By applying Cauchy's theorem on  $\mathbb{C}^+ \setminus D_L^+$ , and Residue's theorem on  $\{\xi \in D^+: |\xi| \leq L\}$ , we have

$$\begin{aligned} &\sum_{k=1}^{n-N} \int_{\partial D_{L,k}^+} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} \left( \int_0^t e^{\omega(\xi)(\tau-t)} \dot{g}_l(\tau) d\tau \right) d\xi \\ &= \sum_{k=1}^{n-N} \mathbf{p.v.} \int_0^\infty \left[ \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \lambda^k \xi) - \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \lambda^{k-1} \xi) \right] \frac{\omega'(\xi)}{\omega(\xi)} \left( \int_0^t e^{\omega(\xi)(\tau-t)} \dot{g}_l(\tau) d\tau \right) d\xi \\ &+ \sum_{k=1}^{n-N} \mathbf{i}^{v_j - u_l + 1} (g_l(t) - g_l(0)) \left[ 2\pi \mathbf{i} \sum_{\omega(\zeta)=0, \arg \zeta \in \Theta_k} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \zeta) + \pi \mathbf{i} \sum_{\omega(\zeta)=0, \arg \zeta \in \Pi_k} \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, \zeta) \right] (|\zeta| > 0) \\ &+ \sum_{k=1}^{n-N} \mathbf{i}^{v_j - u_l + 1} (g_l(t) - g_l(0)) \left( \frac{\pi \mathbf{i}}{n - N} \right) \left( \frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}} \right) (k, 0) \quad (\omega(0) = 0) \end{aligned} \quad (4.27)$$

where

$$\Theta_k = \left\{ \vartheta \in [0, 2\pi]: \frac{(k-1)\pi}{n - N} < \vartheta < \frac{k\pi}{n - N} \right\} \quad (4.28)$$

and

$$\Pi_k = \left\{ \vartheta \in [0, 2\pi]: \vartheta = \frac{(k-1)\pi}{n - N} \text{ and } \vartheta = \frac{k\pi}{n - N} \right\} \quad (4.29)$$

for every  $k = 1, 2, \dots, n - N$ .

**Remark 4.1** In (4.18) and (4.27), we deform the contour  $\partial D_{L,k}^+$  to

$$\left\{ \arg \xi = \frac{k\pi}{n - N} \right\} \cup \left\{ |\xi| = L, \frac{(k-1)\pi}{n - N} < \arg \xi < \frac{k\pi}{n - N} \right\} \cup \left\{ \arg \xi = \frac{(k-1)\pi}{n - N} \right\} \quad (4.30)$$

for every  $k = 1, 2, \dots, n - N$  with the same orientation.

Lastly, recall that  $g_l(0) = q_o(0)$  for every  $u_l \in U$ . By combining all above estimates, we have

$$\begin{aligned}
2\pi(n-N)\partial_x^{v_j}q(t,0) &= -\sum_{k=1}^{n-N}\mathbf{i}^{v_j}\int_{\partial D_{L,k}^+}\sum_{i=1}^N(-1)^{i+j}\left(\frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}}\right)(k,\xi)\omega'(\xi)\widehat{q}_o(z_i^k(\xi))e^{-\omega(\xi)t}d\xi \\
&\quad -\sum_{k=1}^{n-N}\int_{\partial D_{L,k}^+}\sum_{u_l < v_j}\mathbf{i}^{v_j-u_l+1}\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\xi)\frac{\omega'(\xi)}{\omega(\xi)}q_o(0)e^{-\omega(\xi)t}d\xi \\
&\quad +\sum_{u_l > v_j}\mathbf{i}^{v_j-u_l+1}\sum_{k=1}^{n-N}\int_0^\infty\left[\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\lambda^k\xi)-\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\lambda^{k-1}\xi)\right]\omega'(\xi)\left(\int_0^t e^{\omega(\xi)(\tau-t)}g_l(\tau)d\tau\right)d\xi \\
&\quad +\sum_{u_l < v_j}\mathbf{i}^{v_j-u_l+1}\sum_{k=1}^{n-N}\mathbf{p.v}\int_0^\infty\left[\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\lambda^k\xi)-\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\lambda^{k-1}\xi)\right]\frac{\omega'(\xi)}{\omega(\xi)}\left(\int_0^t e^{\omega(\xi)(\tau-t)}\dot{g}_l(\tau)d\tau\right)d\xi \\
&\quad +\sum_{u_l < v_j}\sum_{k=1}^{n-N}\mathbf{i}^{v_j-u_l+1}(g_l(t)-g_l(0))\left[2\pi\mathbf{i}\sum_{\omega(\zeta)=0, \arg \zeta \in \Theta_k}\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\zeta)+\pi\mathbf{i}\sum_{\omega(\zeta)=0, \arg \zeta \in \Pi_k}\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\zeta)\right] (|\zeta| > 0) \\
&\quad +\sum_{u_l < v_j}\sum_{k=1}^{n-N}\mathbf{i}^{v_j-u_l+1}(g_l(t)-g_l(0))\left(\frac{\pi\mathbf{i}}{n-N}\right)\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,0) \quad (\omega(0) = 0).
\end{aligned} \tag{4. 31}$$

□

## 5 Proof of Theorem One

Let  $\omega(\xi) = a_n\xi^n$ . Domain  $D$  in (2. 18) is replaced by the principal domain  $\mathbf{D}$  in (2. 2)-(2. 4), which is an union of  $n$  sectors, as oriented in (2. 7). The following quotients of determinants can be computed explicitly :

$$\left(\frac{\det \mathbf{A}_{ji}}{\det \mathbf{A}}\right)(k,\xi) = a_n^{-1}\left(\frac{\det \mathbf{V}_{ji}}{\det \mathbf{V}}\right)(\rho^{n-N+1-k})\xi^{v_j-n+1}, \quad \xi \in \mathbf{D}_k^+, \tag{5. 1}$$

$$\left(\frac{\det \mathbf{A}^{jl}}{\det \mathbf{A}}\right)(k,\xi) = \left(\frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}}\right)(\rho^{n-N+1-k})\xi^{v_j-u_l}, \quad \xi \in \mathbf{D}_k^+. \tag{5. 2}$$

By permutations, we can assume that  $z_j \in \mathbf{D}_j^-$ . By definition in (2. 22), we have

$$z_j^k(\xi) = \rho^{j+n-N-k}\xi \in \mathbf{D}_j^- \quad \text{whenever} \quad \xi \in \mathbf{D}_k^+ \tag{5. 3}$$

for every  $k = 1, 2, \dots, n-N$ , and every  $j = 1, 2, \dots, N$ .

From (4. 27) in the previous section, the contour  $D_L^+$  is replaced by  $\mathbf{D}^+$ , together with the explicit expressions in (5. 1)-(5. 2) and (5. 3).

In particular, we have

$$\begin{aligned}
& \sum_{k=1}^{n-N} i^{v_j} \int_{\partial D_k^+} \sum_{i=1}^N (-1)^{i+j} \left( \frac{\det A_{ji}}{\det A} \right) (k, \xi) \omega'(\xi) \widehat{q}_o(z_i^k(\xi)) e^{-\omega(\xi)t} d\xi \\
&= \sum_{k=1}^{n-N} \int_{\partial D_k^+} \sum_{i=1}^N (-1)^{i+j} \left( \frac{\det V_{ji}}{\det V} \right) (\rho^{n-N+1-k}) (i\xi)^{v_j} n \widehat{q}_o(\rho^{n-N+i-k} \xi) e^{-\omega(\xi)t} d\xi \\
&\quad \sum_{k=1}^{n-N} \int_{\partial D_k^+} \sum_{u_l < v_j} i^{-u_l+1} \left( \frac{\det A^{jl}}{\det A} \right) (k, \xi) \frac{\omega'(\xi)}{\omega(\xi)} q_o(0) e^{-\omega(\xi)t} d\xi \\
&= - \sum_{k=1}^{n-N} \int_{\partial D^+} \sum_{u_l < v_j} \left( \frac{\det V^{jl}}{\det V} \right) (\rho^{n-N+1-k}) (i\xi)^{v_j-u_l-1} n q_o(0) e^{-\omega(\xi)t} d\xi.
\end{aligned} \tag{5. 4}$$

On the other hand, from (5. 3) we have  $z_j^k(\rho\xi) = z_j^{k-1}(\xi)$  for every  $k = 1, 2, \dots, n - N$  which implies that integrations over every  $\partial D_k^+$  are equivalent. Recall the rays  $\mathcal{R}_k^1$  and  $\mathcal{R}_k^2$  defined below (2. 8).

For  $v_j < u_l$ , we have

$$\begin{aligned}
& \int_0^\infty \left[ \left( \frac{\det A^{jl}}{\det A} \right) (n - N, \xi) - \left( \frac{\det A^{jl}}{\det A} \right) (1, \xi) \right] \omega'(\xi) \left( \int_0^t e^{\omega(\xi)(\tau-t)} g_l(\tau) d\tau \right) d\xi \\
&= \left( \frac{\det V^{jl}}{\det V} \right) (\rho) \int_0^t \left\{ \int_{\mathcal{R}_{n-N}^2} \xi^{v_j-u_l} \omega'(\xi) e^{\omega(\xi)(\tau-t)} d\xi \right\} g_l(\tau) d\tau \\
&+ \left( \frac{\det V^{jl}}{\det V} \right) (\rho^{n-N}) \int_0^t \left\{ \int_{\mathcal{R}_1^1} \xi^{v_j-u_l} \omega'(\xi) e^{\omega(\xi)(\tau-t)} d\xi \right\} g_l(\tau) d\tau
\end{aligned} \tag{5. 5}$$

For  $v_j > u_l$ , we have

$$\begin{aligned}
& \int_0^\infty \left[ \left( \frac{\det A^{jl}}{\det A} \right) (n - N, \xi) - \left( \frac{\det A^{jl}}{\det A} \right) (1, \xi) \right] \frac{\omega'(\xi)}{\omega(\xi)} \left( \int_0^t e^{\omega(\xi)(\tau-t)} \dot{g}_l(\tau) d\tau \right) d\xi \\
&= \left( \frac{\det V^{jl}}{\det V} \right) (\rho) \int_0^t \left\{ \int_{\mathcal{R}_{n-N}^2} \xi^{v_j-u_l} \frac{\omega'(\xi)}{\omega(\xi)} e^{\omega(\xi)(\tau-t)} d\xi \right\} \dot{g}_l(\tau) d\tau \\
&+ \left( \frac{\det V^{jl}}{\det V} \right) (\rho^{n-N}) \int_0^t \left\{ \int_{\mathcal{R}_1^1} \xi^{v_j-u_l} \frac{\omega'(\xi)}{\omega(\xi)} e^{\omega(\xi)(\tau-t)} d\xi \right\} \dot{g}_l(\tau) d\tau.
\end{aligned} \tag{5. 6}$$

Let  $\eta = \omega(\xi)$  and write  $\eta = Re^{i\theta}$  in polar coordinates. From (2. 11),

$$\omega(\xi) = \eta = Re^{i\frac{\pi}{2}} \quad \text{for } \xi \in \mathcal{R}_1^1 \tag{5. 7}$$

and

$$\omega(\xi) = \eta = Re^{i\frac{3\pi}{2}} \quad \text{for } \xi \in \mathcal{R}_{n-N}^2. \tag{5. 8}$$

From the estimation in the previous section, we can deform the contours by rotating  $\mathcal{R}_1^1$  by  $\frac{\pi}{2n}$  clockwise, and by rotating  $\mathcal{R}_{n-N}^2$  by  $\frac{\pi}{2n}$  counterclockwise.

Let

$$\tilde{\vartheta}_1 = \vartheta_1 - \frac{\pi}{2n}, \quad \tilde{\vartheta}_2 = \vartheta_2 + \frac{\pi}{2n}. \quad (5. 9)$$

Equation (5. 5) becomes

$$\begin{aligned} & \left( \frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}} \right) (\rho) e^{i(v_j - u_l)\tilde{\vartheta}_2} \int_0^t \left( \int_{\infty}^0 \eta^{\frac{v_j - u_l}{n}} e^{\eta(\tau-t)} d\eta \right) g_l(\tau) d\tau \\ & + \left( \frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}} \right) (\rho^{n-N}) e^{i(v_j - u_l)\tilde{\vartheta}_1} \int_0^t \left( \int_0^{\infty} \eta^{\frac{v_j - u_l}{n}} e^{\eta(\tau-t)} d\eta \right) g_l(\tau) d\tau. \end{aligned} \quad (5. 10)$$

Similarly, equation (5. 6) becomes

$$\begin{aligned} & \left( \frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}} \right) (\rho) e^{i(v_j - u_l)\tilde{\vartheta}_2} \left\{ \int_0^t \left( \int_{\infty}^0 \eta^{\frac{v_j - u_l - n}{n}} e^{\eta(\tau-t)} d\eta \right) \dot{g}_l(\tau) d\tau \right\} \\ & + \left( \frac{\det \mathbf{V}^{jl}}{\det \mathbf{V}} \right) (\rho^{n-N}) e^{i(v_j - u_l)\tilde{\vartheta}_1} \left\{ \int_0^t \left( \int_0^{\infty} \eta^{\frac{v_j - u_l - n}{n}} e^{\eta(\tau-t)} d\eta \right) \dot{g}_l(\tau) d\tau \right\}. \end{aligned} \quad (5. 11)$$

By letting  $-s = \eta(\tau - t)$ , we find

$$\begin{aligned} & \int_0^t \left( \int_0^{\infty} \eta^{\frac{v_j - u_l}{n}} e^{\eta(\tau-t)} d\eta \right) g_l(\tau) d\tau = \int_0^{\infty} s^{\frac{v_j - u_l}{n}} e^{-s} ds \cdot \int_0^t \frac{g_l(\tau) d\tau}{(t-\tau)^{\frac{v_j - u_l + n}{n}}} \\ & = \Gamma\left(\frac{v_j - u_l + n}{n}\right) \int_0^t \frac{g_l(\tau) d\tau}{(t-\tau)^{\frac{v_j - u_l + n}{n}}} \end{aligned} \quad (5. 12)$$

and simultaneously

$$\begin{aligned} & \int_0^t \left( \int_0^{\infty} \eta^{\frac{v_j - u_l + n}{n}} e^{\eta(\tau-t)} d\eta \right) \dot{g}_l(\tau) d\tau = \int_0^{\infty} s^{\frac{v_j - u_l - n}{n}} e^{-s} ds \cdot \int_0^t \frac{\dot{g}_l(\tau) d\tau}{(t-\tau)^{\frac{v_j - u_l}{n}}} \\ & = \Gamma\left(\frac{v_j - u_l}{n}\right) \int_0^t \frac{\dot{g}_l(\tau) d\tau}{(t-\tau)^{\frac{v_j - u_l}{n}}}. \end{aligned} \quad (5. 13)$$

By bringing the above estimates into (4. 31), we have

$$\begin{aligned}
2\pi\partial_x^{v_j}q(t,0) &= -\frac{1}{n-N}\sum_{k=1}^{n-N}\int_{\partial D_k^+}\sum_{i=1}^N(-1)^{i+j}\left(\frac{\det V_{ji}}{\det V}\right)(\rho^{n-N+1-k})n\widehat{q}_o(\rho^{n-N+i-k}\xi)(i\xi)^{v_j} \\
&+ \frac{1}{n-N}\sum_{k=1}^{n-N}\int_{\partial D_k^+}\sum_{u_l < v_j}\left(\frac{\det V^{jl}}{\det V}\right)(\rho^{n-N+1-k})nq_o(0)(i\xi)^{v_j-u_l-1}e^{-\omega(\xi)t}d\xi \\
&+ \sum_{v_j < u_l}i^{v_j-u_l+1}\Lambda_{jl}\Gamma\left(\frac{v_j-u_l+n}{n}\right)\int_0^t\frac{g_l(\tau)d\tau}{(t-\tau)^{\frac{v_j-u_l+n}{n}}} - \sum_{u_l < v_j}i^{v_j-u_l+1}\Lambda_{jl}\Gamma\left(\frac{v_j-u_l}{n}\right)\int_0^t\frac{\dot{g}_l(\tau)d\tau}{(t-\tau)^{\frac{v_j-u_l}{n}}}
\end{aligned} \tag{5. 14}$$

for every  $v_j \in V$ , where

$$\Lambda_{jl} = \left(\frac{\det V^{jl}}{\det V}\right)(\rho^{n-N})\exp\left(i(v_j-u_l)\left(\vartheta_1^1 - \frac{\pi}{2n}\right)\right) - \left(\frac{\det V^{jl}}{\det V}\right)(\rho)\exp\left(i(v_j-u_l)\left(\vartheta_{n-N}^2 + \frac{\pi}{2n}\right)\right) \tag{5. 15}$$

with  $\vartheta_1$  and  $\vartheta_2$  defined in (2. 11).  $\square$

## 6 Examples

The last section is devoted to a number of examples. For brevity, we assume the zero initial condition:  $q_o(x) \equiv 0$  for  $x \in (0, \infty)$ .

**Example 1.** Our first example is the heat equation:

$$q_t(t, x) - q_{xx}(t, x) = 0, \quad 0 < x < \infty, \quad 0 < t < T. \tag{6. 1}$$

Regarding to (6. 1), we have  $a_n = 1$ ,  $n = 2$  and  $N = 1$ . The principal domain  $D$  is shown as below where the rotation  $\rho = e^{i\pi}$  maps  $D^+$  to  $D^-$ .

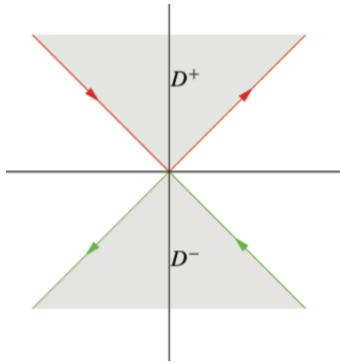


Figure 1:  $\partial D^+$  is represented as the red contour .

From (2. 11), by taking  $n = 2$  and  $\varphi = 0$ , we find

$$\vartheta_1^1 = \frac{\pi}{4}, \quad \vartheta_1^2 = \frac{3\pi}{4}. \tag{6. 2}$$

Suppose that the Dirichlet condition  $q(t, 0) = g(t)$  is given. We aim to find the Neumann boundary value  $q_x(t, 0)$  in terms of  $g(t)$ . In this case, we have  $v_1 = 1, u_1 = 0$ . From (2. 13), the matrix  $\mathbf{V}(\rho)$  is a scalar equal to 1 and  $\mathbf{V}^{11}(\rho) = -1$ .

Bringing all these values into formula (5. 15), we find

$$\begin{aligned}\Lambda_{11} &= \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^1 - \frac{\pi}{4} \right) \right) - \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^2 + \frac{\pi}{4} \right) \right) \\ &= -1 \cdot 1 - (-1)(-1) = -2.\end{aligned}\tag{6. 3}$$

By Theorem One, we have

$$q_x(t, 0) = \frac{-1}{\pi} \Gamma\left(\frac{1}{2}\right) \int_0^t \frac{\dot{g}(\tau) d\tau}{\sqrt{t-\tau}}.\tag{6. 4}$$

On the other hand, let Neumann condition  $q_x(t, 0) = g(t)$  be given. We aim to find  $q(t, 0)$  in terms of  $g(t)$ . We then have  $v_1 = 1, u_1 = 0$ . From (2. 13), the matrix  $\mathbf{V}(\rho)$  is a scalar equal to  $-1$  and  $\mathbf{V}^{11}(\rho) = 1$ . Direct computation shows that  $\Lambda_{11} = -2$  as in (6. 3).

By Theorem One, we have

$$q(t, 0) = \frac{-1}{\pi} \Gamma\left(\frac{1}{2}\right) \int_0^t \frac{g(\tau) d\tau}{\sqrt{t-\tau}}.\tag{6. 5}$$

**Example 2.** Consider a linear evolution equation with a third order derivative:

$$q_t(t, x) + q_{xxx}(t, x) = 0, \quad 0 < x < \infty, \quad 0 < t < T.\tag{6. 6}$$

Regarding to (6. 6), we have  $a_n = -i, n = 3$  and  $N = 2$ . The principal domain  $\mathbf{D}$  is shown as below, where the rotation  $\rho = e^{\frac{2\pi i}{3}}$  maps  $\mathbf{D}^+$  to  $\mathbf{D}_1^-$  and  $\rho^2$  maps  $\mathbf{D}^+$  to  $\mathbf{D}_2^-$ .

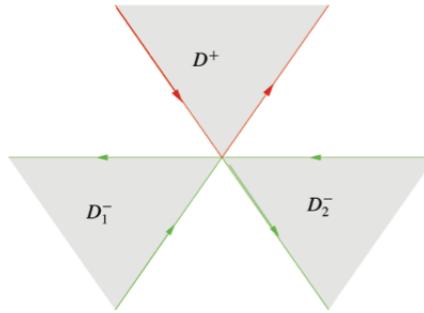


Figure 2:  $\partial\mathbf{D}^+$  is represented as the red contour .

By taking  $n = 3$  and  $\varphi = \frac{3\pi}{2}$  in (2. 11), we find

$$\vartheta_1^1 = \frac{\pi}{3} \quad \text{and} \quad \vartheta_1^2 = \frac{2\pi}{3}.\tag{6. 7}$$

Let  $q(t, 0) = g(t)$  be given. We aim to find  $q_x(t, 0)$  and  $q_{xx}(t, 0)$  in terms of  $g(t)$ . In this case, we have  $u_1 = 0$ ,  $v_1 = 1$  and  $v_2 = 2$ . Therefore,

$$\mathbf{V}(\rho) = \begin{bmatrix} \rho & 1 \\ \rho^2 & 1 \end{bmatrix}, \quad \mathbf{V}^{11}(\rho) = \begin{bmatrix} \rho^2 & 1 \\ \rho & 1 \end{bmatrix}, \quad \mathbf{V}^{21}(\rho) = \begin{bmatrix} \rho & \rho^2 \\ \rho^2 & \rho \end{bmatrix}. \quad (6. 8)$$

From computations, we have

$$\frac{\det \mathbf{V}^{11}}{\det \mathbf{V}}(\rho) = \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}}(\rho) = -1. \quad (6. 9)$$

By (5. 15), we have

$$\begin{aligned} \Lambda_{11} &= \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right)(\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right)(\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= (-1) \exp \left( i \frac{\pi}{6} \right) - (-1) \exp \left( i \frac{5\pi}{6} \right) = -\sqrt{3}, \end{aligned} \quad (6. 10)$$

and

$$\begin{aligned} \Lambda_{21} &= \left( \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}} \right)(\rho) \exp \left( i(v_2 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}} \right)(\rho) \exp \left( i(v_2 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= (-1) \exp \left( i \frac{\pi}{3} \right) - (-1) \exp \left( i \frac{5\pi}{3} \right) = -i\sqrt{3}. \end{aligned} \quad (6. 11)$$

By Theorem One, we have

$$q_x(t, 0) = -\frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{1}{3}\right) \int_0^t \frac{\dot{g}(\tau) d\tau}{(t-\tau)^{\frac{1}{3}}} \quad (6. 12)$$

and

$$q_{xx}(t, 0) = \frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{2}{3}\right) \int_0^t \frac{\dot{g}(\tau) d\tau}{(t-\tau)^{\frac{2}{3}}}. \quad (6. 13)$$

Now, suppose  $q_{xx}(t, 0) = g(t)$  is given. We aim to find  $q(t, 0)$  and  $q_x(t, 0)$  in terms of  $g(t)$ . In this case, we have  $u_1 = 2$ ,  $v_1 = 0$  and  $v_2 = 1$ .

$$\mathbf{V}(\rho) = \begin{bmatrix} \rho^2 & \rho \\ \rho & \rho^2 \end{bmatrix}, \quad \mathbf{V}^{11}(\rho) = \begin{bmatrix} 1 & \rho \\ 1 & \rho^2 \end{bmatrix}, \quad \mathbf{V}^{21}(\rho) = \begin{bmatrix} \rho^2 & 1 \\ \rho & 1 \end{bmatrix}. \quad (6. 14)$$

From computations, we have

$$\frac{\det \mathbf{V}^{11}}{\det \mathbf{V}}(\rho) = \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}}(\rho) = \frac{1}{\rho(\rho+1)} = -1. \quad (6. 15)$$

By (5. 15), we have

$$\begin{aligned} \Lambda_{11} &= \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right)(\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right)(\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= (-1) \exp \left( i \frac{5\pi}{3} \right) - (-1) \exp \left( i \frac{\pi}{3} \right) = \sqrt{3}i, \end{aligned} \quad (6. 16)$$

and

$$\begin{aligned}\Lambda_{21} &= \left( \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_2 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_2 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= (-1) \exp \left( i \frac{11\pi}{6} \right) - (-1) \exp \left( i \frac{7\pi}{6} \right) = -\sqrt{3}.\end{aligned}\tag{6. 17}$$

By Theorem One, we have

$$q(t, 0) = \frac{\sqrt{3}}{2\pi} \Gamma \left( \frac{1}{3} \right) \int_0^t \frac{g(\tau) d\tau}{(t - \tau)^{\frac{1}{3}}}\tag{6. 18}$$

and

$$q_x(t, 0) = -\frac{\sqrt{3}}{2\pi} \Gamma \left( \frac{2}{3} \right) \int_0^t \frac{g(\tau) d\tau}{(t - \tau)^{\frac{2}{3}}}.\tag{6. 19}$$

Lastly, suppose  $q_x(t, 0) = g(t)$  is given. We aim to find  $q(t, 0)$  and  $q_{xx}(t, 0)$  in terms of  $g(t)$ . In this case, we have  $u_1 = 1$ ,  $v_1 = 0$  and  $v_2 = 2$ .

$$\mathbf{V}(\rho) = \begin{bmatrix} \rho^2 & 1 \\ \rho & 1 \end{bmatrix}, \quad \mathbf{V}^{11}(\rho) = \begin{bmatrix} \rho & 1 \\ \rho^2 & 1 \end{bmatrix}, \quad \mathbf{V}^{21}(\rho) = \begin{bmatrix} \rho^2 & \rho \\ \rho & \rho^2 \end{bmatrix}.\tag{6. 20}$$

From computations, we have

$$\frac{\det \mathbf{V}^{11}}{\det \mathbf{V}}(\rho) = \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}}(\rho) = \rho(\rho + 1) = -1.\tag{6. 21}$$

By (5. 15), we have

$$\begin{aligned}\Lambda_{11} &= \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_1 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= (-1) \exp \left( i \frac{11\pi}{6} \right) - (-1) \exp \left( i \frac{7\pi}{6} \right) = -\sqrt{3},\end{aligned}\tag{6. 22}$$

and

$$\begin{aligned}\Lambda_{21} &= \left( \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_2 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{21}}{\det \mathbf{V}} \right) (\rho) \exp \left( i(v_2 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= (-1) \exp \left( i \frac{\pi}{6} \right) - (-1) \exp \left( i \frac{5\pi}{6} \right) = -\sqrt{3}.\end{aligned}\tag{6. 23}$$

By Theorem One, we have

$$q(t, 0) = -\frac{\sqrt{3}}{2\pi} \Gamma \left( \frac{1}{3} \right) \int_0^t \frac{g(\tau) d\tau}{(t - \tau)^{\frac{1}{3}}}\tag{6. 24}$$

and

$$q_{xx}(t, 0) = -\frac{\sqrt{3}}{2\pi} \Gamma \left( \frac{2}{3} \right) \int_0^t \frac{\dot{g}(\tau) d\tau}{(t - \tau)^{\frac{2}{3}}}.\tag{6. 25}$$

**Example 3:** Consider a second linear evolution equation with a third order derivative:

$$q_t(t, x) - q_{xxx}(t, x) = 0, \quad 0 < x < \infty, \quad 0 < t < T. \quad (6. 26)$$

Regarding to (6. 26), we have  $a_n = \mathbf{i}$ ,  $n = 3$  and  $N = 1$ . The principal domain  $\mathbf{D}$  is shown as below, where the rotation  $\rho = e^{\frac{2\pi i}{3}}$  maps  $\mathbf{D}_2^+$  to  $\mathbf{D}^-$  and  $\rho^2$  maps  $\mathbf{D}_1^+$  to  $\mathbf{D}^-$ .

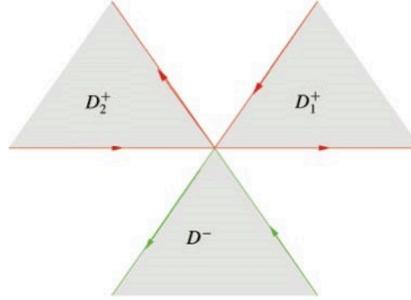


Figure 3:  $\partial\mathbf{D}^+$  is represented as the red contour .

By taking  $n = 3$  and  $\varphi = \frac{3\pi}{2}$  in (2. 11), we find

$$\vartheta_1^1 = 0 \quad \text{and} \quad \vartheta_1^2 = \pi. \quad (6. 27)$$

Let  $q(t, 0) = g_1(t)$  and  $q_x(t, 0) = g_2(t)$  be given. We aim to find  $q_{xx}(t, 0)$  in terms of  $g_1(t)$  and  $g_2(t)$ . In this case, we have  $u_1 = 0$ ,  $u_2 = 1$  and  $v_1 = 2$ . Direct computations show that

$$\begin{aligned} \mathbf{V}(\rho) &= 1, & \mathbf{V}(\rho^2) &= 1, \\ \mathbf{V}^{11}(\rho) &= \rho^2, & \mathbf{V}^{11}(\rho^2) &= \rho, \\ \mathbf{V}^{12}(\rho) &= \rho, & \mathbf{V}^{12}(\rho^2) &= \rho^2. \end{aligned} \quad (6. 28)$$

By (5. 15), we have

$$\begin{aligned} \Lambda_{11} &= \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho^2) \exp \left( \mathbf{i}(v_1 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho) \exp \left( \mathbf{i}(v_1 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= \exp \left( \mathbf{i} \frac{\pi}{3} \right) - \exp \left( \mathbf{i} \frac{5\pi}{3} \right) = \mathbf{i} \sqrt{3}, \end{aligned} \quad (6. 29)$$

and

$$\begin{aligned} \Lambda_{12} &= \left( \frac{\det \mathbf{V}^{12}}{\det \mathbf{V}} \right) (\rho^2) \exp \left( \mathbf{i}(v_2 - u_1) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{12}}{\det \mathbf{V}} \right) (\rho) \exp \left( \mathbf{i}(v_2 - u_1) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= \exp \left( \mathbf{i} \frac{\pi}{6} \right) - \exp \left( \mathbf{i} \frac{5\pi}{6} \right) = -\sqrt{3}. \end{aligned} \quad (6. 30)$$

By Theorem One, we have

$$q_{xx}(t, 0) = -\frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{2}{3}\right) \int_0^t \frac{\dot{g}_1(\tau) d\tau}{(t-\tau)^{\frac{2}{3}}} - \frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{1}{3}\right) \int_0^t \frac{\dot{g}_2(\tau) d\tau}{(t-\tau)^{\frac{1}{3}}}. \quad (6. 31)$$

On the other hand, let  $q_x(t, 0) = g_1(t)$  and  $q_{xx}(t, 0) = g_2(t)$  be given. We aim to find  $q(t, 0)$  in terms of  $g_1(t)$  and  $g_2(t)$ . In this case, we have  $u_1 = 1$ ,  $u_2 = 2$  and  $v_1 = 0$ . Direct computations show that

$$\begin{aligned} \mathbf{V}(\rho) &= \rho^2, & \mathbf{V}(\rho^2) &= \rho, \\ \mathbf{V}^{11}(\rho) &= \rho, & \mathbf{V}^{11}(\rho^2) &= \rho^2, \\ \mathbf{V}^{12}(\rho) &= 1, & \mathbf{V}^{12}(\rho^2) &= 1. \end{aligned} \quad (6. 32)$$

By (5. 15), we have

$$\begin{aligned} \Lambda_{11} &= \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho^2) \exp\left(i(v_1 - u_1)\left(\vartheta_1^1 - \frac{\pi}{6}\right)\right) - \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho) \exp\left(i(v_1 - u_1)\left(\vartheta_1^2 + \frac{\pi}{6}\right)\right) \\ &= \exp\left(i\frac{5\pi}{6}\right) - \exp\left(i\frac{\pi}{6}\right) = -\sqrt{3}, \end{aligned} \quad (6. 33)$$

and

$$\begin{aligned} \Lambda_{12} &= \left( \frac{\det \mathbf{V}^{12}}{\det \mathbf{V}} \right) (\rho^2) \exp\left(i(v_1 - u_2)\left(\vartheta_1^1 - \frac{\pi}{6}\right)\right) - \left( \frac{\det \mathbf{V}^{12}}{\det \mathbf{V}} \right) (\rho) \exp\left(i(v_1 - u_2)\left(\vartheta_1^2 + \frac{\pi}{6}\right)\right) \\ &= \exp\left(i\frac{5\pi}{3}\right) - \exp\left(i\frac{\pi}{3}\right) = -i\sqrt{3}. \end{aligned} \quad (6. 34)$$

By Theorem One, we have

$$q(t, 0) = -\frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{2}{3}\right) \int_0^t \frac{g_1(\tau) d\tau}{(t-\tau)^{\frac{2}{3}}} - \frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{1}{3}\right) \int_0^t \frac{g_2(\tau) d\tau}{(t-\tau)^{\frac{1}{3}}}. \quad (6. 35)$$

Lastly, let  $q(t, 0) = g_1(t)$  and  $q_{xx}(t, 0) = g_2(t)$  be given. We aim to find  $q_x(t, 0)$  in terms of  $g_1(t)$  and  $g_2(t)$ . In this case, we have  $u_1 = 0$ ,  $u_2 = 2$  and  $v_1 = 1$ . Direct computations show that

$$\begin{aligned} \mathbf{V}(\rho) &= \rho, & \mathbf{V}(\rho^2) &= \rho^2, \\ \mathbf{V}^{11}(\rho) &= \rho^2, & \mathbf{V}^{11}(\rho^2) &= \rho, \\ \mathbf{V}^{12}(\rho) &= 1, & \mathbf{V}^{12}(\rho^2) &= 1. \end{aligned} \quad (6. 36)$$

By (5. 15), we have

$$\begin{aligned} \Lambda_{11} &= \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho^2) \exp\left(i(v_1 - u_1)\left(\vartheta_1^1 - \frac{\pi}{6}\right)\right) - \left( \frac{\det \mathbf{V}^{11}}{\det \mathbf{V}} \right) (\rho) \exp\left(i(v_1 - u_1)\left(\vartheta_1^2 + \frac{\pi}{6}\right)\right) \\ &= \exp\left(i\frac{7\pi}{6}\right) - \exp\left(i\frac{11\pi}{6}\right) = -\sqrt{3}, \end{aligned} \quad (6. 37)$$

and

$$\begin{aligned}\Lambda_{12} &= \left( \frac{\det \mathbf{V}^{12}}{\det \mathbf{V}} \right) (\rho^2) \exp \left( \mathbf{i}(v_1 - u_2) \left( \vartheta_1^1 - \frac{\pi}{6} \right) \right) - \left( \frac{\det \mathbf{V}^{12}}{\det \mathbf{V}} \right) (\rho) \exp \left( \mathbf{i}(v_1 - u_2) \left( \vartheta_1^2 + \frac{\pi}{6} \right) \right) \\ &= \exp \left( \mathbf{i} \frac{5\pi}{6} \right) - \exp \left( \mathbf{i} \frac{\pi}{6} \right) = -\sqrt{3}.\end{aligned}\tag{6. 38}$$

By Theorem One, we have

$$q_x(t, 0) = -\frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{1}{3}\right) \int_0^t \frac{\dot{g}_1(\tau) d\tau}{(t-\tau)^{\frac{1}{3}}} - \frac{\sqrt{3}}{2\pi} \Gamma\left(\frac{2}{3}\right) \int_0^t \frac{g_2(\tau) d\tau}{(t-\tau)^{\frac{2}{3}}}.\tag{6. 39}$$

**Example 4:** We next consider an example which is an variance of first third order evolution equation discussed previously, known as the first Stokes equation, with given canonical boundary condition:

$$\begin{cases} q_t(t, x) + q_{xxx}(t, x) + q_x(t, x) = 0, & 0 < x < \infty, \quad 0 < t < T; \\ q(t, 0) = g(t), & 0 < t < T, \end{cases}\tag{6. 40}$$

Regarding to (6. 40), we have  $n = 3, N = 2$  and

$$\omega(\xi) = -\mathbf{i}\xi^3 + \mathbf{i}\xi.\tag{6. 41}$$

The domain  $D$  regarding  $\omega(\xi)$  in (6. 41) is shown as below:

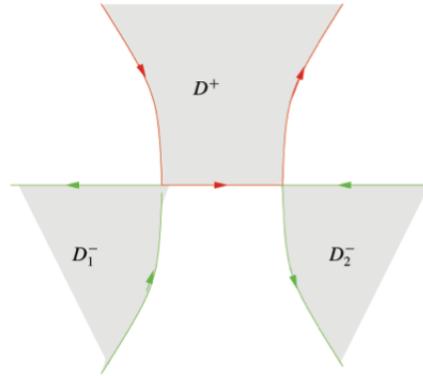


Figure 4:  $\partial D^+$  is represented as the red contour .

We aim to find  $q_x(t, 0)$  and  $q_{xx}(t, 0)$  in terms of  $g(t)$ . Let  $u_1 = 0, v_1 = 1$  and  $v_2 = 2$ . For  $\xi \in \mathbb{C}$  fixed, the roots of (2. 21) can be determined by

$$\frac{\omega(z) - \omega(\xi)}{z - \xi} = -\mathbf{i}(z^2 + \xi z + \xi^2 - 1) = 0.\tag{6. 42}$$

In this case,  $D_L^+ = D_{L,1}^+$ . We abbreviate  $z_i = z_i^1, i = 1, 2$  for which

$$z_1(\xi) = \frac{-\xi + (4 - 3\xi^2)^{\frac{1}{2}}}{2}, \quad z_2(\xi) = \frac{-\xi - (4 - 3\xi^2)^{\frac{1}{2}}}{2}.\tag{6. 43}$$

Notice that as  $|\xi| \rightarrow \infty$  for  $\xi \in D^+$ , we have  $\text{Im}z_1(\xi) \leq 0$  and  $\text{Im}z_2(\xi) \leq 0$ . Recall from (2. 23) and (2. 24), we have

$$\mathbf{A}(z_1, z_2) = \begin{bmatrix} -iz_1 & -i \\ -iz_2 & -i \end{bmatrix}, \quad \mathbf{A}^{11}(z_1, z_2) = \begin{bmatrix} -iz_1^2 + i & -i \\ -iz_2^2 + i & -i \end{bmatrix}, \quad \mathbf{A}^{21}(z_1, z_2) = \begin{bmatrix} -iz_1 & -iz_1^2 + i \\ -iz_2 & -iz_2^2 + i \end{bmatrix}. \quad (6. 44)$$

Direct computations show that

$$\frac{\det \mathbf{A}^{11}}{\det \mathbf{A}}(z_1, z_2) = z_1 + z_2, \quad \frac{\det \mathbf{A}^{21}}{\det \mathbf{A}}(z_1, z_2) = -(1 + z_1 z_2). \quad (6. 45)$$

From (6. 43) we find  $z_1 + z_2 = -\xi$  and  $1 - z_1 z_2 = -\xi^2$  which are both analytic.  $\omega(\xi)$  has zeros at  $-1, 0$  and  $1$ . By Theorem Two, we have

$$2\pi q_x(t, 0) = -\mathbf{p} \cdot \mathbf{v} \int_{-\infty}^{\infty} \frac{3\xi^3 - \xi}{\xi^3 - \xi} \left( \int_0^t e^{i(\xi^3 - \xi)(t-\tau)} \dot{g}(\tau) d\tau \right) d\xi, \quad (6. 46)$$

$$2\pi q_{xx}(t, 0) = -2\pi g(t) - \mathbf{p} \cdot \mathbf{v} \int_{-\infty}^{\infty} i \frac{3\xi^4 - \xi^2}{\xi^3 - \xi} \left( \int_0^t e^{i(\xi^3 - \xi)(t-\tau)} \dot{g}(\tau) d\tau \right) d\xi. \quad (6. 47)$$

**Example 5:** Our last example is the second Stokes equation with given canonical boundary condition:

$$\begin{cases} q_t(t, x) - q_{xxx}(t, x) + q_x(t, x) = 0, & 0 < x < \infty, \quad 0 < t < T; \\ q(t, 0) = g_1(t), & 0 < t < T, \\ q_x(t, 0) = g_2(t), & 0 < t < T. \end{cases} \quad (6. 48)$$

Regarding to (6. 48), we have  $n = 3, N = 1$  and

$$\omega(\xi) = i\xi^3 + i\xi. \quad (6. 49)$$

The domain  $D$  regarding  $\omega(\xi)$  in (6. 49) is shown as below:

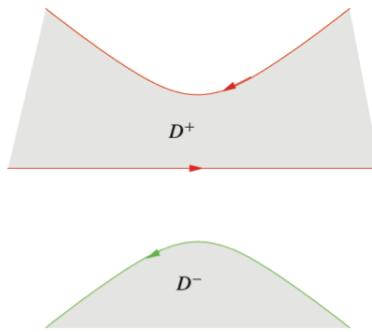


Figure 5:  $\partial D^+$  is represented as the red contour .

Let  $u_1 = 0$ ,  $u_2 = 1$  and  $v_1 = 2$ . For  $\xi \in \mathbb{C}$  fixed, the roots of (2. 21) can be determined by

$$\frac{\omega(z) - \omega(\xi)}{z - \xi} = \mathbf{i}(z^2 + \xi z + \xi^2 + 1) = 0. \quad (6. 50)$$

We find that

$$z_1(\xi) = \frac{-\xi + (-4 - 3\xi^2)^{\frac{1}{2}}}{2}, \quad z_2(\xi) = \frac{-\xi - (-4 - 3\xi^2)^{\frac{1}{2}}}{2} \quad (6. 51)$$

where  $\text{Im}z_1(\xi) \leq 0$  as  $|\xi| \rightarrow \infty$  for  $\xi \in D_{L,1}^+$  and  $\text{Im}z_1^2(\xi) \leq 0$  as  $|\xi| \rightarrow \infty$  for  $\xi \in \mathbf{D}_{L,2}^+$ .

In this case,  $D_L^- = D_{L,1}^-$ . We abbreviate  $z^k = z_1^k$ ,  $k = 1, 2$ . Recall from (2. 23) and (2. 24). We have

$$\begin{aligned} \mathbf{A}(z^k) &= \mathbf{i}, & \mathbf{A}^{11}(z^k) &= \mathbf{i}(z^k)^2 + \mathbf{i}, & \mathbf{A}^{12}(z^k) &= \mathbf{i}z^k \\ \frac{\det \mathbf{A}^{11}}{\det \mathbf{A}}(z^k) &= 1 + (z^k)^2, & \frac{\det \mathbf{A}^{12}}{\det \mathbf{A}}(z^k) &= z^k, & k &= 1, 2. \end{aligned} \quad (6. 52)$$

The quotients of determinants in (6. 52) are analytic for  $|\xi| > 2/\sqrt{3} > 1$ , for which the discriminant in (6. 51) is nonvanishing. On the other hand,  $\omega(\xi) = 0$  in (6. 49) at  $-\mathbf{i}$ ,  $0$  and  $\mathbf{i}$ . By Theorem Two, we have

$$\begin{aligned} 4\pi q_{xx}(t, 0) &= \pi g_2(t) + \mathbf{p.v} \int_0^\infty (z^1(\xi) - z^1(\mathbf{i}\xi) + z^2(\mathbf{i}\xi) - z^2(\xi)) \frac{3\xi^2 + 1}{\xi^3 + \xi} \left( \int_0^t e^{\mathbf{i}(\xi^3 + \xi)(t-\tau)} \dot{g}_2(\tau) d\tau \right) d\xi \\ &\quad + \mathbf{p.v} \int_0^\infty \mathbf{i}((z^1)^2(\xi) - (z^1)^2(\mathbf{i}\xi) + (z^2)^2(\mathbf{i}\xi) - (z^2)^2(\xi)) \frac{3\xi^2 + 1}{\xi^3 + \xi} \left( \int_0^t e^{\mathbf{i}(\xi^3 + \xi)(t-\tau)} \dot{g}_1(\tau) d\tau \right) d\xi. \end{aligned} \quad (6. 53)$$

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