THE θ-BUMP THEOREM FOR PRODUCT FRACTIONAL INTEGRALS

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Abstract. We extend the one parameter θ-bump theorem for fractional integrals of Sawyer and Wheeden to the setting of two parameters, as well as improving the multiparameter result of Tanaka and Yabuta for doubling weights to classical reverse doubling weights.

Contents

1. Introduction

In [SaWh, Theorem 1(A)], Sawyer and Wheeden proved that the fractional integral $I_\alpha f(x) = \int |x-u|^{\alpha-n} f(u)\, du$, $x \in \mathbb{R}^N$, is bounded from one weighted space $L^p(v^p)$ to another $L^q(w^q)$ provided there is $\theta > 1$ such that

$$A^{\alpha,m}_{p,q;\theta}(v,w) \equiv \sup_{I\in\mathcal{D}^N} |I|^{\frac{\alpha}{m}-\frac{1}{p}+\frac{1}{q}} \left(\frac{1}{|I|}\int_I v^{-p'\theta}\right)^{\frac{1}{p'\theta}} \left(\frac{1}{|I|}\int_I w^{q\theta}\right)^{\frac{1}{q\theta}} < \infty.$$

Here $1 < p \le q < \infty$, $0 < \alpha < N$ and v, w are nonnegative measurable functions on $\mathbb{R}^N$, $N \ge 1$. The finiteness of $A^{\alpha,m}_{p,q;1}(v,w)$ when $\theta = 1$ is a well-known necessary condition for the boundedness of I_α, and the above strengthening of that condition is usually referred to as a θ-bump condition. In the same paper [SaWh, the second assertion of Theorem 1(B)], it was shown that in the case $p < q$, if $v^{-p'}$ and w^q are both reverse doubling weights, then the necessary condition $A^{\alpha,m}_{p,q;1}(v,w) < \infty$ is also sufficient for the boundedness of I_α from $L^p(v^p)$ to $L^q(w^q)$. Here a measure μ is *reverse doubling* in $\mathbb{R}^N$ if there are $C, \varepsilon > 0$ such that

$$\left|2^{-s}I\right|_\mu \le C 2^{-\varepsilon s} |I|_\mu\ , \qquad \text{for all } s > 0 \text{ and cubes } I \subset \mathbb{R}^N,$$

where $2^{-s}I$ denotes the cube *concentric* with I and having side length $\ell(2^{-s}I)$ equal to $2^{-s}\ell(I)$.

Recently, H. Tanaka and K. Yabuta [TaYa] used a clever iteration[1] to obtain an n-linear embedding theorem for rectangles that has as a corollary the following result for certain *product* fractional integrals $\widetilde{I}_\alpha^N$ on $\mathbb{R}^N$ given by

$$\widetilde{I}_\alpha^N f(x) \equiv \int_{\mathbb{R}^N} \prod_{j=1}^N |x_j - u_j|^{\alpha-1} f(u)\, du, \qquad x \in \mathbb{R}^N, \qquad 0<\alpha<1.$$

Let $\mathcal{R}^N$ denote the *partial grid* of all rectangles in $\mathbb{R}^N$ with sides parallel to the coordinate axes (which is not a grid). A weight μ is a rectangle doubling weight on $\mathbb{R}^N$ if there is $C>0$ such that

$$|2R|_\mu \le C\,|R|_\mu\,, \qquad \text{for all rectangles } R \in \mathcal{R}^N.$$

Theorem 1.1 (H. Tanaka and K. Yabuta [TaYa, Proposition 5.1]). *Suppose* $1<p<q<\infty$ *and that both* $v^{-p'}$ *and* w^q *are rectangle doubling weights*[2] *on* $\mathbb{R}^N$. *Then* $\widetilde{I}_\alpha^N$ *is bounded from* $L^p(v^p)$ *to* $L^q(w^q)$ *if and only if*

$$\sup_{R\in\mathcal{R}^N} |R|^{\frac{\alpha}{N}-\frac{1}{p}+\frac{1}{q}} \left(\frac{1}{|R|}\int_R v^{-p'}\right)^{\frac{1}{p'}} \left(\frac{1}{|R|}\int_R w^q\right)^{\frac{1}{q}} < \infty.$$

Thus the theorem of Tanaka and Yabuta extends the second assertion in Theorem 1(B) of [SaWh] to product fractional integrals, but with the stronger assumption of rectangle doubling weights instead of rectangle reverse doubling weights. The purpose of this paper is to extend both Theorem 1(A) to product fractional integrals, and to extend the second assertion in Theorem 1(B) of [SaWh] to product fractional integrals and rectangle reverse doubling weights. Here a product fractional integral has the form (more than two factors in the kernel are handled similarly)

$$I_{\alpha,\beta}^{m,n} f(x,y) \equiv \int_{\mathbb{R}^m}\int_{\mathbb{R}^n} |x-u|^{\frac{\alpha}{m}-1} |y-t|^{\frac{\beta}{n}-1} f(u,t)\, dt du, \qquad (x,y)\in\mathbb{R}^m\times\mathbb{R}^n.$$

More precisely, we show in Theorem A below that a rectangle θ-bump condition is always sufficient for the norm inequality, and in Theorem B below that the same condition without a bump is sufficient provided the weights $v^{-p'}$ and w^q are rectangle reverse doubling on $\mathbb{R}^m\times\mathbb{R}^n$ in this sense: a weight μ is *rectangle reverse doubling* on $\mathbb{R}^m\times\mathbb{R}^n$ if there are $C,\varepsilon_1,\varepsilon_2>0$ such that

$$(1.1)\quad \left|\left(2^{-s}I\right)\times\left(2^{-t}J\right)\right|_\mu \le C2^{-\varepsilon_1 s-\varepsilon_2 t}\,|I\times J|_\mu\ , \qquad \text{for all } s,t>0 \text{ and cubes } I\subset\mathbb{R}^m \text{ and } J\subset\mathbb{R}^n.$$

It is well known that rectangle reverse doubling is strictly weaker than rectangle doubling - see the Appendix below.

Theorem A: *Let* $1<p<q<\infty$, $0<\alpha<m$, $0<\beta<n$, $\theta>1$, *and let* v *and* w *be absolutely continuous weights on* $R^m\times R^n$. *Then the product fractional integral* $I_{\alpha,\beta}^{m,n}$ *is bounded from* $L^p(v^p)$

[1] In a nutshell, they use $p<r<q$ and Hölder's inequality with r and r' to separate the measures σ and ω early on, and then use iteration on the resulting 'one weight' Carleson embeddings, the point being that iteration works better with one weight than with two weights.

[2] In [TaYa] the authors use a strong form of reverse doubling on rectangles, which is equivalent to rectangle doubling. See the appendix below.

to $L^q(w^q)$ *if the* θ*-bump rectangle characteristic* $A_{p,q;\theta}^{(\alpha,\beta),(m,n)}(v,w)$ *is finite, where*

$$\mathbb{A}_{p,q;\theta}^{(\alpha,\beta),(m,n)}(v,w) \equiv \sup_{I\times J\in\mathcal{R}^{m,n}} |I|^{\frac{\alpha}{m}-\frac{1}{p}+\frac{1}{q}} |J|^{\frac{\beta}{n}-\frac{1}{p}+\frac{1}{q}} \left(\frac{1}{|I\times J|}\int\int_{I\times J} v^{-p'\theta}\right)^{\frac{1}{p'\theta}} \left(\frac{1}{|I\times J|}\int\int_{I\times J} w^{q\theta}\right)^{\frac{1}{q\theta}} .$$

Theorem B: *Let* $1<p<q<\infty$, $0<\alpha<m$, $0<\beta<n$, *and let* v *and* w *be rectangle reverse doubing weights on* $R^m\times R^n$. *Then the product fractional integral* $I_{\alpha,\beta}^{m,n}$ *is bounded from* $L^p(v^p)$ *to* $L^q(w^q)$ *if and only if the rectangle characteristic* $A_{p,q}^{(\alpha,\beta),(m,n)}(v,w)$ *is finite, where*

$$\mathbb{A}_{p,q}^{(\alpha,\beta),(m,n)}(v,w) \equiv \sup_{I\times J\in\mathcal{R}^{m,n}} |I|^{\frac{\alpha}{m}-\frac{1}{p}+\frac{1}{q}} |J|^{\frac{\beta}{n}-\frac{1}{p}+\frac{1}{q}} \left(\frac{1}{|I\times J|}\int\int_{I\times J} v\right)^{\frac{1}{p'}} \left(\frac{1}{|I\times J|}\int\int_{I\times J} w^{q}\right)^{\frac{1}{q}} .$$

Theorem A and Theorem B are obtained as corollaries of two more general theorems, namely Theorem 2.1 and Theorem 2.5 respectively, concerning the theory of positive bilinear forms, which is developed in Section 2 below.

Our proof of the first result adapts the Tanaka-Yabuta argument to the θ-bump functional used in [SaWh], while the second result regarding rectangle reverse doubling weights adapts the Tanaka-Yabuta argument to the use of NTV good/bad grids in place of the Strömberg $\frac{1}{3}$-trick that was used in [TaYa]. Additional results for the product situation can be found in our paper [SaWa]. See the appendix below for a discussion of the doubling and various reverse doubling conditions.

Acknowledgement: We are grateful to Hitoshi Tanaka for bringing our attention to his beautiful paper [TaYa] with K. Yabuta.

1.1. **Preliminaries.** Let $\mathcal{D}^m$ denote the grid of dyadic cubes in $\mathbb{R}^m$, and let $\mathcal{R}^{m,n}\equiv\mathcal{D}^m\times\mathcal{D}^n$ denote the partial grid of dyadic rectangles in $\mathbb{R}^m\times\mathbb{R}^n$ (which is not actually a grid since it fails the nested property). For $d\mu(x)=u(x)\,dx$ absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^N$, we will use the following θ-bump functional for a cube Q and $\theta>1$ ([SaWh, see page 830]):

$$|Q|_{\mu,\theta} \equiv |Q|^{\frac{1}{\theta'}}\left(\int_Q u^\theta\right)^{\frac{1}{\theta}} .$$

We have $|Q|_\mu \le |Q|_{\mu,\theta}$, and if $P=\dot{\bigcup}_{i=1}^{\infty} Q_i$ is a pairwise disjoint union of the cubes Q_i, then we have

$$\sum_{i=1}^{\infty}|Q_i|_{\mu,\theta} = \sum_{i=1}^{\infty}|Q|^{\frac{1}{\theta'}}\left(\int_Q u^\theta\right)^{\frac{1}{\theta}} \le \left(\sum_{i=1}^{\infty}|Q_i|\right)^{\frac{1}{\theta'}}\left(\sum_{i=1}^{\infty}\int_{Q_i} u^\theta\right)^{\frac{1}{\theta}} = |P|^{\frac{1}{\theta'}}\left(\int_P u^\theta\right)^{\frac{1}{\theta}} = |P|_{\mu,\theta} .$$

The important property of the θ-bump functional on cubes for us is that, when taken to a power larger than 1, it automatically satisfies a Carleson condition taken over all dyadic subcubes. More precisely, if $\rho > 1$, then

$$\sum_{Q\in\mathcal{D}^N:\ Q\subset P} |Q|_{\mu,\theta}^{\rho} = \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{D}^N:\ \ell(Q)=2^{-k}\ell(P)} |Q|^{\frac{\rho-1}{\theta'}} \left(\int_Q u^\theta\right)^{\frac{\rho-1}{\theta}} |Q|_{\mu,\theta} \tag{1.2}$$
$$\leq \sum_{k=-\infty}^{\infty} \sum_{Q\in\mathcal{D}^N:\ \ell(Q)=2^{-k}\ell(P)} \left(C2^{-kN\varepsilon}|P|\right)^{\frac{\rho-1}{\theta'}} \left(\int_P u^\theta\right)^{\frac{\rho-1}{\theta}} |Q|_{\mu,\theta}$$
$$\leq \sum_{k=-\infty}^{\infty} \left(C2^{-kN\varepsilon}|P|\right)^{\frac{\rho-1}{\theta'}} \left(\int_P u^\theta\right)^{\frac{\rho-1}{\theta}} |P|_{\mu,\theta} = C_{N\varepsilon\frac{\rho-1}{\theta'}} |P|_{\mu,\theta}^{\rho} .$$

This automatic Carleson condition leads to a corresponding automatic Carleson embedding lemma.

Lemma 1.2. *Suppose that* $1 < s < r < \infty$, $\theta > 1$, *and that* $d\mu(x) = u(x)\,dx$ *is a locally* L^θ *absolutely continuous measure on* $\mathbb{R}^N$. *Then we have*

$$\left\{\sum_{Q\in\mathcal{D}^N} |Q|_{\mu,\theta}^{\frac{r}{s}} \left(\frac{1}{|Q|_{\mu,\theta}}\int_Q f d\mu\right)^r\right\}^{\frac{1}{r}} \leq C_{r,s,\theta}\|f\|_{L^s(\mu)} , \qquad f \geq 0.$$

Proof. The cubes in $\mathcal{D}^N$ form a grid, and so for each integer $k\in\mathbb{Z}$, we can consider the maximal dyadic cubes $\left\{M_i^k\right\}_{i=1}^{\infty}$ from $\mathcal{D}^N$ such that

$$\frac{1}{\left|M_i^k\right|_{\mu,\theta}}\int_{M_i^k} f d\mu > 2^k.$$

Then we can estimate using (1.2) that

$$\sum_{Q\in\mathcal{D}^N} |Q|_{\mu,\theta}^{\frac{r}{s}} \left(\frac{1}{|Q|_{\mu,\theta}}\int_Q f d\mu\right)^r \leq \sum_{k=-\infty}^{\infty} \sum_{\substack{Q\in\mathcal{D}^N\\ 2^k<\frac{1}{|Q|_{\mu,\theta}}\int_Q fd\mu\leq 2^{k+1}}} |Q|_{\mu,\theta}^{\frac{r}{s}} \left(2^{k+1}\right)^r$$
$$\leq \sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty}\sum_{Q\in\mathcal{D}^N:\ Q\subset M_i^k} |Q|_{\mu,\theta}^{\frac{r}{s}} \left(2^{k+1}\right)^r$$
$$= 2^r \sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty}\left\{\sum_{Q\in\mathcal{D}^N:\ Q\subset M_i^k} |Q|_{\mu,\theta}^{\frac{r}{s}}\right\} 2^{kr} \leq 2^r C_{N\varepsilon\frac{\frac{r}{s}-1}{\theta'}} \sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty} \left|M_i^k\right|_{\mu,\theta}^{\frac{r}{s}} 2^{kr} .$$

Now we use the fact that

$$\frac{1}{\left|M_i^k\right|_{\mu,\theta}}\int_{M_i^k\cap\left\{f>2^{k-1}\right\}} f d\mu = \frac{1}{\left|M_i^k\right|_{\mu,\theta}}\int_{M_i^k} f d\mu - \frac{1}{\left|M_i^k\right|_{\mu,\theta}}\int_{M_i^k\cap\left\{f\leq 2^{k-1}\right\}} f d\mu$$
$$\geq \frac{1}{\left|M_i^k\right|_{\mu,\theta}}\int_{M_i^k} f d\mu - \frac{1}{\left|M_i^k\right|_{\mu,\theta}}\int_{M_i^k} 2^{k-1} d\mu$$
$$> 2^k - 2^{k-1}\frac{\left|M_i^k\right|_{\mu}}{\left|M_i^k\right|_{\mu,\theta}} \geq 2^{k-1},$$

to obtain

$$\begin{aligned}\sum_{Q\in\mathcal{D}^N}|Q|_{\mu,\theta}^{\frac{r}{s}}\left(\frac{1}{|Q|_{\mu,\theta}}\int_Q fd\mu\right)^r &\leq 2^r C_{N\varepsilon\frac{\frac{r}{s}-1}{\theta'}}\sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty}\left|M_i^k\right|_{\mu,\theta}^{\frac{r}{s}}2^{kr}\\ &\leq C_{r,s,\theta}^r\sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty}\left(2^{-k}\int_{M_i^k\cap\left\{f>2^{k-1}\right\}}fd\mu\right)^{\frac{r}{s}}2^{kr}\\ &\leq C_{r,s,\theta}^r\left(\sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty}2^{k(s-1)}\int_{M_i^k\cap\left\{f>2^{k-1}\right\}}fd\mu\right)^{\frac{r}{s}}.\end{aligned}$$

We now use that the cubes $\left\{M_i^k\right\}_{i=1}^{\infty}$ are pairwise disjoint in i to continue with the estimate

$$\begin{aligned}&\left(\sum_{k=-\infty}^{\infty}\sum_{i=1}^{\infty}2^{k(s-1)}\int_{M_i^k\cap\left\{f>2^{k-1}\right\}}fd\mu\right)^{\frac{r}{s}}\leq\left(\sum_{k=-\infty}^{\infty}2^{k(s-1)}\int_{\left\{f>2^{k-1}\right\}}fd\mu\right)^{\frac{r}{s}}\\ =&\left(\int\left\{\sum_{k\in\mathbb{Z}:\ 2^k<2f(x)}2^{k(s-1)}\right\}f(x)\,d\mu(x)\right)^{\frac{r}{s}}\leq C_s\left(\int f(x)^{(s-1)}f(x)\,d\mu(x)\right)^{\frac{r}{s}}\\ =&\ C_s\left(\int f(x)^s\,d\mu(x)\right)^{\frac{r}{s}}=C_s\left\|f\right\|_{L^s(\mu)}^r\ .\end{aligned}$$

□

2. The 2-parameter theory

Here we state and prove our extensions of Theorem 1(A) and the second assertion of Theorem 1(B) in [SaWh]. We begin with the θ-bump condition.

2.1. The θ-bump condition for bilinear embeddings.

Here is a variation on the Tanaka-Yabuta theorem [TaYa, Theorem 1.1] involving general weights that satisfy a θ-bump analogue of the 'rectangle testing' condition in [TaYa]. We extend the definition of the θ-bump functional to rectangles in the obvious way,

$$|R|_{\mu,\theta}\equiv|R|^{\frac{1}{\theta'}}\left(\int_R u^\theta\right)^{\frac{1}{\theta}},$$

for $d\mu(x,y)=u(x,y)\,dxdy$ absolutely continuous and R a rectangle in $\mathbb{R}^m\times\mathbb{R}^n$.

Theorem 2.1. *Suppose $1<p<q<\infty$. Let $d\sigma=v^{-p'}dx$ and $d\omega=w^q dx$ be locally finite absolutely continuous weights on $\mathbb{R}^m\times\mathbb{R}^n$, let $\theta>1$, and let $K:\mathcal{R}^{m,n}\to[0,\infty)$. Then the norm $\mathbb{N}_K(\sigma,\omega)$ of the positive bilinear inequality,*

$$\sum_{R\in\mathcal{R}^{m,n}}K(R)\left(\int_R fd\sigma\right)\left(\int_R gd\omega\right)\leq\mathbb{N}_K(\sigma,\omega)\ \left\|f\right\|_{L^p(\sigma)}\left\|g\right\|_{L^{q'}(\omega)}\ ,\qquad f,g\geq 0,$$

is finite independent of all partial grids $\mathcal{R}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n$ *if the* θ*-bump product characteristic* $\mathbb{A}_{K,\theta}(\sigma,\omega)$ *is finite, where*

$$\begin{aligned}\mathbb{A}_{K,\theta}(\sigma,\omega) &\equiv \sup_{R\in\mathcal{R}^{m,n}} K(R) \left[|R|^{\frac{1}{p'\theta'}}\left(\int_R v^{-p'\theta}d\sigma\right)^{\frac{1}{p'\theta}}\right]\left[|R|^{\frac{1}{q\theta'}}\left(\int_R w^{q\theta}d\omega\right)^{\frac{1}{q\theta}}\right]\\ &= \sup_{R\in\mathcal{R}^{m,n}} K(R)\ |R|_{\omega,\theta}^{\frac{1}{q}}\ |R|_{\sigma,\theta}^{\frac{1}{p'}}\ .\end{aligned}$$

Proof. As in [TaYa], we choose $p<r<q$. Then the definition of the θ-bump characteristic, followed by Hölder's inequality with exponents r and r', gives

$$\begin{aligned}&\sum_{R\in\mathcal{R}^{m,n}} K(R)\left(\int_R fd\sigma\right)\left(\int_R gd\omega\right)\\ &= \sum_{R\in\mathcal{R}^{m,n}}\left\{K(R)\ |R|_{\sigma,\theta}^{\frac{1}{p'}}|R|_{\omega,\theta}^{\frac{1}{q}}\right\}|R|_{\sigma,\theta}^{\frac{1}{p}}|R|_{\omega,\theta}^{\frac{1}{q'}}\left(\frac{1}{|R|_{\sigma,\theta}}\int_R fd\sigma\right)\left(\frac{1}{|R|_{\omega,\theta}}\int_R gd\omega\right)\\ &\leq \mathbb{A}_{K,\theta}(\sigma,\omega)\left\{\sum_{R\in\mathcal{R}^{m,n}}|R|_{\sigma,\theta}^{\frac{r}{p}}\left(\frac{1}{|R|_{\sigma,\theta}}\int_R fd\sigma\right)^r\right\}^{\frac{1}{r}}\left\{\sum_{R\in\mathcal{R}^{m,n}}|R|_{\omega,\theta}^{\frac{r'}{q'}}\left(\frac{1}{|R|_{\omega,\theta}}\int_R gd\omega\right)^{r'}\right\}^{\frac{1}{r'}},\end{aligned}$$

and the theorem now follows from the following proposition. □

Proposition 2.2. *Suppose that* $1<s<r<\infty$, $\theta>1$, *and that* μ *is a locally finite absolutely continuous measure on* $\mathbb{R}^m\times\mathbb{R}^n$. *Then we have*

$$\left\{\sum_{R\in\mathcal{R}^{m,n}}|R|_{\mu,\theta}^{\frac{r}{s}}\left(\frac{1}{|R|_{\mu,\theta}}\int_R fd\mu\right)^r\right\}^{\frac{1}{r}} \leq C_{s,r,\theta}\|f\|_{L^s(\mu)}\ , \qquad f\geq 0.$$

Proof. We follow the outline of the iteration argument in H. Tanaka and K. Yabuta [TaYa], but adapted to θ-bump functionals. Let $d\mu(x,y)=u(x,y)\,dxdy$ and define

$$\begin{aligned}&u^y(x)\equiv u(x,y)\ \text{ and } u_x(y)\equiv u(x,y),\\ &d\mu^y(x)=u^y(x)\,dx \text{ and } d\mu_x(y)=u_x(y)\,dy\\ &\qquad\text{for a.e. } x\in\mathbb{R}^m,\ \text{a.e. } y\in\mathbb{R}^n,\end{aligned}$$

and note that

$$\begin{aligned}&|J|_{\mu_x,\theta}\equiv|J|^{\frac{1}{\theta'}}\left(\int_J u_x(y)^\theta\,dy\right)^{\frac{1}{\theta}} \text{ and } |I|_{\mu^y,\theta}\equiv|I|^{\frac{1}{\theta'}}\left(\int_I u^y(y)^\theta\,dx\right)^{\frac{1}{\theta}}\\ &\text{for a.e. } x\in\mathbb{R}^m,\ \text{a.e. } y\in\mathbb{R}^n.\end{aligned}$$

Now take $f\in L^p(\mu)$ and let

$$F^J(x)\equiv\frac{1}{|J|_{\mu_x,\theta}}\int_J f(x,y)\,u(x,y)\,dy \qquad \text{for a.e. } x\in\mathbb{R}^m.$$

Note that

$$\left|I\times J\right|_{\mu,\theta}=\left|I\times J\right|^{\frac{1}{\theta'}}\left(\int_I\left\{\int_J u\left(x,y\right)^{\theta}dy\right\}dx\right)^{\frac{1}{\theta}}=\left|I\right|^{\frac{1}{\theta'}}\left(\int_I\left\{\left|J\right|^{\frac{1}{\theta'}}\left(\int_J u\left(x,y\right)^{\theta}dy\right)^{\frac{1}{\theta}}\right\}^{\theta}dx\right)^{\frac{1}{\theta}}$$

where we can interpret the term in braces as

$$\left|J\right|^{\frac{1}{\theta'}}\left(\int_J u_x\left(y\right)^{\theta}dy\right)^{\frac{1}{\theta}}=\left|J\right|_{\mu_x,\theta}$$

so that we have

$$\left|I\times J\right|_{\mu,\theta}=\left|I\right|^{\frac{1}{\theta'}}\left(\int_I\left|J\right|^{\theta}_{\mu_x,\theta}dx\right)^{\frac{1}{\theta}}\equiv\left|I\right|^{\frac{1}{\theta'}}\left(\int_I\left(J_{\mu,\theta}\left(x\right)\right)^{\theta}dx\right)^{\frac{1}{\theta}}=\left|I\right|_{J_{\mu,\theta},\theta}$$

where we have defined the absolutely continuous measure $J_{\mu,\theta}$ by $dJ_{\mu,\theta}\left(x\right)=J_{\mu,\theta}\left(x\right)dx$ and where its density function, which with a small abuse of notation we also denote by $J_{\mu,\theta}$, is given by

$$J_{\mu,\theta}\left(x\right)\equiv\left|J\right|_{\mu_x,\theta}\,,\qquad x\in\mathbb{R}^m.$$

We then estimate

$$\begin{aligned}
&\sum_{R\in\mathcal{R}^{m,n}}\left|R\right|^{\frac{r}{s}}_{\mu,\theta}\left(\frac{1}{\left|R\right|_{\mu,\theta}}\int_R f\left(x,y\right)u\left(x,y\right)dxdy\right)^{r}\\
=&\sum_{I\times J\in\mathcal{R}^{m,n}}\left|I\times J\right|^{\frac{r}{s}}_{\mu,\theta}\left(\frac{1}{\left|I\times J\right|_{\mu,\theta}}\int_{I\times J}f\left(x,y\right)u\left(x,y\right)dxdy\right)^{r}\\
=&\sum_{J\in\mathcal{D}^n}\sum_{I\in\mathcal{D}^m}\left|I\right|^{\frac{r}{s}}_{J_{\mu,\theta},\theta}\left(\frac{1}{\left|I\right|_{J_{\mu,\theta},\theta}}\int_I\left(\int_J f\left(x,y\right)u\left(x,y\right)dy\ \frac{1}{J_{\mu,\theta}\left(x\right)}\right)J_{\mu,\theta}\left(x\right)dx\right)^{r}\\
=&\sum_{J\in\mathcal{D}^n}\left\{\sum_{I\in\mathcal{D}^m}\left|I\right|^{\frac{r}{s}}_{J_{\mu,\theta},\theta}\left(\frac{1}{\left|I\right|_{J_{\mu,\theta},\theta}}\int_I F^J\left(x\right)J_{\mu,\theta}\left(x\right)dx\right)^{r}\right\}\lesssim\sum_{J\in\mathcal{D}^n}\left(\int_{\mathbb{R}^m}F^J\left(x\right)^{s}J_{\mu,\theta}\left(x\right)dx\right)^{\frac{r}{s}},
\end{aligned}$$

by Lemma 1.2 above applied with the locally finite absolutely continuous measures $J_{\mu,\theta}$ on $\mathbb{R}^m$, $J\in\mathcal{D}^n$. Now we continue to estimate the latter sum raised to the power $\frac{s}{r}$ by Minkowski's inequality,

$$\left\{\sum_{J\in\mathcal{D}^n}\left(\int_{\mathbb{R}^m}F^J\left(x\right)^{s}J_{\mu,\theta}\left(x\right)dx\right)^{\frac{r}{s}}\right\}^{\frac{s}{r}}\leq\int_{\mathbb{R}^m}\left\{\sum_{J\in\mathcal{D}^n}\left(F^J\left(x\right)^{s}\right)^{\frac{r}{s}}\right\}^{\frac{s}{r}}J_{\mu,\theta}\left(x\right)dx=\int_{\mathbb{R}^m}\left\{\sum_{J\in\mathcal{D}^n}J_{\mu,\theta}\left(x\right)^{\frac{r}{s}}F^J\left(x\right)^{r}\right\}^{\frac{s}{r}}$$

Now apply Lemma 1.2 above with the locally finite absolutely continuous measures μ_x on $\mathbb{R}^n$ for a.e. $x\in\mathbb{R}^m$ to obtain

$$\begin{aligned}
\sum_{J\in\mathcal{D}^n}J_{\mu,\theta}\left(x\right)^{\frac{r}{s}}F^J\left(x\right)^{r}&=\sum_{J\in\mathcal{D}^n}J_{\mu,\theta}\left(x\right)^{\frac{r}{s}}\left(\frac{1}{\left|J\right|_{\mu_x,\theta}}\int_J f_x\left(y\right)u_x\left(y\right)dy\right)^{r}\\
&=\sum_{J\in\mathcal{D}^n}\left|J\right|^{\frac{r}{s}}_{\mu_x,\theta}\left(\frac{1}{\left|J\right|_{\mu_x,\theta}}\int_J f_x\left(y\right)u_x\left(y\right)dy\right)^{r}\\
&\lesssim\left(\int_{\mathbb{R}^n}f_x\left(y\right)^{s}u_x\left(y\right)dy\right)^{\frac{r}{s}}=\left(\int_{\mathbb{R}^n}f\left(x,y\right)^{s}u\left(x,y\right)dy\right)^{\frac{r}{s}},
\end{aligned}$$

uniformly for a.e. $x \in \mathbb{R}^m$. Plugging this into the previous display gives

$$\begin{aligned}\left\{ \sum_{J\in\mathcal{D}^n} \left(\int_{\mathbb{R}^m} F^J(x)^s J_{\mu,\theta}(x)\,dx \right)^{\frac{r}{s}} \right\}^{\frac{s}{r}} &\lesssim \int_{\mathbb{R}^m} \left\{ \left(\left(\int_{\mathbb{R}^n} f(x,y)^s u(x,y)\,dy \right)^{\frac{r}{s}} \right\}^{\frac{s}{r}} dx \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^n} f(x,y)^s u(x,y)\,dydx = \|f\|_{L^s(\mu)}^s .\end{aligned}$$

Altogether then we have

$$\begin{aligned}&\sum_{R\in\mathcal{R}^{m,n}} |R|_{\mu,\theta}^{\frac{r}{s}} \left(\frac{1}{|R|_{\mu,\theta}} \int_R f(x,y)\,u(x,y)\,dxdy \right)^r \\ \lesssim &\sum_{J\in\mathcal{D}^n} \left(\int_{\mathbb{R}^m} F^J(x)^s J_{\mu,\theta}(x)\,dx \right)^{\frac{r}{s}} \lesssim \|f\|_{L^s(\mu)}^r .\end{aligned}$$

□

2.1.1. *Product fractional integrals.* The Tanaka-Yabuta theorem [TaYa, Theorem 1.1], as well as the variant in Theorem 2.1 above, uses an arbitrary nonnegative function $K(R)$ defined on dyadic rectangles $R \in \mathcal{R}^{m,n}$. If for $0 < \frac{\alpha}{m}, \frac{\beta}{n} < 1$, we define

$$K_{\alpha,\beta}^{m,n}(R) = K(I \times J) \equiv |I|^{\frac{\alpha}{m}-1} |J|^{\frac{\beta}{n}-1}, \tag{2.1}$$

for $R = I \times J \in \mathcal{R}^{m,n}$, then in the special case $K = K_{\alpha,\beta}^{m,n}$ we have the following pointwise estimate,

$$\begin{aligned}&\sum_{R\in\mathcal{R}^{m,n}} K_{\alpha,\beta}^{m,n}(R)\,\mathbf{1}_R(x,y)\,\mathbf{1}_R(u,v) = \sum_{I\times J\in\mathcal{R}^{m,n}} \{K(I\times J) : x,u\in I \text{ and } y,v\in J\} \\ = &\sum_{I\times J\in\mathcal{R}^{m,n}} \left\{ |I|^{\frac{\alpha}{m}-1} |J|^{\frac{\beta}{n}-1} : x,u\in I \text{ and } y,v\in J \right\} = \sum_{I\in\mathcal{D}^m} \left\{ |I|^{\frac{\alpha}{m}-1} : x,u\in I \right\} \times \sum_{J\in\mathcal{D}^n} \left\{ |J|^{\frac{\beta}{n}-1} : y,v\in J \right\} \\ \approx &\ d(x,u)^{\frac{\alpha}{m}-1} d(y,v)^{\frac{\beta}{n}-1} \lesssim |x-u|^{\frac{\alpha}{m}-1} |y-v|^{\frac{\beta}{n}-1},\end{aligned}$$

where $d_{dy}(x,u)$ denotes the *dyadic* distance between x and u in $\mathbb{R}^m$, and $d_{dy}(y,v)$ denotes the *dyadic* distance between y and v in $\mathbb{R}^n$. Here the dyadic distance between two points p and q in $\mathbb{R}^k$ is defined to be the side length of the smallest dyadic cube containing p and q. Note that the dyadic distance is at least $\frac{1}{\sqrt{k}}$ times the Euclidean distance since any dyadic cube Q containing x and y must satisfy

$$\ell(Q) \geq \max_{1\leq i\leq k} |x_i - y_i| \geq \sqrt{\frac{1}{k} \sum_{i=1}^k |x_i - y_i|^2} = \frac{1}{\sqrt{k}} |x-y| .$$

So in order to apply the above theorem to the product fractional integral operator with kernel $|x-u|^{\frac{\alpha}{m}-1} |y-v|^{\frac{\beta}{n}-1}$ it suffices to appeal to Strömberg's well-known $\frac{1}{3}$-trick for the dyadic grids $\{\mathcal{D}_i^m\}_{i=1}^{3^m}$ and $\{\mathcal{D}_j^n\}_{j=1}^{3^n}$, to obtain

$$\sum_{i=1}^{3^m} \sum_{j=1}^{3^n} \left[\sum_{R=I\times J\in\mathcal{D}_i^m\times\mathcal{D}_j^n} K(R)\,\mathbf{1}_R(x,y)\,\mathbf{1}_R(u,v) \right] \approx |x-u|^{\frac{\alpha}{m}-1} |y-v|^{\frac{\beta}{n}-1} . \tag{2.2}$$

Variants of the following lemma can be found many times over in the literature, too numerous to mention here. Let $\mathcal{P}^N$ denote the collection of all cubes in $\mathbb{R}^N$ with sides parallel to the coordinate axes.

Lemma 2.3. *For $K(R)$ defined as in (2.1) we have (2.2).*

Proof. For convenience we recall a variation on the $\frac{1}{3}$-trick given in Lemma 2.5 of [HyLaPe]. For a given dyadic grid $\mathcal{D} \subset \mathcal{P}^N$ with side lengths in $\left\{\frac{2^m}{3}\right\}_{m\in\mathbb{Z}}$, partition the collection of tripled cubes $\{3I\}_{I\in\mathcal{D}}$ into 3^N subcollections $\{S_u\}_{u=1}^{3^N}$, with the property that for each subcollection S_u there exists a dyadic grid $\mathcal{D}_u$ with side lengths in $\{2^m\}_{m\in\mathbb{Z}}$, such that $S_u \subset \mathcal{D}_u$. With these grids $\{\mathcal{D}_u\}_{u=1}^{3^N}$ fixed, we have the following sandwiching property. For each cube $P \in \mathcal{P}^N$ and each integer $j \in \mathbb{N}$, there is a choice of $u = u(P,j)$ with $1 \le u \le 3^n$ and a cube $I = I_{u(P,j)} \in \mathcal{D}_u$ such that

$$\begin{aligned} \ell(I) &\le 18\,\ell(P), \\ 3P &\subset I, \\ 2^j P &\subset \pi_{\mathcal{D}_u}^{(j)} I, \end{aligned} \tag{2.3}$$

where $\pi_{\mathcal{D}_u}^{(j)} I$ denotes the j^{th} grandparent of I in the grid $\mathcal{D}_u$.

Now fix $(x,y) \in \mathbb{R}^m \times \mathbb{R}^n$. For $x \in \mathbb{R}^N$, let $P(x,\ell)$ denote the cube centered at x with side length $\ell \in \left\{2^k\right\}_{k\in\mathbb{Z}}$. Then with $R_{a,b}(x,y) \equiv P(x,2^a) \times Q\left(y,2^b\right)$ for $a,b \in \mathbb{Z}$, we note that the right hand side of (2.2) is equivalent to

$$\sum_{a,b\in\mathbb{Z}} K\left(R_{a,b}(x,y)\right) \mathbf{1}_{R_{a,b}(x,y)}(x,y)\, \mathbf{1}_{R_{a,b}(x,y)}(u,v), \qquad (u,v) \in \mathbb{R}^m \times \mathbb{R}^n.$$

The first two lines in (2.3) now prove (2.2), since for each rectangle $R_{a,b}(x,y) \equiv P(x,2^a) \times Q\left(y,2^b\right)$ there is $I \times J \in \bigcup_{i=1}^{3^m} \bigcup_{j=1}^{3^n} \left(\mathcal{D}_i^m \times \mathcal{D}_j^n\right)$ such that

$$3R_{a,b}(x,y) \subset I \times J \subset 18 R_{a,b}(x,y),$$

and moreover, by the definition of K in (2.1), we then have $K\left(R_{a,b}(x,y)\right) \approx K(I \times J)$. We do not need the third line in (2.3) here. □

Armed with Lemma 2.3, we can now apply Theorem 2.1 to complete the proof of Theorem A in the introduction.

Remark 2.4. The above proof of Theorem A, when restricted to the 1-parameter case, gives a short and elegant proof of Theorem 1(A) in [SaWh] in the special case $p < q$.

2.2. **Reverse doubling weights for bilinear embeddings.** Here is a slight improvement of the theorem of Tanaka and Yabuta [TaYa], valid for the product fractional integral kernel, as well as more general kernels K satisfying property (2.5) below regarding expectations taken over partial grids $\mathcal{R}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n$. Recall that μ is a rectangle reverse doubling weight on $\mathbb{R}^m \times \mathbb{R}^n$ if (1.1) holds.

Theorem 2.5. *Suppose $1 < p < q < \infty$. Let σ and ω be rectangle reverse doubling weights on $\mathbb{R}^m \times \mathbb{R}^n$, and let $K = K_{\alpha,\beta}^{m,n} : \mathcal{R}^{m,n} \to [0,\infty)$ be as in (2.1), or more generally satisfy the expectation inequality (2.5) below. Then the norm $\mathbb{N}_K(\sigma,\omega)$ of the positive bilinear inequality,*

$$\sum_{R\in\mathcal{R}^{m,n}} K(R)\left(\int_R f d\sigma\right)\left(\int_R g d\omega\right) \leq \mathbb{N}_K(\sigma,\omega)\ \|f\|_{L^p(\sigma)}\|g\|_{L^{q'}(\omega)}\ , \qquad f,g\geq 0,$$

is finite for all partial grids $\mathcal{R}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n$ if and only if

$$\mathbb{A}_K(\sigma,\omega) \equiv \sup_{R\in\mathcal{P}^m\times\mathcal{P}^n} K(R)\ |R|_\omega^{\frac{1}{q}}\ |R|_\sigma^{\frac{1}{p'}} < \infty, \qquad \textit{for all rectangles } R\in\mathcal{P}^n\times\mathcal{P}^m\ .$$

Proof. We begin the proof with a brief review of the good/bad grid technology of Nazarov, Treil and Volberg. See [NTV2], [NTV4], or [Vol] for more detail. We restrict to dimension $n=1$ for the moment. Let $0<\varepsilon<1$ and $\mathbf{r}\in\mathbb{N}$ to be chosen later. Define J to be $\varepsilon-$ *good* in an interval K if

$$d(J, skel\ K) > 2|J|^{\varepsilon}|K|^{1-\varepsilon},$$

where the skeleton *skel* K of an interval K consists of its two endpoints and its midpoint. Define $\mathcal{D}_{(\mathbf{r},\varepsilon)-good}$ to consist of those $J\in\mathcal{D}$ such that J is good in every superinterval $K\in\mathcal{D}$ that lies at least $\mathbf{r}$ levels above J. As the goodness parameters ε and $\mathbf{r}$ will eventually be fixed throughout the proof, we sometimes suppress the parameters, and simply write $\mathcal{D}_{good}$ in place of $\mathcal{D}_{(\mathbf{r},\varepsilon)-good}$, and say "$J$ is *good*" instead of "J is good in every superinterval $K\in\mathcal{D}$ that lies at least $\mathbf{r}$ levels above J". We also define $\mathcal{D}_{bad}\equiv\mathcal{D}\setminus\mathcal{D}_{good}$.

Parameterizations of dyadic grids: Here we recall a construction from [SaShUr10] that was in turn based on that of Hytönen in [Hyt2]. Momentarily fix a large positive integer $M\in\mathbb{N}$, and consider the tiling of $\mathbb{R}$ by the family of intervals $\mathbb{D}_M\equiv\left\{I_\alpha^M\right\}_{\alpha\in\mathbb{Z}}$ having side length 2^{-M} and given by $I_\alpha^M\equiv I_0^M+2^{-M}\alpha$ where $I_0^M=\left[0,2^{-M}\right)$. A *dyadic grid* $\mathcal{D}$ built on $\mathbb{D}_M$ is defined to be a family of intervals $\mathcal{D}$ satisfying:

(**1**) Each $I\in\mathcal{D}$ has side length $2^{-\ell}$ for some $\ell\in\mathbb{Z}$ with $\ell\leq M$, and I is a union of $2^{M-\ell}$ intervals from the tiling $\mathbb{D}_M$,

(**2**) For $\ell\leq M$, the collection $\mathcal{D}_{[\ell]}$ of intervals in $\mathcal{D}$ having side length $2^{-\ell}$ forms a pairwise disjoint decomposition of the space $\mathbb{R}$,

(**3**) Given $I\in\mathcal{D}_{[i]}$ and $J\in\mathcal{D}_{[j]}$ with $j\leq i\leq M$, it is the case that either $I\cap J=\emptyset$ or $I\subset J$.

We denote by $\boldsymbol{A}_M$ the collection of all dyadic grids built on $\mathbb{D}_M$. We now momentarily fix an integer $N\in\mathbb{Z}$ with $N\leq M$, and consider the collection $\boldsymbol{A}_M^N$ of dyadic grids obtained by restricting the grids in $\boldsymbol{A}_M$ to contain only intervals of side length at most 2^{-N}. We refer to the dyadic grids in $\boldsymbol{A}_M^N$ as *special truncated* dyadic grids built on $\mathbb{D}_M$ of size 2^{-N} - *special* because any interval I from one of these grids that contains 0, will have 0 as an endpoint, and *truncated* because the side length of I is restricted to lie between 2^{-M} and 2^{-N}.

There are now two traditional means of constructing probability measures on collections of such dyadic grids, namely parameterization by choice of parent, and parameterization by translation. We will only need the former parameterization here. For any

$$\beta = \{\beta_i\}_{i\in\mathbb{Z}_M^N} \in \omega_M^N \equiv \{0,1\}^{\mathbb{Z}_M^N},$$

where $\mathbb{Z}_M^N \equiv \{\ell \in \mathbb{Z} : N \leq \ell \leq M\}$, define the dyadic grid $\mathcal{D}_\beta$ built on $\mathbb{D}_M$ of size 2^{-N} by

$$\mathcal{D}_\beta = \left\{ 2^{-\ell}\left([0,1) + k + \sum_{i:\ \ell<i\leq M} 2^{-i+\ell}\beta_i \right) \right\}_{N\leq\ell\leq M,\, k\in\mathbb{Z}} . \tag{2.4}$$

Place the uniform probability measure ρ_M^N on the finite index space $\omega_M^N = \{0,1\}^{\mathbb{Z}_M^N}$, namely that which charges each $\beta \in \omega_M^N$ equally, so that $\boldsymbol{A}_M^N \equiv \{\mathcal{D}_\beta\}_{\beta\in\omega_M^N}$.

This construction may be thought of as being *parameterized by scales* - each component β_i in $\beta = \{\beta_i\}_{i\in\mathbb{Z}_M^N} \in \omega_M^N$ amounting to a choice of the two possible tilings at level i that respect the choice of tiling at the level below. To limit cluttering of notation, we sometimes suppress reference to M and N in our families of grids, and use the notation Ω instead of Ω_M^N for the set of grids, and then use $\boldsymbol{P}_\Omega$ and $\boldsymbol{E}_\Omega$ to denote probability and expectation with respect to families of grids. However, the integers $N \leq M$ are always understood to be present. Finally, by taking products, we can extend these definitions to Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ in the obvious way using $\Omega_M^N \equiv \omega_M^N \times ... \times \omega_M^N$ with m and n factors respectively. We thus obtain a finite collection of special truncated grids $\boldsymbol{A}_M^N$ of cubes in either $\mathbb{R}^m$ or $\mathbb{R}^n$ indexed by Ω_M^N (we suppress dependence on the dimensions m and n). Finally, we identify the collection of special truncated grids $\boldsymbol{A}_M^N$ with its index set Ω_M^N, and we will write Ω_M^N for the finite collection such grids in the sequel.

Given a pair of dyadic grids $\mathcal{D}^m \in \Omega_M^N$ and $\mathcal{D}^n \in \Omega_M^N$ in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, form the corresponding *partial* dyadic grid $\mathcal{R}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n$ of rectangles (partial because $\mathcal{R}^{m,n}$ does not have the nested property). We say that a rectangle $R = I \times J \in \mathcal{R}_{good}^{m,n}$ (and say R is good) if both $I \in \mathcal{D}_{good}^m$ and $J \in \mathcal{D}_{good}^n$. Given a positive bilinear form $\mathcal{B}_{\mathcal{R}^{m,n}}$ restricted to one of these partial dyadic grids $\mathcal{R}^{m,n}$, i.e.

$$\mathcal{B}_{\mathcal{R}^{m,n}}(f,g) \equiv \sum_{R\in\mathcal{R}^{m,n}} K(R)\left(\int_R f d\sigma\right)\left(\int_R g d\omega\right), \qquad f \in L^p(\sigma), g \in L^{q'}(\omega),$$

we follow the NTV idea and dominate $B_{\mathcal{R}^{m,n}}(f,g) = B_{\mathcal{D}^m\times\mathcal{D}^n}(f,g)$ as follows:

$$\begin{aligned}
\mathcal{B}_{\mathcal{D}^m\times\mathcal{D}^n}(f,g) &\leq \left\{ \sum_{I\times J\in\mathcal{D}_{good}^m\times\mathcal{D}_{good}^n} + \sum_{I\times J\in\mathcal{D}^m\times\mathcal{D}_{bad}^n} + \sum_{I\times J\in\mathcal{D}_{bad}^m\times\mathcal{D}^n} \right\} K(I\times J)\left(\int_{I\times J} f d\sigma\right)\left(\int_{I\times J} g d\omega\right) \\
&\equiv \mathcal{B}_{\mathcal{D}_{good}^m\times\mathcal{D}_{good}^n}(f,g) + \mathcal{B}_{\mathcal{D}^m\times\mathcal{D}_{bad}^n}(f,g) + \mathcal{B}_{\mathcal{D}_{bad}^m\times\mathcal{D}^n}(f,g).
\end{aligned}$$

We now define a *shifted dyadic grid* in the original way, namely to be the collection of dyadic cubes

$$\mathcal{D}_\alpha \equiv \left\{ 2^j\left(k + [0,1)^m + (-1)^j\alpha\right) : j \in \mathbb{Z} \text{ and } k \in \mathbb{Z}^m \right\}, \qquad \text{for } \alpha \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}^m .$$

Then each $\mathcal{D}_\alpha$ is a dyadic grid on $\mathbb{R}^m$. Moreover, the crucial property of these shifted grids for us is this.

Shifted Property: If Q is any cube in $\mathbb{R}^m$ whose side length is an integral power of 2, then there is $\alpha \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}^m$, and a cube $Q_\alpha \in \mathcal{D}_\alpha$, such that $Q \subset Q_\alpha$ and $\ell(Q_\alpha) = 2\ell(Q)$.

Indeed, we may assume $m = 1$, $\ell(Q) = 1$ and that $Q \cap [-1, 1) \neq \emptyset$. Then $Q \subset [-2, 0)$ if $-1 \in Q$, $Q \subset [0, 2)$ if $1 \in Q$, $Q \subset [-2, 0) + \frac{2}{3}$ if $0 \in Q$ and $|Q \cap [-1, 0)| \geq \frac{1}{2}$, and finally $Q \subset [-2, 0) + \frac{4}{3}$ if $0 \in Q$ and $|Q \cap [0, 1)| \geq \frac{1}{2}$. We enumerate these dyadic grids by $\{\mathcal{D}_i^m\}_{i=1}^{3^m}$, and of course by $\left\{\mathcal{D}_j^n\right\}_{j=1}^{3^n}$ in dimension n. A simple adaptation of Lemma 2.3, using the Shifted Property for this collection of shifted dyadic grids, shows that the positive bilinear form

$$\mathcal{I}(f, g) \equiv \int_{\mathbb{R}^m \times \mathbb{R}^n} I_{\alpha,\beta}^{m,n}(f\sigma)\, g\omega$$

satisfies

$$\mathcal{I}(f, g) \approx \sup_{\substack{1 \leq i \leq 3^m \\ 1 \leq j \leq 3^n}} \mathcal{B}_{\mathcal{D}_i^m \times \mathcal{D}_j^n}(f, g) \approx \sup_{N < 0 < M} \sup_{\mathcal{D}^m \times \mathcal{D}^n \in \Gamma_M^N \times \Gamma_M^N} \mathcal{B}_{\mathcal{D}^m \times \mathcal{D}^n}(f, g),$$

where Γ_M^N consists of the truncated (but not necessarily special) grids obtained by restricting the dyadic cubes in $\mathcal{D}_i^m$ to have side length between 2^{-M} and 2^{-N}. Thus we need only treat the bilinear forms $\mathcal{B}_{\mathcal{D}^m \times \mathcal{D}^n}(f, g)$ with $\mathcal{D}^m \times \mathcal{D}^n \in \Gamma_M^N \times \Gamma_M^N$, and obtain estimates independent of $N < 0 < M$. In fact, for technical reasons revealed below, we further restrict the forms to $\mathcal{D}^m \times \mathcal{D}^n \in \Gamma_M^{N+1} \times \Gamma_M^{N+1}$ where $N < 0 < M$.

We emphasize that our partial special truncated dyadic grids $\mathcal{R}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n \in \Omega_M^N \times \Omega_M^N$ have factors $\mathcal{D}^m$ and $\mathcal{D}^n$ that are restricted to lie in the finite collections of *special* dyadic truncated grids Ω_M^N for dimensions m and n respectively, while our bilinear forms $\mathcal{B}_{\mathcal{D}^m \times \mathcal{D}^n}$ are taken over partial truncated dyadic grids $\mathcal{D}^m \times \mathcal{D}^n$ in $\Gamma_M^{N+1} \times \Gamma_M^{N+1}$ which fail in general to be special. We denote by $\left(\mathcal{S}_M^N\right)^{m,n}$ the collection of all *special* rectangles $R = I \times J$ that are contained in some partial *special* truncated dyadic grid $\mathcal{R}^{m,n} \in \Omega_M^N \times \Omega_M^N$. In particular, these rectangles are of the form $I \times J$ where the cubes I and J share their vertices with the vertices of cubes in $\mathbb{D}_M$ for dimensions m and n respectively. Thus for $\mathcal{D}^m \times \mathcal{D}^n \in \Omega_M^N \times \Omega_M^N$, the bilinear forms $\mathcal{B}_{\mathcal{D}^m \times \mathcal{D}^n} \equiv \sum_{R \in \mathcal{R}^{m,n}} K(R) \left(\int_R f d\sigma\right) \left(\int_R g d\omega\right)$ are restricted to summing over certain collections of rectangles in $\left(\mathcal{S}_M^N\right)^{m,n}$. With this in mind, we continue to sometimes suppress the integers M and N from the notation, and write simply $\Omega \times \Omega$ for $\Omega_M^N \times \Omega_M^N$, and $\mathcal{S}^{m,n}$ for $\left(\mathcal{S}_M^N\right)^{m,n}$.

Let $\boldsymbol{P}_{(\Omega \times \Omega)_R}$ denote the uniform probability measure on the set $(\Omega \times \Omega)_R$ of grids in $\Omega \times \Omega$ that contain the rectangle R (each such grid is assigned equal measure). One should keep in mind that while we write $\int_{\Omega \times \Omega} h(\mathcal{R}^{m,n})\, d\boldsymbol{P}_{\Omega \times \Omega}(\mathcal{R}^{m,n})$ as an integral, it is really just a finite sum, as the product set $\Omega \times \Omega = \Omega_M^N \times \Omega_M^N$ has only finitely many partial special truncated grids in it - namely $2^{(m+n)(M-N)}$ of them.

Now suppose that $K(R) = K_{\alpha,\beta}^{m,n}(R)$ as in (2.1), or more generally that K is decreasing, i.e. $K(R') \leq K(R)$ for $R' \supset R$, and 'tripling', i.e. $K(R) \leq CK(3R)$ for all rectangles R. We claim that there is then a positive constant $c > 0$ such that for any **fixed** partial dyadic grid $\mathcal{G}^m \times \mathcal{G}^n \in \Gamma_M^{N+1} \times \Gamma_M^{N+1}$, we have

$$\boldsymbol{E}_{\Omega \times \Omega} \mathcal{B}_{\mathcal{D}^m \times \mathcal{D}^n}(f, g) \geq c \mathcal{B}_{\mathcal{G}^m \times \mathcal{G}^n}(f, g), \qquad \text{for } K(R) \text{ as above.} \tag{2.5}$$

Indeed, if we let $\mathcal{A} \equiv \{(\mathcal{R}^{m,n}, R) \in (\Omega \times \Omega) \times \mathcal{S}^{m,n} : R \in \mathcal{R}^{m,n}\}$ denote the set of pairs of grids and rectangles such that the rectangle is a member of the grid, then the left hand side of (2.5) is given by

$$\begin{aligned}
&\int_{\Omega\times\Omega} \left\{ \sum_{R\in\mathcal{R}^{m,n}} K(R) \left(\int_R f d\sigma\right) \left(\int_R g d\omega\right) \right\} d\boldsymbol{P}_{\Omega\times\Omega}(\mathcal{R}^{m,n}) \\
&= \left(\int_{\Omega\times\Omega} \sum_{R\in\mathcal{S}^{m,n}} \right) \mathbf{1}_{\mathcal{A}}((\mathcal{R}^{m,n}, R)) K(R) \left(\int_R f d\sigma\right) \left(\int_R g d\omega\right) d\boldsymbol{P}_{\Omega\times\Omega}(\mathcal{R}^{m,n}) \\
&= \left(\sum_{R\in\mathcal{S}^{m,n}} \int_{\Omega\times\Omega} \right) \mathbf{1}_{\mathcal{A}}((\mathcal{R}^{m,n}, R)) K(R) \left(\int_R f d\sigma\right) \left(\int_R g d\omega\right) d\boldsymbol{P}_{\Omega\times\Omega}(\mathcal{R}^{m,n}) \\
&= \sum_{R\in\mathcal{S}^{m,n}} \left\{ K(R) \left(\int_R f d\sigma\right) \left(\int_R g d\omega\right) \int_{\Omega\times\Omega} \mathbf{1}_{\mathcal{A}}((\mathcal{R}^{m,n}, R)) d\boldsymbol{P}_{(\Omega\times\Omega)}(\mathcal{R}^{m,n}) \right\} \\
&= \sum_{R\in\mathcal{S}^{m,n}} \left\{ K(R) \left(\int_R f d\sigma\right) \left(\int_R g d\omega\right) \boldsymbol{P}_{(\Omega\times\Omega)}\left[(\Omega\times\Omega)_R\right] \right\}.
\end{aligned}$$

The reason for restricting our attention to a finite collection of partial dyadic grids here is in order to apply Fubini's theorem to $\int_{\Omega\times\Omega} \sum_{R\in\mathcal{S}^{m,n}} ...d\boldsymbol{P}_{(\Omega\times\Omega)} = \sum_{R\in\mathcal{S}^{m,n}} \int_{\Omega\times\Omega} ...d\boldsymbol{P}_{(\Omega\times\Omega)}$ using that counting measure on $\mathcal{S}^{m,n}$ is σ-finite. If instead, we were to consider all possible grids in $\Omega \times \Omega$, then $\mathcal{S}^{m,n}$ would consist of **all** possible rectangles, and counting measure on $\mathcal{S}^{m,n}$ would no longer be σ-finite, thus invalidating the application of Fubini's theorem (which is actually false in this instance since $\boldsymbol{P}_{(\Omega\times\Omega)}\left[(\Omega\times\Omega)_R\right]$ would then vanish for each $R \in \mathcal{S}^{m,n}$).

The bilinear form on the right hand side of (2.5) is given by

$$\mathcal{B}_{\mathcal{G}^m\times\mathcal{G}^n}(f,g) = \sum_{P\in\mathcal{G}^m\times\mathcal{G}^n} K(P) \left(\int_P f d\sigma\right) \left(\int_P g d\omega\right),$$

where $P = I \times J$ and the cubes $I \in \mathcal{G}^m$ and $J \in \mathcal{G}^n$ satisfy

$$2^{-M} \le \ell(I) = 2^i \le 2^{-N-1} \text{ and } 2^{-M} \le \ell(J) = 2^j \le 2^{-N-1}, \tag{2.6}$$

but do not necessarily share their vertices with those of the cubes in the set $\mathbb{D}_M$. We refer to a rectangle $P = I \times J$ satisfying $\ell(I) = 2^i$ and $\ell(J) = 2^j$ as an (i,j)-rectangle. Of course the rectangles belonging to the partial grids $\mathcal{G}^m \times \mathcal{G}^n$ in $\Gamma_M^N \times \Gamma_M^N$ are further restricted by the $\frac{1}{3}$-shift, but we will not need this in what follows.

Fix an (i,j)-rectangle $P = I \times J \in \mathcal{G}^m \times \mathcal{G}^n$ with $\mathcal{G}^m \times \mathcal{G}^n \in \Gamma_M^N \times \Gamma_M^N$. We claim there is a subset $(\Omega \times \Omega)[P]$ of $\Omega\times\Omega$ of measure $\frac{1}{2^m}\frac{1}{2^n}$ such that each partial dyadic grid $\mathcal{S}^{m,n} = \mathcal{D}^m \times \mathcal{D}^n \in (\Omega \times \Omega)[P]$ has an $(i+1, j+1)$-rectangle $R = K \times L \in \mathcal{S}^{m,n}$ that contains P. Indeed, given an arbitrary grid $\mathcal{D}^m \in \Omega$, if $K' \in \mathcal{D}^m$ is such that $\ell(K') = \ell(I)$ and $K' \cap I \neq \emptyset$, then one of the 2^m possible parents K of K' will contain I. The same construction can be carried out for J, and our claim follows easily. Denote this $(i+1, j+1)$-rectangle $R = K \times L$ by R_P to indicate its dependence on P.

Thus using that $K(P) \leq CK(3P) \leq CK(R_P)$ since $R_P \subset 3P$, we have both

$$\begin{aligned}
K(P)\left(\int_P f d\sigma\right)\left(\int_P g d\omega\right) &\leq CK(R_P)\left(\int_{R_P} f d\sigma\right)\left(\int_{R_P} g d\omega\right), \\
\boldsymbol{P}_{\Omega\times\Omega}\left((\Omega\times\Omega)[P]\right) &= \frac{1}{2^{m+n}},
\end{aligned}$$

for all (i,j)-rectangles $P \in \mathcal{G}^m \times \mathcal{G}^n$. We conclude that $\frac{1}{\#(\Omega\times\Omega)[P]} = \frac{2^{m+n}}{\#(\Omega\times\Omega)}$ for all such P, and that

$$\begin{aligned}
\mathcal{B}_{\mathcal{G}^m\times\mathcal{G}^n}(f,g) &= \sum_{P\in\mathcal{G}^m\times\mathcal{G}^n} K(P)\left(\int_P f d\sigma\right)\left(\int_P g d\omega\right) \\
&\leq \sum_{P\in\mathcal{G}^m\times\mathcal{G}^n} CK(R_P)\left(\int_{R_P} f d\sigma\right)\left(\int_{R_P} g d\omega\right) \\
&= \sum_{P\in\mathcal{G}^m\times\mathcal{G}^n} C\frac{2^{m+n}}{\#(\Omega\times\Omega)} \sum_{\mathcal{R}^{m,n}\in(\Omega\times\Omega)[P]} \sum_{R\in\mathcal{R}^{mn}} K(R)\left(\int_R f d\sigma\right)\left(\int_R g d\omega\right) \\
&\leq 2^{m+n}C\frac{1}{\#(\Omega\times\Omega)} \sum_{\mathcal{R}^{m,n}\in(\Omega\times\Omega)} \sum_{R\in\mathcal{R}^{m,n}} K(R)\left(\int_R f d\sigma\right)\left(\int_R g d\omega\right) \\
&= 2^{m+n}C\boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m\times\mathcal{D}^n}(f,g),
\end{aligned}$$

so that (2.5) holds with $c = \frac{1}{2^{m+n}C}$.

Recalling that our collections of partial grids $\Omega \times \Omega$ are actually $\Omega_M^N \times \Omega_M^N$ for $N < 0 < M$, and that our bilinear form $\mathcal{B}_{\mathcal{D}^m\times\mathcal{D}^n}$ is restricted to summing over (i,j)-rectangles satisfying (2.6), it then follows that the norm $\mathfrak{N}_{\mathcal{I}}$ of the bilinear form $\mathcal{I}$ can be estimated using $\|f\|_{L^p(\sigma)} = \|g\|_{L^{q'}(\omega)} = 1$ chosen so that $\mathfrak{N}_{\mathcal{I}} = \mathcal{I}(f,g)$:

$$\begin{aligned}
\mathfrak{N}_{\mathcal{I}} &= \mathcal{I}(f,g) \lesssim \boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m\times\mathcal{D}^n}(f,g) \\
&\leq \boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m_{good}\times\mathcal{D}^n_{good}}(f,g) + \boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m\times\mathcal{D}^n_{bad}}(f,g) + \boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m_{bad}\times\mathcal{D}^n}(f,g).
\end{aligned}$$

Now the conditional probability that a given cube I is bad in a grid $\mathcal{D}^m$ that contains it is small, in fact (see e.g. [NTV2], [NTV4], [Vol] or [SaShUr, Subsubsection 3.1.1]) we have

$$\boldsymbol{P}_{\Omega}\left\{\mathcal{D}^m : I \text{ is bad in } \mathcal{D}^m \mid \text{conditioned on } I \in \mathcal{D}^m\right\} \leq C2^{-\varepsilon\mathbf{r}}.$$

Thus we obtain

$$\begin{aligned}
\boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m\times\mathcal{D}^n_{bad}}(f,g) &\leq C2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{\mathcal{I}}\|f\|_{L^p(\sigma)}\|g\|_{L^{q'}(\omega)} = C2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{\mathcal{I}}, \\
\boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m_{bad}\times\mathcal{D}^n}(f,g) &\leq C2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{\mathcal{I}}\|f\|_{L^p(\sigma)}\|g\|_{L^{q'}(\omega)} = C2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{\mathcal{I}},
\end{aligned}$$

and hence

$$\mathfrak{N}_{\mathcal{I}} \leq C\boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m_{good}\times\mathcal{D}^n_{good}}(f,g) + 2C2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{\mathcal{I}},$$

which gives

$$\mathfrak{N}_{\mathcal{I}} \leq \frac{C}{1-2C2^{-\varepsilon\mathbf{r}}}\boldsymbol{E}_{\Omega\times\Omega}\mathcal{B}_{\mathcal{D}^m_{good}\times\mathcal{D}^n_{good}}(f,g)$$

if $\varepsilon\mathbf{r}$ is chosen sufficiently small.

Thus we see that in order to prove Theorem 2.5, we need only consider the 'good' bilinear form $\mathcal{B}_{\mathcal{D}^m_{good}\times\mathcal{D}^n_{good}}(f,g)$ and estimate it independently of the partial grid of good rectangles $\mathcal{D}^m_{good}\times\mathcal{D}^n_{good}$. We can then let $N\to-\infty$ and $M\to\infty$ to capture the full bilinear form. Finally, using arguments as in [TaYa] or above, the proof of Theorem 2.5 is reduced to the following Carleson embedding for 'good' rectangles.

Carleson embedding: Suppose that $1<s<r<\infty$ and that μ is a product reverse doubling measure on $\mathbb{R}^m\times\mathbb{R}^n$. Then we have

$$\left\{\sum_{R\in\mathcal{D}^m_{good}\times\mathcal{D}^n_{good}}|R|_\mu^{\frac{r}{s}}\left(\frac{1}{|R|_\mu}\int_R f d\mu\right)^r\right\}^{\frac{1}{r}}\leq C_{s,r}\left\|f\right\|_{L^s(\mu)}\ ,\qquad f\geq 0,$$

where $C_{s,r}$ depends only on s, r, the reverse doubling constants for μ, and the goodness parameters $\varepsilon,\mathbf{r}$. In particular, $C_{s,r}$ is independent of the partial grid $\mathcal{D}^m_{good}\times\mathcal{D}^n_{good}$.

Continuing to follow the iteration argument of Tanaka and Yabuta as in [TaYa] or above, further reduces matters to proving the following Carleson condition on **cubes** for a reverse doubling measure μ on $\mathbb{R}^N$ with exponent $\eta>0$, and a power $\rho>1$:

$$\sum_{Q\in\mathcal{D}^N_{good}:\ Q\subset P}|Q|_\mu^\rho\leq C_{N,\mathbf{r},\varepsilon,\rho}\left|P\right|_{\mu,\theta}^\rho\ . \tag{2.7}$$

Indeed, the reader can easily verify that the arguments work just as well for the subgrids $\mathcal{D}^m_{good}$ and $\mathcal{D}^n_{good}$ in place of the grids $\mathcal{D}^m$ and $\mathcal{D}^n$.

It is now at this point that the goodness of the cubes Q plays a crucial role in conjuction with the reverse doubling property. To see (2.7), recall the goodness parameters $0<\varepsilon<1$ and $\mathbf{r}\in\mathbb{N}$ and observe that if Q is a good cube contained in P then
either $\ell(Q)\geq\ell(P)-\mathbf{r}$ and we can use the trivial estimate $|Q|_\mu^\rho\leq|P|_\mu^\rho$,
or $\ell(Q)<\ell(P)-\mathbf{r}$ in which case $dist(Q,\partial P)\geq 2\ell(Q)^\varepsilon\,\ell(P)^{1-\varepsilon}$.

In this latter case we note that if $\ell(Q)=2^{-k}\ell(P)$ then

$$2^{k(1-\varepsilon)}Q=\left(\frac{\ell(P)}{\ell(Q)}\right)^{1-\varepsilon}Q\subset\frac{2\ell(Q)^\varepsilon\,\ell(P)^{1-\varepsilon}}{\ell(Q)}Q\subset\frac{dist(Q,\partial P)}{\ell(Q)}Q\subset P$$

and so by reverse doubling we have

$$|Q|_\mu\leq C2^{-\eta k(1-\varepsilon)}\left|\left(\frac{\ell(P)}{\ell(Q)}\right)^{1-\varepsilon}Q\right|_\mu\leq C2^{-\eta(1-\varepsilon)k}\left|P\right|_\mu\ .$$

Thus we can estimate

$$\begin{aligned}\sum_{Q\in\mathcal{D}_{good}^{N}:\ Q\subset P}|Q|_{\mu}^{\rho} &\leq \sum_{k=0}^{\mathbf{r}}2^{Nk}|P|_{\mu}^{\rho}+\sum_{k=\mathbf{r}+1}^{\infty}\sum_{Q\in\mathcal{D}_{good}^{N}:\ \ell(Q)=2^{-k}\ell(P)}|Q|_{\mu}^{\rho-1}|Q|_{\mu}\\ &\leq C_{N,\mathbf{r}}|P|_{\mu}^{\rho}+\sum_{k=\mathbf{r}+1}^{\infty}\sum_{Q\in\mathcal{D}_{good}^{N}:\ \ell(Q)=2^{-k}\ell(P)}\left(C2^{-\eta(1-\varepsilon)k}|P|_{\mu}\right)^{\rho-1}|Q|_{\mu}\\ &\leq C_{N,\mathbf{r}}|P|_{\mu}^{\rho}+\left\{\sum_{k=0}^{\infty}\left(C2^{-\eta(1-\varepsilon)(\rho-1)k}\right)\right\}|P|_{\mu}^{\rho}=C_{N,\mathbf{r},\varepsilon,\rho}|P|_{\mu}^{\rho}\ .\end{aligned}$$

This completes the proof of (2.7), and hence also that of Theorem 2.5. $\square$

Theorem B in the introduction is an immediate consequence of Theorem 2.5.

2.3. **Concluding remarks.** In the case of kernels $K=K_{\alpha,\beta}^{m,n}$ given by (2.1), or more generally that satisfy (2.5), one can assume for each weight separately, either rectangle reverse doubling, or a half θ-bump condition, in order to obtain norm boundedness. For example, the following hybrid theorem holds.

Theorem 2.6. *Suppose $1<p<q<\infty$. Let σ be a product reverse doubling weight on $\mathbb{R}^n$, let $d\omega(x)=w(x)^q dx$ be absolutely continuous with respect to Lebesgue measure, and let $K=K_{\alpha,\beta}^{m,n}:\mathcal{R}^{m,n}\to[0,\infty)$ be as in (2.1), or more generally satisfy (2.5). Then the norm $\mathbb{N}_K(\sigma,\omega)$ of the positive bilinear inequality,*

$$\sum_{R\in\mathcal{R}^{m,n}}K(R)\left(\int_R fd\sigma\right)\left(\int_R gd\omega\right)\leq\mathbb{N}_K(\sigma,\omega)\ \|f\|_{L^p(\sigma)}\|g\|_{L^{q'}(\omega)}\ ,\qquad f,g\geq 0,$$

is finite for all products of grids $\mathcal{R}^{m,n}=\mathcal{D}^m\times\mathcal{D}^n$ if the half θ-bump rectangle characteristic $\mathbb{A}_{K,\theta}^{\omega}(\sigma,\omega)$ is finite, where

$$\begin{aligned}\mathbb{A}_{K,\theta}^{\omega}(\sigma,\omega) &\equiv \sup_{R\in\mathcal{R}^{m,n}}K(R)\ \left(\int_R v^{-p'}d\sigma\right)^{\frac{1}{p'}}\ \left[|R|^{\frac{1}{q\theta'}}\left(\int_R w^{q\theta}d\omega\right)^{\frac{1}{q\theta}}\right]\\ &= \sup_{R\in\mathcal{R}^{m,n}}K(R)\ |R|_{\omega,\theta}^{\frac{1}{q}}\ |R|_{\sigma}^{\frac{1}{p'}}\ .\end{aligned}$$

The proof is an easy exercise in combining the proofs of Theorems 2.1 and 2.5 above.

3. Appendix

We say that a weight μ on the real line is **strongly** reverse doubling if there is $\beta<1$ such that

$$|I_{left}|_{\mu},|I_{right}|_{\mu}\leq\beta|I|_{\mu}\ \text{ for all intervals } I,$$

where if $I=[a,b)$, then $I_{left}=\left[a,\frac{a+b}{2}\right)$ and $I_{right}=\left[\frac{a+b}{2},b\right)$ are the left and right halves of I respectively. A strongly reverse doubling weight on $\mathbb{R}$ is a doubling weight on $\mathbb{R}$, since if we choose

N so large that $\beta^N < \frac{1}{4}$, then for $I = [a, b)$, we have

$$\left| \left[a, a + \frac{b-a}{2^N} \right) \right|_\mu , \quad \left| \left[b - \frac{b-a}{2^N}, b \right) \right|_\mu \leq \beta^N |I|_\mu < \frac{1}{4} |I|_\mu \ .$$

Hence

$$\begin{aligned} \left| \left[a + \frac{b-a}{2^N}, b - \frac{b-a}{2^N} \right) \right|_\mu &= |[a,b)|_\mu - \left| \left[a, a + \frac{b-a}{2^N} \right) \right|_\mu - \left| \left[b - \frac{b-a}{2^N}, b \right) \right|_\mu \\ &\geq \left(1 - \frac{1}{4} - \frac{1}{4} \right) |I|_\mu = \frac{1}{2} |I|_\mu \ , \end{aligned}$$

where the length of the interval $\left[a + \frac{b-a}{2^N}, b - \frac{b-a}{2^N} \right)$ is $\frac{2^{N-1}-1}{2^{N-1}} \ell(I)$. Thus with $\gamma = \frac{2^{N-1}}{2^{N-1}-1} > 1$, we have for every interval K,

$$|\gamma K|_\mu \leq 2 |K|_\mu \ , \text{ hence } |2K|_\mu \leq 2^M |K|_\mu \text{ if } \gamma^M \geq 2,$$

which shows that μ is doubling. Similarly we see that a strongly rectangle reverse doubling weight on $\mathbb{R}^N$ is a rectangle doubling weight on $\mathbb{R}^N$. Here μ is strongly rectangle reverse doubling if there is $\beta < 1$ such that

$$\begin{aligned} & \left| I^1 \times ... \times I^k_{left} \times ... \times I^N \right|_\mu , \left| I^1 \times ... \times I^k_{right} \times ... \times I^N \right|_\mu \\ \leq \quad & \beta \left| I^1 \times ... \times I^N_\mu \right|_\mu \text{ for all rectangles } I^1 \times ... \times I^N \text{ and } 1 \leq k \leq N, \end{aligned}$$

and μ is rectangle doubling if there is $C > 0$ such that

$$\left| \left(2I^1 \right) \times ... \times \left(2I^N \right) \right|_\mu \leq C \left| I^1 \times ... \times I^N \right|_\mu \text{ for all rectangles } I^1 \times ... \times I^N.$$

Example 3.1. Suppose that μ is a rectangle doubling weight on $\mathbb{R}^N$. Then $d\nu(x) \equiv \mathbf{1}_{[0,\infty)^N}(x) \mu(x)$ is a rectangle reverse doubling weight on $\mathbb{R}^N$ that is not a rectangle doubling weight on $\mathbb{R}^N$.

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