On the end-point of Stein-Weiss inequality

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Abstract

This paper has two purposes. First, we show that the classical Stein-Weiss inequality is true for $p = 1$. Second, by considering a family of strong fractional integral operators whose kernels have singularity on every coordinate subspace, we extend this end-point result to the multi-parameter settings.

1 Introduction

Let $0 < \alpha < \mathbf{N}$ and $\gamma, \delta < \mathbf{N}$. We define

$$\mathbf{I}_{\alpha\gamma\delta} f(x) = \int_{\mathbb{R}^\mathbf{N}} f(y) \left(\frac{1}{|x|}\right)^\gamma \left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha} \left(\frac{1}{|y|}\right)^\delta dy, \qquad x \neq 0. \tag{1. 1}$$

In 1928, Hardy and Littlewood [1] first investigated the $\mathbf{L}^p - \mathbf{L}^q$-regularity of $\mathbf{I}_{\alpha\gamma\delta}$ at $\mathbf{N} = 1$. Thirty years later, Stein and Weiss [3] extended this result to every higher dimensional space. Today, it bears the name of Stein-Weiss inequality.

$\diamond$ Throughout, $\mathfrak{C} > 0$ is regarded as a generic constant depending on its subindices.

Theorem A: Stein and Weiss, 1958

Let $\mathbf{I}_{\alpha\gamma\delta}$ defined in (1. 1) for $0 < \alpha < \mathbf{N}$ and $\gamma, \delta < \mathbf{N}$. We have

$$\left\|\mathbf{I}_{\alpha\gamma\delta} f\right\|_{\mathbf{L}^q(\mathbb{R}^\mathbf{N})} \leq \mathfrak{C}_{\alpha\,\gamma\,\delta\,p\,q} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1 < p \leq q < \infty \tag{1. 2}$$

if and only if

$$\gamma < \frac{\mathbf{N}}{q}, \qquad \delta < \mathbf{N}\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \geq 0, \qquad \frac{\alpha}{\mathbf{N}} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}. \tag{1. 3}$$

Remark 1.1. *In the original paper of Stein and Weiss [3], (1. 3) is given as a sufficient condition. These constraints of $\alpha, \gamma, \delta, p, q$ in (1. 3) are in fact necessary. See subsection* **3.1**.

When $\gamma = \delta = 0$, **Theorem A** was proved in $\mathbb{R}^\mathbf{N}$ by Sobolev [2]. This is also known as Hardy-Littlewood-Sobolev inequality.

The theory of fractional integration in weighted norms has been substantially developed during the second half of 20th century. See Fefferman and Muckenhoupt [7], Muckenhoupt and Wheeden [8], Pérez [9] and Sawyer and Wheeden [10].

Our first main result is an improvement of **Theorem A** to include the end-point $p = 1$.

Theorem 1.1. *Let $\mathbf{I}_{\alpha\gamma\delta}$ defined in (1. 1) for $0<\alpha<\mathbf{N}$ and $\gamma,\delta<\mathbf{N}$. We have*

$$\left\|\mathbf{I}_{\alpha\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^\mathbf{N})} ~\le~ \mathfrak{C}_{\alpha~\gamma~\delta~p~q}~\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1\le p\le q<\infty \tag{1. 4}$$

if and only if

$$\gamma ~<~ \frac{\mathbf{N}}{q}, \qquad \delta ~<~ \mathbf{N}\left(\frac{p-1}{p}\right), \qquad \gamma+\delta ~\ge~ 0, \qquad \frac{\alpha}{\mathbf{N}} ~=~ \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}}. \tag{1. 5}$$

Remark 1.2. *For $p=1$ and $\gamma+\delta>0$, Theorem 1.1 is recently proved by Nápoli and Picon [11] whereas (1. 4) is equivalently considered as a two-weight $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality for $f*|x|^{\alpha-\mathbf{N}}$ by taking into account $|x|^{-\gamma},|x|^{\delta}$ as the weights. The regarding estimates are developed in analogue to the work of Sawyer and Wheeden [10]. See Section 2 of [11]. In particular, the 'good kernel' approximation used there is nice but cannot handle the case when $\gamma+\delta=0$.*

Next, consider $\mathbb{R}^\mathbf{N}=\mathbb{R}^{\mathbf{N}_1}\times\mathbb{R}^{\mathbf{N}_2}\times\cdots\times\mathbb{R}^{\mathbf{N}_n}$. Let $\gamma,\delta<\mathbf{N}$ and

$$\alpha ~=~ \alpha_1+\alpha_2+\cdots+\alpha_n, \qquad 0<\alpha_i<\mathbf{N}_i, \qquad i=1,2,\ldots,n. \tag{1. 6}$$

We define

$$\mathbf{II}_{\alpha\gamma\delta}f(x) ~=~ \int_{\mathbb{R}^\mathbf{N}} f(y)\left(\frac{1}{|x|}\right)^{\gamma}\prod_{i=1}^n\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y|}\right)^{\delta}dy, \qquad x\neq 0. \tag{1. 7}$$

Observe that the kernel of $\mathbf{II}_{\alpha\gamma\delta}$ has singularity on every coordinate subspace. The study of certain operators dates back to the time of Jessen, Marcinkiewicz and Zygmund. Over the several past decades, a number of remarkable results have been accomplished for the $\mathbf{L}^p$-theory of singular integrals. See Fefferman and Stein [13], Journé [14], Fefferman [12] and Müller, Ricci and Stein [6]. This area remains largely open for fractional integrals. Recently, a characterization is established in [15] between the $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality

$$\left\|\mathbf{II}_{\alpha\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^\mathbf{N})} ~\le~ \mathfrak{C}_{p~q~\alpha~\gamma~\delta}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1<p\le q<\infty$$

and the necessary constraints consisting of γ,δ,p,q and α_i, $i=1,2,\ldots,n$.

Our second result gives such an extension to Theorem 1.1 when $p=1$.

Theorem 1.2. *Let $\mathbf{II}_{\alpha\gamma\delta}$ defined in (1. 7) for $0<\alpha_i<\mathbf{N}_i$, $i=1,2,\ldots,n$ and $\gamma,\delta<\mathbf{N}$. We have*

$$\left\|\mathbf{II}_{\alpha\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^\mathbf{N})} ~\le~ \mathfrak{C}_{\alpha~\gamma~\delta~q}~\left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^\mathbf{N})}, \qquad 1\le q<\infty \tag{1. 8}$$

if and only if

$$\gamma ~<~ \frac{\mathbf{N}}{q}, \qquad \delta ~<~ 0, \qquad \gamma+\delta ~\ge~ 0, \qquad \frac{\alpha}{\mathbf{N}} ~=~ 1-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}} \tag{1. 9}$$

and

$$\alpha_i-\mathbf{N}_i ~<~ \delta, \qquad i ~=~ 1,2,\ldots,n. \tag{1. 10}$$

In order to prove Theorem 1.2, we develop a new framework where the product space $\mathbb{R}^{\mathbf{N}_1}\times\mathbb{R}^{\mathbf{N}_2}\times\cdots\times\mathbb{R}^{\mathbf{N}_n}$ is decomposed into an infinitely many dyadic cones. Every partial operator defined on one of these cones is essentially an one-parameter fractional integral operator, satisfying the regarding $\mathbf{L}^1 \longrightarrow \mathbf{L}^q$-norm inequality. Furthermore, the operator's norm decays exponentially as the eccentricity of the cone getting large.

This paper is organized as follows. The next section is devoted to two fundamental lemmas initially given by Stein and Weiss [3] for $p > 1$. These results are improved now to become applicable for $p \geq 1$. We prove Theorem 1.1 and Theorem 1.2 in Section 3 and Section 4 respectively.

2 Two fundamental lemmas

Lemma 2.1. *Let $\Omega(u,v) \geq 0$ defined in the quadrant $\{(u,v) : u \geq 0,\ v \geq 0\}$ which is homogeneous of degree $-\mathbf{N}$ and*

$$\mathbf{A} \doteq \int_0^\infty \Omega(1,t) t^{\mathbf{N}\left(\frac{p-1}{p}\right)-1} dt < \infty. \tag{2. 1}$$

Consider

$$\mathbf{U}f(x) = \int_{\mathbb{R}^\mathbf{N}} \Omega\left(|x|,|y|\right) f(y) dy. \tag{2. 2}$$

We have

$$\left\|\mathbf{U}f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})} \leq \mathfrak{C}_\mathbf{A} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1 \leq p < \infty. \tag{2. 3}$$

Proof. Let $R = |x|$ and $r = |y|$. For $\mathbf{N} \geq 2$, write $x = R\xi$ and $y = r\eta$ of which ξ, η are unit vectors. We have

$$\mathbf{U}f(x) = \int_{\mathbb{S}^{\mathbf{N}-1}} \int_0^\infty \Omega(R,r) f(r\eta) r^{\mathbf{N}-1} dr d\sigma(\eta) \tag{2. 4}$$

where $d\sigma$ denotes the surface measure on $\mathbb{S}^{\mathbf{N}-1}$.

Recall that Ω is homogeneous of degree $-\mathbf{N}$. Consider

$$\begin{aligned}
&\left\{\int_0^\infty \left|\int_0^\infty \Omega(R,r) f(r\eta) r^{\mathbf{N}-1} dr\right|^p R^{\mathbf{N}-1} dR\right\}^{\frac{1}{p}} \\
&= \left\{\int_0^\infty \left|\int_0^\infty \Omega(1,t) f(tR\eta) t^{\mathbf{N}-1} dt\right|^p R^{\mathbf{N}-1} dR\right\}^{\frac{1}{p}} \\
&\leq \int_0^\infty \Omega(1,t) t^{\mathbf{N}-1} \left\{\int_0^\infty \left|f(tR\eta)\right|^p R^{\mathbf{N}-1} dR\right\}^{\frac{1}{p}} dt \qquad \text{by Minkowski integral inequality} \qquad (2.\ 5) \\
&= \int_0^\infty \Omega(1,t) t^{\mathbf{N}\left[1-\frac{1}{p}\right]-1} \left\{\int_0^\infty \left|f(r\eta)\right|^p r^{\mathbf{N}-1} dr\right\}^{\frac{1}{p}} dt \\
&= \mathbf{A} \left\{\int_0^\infty \left|f(r\eta)\right|^p r^{\mathbf{N}-1} dr\right\}^{\frac{1}{p}}.
\end{aligned}$$

We find

$$
\begin{aligned}
\left\|\mathbf{U}f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})} &= \left\{\int_{\mathbb{S}^{\mathbf{N}-1}}\int_0^\infty \left|\int_{\mathbb{S}^{\mathbf{N}-1}}\int_0^\infty \Omega(R,r)f(r\eta)r^{\mathbf{N}-1}drd\sigma(\eta)\right|^p R^{\mathbf{N}-1}dRd\sigma(\xi)\right\}^{\frac{1}{p}} \\
&= \omega_{\mathbf{N}-1}^{\frac{1}{p}}\left\{\int_0^\infty \left|\int_{\mathbb{S}^{\mathbf{N}-1}}\int_0^\infty \Omega(R,r)f(r\eta)r^{\mathbf{N}-1}drd\sigma(\eta)\right|^p R^{\mathbf{N}-1}dR\right\}^{\frac{1}{p}} \\
&\le \omega_{\mathbf{N}-1}^{\frac{1}{p}}\int_{\mathbb{S}^{\mathbf{N}-1}}\left\{\int_0^\infty \left|\int_0^\infty \Omega(R,r)f(r\eta)r^{\mathbf{N}-1}dr\right|^p R^{\mathbf{N}-1}dR\right\}^{\frac{1}{p}} d\sigma(\eta) \\
&\qquad\qquad \text{by Minkowski integral inequality} \\
&\le \omega_{\mathbf{N}-1}^{\frac{1}{p}}\mathbf{A}\int_{\mathbb{S}^{\mathbf{N}-1}}\left\{\int_0^\infty \left|f(r\eta)\right|^p r^{\mathbf{N}-1}dr\right\}^{\frac{1}{p}} d\sigma(\eta) \qquad \text{by (2. 5)} \\
&\le \mathbf{A}\omega_{\mathbf{N}-1}^{\frac{1}{p}}\left\{\int_{\mathbb{S}^{\mathbf{N}-1}}\int_0^\infty \left|f(r\eta)\right|^p r^{\mathbf{N}-1}drd\sigma(\eta)\right\}^{\frac{1}{p}}\left\{\int_{\mathbb{S}^{\mathbf{N}-1}} d\sigma(\eta)\right\}^{\frac{p-1}{p}} \\
&\qquad\qquad \text{by Hölder inequality} \\
&= \mathbf{A}\omega_{\mathbf{N}-1}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}
\end{aligned}
\tag{2. 6}
$$

where $\omega_{\mathbf{N}-1} = 2\pi^{\frac{\mathbf{N}}{2}}\Gamma^{-1}\left(\frac{\mathbf{N}}{2}\right)$ is the area of $\mathbb{S}^{\mathbf{N}-1}$.

When $\mathbf{N}=1$, simply take $d\sigma$ to be the point measure on 1 and -1. The same estimates hold in (2. 5)-(2. 6). □

Lemma 2.2. *Define $\Delta(t,\xi,\eta) = \left|1-2t\xi\cdot\eta+t^2\right|^{\frac{1}{2}}$ for $t>0$ and $\xi,\eta\in\mathbb{S}^{\mathbf{N}-1}$. We have*

$$
\begin{aligned}
\int_{\mathbb{S}^{\mathbf{N}-1}} \frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)}d\sigma(\xi) &= \int_{\mathbb{S}^{\mathbf{N}-1}} \frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)}d\sigma(\eta) \\
&\le \mathfrak{C}\ |1-t|^{-\frac{\mathbf{N}-\alpha}{\mathbf{N}}}, \qquad t\neq 1, \qquad \xi,\eta\in\mathbb{S}^{\mathbf{N}-1}.
\end{aligned}
\tag{2. 7}
$$

Proof. When $\mathbf{N}=1$, $d\sigma$ becomes the point measure on 1 or -1. The two integrals in (2. 7) are clearly bounded by $\mathfrak{C}|1-t|^{\alpha-1}$ for $t>0, t\neq 1$.

Let $\mathbf{N}\ge 2$. Note that $\Delta(t,\xi,\eta)$ is symmetric *w.r.t* ξ and η. For $0<t<1$, we have

$$
\mathbf{P}(\xi,t\eta) = \frac{1-|t\eta|^2}{|\xi-t\eta|^\mathbf{N}} = \frac{1-t^2}{\Delta^\mathbf{N}(t,\xi,\eta)}, \qquad \xi,\eta\in\mathbb{S}^{\mathbf{N}-1} \tag{2. 8}
$$

which is the Poisson kernel on the unit sphere $\mathbb{S}^{\mathbf{N}-1}$.

A direct computation shows

$$
\Delta_\eta\mathbf{P}(\xi,t\eta) = 0, \qquad \xi\in\mathbb{S}^{\mathbf{N}-1} \tag{2. 9}
$$

where Δ_η is the Laplacian operator *w.r.t* η.

By using the mean value property of harmonic functions, we find

$$
1 = \mathbf{P}(\xi,0) = \frac{1}{\omega_{\mathbf{N}-1}}\int_{\mathbb{S}^{\mathbf{N}-1}}\mathbf{P}(\xi,t\eta)d\sigma(\eta), \qquad 0<t<1. \tag{2. 10}
$$

This further implies

$$\frac{1}{\omega_{\mathbf{N}-1}}\int_{\mathbb{S}^{\mathbf{N}-1}}\frac{1-t^2}{\Delta^{\mathbf{N}}(t,\xi,\eta)}d\sigma(\eta) = 1, \qquad 0<t<1. \tag{2. 11}$$

On the other hand, write $0<s=t^{-1}<1$ for $t>1$. From (2. 8), we have

$$\begin{aligned}\frac{1-t^2}{\Delta^{\mathbf{N}}(t,\xi,\eta)} &= \frac{1-t^2}{\left|1+2t\xi\cdot\eta+t^2\right|^{\frac{\mathbf{N}}{2}}} = \frac{t^2(s^2-1)}{t^{\mathbf{N}}\left|s^2+2s\xi\cdot\eta+1\right|^{\frac{\mathbf{N}}{2}}}\\ &= -t^{2-\mathbf{N}}\frac{1-s^2}{\Delta^{\mathbf{N}}(s,\xi,\eta)} = -t^{2-\mathbf{N}}\mathbf{P}(\xi,s\eta), \qquad \xi,\eta\in\mathbb{S}^{\mathbf{N}-1}.\end{aligned} \tag{2. 12}$$

By using (2. 10) and (2. 12), we find

$$\begin{aligned}\frac{1}{\omega_{\mathbf{N}-1}}\int_{\mathbb{S}^{\mathbf{N}-1}}\frac{1-t^2}{\Delta^{\mathbf{N}}(t,\xi,\eta)}d\sigma(\eta) &= -t^{2-\mathbf{N}}\frac{1}{\omega_{\mathbf{N}-1}}\int_{\mathbb{S}^{\mathbf{N}-1}}\mathbf{P}(\xi,s\eta)d\sigma(\eta)\\ &= -t^{2-\mathbf{N}}, \qquad t>1.\end{aligned} \tag{2. 13}$$

By putting together (2. 11) and (2. 13), we obtain

$$\frac{1}{\omega_{\mathbf{N}-1}}\int_{\mathbb{S}^{\mathbf{N}-1}}\frac{1}{\Delta^{\mathbf{N}}(t,\xi,\eta)}d\sigma(\eta) \le \frac{1}{|1-t^2|} < \frac{1}{|1-t|}. \tag{2. 14}$$

Lastly, by applying Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{S}^{\mathbf{N}-1}}\frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)}d\sigma(\eta) &\le \left\{\int_{\mathbb{S}^{\mathbf{N}-1}}\left[\frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)}\right]^{\frac{\mathbf{N}}{\mathbf{N}-\alpha}}d\sigma(\eta)\right\}^{\frac{\mathbf{N}-\alpha}{\mathbf{N}}}\left\{\int_{\mathbb{S}^{\mathbf{N}-1}}d\sigma(\eta)\right\}^{\frac{\alpha}{\mathbf{N}}}\\ &= \left\{\int_{\mathbb{S}^{\mathbf{N}-1}}\frac{1}{\Delta^{\mathbf{N}}(t,\xi,\eta)}d\sigma(\eta)\right\}^{\frac{\mathbf{N}-\alpha}{\mathbf{N}}}(\omega_{\mathbf{N}-1})^{\frac{\alpha}{\mathbf{N}}}\\ &\le \left(\frac{1}{|1-t|}\right)^{\frac{\mathbf{N}-\alpha}{\mathbf{N}}}(\omega_{\mathbf{N}-1})^{\frac{\mathbf{N}-\alpha}{\mathbf{N}}}(\omega_{\mathbf{N}-1})^{\frac{\alpha}{\mathbf{N}}} \qquad \text{by (2. 14)}\\ &= \omega_{\mathbf{N}-1}\,|1-t|^{-\frac{\mathbf{N}-\alpha}{\mathbf{N}}}.\end{aligned} \tag{2. 15}$$

3 Proof of Theorem 1.1

3.1 The $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 4) implies the constraints in (1. 5)

Case 1 Let $p=1$. We first carry out an implication given by Sawyer and Wheeden [10]. Denote Q as any cube in $\mathbb{R}^{\mathbf{N}}$. Choose $f=\chi_Q$ which is an indicator function supported in Q. The $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 4) implies

$$\sup_{Q\subset\mathbb{R}^{\mathbf{N}}} |Q|^{\frac{\alpha}{\mathbf{N}}-1+\frac{1}{q}}\left\{\frac{1}{|Q|}\int_Q\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|Q|}\int_Q\left(\frac{1}{|x|}\right)^{\delta}dx\right\} < \infty. \tag{3. 1}$$

A standard exercise of changing dilations inside (3. 1) shows that $\frac{\alpha}{\mathbf{N}} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$ is an necessary condition. Moreover, it is essential to have $\gamma < \frac{\mathbf{N}}{q}$ for the local integrability of $|x|^{-\gamma q}$. We claim $\frac{\alpha}{\mathbf{N}} - 1 + \frac{1}{q} \geq 0$. Together with $\frac{\alpha}{\mathbf{N}} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$, we must have $\gamma + \delta \geq 0$.

Suppose $\frac{\alpha}{\mathbf{N}} - 1 + \frac{1}{q} < 0$. Let Q centered on some $x_o \neq 0$. By shrinking Q to x_o and applying Lebesgue's Differentiation Theorem, we find

$$\left\{\frac{1}{|Q|}\int_Q \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|Q|}\int_Q \left(\frac{1}{|x|}\right)^{\delta} dx\right\} = |x_o|^{-(\gamma+\delta)} > 0.$$

On the other hand, $|Q|^{\frac{\alpha}{\mathbf{N}}-1+\frac{1}{q}} \longrightarrow \infty$. This contradicts to (3. 1).

Let $\mathbf{I}_{\alpha\gamma\delta}f$ defined in (1. 1). Assert $f \geq 0$, $f \in \mathbf{L}^1(\mathbb{R}^{\mathbf{N}})$ supported in the unit ball, denoted by $\mathbf{B}$. We have

$$\begin{aligned}\mathbf{I}_{\alpha\gamma\delta}f(x) &\geq \chi(|x|>10)\int_{\mathbf{B}} f(y)\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha}\left(\frac{1}{|y|}\right)^{\delta} dy \\ &> 2^{\alpha-\mathbf{N}}\left(\frac{1}{|x|}\right)^{\mathbf{N}-\alpha+\gamma}\chi(|x|>10)\int_{\mathbf{B}} f(y)\left(\frac{1}{|y|}\right)^{\delta} dy.\end{aligned} \tag{3. 2}$$

Observe that if $\delta \geq 0$, then $\frac{\alpha}{\mathbf{N}} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$ implies $\mathbf{N} - \alpha + \gamma \leq \frac{\mathbf{N}}{q}$. Consequently, $[\mathbf{I}_{\alpha\gamma\delta}f]^q$ cannot be integrable in $\mathbb{R}^{\mathbf{N}}$. Therefore, we also need $\delta < 0$.

Case 2 Let $p > 1$. Consider $f(x) = \chi_Q(x)|x|^{-\delta\left(\frac{p}{p-1}\right)}$. As a well known, (1. 4) implies

$$\sup_{Q\subset\mathbb{R}^{\mathbf{N}}} |Q|^{\frac{\alpha}{\mathbf{N}}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{|Q|}\int_Q\left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|Q|}\int_Q\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx\right\}^{\frac{p-1}{p}} < \infty. \tag{3. 3}$$

We essentially need $\gamma < \mathbf{N}/q$ and $\delta < \mathbf{N}\left(\frac{p-1}{p}\right)$ for which $|x|^{-\gamma q}$ and $|x|^{-\delta\left(\frac{p}{p-1}\right)}$ are locally integrable. By changing dilations inside (3. 3), we find $\frac{\alpha}{\mathbf{N}} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$ is an necessity. Moreover, we claim $\frac{\alpha}{\mathbf{N}} \geq \frac{1}{p} - \frac{1}{q}$. Together with $\frac{\alpha}{\mathbf{N}} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$, we must have $\gamma + \delta \geq 0$.

Suppose $\frac{\alpha}{\mathbf{N}} < \frac{1}{p} - \frac{1}{q}$. Let Q centered on some $x_o \neq 0$. By shrinking Q to x_o and applying Lebesgue's Differentiation Theorem, we find

$$\left\{\frac{1}{|Q|}\int_Q\left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|Q|}\int_Q\left(\frac{1}{|x|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx\right\}^{\frac{p-1}{p}} = |x_o|^{-(\gamma+\delta)} > 0.$$

On the other hand, $|Q|^{\frac{\alpha}{\mathbf{N}}-\frac{1}{p}+\frac{1}{q}} \longrightarrow \infty$. We reach a contradiction to (3. 3).

3.2 The constraints in (1. 5) imply the $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 4)

Consider

$$\mathbf{I}_{\alpha\gamma\delta}f(x) = \mathbf{U}_1 f(x) + \mathbf{U}_2 f(x) + \mathbf{U}_3 f(x), \qquad f \geq 0 \tag{3. 4}$$

where

$$\mathbf{U}_1 f(x) = \int_{\mathbb{R}^\mathbf{N}} f(y)\Omega_1(x,y)dy,$$

$$\Omega_1(x,y) = \begin{cases} \left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha}\left(\frac{1}{|y|}\right)^{\delta}, & 0 \le \frac{|y|}{|x|} \le \frac{1}{2}, \\ 0, & \frac{|y|}{|x|} > \frac{1}{2}; \end{cases} \tag{3. 5}$$

$$\mathbf{U}_2 f(x) = \int_{\mathbb{R}^\mathbf{N}} f(y)\Omega_2(x,y)dy,$$

$$\Omega_2(x,y) = \begin{cases} \left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha}\left(\frac{1}{|y|}\right)^{\delta}, & \frac{|y|}{|x|} \ge 2, \\ 0, & 0 \le \frac{|y|}{|x|} < 2 \end{cases} \tag{3. 6}$$

and

$$\mathbf{U}_3 f(x) = \int_{\mathbb{R}^\mathbf{N}} f(y)\Omega_3(x,y)dy,$$

$$\Omega_3(x,y) = \begin{cases} \left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha}\left(\frac{1}{|y|}\right)^{\delta}, & \frac{1}{2} < \frac{|y|}{|x|} < 2, \\ 0, & 0 \le \frac{|y|}{|x|} \le \frac{1}{2} \text{ or } \frac{|y|}{|x|} \ge \frac{1}{2}. \end{cases} \tag{3. 7}$$

We aim to show that each $\mathbf{U}_i f, i = 1,2,3$ satisfies the $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 4). The first two can be obtained by refining the proof of Stein and Weiss [3] and using the two lemmas from the previous section. The crucial part of our estimates comes when dealing with $\mathbf{U}_3 f$ for which we go through an interpolation argument of changing measures.

Case One Let $1 \le p = q < \infty$. The homogeneity condition $\frac{\alpha}{\mathbf{N}} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$ implies $\alpha = \gamma + \delta$.

Note that $|x-y| \ge \frac{1}{2}|x|$ if $\frac{|y|}{|x|} \le \frac{1}{2}$. From (3. 5), we find

$$\Omega_1(x,y) \le 2^{\mathbf{N}-\alpha} \begin{cases} \left(\frac{1}{|x|}\right)^{\mathbf{N}-\alpha+\gamma}\left(\frac{1}{|y|}\right)^{\delta}, & 0 \le \frac{|y|}{|x|} \le \frac{1}{2}, \\ 0, & \frac{|y|}{|x|} > \frac{1}{2}. \end{cases} \tag{3. 8}$$

Because $\delta < \mathbf{N}\left(\frac{p-1}{p}\right)$, we have

$$\mathbf{A}_1 \doteq \int_0^\infty \Omega_1(1,t) t^{\mathbf{N}\left(\frac{p-1}{p}\right)-1} dt \le 2^{\mathbf{N}-\alpha}\int_0^{\frac{1}{2}} t^{\mathbf{N}\left(\frac{p-1}{p}\right)-\delta-1}dt < \infty. \tag{3. 9}$$

By applying Lemma 2.1, we obtain

$$\left\|\mathbf{U}_1 f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})} \le \mathfrak{C}_{\alpha\,\delta\,p}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1 \le p < \infty. \tag{3. 10}$$

On the other hand, $|x-y|\geq\frac{1}{2}|y|$ if $\frac{|y|}{|x|}\geq 2$. From (3. 6), we find

$$\Omega_2(x,y) \ \leq \ 2^{\mathbf{N}-\alpha}\begin{cases} \left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|y|}\right)^{\mathbf{N}-\alpha+\delta}, & \frac{|y|}{|x|}\geq 2, \\ 0, & 0\leq\frac{|y|}{|x|}<2 \end{cases} \tag{3. 11}$$

Because $\gamma<\mathbf{N}/q=\mathbf{N}/p$ and $\alpha=\gamma+\delta$, we have

$$\begin{aligned} \mathbf{A}_2 \ &\doteq \ \int_0^\infty \Omega_2(1,t)t^{\mathbf{N}\left(\frac{p-1}{p}\right)-1}dt \\ &\leq \ 2^{\mathbf{N}-\alpha}\int_2^\infty t^{\mathbf{N}\left(\frac{p-1}{p}\right)-\mathbf{N}+\alpha-\delta-1}dt \ = \ 2^{\mathbf{N}-\alpha}\int_2^\infty t^{-\frac{\mathbf{N}}{p}+\gamma-1}dt \ < \ \infty. \end{aligned} \tag{3. 12}$$

By applying Lemma 2.1, we obtain

$$\left\|\mathbf{U}_2 f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})} \ \leq \ \mathfrak{C}_{\alpha\ \gamma\ p}\ \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1\leq p<\infty. \tag{3. 13}$$

Write $x=R\xi$ and $y=r\eta$ for $\xi,\eta\in\mathbb{S}^{\mathbf{N}-1}$. Recall Ω_3 defined in (3. 7) which is homogeneous of degree $-\mathbf{N}$. We have

$$\begin{aligned} \mathbf{U}_3 f(x) \ &= \ \int_{\mathbb{R}^\mathbf{N}}\Omega_3(x,y)f(y)dy \\ &= \ \int_{\mathbb{S}^{\mathbf{N}-1}}\int_0^\infty \Omega_3(R\xi,r\eta)f(r\eta)r^{\mathbf{N}-1}drd\sigma(\eta) \\ &= \ \int_{\mathbb{S}^{\mathbf{N}-1}}\int_0^\infty \Omega_3(\xi,t\eta)f(tR\eta)t^{\mathbf{N}-1}dtd\sigma(\eta) \\ &= \ \int_{\mathbb{S}^{\mathbf{N}-1}}\int_{\frac{1}{2}}^2 \frac{1}{|\xi-t\eta|^{\mathbf{N}-\alpha}}f(tR\eta)t^{\mathbf{N}-1-\delta}dtd\sigma(\eta) \qquad \text{by (3. 7)} \\ &= \ \int_{\mathbb{S}^{\mathbf{N}-1}}\int_{\frac{1}{2}}^2 \frac{1}{\left[(\xi-t\eta)\cdot(\xi-t\eta)\right]^{\frac{\mathbf{N}-\alpha}{2}}}f(tR\eta)t^{\mathbf{N}-1-\delta}dtd\sigma(\eta) \\ &= \ \int_{\mathbb{S}^{\mathbf{N}-1}}\int_{\frac{1}{2}}^2 \frac{1}{|1-2t\xi\cdot\eta+t^2|^{\frac{\mathbf{N}-\alpha}{2}}}f(tR\eta)t^{\mathbf{N}-1-\delta}dtd\sigma(\eta) \\ &\lesssim \ \int_{\mathbb{S}^{\mathbf{N}-1}}\int_{\frac{1}{2}}^2 \frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)}f(tR\eta)dtd\sigma(\eta). \end{aligned} \tag{3. 14}$$

In particular, for $\mathbf{N}=1$, $d\sigma$ is a point measure on 1 and -1 inside (3. 14). We find

$$\mathbf{U}_3 f(x) \ \lesssim \ \int_{\frac{1}{2}}^2 |1-t|^{\alpha-1}\Big[f(tR)+f(-tR)\Big]dt. \tag{3. 15}$$

From (3. 14), we have

$$
\begin{aligned}
\left\|\mathbf{U}_3 f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})} \ &\lesssim \ \left\{\int_{\mathbb{S}^{\mathbf{N}-1}}\int_0^\infty \left\{\int_{\mathbb{S}^{\mathbf{N}-1}}\int_{\frac{1}{2}}^2 \frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)} f(tR\eta)dt d\sigma(\eta)\right\}^p R^{\mathbf{N}-1}dR d\sigma(\xi)\right\}^{\frac{1}{p}} \\
\lesssim &\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{\mathbf{N}-1}} \left\{\int_{\mathbb{S}^{\mathbf{N}-1}} \frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)} f(tR\eta) d\sigma(\eta)\right\}^p d\sigma(\xi) R^{\mathbf{N}-1}dR\right\}^{\frac{1}{p}} dt \\
&\qquad \text{by Minkowski integral inequality} \\
\lesssim &\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{\mathbf{N}-1}} \left\{\int_{\mathbb{S}^{\mathbf{N}-1}} \frac{[f(tR\eta)]^p}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)} d\sigma(\eta)\right\}\left\{\int_{\mathbb{S}^{\mathbf{N}-1}} \frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)} d\sigma(\eta)\right\}^{p-1} d\sigma(\xi) R^{\mathbf{N}-1}dR\right\}^{\frac{1}{p}} dt \\
&\qquad \text{by Hölder inequality} \\
\lesssim &\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{\mathbf{N}-1}} \left\{\int_{\mathbb{S}^{\mathbf{N}-1}} \frac{[f(tR\eta)]^p}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)} d\sigma(\eta)\right\} |1-t|^{\left[\frac{\alpha-\mathbf{N}}{\mathbf{N}}\right](p-1)} d\sigma(\xi) R^{\mathbf{N}-1}dR\right\}^{\frac{1}{p}} dt \\
&\qquad \text{by Lemma 2.2} \\
= &\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{\mathbf{N}-1}} \left\{|1-t|^{\left[\frac{\alpha-\mathbf{N}}{\mathbf{N}}\right](p-1)} \int_{\mathbb{S}^{\mathbf{N}-1}} \frac{1}{\Delta^{\mathbf{N}-\alpha}(t,\xi,\eta)} d\sigma(\xi)\right\} [f(tR\eta)]^p R^{\mathbf{N}-1}dR d\sigma(\eta)\right\}^{\frac{1}{p}} dt \\
\lesssim &\int_{\frac{1}{2}}^2 \left\{|1-t|^{\left[\frac{\alpha-\mathbf{N}}{\mathbf{N}}\right]p} \int_0^\infty \int_{\mathbb{S}^{\mathbf{N}-1}} [f(tR\eta)]^p R^{\mathbf{N}-1}dR d\sigma(\eta)\right\}^{\frac{1}{p}} dt \qquad \text{by Lemma 2.2} \\
= &\int_{\frac{1}{2}}^2 \left\{\int_0^\infty \int_{\mathbb{S}^{\mathbf{N}-1}} [f(tR\eta)]^p d\sigma(\eta) R^{\mathbf{N}-1}dR\right\}^{\frac{1}{p}} |1-t|^{\frac{\alpha-\mathbf{N}}{\mathbf{N}}} dt \\
= &\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})} \int_{\frac{1}{2}}^2 |1-t|^{\frac{\alpha-\mathbf{N}}{\mathbf{N}}} dt \ \leq \ \mathfrak{C}_\alpha \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1\leq p<\infty.
\end{aligned}
\tag{3. 16}
$$

Moreover, by using (3. 15), we find

$$
\begin{aligned}
\left\|\mathbf{U}_3 f\right\|_{\mathbf{L}^p(\mathbb{R})} \ &\lesssim \ \left\{\int_0^\infty \left\{\int_{\frac{1}{2}}^2 |1-t|^{\alpha-1}\Big[f(tR)+f(-tR)\Big]dt\right\}^p dR\right\}^{\frac{1}{p}} \\
&\leq \ \int_{\frac{1}{2}}^2 \left\{\int_0^\infty \Big[f(tR)+f(-tR)\Big]^p dR\right\}^p |1-t|^{\alpha-1} dt \\
&\qquad \text{by Minkowski integral inequality} \\
&= \ \left\|f\right\|_{\mathbf{L}^p(\mathbb{R})} \int_{\frac{1}{2}}^2 |1-t|^{\alpha-1} dt \\
&\leq \ \mathfrak{C}_\alpha \left\|f\right\|_{\mathbf{L}^p(\mathbb{R})}, \qquad 1\leq p<\infty.
\end{aligned}
\tag{3. 17}
$$

Case Two Consider $1 \le p < q < \infty$. Assert

$$\mathbf{V}_\delta f(x) = |x|^{-\mathbf{N}+\delta} \int_{|y|<|x|} |y|^{-\delta} f(y) dy, \qquad \delta < \mathbf{N}\left(\tfrac{p-1}{p}\right). \tag{3. 18}$$

We claim

$$\left\|\mathbf{V}_\delta f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})} \le \mathfrak{C}_{\delta\, p} \, \|f\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1 \le p < \infty. \tag{3. 19}$$

Write

$$\mathbf{V}_\delta f(x) = \int_{\mathbb{R}^n} \Omega(|x|,|y|) f(y) dy, \qquad \Omega(u,v) = \begin{cases} u^{-\mathbf{N}+\delta} v^{-\delta} & \text{if } v < u \\ 0 & \text{otherwise.} \end{cases} \tag{3. 20}$$

Observe that Ω in (3. 20) is homogeneous of degree $-\mathbf{N}$. Moreover,

$$\int_0^\infty \Omega(1,t) t^{\mathbf{N}(\frac{p-1}{p})-1} dt = \int_0^1 t^{\mathbf{N}(\frac{p-1}{p})-\delta-1} dt < \infty \tag{3. 21}$$

provided by $\delta < \mathbf{N}\left(\frac{p-1}{p}\right)$. Lemma 2.1 implies (3. 19).

On the other hand, $\mathbf{V}_\delta f$ defined in (3. 18) satisfies

$$\begin{aligned} \mathbf{V}_\delta f(x) &\le |x|^{-\mathbf{N}+\delta} \left\{ \int_{|y|<|x|} |y|^{-\delta(\frac{p}{p-1})} dy \right\}^{\frac{p-1}{p}} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})} \qquad \text{by Hölder inequality} \\ &\le \mathfrak{C}_{\delta\, p} \, |x|^{-\frac{\mathbf{N}}{p}} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1 \le p < \infty. \end{aligned} \tag{3. 22}$$

Recall $\mathbf{U}_1 f$ defined in (3. 5). Note that $0 \le \frac{|y|}{|x|} \le \frac{1}{2}$ implies $\frac{1}{2}|x| \le |x| - |y| \le |x-y|$. Let $f, g \ge 0$ and $f \in \mathbf{L}^p(\mathbb{R}^\mathbf{N})$, $g \in \mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^\mathbf{N})$. We have

$$\begin{aligned} \int_{\mathbb{R}^\mathbf{N}} \mathbf{U}_1 f(x) g(x) dx &= \int_{\mathbb{R}^\mathbf{N}} \left\{ \int_{|y| \le \frac{1}{2}|x|} \frac{f(y) g(x)}{|x|^\gamma \, |x-y|^{\mathbf{N}-\alpha} \, |y|^\delta} dy \right\} dx \\ &\lesssim \int_{\mathbb{R}^\mathbf{N}} \left\{ \int_{|y|<|x|} \frac{f(y) g(x)}{|x|^{\gamma+\mathbf{N}-\alpha} \, |y|^\delta} dy \right\} dx \\ &= \int_{\mathbb{R}^\mathbf{N}} |x|^{\alpha-(\gamma+\delta)} g(x) \left\{ |x|^{-\mathbf{N}+\delta} \int_{|y|<|x|} f(y) |y|^{-\delta} dy \right\} dx \\ &= \int_{\mathbb{R}^\mathbf{N}} |x|^{\alpha-(\gamma+\delta)} g(x) \mathbf{V}_\delta f(x) dx \\ &\le \left\{ \int_{\mathbb{R}^\mathbf{N}} |x|^{[\alpha-(\gamma+\delta)]q} \left(\mathbf{V}_\delta f\right)^q (x) dx \right\}^{\frac{1}{q}} \left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^\mathbf{N})} \quad \text{by Hölder inequality.} \end{aligned} \tag{3. 23}$$

Let $\frac{\alpha}{\mathbf{N}}=\frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}}, 1\le p<q<\infty$. We find

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^{\mathbf{N}}} |x|^{\left[\alpha-(\gamma+\delta)\right]q}\Big(\mathbf{V}_\delta f\Big)^q(x)dx\right\}^{\frac{1}{q}}\\
&\le \left\{\int_{\mathbb{R}^{\mathbf{N}}} |x|^{\left[\alpha-(\gamma+\delta)\right]q}|x|^{-\mathbf{N}\left[\frac{q}{p}-1\right]}\left\|f\right\|^{q-p}_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}\Big(\mathbf{V}_\delta f\Big)^p(x)dx\right\}^{\frac{1}{q}} \qquad \text{by (3. 22)} \qquad (3.\ 24)\\
&= \left\|f\right\|^{1-\frac{p}{q}}_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(\mathbf{V}_\delta f\Big)^p(x)dx\right\}^{\frac{1}{q}} \le \mathfrak{C}_{\delta\ p}\ \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})} \qquad \text{by (3. 19).}
\end{aligned}$$

From (3. 23)-(3. 24), we conclude

$$\left\|\mathbf{U}_1 f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \le \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1\le p<q<\infty. \qquad (3.\ 25)$$

Consider

$$\mathbf{V}_\gamma g(x) = |x|^{-\mathbf{N}+\gamma}\int_{|y|<|x|}|y|^{-\gamma}g(y)dy, \qquad \gamma<\tfrac{\mathbf{N}}{q}. \qquad (3.\ 26)$$

We claim

$$\left\|\mathbf{V}_\gamma g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})} \le \mathfrak{C}_{\gamma\ q}\ ||g||_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})}, \qquad 1<q<\infty. \qquad (3.\ 27)$$

Write

$$\mathbf{V}_\gamma g(x) = \int_{\mathbb{R}^n}\Omega(|x|,|y|)g(y)dy, \qquad \Omega(u,v) = \begin{cases} u^{-\mathbf{N}+\gamma}v^{-\gamma} & \text{if } v<u\\ 0 & \text{otherwise.}\end{cases} \qquad (3.\ 28)$$

Observe that Ω in (3. 28) is homogeneous of degree $-\mathbf{N}$. Moreover,

$$\int_0^\infty \Omega(1,t)t^{\frac{\mathbf{N}}{q}-1}dt = \int_0^1 t^{\frac{\mathbf{N}}{q}-\gamma-1}dt < \infty \qquad (3.\ 29)$$

provided by $\gamma<\frac{\mathbf{N}}{q}$. Lemma 2.1 implies (3. 27).

On the other hand, $\mathbf{V}_\gamma g$ defined in (3. 26) satisfies

$$\begin{aligned}
\mathbf{V}_\gamma g(y) &= |y|^{-\mathbf{N}+\gamma}\int_{|x|<|y|}|x|^{-\gamma}g(x)dx\\
&\le |y|^{-\mathbf{N}+\gamma}\left\{\int_{|x|<|y|}|x|^{-\gamma q}dx\right\}^{\frac{1}{q}}\left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})} \qquad \text{by Hölder inequality} \qquad (3.\ 30)\\
&\le \mathfrak{C}_{\gamma\ q}\ |y|^{-\mathbf{N}\left(\frac{q-1}{q}\right)}\left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})}.
\end{aligned}$$

Recall $\mathbf{U}_2 f$ defined in (3. 6). Note that $\frac{|y|}{|x|}\ge 2$ implies $\frac{1}{2}|y|\le|y|-|x|\le|x-y|$. We have

$$\begin{aligned}
\int_{\mathbb{R}^{\mathbf{N}}}\mathbf{U}_2 f(x)g(x)dx &= \int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{|y|\ge 2|x|}\frac{f(y)}{|x|^\gamma\ |x-y|^{\mathbf{N}-\alpha}|y|^\delta}dy\right\}g(x)dx\\
&\lesssim \int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{|y|>|x|}\frac{f(y)g(x)}{|x|^\gamma\ |y|^{\mathbf{N}-\alpha+\delta}}dy\right\}dx.
\end{aligned} \qquad (3.\ 31)$$

By using (3. 31) and Tonelli's theorem, we have

$$\begin{aligned}\int_{\mathbb{R}^{\mathbf{N}}} \mathbf{U}_2 f(x) g(x) dx &\lesssim \int_{\mathbb{R}^{\mathbf{N}}} |y|^{\alpha-(\gamma+\delta)} f(y) \left\{ |y|^{-\mathbf{N}+\gamma} \int_{|x|<|y|} g(x)|x|^{-\gamma} dx \right\} dy \\ &= \int_{\mathbb{R}^{\mathbf{N}}} |y|^{\alpha-(\gamma+\delta)} f(y) \mathbf{V}_\gamma g(y) dy \\ &\leq \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})} \left\{ \int_{\mathbb{R}^{\mathbf{N}}} |y|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}} \Big(\mathbf{V}_\gamma g\Big)^{\frac{p}{p-1}}(y) dy \right\}^{\frac{p-1}{p}} \qquad \text{by Hölder inequality.}\end{aligned} \tag{3. 32}$$

Let $\frac{\alpha}{\mathbf{N}} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}, 1 \leq p < q < \infty$. For $p = 1$, we find

$$\begin{aligned}\left\| |y|^{\alpha-(\gamma+\delta)} \mathbf{V}_\gamma g(y) \right\|_{\mathbf{L}^\infty(\mathbb{R}^{\mathbf{N}})} &\leq \mathfrak{C}_{\gamma\ q}\ |y|^{\alpha-(\gamma+\delta)} |y|^{-\mathbf{N}\left(\frac{q-1}{q}\right)} \left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})} \qquad \text{by (3. 30)} \\ &= \mathfrak{C}_{\gamma\ q}\ \left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})}.\end{aligned} \tag{3. 33}$$

For $p > 1$, write

$$\begin{aligned}&\left\{ \int_{\mathbb{R}^{\mathbf{N}}} |y|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}} \Big(\mathbf{V}_\gamma g\Big)^{\frac{p}{p-1}}(y) dy \right\}^{\frac{p-1}{p}} \\ &= \left\{ \int_{\mathbb{R}^{\mathbf{N}}} |y|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}} \Big(\mathbf{V}_\gamma g\Big)^{\left[\frac{p}{p-1}-\frac{q}{q-1}\right]}(y) \Big(\mathbf{V}_\gamma g\Big)^{\frac{q}{q-1}}(y) dy \right\}^{\frac{p-1}{p}}.\end{aligned}$$

By using (3. 30) again, we have

$$\begin{aligned}|y|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}} \Big(\mathbf{V}_\gamma g\Big)^{\left[\frac{p}{p-1}-\frac{q}{q-1}\right]}(y) &\leq \mathfrak{C}_{\gamma\ q}\ |y|^{\left[\alpha-(\gamma+\delta)\right]\frac{p}{p-1}} |y|^{-\mathbf{N}\left(\frac{q-1}{q}\right)\left[\frac{p}{p-1}-\frac{q}{q-1}\right]} \left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})}^{\frac{p}{p-1}-\frac{q}{q-1}} \\ &= \mathfrak{C}_{\gamma\ q}\ |y|^{\mathbf{N}\left[\frac{1}{p}-\frac{1}{q}\right]\frac{p}{p-1}-\mathbf{N}\left(\frac{q-1}{q}\right)\left[\frac{p}{p-1}-\frac{q}{q-1}\right]} \left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})}^{\frac{p}{p-1}-\frac{q}{q-1}} \\ &= \mathfrak{C}_{\gamma\ q}\ \left\|g\right\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})}^{\frac{p}{p-1}-\frac{q}{q-1}}.\end{aligned} \tag{3. 34}$$

From (3. 32)-(3. 34), together with the $\mathbf{L}^{\frac{q}{q-1}}$-estimate in (3. 27), we conclude

$$\left\|\mathbf{U}_2 f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \leq \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1 \leq p < q < \infty. \tag{3. 35}$$

Recall $\mathbf{U}_3 f$ defined in (3. 7). We have

$$\begin{aligned}\mathbf{U}_3 f(x) &= \int_{\frac{1}{2}|x|<|y|<2|x|} f(y) \left(\frac{1}{|x|}\right)^\gamma \left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha} \left(\frac{1}{|y|}\right)^\delta dy \\ &\lesssim \int_{\mathbb{R}^{\mathbf{N}}} f(y) \left(\frac{1}{|x-y|}\right)^{\mathbf{N}-(\alpha-\gamma-\delta)} dy.\end{aligned} \tag{3. 36}$$

Note that $\frac{\alpha-\gamma-\delta}{\mathbf{N}}=\frac{1}{p}-\frac{1}{q}, 1\leq p<q<\infty$. Define

$$\mathbf{I}_{\alpha-\gamma-\delta}f(x) ~=~ \int_{\mathbb{R}^{\mathbf{N}}} f(y)\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-(\alpha-\gamma-\delta)} dy. \tag{3. 37}$$

For $p>1$, we have

$$\left\|\mathbf{I}_{\alpha-\gamma-\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} ~\leq~ \mathfrak{C}_{p~q}~\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}. \tag{3. 38}$$

This is a classical result due to Hardy, Littlewood and Sobolev [1]-[2].

Remark 3.1. *For $p=1$, we have $\mathbf{I}_{\alpha-\gamma-\delta}\colon \mathbf{L}^1(\mathbb{R}^{\mathbf{N}})\longrightarrow \mathbf{L}^{q,\infty}(\mathbb{R}^{\mathbf{N}})$ for $\frac{\alpha-\gamma-\delta}{\mathbf{N}}=1-\frac{1}{q}, 1<q<\infty$. See chapter V of Stein [5].*

Given $E\subset\mathbb{R}^{\mathbf{N}}$, denote $\mathbf{vol}\{E\}=\int_E dx$. From (3. 36)-(3. 37) and **Remark 3.1**, we have

$$\begin{aligned}\lambda^q\mathbf{vol}\left\{x\in\mathbb{R}^{\mathbf{N}}\colon \mathbf{U}_3 f(x)>\lambda\right\} ~&\leq~ \lambda^q\mathbf{vol}\left\{x\in\mathbb{R}^{\mathbf{N}}\colon \mathbf{I}_{\alpha-\gamma-\delta}f(x)>\lambda\right\}\\ &\lesssim~ \left\|f\right\|^q_{\mathbf{L}^1(\mathbb{R}^{\mathbf{N}})}, \qquad \lambda>0.\end{aligned} \tag{3. 39}$$

By replacing $f(x)$ with $f(x)|x|^\delta$ inside (3. 39), we obtain

$$\lambda~\mathbf{vol}\left\{x\in\mathbb{R}^{\mathbf{N}}\colon \int_{\frac{1}{2}|x|<|y|<2|x|} f(y)\left(\frac{1}{|x|}\right)^\gamma\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha} dy>\lambda\right\}^{\frac{1}{q}} ~\lesssim~ \int_{\mathbb{R}^{\mathbf{N}}} f(x)|x|^\delta dx \tag{3. 40}$$

for every $\lambda>0$.

Recall $\delta<0$. Let $\delta_1<\delta<\delta_2<0$ of which $\delta_i, i=1,2$ are close to δ. We find

$$\frac{\alpha}{\mathbf{N}} ~=~ 1-\frac{1}{q_i}+\frac{\gamma+\delta_i}{\mathbf{N}}, \qquad i ~=~ 1,2 \tag{3. 41}$$

for some $q_1>q>q_2>1$.

By carrying out the same argument as (3. 36)-(3. 40), we simultaneously have

$$\lambda~\mathbf{vol}\left\{x\in\mathbb{R}^{\mathbf{N}}\colon \int_{\frac{1}{2}|x|<|y|<2|x|} f(y)\left(\frac{1}{|x|}\right)^\gamma\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha} dy>\lambda\right\}^{\frac{1}{q_i}} ~\lesssim~ \int_{\mathbb{R}^{\mathbf{N}}} f(x)|x|^{\delta_i} dx, \quad i ~=~ 1,2 \tag{3. 42}$$

for every $\lambda>0$.

Next, we need to apply a Marcinkiewicz interpolation theorem of changing measures, due to Stein and Weiss [4].

Let $\mu_i, i=1,2$ be two absolutely continuous measures satisfying

$$\mu_i(E) ~=~ \int_E |x|^{\delta_i}dx, \qquad i ~=~ 1,2. \tag{3. 43}$$

Define

$$\mu_t(E)=\int_E |x|^{\delta_1(1-t)}|x|^{\delta_2 t}dx, \qquad \frac{1}{q_t} ~=~ \frac{1-t}{q_1}+\frac{t}{q_2}, \qquad 0 ~\leq~ t ~\leq~ 1. \tag{3. 44}$$

Theorem B: Stein and Weiss, 1958
Let $\mathbf{T}$ *be a sub-linear operator, having the following properties:*

(1) The domain of $\mathbf{T}$ *includes* $\mathbf{L}^1(\mathbb{R}^\mathbf{N}, d\mu_1) \cap \mathbf{L}^1(\mathbb{R}^\mathbf{N}, d\mu_2)$.

(2) If $f \in \mathbf{L}^1(\mathbb{R}^\mathbf{N}, d\mu_i)$, $i = 1, 2$, *we have*

$$\lambda\, \mathbf{vol}\left\{x \in \mathbb{R}^\mathbf{N} : \left|\mathbf{T}f(x)\right| > \lambda\right\}^{\frac{1}{q_i}} \;\leq\; \mathfrak{C} \int_{\mathbb{R}^\mathbf{N}} |f(x)| d\mu_i(x), \qquad i = 1, 2. \tag{3. 45}$$

Then,

$$\left\|\mathbf{T}f\right\|_{\mathbf{L}^{q_t}(\mathbb{R}^\mathbf{N})} \;\leq\; \mathfrak{C} \int_{\mathbb{R}^\mathbf{N}} |f(x)| d\mu_t(x), \qquad 0 < t < 1. \tag{3. 46}$$

Remark 3.2. Theorem B *in the more general setting of measurable spaces can be found in the paper by Stein and Weiss [4].*

Recall $\frac{\alpha}{\mathbf{N}} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$ and (3. 41). There is a $0 < t < 1$ such that $\delta = (1-t)\delta_1 + t\delta_2$ and $\frac{1}{q} = \frac{1-t}{q_1} + \frac{t}{q_2}$. By using (3. 42) and applying **Theorem B**, we obtain

$$\left\|\mathbf{U}_3 f\right\|_{\mathbf{L}^q(\mathbb{R}^\mathbf{N})} \;\leq\; \left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^\mathbf{N})}. \tag{3. 47}$$

4 Proof of Theorem 1.2

4.1 Norm inequality in (1. 8) implies constraints in (1. 9)-(1. 10)

Let $\mathbf{II}_{\alpha\gamma\delta}$ defined in (1. 7) for $0 < \alpha_i < \mathbf{N}_i, i = 1, 2, \ldots, n$ and $\gamma, \delta \in \mathbb{R}$. First, it is clear that $\gamma < \frac{\mathbf{N}}{q}, \delta < 0, \gamma + \delta \geq 0, \frac{\alpha}{\mathbf{N}} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$ are still required because $\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha} \leq \prod_{i=1}^n \left(\frac{1}{|x_i - y_i|}\right)^{\mathbf{N}_i - \alpha_i}$. Next, we show $\alpha_i - \mathbf{N}_i < \delta, i = 1, 2, \ldots, n$ as an extra necessary condition.

Let $\mathbf{Q} \doteq \mathbf{Q}_1 \times \mathbf{Q}_2 \times \cdots \times \mathbf{Q}_n$ where $\mathbf{Q}_i \subset \mathbb{R}^{\mathbf{N}_i}, i = 1, 2, \ldots, n$ is a cube. We write $\mathbf{Q} = \mathbf{Q}_i \times \mathbf{Q}'_i$ where $\mathbf{Q}'_i = \bigotimes_{j \neq i} \mathbf{Q}_j$. Denote $x = (x_i, x'_i) \in \mathbb{R}^{\mathbf{N}_i} \times \mathbb{R}^{\mathbf{N}-\mathbf{N}_i}$.

Let f be an indicator function supported in $\mathbf{Q}$. The $\mathbf{L}^1 \longrightarrow \mathbf{L}^q$-norm inequality in (1. 8) implies

$$\begin{aligned} \mathbf{A}_q^{\alpha\,\gamma\,\delta}(\mathbf{Q}) \;&=\; \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - 1 + \frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|} \int_\mathbf{Q} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|} \int_\mathbf{Q} \left(\frac{1}{|x|}\right)^{\delta} dx\right\} \\ &\doteq\; \prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - 1 + \frac{1}{q}}\; \mathbf{B}_q^{\alpha\,\gamma\,\delta}(\mathbf{Q}) \;\leq\; \mathfrak{C}_{\alpha\,\gamma\,\delta\,q}, \qquad \mathbf{Q} \subset \mathbb{R}^\mathbf{N}. \end{aligned} \tag{4. 1}$$

We claim $\frac{\alpha_i}{\mathbf{N}_i} - 1 + \frac{1}{q} \geq 0, i = 1, 2, \ldots, n$. Suppose $\frac{\alpha_i}{\mathbf{N}_i} - 1 + \frac{1}{q} < 0$ for some $i \in \{1, 2, \ldots, n\}$. Consider $\mathbf{Q}$ centered on the origin of $\mathbb{R}^\mathbf{N}$. Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = \lambda$, $0 < \lambda < 1$ and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}} = 1$ for $j \neq i$. By shrinking $\mathbf{Q}_i$ to $0 \in \mathbb{R}^{\mathbf{N}_i}$ and applying Lebesgue's Differentiation Theorem, we find

$$\lim_{\lambda \longrightarrow 0} \mathbf{B}_q^{\alpha\,\gamma\,\delta}(\mathbf{Q}) \;=\; \left\{\frac{1}{|\mathbf{Q}'_i|} \int_{\mathbf{Q}'_i} \left(\frac{1}{|x'_i|}\right)^{\gamma q} dx'_i\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}'_i|} \int_{\mathbf{Q}'_i} \left(\frac{1}{|x'_i|}\right)^{\delta} dx'_i\right\} \;>\; 0. \tag{4. 2}$$

Consequently, $\mathbf{A}_q^{\alpha\,\gamma\,\delta}(\mathbf{Q})$ in (4. 1) diverges to infinity as $\lambda \longrightarrow 0$.

Let

$$\mathbf{Q}_i^k \;=\; \mathbf{Q}_i \cap \left\{2^{-k-1} \le |x_i| < 2^{-k}\right\}, \qquad k \ge 0, \qquad i = 1, 2, \ldots, n. \tag{4. 3}$$

Given $i \in \{1, 2, \ldots, n\}$, we assert $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}} = \lambda$ for $j \ne i$. Write

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{q\left[\frac{\alpha_i}{\mathbf{N}_i}-1+\frac{1}{q}\right]} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta} dx\right\}^q \;= \\
&\sum_{k\ge0}\prod_{j\ne i} |\mathbf{Q}_j|^{q\left[\frac{\alpha_j}{\mathbf{N}_j}-1+\frac{1}{q}\right]} \left\{\prod_{j\ne i}\frac{1}{|\mathbf{Q}_j|}\iint_{\mathbf{Q}_i^k\times\bigotimes_{j\ne i}\mathbf{Q}_j} \sqrt{|x_i|^2+\sum_{j\ne i}|x_j|^2}^{\,-\gamma q} dx_i \prod_{j\ne i} dx_j\right\}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta} dx\right\}^q \\
&\doteq\; \sum_{k\ge0} \mathbf{A}_k(\lambda).
\end{aligned} \tag{4. 4}$$

By applying Lebesgue's Differentiation Theorem, we have

$$\lim_{\lambda\longrightarrow0} \prod_{j\ne i}\frac{1}{|\mathbf{Q}_j|}\iint_{\mathbf{Q}_i^k\times\bigotimes_{j\ne i}\mathbf{Q}_j} \sqrt{|x_i|^2+\sum_{j\ne i}|x_j|^2}^{\,-\gamma q} dx_i \prod_{j\ne i} dx_j \;=\; \int_{\mathbf{Q}_i^k} \left(\frac{1}{|x_i|}\right)^{\gamma q} dx_i. \tag{4. 5}$$

Suppose that there is a $j \ne i$ such that $\frac{\alpha_j}{\mathbf{N}_j} - 1 + \frac{1}{q} > 0$. We find $\mathbf{A}_k(0) = 0$ for every $k \ge 0$. Moreover, each $\mathbf{A}_k(\lambda)$ is Hölder continuous for $\lambda \ge 0$ whose exponent is strict positive depending on $\frac{\alpha_j}{\mathbf{N}_j} - 1 + \frac{1}{q}$. Recall the inequality in (4. 1). For every $\lambda > 0$, $\sum_{k\ge0}\mathbf{A}_k(\lambda) \le \mathfrak{C}_{\alpha\,\gamma\,\delta\,q}$. Consequently, $\sum_{k\ge0}\mathbf{A}_k(\lambda)$ is continuous at $\lambda = 0$. We have

$$\lim_{\lambda\longrightarrow0}\sum_{k\ge0}\mathbf{A}_k(\lambda) \;=\; 0. \tag{4. 6}$$

A direct computation shows

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{q\left[\frac{\alpha_i}{\mathbf{N}_i}-1+\frac{1}{q}\right]} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta} dx\right\}^q \\
&\ge\; \mathfrak{C}_{\delta\,q}\prod_{j\ne i}\lambda^{q\left[\alpha_j-\mathbf{N}_j+\frac{\mathbf{N}_j}{q}\right]}\int_{\mathbf{Q}_i}\sqrt{|x_i|^2+\sum_{j\ne i}\lambda^2}^{\,-\gamma q} dx_i \\
&\ge\; \mathfrak{C}_{\delta\,q}\prod_{j\ne i}\lambda^{q\left[\alpha_j-\mathbf{N}_j+\frac{\mathbf{N}_j}{q}\right]}\int_{0<|x_i|\le\lambda}\left(\frac{1}{\lambda}\right)^{\gamma q} dx_i \;=\; \mathfrak{C}_{\gamma\,\delta\,q}\,\lambda^{\mathbf{N}_i-\gamma q+\sum_{j\ne i} q\left[\alpha_j-\mathbf{N}_j+\frac{\mathbf{N}_j}{q}\right]}.
\end{aligned} \tag{4. 7}$$

From (4. 6)-(4. 7), by using $\frac{\alpha}{\mathbf{N}} = 1 - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}$, we find

$$\frac{\mathbf{N}_i}{q} - \gamma + \sum_{j\ne i}\alpha_j - \mathbf{N}_j + \frac{\mathbf{N}_j}{q} \;>\; 0 \qquad\Longrightarrow\qquad \alpha_i \;<\; \frac{\mathbf{N}}{q} - \gamma + \alpha - \mathbf{N} + \mathbf{N}_i \;=\; \mathbf{N}_i + \delta. \tag{4. 8}$$

On the other hand, suppose $\frac{\alpha_j}{\mathbf{N}_j}-1+\frac{1}{q}=0$ for every $j\neq i$. Similar to (4. 7), we find

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{q\left[\frac{\alpha_i}{\mathbf{N}_i}-1+\frac{1}{q}\right]}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\delta}dx\right\}^q\\
&\geq\ \mathfrak{C}_{\delta\, q}\prod_{j\neq i}\lambda^{q\left[\alpha_j-\mathbf{N}_j+\frac{\mathbf{N}_j}{q}\right]}\int_{\mathbf{Q}_i}\sqrt{|x_i|^2+\sum_{j\neq i}\lambda^2}^{\,-\gamma q}dx_i \qquad (4.\ 9)\\
&\geq\ \mathfrak{C}_{\delta\, q}\int_{\lambda<|x_i|\leq 1}\left(\frac{1}{|x_i|}\right)^{\gamma q}dx_i.
\end{aligned}$$

The last integral in (4. 9) converges as $\lambda\longrightarrow 0$. We must have $\gamma q<\mathbf{N}_i$. By using $\frac{\alpha}{\mathbf{N}}=1-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}}$ and taking into account $\frac{\alpha_j}{\mathbf{N}_j}-1+\frac{1}{q}=0$ for every $j\neq i$, we find

$$\alpha_i\ =\ \mathbf{N}_i-\frac{\mathbf{N}_i}{q}+\gamma+\delta \qquad\Longrightarrow\qquad \alpha_i\ <\ \mathbf{N}_i+\delta. \tag{4. 10}$$

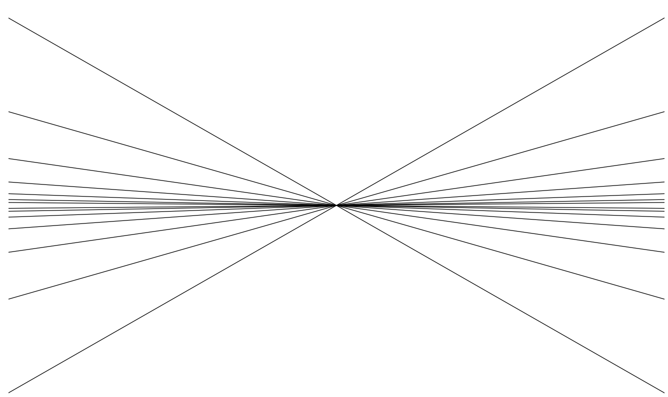
4.2 Constraints in (1. 9)-(1. 10) imply the norm inequality in (1. 8)

Now, we prove the $\mathbf{L}^1\longrightarrow\mathbf{L}^q$-norm inequality in (1. 8) for α,γ,δ satisfying (1. 9)-(1. 10). By symmetry, we assume $|x_1-y_1|\geq|x_i-y_i|, i=2,\ldots,n$. Denote $\mathbf{t}$ to be an $(n-1)$-tuple: $\left(2^{-t_2},\ldots,2^{-t_n}\right)$ where $t_i\geq 0,\ i=2,\ldots,n$. Let $f\geq 0$. We define

$$\Delta_{\mathbf{t}}\mathbf{II}_{\alpha\gamma\delta}f(x)\ =\ \int_{\Gamma_{\mathbf{t}}(x)}f(y)\left(\frac{1}{|x|}\right)^{\gamma}\prod_{i=1}^n\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y|}\right)^{\delta}dy,\qquad x\neq 0 \tag{4. 11}$$

where

$$\Gamma_{\mathbf{t}}(x)\ =\ \left\{y\in\mathbb{R}^{\mathbf{N}}:\ 2^{-t_i-1}<\frac{|x_i-y_i|}{|x_1-y_1|}\leq 2^{-t_i},\ i=2,\ldots,n\right\}. \tag{4. 12}$$



In particular, we write $\Gamma_o(x)=\Gamma_{\mathbf{t}}(x)$ for $t_2=\cdots=t_n=0$.

Let

$$\mathbf{t}x \;=\; \Big(x_1, 2^{-t_2}x_2, \ldots, 2^{-t_n}x_n\Big), \qquad \mathbf{t}^{-1}x \;=\; \Big(x_1, 2^{t_2}x_2, \ldots, 2^{t_n}x_n\Big). \tag{4. 13}$$

We have

$$\begin{array}{lr}
\Delta_{\mathbf{t}}\mathbf{II}_{\alpha\gamma\delta}f(x) \;=\; \displaystyle\int_{\Gamma_{\mathbf{t}}(x)} f(y)\left(\frac{1}{|x|}\right)^{\gamma}\prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y|}\right)^{\delta}dy & \\
\;=\; \displaystyle\int_{\Gamma_o(x)} f(\mathbf{t}^{-1}y)\left(\frac{1}{|\mathbf{t}^{-1}x|}\right)^{\gamma}\prod_{i=1}^{n}\left(\frac{1}{|2^{t_i}x_i-2^{t_i}y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|\mathbf{t}^{-1}y|}\right)^{\delta}2^{\mathbf{N}_2t_2+\cdots+\mathbf{N}_nt_n}dy & (4.\ 14)\\
\qquad\qquad\qquad x\to\mathbf{t}^{-1}x,\, y\to\mathbf{t}^{-1}y & \\
\;=\; 2^{\alpha_2t_2+\cdots+\alpha_nt_n}\displaystyle\int_{\Gamma_o(x)} f(\mathbf{t}^{-1}y)\left(\frac{1}{|\mathbf{t}^{-1}x|}\right)^{\gamma}\prod_{i=1}^{n}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|\mathbf{t}^{-1}y|}\right)^{\delta}dy. &
\end{array}$$

Denote

$$t_\mu \;\doteq\; \max\{t_i : i=2,\ldots,n\}. \tag{4. 15}$$

Recall $\gamma+\delta\geq 0$ and $\delta<0$. Because $t_i\geq 0$, $i=2,\ldots,n$, we find

$$\left|\mathbf{t}^{-1}x\right|^{\gamma} \;\geq\; |x|^{\gamma}, \qquad \left|\mathbf{t}^{-1}y\right|^{\delta} \;\geq\; 2^{t_\mu\delta}\left|y\right|^{\delta}. \tag{4. 16}$$

From (4. 14) to (4. 16), we further have

$$\begin{array}{lr}
\Delta_{\mathbf{t}}\mathbf{II}_{\alpha\gamma\delta}f(x) \;\leq\; 2^{\alpha_1t_1+\alpha_2t_2+\cdots+\alpha_nt_n}2^{-t_\mu\delta}\displaystyle\int_{\Gamma_o(x)} f(\mathbf{t}^{-1}y)\prod_{i=1}^{n}\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x_i-y_i|}\right)^{\mathbf{N}_i-\alpha_i}\left(\frac{1}{|y|}\right)^{\delta}dy & \\
 & (4.\ 17)\\
\qquad\lesssim\; 2^{\alpha_1t_1+\alpha_2t_2+\cdots+\alpha_nt_n}2^{-t_\mu\delta}\displaystyle\int_{\mathbb{R}^{\mathbf{N}}} f(\mathbf{t}^{-1}y)\left(\frac{1}{|x|}\right)^{\gamma}\left(\frac{1}{|x-y|}\right)^{\mathbf{N}-\alpha}\left(\frac{1}{|y|}\right)^{\delta}dy. &
\end{array}$$

By using (4. 17) and applying Theorem 1.1, we obtain

$$\begin{array}{lr}
\left\|\Delta_{\mathbf{t}}\mathbf{II}_{\alpha\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \;\leq\; \mathfrak{C}_{\alpha\ \gamma\ \delta\ q}\,2^{\alpha_2t_2+\cdots+\alpha_nt_n}2^{-t_\mu\delta}\displaystyle\int_{\mathbb{R}^{\mathbf{N}}} f(\mathbf{t}^{-1}y)dy & \\
 & (4.\ 18)\\
\qquad =\; \mathfrak{C}_{\alpha\ \gamma\ \delta\ q}\,2^{(\alpha_2-\mathbf{N}_2)t_2+\cdots+(\alpha_n-\mathbf{N}_n)t_n}2^{-t_\mu\delta}\left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^{\mathbf{N}})}. &
\end{array}$$

Recall $\alpha_i-\mathbf{N}_i<\delta, i=1,2,\ldots,n$. Let $\varepsilon=\min\Big\{\mathbf{N}_\mu-\alpha_\mu+\delta,\ \mathbf{N}_i-\alpha_i,\ i\neq\mu\Big\}>0$. From (4. 18), we find

$$\left\|\Delta_{\mathbf{t}}\mathbf{II}_{\alpha\gamma\delta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \;\leq\; \mathfrak{C}_{\alpha\ \gamma\ \delta\ q}\,2^{-\varepsilon\sum_{i=2}^n t_i}\left\|f\right\|_{\mathbf{L}^1(\mathbb{R}^{\mathbf{N}})}. \tag{4. 19}$$

Lastly, by applying Mikowski inequality, we finish the proof of Theorem 1.2.

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