Multi-parameter fractional integration on Heisenberg group

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Abstract

First, we study a family of fractional integral operator defined as

$$\mathbf{I}_{\alpha\beta\vartheta} f(u,v,t) \ = \ \iiint_{\mathbb{R}^{2n+1}} f(\xi,\eta,\tau)\mathbf{V}^{\alpha\beta\vartheta}[(u,v,t)\odot(\xi,\eta,\tau)^{-1}]d\xi d\eta d\tau$$

where $\odot$ denotes the multiplication law.

$\mathbf{V}^{\alpha\beta\vartheta}$ is a distribution in $\mathbb{R}^{2n+1}$ satisfying Zygmund dilations. A characterization is established between $\mathbf{I}_{\alpha\beta\vartheta}\colon \mathbf{L}^p(\mathbb{R}^{2n+1}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{2n+1})$ and necessary constraints consisting of $\alpha,\beta\in\mathbb{R}$ and $\vartheta\geq 0$ for $1<p<q<\infty$.

For $0\leq\gamma<1$, define

$$\mathbf{M}_\gamma f(u,v,t) \ = \ \sup_{\mathbf{R}\subset\mathbb{R}^{2n+1}} \mathbf{vol}\{\mathbf{R}\}^{\gamma-1} \iiint_{\mathbf{R}} \Big| f[(u,v,t)\odot(\xi,\eta,\tau)^{-1}]\Big| d\xi d\eta d\tau.$$

where $\mathbf{R}\subset\mathbb{R}^{2n+1}$ is a rectangle centered on the origin with sides parallel to the coordinates. We show $\mathbf{M}_\gamma\colon \mathbf{L}^p(\mathbb{R}^{2n+1}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{2n+1})$ for $1<p\leq q<\infty$ if and only if $\gamma=\frac{1}{p}-\frac{1}{q}$.

1 Introduction

A fractional integral operator $\mathbf{T_a}$ is initially defined on $\mathbb{R}^{\mathbf{N}}$ as

$$\mathbf{T_a} f(x) \ = \ \int_{\mathbb{R}^{\mathbf{N}}} f(y)\left[\frac{1}{|x-y|}\right]^{\mathbf{N}-\mathbf{a}} dy, \qquad 0<\mathbf{a}<\mathbf{N}. \tag{1. 1}$$

In 1928, Hardy and Littlewood [1] have obtained an regularity theorem for $\mathbf{T_a}$ when $\mathbf{N}=1$. Ten years later, Sobolev [2] made extensions on every higher dimensional space.

$\diamond$ Throughout, $\mathfrak{B}>0$ is regarded as a generic constant depending on its sub-indices.

Hardy-Littlewood-Sobolev theorem *Let* $\mathbf{T_a}$ *defined in (1. 1) for* $0<\mathbf{a}<\mathbf{N}$. *We have*

$$\begin{gathered}\left\|\mathbf{T_a} f\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \ \leq \ \mathfrak{B}_{p\ q}\ \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1<p<q<\infty \\ \textit{if and only if} \qquad \frac{\mathbf{a}}{\mathbf{N}} \ = \ \frac{1}{p}-\frac{1}{q}.\end{gathered} \tag{1. 2}$$

This classical result was first re-investigated by Folland and Stein [4] on Heisenberg group. We shall work on its real variable representation with a multiplication law:

$$\begin{gathered}(u,v,t)\odot(\xi,\eta,\tau) \ = \ \Big[u+\xi, v+\eta, t+\tau+\mu(u\cdot\eta-v\cdot\xi)\Big], \qquad \mu\in\mathbb{R} \\ (u,v,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}, \qquad (\xi,\eta,\tau)^{-1}=(-\xi,-\eta,-\tau)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}.\end{gathered} \tag{1. 3}$$

Let $0 < \delta < n+1$. Consider

$$\mathbf{S}_\delta f(u,v,t) \; = \; \iiint_{\mathbb{R}^{2n+1}} f(\xi,\eta,\tau)\Omega^\delta\Big[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\Big]d\xi d\eta d\tau. \tag{1. 4}$$

Ω^δ is a distribution in $\mathbb{R}^{2n+1}$ agree with

$$\Omega^\delta(u,v,t) \; = \; \left[\frac{1}{|u|^2+|v|^2+|t|}\right]^{n+1-\delta}, \qquad (u,v,t)\neq(0,0,0). \tag{1. 5}$$

Folland-Stein theorem *Let* $\mathbf{S}_\delta$ *defined in (1. 4)-(1. 5) for* $0<\delta<n+1$. *We have*

$$\begin{array}{c} \Big\|\mathbf{S}_\delta f\Big\|_{\mathbf{L}^q(\mathbb{R}^{n+1})} \; \leq \; \mathfrak{B}_{p\;q}\;\Big\|f\Big\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1<p<q<\infty \\ \textit{if and only if} \qquad \dfrac{\delta}{n+1} \; = \; \dfrac{1}{p}-\dfrac{1}{q}. \end{array} \tag{1. 6}$$

The best constant for the $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 6) is found by Frank and Lieb [13]. A discrete analogue of this result has been obtained by Pierce [14]. Recently, the regarding commutator estimates are established by Fanelli and Roncal [15].

In this paper, we give a multi-parameter extension to **Folland-Stein theorem** by replacing Ω^δ with a larger kernel having singularity on every coordinate subspace. First, it is clear

$$\Omega^\delta(u,v,t) \; \leq \; \left[\frac{1}{|u||v|+|t|}\right]^{n+1-\delta}, \qquad (u,t)\neq(0,0) \;\; \text{or} \;\; (v,t)\neq(0,0).$$

A direct computation shows

$$\begin{aligned} \left[\frac{1}{|u||v|+|t|}\right]^{n+1-\delta} \; &\approx \; \left[\frac{1}{|u|^2|v|^2+t^2}\right]^{\frac{n+1}{2}-\frac{\delta}{2}} \; = \; |u|^{\frac{\delta}{2}-\frac{n+1}{2}}|v|^{\frac{\delta}{2}-\frac{n+1}{2}}|t|^{\frac{\delta}{2}-\frac{n+1}{2}}\left[\frac{|u||v||t|}{|u|^2|v|^2+t^2}\right]^{\frac{n+1}{2}-\frac{\delta}{2}} \\ &= \; |u|^{\left[\frac{\delta}{2}+\frac{n-1}{2}\right]-n}|v|^{\left[\frac{\delta}{2}+\frac{n-1}{2}\right]-n}|t|^{\left[\frac{\delta}{2}-\frac{n-1}{2}\right]-1}\left[\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right]^{-\left[\frac{n+1}{2}-\frac{\delta}{2}\right]}, \\ &\qquad u\neq0,\, v\neq0,\, t\neq0. \end{aligned}$$

Above estimates lead us to the following assertion. Let $\alpha,\beta\in\mathbb{R}$ and $\vartheta\geq0$. $\mathbf{V}^{\alpha\beta\vartheta}$ is a distribution in $\mathbb{R}^{2n+1}$ agree with

$$\mathbf{V}^{\alpha\beta\vartheta}(u,v,t) \; = \; |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right]^{-\vartheta}, \qquad u\neq0, v\neq0,\, t\neq0. \tag{1. 7}$$

Remark 1.1. *By taking into account* $\alpha=\frac{\delta}{2}+\frac{n-1}{2}$, $\beta=\frac{\delta}{2}-\frac{n-1}{2}$ *and* $\vartheta=\frac{n+1}{2}-\frac{\delta}{2}$ *for* $0<\delta<n+1$, *we find* $\alpha>n\beta$ *and* $\vartheta=\frac{n+1}{2}-\frac{\delta}{2}>\frac{\alpha-n\beta}{n+1}$. *Hence* $\Omega^\delta(u,v,t)\leq\mathbf{V}^{\alpha\beta\vartheta}(u,v,t)$ *for* $u\neq0, v\neq0,\, t\neq0$.

Define

$$\mathbf{I}_{\alpha\beta\vartheta}f(u,v,t) \; = \; \iiint_{\mathbb{R}^{2n+1}} f\,(\xi,\eta,\tau)\,\mathbf{V}^{\alpha\beta\vartheta}\Big[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\Big]d\xi d\eta d\tau. \tag{1. 8}$$

Observe that

$$\mathbf{V}^{\alpha\beta\vartheta}\Big[(ru,sv,rst)\odot(r\xi,s\eta,rs\tau)^{-1}\Big] \; = \; r^{\alpha+\beta-n-1}s^{\alpha+\beta-n-1}\mathbf{V}^{\alpha\beta\vartheta}\Big[(u,v,t)\odot(\xi,\eta,\tau)^{-1}\Big], \;\; r,s>0. \tag{1. 9}$$

A convolution operator of this type is said to be associated with Zygmund dilation. Singular integral operators carrying certain multi-parameter structures defined on Heisenberg group have been systematically studied, for instance by Phong and Stein [5], Ricci and Stein [6] and Müller, Ricci and Stein [7]. Much less is known for fractional integration in this direction.

Theorem One *Let $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 7)-(1. 8) for $\alpha,\beta\in\mathbb{R}$ and $\vartheta\geq 0$. We have*

$$\begin{array}{c}\left\|\mathbf{I}_{\alpha\beta\vartheta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})}~\leq~\mathfrak{B}_{p~q}~\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})},\qquad 1<p<q<\infty\\[2ex] \textit{if and only if}\qquad \vartheta\geq\frac{|\alpha-n\beta|}{n+1},\qquad \frac{\alpha+\beta}{n+1}~=~\frac{1}{p}-\frac{1}{q}.\end{array}\tag{1. 10}$$

Next, denote $\mathbf{R}\subset\mathbb{R}^{2n+1}$ to be a rectangle parallel to the coordinates. Let $0\leq\gamma<1$. A strong fractional maximal operator is defined on Heisenberg group as

$$\mathbf{M}_\gamma f(u,v,t)~=~\sup_{\mathbf{R}\ni(0,0,0)}\mathbf{vol}\{\mathbf{R}\}^{\gamma-1}\iiint_{\mathbf{R}}\left|f[(u,v,t)\odot(\xi,\eta,\tau)^{-1}]\right|d\xi d\eta d\tau.\tag{1. 11}$$

Theorem Two *Let $\mathbf{M}_\gamma$ defined in (1. 11) for $0\leq\gamma<1$. We have*

$$\begin{array}{c}\left\|\mathbf{M}_\gamma f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})}~\leq~\mathfrak{B}_{p~q}~\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})},\qquad 1<p\leq q<\infty\\[2ex] \textit{if and only if}\qquad \gamma~=~\frac{1}{p}-\frac{1}{q}.\end{array}\tag{1. 12}$$

For $\gamma=0$, $\mathbf{M}_0\doteq\mathbf{M}$ in (1. 11) is the strong maximal operator defined on Heisenberg group. The $\mathbf{L}^p$-boundedness of the strong maximal operator defined on more general Nilpotent Lie groups is proved by Christ [10]. Thereby, the elegant work is done by using a number of 'ingredients' developed previously by Ricci and Stein [8] and Christ [11]-[12]. We prove **Theorem Two** with a more direct approach by applying a multi-parameter covering lemma due to Córdoba and Fefferman [3].

As a special case, consider $\mathbf{R}=\mathbf{Q}_1\times\mathbf{Q}_2\times\mathbf{Q}_3\subset\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$: $\mathbf{Q}_1,\mathbf{Q}_2$ and $\mathbf{Q}_3$ are cubes centered on the origin of regarding subspaces. For $\alpha,\beta\in\mathbb{R}$, we define

$$\begin{array}{lr}\mathbf{M}_{\alpha\beta}f(u,v,t)~=~\displaystyle\sup_{\mathbf{R}\ni(0,0,0):\mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}} & \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_3\}^{\beta-1}\\ & \displaystyle\iiint_{\mathbf{R}}\left|f[(u,v,t)\odot(\xi,\eta,\tau)^{-1}]\right|d\xi d\eta d\tau.\end{array}\tag{1. 13}$$

This is known as the fractional maximal function associated with Zygmund dilation defined on Heisenberg group. For $\mathbf{M}_{\alpha\beta}$ defined on Euclidean space, in particular for $\alpha=\beta=0$, the regarding $\mathbf{L}^p$-theorem and its weighted analogue have been well established. See the paper by Ricci and Stein [6] and Fefferman and Pipher [9].

Later, we shall find

$$\mathbf{M}_{\alpha\beta}f(u,v,t)~\leq~\mathbf{M}_\gamma f(u,v,t),\qquad \gamma=\tfrac{\alpha+\beta}{n+1}.\tag{1. 14}$$

Corollary One *Let $\mathbf{M}_{\alpha\beta}$ defined in (1. 13) for $\alpha, \beta \in \mathbb{R}$. We have*

$$\left\|\mathbf{M}_{\alpha\beta} f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})} \leq \mathfrak{B}_{p\,q} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}, \qquad 1 < p \leq q < \infty$$
$$\textit{if and only if} \qquad \frac{\alpha+\beta}{n+1} = \frac{1}{p} - \frac{1}{q}. \tag{1. 15}$$

The remaining paper is organized as follows. In the next section, we prove some necessary constraints consisting of α, β, p, q. These include **Remark 1.1** and the homogeneity condition in (1. 10). In section 3, we prove **Theorem One**. In section 4, we prove **Theorem Two**.

2 Some necessary constraints

Let $\mathbf{I}_{\alpha\beta\vartheta}$ defined in (1. 7)-(1. 8). By changing variable $\tau \longrightarrow \tau + \mu(u\cdot\eta - v\cdot\xi)$, we find

$$\begin{aligned}
\mathbf{I}_{\alpha\beta\vartheta} f(u,v,t) &= \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right) \mathbf{V}^{\alpha\beta\vartheta}(u-\xi, v-\eta, t-\tau) d\xi d\eta d\tau \\
&= \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right) \\
&\qquad |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau.
\end{aligned} \tag{2. 1}$$

Consider a more general situation by replacing $\mathbf{V}^{\alpha\beta\vartheta}(u,v,t)$ with

$$|u|^{\alpha_1-n}|v|^{\alpha_2-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|}+\frac{|t|}{|u||v|}\right]^{-\vartheta}, \qquad \alpha_1,\alpha_2,\beta\in\mathbb{R}, \qquad \vartheta \geq 0. \tag{2. 2}$$

By changing dilations $(u,v,t) \longrightarrow (ru, sv, rs\lambda t)$ and $(\xi,\eta,\tau) \longrightarrow (r\xi, s\eta, rs\lambda\tau)$ for $r,s>0$ and $0<\lambda<1$ or $\lambda>1$, we have

$$\begin{aligned}
&\left\{\iiint_{\mathbb{R}^{2n+1}}\left\{\iiint_{\mathbb{R}^{2n+1}} f\left[r^{-1}\xi, s^{-1}\eta, r^{-1}s^{-1}\lambda^{-1}[\tau-\mu\lambda(u\cdot\eta-v\cdot\xi)]\right]\right.\right. \\
&\left.\left. |u-\xi|^{\alpha_1-n}|v-\eta|^{\alpha_2-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}} \\
&= r^{\alpha_1+\beta}s^{\alpha_2+\beta}r^{\frac{n+1}{q}}s^{\frac{n+1}{q}}\lambda^{\beta}\lambda^{\frac{1}{q}}\left\{\iiint_{\mathbb{R}^{2n+1}}\left\{\iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\right.\right. \\
&\left.\left. |u-\xi|^{\alpha_1-n}|v-\eta|^{\alpha_2-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{\lambda|t-\tau|}+\frac{\lambda|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}} \qquad (2.\ 3) \\
&\geq r^{\alpha_1+\beta}s^{\alpha_2+\beta}r^{\frac{n+1}{q}}s^{\frac{n+1}{q}}\lambda^{\beta}\lambda^{\frac{1}{q}}\begin{cases}\lambda^{\vartheta}, & 0<\lambda<1, \\ \lambda^{-\vartheta}, & \lambda>1\end{cases} \\
&\quad\left\{\iiint_{\mathbb{R}^{2n+1}}\left\{\iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right)\right.\right. \\
&\left.\left. |u-\xi|^{\alpha_1-n}|v-\eta|^{\alpha_2-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau\right\}^q dudvdt\right\}^{\frac{1}{q}}.
\end{aligned}$$

The $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 12) implies that the last line of (2. 3) is bounded by

$$\left\{\iiint_{\mathbb{R}^{2n+1}} \left|f\left(r^{-1}\xi, s^{-1}\eta, r^{-1}s^{-1}\lambda^{-1}\tau\right)\right|^p d\xi d\eta d\tau\right\}^{\frac{1}{p}} = r^{\frac{n+1}{p}} s^{\frac{n+1}{p}} \lambda^{\frac{1}{p}} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}. \tag{2. 4}$$

This must be true for every $r, s > 0$ and $0 < \lambda < 1$ or $\lambda > 1$. We necessarily have

$$\frac{\alpha_1 + \beta}{n+1} = \frac{1}{p} - \frac{1}{q} = \frac{\alpha_2 + \beta}{n+1}, \qquad \beta + \vartheta \ge \frac{1}{p} - \frac{1}{q} \qquad \text{or} \qquad \beta - \vartheta \le \frac{1}{p} - \frac{1}{q}. \tag{2. 5}$$

The first constraint in (2. 5) forces us to have $\alpha_1 = \alpha_2$. Therefore, write

$$\frac{\alpha + \beta}{n+1} = \frac{1}{p} - \frac{1}{q}, \qquad \alpha = \alpha_1 = \alpha_2. \tag{2. 6}$$

By bringing (2. 6) to the two inequalities in (2. 5), we find

$$\vartheta \ge \beta - \frac{\alpha+\beta}{n+1} = \frac{n\beta - \alpha}{n+1} \qquad \text{or} \qquad \vartheta \ge \frac{\alpha+\beta}{n+1} - \beta = \frac{\alpha - n\beta}{n+1}. \tag{2. 7}$$

Together, we conclude $\vartheta \ge \frac{|\alpha - n\beta|}{n+1}$.

3 Proof of Theorem One

Given $\alpha, \beta \in \mathbb{R}$ and $\vartheta \ge \frac{|\alpha-n\beta|}{n+1}$, $\mathbf{V}^{\alpha\beta\vartheta}$ is a distribution in $\mathbb{R}^{2n+1}$ agree with $\mathbf{V}^{\alpha\beta\vartheta}(u, v, t)$ in (1. 7) whenever $u \neq 0, v \neq 0, t \neq 0$.

Suppose $\alpha \ge n\beta$. We have $\frac{|\alpha-n\beta|}{n+1} = \frac{\alpha-n\beta}{n+1}$ and

$$\begin{aligned} \mathbf{V}^{\alpha\beta\vartheta}(\xi, \eta, \tau) &\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|} + \frac{|t|}{|u||v|}\right]^{-\frac{\alpha-n\beta}{n+1}} \\ &\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|}\right]^{-\frac{\alpha-n\beta}{n+1}} \\ &= |u|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|t|^{\frac{\alpha+\beta}{n+1}-1}, \qquad u \neq 0, v \neq 0, t \neq 0. \end{aligned} \tag{3. 1}$$

Suppose $\alpha \le n\beta$. We find $\frac{|\alpha-n\beta|}{n+1} = \frac{n\beta-\alpha}{n+1}$ and

$$\begin{aligned} \mathbf{V}^{\alpha\beta\vartheta}(u, v, t) &\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|u||v|}{|t|} + \frac{|t|}{|u||v|}\right]^{\frac{\alpha-n\beta}{n+1}} \\ &\le |u|^{\alpha-n}|v|^{\alpha-n}|t|^{\beta-1}\left[\frac{|t|}{|u||v|}\right]^{\frac{\alpha-n\beta}{n+1}} \\ &= |u|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|v|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|t|^{\frac{\alpha+\beta}{n+1}-1}, \qquad u \neq 0, v \neq 0, t \neq 0. \end{aligned} \tag{3. 2}$$

Let $\mathbf{I}_{\alpha\beta\vartheta} f$ defined in (1. 7)-(1. 8) and

$$\frac{\alpha+\beta}{n+1} = \frac{1}{p} - \frac{1}{q}, \qquad 1<p<q<\infty. \tag{3. 3}$$

By changing variable $\tau \longrightarrow \tau + \mu(u\cdot\eta - v\cdot\xi)$, we have

$$\begin{aligned} \mathbf{I}_{\alpha\beta\vartheta} f(u,v,t) &= \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right) \\ &\qquad |u-\xi|^{\alpha-n}|v-\eta|^{\alpha-n}|t-\tau|^{\beta-1}\left[\frac{|u-\xi||v-\eta|}{|t-\tau|}+\frac{|t-\tau|}{|u-\xi||v-\eta|}\right]^{-\vartheta} d\xi d\eta d\tau \\ &\le \iiint_{\mathbb{R}^{2n+1}} f\left(\xi,\eta,\tau-\mu(u\cdot\eta-v\cdot\xi)\right) \\ &\qquad |u-\xi|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|t-\tau|^{\frac{\alpha+\beta}{n+1}-1} d\xi d\eta d\tau \qquad \text{by (3. 1)-(3. 2).} \end{aligned} \tag{3. 4}$$

Because $\mathbf{V}^{\alpha\beta\vartheta}$ is positive definite, it is suffice to assert $f\ge 0$. Define

$$\mathbf{F}_{\alpha\beta}(\xi,\eta,u,v,t) = \int_{\mathbb{R}} f\left(\xi,\eta,\tau-\mu\left(u\cdot\eta-v\cdot\xi\right)\right)|t-\tau|^{\frac{\alpha+\beta}{n+1}-1} d\tau. \tag{3. 5}$$

From (3. 4)-(3. 5), we find

$$\mathbf{I}_{\alpha\beta\vartheta} f(u,v,t) \le \iint_{\mathbb{R}^{2n}} |u-\xi|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\mathbf{F}_{\alpha\beta}(\xi,\eta,u,v,t) d\xi d\eta. \tag{3. 6}$$

Recall the **Hardy-Littlewood-Sobolev theorem** stated in the beginning of this paper. By applying (1. 2) with $\mathbf{a}=\frac{\alpha+\beta}{n+1}$ and $\mathbf{N}=1$, we have

$$\begin{aligned} \left\{\int_{\mathbb{R}} \mathbf{F}^q_{\alpha\beta}(\xi,\eta,u,v,t)dt\right\}^{\frac{1}{q}} &\le \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}}\left[f\left(\xi,\eta,t-\mu\left(u\cdot\eta-v\cdot\xi\right)\right)\right]^p dt\right\}^{\frac{1}{p}} \\ &= \mathfrak{B}_{p\,q}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})} \end{aligned} \tag{3. 7}$$

regardless of $(u,v)\in\mathbb{R}^n\times\mathbb{R}^n$.

On the other hand, by applying (1. 2) with $\mathbf{a}=n[\frac{\alpha+\beta}{n+1}]$ and $\mathbf{N}=n$, we find

$$\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}|u-\xi|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})} d\xi\right\}^q du\right\}^{\frac{1}{q}} \le \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}^n}\left\|f(u,\eta,\cdot)\right\|^p_{\mathbf{L}^p(\mathbb{R})} du\right\}^{\frac{1}{p}} \tag{3. 8}$$

and

$$\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}|v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})} d\eta\right\}^q dv\right\}^{\frac{1}{q}} \le \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}^n}\left\|f(\xi,v,\cdot)\right\|^p_{\mathbf{L}^p(\mathbb{R})} dv\right\}^{\frac{1}{p}}. \tag{3. 9}$$

From (3. 6), we have

$$
\begin{aligned}
&\left\|\mathbf{I}_{\alpha\beta\vartheta} f\right\|_{\mathbf{L}^q(\mathbb{R}^{2n+1})}\\
&\le \left\{\iiint_{\mathbb{R}^{2n+1}}\left\{\iint_{\mathbb{R}^{2n}} |u-\xi|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\mathbf{F}_{\alpha\beta}(\xi,\eta,u,v,t)d\xi d\eta\right\}^q dudvdt\right\}^{\frac{1}{q}}\\
&\le \left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\mathbb{R}^{2n}} |u-\xi|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\{\int_{\mathbb{R}}\mathbf{F}^q_{\alpha\beta}(\xi,\eta,u,v,t)dt\right\}^{\frac{1}{q}} d\xi d\eta\right\}^q dudv\right\}^{\frac{1}{q}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{by Minkowski integral inequality}\\
&\le \mathfrak{B}_{p\,q}\left\{\iint_{\mathbb{R}^{2n}}\left\{\iint_{\mathbb{R}^{2n}} |u-\xi|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}|v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})} d\xi d\eta\right\}^q dudv\right\}^{\frac{1}{q}} \qquad \text{by (3. 7)}\\
&\le \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} |v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(u,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})} d\eta\right\}^p du\right\}^{\frac{q}{p}} dv\right\}^{\frac{1}{q}} \qquad \text{by (3. 8)}\\
&\le \mathfrak{B}_{p\,q}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^n} |v-\eta|^{n\left[\frac{\alpha+\beta}{n+1}\right]-n}\left\|f(u,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})} d\eta\right\}^q dv\right\}^{\frac{p}{q}} du\right\}^{\frac{1}{p}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{by Minkowski integral inequality}\\
&\le \mathfrak{B}_{p\,q}\left\{\iint_{\mathbb{R}^{2n}}\left\|f(u,v,\cdot)\right\|^p_{\mathbf{L}^p(\mathbb{R})} dudv\right\}^{\frac{1}{p}} \qquad \text{by (3. 9)}\\
&= \mathfrak{B}_{p\,q}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}.
\end{aligned}
\tag{3. 10}
$$

4 Proof of Theorem Two

Recall $\mathbf{M}_\gamma$ defined in (1. 11) for $0 \le \gamma < 1$. By taking $\xi \longrightarrow u-\xi$, $\eta \longrightarrow v-\eta$ and $\tau \longrightarrow t-\tau$, $\mathbf{M}_\gamma$ can be equivalently defined as

$$
\mathbf{M}_\gamma f(u,v,t) \;=\; \sup_{\mathbf{R}\ni(u,v,t)} \mathbf{vol}\{\mathbf{R}\}^{\gamma-1}\iiint_{\mathbf{R}}\Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|d\xi d\eta d\tau. \tag{4. 11}
$$

Similarly, $\mathbf{M}_{\alpha\beta}$ defined in (1. 13) is equivalent to

$$
\begin{aligned}
&\mathbf{M}_{\alpha\beta} f(u,v,t) \;=\\
&\sup_{\substack{\mathbf{R}\ni(u,v,t)\\ \mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}}} \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{\alpha}{n}-1}\mathbf{vol}\{\mathbf{Q}_3\}^{\beta-1}\iiint_{\mathbf{R}}\Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|d\xi d\eta d\tau
\end{aligned}
\tag{4. 12}
$$

where $\mathbf{R} = \mathbf{Q}_1\times\mathbf{Q}_2\times\mathbf{Q}_3 \subset \mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$. Moreover, $\mathbf{vol}\{\mathbf{Q}_3\} = \mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}$ implies $\mathbf{vol}\{\mathbf{R}\} = \mathbf{vol}\{\mathbf{Q}_1\}^{1+\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{1+\frac{1}{n}}$.

From (4. 12), we find

$$
\begin{aligned}
&\mathbf{M}_{\alpha\beta} f(u,v,t) \ = \\
&\sup_{\substack{\mathbf{R}\ni(u,v,t)\\ \mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}}} \mathbf{vol}\{\mathbf{Q}_1\}^{\left[\frac{\alpha+\beta}{n+1}-1\right](1+\frac{1}{n})}\mathbf{vol}\{\mathbf{Q}_2\}^{\left[\frac{\alpha+\beta}{n+1}-1\right](1+\frac{1}{n})} \iiint_{\mathbf{R}} \Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big| d\xi d\eta d\tau \\
&= \sup_{\substack{\mathbf{R}\ni(u,v,t)\\ \mathbf{vol}\{\mathbf{Q}_3\}=\mathbf{vol}\{\mathbf{Q}_1\}^{\frac{1}{n}}\mathbf{vol}\{\mathbf{Q}_2\}^{\frac{1}{n}}}} \mathbf{vol}\{\mathbf{R}\}^{\frac{\alpha+\beta}{n+1}-1} \iiint_{\mathbf{R}} \Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big| d\xi d\eta d\tau \\
&\le \sup_{\mathbf{R}\ni(u,v,t)} \mathbf{vol}\{\mathbf{R}\}^{\gamma-1} \iiint_{\mathbf{R}} \Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big| d\xi d\eta d\tau \qquad (\ \gamma = \tfrac{\alpha+\beta}{n+1}\ ) \\
&= \mathbf{M}_\gamma f(u,v,t).
\end{aligned} \tag{4. 13}
$$

Hence, $\mathbf{M}_{\alpha\beta}$ is controlled by the strong fractional maximal operator $\mathbf{M}_\gamma$ whenever $\gamma = \frac{\alpha+\beta}{n+1}$. Let $\gamma = \frac{1}{p} - \frac{1}{q}$, $1 < p \le q < \infty$. This required homogeneity condition can be found by changing dilation in (1. 12). In order to prove the converse, we need the following multi-parameter covering lemma.

Córdoba-Fefferman covering lemma

Let $\left\{\mathbf{R}_j\right\}_{j=1}^{\infty}$ be a collection of rectangles in $\mathbb{R}^{2n+1}$ parallel to the coordinates. There is a subsequence $\left\{\widehat{\mathbf{R}}_k\right\}_{k=1}^{\infty}$ such that

$$
\mathbf{vol}\left\{\bigcup_j \mathbf{R}_j\right\} \lesssim \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\} \tag{4. 14}
$$

and

$$
\left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|^p_{\mathbf{L}^p(\mathbb{R}^{2n+1})} \lesssim \mathbf{vol}\left\{\bigcup_k \widehat{\mathbf{R}}_k\right\}, \qquad 1 < p < \infty \tag{4. 15}
$$

where χ is an indicator function.

Remark 4.1. *This covering lemma is established by Córdoba and Fefferman [3] within a much more general setting. Namely, the Lebesgue measure can be replaced by an absolutely continuous measure whose Nikodym derivative satisfies the rectangle A_∞-property.*

Define

$$
\mathbf{U}_\lambda = \left\{(u,v,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \colon \mathbf{M}_\gamma f(u,v,t) > \lambda\right\}. \tag{4. 16}
$$

Given any $(u,v,t) \in \mathbf{U}_\lambda$, there is a rectangle $\mathbf{R}_j \ni (u,v,t)$ such that

$$
\mathbf{vol}\{\mathbf{R}_j\}^{\gamma-1} \iiint_{\mathbf{R}_j} \Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big| d\xi d\eta d\tau > \frac{1}{2}\lambda. \tag{4. 17}
$$

Let (u,v,t) run through the set $\mathbf{U}_\lambda$. We have $\mathbf{U}_\lambda \subset \bigcup_j \mathbf{R}_j$.

By applying the covering lemma, we select a subsequence $\{\widehat{\mathbf{R}}_k\}_{k=1}^{\infty}$ from the union above and

$$
\begin{aligned}
\mathbf{vol}\Big\{\mathbf{U}_\lambda\Big\} &\lesssim \mathbf{vol}\Big\{\bigcup_j \mathbf{R}_j\Big\} \lesssim \mathbf{vol}\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\} \qquad \text{by (4. 14)} \\
&\leq \sum_k \mathbf{vol}\left\{\widehat{\mathbf{R}}_k\right\} \\
&\leq \sum_k \left\{\frac{2}{\lambda}\iiint_{\widehat{\mathbf{R}}_k}\Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|d\xi d\eta d\tau\right\}^{\frac{1}{1-\gamma}} \qquad \text{by (4. 17).}
\end{aligned}
\tag{4. 18}
$$

Because $0\leq\gamma<1$, we further have

$$
\begin{aligned}
\mathbf{vol}\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\} &\lesssim \lambda^{-\frac{1}{1-\gamma}}\left\{\sum_k \iiint_{\widehat{\mathbf{R}}_k}\Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|d\xi d\eta d\tau\right\}^{\frac{1}{1-\gamma}} \\
&= \lambda^{-\frac{1}{1-\gamma}}\left\{\iiint_{\mathbb{R}^{2n+1}}\left|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\sum_k \chi_{\widehat{\mathbf{R}}_k}(\xi,\eta,\tau)\right|d\xi d\eta d\tau\right\}^{\frac{1}{1-\gamma}} \\
&\leq \lambda^{-\frac{1}{1-\gamma}}\left\{\iiint_{\mathbb{R}^{2n+1}}\Big|f(\xi,\eta,\tau+\mu(u\cdot\eta-v\cdot\xi))\Big|^p d\xi d\eta d\tau\right\}^{\frac{1}{p}\frac{1}{1-\gamma}}\left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1})}^{\frac{1}{1-\gamma}} \\
&\qquad\qquad \text{by Hölder inequality} \\
&= \lambda^{-\frac{1}{1-\gamma}}\left\{\iint_{\mathbb{R}^{2n}}\left\|f(\xi,\eta,\cdot)\right\|_{\mathbf{L}^p(\mathbb{R})}^p d\xi d\eta\right\}^{\frac{1}{p}\frac{1}{1-\gamma}}\left\|\sum_k \chi_{\widehat{\mathbf{R}}_k}\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n+1})}^{\frac{1}{1-\gamma}} \\
&\leq \lambda^{-\frac{1}{1-\gamma}}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}^{\frac{1}{1-\gamma}}\mathbf{vol}\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\}^{\frac{p-1}{p}\frac{1}{1-\gamma}} \qquad \text{by (4. 15).}
\end{aligned}
\tag{4. 19}
$$

By raising both sides of (4. 19) to the $(1-\gamma)$-th power and then taking into account for $1-\gamma-\frac{p-1}{p}=\frac{1}{p}-\left[\frac{1}{p}-\frac{1}{q}\right]=\frac{1}{q}$, we find

$$
\mathbf{vol}\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\}^{\frac{1}{q}} \lesssim \frac{1}{\lambda}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}. \tag{4. 20}
$$

Let $\mathbf{U}_\lambda$ defined in (4. 16). From (4. 18) and (4. 20), we obtain

$$
\begin{aligned}
\mathbf{vol}\Big\{(u,v,t)\in\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}\colon \mathbf{M}_\gamma f(u,v,t)>\lambda\Big\}^{\frac{1}{q}} &\lesssim \mathbf{vol}\Big\{\bigcup_k \widehat{\mathbf{R}}_k\Big\}^{\frac{1}{q}} \\
&\lesssim \frac{1}{\lambda}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{2n+1})}.
\end{aligned}
\tag{4. 21}
$$

By using this weak type (p,q)-estimate and applying Marcinkiewicz interpolation theorem, we conclude that $\mathbf{M}_\gamma$ is bounded from $\mathbf{L}^p(\mathbb{R}^{2n+1})$ to $\mathbf{L}^q(\mathbb{R}^{2n+1})$ if $\gamma=\frac{1}{p}-\frac{1}{q}, 1<p\leq q<\infty$.

4.1 Proof of the covering lemma

We re-arrange the order of $\{\mathbf{R}_j\}_{j=1}^{\infty}$ if necessary so that the side length of $\mathbf{R}_j$ parallel to the t-coordinate is decreasing as $j \longrightarrow \infty$. For brevity, we call it t-side length. Denote $\mathbf{R}_j^*$ to be the rectangle co-centered with $\mathbf{R}_j$ having its t-side length tripled and keeping the others same. We select $\widehat{\mathbf{R}}_k$ from $\left\{\mathbf{R}_j\right\}_{j=1}^{\infty}$ as follows.

Let $\widehat{\mathbf{R}}_1 = \mathbf{R}_1$. Having chosen $\widehat{\mathbf{R}}_1, \widehat{\mathbf{R}}_2, \ldots, \widehat{\mathbf{R}}_{N-1}$, we pick $\widehat{\mathbf{R}}_N$ as the first rectangle $\mathbf{R}$ on the list of $\mathbf{R}_j$'s after $\widehat{\mathbf{R}}_{N-1}$ so that

$$\mathbf{vol}\left\{ \mathbf{R} \cap \left[\bigcup_{\substack{k=1 \\ \widehat{\mathbf{R}}_k^* \cap \mathbf{R} \neq \emptyset}}^{N-1} \widehat{\mathbf{R}}_k^* \right] \right\} < \frac{1}{2}\mathbf{vol}\left\{\mathbf{R}\right\}. \tag{4. 22}$$

Suppose $\mathbf{R}$ is an unselected rectangle. There is a positive number M such that $\mathbf{R}$ is on the list of $\mathbf{R}_j$'s after $\widehat{\mathbf{R}}_M$ and

$$\mathbf{vol}\left\{ \mathbf{R} \cap \left[\bigcup_{\substack{k=1 \\ \widehat{\mathbf{R}}_k^* \cap \mathbf{R} \neq \emptyset}}^{M} \widehat{\mathbf{R}}_k^* \right] \right\} \geq \frac{1}{2}\mathbf{vol}\left\{\mathbf{R}\right\}. \tag{4. 23}$$

Recall $\widehat{\mathbf{R}}_k^*$ whose t-side length is tripled. Moreover, the t-side length of $\{\mathbf{R}_j\}_{j=1}^{\infty}$ is decreasing as $j \longrightarrow \infty$. On the t-coordinate, the projection of $\mathbf{R}$ is covered by the projection of the union inside (4. 23).

Slice all rectangles with a plane perpendicular to the t-axis. Denote $\mathbf{S}$, $\widehat{\mathbf{S}}_k$ and $\widehat{\mathbf{S}}_k^*$ to be the slices regarding to $\mathbf{R}$, $\widehat{\mathbf{R}}_k$ and $\widehat{\mathbf{R}}_k^*$. Consequently, (4. 23) implies

$$\mathbf{vol}\left\{ \mathbf{S} \cap \left[\bigcup_{\substack{k \\ \widehat{\mathbf{S}}_k^* \cap \mathbf{S} \neq \emptyset}}^{M} \widehat{\mathbf{S}}_k^* \right] \right\} \geq \frac{1}{2}\mathbf{vol}\left\{\mathbf{S}\right\}. \tag{4. 24}$$

Let $\mathbf{M}$ be the strong maximal operator defined in $\mathbb{R}^{2n}$. Observe that (4. 24) further implies

$$\mathbf{M}\chi_{\bigcup_k \widehat{\mathbf{S}}_k^*}(u,v) > \frac{1}{2}, \qquad (u,v) \in \bigcup\nolimits_j \mathbf{S}_j. \tag{4. 25}$$

From (4. 24)-(4. 25), by applying the $\mathbf{L}^p$-boundedness of $\mathbf{M}$, we find

$$\mathbf{vol}\left\{\bigcup\nolimits_j \mathbf{S}_j\right\} \lesssim \mathbf{vol}\left\{\bigcup\nolimits_k \widehat{\mathbf{S}}_k^*\right\}. \tag{4. 26}$$

By using (4. 26) and integrating in the t-coordinate, we have

$$\mathbf{vol}\left\{\bigcup\nolimits_j \mathbf{R}_j\right\} \lesssim \mathbf{vol}\left\{\bigcup\nolimits_k \widehat{\mathbf{R}}_k^*\right\} \lesssim \mathbf{vol}\left\{\bigcup\nolimits_k \widehat{\mathbf{R}}_k\right\} \tag{4. 27}$$

which is (4. 14).

On the other hand, (4. 22) implies

$$\mathbf{vol}\left\{\widehat{\mathbf{S}}_N\cap\left[\bigcup_{\substack{k=1\\ \widehat{\mathbf{S}}^*_k\cap\mathbf{S}\neq\emptyset}}^{N-1}\widehat{\mathbf{S}}^*_k\right]\right\} < \frac{1}{2}\mathbf{vol}\left\{\widehat{\mathbf{S}}_N\right\}. \tag{4. 28}$$

Denote $\widehat{\mathbf{E}}_N=\widehat{\mathbf{S}}_N\setminus\bigcup_{k<N}\widehat{\mathbf{S}}_k$. From (4. 28), we find

$$\mathbf{vol}\left\{\widehat{\mathbf{E}}_N\right\} > \frac{1}{2}\mathbf{vol}\left\{\widehat{\mathbf{S}}_N\right\}. \tag{4. 29}$$

Let $\phi\in\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n})$ and $\left\|\phi\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n})}=1$. We have

$$\begin{aligned}\iint_{\mathbb{R}^{2n}}\phi(u,v)\sum_k\chi_{\widehat{\mathbf{S}}_k}(u,v)dudv &= \sum_k\iint_{\widehat{\mathbf{S}}_k}\phi(u,v)dudv\\ &= \sum_k\left\{\frac{1}{\mathbf{vol}\{\widehat{\mathbf{S}}_k\}}\iint_{\widehat{\mathbf{S}}_k}\phi(u,v)dudv\right\}\mathbf{vol}\left\{\widehat{\mathbf{S}}_k\right\}\\ &< 2\sum_k\left\{\frac{1}{\mathbf{vol}\{\widehat{\mathbf{S}}_k\}}\iint_{\widehat{\mathbf{S}}_k}\phi(u,v)dudv\right\}\mathbf{vol}\left\{\widehat{\mathbf{E}}_k\right\} \qquad \text{by (4. 29)}\\ &\lesssim \sum_k\iint_{\widehat{\mathbf{E}}_k}\mathbf{M}\phi(u,v)dudv\\ &= \iint_{\bigcup_k\widehat{\mathbf{S}}_k}\mathbf{M}\phi(u,v)dudv.\end{aligned} \tag{4. 30}$$

By applying Hölder inequality and the $\mathbf{L}^p$-boundedness of $\mathbf{M}$, we find

$$\begin{aligned}\iint_{\bigcup_k\widehat{\mathbf{S}}_k}\mathbf{M}\phi(u,v)dudv &\le \left\|\mathbf{M}\phi\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{2n})}\mathbf{vol}\left\{\bigcup_k\widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}}\\ &\le \mathfrak{B}_p\,\mathbf{vol}\left\{\bigcup_k\widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}}.\end{aligned} \tag{4. 31}$$

By substituting (4. 31) to (4. 30) and taking the supremum of ϕ, we arrive at

$$\left\|\sum_k\chi_{\widehat{\mathbf{S}}_k}\right\|_{\mathbf{L}^p(\mathbb{R}^{2n})} \le \mathfrak{B}_p\,\mathbf{vol}\left\{\bigcup_k\widehat{\mathbf{S}}_k\right\}^{\frac{1}{p}}. \tag{4. 32}$$

Raising both sides of (4. 32) to the p^{th} power and integrating over t give us (4. 15).

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