On the Controllability of Lagrangian Systems by Active Constraints

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Abstract

We consider a mechanical system which is controlled by means of moving constraints. Namely, we assume that some of the coordinates can be directly assigned as functions of time by means of frictionless constraints. This leads to a system of ODE's whose right hand side depends quadratically on the time derivative of the control. In this paper we introduce a simplified dynamics, described by a differential inclusion. We prove that every trajectory of the differential inclusion can be uniformly approximated by a trajectory of the original system, on a sufficiently large time interval, starting at rest. Under a somewhat stronger assumption, we show this second trajectory reaches exactly the same terminal point.

1 Introduction

Consider a system whose state is described by N Lagrangian variables $q^1, \ldots, q^N$. Let the kinetic energy $T = T(q, \dot{q})$ be given by a positive definite quadratic form of the time derivatives $\dot{q}^i$, namely

$$T(q, \dot{q}) \;=\; \frac{1}{2}\dot{q}^\dagger G \dot{q} \;=\; \frac{1}{2}\sum_{i,j=1}^{N} g_{ij}(q)\,\dot{q}^i\dot{q}^j\,. \tag{1.1}$$

Let the coordinates be split in two groups: $\{q^1, \ldots, q^n\}$ and $\{q^{n+1}, \ldots, q^{n+m}\}$, with $N = n+m$. The $(n+m)\times(n+m)$ symmetric matrix G in (7.1) will thus take the corresponding block form

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (g_{ij}) & (g_{i,n+\beta}) \\ (g_{n+\alpha,j}) & (g_{n+\alpha,\,n+\beta}) \end{pmatrix}. \tag{1.2}$$

We assume that a controller can prescribe the values of the last m coordinates as functions of time, say

$$q^{n+\alpha}(t) = u_\alpha(t) \qquad\qquad \alpha = 1, \ldots, m\,, \tag{1.3}$$

by implementing m frictionless constraints. Here **frictionless** means that the forces produced by the constraints make zero work in connection with any virtual displacement of the remaining free coordinates $q^1, \ldots, q^n$. In the absence of external forces, the motion is thus governed by the equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q^i}(q, \dot q) - \frac{\partial T}{\partial q^i}(q, \dot q) \;=\; \Phi_i(t) \qquad\qquad i = 1, \ldots, n+m\,. \tag{1.4}$$

Here Φ_i are the components of the forces generated by the constraints. The assumption that these constraints are frictionless is expressed by the identities

$$\Phi_i(t) \equiv 0 \qquad\qquad i = 1, \ldots, n\,. \tag{1.5}$$

By introducing the conjugate momenta

$$p_i \;=\; p_i(q, \dot q) \;\doteq\; \frac{\partial T}{\partial \dot q^i} \;=\; \sum_{i=1}^{n+m} g_{ij}(q)\,\dot q^j\,, \tag{1.6}$$

it is well known that the evolution of the first n variables $(q^1, \ldots, q^n)$ and of the corresponding momenta $(p_1, \ldots, p_n)$ can be described by the system

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p \end{pmatrix} + \begin{pmatrix} K \\ \\ -p^\dagger \frac{\partial K}{\partial q} \end{pmatrix} \dot u \,+\, \dot u^\dagger \begin{pmatrix} 0 \\ \\ \frac{1}{2}\frac{\partial E}{\partial q} \end{pmatrix} \dot u\,. \tag{1.7}$$

Here A, K, E are functions of q, u, defined as

$$A = \left(a^{ij}\right) \doteq (G_{11})^{-1}\,, \qquad\qquad E = G_{22} - G_{21} A G_{12}\,, \qquad\qquad K = -AG_{12}\,. \tag{1.8}$$

For convenience, in (1.7) the vectors $q, p \in I\!R^n$ are written as column vectors, while the symbol † denotes transposition.

In general, (1.7) is a system of equations whose right hand side depends quadratically on the time derivatives of the control function $u = (u_1, \ldots, u_m)$. A detailed description of all trajectories of this system is difficult, because of the interplay between linear and quadratic terms. In this paper, to study (1.7) we introduce a simplified system, described by a differential inclusion. For each q, u, we define the convex cone

$$\Gamma(q, u) \doteq \overline{co}\left\{ A(q, u)\Big(w^\dagger \frac{\partial E(q, u)}{\partial q}\, w\Big)\,; \quad w \in I\!R^m \right\},$$

where $\overline{co}$ denotes a closed convex hull. Intuitively, one can think of $\Gamma(q, u)$ as the set of velocities which can be instantaneously produced at (q, u), by small vibrations of the active constraint $u(\cdot)$. We then consider the differential inclusion

$$\dot q \in K(q, u)\,\dot u + \Gamma(q, u) \qquad\qquad q(0) = \bar q, \quad u(0) = \bar u. \tag{1.9}$$

Trajectories of (1.9) will be compared with trajectories of the original system (1.7), with initial data

$$q(0) = \bar{q}, \qquad u(0) = \bar{u}, \qquad p(0) = 0. \tag{1.10}$$

Our main results show that, for every solution $s \mapsto q^*(s)$ of (1.9), say defined for $s \in [0,1]$, there exists a smooth solution $t \mapsto (q(t),\, p(t))$ of the Cauchy problem (1.7), (1.10), defined on a suitably long time interval $[0,T]$, following almost the same path. Namely, given $\varepsilon > 0$, a solution (q,p) of (1.7), (1.10) can be found such that

$$\Big|q(t) - q^*(\psi(t))\Big| < \varepsilon\,, \qquad |p(t)| < \varepsilon \qquad \text{for all } t \in [0,T], \tag{1.11}$$

for a suitable time rescaling $\psi : [0,T] \mapsto [0,1]$. Under a somewhat stronger assumption, the terminal values of the two trajectories can be made equal, namely

$$q(T) = q^*(1).$$

Remark 1. Since the components p_i bear a linear relation to the velocities $\dot{q}^j$, the system (1.7) describes a "second order" dynamics, which could be equivalently written in terms of the second derivatives $\ddot{q}^j$. On the other hand, the reduced system (1.9) contains no inertial term, and is essentially of first order. The inequalities (1.11) show that, keeping $p(t) \approx 0$, the two dynamics can be related. We remark that the present results are entirely different in nature from those in [4, 7, 8], where the impulsive control system is approximated by a differential inclusion living in the $2n$-dimensional space described by the (q,p)-variables.

The paper is organized as follows. Section 2 contains precise statements of the main results. The proofs are then worked out in Sections 3–5. Section 6 contains two examples. The first one shows the necessity of a technical assumption. The second one provides a simple application to the control of a bead sliding without friction along a rotating bar. The last section is the derivation of evolution equations in (1.7).

For the theory of multifunctions and differential inclusions we refer to [2] or [16]. Earlier results on impulsive control systems were provided in [6, 7, 9, 10]. A general introduction to the theory of control can be found in [5, 11, 14] and in [17]. We remark that the idea of averaging, used in the proof of our main theorem, is widespread in the analysis of mechanical systems with oscillatory behavior. Several results in this direction can be found in [1, 3].

2 Statement of Main Results

Motivated by the model (1.7), from now on we consider a system of the form

$$\begin{pmatrix} \dot{q} \\ \\ \dot{p} \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2}p^\dagger Bp \end{pmatrix} + \begin{pmatrix} K \\ \\ -p^\dagger C \end{pmatrix} \dot{u} \,+\, \dot{u}^\dagger \begin{pmatrix} 0 \\ \\ D \end{pmatrix} \dot{u}\,. \tag{2.1}$$

Given an initial data

$$q(0) = \bar{q}, \qquad p(0) = \bar{p}, \qquad u(0) = \bar{u}, \tag{2.2}$$

we shall study the set of trajectories of (2.1).

The difficulty in analyzing (2.1) stems from the fact that the right hand side contains both linear and quadratic terms w.r.t. the time derivative $\dot{u}$. A simplification can be achieved by considering separately the contributions of these terms. If $D \equiv 0$, we have the reduced system

$$\begin{pmatrix} \dot{q} \\ \\ \dot{p} \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2} p^\dagger B p \end{pmatrix} + \begin{pmatrix} K \\ \\ -p^\dagger C \end{pmatrix} \dot{u} \,. \tag{2.3}$$

Notice that, if $\bar{p} = 0$, then $p(t) \equiv 0$ for every time t. In this case, the trajectory of the system (2.3) is entirely determined by solving the reduced equation

$$\dot{q} = K(q,u)\dot{u} \,, \qquad\qquad q(0) = \bar{q} \,. \tag{2.4}$$

We claim that, even in the case $D \neq 0$, given a sufficiently long time interval, every trajectory of (2.4) can be uniformly approximated by a trajectory of the original system (2.1). More generally, if the initial speed is sufficiently small, then one can track every solution to the differential inclusion

$$\dot{q} \in K(q,u)\dot{u} + \Gamma(q,u) \,, \qquad\qquad q(0) = \bar{q} \,. \tag{2.5}$$

Here Γ is the cone defined by

$$\Gamma(q,u) \doteq \overline{co} \Big\{ A(q,u) \left(w^\dagger D(q,u)\, w \right) ; \quad w \in I\!R^m \Big\}, \tag{2.6}$$

where $\overline{co}$ denotes the closed convex hull of a set.

Definition 1. Given an absolutely continuous control function $t \mapsto u(t)$, defined for $t \in [0,T]$, by a Carathéodory solution of the differential inclusion (2.5) we mean an absolutely continuous map $t \mapsto q(t)$ such that

$$\dot{q}(t) - K(q(t),u(t))\dot{u}(t) \in \Gamma(q(t),u(t)) \qquad\qquad \text{for a.e.} \quad t \in [0,T]. \tag{2.7}$$

Our main result is concerned with approximation of trajectories of (2.5) with solutions of the full system (2.1). Our basic hypotheses are as follows.

(H) The matrices A, B, K, C in (2.1) are locally Lipschitz continuous functions of the variables q, u, and the same is true of D and of the partial derivatives K_q, K_u. Moreover, the cone Γ in (2.6) depends continuously on (q,u); namely, the compact, convex valued multifunction

$$(q,u) \ \mapsto \ \Gamma_1(q,u) \ \doteq \ \Big\{ p \in \Gamma(q,u) \,; \quad |p| \leq 1 \Big\} \tag{2.8}$$

is continuous w.r.t. the Hausdorff distance.

Theorem 1. *Let the assumptions (H) hold, and let $s \mapsto q^*(s)$ be any Carathéodory solution of differential inclusion (2.5) defined for $s \in [0,1]$, corresponding to an absolutely continuous control $u^*(\cdot)$.*

Then, for every $\varepsilon > 0$, there exists $\delta > 0$, an interval $[0,T]$ and a smooth control $u(\cdot)$ defined on $[0,T]$ such that the following holds. If $|\bar{p}| < \delta$, then the corresponding solution of (2.1) with initial data (2.2) satisfies

$$\sup_{t\in[0,T]} |p(t)| < \varepsilon, \qquad \sup_{t\in[0,T]} \Big| q(t) - q^*(\psi(t)) \Big| < \varepsilon, \qquad \sup_{t\in[0,T]} \Big| u(t) - u^*(\psi(t)) \Big| < \varepsilon, \tag{2.9}$$

for some increasing diffeomorphism $\psi : [0,T] \mapsto [0,1]$.

Remark 2. Assume that, more generally, the control u^* and the trajectory q^* are defined on an interval $[0,T^*]$. Since $\dot u$ enters linearly in the equation (2.5), the rescaled function $\hat q(s) = q^*(T^* s)$ provides another solution of (2.5), corresponding to the control $\hat u(s) \doteq u^*(T^* s)$. By a linear rescaling of time , it is thus not restrictive to assume that q^*, u^* are defined for $s \in [0,1]$.

Next, we consider the problem of exactly reaching a state $(\mathbf{q},\mathbf{u})$ at some (possibly large) time T, with small terminal speed. As a preliminary, we introduce a notion of normal reachability. As in [12], this means that there exists a family of trajectories whose terminal points nicely cover a whole neighborhood of the target point $(\mathbf{q},\mathbf{u})$. More precisely:

Definition 2. Given the differential inclusion (2.5), the state $(\mathbf{q},\mathbf{u}) \in R^{n+m}$ is *normally reachable* from the initial state $(\bar q, \bar u)$ if there exists a parameterized family of trajectories

$$s \mapsto (q^\lambda(s),\, u^\lambda(s)), \qquad\qquad \lambda \in \Lambda \subset I\!\!R^{n+m}, \quad s \in [0,1]$$

with the following properties.

(i) The parameter λ ranges in a neighborhood Λ of the origin in $I\!\!R^{n+m}$. The map $\lambda \mapsto (q^\lambda(\cdot),\, u^\lambda(\cdot))$ is continuous from Λ into $W^{1,1}\left([0,1]\,;\, I\!\!R^{n+m}\right)$.

(ii) For every $\lambda \in \Lambda$ we have $(q^\lambda(0),\, u^\lambda(0)) = (\bar q, \bar u)$. Moreover, when $\lambda = 0 \in I\!\!R^{n+m}$ we have $(q^0(1),\, u^0(1)) = (\mathbf{q},\mathbf{u})$ and the $(n+m)\times(n+m)$ Jacobian matrix

$$\left(\frac{\partial (q^\lambda(1),\, u^\lambda(1))}{\partial \lambda}\right)$$

has full rank, i.e. it is invertible.

Theorem 2. *Let (H) hold, and assume that the state $(\mathbf{q},\mathbf{u}) \in R^{n+m}$ is* normally reachable *from the initial state $(\bar q, \bar u)$, for the differential inclusion (2.5). Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. If $|\bar p| < \delta$, there exists a time T and a control function u defined on $[0,T]$ such that the corresponding solution of (2.1) satisfies (2.9) together with*

$$(q(T),\, u(T)) = (\mathbf{q},\mathbf{u})\,. \tag{2.10}$$

The proof of Theorem 2 relies on a topological argument. The key ingredient is the following continuous approximation lemma. By $\mathcal{AC}([0,T])$ we denote here the space of absolutely continuous functions on $[0,T]$, with norm

$$\|f\|_{\mathcal{AC}} \;\doteq\; \int_0^T |\dot f(t)|\, dt + \sup_{t\in[0,T]} \;|f(t)|\,.$$

Lemma 1. *Let (H) hold. Consider a family of solutions (q^λ, u^λ) of the differential inclusion (2.5), assuming that the map $\lambda \mapsto (q^\lambda(\cdot),\, u^\lambda(\cdot))$ is continuous from a compact set $\Lambda \subset I\!\!R^d$ into $\mathcal{AC}\left([0,1]\,;\, I\!\!R^{n+m}\right)$. Then, given any $\varepsilon > 0$, there exists a map $(\lambda, s) \mapsto (\tilde u^\lambda(s),\, \tilde w^\lambda(s))$*

from $\Lambda \times [0,1]$ into $I\!R^m \times I\!R^m$, which is continuous w.r.t. λ and $\mathcal{C}^\infty$ in the variable s, such that the following holds. Calling $\tilde q^\lambda(\cdot)$ the solution to

$$\frac{d}{ds}q(s) \;=\; K(q(s),\tilde u^\lambda(s))\cdot\frac{d}{ds}\tilde u^\lambda(s)+A(q(s),\tilde u^\lambda(s))\Big(\tilde w^\lambda(s)^\dagger D(q(s),\tilde u^\lambda(s))\tilde w^\lambda(s)\Big), \qquad q(0)=\bar q, \tag{2.11}$$

for every $\lambda\in\Lambda$ one has

$$\sup_{s\in[0,1]}\Big|\tilde q^\lambda(s)-q^\lambda(s)\Big|<\varepsilon, \qquad\qquad \sup_{s\in[0,1]}|\tilde u^\lambda(s)-u^\lambda(s)|<\varepsilon. \tag{2.12}$$

Remark 3. The assumption (H) requires that the maps A,B,K,C,D be locally Lipschitz continuous. We observe that, toward the proof of Lemma 1, it is not restrictive to assume that all these maps have compact support, and are therefore globally Lipschitz continuous. Indeed, the set

$$\Omega_0\doteq\Big\{(q^\lambda(s),\,u^\lambda(s))\,;\quad s\in[0,1]\,,\quad \lambda\in\Lambda\Big\}\,, \tag{2.13}$$

is compact, and the same is true for its closed neighborhood

$$\Omega_\rho\doteq\Big\{(q,u)\in I\!R^{n+m}\,;\quad |q-q^\lambda(s)|\le\rho\,,\quad |u-u^\lambda(s)|\le\rho\qquad\text{for some}\;\; s\in[0,1]\,,\quad\lambda\in\Lambda\Big\}, \tag{2.14}$$

for any $\rho>0$. Let $\varphi: I\!R^{n+m}\mapsto I\!R_+$ be a smooth cutoff function such that

$$\varphi(q,u)\;=\left\{\begin{array}{lll} 1 & \text{if} & (q,u)\in\Omega_1\,,\\ 0 & \text{if} & (q,u)\notin\Omega_2\,.\end{array}\right.$$

The functions $\widehat A\doteq\varphi\cdot A,\ldots,\widehat D\doteq\varphi\cdot D$ have compact support and are thus globally Lipschitz continuous. If the conclusion of Lemma 1 holds for $\widehat A,\widehat B,\widehat K,\widehat C,\widehat D$, then it also holds for the original functions A,B,K,C,D. Indeed, when $0<\varepsilon<1$, the inequalities (2.12) imply that $(q^\lambda(s),p^\lambda(s))\in\Omega_1$. Restricted to Ω_1, one has the identities $A=\widehat A,\ldots,D=\widehat D$. This same remark applies to the proofs of Theorems 1 and 2.

3 Proof of the approximation lemma

We first prove two auxiliary results. Recall that the convex sets $\Gamma_1(q,u)$ were defined at (2.8). For notational convenience, we introduce the set of coefficients of convex combinations

$$\Delta_\nu\doteq\Big\{\theta=(\theta_1,\ldots,\theta_\nu)\,;\quad \theta_i\in[0,1]\,,\quad \sum_{i=1}^\nu\theta_i=1\Big\}.$$

Lemma 2. *Given $\varepsilon'>0$ and a compact set $\Omega\subset I\!R^{n+m}$, there exist finitely many vectors $\mathbf{w}_1,\ldots,\mathbf{w}_\nu$ such that the following holds. Given any $(q,u)\in\Omega$ and any $p\in\Gamma_1(q,u)$, there exist coefficients $(\theta_1,\ldots,\theta_\nu)\in\Delta_\nu$ such that*

$$\left|p-A(q,u)\cdot\sum_{i=1}^\nu\theta_i\mathbf{w}_i^\dagger D(q,u)\mathbf{w}_i\right|\le\varepsilon'\,. \tag{3.1}$$

Proof. Consider the domain

$$\mathcal{D} \doteq \Big\{(q,u,p)\,;\quad (q,u)\in\Omega\,,\quad p\in\Gamma_1(q,u)\Big\}\,. \tag{3.2}$$

Notice that $\mathcal{D}$ is compact, because of the assumption (H). For each $(\bar q,\bar u,\bar p)\in\mathcal{D}$, choose finitely many vectors $\mathbf{w}_i=\mathbf{w}_i^{(\bar q,\bar u,\bar p)}$ and coefficients $\theta_i=\theta_i^{(\bar q,\bar u,\bar p)}$, $i=1,\ldots,M$, such that

$$\sum_{i=1}^M\theta_i=1\,,\qquad\qquad \left|\bar p-A(\bar q,\bar u)\cdot\sum_{i=1}^M\theta_i\mathbf{w}_i^\dagger D(\bar q,\bar u)\mathbf{w}_i\right|\le\frac{\varepsilon'}{2}\,.$$

By continuity, we still have

$$\left|p-A(q,u)\cdot\sum_{i=1}^M\theta_i\mathbf{w}_i^\dagger D(q,u)\mathbf{w}_i\right|\le\varepsilon'$$

for all (q,u,p) in a neighborhood $V^{(\bar q,\bar u,\bar p)}$ of the point $(\bar q,\bar u,\bar p)$. Covering the compact domain $\mathcal{D}$ with finitely many neighborhoods $V_\ell=V^{(q_\ell,u_\ell,p_\ell)}$, $\ell=1,\ldots,\kappa$, and choosing

$$\{\mathbf{w}_1,\ldots,\mathbf{w}_\nu\}\ \doteq\ \Big\{\mathbf{w}_1^{(q_1,u_1,p_1)},\ldots,\mathbf{w}_{M(1)}^{(q_1,u_1,p_1)},\quad\ldots\quad,\mathbf{w}_1^{(q_\kappa,u_\kappa,p_\kappa)},\ldots,\mathbf{w}_{M(\kappa)}^{(q_\kappa,u_\kappa,p_\kappa)}\Big\},$$

we achieve the conclusion of the lemma. □

The next lemma states that, if we relax the inequality in (3.1), the coefficients θ_i can be chosen depending continuously on q,u,p.

Lemma 3. *Given a compact set $\Omega\subset I\!R^{n+m}$, define the compact domain $\mathcal{D}$ as in (3.2). Then, for any $\varepsilon'>0$, there exists a continuous mapping $\Theta=(\Theta_1,\ldots,\Theta_\nu):\mathcal{D}\mapsto\Delta_\nu$, such that*

$$\left|p-A(q,u)\cdot\sum_{i=1}^\nu\Theta_i(q,u,p)\mathbf{w}_i^\dagger D(q,u)\mathbf{w}_i\right|\le 2\varepsilon'\,. \tag{3.3}$$

for all $(q,u,p)\in\mathcal{D}$.

Proof. By continuity and compactness, there exists $\delta>0$ such that the following holds. If

$$|q-\tilde q|<\delta\,,\qquad |u-\tilde u|<\delta\,,\qquad |p-\tilde p|<\delta,\qquad |\theta_i-\tilde\theta_i|<\delta\qquad \text{for } i=1,\ldots,\nu\,, \tag{3.4}$$

and if

$$\left|\tilde p-A(\tilde q,\tilde u)\cdot\sum_{i=1}^\nu\tilde\theta_i\mathbf{w}_i^\dagger D(\tilde q,\tilde u)\mathbf{w}_i\right|\le\varepsilon'\,, \tag{3.5}$$

then

$$\left|p-A(q,u)\cdot\sum_{i=1}^\nu\theta_i\mathbf{w}_i^\dagger D(q,u)\mathbf{w}_i\right|\le 2\varepsilon'\,. \tag{3.6}$$

Next, consider the set-valued function

$$\widehat\Theta(q,u,p)\doteq\left\{\theta=(\theta_1,\ldots,\theta_\nu)\in\Delta_\nu\,,\qquad \left|p-A(q,u)\cdot\sum_{i=1}^\nu\theta_i\,\mathbf{w}_i^\dagger D(q,u)\mathbf{w}_i\right|\le\varepsilon'\right\}.$$

Observe that the multifunction $\widehat{\Theta} : \mathcal{D} \mapsto \Delta_\nu$ has closed graph, and non-empty, compact, convex values. By a selection theorem in [2], for every $\delta > 0$, this multifunction admits a continuous, δ-approximate selection $\Theta : \mathcal{D} \mapsto \Delta_\nu$, in the sense of graph. Calling $\mathcal{N}(S, \delta)$ the δ-neighborhood around a set S, this means that

$$\text{graph}\,\Theta \;\subset\; \mathcal{N}\Big(\text{graph}\,\widehat{\Theta}\,,\,\delta\Big)\,.$$

If $\delta > 0$ was chosen sufficiently small, so that (3.4)-(3.5) imply (3.6), then the continuous function Θ satisfies the conclusion of the lemma. □

Proof of Lemma 1. According to Remark 3, we can assume that all functions A, B, K, C, D have compact support, hence they are all globally Lipschitz continuous and uniformly bounded. The proof of the continuous approximation lemma will be given in several steps.

1. By assumption, for every $\lambda \in \Lambda$ we have

$$\dot{q}^\lambda(s) = K\left(q^\lambda(s), u^\lambda(s)\right)\dot{u}^\lambda(s) + \gamma^\lambda(s)\,, \tag{3.7}$$

where $s \mapsto \gamma^\lambda(s) \in \Gamma(q^\lambda(s),\, u^\lambda(s))$ is some measurable map, depending continuously on λ in the $\mathbf{L}^1$ norm.

We claim that it is not restrictive to assume that the functions $\dot{q}^\lambda(\cdot)$, $\dot{u}^\lambda(\cdot)$, and $\gamma^\lambda(\cdot)$ are uniformly bounded. Indeed, fix an integer N and define the times $s_i \doteq i/N$. For each λ, consider the time rescaling

$$t^\lambda(s) \doteq s_{i-1} + \frac{\displaystyle\int_{s_{i-1}}^{s}\Big(|\dot{u}^\lambda(t)| + |\gamma^\lambda(t)| + N^{-1}\Big)\,dt}{N\cdot\displaystyle\int_{s_{i-1}}^{s_i}\Big(|\dot{u}^\lambda(t)| + |\gamma^\lambda(t)| + N^{-1}\Big)\,dt} \qquad \text{if} \quad s \in [s_{i-1},\, s_i]. \tag{3.8}$$

Observe that the map $s \mapsto t^\lambda(s)$ is strictly increasing, satisfies

$$t^\lambda(s_i) = s_i \qquad\qquad \text{for all } i = 0, 1, \ldots, N,$$

and has a Lipschitz continuous inverse which we denote by $t \mapsto s^\lambda(t)$. We now define

$$q_N^\lambda(t) \doteq q^\lambda(s^\lambda(t)), \qquad u_N^\lambda(t) \doteq u^\lambda(s^\lambda(t)), \qquad \gamma_N^\lambda(t) \doteq \gamma^\lambda(s^\lambda(t))\cdot\left(\frac{d}{dt}s^\lambda(t)\right).$$

By (3.8), the above definitions yield

$$\frac{d}{dt}q_N^\lambda(t) = K\left(q_N^\lambda(t), u_N^\lambda(t)\right)\cdot\frac{d}{dt}u_N^\lambda(t) + \gamma_N^\lambda(t). \tag{3.9}$$

Moreover, for a.e. $t \in [s_{i-1}, s_i]$, (3.8) implies

$$\left|\frac{d}{dt}u_N^\lambda(t)\right| + |\gamma_N^\lambda(t)| \le N\cdot\int_{s_{i-1}}^{s_i}\Big(|\dot{u}^\lambda(t)| + |\gamma^\lambda(t)| + N^{-1}\Big)\,dt\,,$$

showing that $\dot{u}_N^\lambda$ and γ_N^λ remain uniformly bounded. The continuity w.r.t. the parameter λ implies that these bounds are uniform as λ ranges in the compact set Λ. Moreover, the maps $\lambda \mapsto q_N^\lambda(\cdot)$ and $\lambda \mapsto u_N^\lambda(\cdot)$ are continuous from Λ into $\mathcal{AC}([0,1])$.

Finally, for any given $\varepsilon > 0$, by choosing the integer N sufficiently large we can achieve the inequalities

$$\sup_{s\in[0,1]} \Big|q_N^\lambda(s) - q^\lambda(s)\Big| < \varepsilon, \qquad \sup_{s\in[0,1]} \Big|u_N^\lambda(s) - u^\lambda(s)\Big| < \varepsilon. \tag{3.10}$$

Since q_N^λ satisfies (3.9) and K is bounded, we conclude that the derivative $\dot q_N^\lambda$ is uniformly bounded as well. This completes the proof of our claim.

2. From now on, we can thus assume that

$$|\dot q^\lambda(s)| + |\dot u^\lambda(s)| + |\gamma^\lambda(s)| \ \leq\ \overline{M} \qquad \text{for a.e. } s\in[0,1], \tag{3.11}$$

for some constant $\overline{M}$ and every $\lambda\in\Lambda$.

Consider the compact set $\Omega \doteq \Omega_1$ defined as in (2.14), and the corresponding domain $\mathcal{D}$ as in (3.2).

For a given $\varepsilon' > 0$, whose precise value will be determined later, we can choose vectors $\mathbf{w}_1,\ldots,\mathbf{w}_\nu$ according to Lemma 2. Let $\Theta : \mathcal{D} \mapsto \Delta_\nu$ be the continuous map constructed in Lemma 3, and define the measurable coefficients

$$\theta_i^\lambda(s) \doteq \Theta\left(q^\lambda(s),\, u^\lambda(s),\, \frac{\gamma^\lambda(s)}{|\gamma^\lambda(s)|}\right). \tag{3.12}$$

By (3.3) we have

$$\left|\gamma^\lambda(s) - A\Big(q^\lambda(s),\, u^\lambda(s)\Big)\cdot \sum_{i=1}^{\nu} |\gamma^\lambda(s)|\,\theta_i^\lambda(s)\mathbf{w}_i^\dagger D\Big(q^\lambda(s),\, u^\lambda(s)\Big)\,\mathbf{w}_i\right| \ \leq\ 2\varepsilon'\,|\gamma^\lambda(s)| \ \leq\ 2\varepsilon'\overline{M} \tag{3.13}$$

for a.e. $s\in[0,1]$.

3. Next, we divide the interval $[0,1]$ into $k\,\nu$ equal subintervals, choosing k very large. For notational convenience we set

$$\tau_j \doteq \frac{j}{k}\,, \qquad \tau_{j,\ell} \doteq \frac{j}{k} + \frac{\ell}{k\,\nu}\,.$$

Here $j = 0,\ldots,k$, while $\ell = 0,\ldots,\nu$. For each $\lambda\in\Lambda$, we now define a continuous, piecewise affine control function $s\mapsto \tilde u^\lambda(s)$ by setting

$$\tilde u^\lambda(\tau_j) \ \doteq\ u^\lambda(\tau_j) \qquad j = 0,\ldots,k, \tag{3.14}$$

and extending $\tilde u^\lambda$ to an affine map on each interval $[\tau_{j-1}\,,\ \tau_j]$. Since by (3.11) the functions $u^\lambda(\cdot)$ are uniformly Lipschitz continuous, by choosing k large enough we can achieve the bounds

$$\Big|\tilde u^\lambda(s) - u^\lambda(s)\Big| < \varepsilon \qquad \text{for all } s\in[0,1],\quad \lambda\in\Lambda. \tag{3.15}$$

Moreover, we define

$$\tilde w^\lambda(s) \ \doteq\ \left(k\,\nu\cdot\int_{\tau_j}^{\tau_{j+1}} |\gamma^\lambda(s)|\,\theta_\ell^\lambda(s)\,ds\right)^{1/2} \mathbf{w}_\ell \qquad \text{for} \qquad s\in\,]\tau_{j,\ell-1},\ \tau_{j,\ell}]\,. \tag{3.16}$$

Here $j = 0, \ldots, k-1$, while $\ell = 1, \ldots, \nu$. Call $\tilde{q}^\lambda(\cdot)$ the corresponding solution of (2.11). In the next step we will prove that, by choosing first $\varepsilon' > 0$ sufficiently small and then the integer k large enough, the inequalities in (2.12) are satisfied.

4. To compare the two functions $q^\lambda(\cdot)$ and $\tilde{q}^\lambda(\cdot)$, we introduce a third function $Q^\lambda(\cdot)$, defined as the solution to the Cauchy problem

$$\begin{cases} \dot{Q}^\lambda(s) = K\left(Q^\lambda(s), u^\lambda(s)\right)\dot{u}^\lambda(s) + A\left(Q^\lambda(s), u^\lambda(s)\right) \cdot \sum_{i=1}^{\nu} |\gamma^\lambda(s)|\, \theta_i^\lambda(s)\mathbf{w}_i^\dagger D\left(Q^\lambda(s),\, u^\lambda(s)\right)\mathbf{w}_i\,, \\ Q^\lambda(0) = \bar{q}\,. \end{cases} \tag{3.17}$$

To estimate the difference $|q^\lambda - Q^\lambda|$, consider the Picard map $y \mapsto \mathcal{P}y$ (depending on $\lambda \in \Lambda$), defined as

$$(\mathcal{P}y)(t) \doteq \bar{q} + \int_0^t \left\{K(y(s), u^\lambda(s))\,\dot{u}^\lambda(s) + \gamma^\lambda(s)\right\} ds\,. \tag{3.18}$$

By Remark 3 and by step **1** in this proof, we can assume that K is globally Lipschitz continuous and that the functions $\dot{u}^\lambda$ are uniformly bounded. Therefore there exists a constant L, independent of $\lambda \in \Lambda$, such that each Picard map $\mathcal{P}$ is a strict contraction w.r.t. the weighted norm

$$\|y\|_* \doteq \sup_{s\in[0,1]} e^{-Ls}|y(s)|\,.$$

More precisely, for every continuous functions $y, \tilde{y}$,

$$\|\mathcal{P}y - \mathcal{P}\tilde{y}\|_* \le \frac{1}{2}\|y - \tilde{y}\|_*\,. \tag{3.19}$$

In turn (see for example the Appendix in [5]), since $q^\lambda(\cdot)$ is the fixed point of $\mathcal{P}$, for every $y(\cdot)$ this implies the estimate

$$\|y - q^\lambda\|_* \le 2\|y - \mathcal{P}y\|_*\,. \tag{3.20}$$

We now have

$$(\mathcal{P}Q^\lambda)(t) - Q^\lambda(t) = \int_0^t \left\{\gamma^\lambda(s) - A\left(q^\lambda(s),\, u^\lambda(s)\right) \cdot \sum_{i=1}^{\nu} |\gamma^\lambda(s)|\, \theta_i^\lambda(s)\mathbf{w}_i^\dagger D\left(q^\lambda(s),\, u^\lambda(s)\right)\mathbf{w}_i\right\} ds\,.$$

By (3.13), this yields

$$\sup_{t\in[0,1]} \left|q^\lambda(t) - Q^\lambda(t)\right| \;\le\; e^L\|q^\lambda - Q^\lambda\|_* \;\le\; 2e^L\|q^\lambda - \mathcal{P}q^\lambda\|_* \;\le\; 4e^L\varepsilon'\overline{M}\,. \tag{3.21}$$

Notice that the constant L depends only on the Lipschitz norm of K and on the upper bound on $|\dot{u}^\lambda|$ at (3.11). Therefore, we can assume that $\varepsilon' > 0$ in (3.11) was chosen so that

$$\sup_{t\in[0,1]} \left|q^\lambda(t) - Q^\lambda(t)\right| \;\le\; 4e^L\varepsilon'\overline{M} \;<\; \frac{\varepsilon}{2}\,. \tag{3.22}$$

Next, to estimate the difference $|\tilde{q}^\lambda - Q^\lambda|$, we consider a second Picard map $y \mapsto \mathcal{P}y$, with

$$(\mathcal{P}y)(t) \doteq \bar{q} + \int_0^t \left\{K(y(s), u^\lambda(s))\dot{u}^\lambda(s) + A(y(s), u^\lambda(s)) \cdot \sum_{i=1}^{\nu} |\gamma^\lambda(s)|\, \theta_i^\lambda(s)\mathbf{w}_i^\dagger D(y(s),\, u^\lambda(s))\mathbf{w}_i\right\} ds \tag{3.23}$$

By the boundedness of $\dot{u}^\lambda$, γ^λ, and by the Lipschitz continuity of K, A, D, this map will be a strict contraction and satisfy (3.19) w.r.t. some weighted norm of the form

$$\|y\|_* \doteq \sup_{s\in[0,1]} e^{-L's}|y(s)|\,.$$

Notice that in this case the constant L' may depend also on $\max\{|\mathbf{w}_1|, \ldots, |\mathbf{w}_\nu|\}$, and hence on the earlier choice of ε'.

In addition to (3.14), for every j and every choice of the constants y_j, u_j, the definition (3.16) yields

$$\int_{\tau_j}^{\tau_{j+1}} \Big[\tilde{w}^\lambda(s)\Big]^\dagger D(y_j, u_j)\, \tilde{w}^\lambda(s)\, ds \;=\; \int_{\tau_j}^{\tau_{j+1}} \sum_{\ell=1}^{\nu} |\gamma^\lambda(s)|\, \theta_\ell^\lambda(s)\, \mathbf{w}_\ell^\dagger\, D(y_j, u_j)\, \mathbf{w}_\ell\, ds.$$

Therefore, by the uniform Lipschitz continuity of the maps K, A, D and $\tilde{q}^\lambda, u^\lambda$, it follows the estimate

$$\Big|(\mathcal{P}\tilde{q}^\lambda)(\tau_j) - \tilde{q}^\lambda(\tau_j)\Big| \;\leq\; C_1 \cdot \sup_{i=1,\ldots,j} \;\sup_{t,t'\in[\tau_{i-1},\tau_i]} \Big(\Big|\tilde{q}^\lambda(t) - \tilde{q}^\lambda(t')\Big| + \Big|u^\lambda(t) - u^\lambda(t')\Big|\Big) \;\leq\; \frac{C_2}{k}\,,$$

for suitable constants C_1, C_2, depending on $\overline{M}$ and $\max\{|\mathbf{w}_1|, \ldots, |\mathbf{w}_\nu|\}$ but not on λ, k. More generally, for $t \in [\tau_j, \tau_{j+1}]$ we have

$$\Big|(\mathcal{P}\tilde{q}^\lambda)(t) - \tilde{q}^\lambda(t)\Big| \;\leq\; \Big|(\mathcal{P}\tilde{q}^\lambda)(t) - (\mathcal{P}\tilde{q}^\lambda)(\tau_j)\Big| + \Big|(\mathcal{P}\tilde{q}^\lambda)(\tau_j) - \tilde{q}^\lambda(\tau_j)\Big| + \Big|\tilde{q}^\lambda(\tau_j) - \tilde{q}^\lambda(t)\Big| \;\leq\; \frac{C_3}{k}\,,$$

for a suitable constant C_3. Observing that Q^λ is the fixed point of the Picard map $\mathcal{P}$ in (3.23), we can thus choose k large enough so that

$$\sup_{t\in[0,1]} \Big|\tilde{q}^\lambda(t) - Q^\lambda(t)\Big| \;\leq\; e^{L'}\|\tilde{q}^\lambda - Q^\lambda\|_* \;\leq\; 2e^{L'}\|\tilde{q}^\lambda - \mathcal{P}\tilde{q}^\lambda\|_* \;\leq\; 2e^{L'} \cdot \frac{C_3}{k} \;<\; \frac{\varepsilon}{2}. \tag{3.24}$$

5. At this stage we have constructed functions $\tilde{u}^\lambda, \tilde{w}^\lambda$ which satisfy (2.12). However, the maps

$$(\lambda, s) \mapsto \frac{d}{ds}\tilde{u}^\lambda(s), \qquad\qquad (\lambda, s) \mapsto \tilde{w}^\lambda(s)$$

are continuous as functions of λ, but piecewise constant with jumps at the points $\tau_{j,\ell}$ as functions of the time variable $s \in [0,1]$. To complete the proof, we need to achieve smoothness w.r.t. the variable s. This is obtained by a standard mollification procedure.

We first extend each the functions $\tilde{u}^\lambda$ by setting $\tilde{u}^\lambda(s) = \tilde{u}^\lambda(0)$ if $s < 0$, $\tilde{u}^\lambda(s) = \tilde{u}^\lambda(1)$ if $s > 1$, and similarly for $\tilde{w}^\lambda$. Then we perform a mollification in the s-variable:

$$U^\lambda(s) \doteq \int u^\lambda(s-\sigma)\phi_\rho(\sigma)\, d\sigma\,, \qquad W^\lambda(s) \doteq \int w^\lambda(s-\sigma)\phi_\rho(\sigma)\, d\sigma\,.$$

Here ϕ_ρ is a standard mollification kernel, so that $\phi_\rho(\sigma) \doteq \rho^{-1}\phi(\rho^{-1}\sigma)$ for some smooth function with compact support $\phi \in \mathcal{C}_c^\infty$, with $\phi \geq 0$ and $\int \phi(\sigma)\, d\sigma = 1$.

By choosing $\rho > 0$ sufficiently small, it is clear that the functions U^λ and W^λ, in place of $\tilde{u}^\lambda$ and $\tilde{q}^\lambda$, satisfy all conclusions of Lemma 1. □

Remark 4. Since the solution of (2.11) depends continuously on $\tilde{w}^\lambda$, we can slightly perturb these functions in $\mathbf{L}^1$ and still achieve the pointwise inequalities (2.12). In particular, on the smooth functions $\tilde{w}^\lambda$ we can impose the additional requirement that

$$\tilde{w}^\lambda(s) = 0 \qquad\qquad \text{for all } \lambda \in \Lambda\,, \quad s \in [0, \varepsilon_0]\,, \tag{3.25}$$

for some $\varepsilon_0 > 0$ sufficiently small.

4 Proof of Theorem 1

Using Lemma 1 in the special case where the parameter set Λ is a singleton, we can assume that $q^*(\cdot)$ and $u^*(\cdot)$ are smooth, and that there exists a smooth function $w^*(\cdot)$ such that

$$\dot{q}^*(s) \;=\; K(q^*(s), u^*(s))\dot{u}^*(s) + A(q^*(s), u^*(s))\Big(w^*(s)^\dagger D(q^*(s), u^*(s))w^*(s)\Big). \tag{4.1}$$

Moreover, by Remark 4, for some $\varepsilon_0 > 0$ sufficiently small we can assume that

$$w^*(s) = 0 \qquad\qquad \text{for all } s \in [0, \varepsilon_0]. \tag{4.2}$$

Define the nonlinear time rescaling $\psi : [0, T] \mapsto [0, 1]$,

$$s \;=\; \psi(t) \;\doteq\; 1 - \frac{\ln(1 + T - t)}{\ln(1 + T)}\,. \tag{4.3}$$

In the following, a prime will denote differentiation w.r.t. $t \in [0, T]$, while the upper dot means a derivative w.r.t. $s \in [0, 1]$. We claim that, by setting $\alpha \doteq \sqrt{\ln(1 + T)}$ and defining

$$u(t) \doteq u^*(\psi(t)) + \frac{\sqrt{2}}{\alpha^2}\psi'(t)\sin(\alpha^3 t) \cdot w^*(\psi(t))\,, \tag{4.4}$$

the corresponding solution $t \mapsto (q(t), p(t))$ of (2.1), (2.2) satisfies the estimates (2.9), provided that $|\bar{p}|$ is small and T is sufficiently large. This will be proved in several steps.

1. It will be convenient to work with the variable $s = \psi(t) \in [0, 1]$, and derive an evolution equation for q, p as functions of s. By the definition of ψ in (4.3) it follows

$$t(s) \;=\; \psi^{-1}(s) \;=\; (1 + T)\Big(1 - e^{-s\cdot\ln(1+T)}\Big), \tag{4.5}$$

$$\frac{ds}{dt} \;=\; \psi'(t) \;=\; \frac{1}{\ln(1+T)} \cdot \frac{1}{1 + T - t} \;=\; \frac{e^{s\cdot\ln(1+T)}}{(1+T)\ln(1+T)} \;\doteq\; \eta(s)\,. \tag{4.6}$$

In turn, the functions

$$s \mapsto (\tilde{q}(s), \tilde{p}(s), \tilde{u}(s)) \doteq \Big(q(\psi^{-1}(s)),\, p(\psi^{-1}(s)),\, u(\psi^{-1}(s))\Big)$$

satisfy

$$\frac{d}{ds}\begin{pmatrix} \tilde{q} \\ \\ \tilde{p} \end{pmatrix} \;=\; \frac{1}{\eta(s)}\left[\begin{pmatrix} A\tilde{p} \\ \\ -\frac{1}{2}\tilde{p}^\dagger B\tilde{p} \end{pmatrix} \;+\; \begin{pmatrix} K \\ \\ -\tilde{p}^\dagger C \end{pmatrix}\frac{du}{dt} \;+\; \left(\frac{du}{dt}\right)^\dagger \begin{pmatrix} 0 \\ \\ D \end{pmatrix}\frac{du}{dt}\right].$$

Differentiating (4.4) and recalling that $\psi' = \eta$, we find

$$\frac{du}{dt} = \dot u^*(s)\,\eta(s) + \tfrac{\sqrt{2}}{\alpha^2}\psi''\,\sin(\alpha^3\psi^{-1}(s))w^*(s) + \sqrt{2}\,\alpha\,\eta(s)\,\cos(\alpha^3\psi^{-1}(s))\,w^*(s)$$
$$+\tfrac{\sqrt{2}}{\alpha^2}\sin(\alpha^3\psi^{-1}(s))\,\dot w^*(s)\,\eta^2(s).$$

Notice that (4.6) yields

$$\psi''(t) = \frac{1}{\ln(1+T)}\frac{1}{(1+T-t)^2} = \frac{e^{2s\cdot\ln(1+T)}}{(1+T)^2\ln(1+T)}\,.$$

Putting together the above computations, we finally obtain

$$\frac{d}{ds}\begin{pmatrix}\tilde q\\ \\ \tilde p\end{pmatrix} = \begin{pmatrix}A\tilde p\\ \\ -\frac{1}{2}\tilde p^\dagger B\tilde p\end{pmatrix}\frac{1}{\eta(s)} + \begin{pmatrix}K\\ \\ -\tilde p^\dagger C\end{pmatrix}(\dot u^*(s)+\phi_1(s)+\phi_2(s))$$
$$+\ \Big[\sqrt{2}\alpha\cos(\alpha^3\psi^{-1}(s))\,w^*(s)+\zeta(s)\Big]^\dagger\begin{pmatrix}0\\ \\ D\end{pmatrix}\Big[\sqrt{2}\alpha\cos(\alpha^3\psi^{-1}(s))\,w^*(s)+\zeta(s)\Big]\cdot\eta(s)\,, \tag{4.7}$$

where the functions ϕ_1, ϕ_2, ζ are given respectively by

$$\begin{cases}\phi_1(s) = \sqrt{2}\,\alpha\cos(\alpha^3\psi^{-1}(s))\,w^*(s)\,,\\[2ex] \phi_2(s) = \dfrac{\sqrt{2}}{\alpha^2}\sin(\alpha^3\psi^{-1}(s))\left(\dfrac{\dot w^*(s)}{\ln(1+T)}+w^*(s)\right)\dfrac{e^{s\cdot\ln(1+T)}}{1+T}\,,\\[2ex] \zeta(s) = \dot u^*(s) + \dfrac{\sqrt{2}}{\alpha^2}\sin(\alpha^3\psi^{-1}(s))\left(\dfrac{\dot w^*(s)}{\ln(1+T)}+w^*(s)\right)\dfrac{e^{s\cdot\ln(1+T)}}{1+T}\,.\end{cases} \tag{4.8}$$

Before we derive the basic estimates, it is convenient to introduce two more variables, namely

$$p^*(s) \doteq w^*(s)^\dagger D(q^*(s),u^*(s))w^*(s)\,, \qquad \mathbf{p}(s) \doteq \frac{\tilde p(s)}{\eta(s)}\,. \tag{4.9}$$

We observe that

$$\frac{d}{ds}\mathbf{p} = \frac{1}{\eta}\frac{d\tilde p}{ds} - \frac{\dot\eta}{\eta^2}\tilde p = \frac{1}{\eta}\frac{d\tilde p}{ds} - \alpha^2\mathbf{p}\,.$$

In term of $\mathbf{p}$, the system (4.7) takes the form

$$\frac{d}{ds}\begin{pmatrix}\tilde q\\ \\ \mathbf{p}\end{pmatrix} = \begin{pmatrix}A\mathbf{p}\\ \\ -\frac{\eta(s)}{2}\mathbf{p}^\dagger B\mathbf{p}\end{pmatrix} + \begin{pmatrix}K\\ \\ -\eta(s)\mathbf{p}^\dagger C\end{pmatrix}(\dot u^*(s)+\phi_1(s)+\phi_2(s)) - \begin{pmatrix}0\\ \\ \alpha^2\mathbf{p}\end{pmatrix}$$
$$+\ \Big[\sqrt{2}\alpha\cos(\alpha^3\psi^{-1}(s))\,w^*(s)+\zeta(s)\Big]^\dagger\begin{pmatrix}0\\ \\ D\end{pmatrix}\Big[\sqrt{2}\alpha\cos(\alpha^3\psi^{-1}(s))\,w^*(s)+\zeta(s)\Big]\,. \tag{4.10}$$

Notice that all functions A, K, B, C, D here depend on $\tilde{q}, u^*$.

2. To help the reader, we give here a heuristic argument motivating our key estimate.

By (4.6) it follows

$$0 < \eta(s) \leq \frac{1}{\ln(1+T)} \,. \tag{4.11}$$

From the second equation in (4.10) one obtains

$$\begin{aligned}
\frac{d}{ds}\mathbf{p}(s) \quad &= \quad -\,\alpha^2 \mathbf{p}(s) + 2\alpha^2 \cos^2(\alpha^3\, \psi^{-1}(s))\, w^*(s)^\dagger D w^*(s) + \mathcal{O}(1)\cdot\alpha \\
&\approx \quad -\,\alpha^2 \mathbf{p}(s) + \alpha^2\, w^*(s)^\dagger D w^*(s) + \mathcal{O}(1)\cdot\alpha\,.
\end{aligned}$$

Notice that last approximation follows from the fact that the function $s \mapsto \cos^2(\alpha^3\psi^{-1}(s))$ is rapidly oscillating and has average $1/2$.

Performing an integration by parts, the solution to the Cauchy problem

$$\dot{P}(s) \quad = \quad -\,\alpha^2 \cdot P(s) + \alpha^2\, w^*(s)^\dagger D(s) w^*(s)\,, \qquad\qquad P(0) = 0,$$

can be written as

$$\begin{aligned}
P(s) \quad &= \quad \int_0^s e^{-\alpha^2(s-\sigma)} \alpha^2 \Big(w^*(\sigma)^\dagger D(\sigma) w^*(\sigma) \Big)\, d\sigma \\
&= \quad w^*(s)^\dagger D(s) w^*(s) - e^{-\alpha^2 s} \Big(w^*(0)^\dagger D(0) w^*(0) \Big) \\
&\qquad - \int_0^s e^{-\alpha^2(s-\sigma)} \cdot \left[\frac{d}{d\sigma} \Big(w^*(\sigma)^\dagger D(\sigma) w^*(\sigma) \Big) \right] d\sigma \\
&= \quad w^*(s)^\dagger D(s) w^*(s) + \mathcal{O}(1)\cdot\alpha^{-2}.
\end{aligned}$$

Since $\alpha = \sqrt{\ln(1+T)} \to \infty$ as $T \to \infty$, we thus expect the convergence $\mathbf{p}(s) \to p^*(s)$ uniformly for $s \in [0,1]$, where p^* is the function introduced in (4.9). In turn, the first equation in (4.10) yields

$$\frac{d}{ds}\tilde{q}(s) \quad \approx \quad A p^*(s) + K \dot{u}^*(s)\,.$$

Indeed, in the computation of $K\phi$, the rapidly oscillating terms cancel out in the limit.

As $T \to \infty$, we thus expect $\tilde{q}(s) \to q^*(s)$ uniformly for $s \in [0,1]$. Moreover, by (4.6) and (4.9), $\tilde{p}(s) = \mathbf{p}(s)\eta(s) \to 0$ as $T \to \infty$. The remaining steps of the proof will render entirely rigorous the above argument.

3. In this section, for future use, we provide estimates on two types of rapidly oscillating integrals. In both cases the key ingredient is an integration by parts. We assume that the functions h, β are $\mathcal{C}^2$ on the closed interval $[0,1]$, with $h'(s) > 0$.

First, multiplying and dividing by $h'(s)$ we compute

$$\begin{aligned}
&\int_0^\tau \cos(\alpha^3 h(s))\beta(s)\,ds \\
&= \left(\int_0^\tau \cos(\alpha^3 h(r))\,h'(r)\,dr\right)\cdot \frac{\beta(\tau)}{h'(\tau)} - \int_0^\tau \left(\int_0^s \cos(\alpha^3 h(r))h'(r)\,dr\right)\cdot\left(\frac{d}{ds}\,\frac{\beta(s)}{h'(s)}\right)ds \\
&= \frac{\sin(\alpha^3 h(\tau)) - \sin(\alpha^3 h(0))}{\alpha^3}\cdot\frac{\beta(\tau)}{h'(\tau)} \\
&\quad - \int_0^\tau \frac{\sin(\alpha^3 h(s)) - \sin(\alpha^3 h(0))}{\alpha^3}\cdot\frac{\beta'(s)h'(s)-\beta(s)h''(s)}{[h'(s)]^2}\,ds\,.
\end{aligned} \tag{4.12}$$

Of course, an entirely similar estimate is valid replacing the cosine with a sine function. Next, by similar methods we compute

$$\begin{aligned}
&\int_0^\tau e^{-\alpha^2(\tau-s)}2\alpha^2\cos^2(\alpha^3 h(s))\beta(s)\,ds \\
&= \int_0^\tau e^{-\alpha^2(\tau-s)}\alpha^2\beta(s)\,ds + \int_0^\tau \alpha^2 e^{-\alpha^2(\tau-s)}\Big(2\cos^2(\alpha^3 h(s))-1\Big)h'(s)\frac{\beta(s)}{h'(s)}\,ds \qquad (4.13)\\
&= I_1 + I_2\,.
\end{aligned}$$

$$\begin{aligned}
I_1 &= \alpha^2 e^{-\alpha^2\tau}\left(\int_0^\tau e^{\alpha^2 s}\,ds\right)\beta(\tau) - \alpha^2 e^{-\alpha^2\tau}\int_0^\tau\left(\int_0^s e^{\alpha^2 r}\,dr\right)\beta'(s)\,ds \\
&= (1-e^{-\alpha^2\tau})\beta(\tau) - e^{-\alpha^2\tau}\int_0^\tau\Big(e^{\alpha^2 s}-1\Big)\beta'(s)\,ds\,,
\end{aligned}$$

$$|I_1-\beta(\tau)| \;\le\; e^{-\alpha^2\tau}|\beta(0)| + \frac{1}{\alpha^2}\,\|\beta'\|_{\mathbf{L}^\infty}\,. \tag{4.14}$$

$$\begin{aligned}
I_2 &= \alpha^2 e^{-\alpha^2\tau}\Bigg\{\int_0^\tau \Big(2\cos^2(\alpha^3 h(s))-1\Big)h'(s)\,ds\cdot e^{\alpha^2\tau}\,\frac{\beta(\tau)}{h'(\tau)} \\
&\quad -\int_0^\tau\left(\int_0^s\Big(2\cos^2(\alpha^3 h(r))-1\Big)h'(r)\,dr\right)\cdot e^{\alpha^2 s}\left(\frac{\alpha^2\beta(s)+\beta'(s)}{h'(s)} - \frac{\beta(s)h''(s)}{[h'(s)]^2}\right)ds\Bigg\} \\
&= \alpha^2 e^{-\alpha^2\tau}\Bigg\{\frac{\sin(2\alpha^3 h(\tau))-\sin(2\alpha^3 h(0))}{2\alpha^3}\cdot e^{\alpha^2\tau}\,\frac{\beta(\tau)}{h'(\tau)} \\
&\quad -\int_0^\tau \frac{\sin(2h(s))-\sin(2\alpha^3 h(0))}{2\alpha^3}\cdot e^{\alpha^2 s}\cdot\left(\frac{\alpha^2\beta(s)+\beta'(s)}{h'(s)} - \frac{\beta(s)h''(s)}{[h'(s)]^2}\right)ds\Bigg\}
\end{aligned}$$

$$|I_2| \;\le\; \frac{1}{\alpha}\Big(\alpha^2\|\beta\|_{\mathbf{L}^\infty} + \|\beta'\|_{\mathbf{L}^\infty}\Big)\cdot\left\{\min_{s\in[0,1]} h'(s)\right\}^{-1} + \frac{1}{\alpha}\,\|\beta\|_{\mathbf{L}^\infty}\left\|\frac{h''}{[h']^2}\right\|_{\mathbf{L}^\infty}. \tag{4.15}$$

4. For a fixed $T > 0$, consider the solution to the Cauchy problem (4.10) with initial data

$$\tilde{q}(0) = \bar{q}, \qquad\qquad \mathbf{p}(0) = 0\,. \tag{4.16}$$

Its solution $s \mapsto (\tilde{q}(s), \mathbf{p}(s))$ can be obtained as the fixed point of a Picard transformation. Namely, the transformation $(q,p) \mapsto (\mathcal{Q}(q,p),\ \mathcal{P}(q,p))$ whose components are

$$\begin{aligned}\mathcal{Q}(q,p)(\tau) \ &= \ \bar{q} + \int_0^\tau A(q(s),u^*(s))p(s)\,ds + \int_0^\tau K(q(s),u^*(s))(\dot{u}^*(s)+\phi_2(s))\,ds \\ &- \int_0^\tau \Big[K_q(q(s),u^*(s))\cdot\Big(A(q(s),u^*(s))p(s) + K(q(s),u^*(s))(\dot{u}^*(s)+\phi_1(s)+\phi_2(s))\Big) \\ &+ K_u(q(s),u^*(s))\dot{u}^*(s)\Big]\cdot\Big(\int_0^s \phi_1(r)\,dr\Big)\ ds \ + \ K(q(\tau),u^*(\tau))\int_0^\tau \phi_1(s)\,ds\,,\end{aligned} \tag{4.17}$$

$$\begin{aligned}\mathcal{P}(q,p)(\tau) \ &= \ \int_0^\tau e^{-\alpha^2(\tau-s)}\cdot 2\alpha^2\cos^2(\alpha^3\psi^{-1}(s))\, w^*(s)^\dagger D(q(s),u^*(s))w^*(s)\,ds \\ &- \int_0^\tau e^{-\alpha^2(\tau-s)}\cdot\eta(s)p(s)^\dagger\left(B(q(s),u^*(s))\frac{p(s)}{2} + C(q(s),u^*(s))(\dot{u}^*(s)+\phi_1(s)+\phi_2(s))\right)\,ds \\ &+ \int_0^\tau e^{-\alpha^2(\tau-s)}\cdot\Big[\sqrt{2}\alpha\cos(\alpha^3\psi^{-1}(s))\,w^*(s)^\dagger D(q(s),u^*(s))\zeta(s) \\ &+ \zeta(s)^\dagger D(q(s),u^*(s))\sqrt{2}\alpha\cos(\alpha^3\psi^{-1}(s))\,w^*(s) + \zeta(s)^\dagger D(q(s),u^*(s))\zeta(s)\Big]\,ds\,.\end{aligned} \tag{4.18}$$

Notice that the last two integral terms in (4.17) are obtained from

$$\int_0^\tau K(q(s),u^*(s))\,\phi_1(s)\,ds\,,$$

after an integration by parts.

On the family of couples of continuous functions $(q,p) : [0,1] \mapsto I\!R^{n+n}$ we consider the equivalent norm

$$\Big\|(q,p)\Big\|_* \doteq \sup_{s\in[0,1]}\ \max\left\{e^{-\rho s}|q(s)|\,,\ \frac{e^{-\rho s}}{\kappa}\,|p(s)|\right\}. \tag{4.19}$$

We claim that, if the constants ρ, κ are chosen sufficiently large, depending on the functions A, B, C, D, K but not on α, then the Picard transformation $(\mathcal{Q},\mathcal{P})$ is a strict contraction w.r.t. this equivalent norm. Namely,

$$\Big\|\Big(\mathcal{Q}(q,p) - \mathcal{Q}(\hat{q},\hat{p})\,,\ \mathcal{P}(q,p) - \mathcal{P}(\hat{q},\hat{p})\Big)\Big\|_* \ < \ \frac{1}{2}\,\Big\|(q-\hat{q},\,p-\hat{p})\Big\|_*\,. \tag{4.20}$$

Moreover, we claim that, as $\alpha = \sqrt{1+T} \to \infty$, one has

$$\Big\|\Big(\mathcal{Q}(q^*,p^*),\mathcal{P}(q^*,p^*)\Big) - (q^*,p^*)\Big\|_* \ \to \ 0. \tag{4.21}$$

The two claims (4.20)-(4.21) will be proved in the next two sections. In turn, they yield

$$\|(\tilde{q}, \mathbf{p}) - (q^*, p^*)\|_* \;<\; 2\left\|\Big(\mathcal{Q}(q^*, p^*), \mathcal{P}(q^*, p^*)\Big) - (q^*, p^*)\right\|_* \;\to\; 0 \tag{4.22}$$

as $\alpha \to \infty$. From (4.22), the conclusions in (2.9) will follow easily.

5. In this step we establish the strict contraction property (4.20). As in Remark 3, it suffices to prove (4.20) assuming that all functions $q, \hat{q}, p, \hat{p}$ take values within some (possibly large) bounded set.

Assume that $\delta \doteq \left\|(q - \hat{q},\, p - \hat{p})\right\|_*$, so that

$$|q(s) - \hat{q}(s)| \leq \delta e^{\rho s}\,, \qquad\qquad |p(s) - \hat{p}(s)| \leq \delta\kappa\, e^{\rho s} \qquad\qquad \text{for all } s \in [0, 1]. \tag{4.23}$$

By (4.6) and (4.8) we have

$$0 < \eta(s) \leq \frac{1}{\alpha^2}\,, \qquad |\phi_1(s)| \leq C_1\alpha\,, \qquad |\phi_2(s)| \leq \frac{C_1}{\alpha^2}\,, \qquad |\zeta(s)| \leq C_1\,. \tag{4.24}$$

Here and in the following, by $C_1, C_2, \ldots$ we denote constants depending on the functions A, B, C, D, K, u^*, w^*, but not on $\alpha \doteq \sqrt{\ln(1+T)}$. Applying (4.12) to the case where

$$h(s) = \psi^{-1}(s) = (1+T)(1 - e^{-s\cdot\ln(1+T)}), \qquad\qquad \beta(s) = \sqrt{2}\,\alpha\, w^*(s), \tag{4.25}$$

$$h'(s) \;=\; (1+T)\ln(1+T)e^{-s\cdot\ln(1+T)} \;\geq\; \ln(1+T)\,, \qquad\qquad \frac{h''(s)}{[h'(s)]^2} \;\leq\; 1\,, \tag{4.26}$$

one finds

$$\left|\int_0^\tau \phi_1(s)\, ds\right| \leq \frac{C_2}{\alpha^2} \tag{4.27}$$

for all $\tau \in [0, 1]$. Recalling (4.17) and using (4.27) we obtain

$$\Big|\mathcal{Q}(q, p)(s) - \mathcal{Q}(\hat{q}, \hat{p})(s)\Big| \;\leq\; \int_0^s C_3\Big(|p(s) - \hat{p}(s)| + |q(s) - \hat{q}(s)|\Big)\, ds\,. \tag{4.28}$$

By (4.6) we have $\eta(s) \leq \alpha^{-2}$. From (4.18) it thus follows

$$\Big|\mathcal{P}(q, p)(\tau) - \mathcal{P}(\hat{q}, \hat{p})(\tau)\Big| \;\leq\; \int_0^\tau e^{-\alpha^2(\tau - s)} C_4\Big(\alpha^2|q(s) - \hat{q}(s)| + \frac{1}{\alpha}|p(s) - \hat{p}(s)|\Big)\, ds\,. \tag{4.29}$$

The bounds (4.23) and (4.29) imply

$$\begin{aligned} \frac{e^{-\rho\tau}}{\kappa}\Big|\mathcal{P}(q, p)(\tau) - \mathcal{P}(\hat{q}, \hat{p})(\tau)\Big| \;&\leq\; \frac{e^{-\rho\tau}}{\kappa}\cdot\int_0^\tau e^{-\alpha^2(\tau - s)} C_4\Big(\alpha^2\delta e^{\rho s} + \frac{\kappa}{\alpha}\delta e^{\rho s}\Big)\, ds\,, \\ &< \;\frac{e^{-(\alpha^2+\rho)\tau}}{\kappa}\cdot C_4\delta\, e^{(\alpha^2+\rho)\tau}\cdot\frac{\alpha^2 + \frac{\kappa}{\alpha}}{\alpha^2 + \rho} \;\leq\; \frac{\delta}{2}\,, \end{aligned} \tag{4.30}$$

provided that $\kappa > 2C_4$ and α is suitably large.

In a similar way, the bounds (4.23) and (4.28) imply

$$\begin{aligned} e^{-\rho s}\Big|\mathcal{Q}(q,p)(\tau)-\mathcal{Q}(\hat q,\hat p)(\tau)\Big| &\leq e^{-\rho s}\cdot\int_0^\tau C_3\Big(\kappa\delta e^{\rho s}+\delta e^{\rho s}\Big)\,ds\,, \\ &\leq C_3\delta\cdot\frac{\kappa+1}{\rho} \leq \frac{\delta}{2}\,, \end{aligned} \tag{4.31}$$

provided that $\rho\geq 2C_3(\kappa+1)$.

6. In this step we estimate the distance between (q^*,p^*) and the fixed point $(\tilde q,\mathbf{p})$ of the transformation $(\mathcal{Q},\mathcal{P})$. We recall that q^* satisfies

$$q^*(\tau)=\bar q+\int_0^\tau A(q^*(s),u^*(s))p^*(s)\,ds+\int_0^\tau K(q(s),u^*(s))\,\dot u^*(s)\,ds\,,$$

with p^* defined at (4.9). Comparing this with (4.17), we obtain

$$\begin{aligned} \Big|\mathcal{Q}(q^*,p^*)(\tau)-q^*(\tau)\Big| \leq{}& \left|\int_0^\tau K(q(s),u^*(s))\phi_2(s)\,ds\right| \\ &+\int_0^\tau\Big|K_q(q(s),u^*(s))\cdot\Big(A(q(s),u^*(s))p(s)+K(q(s),u^*(s))(\dot u^*(s)+\phi_1(s)+\phi_2(s))\Big) \\ &+K_u(q(s),u^*(s))\dot u^*(s)\Big|\cdot\left|\int_0^s\phi_1(r)\,dr\right|\,ds \;+\; \Big|K(q(\tau),u^*(\tau))\Big|\cdot\left|\int_0^\tau\phi_1(s)\,ds\right|\,. \end{aligned} \tag{4.32}$$

The definition of ϕ_1,ϕ_2 at (4.8) implies

$$|\phi_1(s)|\leq C_5\,\alpha\,,\qquad\qquad |\phi_2(s)|\leq C_5\,\alpha^{-2}.$$

Using the estimate (4.27) we thus obtain

$$\Big|\mathcal{Q}(q^*,p^*)(\tau)-q^*(\tau)\Big| \leq C_6\,\alpha^{-1}. \tag{4.33}$$

Next, comparing (4.9) with (4.18), we obtain

$$\begin{aligned} |p^*(\tau)-\mathcal{P}(q^*,p^*)(\tau)| \leq{}& \left|p^*(\tau)-\int_0^\tau e^{-\alpha^2(\tau-s)}2\alpha^2\cos^2(\alpha^3\psi^{-1}(s))\cdot p^*(s)ds\right| \\ &+\int_0^\tau e^{-\alpha^2(\tau-s)}C_7\,\alpha\,ds \doteq J_1+J_2\,. \end{aligned} \tag{4.34}$$

A straightforward computation yields

$$|J_2| \leq C_7\,\alpha^{-1}. \tag{4.35}$$

To estimate J_1, we use (4.14)-(4.15) with $\beta(s)=p^*(s)$, $h(s)=\psi^{-1}(s)$. By (4.5), this implies

$$h'(s) = (1+T)\ln(1+T)e^{-s\cdot\ln(1+T)} \geq \ln(1+T) = \alpha^2,\qquad\qquad \left|\frac{h''(s)}{[h'(s)]^2}\right|\leq 1\,.$$

Recalling that $p^*(s) = w^*(s) = 0$ for $s \in [0, \varepsilon_0]$, we thus obtain

$$|J_1| \;\leq\; C_8\,\alpha^{-1}. \tag{4.36}$$

7. By choosing T, and hence also $\alpha = \sqrt{\ln(1+T)}$, sufficiently large, the difference between (q^*, p^*) and the fixed point $(\tilde{q}, \mathbf{p})$ of the transformation $(\mathcal{Q}, \mathcal{P})$ can thus be rendered arbitrarily small, in the norm $\|\cdot\|_*$ introduced at (4.19). Since the constant κ is independent of T, the norm $\|\cdot\|_*$ is uniformly equivalent to the $\mathcal{C}^0$ norm. This establishes the last two estimates in (2.9) when $p(0) = \bar{p} = 0$. By (4.6) and (4.9), we have

$$|\tilde{p}(s)| \;=\; |\mathbf{p}(s)|\,\eta(s) \;\leq\; \frac{|\mathbf{p}(s)|}{\ln(1+T)}\,.$$

Since $\mathbf{p}(s) \to p^*(s)$ as $T \to \infty$, uniformly for $s \in [0, 1]$, this implies the uniform convergence $\tilde{p}(s) \to 0$. By continuity, all the estimates in (2.9) remain valid whenever $|p(0)| \leq \delta$ for some $\delta > 0$ small enough. $\square$

5 Proof of Theorem 2

By a translation of coordinates, it is not restrictive to assume that $(\mathbf{q}, \mathbf{u}) = 0 \in I\!R^{n+m}$. By assumption, when $\lambda = 0 \in I\!R^{n+m}$ we thus have $(q^0(1),\, u^0(1)) = (\mathbf{q}, \mathbf{u}) = 0 \in I\!R^{n+m}$. Moreover the $(n+m) \times (n+m)$ Jacobian matrix

$$J \;=\; \left(\frac{\partial(q^\lambda(1),\, u^\lambda(1))}{\partial\lambda}\right), \tag{5.1}$$

computed at the point $\lambda = 0 \in I\!R^{n+m}$, has maximum rank. For notational convenience, we denote by $z = (q, u)$ the variable in $I\!R^{n+m}$ and call J^{-1} the inverse of the matrix J in (5.1). Taking $\lambda = J^{-1}z$, we thus have

$$\lim_{z \to 0} \frac{z - \Big(q^{J^{-1}z}(1)\;,\; u^{J^{-1}z}(1)\Big)}{|z|} \;=\; 0. \tag{5.2}$$

Choosing $\rho > 0$ sufficiently small, from (5.2) we deduce

$$\Big|z - \Big(q^{J^{-1}z}(1)\;,\; u^{J^{-1}z}(1)\Big)\Big| \;\leq\; \frac{|z|}{3}\,, \qquad \text{for all}\;\; z \in \mathcal{B}_\rho\,. \tag{5.3}$$

where $\mathcal{B}_\rho$ is the closed ball in $I\!R^{n+m}$, centered at tho origin with radius ρ.

Next, we apply Lemma 1 and obtain a continuous map $(s, \lambda) \mapsto (\tilde{u}^\lambda(s), \tilde{w}^\lambda(s))$ such that the corresponding solutions of (2.11) satisfy (2.12) with $\varepsilon = \rho/3$. Together with (5.3), this implies

$$\Big|z - \Big(\tilde{q}^{J^{-1}z}(1)\;,\; \tilde{u}^{J^{-1}z}(1)\Big)\Big| \;\leq\; \frac{2\rho}{3}\,, \qquad \text{for all}\;\; z \in \mathcal{B}_\rho\,. \tag{5.4}$$

Finally, as in (4.4), we define $\alpha = \sqrt{1+T}$ and the controls

$$U^\lambda(t) \doteq \tilde{u}^\lambda(\psi(t)) + \sqrt{2}\psi'(t)\sin(\alpha\, t)\cdot \tilde{w}^\lambda(\psi(t)) \qquad\qquad t\in[0,T]\,. \tag{5.5}$$

If $|\bar{p}|$ is sufficiently small, choosing T sufficiently large the proof of Theorem 1 shows that the corresponding solutions $t\mapsto \big(Q^\lambda(t), P^\lambda(t)\big)$ of (2.1)-(2.2) satisfy

$$\Big|\Big(Q^\lambda(\psi^{-1}(1)),\, U^\lambda(\psi^{-1}(1))\Big) - \Big(\tilde{q}^\lambda(1),\, \tilde{u}^\lambda(1)\Big)\Big| \le \frac{\rho}{3}\,, \qquad \text{for all } \ \lambda\in\Lambda\,. \tag{5.6}$$

We now consider the map

$$z \ \stackrel{\Phi}{\longrightarrow}\ z \ - \ \Big(Q^{J^{-1}z}(1)\ ,\ U^{J^{-1}z}(1)\Big) \qquad\qquad z\in\mathcal{B}_\rho\,.$$

By (5.4) and (5.6), Φ is a continuous map of the closed ball $\mathcal{B}_\rho$ into itself. Hence, by Brouwer's theorem, it has a fixed point z^*. This implies that exist $\lambda^* = J^{-1}z^*$ such that $(Q^{\lambda^*}(1)\ ,\ U^{\lambda^*}(1)) \ = \ 0\in I\!R^{n+m}$, completing the proof. $\square$

6 Examples

In Lemma 1, the assumption (H) on the continuity of the cone Γ plays a key role. Indeed, if the map $(q,u)\mapsto\Gamma(q,u)$ is not continuous the conclusion may be false.

Example 1. Let $q,u\in I\!R$, and consider the Cauchy problem

$$\dot{q} = q^2\,\dot{u}^2\,, \qquad\qquad q(0) = -1. \tag{6.1}$$

This corresponds to (2.5)-(2.6), taking

$$K(q,u)\equiv 0, \qquad\quad A(q,u) = q^2\,, \qquad\quad D(q,u)\equiv 1.$$

In this case we have

$$\Gamma(q,u) = \begin{cases} \{p\in I\!R\,;\ p\ge 0\} & \text{if } \ q\neq 0\,,\\ \{0\} & \text{if } \ q=0\,,\end{cases}$$

Hence, the map $\tilde{q}(t) = t-1$ provides a solution to (2.5). However, for every $\mathcal{C}^1$ map $t\mapsto u(t)$ the corresponding solution of (6.1) satisfies $q(t)<0$ for all $t\ge 0$. Hence the map $\tilde{q}$ cannot be approximated by smooth solutions of (6.1).

Next, we illustrate a simple application of Theorems 1 and 2.

Example 2. Consider a bead with mass m, sliding without friction along a bar. We assume that the bar can be rotated around the origin on a horizontal plane (see fig. 1).

This system can be described by two Lagrangian parameters: the distance r of the bead from the origin, and the angle θ formed by the bar and a fixed line through the origin. The kinetic energy of the bead is given by

$$T(r,\theta,\dot{r},\dot{\theta}) = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)\,. \tag{6.2}$$

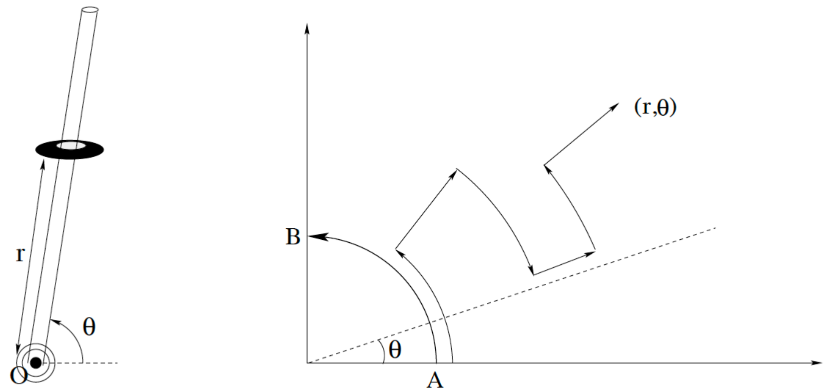

Figure 1: Left: A bead sliding without friction along a rotating bar. Right: trajectories with non-decreasing radius.

We assign the angle $\theta = u(t)$ as a function of time, while the radius r is the remaining free coordinate. Setting $p = \partial T / \partial \dot{r} = m\dot{r}$, the motion is thus described by the equations

$$\begin{cases} \dot{r} &= p/m\,, \\ \dot{p} &= m r \dot{u}^2\,. \end{cases} \tag{6.3}$$

Observe that in this case the right hand side of the equation contains the square of the derivative of the control.

Consider the problem of steering the bead from $A = (r_A, \theta_A) = (1, 0)$ to a point very close to $B = (r_B, \theta_B) = (1, \pi/2)$, during an interval of time $[0, T]$ possibly very large. Observe that this goal cannot be achieved by rotating the bar with small but constant angular velocity. Indeed, choosing $\theta(t) = u(t) = \pi t/2T$, the trajectory of (6.3) corresponding to the initial data $r(0) = 1$, $p(0) = 0$ is obtained by solving

$$\ddot{r} = \left(\frac{\pi}{2T}\right)^2 r\,, \qquad\qquad r(0) = 1, \quad \dot{r}(0) = 0.$$

Hence $r(t) = \frac{1}{2}(e^{\pi t/2T} + e^{-\pi t/2T})$. In particular, $r(T) = \frac{1}{2}(e^{\pi/2} - e^{-\pi/2})$ for every choice of T. Of course, this value does not converge to 1 as $T \to \infty$.

We observe that, in the present case, the differential inclusion (2.5) reduces to $\dot{r} \geq 0$. By Theorem 1, every continuous trajectory of the form $t \mapsto (r(t), \theta(t))$, with r a non-decreasing function of time, can be tracked by solutions of the full system (6.3). In particular, according to (4.3), the trajectory

$$s \mapsto (r(s), \theta(s)) = (1, \pi s/2) \qquad\qquad s \in [0,1]$$

can be traced by using the control

$$u(t) = \left(1 - \frac{\ln(1+T-t)}{\ln(1+T)}\right)\frac{\pi}{2} \qquad\qquad t \in [0,T].$$

Next, we observe that, if $r^* > r_0$, then the point (r^*, θ^*) is normally reachable from the initial point (r_0, θ_0) by solutions of the differential inclusion $(\dot{r}, \dot{\theta}) \in I\!R_+ \times I\!R$. Hence, by Theorem 2, for each (r^*, θ^*) with $r^* > r_0$ there exists $T > 0$ sufficiently large and a control $u : [0,T] \mapsto I\!R$ with $u(0) = \theta_0$, $u(T) = \theta^*$, such that the solution of (6.3) with initial data

$$r(0) = r_0, \qquad\qquad p(0) = 0$$

satisfies $r(T) = r^*$.

7 Derivation of the evolution equations

Consider a system whose state is described by N Lagrangian variables $q^1, \ldots, q^N$. Let the kinetic energy $T = T(q, \dot{q})$ be given by a positive definite quadratic form of the time derivatives $\dot{q}^i$, namely

$$T(q,\dot{q}) \;=\; \frac{1}{2}\dot{q}^\dagger G \dot{q} \;=\; \frac{1}{2}\sum_{i,j=1}^{N} g_{ij}(q)\,\dot{q}^i \dot{q}^j\,. \tag{7.1}$$

Let the coordinates be split in two groups: $\{q^1, \ldots, q^n\}$ and $\{q^{n+1}, \ldots, q^{n+m}\}$, with $N = n+m$. The $(n+m) \times (n+m)$ symmetric matrix G in (7.1) will thus take the corresponding block form

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (g_{ij}) & (g_{i,n+\beta}) \\ (g_{n+\alpha,j}) & (g_{n+\alpha,\,n+\beta}) \end{pmatrix} \tag{7.2}$$

and denote its inverse by

$$\widehat{G} \doteq G^{-1} = \begin{pmatrix} \widehat{G}_{11} & \widehat{G}_{12} \\ \widehat{G}_{21} & \widehat{G}_{22} \end{pmatrix} = \begin{pmatrix} (g^{ij}) & (g^{i,n+\beta}) \\ (g^{n+\alpha,j}) & (g^{n+\alpha,\,n+\beta}) \end{pmatrix}.$$

Introduce the matrices

$$A = \left(a^{ij}\right) \doteq (G_{11})^{-1}, \qquad\qquad E = (e_{\alpha,\beta}) = (\widehat{G}_{22})^{-1}, \qquad\qquad K = -AG_{12}\,, \tag{7.3}$$

Since $\widehat{G} \;=\; G^{-1}$, we observe that

$$\begin{aligned} &\widehat{G}_{11}G_{11} + \widehat{G}_{12}G_{21} = Id\,, \qquad \widehat{G}_{22}G_{22} + \widehat{G}_{21}G_{12} = Id\,, \\ &\widehat{G}_{21}G_{11} + \widehat{G}_{22}G_{21} = 0\,, \qquad G_{11}\widehat{G}_{12} + G_{12}\widehat{G}_{22} = 0. \end{aligned} \tag{7.4}$$

Therefore, a straight forward rewriting of the above equations:

$$
\begin{cases}
G_{11}^{-1} = \widehat{G}_{11} + \widehat{G}_{12} G_{21} \widehat{G}_{11}^{-1}\,, & \widehat{G}_{21} G_{11}^{-1} = -\widehat{G}_{22}^{-1} \widehat{G}_{21}\,, \\[2ex]
\widehat{G}_{22}^{-1} = G_{22} + \widehat{G}_{22}^{-1} \widehat{G}_{21} G_{12}\,, & \widehat{G}_{22}^{-1} \widehat{G}_{21} = -G_{21} G_{11}^{-1} \\[2ex]
G_{11} \widehat{G}_{12} = -G_{12} \widehat{G}_{22}\,, \qquad A = G_{11}^{-1}\,, & E = \widehat{G}_{22}^{-1}
\end{cases}
\tag{7.5}
$$

shows that the following identities hold

$$
A = \widehat{G}_{11} - \widehat{G}_{12} E \widehat{G}_{21}\,, \qquad E = G_{22} - G_{21} A G_{12}\,, \qquad K = \widehat{G}_{12} E\,. \tag{7.6}
$$

We assume that a controller can prescribe the values of the last m coordinates as functions of time, say

$$
q^{n+\alpha}(t) = u_\alpha(t) \qquad \alpha = 1, \ldots, m\,, \tag{7.7}
$$

by implementing m frictionless constraints. Here **frictionless** means that the forces produced by the constraints make zero work in connection with any virtual displacement of the remaining free coordinates $q^1, \ldots, q^n$. In the absence of external forces, the motion is thus governed by the equations

$$
\frac{d}{dt} \frac{\partial T}{\partial \dot q^i}(q, \dot q) - \frac{\partial T}{\partial q^i}(q, \dot q) \;=\; \Phi_i(t) \qquad i = 1, \ldots, n+m\,. \tag{7.8}
$$

Here Φ_i are the components of the forces generated by the constraints. The assumption that these constraints are frictionless is expressed by the identities

$$
\Phi_i(t) \equiv 0 \qquad i = 1, \ldots, n\,. \tag{7.9}
$$

Introducing the conjugate momenta

$$
p_i \;=\; p_i(q, \dot q) \;\doteq\; \frac{\partial T}{\partial \dot q^i} \;=\; \sum_{i=1}^{n+m} g_{ij}(q)\, \dot q^j\,, \tag{7.10}
$$

We now consider the system of Hamiltonian equations for the first n variables

$$
\begin{cases}
\dot q^i &= \frac{\partial H}{\partial p_i}(q, p) \\
\dot p_i &= -\frac{\partial H}{\partial q^i}(q, p)
\end{cases}
\qquad i = 1, \ldots, n\,. \tag{7.11}
$$

Notice that (1.6) is a system of $2n$ equations for $q^1, \ldots, q^n, p_1, \ldots, p_n$, where the right hand side also depends on the remaining components q^i, p_i, $i = n+1, \ldots, n+m$. We can remove this explicit dependence by inserting the values

$$
\begin{cases}
q^{n+i} = u_i(t)\,, \qquad \dot q^{n+i} = \dot u_i(t) & i = 1, \ldots, m\,, \\
p_j = p_j(p_1, \ldots, p_n,\, \dot q^{n+1}, \ldots, \dot q^{n+m}) & j = n+1, \ldots, n+m\,.
\end{cases}
\tag{7.12}
$$

From now on it will be more convenient to use vector notations. We thus write $(q, u) = (q^1, \ldots, q^n, u_1, \ldots, u_m)$, $(p, \eta) = (p_1, \ldots, p_n, p_{n+1}, \ldots, p_{n+m})$. Recalling that $\widehat{G} = G^{-1}$, we thus have

$$
\begin{pmatrix} p \\ \eta \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \dot q \\ \dot u \end{pmatrix}, \qquad \begin{pmatrix} \dot q \\ \dot u \end{pmatrix} = \begin{pmatrix} \widehat{G}_{11} & \widehat{G}_{12} \\ \widehat{G}_{21} & \widehat{G}_{22} \end{pmatrix} \begin{pmatrix} p \\ \eta \end{pmatrix}. \tag{7.13}
$$

Multiplying by $A \doteq (G_{11})^{-1}$ the identity

$$p = G_{11}\dot{q} + G_{12}\dot{u}$$

we obtain

$$\dot{q} \;=\; (G_{11})^{-1}p - (G_{11})^{-1}G_{12}\dot{u} \;=\; Ap - AG_{12}\,\dot{u} \;= Ap + K\,\dot{u}\,. \tag{7.14}$$

Similarly, multiplying by $E = (\widehat{G}_{22})^{-1}$ the identity

$$\dot{u} = \widehat{G}_{21}p + \widehat{G}_{22}\eta$$

we obtain

$$\eta \;=\; (\widehat{G}_{22})^{-1}\dot{u} - (\widehat{G}_{22})^{-1}\widehat{G}_{21}p \;=\; E\dot{u} - E\widehat{G}_{21}\,p\,. \tag{7.15}$$

From the equation

$$\dot{p} \;=\; -\frac{1}{2}p^{\dagger}\frac{\partial\widehat{G}_{11}}{\partial q}p - p^{\dagger}\frac{\partial\widehat{G}_{12}}{\partial q}\eta - \frac{1}{2}\eta^{\dagger}\frac{\partial\widehat{G}_{22}}{\partial q}\eta\,,$$

using (7.15) we obtain

$$\dot{p} \;=\; -\frac{1}{2}p^{\dagger}\frac{\partial\widehat{G}_{11}}{\partial q}p - p^{\dagger}\frac{\partial\widehat{G}_{12}}{\partial q}\Big(E\dot{u} - E\widehat{G}_{21}\,p\Big) - \frac{1}{2}\Big(E\dot{u} - E\widehat{G}_{21}\,p\Big)^{\dagger}\frac{\partial\widehat{G}_{22}}{\partial q}\Big(E\dot{u} - E\widehat{G}_{21}\,p\Big)\,. \tag{7.16}$$

Here and in the sequel, we use the notation $p^{\dagger}, \eta^{\dagger}\ldots$ to denote the transpose of column vectors such as p, η.

Now,we rewrite equation (6.15) into the form

$$\begin{aligned}\dot{p} = -\frac{1}{2}p^{\dagger}\Big(\frac{\partial\widehat{G}_{11}}{\partial q} - \frac{\partial\widehat{G}_{12}}{\partial q}E\widehat{G}_{21} + \frac{1}{2}(E\widehat{G}_{21})^{\dagger}\frac{\partial\widehat{G}_{22}}{\partial q}E\widehat{G}_{21}\Big)p \\ -p^{\dagger}\Big(+ \frac{\partial\widehat{G}_{12}}{\partial q}E - (E\widehat{G}_{21})^{\dagger}\frac{\partial\widehat{G}_{22}}{\partial q}E\Big)\dot{u} - \frac{1}{2}\dot{u}^{\dagger}\Big(E^{\dagger}\frac{\partial\widehat{G}_{22}}{\partial q}E\Big)\dot{u}\end{aligned} \tag{7.17}$$

since $E = \widehat{G}_{22}^{-1}$,together with $\widehat{G}$ is symmetric,we have

$$\frac{\partial\widehat{G}_{22}}{\partial q}E \;=\; -\,\widehat{G}_{22}\frac{\partial E}{\partial q} \qquad ; \qquad \widehat{G}_{12}^{\dagger} = \widehat{G}_{21}$$

Hence equation (6.16) can be further rewrite as

$$\begin{aligned}\dot{p} = -\frac{1}{2}p^{\dagger}\Big(\frac{\partial\widehat{G}_{11}}{\partial q} - \frac{\partial\widehat{G}_{12}}{\partial q}E\widehat{G}_{21} - \frac{1}{2}\widehat{G}_{12}\frac{\partial E}{\partial q}\widehat{G}_{21}\Big)p \\ -p^{\dagger}\Big(\frac{\partial\widehat{G}_{12}}{\partial q}E + (\widehat{G}_{12})\frac{\partial E}{\partial q}\Big)\dot{u} + \frac{1}{2}\dot{u}^{\dagger}\frac{\partial E}{\partial q}\dot{u}\end{aligned} \tag{7.18}$$

Recall that $A = \widehat{G}_{11} - \widehat{G}_{12}E\widehat{G}_{21}$ and $K = \widehat{G}_{12}E$, we have

$$\frac{\partial A}{\partial q} = \frac{\partial\widehat{G}_{11}}{\partial q} - 2\frac{\partial\widehat{G}_{12}}{\partial q}E\widehat{G}_{21} - \widehat{G}_{12}\frac{\partial E}{\partial q}\widehat{G}_{21} \qquad\qquad \frac{\partial K}{\partial q} = \frac{\partial\widehat{G}_{12}}{\partial q}E + \widehat{G}_{12}\frac{\partial E}{\partial q}.$$

Together with (6.13), we finally obtain that the evolution of the first n variables $(q^1, \ldots, q^n)$ and of the corresponding momenta $(p_1, \ldots, p_n)$ can be described by the system

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2} p^\dagger \frac{\partial A}{\partial q} p \end{pmatrix} + \begin{pmatrix} K \\ \\ -p^\dagger \frac{\partial K}{\partial q} \end{pmatrix} \dot u + \dot u^\dagger \begin{pmatrix} 0 \\ \\ \frac{1}{2} \frac{\partial E}{\partial q} \end{pmatrix} \dot u \,. \tag{7.19}$$

Here A, K, E are functions of q, u, defined as

$$A = \left(a^{ij}\right) \doteq (G_{11})^{-1}\,, \qquad E = G_{22} - G_{21} A G_{12}\,, \qquad K = -AG_{12}\,. \tag{7.20}$$

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