$\mathbf{L}^p$-boundedness of the Bochner-Riesz operator

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Abstract

In this paper, we give a new approach to the Bochner-Riesz summability. As a result, we show that the Bochner-Riesz operator $\mathbf{S}^\delta, 0 < \mathbf{Re}\delta < \frac{1}{2}$ is bounded on $\mathbf{L}^p(\mathbb{R}^n)$ for $\frac{n-1}{2n} \le \frac{1}{p} \le \frac{n+1}{2n}$.

Contents

1 Introduction

A classical problem in harmonic analysis is to make precise the sense of Fourier inversion formulae, for given $f \in \mathbf{L}^p(\mathbb{R}^n)$. One natural way of formulating such identities is via the summability method due to Bochner and Riesz. This assertion leads us to study for the $\mathbf{L}^p$-boundedness of Bochner-Riesz operator, defined as

$$\mathbf{S}^\delta f(x) = \int_{\mathbb{R}^n} e^{2\pi \mathbf{i} x\cdot\xi} \widehat{f}(\xi) \left(1 - |\xi|^2\right)_+^\delta d\xi, \qquad \mathbf{Re}\delta \ge 0. \tag{1. 1}$$

At $\delta = 0$, we revisit on the famous ball multiplier problem. $\mathbf{S}^0$ is bounded only on $\mathbf{L}^2(\mathbb{R}^n)$ proved by Fefferman [7].

⋄ Throughout, $\mathfrak{B} > 0$ is regarded as a generic constant whose value depends on its sub-indices.

⋄ We always write $\mathbf{c} > 0$ for some large constant.

$\mathbf{S}^\delta f$ in (1. 1) can be written as a convolution

$$\begin{aligned}\mathbf{S}^\delta f(x) &= \pi^{-\delta}\Gamma(1+\delta)\int_{\mathbb{R}^n} f(x-u)\Omega^\delta(u)du,\\ \Omega^\delta(u) &= \left(\frac{1}{|u|}\right)^{\frac{n}{2}+\delta}\mathbf{J}_{\frac{n}{2}+\delta}\Big(2\pi|u|\Big)\end{aligned}\tag{1. 2}$$

where Γ and $\mathbf{J}$ denote Gamma and Bessel functions. See (A. 7)-(A. 8).

Moreover, the estimate in (A. 5) implies

$$\left|\Omega^\delta(u)\right| \leq \mathfrak{B}_{\mathbf{Re}\delta}\, e^{\mathbf{c}|\mathbf{Im}\delta|}\left(\frac{1}{1+|u|}\right)^{\frac{n+1}{2}+\mathbf{Re}\delta}.$$

Observe that for $\mathbf{Re}\delta > \frac{n-1}{2}$, the kernel of $\mathbf{S}^\delta$ is an $\mathbf{L}^1$-function in $\mathbb{R}^n$. We thus have

$$\left\|\mathbf{S}^\delta f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\delta}\, e^{\mathbf{c}|\mathbf{Im}\delta|}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1\leq p\leq\infty. \tag{1. 3}$$

When $0<\delta\leq\frac{n-1}{2}$, a counter example given by Herz [15] shows that $\mathbf{S}^\delta$ is not bounded on $\mathbf{L}^p(\mathbb{R}^n)$ for either $p\leq\frac{2n}{n+1+2\delta}$ or $p\geq\frac{2n}{n-1-2\delta}$.

Bochner-Riesz Conjecture *Let* $\mathbf{S}^\delta$ *defined in (1. 1) for* $0<\delta\leq\frac{n-1}{2}$. *We have*

$$\left\|\mathbf{S}^\delta f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\delta\, p}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \tag{1. 4}$$

if and only if

$$\frac{n-1-2\delta}{2n} < \frac{1}{p} < \frac{n+1+2\delta}{2n}. \tag{1. 5}$$

In 1972, Carlson and Sjölin [4] proved the conjecture for $n=2$. A year later, Fefferman [6] observes the restriction estimate:

$$\left\{\int_{\mathbb{S}^{n-1}}\left|\widehat{f}(\xi)\right|^2 d\sigma(\xi)\right\}^{\frac{1}{2}} \leq \mathfrak{B}_p\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1<p<\infty \tag{1. 6}$$

implying $\mathbf{S}^\delta$, $\delta>0$ bounded on $\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^n)$. Here, $d\sigma$ denotes the Lebesgue measure on $\mathbb{S}^{n-1}$.

Consequently, the Tomas-Stein restriction theorem implies the $\mathbf{L}^p$-norm inequality in (1. 4) for p belonging to (1. 5) with an extra condition of $p\geq\frac{2n+2}{n-1}$ or $p\leq\frac{2n+2}{n+3}$. See Tomas [23] or Chapter IX of Stein [18]. Today, the connection between Fourier restriction theorem and Bochner-Riesz summability is extensively explored. For instance, Tao [21] proves (1. 4)-(1. 5) implying the Restriction conjecture on $\mathbb{S}^{n-1}$. On the other hand, Carbery [5] shows that such implication can be reversed if the unit sphere is replaced by a paraboloid.

A number of remarkable results have been accomplished in this broad area, for example by Bourgain [1]-[2], Bourgain and Guth [3], Guth [8]-[9], Tao and Vargas [22], Guo-et-al [11], Guo, Wang and Zhang [12], Gressman [13]-[14], Lee [16] and Wolff [25].

Our main result is stated in below.

Theorem One *Let* $\mathbf{S}^\delta$ *defined in (1. 1) for* $0<\mathbf{Re}\delta<\frac{1}{2}$. *We have*

$$\left\|\mathbf{S}^\delta f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\delta}\, e^{\mathbf{c}|\mathbf{Im}\delta|} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n}. \tag{1. 7}$$

From (1. 3) and (1. 7), by applying Stein interpolation theorem [19], we obtain (1. 4)-(1. 5).

In order to prove **Theorem One**, we next introduce another convolution operator, denoted by $\mathbf{I}^\alpha$ for $0<\mathbf{Re}\alpha<1$ whose Fourier transform is given in terms of cone multipliers having negative orders. The $\mathbf{L}^p$-boundedness this operator implies **Theorem One**.

Furthermore, we develop two different types of dyadic decomposition respectively on the frequency space and the physical space. Every consisting partial operator will be split into two after a **pairing formulation** between these dyadic decompositions. Each one of them satisfies the desired $\mathbf{L}^p$-estimates.

1.1 Cone multiplier of negative orders

Let $\mathbf{S}^\delta, \mathbf{Re}\delta>0$ defined in (1. 1). Our key observation is

$$\left(1-|\xi|^2\right)_+^\delta = 2\delta\int_0^1 \left(\frac{1}{\tau^2-|\xi|^2}\right)_+^{1-\delta}\tau d\tau, \qquad \mathbf{Re}\delta>0.$$

For $0<\mathbf{Re}\delta<1$, the cone multiplier $\left(\frac{1}{\tau^2-|\xi|^2}\right)_+^{1-\delta}$ having a negative order has been previously investigated by Gelfand and Shilov [10].

Denote $1-\delta=\alpha$ for $\alpha\in\mathbb{C}$. Λ^α is a distribution defined in $\mathbb{R}^{n+1}$ by analytic continuation from

$$\mathbf{Re}\alpha>\tfrac{n-1}{2}, \qquad \Lambda^\alpha(x,t) = \pi^{\alpha-\frac{n+1}{2}}\Gamma^{-1}\left(\alpha-\tfrac{n-1}{2}\right)\left(\frac{1}{t^2-|x|^2}\right)_+^{\frac{n+1}{2}-\alpha}.$$

For $0<\mathbf{Re}\alpha<1$, the Fourier transform of Λ^α agrees with the function

$$\widehat{\Lambda}^\alpha(\xi,\tau) = \pi^{\frac{n-1}{2}-2\alpha}\Gamma(\alpha)\left\{\left(\frac{1}{\tau^2-|\xi|^2}\right)_-^\alpha - \sin\pi\left(\alpha-\tfrac{1}{2}\right)\left(\frac{1}{\tau^2-|\xi|^2}\right)_+^\alpha\right\} \tag{1. 8}$$

whenever $|\tau|\neq|\xi|$. See appendix B.

Let $\varphi\in\mathcal{C}_o^\infty(\mathbb{R})$ be a smooth cut-off function such that $\varphi(t)=1$ if $|t|\leq 1$ and $\varphi(t)=0$ for $|t|>2$. Define

$$\widehat{\phi}(\xi) = \varphi\left(\frac{2}{3}|\xi|\right)-\varphi\left(3|\xi|\right). \tag{1. 9}$$

Note that $\mathbf{supp}\widehat{\phi}\subset\left\{\xi\in\mathbb{R}^n\colon \frac{1}{3}<|\xi|\leq 3\right\}$ and $\widehat{\phi}(\xi)=1$ for $\frac{2}{3}<|\xi|<\frac{3}{2}$.

In order to prove **Theorem One**, we consider

$$\mathbf{I}^\alpha f(x) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\phi}(\xi)\left\{\int_0^1 \widehat{\Lambda}^\alpha(\xi,\tau)\tau d\tau\right\}d\xi. \tag{1. 10}$$

Theorem Two *Let* $\mathbf{I}^\alpha$ *defined in (1. 8)-(1. 10) for* $\frac{1}{2} < \mathbf{Re}\alpha < 1$. *We have*

$$\left\|\mathbf{I}^\alpha f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n}. \tag{1. 11}$$

Remark 1.1 Theorem Two *implies* **Theorem One**.

As investigated by Strichartz [20], Ω^α is a distribution defined in $\mathbb{R}^n$ by analytic continuation from

$$\mathbf{Re}\alpha > \tfrac{n-1}{2}, \qquad \Omega^\alpha(x) = \pi^{\alpha-\frac{n+1}{2}}\Gamma^{-1}\left(\alpha - \tfrac{n-1}{2}\right)\left(\frac{1}{1-|x|^2}\right)_+^{\frac{n+1}{2}-\alpha}.$$

Equivalently, it can be defined by

$$\begin{aligned}\widehat{\Omega}^\alpha(\xi) &= \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-\left[\frac{n+1}{2}-\alpha\right]} \mathbf{J}_{\frac{n}{2}-\left[\frac{n+1}{2}-\alpha\right]}\big(2\pi|\xi|\big)\\ &= \left(\frac{1}{|\xi|}\right)^{\alpha-\frac{1}{2}} \mathbf{J}_{\alpha-\frac{1}{2}}\big(2\pi|\xi|\big), \qquad \alpha \in \mathbb{C}.\end{aligned}$$

In section 3, we show that $\mathbf{I}^\alpha$ defined in (1. 10) for $\frac{1}{2} < \mathbf{Re}\alpha < 1$ can be expressed as

$$\begin{aligned}\mathbf{I}^\alpha f(x) &= \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\phi}(\xi)\left\{\int_{\mathbb{R}} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1}dr\right\}d\xi,\\ \omega(r) &= e^{2\pi\mathbf{i}r}\int_0^1 e^{-2\pi\mathbf{i}\tau r}\tau d\tau = \frac{-1}{2\pi\mathbf{i}}\frac{1}{r} - \frac{1}{4\pi^2r^2}\left[1-e^{-2\pi\mathbf{i}r}\right].\end{aligned} \tag{1. 12}$$

Moreover, define

$$\begin{aligned}\mathbf{I}^\alpha_< f(x) &= \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\mathbf{P}}^\alpha_<(\xi)d\xi,\\ \widehat{\mathbf{P}}^\alpha_<(\xi) &= \widehat{\phi}(\xi)\int_{-1}^1 e^{-2\pi\mathbf{i}r}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1}dr\end{aligned} \tag{1. 13}$$

and

$$\begin{aligned}\mathbf{I}^\alpha_j f(x) &= \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\mathbf{P}}^\alpha_j(\xi)d\xi,\\ \widehat{\mathbf{P}}^\alpha_j(\xi) &= \widehat{\phi}(\xi)\int_{2^{j-1}\leq|r|<2^j} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1}dr, \qquad j>0.\end{aligned} \tag{1. 14}$$

Later, we shall see $\mathbf{P}^\alpha_< \in \mathbf{L}^1(\mathbb{R}^n)$. Hence that $\mathbf{I}^\alpha_<$ is bounded on $\mathbf{L}^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$.

Our aim is to show

$$\left\|\mathbf{I}^\alpha_j f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-\varepsilon j} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n} \tag{1. 15}$$

for every $j > 0$ and some $\varepsilon = \varepsilon(\mathbf{Re}\alpha) > 0$.

Given $\frac{1}{2} < \mathbf{Re}\alpha < 1$, $\mathbf{I}_j^{\alpha\ z}, 0 \leq \mathbf{Re}z \leq 1$ is a family of analytic operators which will be defined explicitly in section 4. In particular, we have $\mathbf{I}_j^{\alpha\ \frac{1}{n}} f = \mathbf{I}_j^{\alpha} f$.

Ideally, we hope to establish (1. 15) by using complex interpolation as follows. Suppose

$$\left\| \mathbf{I}_j^{\alpha\ 0+\mathbf{i}\mathbf{Im}z} f \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \tag{1. 16}$$

and

$$\left\| \mathbf{I}_j^{\alpha\ 1+\mathbf{i}\mathbf{Im}z} f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p \leq \infty \tag{1. 17}$$

for every $j > 0$ and some $\varepsilon = \varepsilon(\mathbf{Re}\alpha) > 0$.

We obtain (1. 15) by applying Stein interpolation theorem [19] and taking into account for $\frac{n-1}{2n} = 0\left(\frac{1}{n}\right) + \frac{1}{2}\left(\frac{n-1}{n}\right)$ and $\frac{n+1}{2n} = 1\left(\frac{1}{n}\right) + \frac{1}{2}\left(\frac{n-1}{n}\right)$.

Unfortunately, the $\mathbf{L}^2$-estimate in (1. 16) is NOT true. The best result can be concluded is

$$\left\| \mathbf{I}_j^{\alpha\ 0+\mathbf{i}\mathbf{Im}z} f \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{j/2} 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^2(\mathbb{R}^n)}, \qquad j > 0. \tag{1. 18}$$

This is consequence of which the Fourier multiplier regarding every $\mathbf{I}_j^{\alpha\ 0+\mathbf{i}\mathbf{Im}z}$ has its norm bounded by

$$\mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \left| \frac{1}{1-|\xi|} \right|^{\frac{1}{2}-\varepsilon}, \qquad |\xi| \approx 1 \pm 2^{-j}.$$

To obtain the $\mathbf{L}^p$-estimate in (1. 15), we first write $\mathbf{I}_j^{\alpha}$ into a group of partial operators.

Given $\frac{1}{2} < \mathbf{Re}\alpha < 1$, denote $0 < \sigma = \sigma(\mathbf{Re}\alpha) < \frac{1}{2}$ for some constant which can be chosen sufficiently small. In particular, we shall find $\sigma \longrightarrow 0$ as $\mathbf{Re}\alpha \longrightarrow 1$.

For every $j > 0$, assert $\lambda_m \in [2^{j-1}, 2^j]$ where $\lambda_0 = 2^{j-1}$, $\lambda_M = 2^j$ and

$$2^{\sigma j-1} \leq \lambda_m - \lambda_{m-1} < 2^{\sigma j}. \tag{1. 19}$$

Remark 1.2 *There are at most a constant multiple of* $2^{(1-\sigma)j}$ *many* λ_m*'s inside* $[2^{j-1}, 2^j]$.

Let $\mathbf{I}_j^{\alpha}$ defined in (1. 14). Consider

$$\begin{gathered}\mathbf{I}_j^{\alpha} f(x) = \sum_{m=1}^{M} \mathbf{I}_{j\,m}^{\alpha} f(x), \qquad \mathbf{I}_{j\,m}^{\alpha} f(x) = \int_{\mathbb{R}^n} e^{2\pi \mathbf{i} x\cdot\xi} \widehat{f}(\xi) \widehat{\mathbf{P}}_{j\,m}^{\alpha}(\xi) d\xi, \\ \widehat{\mathbf{P}}_{j\,m}^{\alpha}(\xi) = \widehat{\phi}(\xi) \int_{\lambda_{m-1} \leq |r| < \lambda_m} e^{-2\pi \mathbf{i} r} \widehat{\Omega}^{\alpha}(r\xi) \omega(r) |r|^{2\alpha-1} dr.\end{gathered} \tag{1. 20}$$

We claim

$$\left\| \mathbf{I}_{j\,m}^{\alpha} f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j}\, 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n} \tag{1. 21}$$

for every $j > 0$, $m = 1, \ldots, M$ and some $\varepsilon = \varepsilon(\mathbf{Re}\alpha) > 0$.

Clearly, (1. 21) and **Remark 1.2** together with Minkowski inequality imply (1. 15).

1.2 A pairing formulation

First, we introduce an ingenious dyadic decomposition initially constructed by Fefferman [6] and later refined by Seeger, Sogge and Stein [17] in their study of Fourier integral operators. Given $j > 0$, $\left\{\xi_j^\nu\right\}_\nu$ is a collection of points almost equally distributed on $\mathbb{S}^{n-1}$ having a grid length between $2^{-j/2-1}$ and $2^{-j/2}$: **(1)** $|\xi_j^\mu - \xi_j^\nu| \geq 2^{-j/2-1}$ for every $\xi_j^\mu \neq \xi_j^\nu$. **(2)** For any $u \in \mathbb{S}^{n-1}$, there is a ξ_j^ν in the open set $\{\xi \in \mathbb{S}^{n-1} : |\xi - u| < 2^{-j/2+1}\}$.

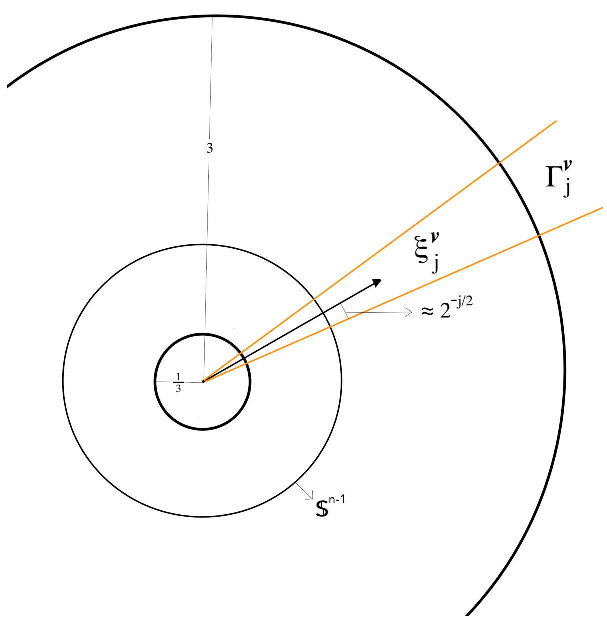
Let $\varphi \in C_o^\infty(\mathbb{R})$ be a smooth cut-off function such that $\varphi(t) = 1$ if $|t| \leq 1$ and $\varphi(t) = 0$ for $|t| > 2$. Define

$$\varphi_j^\nu(\xi) = \frac{\varphi\left[2^{j/2}\left|\frac{\xi}{|\xi|} - \xi_j^\nu\right|\right]}{\sum_{\nu:\xi_j^\nu \in \mathbb{S}^{n-1}} \varphi\left[2^{j/2}\left|\frac{\xi}{|\xi|} - \xi_j^\nu\right|\right]} \tag{1. 22}$$

whose support is contained in the cone

$$\Gamma_j^\nu = \left\{\xi \in \mathbb{R}^n : \left|\frac{\xi}{|\xi|} - \xi_j^\nu\right| < 2^{-j/2+1}\right\}. \tag{1. 23}$$

Observe that by the construction in **(1)-(2)**, the support of $\sum_{\nu:\xi_j^\nu \in \mathbb{S}^{n-1}} \varphi\left[2^{j/2}\left|\frac{\xi}{|\xi|} - \xi_j^\nu\right|\right]$ is $\mathbb{R}^n$. Furthermore, there are at most a constant multiple of $2^{\left(\frac{n-1}{2}\right)j}$ many ξ_j^ν s.



Next, consider

$$\left\{\xi_j^\nu\right\}_\nu = \bigcup_{\ell=1}^{L} \mathcal{Z}_\ell. \tag{1. 24}$$

We select each $\mathcal{Z}_\ell$ from $\left\{\xi_j^\nu\right\}_\nu \setminus \{\mathcal{Z}_1 \cup \cdots \cup \mathcal{Z}_{\ell-1}\}$ as follows.

(a) $|\xi_j^\mu - \xi_j^\nu| \geq \mathbf{c}2^{-j/2+\sigma j}$ whenever $\xi_j^\mu \neq \xi_j^\nu$ belonging $\mathcal{Z}_\ell$. **(b)** If ξ_j^μ satisfies $|\xi_j^\mu - \xi_j^\nu| \geq 2\mathbf{c}2^{-j/2+\sigma j}$ for every selected ξ_j^ν, then ξ_j^μ shall be selected.

Remark 1.3 *Each $\mathcal{Z}_\ell$, $\ell = 1, \ldots, L-1$ consists of approximately $2^{\left[\frac{n-1}{2}-(n-1)\sigma\right]j}$ elements. Moreover, $\mathcal{Z}_L$ at most has this many. There are a constant multiple of $2^{(n-1)\sigma j}$ such $\mathcal{Z}_\ell \subset \left\{\xi_j^\nu\right\}_\nu$.*

Given $\xi_j^\mu \in \mathcal{Z}_\ell$, $\mathbf{L}_\mu$ is an $n\times n$-orthogonal matrix with $\det \mathbf{L}_\mu = 1$ and $\mathbf{L}_\mu^T \xi_j^\mu = (1,0)^T \in \mathbb{R}\times\mathbb{R}^{n-1}$. We define

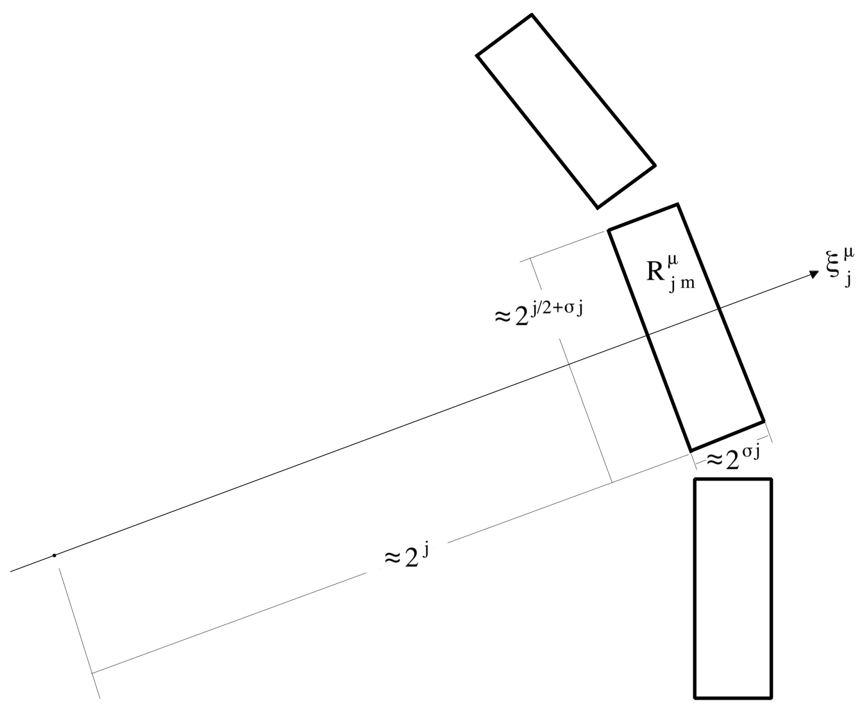
$$\Psi^{\alpha\ \ell}_{j\ m}(x) = \sum_{\mu:\xi_j^\mu \in \mathcal{Z}_\ell} \Psi^{\alpha\ \mu}_{j\ m}(x),$$

$$\Psi^{\alpha\ \mu}_{j\ m}(x) = \varphi\left[2^{-\sigma j-1}\left|\lambda_m - \left(\mathbf{L}_\mu^T x\right)_1\right|\right] \prod_{i=2}^n \varphi\left[2^{-\left[\frac{1}{2}+\sigma\right]j}\left|\left(\mathbf{L}_\mu^T x\right)_i\right|\right]. \tag{1. 25}$$

Each $\Psi^{\alpha\ \mu}_{j\ m}$ is supported in the rectangle

$$\mathrm{R}^\mu_{j\ m} = \left\{x \in \mathbb{R}^n \colon \lambda_m - 2^{\sigma j+2} < \left(\mathbf{L}_\mu^T x\right)_1 < \lambda_m + 2^{\sigma j+2},\ \left|\left(\mathbf{L}_\mu^T x\right)_i\right| < 2^{\left[\frac{1}{2}+\sigma\right]j+1},\ i = 2, \ldots, n\right\}. \tag{1. 26}$$

Moreover, $\mathrm{R}^\mu_{j\ m} \cap \mathrm{R}^\nu_{j\ m} = \emptyset$ if $\xi_j^\mu \neq \xi_j^\nu$ inside $\mathcal{Z}_\ell$ whereas $\left|\xi_j^\mu - \xi_j^\nu\right| \geq \mathbf{c}2^{-j/2}2^{\sigma j}$ for $\mathbf{c}$ large.



For every $j>0$ and $m=1,\ldots,M$, we assert

$$\widehat{\mathbf{P}}^{\alpha}_{j\,m}(\xi) = \sum_{\ell=1}^{L}\widehat{\mathbf{P}}^{\alpha\,\ell}_{j\,m}(\xi), \qquad \widehat{\mathbf{P}}^{\alpha\,\ell}_{j\,m}(\xi) = \sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell}}\widehat{\mathbf{P}}^{\alpha\,\nu}_{j\,m}(\xi),$$
$$\widehat{\mathbf{P}}^{\alpha\,\nu}_{j\,m}(\xi) = \varphi^{\nu}_{j}(\xi)\widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^{\alpha}(r\xi)\omega(r)|r|^{2\alpha-1}dr. \tag{1. 27}$$

Recall $\mathbf{I}^{\alpha}_{j\,m}f=f*\mathbf{P}^{\alpha}_{j\,m}$ from (1. 20). We have

$$\mathbf{I}^{\alpha}_{j\,m}f(x) = \sum_{\ell=1}^{L}\mathbf{I}^{\alpha\,\ell}_{j\,m}f(x), \qquad \mathbf{I}^{\alpha\,\ell}_{j\,m}f(x) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\mathbf{P}}^{\alpha\,\ell}_{j\,m}(\xi)d\xi. \tag{1. 28}$$

Let $\Psi^{\alpha\,\ell}_{j\,m}$ defined in (1. 25). Consider

$$\mathbf{II}^{\alpha\,\ell}_{j\,m}f(x) = \int_{\mathbb{R}^n} f(x-u)\mathbf{U}^{\alpha\,\ell}_{j\,m}(u)du,$$
$$\mathbf{U}^{\alpha\,\ell}_{j\,m}(u) = \mathbf{P}^{\alpha\,\ell}_{j\,m}(u)\left[1-\Psi^{\alpha\,\ell}_{j\,m}(u)\right] + \sum_{\mu:\xi^{\mu}_{j}\in\mathcal{Z}_{\ell}}\Psi^{\alpha\,\mu}_{j\,m}(u)\sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell},\xi^{\nu}_{j}\neq\xi^{\mu}_{j}}\mathbf{P}^{\alpha\,\nu}_{j\,m}(u) \tag{1. 29}$$

and

$$\mathbf{III}^{\alpha\,\ell}_{j\,m}f(x) = \int_{\mathbb{R}^n} f(x-u)\mathbf{V}^{\alpha\,\ell}_{j\,m}(u)du,$$
$$\mathbf{V}^{\alpha\,\ell}_{j\,m}(u) = \sum_{\mu:\xi^{\mu}_{j}\in\mathcal{Z}_{\ell}}\Psi^{\alpha\,\mu}_{j\,m}(u)\sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell},\xi^{\nu}_{j}=\xi^{\mu}_{j}}\mathbf{P}^{\alpha\,\nu}_{j\,m}(u). \tag{1. 30}$$

From (1. 29)-(1. 30), we find

$$\begin{aligned}\mathbf{U}^{\alpha\,\ell}_{j\,m}(x)+\mathbf{V}^{\alpha\,\ell}_{j\,m}(x) &= \mathbf{P}^{\alpha\,\ell}_{j\,m}(x)\left[1-\Psi^{\alpha\,\ell}_{j\,m}(x)\right] + \sum_{\mu:\xi^{\mu}_{j}\in\mathcal{Z}_{\ell}}\Psi^{\alpha\,\mu}_{j\,m}(x)\sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell}}\mathbf{P}^{\alpha\,\nu}_{j\,m}(x)\\ &= \mathbf{P}^{\alpha\,\ell}_{j\,m}(x)\left[1-\Psi^{\alpha\,\ell}_{j\,m}(x)\right] + \Psi^{\alpha\,\ell}_{j\,m}(x)\mathbf{P}^{\alpha\,\ell}_{j\,m}(x)\\ &= \mathbf{P}^{\alpha\,\ell}_{j\,m}(x)\end{aligned}$$

and therefore $\mathbf{I}^{\alpha\,\ell}_{j\,m}f=\mathbf{II}^{\alpha\,\ell}_{j\,m}f+\mathbf{III}^{\alpha\,\ell}_{j\,m}f$.

In section 4, we will explicitly define

$$\mathbf{I}^{\alpha\,\ell\,z}_{j\,m}f(x) = \mathbf{II}^{\alpha\,\ell\,z}_{j\,m}f(x)+\mathbf{III}^{\alpha\,\ell\,z}_{j\,m}f(x), \qquad 0\leq\mathbf{Re}z\leq 1$$

of which $\mathbf{II}^{\alpha\,\ell\,z}_{j\,m}$ and $\mathbf{III}^{\alpha\,\ell\,z}_{j\,m}$ are two families of analytic operators . In particular,

$$\mathbf{II}^{\alpha\,\ell\,\frac{1}{n}}_{j\,m}f(x) = \mathbf{II}^{\alpha\,\ell}_{j\,m}f(x), \qquad \mathbf{III}^{\alpha\,\ell\,\frac{1}{n}}_{j\,m}f(x) = \mathbf{III}^{\alpha\,\ell}_{j\,m}f(x).$$

• Given $j > 0$, $m = 1, \ldots, M$ and $\ell = 1, \ldots, L$, we have

$$\left\| \mathbf{II}_{j\,m}^{\alpha\,\ell\,0+\mathbf{iIm}z} f \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{j/2} 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \tag{1. 31}$$

and

$$\left\| \mathbf{III}_{j\,m}^{\alpha\,\ell\,0+\mathbf{iIm}z} f \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^2(\mathbb{R}^n)} \tag{1. 32}$$

for some $\varepsilon = \varepsilon(\mathbf{Re}\alpha) > 0$.

Observe that at $\mathbf{Re}z = 0$, $\mathbf{III}_{j\,m}^{\alpha\,\ell\,z}$ is a *good* partial operator of $\mathbf{I}_{j\,m}^{\alpha\,\ell\,z}$ whose $\mathbf{L}^2$-norm is bound by $\mathfrak{B}_{\mathbf{Re}\alpha} e^{\mathbf{c}|\mathbf{Im}\alpha|} 2^{-(1-\sigma)j} 2^{-\varepsilon j}$. On the other hand, $\mathbf{II}_{j\,m}^{\alpha\,\ell\,z}$ is the *bad* one, satisfying (1. 31).

• Given $j > 0$, $m = 1, \ldots, M$ and $\ell = 1, \ldots, L$, we find

$$\left\| \mathbf{II}_{j\,m}^{\alpha\,\ell\,1+\mathbf{iIm}z} f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{N\,\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-Nj}\, 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p \leq \infty \tag{1. 33}$$

and

$$\left\| \mathbf{III}_{j\,m}^{\alpha\,\ell\,1+\mathbf{iIm}z} f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1 \leq p \leq \infty \tag{1. 34}$$

for every $N \geq 0$ and some $\varepsilon = \varepsilon(\mathbf{Re}\alpha) > 0$.

At $\mathbf{Re}z = 1$, $\mathbf{III}_{j\,m}^{\alpha\,\ell\,z}$ is a *bad* partial operator of $\mathbf{I}_{j\,m}^{\alpha\,\ell\,z}$. The best can be concluded is that it satisfies (1. 34). On the other hand, $\mathbf{II}_{j\,m}^{\alpha\,\ell\,z}$ becomes the *good* one whose $\mathbf{L}^p$-norm is bounded by $\mathfrak{B}_{N\,\mathbf{Re}\alpha} e^{\mathbf{c}|\mathbf{Im}\alpha|} 2^{-(1-\sigma)j} 2^{-Nj} 2^{-\varepsilon j}$ for every $N \geq 0$.

Recall $\mathbf{II}_{j\,m}^{\alpha\,\ell\,\frac{1}{n}} = \mathbf{II}_{j\,m}^{\alpha\,\ell}$ and $\mathbf{III}_{j\,m}^{\alpha\,\ell\,\frac{1}{n}} = \mathbf{III}_{j\,m}^{\alpha\,\ell}$. Take into account for $\frac{n-1}{2n} = 0\left(\frac{1}{n}\right) + \frac{1}{2}\left(\frac{n-1}{n}\right)$ and $\frac{n+1}{2n} = 1\left(\frac{1}{n}\right) + \frac{1}{2}\left(\frac{n-1}{n}\right)$.

From (1. 31) and (1. 33) with N chosen sufficiently large, we have

$$\left\| \mathbf{II}_{j\,m}^{\alpha\,\ell} f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n} \tag{1. 35}$$

by applying Stein interpolation theorem [19].

From (1. 32) and (1. 34), Stein interpolation theorem [19] again implies

$$\left\| \mathbf{III}_{j\,m}^{\alpha\,\ell} f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n}. \tag{1. 36}$$

Recall (1. 28) and **Remark 1.3**: $\mathbf{I}_{j\,m}^{\alpha} = \sum_{\ell=1}^{L} \mathbf{I}_{j\,m}^{\alpha\,\ell}$ and L is at most a constant multiple of $2^{(n-1)\sigma j}$. By putting together (1. 35)-(1. 36) and applying Minkowski inequality, we find

$$\begin{gathered} \left\| \mathbf{I}_{j\,m}^{\alpha} f \right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{(n-1)\sigma j} 2^{-\varepsilon j}\, \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \\ \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n}. \end{gathered} \tag{1. 37}$$

For σ chosen sufficiently small, i.e: $n\sigma < \frac{1}{2}\varepsilon$, we obtain (1. 21) as desired.

The remaining paper is organized as follows. In section 2, we show **Theorem Two** implying **Theorem One**. In section 3, after a formal derivation of $\mathbf{I}^{\alpha\,\ell}_{j\,m} = \mathbf{II}^{\alpha\,\ell}_{j\,m} + \mathbf{III}^{\alpha\,\ell}_{j\,m}$, we introduce two end-point estimates associated with $\mathbf{Re}z = 0$ and $\mathbf{Re}z = 1$ in (1. 31)-(1. 34), stated as **Proposition One** and **Proposition Two**. By using these results, we finish the proof of **Theorem Two** in section 4. We prove **Proposition One** in section 5. Section 6 is devoted to certain preliminaries regarding **Proposition Two**. We prove **Proposition Two** in section 7. Two appendices added in the end for the sake of self-containedness.

2 Theorem Two implies Theorem One

Let $0 < \mathbf{Re}\delta < \frac{1}{2}$. Consider

$$\mathbf{S}^{\delta}_{\psi} f(x) = \int_{\mathbb{R}^n} e^{2\pi \mathbf{i} x\cdot\xi} \widehat{f}(\xi)\widehat{\psi}(\xi)\left(1-|\xi|^2\right)^{\delta}_{+} d\xi \tag{2. 1}$$

where $\widehat{\psi}(\xi) = \left[\widehat{\phi}(\xi)\right]^2$ and $\widehat{\phi}$ is defined in (1. 9).

Observe that $\left[1-\widehat{\psi}(\xi)\right]\left(1-|\xi|^2\right)^{\delta}_{+}$ is a $\mathbf{L}^p$-Fourier multiplier for $1 < p < \infty$. In order to prove **Theorem One**, it is suffice to show

$$\left\|\mathbf{S}^{\delta}_{\psi} f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\delta}\, e^{\mathbf{c}|\mathbf{Im}\delta|} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} \leq \frac{1}{p} \leq \frac{n+1}{2n}. \tag{2. 2}$$

Let $1-\delta = \alpha \in \mathbb{C}$. We define

$$\begin{aligned}
\mathbf{m}^{\alpha}_{+}(\xi) &= \widehat{\phi}(\xi) \int_0^1 \left(\frac{1}{\tau^2-|\xi|^2}\right)^{\alpha}_{+} \tau d\tau \\
&= \widehat{\phi}(\xi) \begin{cases} \displaystyle\int_{|\xi|}^1 \left(\frac{1}{\tau^2-|\xi|^2}\right)^{\alpha} \tau d\tau, & |\xi| < 1 \\ 0 & |\xi| \geq 1 \end{cases} \\
&= \frac{1}{2}(1-\alpha)^{-1}\widehat{\phi}(\xi)\left(1-|\xi|^2\right)^{1-\alpha}_{+}
\end{aligned} \tag{2. 3}$$

and

$$\begin{aligned}
\mathbf{m}^{\alpha}_{-}(\xi) &= \widehat{\phi}(\xi) \int_0^1 \left(\frac{1}{\tau^2-|\xi|^2}\right)^{\alpha}_{-} \tau d\tau \\
&= (-1)^{-\alpha}\widehat{\phi}(\xi) \begin{cases} \displaystyle\int_0^{|\xi|} \left(\frac{1}{\tau^2-|\xi|^2}\right)^{\alpha} \tau d\tau, & |\xi| \leq 1 \\ \displaystyle\int_0^1 \left(\frac{1}{\tau^2-|\xi|^2}\right)^{\alpha} \tau d\tau, & |\xi| > 1 \end{cases} \\
&= \frac{1}{2}(1-\alpha)^{-1}\widehat{\phi}(\xi)\left[|\xi|^{2(1-\alpha)} - \left(1-|\xi|^2\right)^{1-\alpha}_{-}\right].
\end{aligned} \tag{2. 4}$$

Recall $\widehat{\Lambda}^\alpha(\xi,\tau)$ given at (1. 8). From (2. 3)-(2. 4), we find

$$\widehat{\phi}(\xi)\int_0^1 \widehat{\Lambda}^\alpha(\xi,\tau)\tau d\tau ~=~ \pi^{\frac{n-1}{2}-2\alpha}\Gamma(\alpha)\Big\{\mathbf{m}_-^\alpha(\xi)-\sin\pi\left(\alpha-\tfrac{1}{2}\right)\mathbf{m}_+^\alpha(\xi)\Big\}. \tag{2. 5}$$

Moreover, define

$$\mathbf{m}^\alpha(\xi) ~=~ \widehat{\phi}(\xi)\Big\{-\left(1-|\xi|^2\right)_-^{1-\alpha}-\sin\pi\left(\alpha-\tfrac{1}{2}\right)\left(1-|\xi|^2\right)_+^{1-\alpha}\Big\}. \tag{2. 6}$$

From (2. 3)-(2. 4), (2. 5) and (2. 6), we have

$$\widehat{\phi}(\xi)\int_0^1 \widehat{\Lambda}^\alpha(\xi,\tau)\tau d\tau ~=~ \pi^{\frac{n-2}{2}-2\alpha}\frac{1}{2}(1-\alpha)^{-1}\Gamma(\alpha)\Big[\mathbf{m}^\alpha(\xi)+\widehat{\phi}(\xi)|\xi|^{2(1-\alpha)}\Big]. \tag{2. 7}$$

Remark 2.1 *Let $\mathbf{T}_z f(x) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\mathbf{m}^z(\xi)d\xi$, $z\in\mathbb{C}$. We call $\mathbf{m}^z(\xi)$ a restricted $\mathbf{L}^p$-Fourier multiplier if*

$$\left\|\mathbf{T}_z f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} ~\leq~ \mathfrak{B}_{\mathbf{Re}z}\, e^{\mathbf{c}|\mathbf{Im}z|}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad \frac{n-1}{2n} ~\leq~ \frac{1}{p} ~\leq~ \frac{n+1}{2n}.$$

Recall $\mathbf{I}^\alpha$ defined in (1. 10). Let $\frac{1}{2}<\mathbf{Re}\alpha<1$. **Theorem Two** states that $\mathbf{I}^\alpha f$ satisfies the $\mathbf{L}^p$-norm inequality in (1. 11). Therefore, the left-hand-side of the equation in (2. 7) is a restricted $\mathbf{L}^p$-Fourier multiplier. Because $\widehat{\phi}(\xi)|\xi|^{2(1-\alpha)}$ is a $\mathbf{L}^p$-Fourier multiplier for $1<p<\infty$, we find $\mathbf{m}^\alpha(\xi)$ as a restricted $\mathbf{L}^p$-Fourier multiplier.

Consider

$$\widehat{\phi}(\xi)\mathbf{m}^\alpha(\xi) ~=~ \widehat{\psi}(\xi)\Big\{-\left(1-|\xi|^2\right)_-^{1-\alpha}-\sin\pi\left(\alpha-\tfrac{1}{2}\right)\left(1-|\xi|^2\right)_+^{1-\alpha}\Big\}$$

and

$$\mathbf{m}^{\frac{1}{2}+\frac{\alpha}{2}}(\xi) ~=~ \widehat{\phi}(\xi)\Big\{-\left(1-|\xi|^2\right)_-^{\frac{1}{2}-\frac{\alpha}{2}}-\sin\pi\left(\tfrac{\alpha}{2}\right)\left(1-|\xi|^2\right)_+^{\frac{1}{2}-\frac{\alpha}{2}}\Big\}.$$

Clearly, both $\widehat{\phi}(\xi)\mathbf{m}^\alpha(\xi)$ and $\mathbf{m}^{\frac{1}{2}+\frac{\alpha}{2}}(\xi)$ are restricted $\mathbf{L}^p$-Fourier multipliers. Furthermore,

$$\Big[\mathbf{m}^{\frac{1}{2}+\frac{\alpha}{2}}(\xi)\Big]^2 ~=~ \widehat{\psi}(\xi)\Big\{\left(1-|\xi|^2\right)_-^{1-\alpha}+\sin^2\pi\left(\tfrac{\alpha}{2}\right)\left(1-|\xi|^2\right)_+^{1-\alpha}\Big\} \tag{2. 8}$$

is indeed another restricted $\mathbf{L}^p$-Fourier multiplier.

By adding $\Big[\mathbf{m}^{\frac{1}{2}+\frac{\alpha}{2}}(\xi)\Big]^2$ and $\widehat{\phi}(\xi)\mathbf{m}^\alpha(\xi)$ together, we obtain

$$\widehat{\psi}(\xi)\Big[\sin^2\pi\left(\tfrac{\alpha}{2}\right)-\sin\pi\left(\alpha-\tfrac{1}{2}\right)\Big]\left(1-|\xi|^2\right)_+^{1-\alpha}. \tag{2. 9}$$

From direct computation, we find

$$\begin{aligned}\sin^2\pi\left(\tfrac{\alpha}{2}\right) ~&=~ \frac{1}{2}-\frac{1}{4}\left[e^{-\pi\mathbf{Im}\alpha}+e^{\pi\mathbf{Im}\alpha}\right]\cos\pi\mathbf{Re}\alpha-\frac{\mathbf{i}}{4}\left[e^{-\pi\mathbf{Im}\alpha}-e^{\pi\mathbf{Im}\alpha}\right]\sin\pi\mathbf{Re}\alpha,\\ \sin\pi\left(\alpha-\tfrac{1}{2}\right) ~&=~ -\frac{1}{2}\left[e^{-\pi\mathbf{Im}\alpha}+e^{\pi\mathbf{Im}\alpha}\right]\cos\pi\mathbf{Re}\alpha-\frac{\mathbf{i}}{2}\left[e^{-\pi\mathbf{Im}\alpha}-e^{\pi\mathbf{Im}\alpha}\right]\sin\pi\mathbf{Re}\alpha.\end{aligned} \tag{2. 10}$$

Observe that $\sin^2\pi\left(\frac{\alpha}{2}\right)-\sin\pi\left(\alpha-\frac{1}{2}\right)\neq 0$ for $0<\mathbf{Re}\alpha<1$ as a consequence of (2. 10). Together, we conclude $\widehat{\psi}(\xi)\left(1-|\xi|^2\right)_+^{1-\alpha}$ to be a restricted $\mathbf{L}^p$-Fourier multiplier as desired.

3 End-point estimates regarding $\mathbf{II}^{\alpha\,\ell}_{j\,m}$ and $\mathbf{III}^{\alpha\,\ell}_{j\,m}$

Recall $\mathbf{I}^\alpha$ defined in (1. 10). First, we show $\mathbf{I}^\alpha f$ for $\frac{1}{2}<\mathbf{Re}\alpha<1$ can be expressed as (1. 12). Ω^α is a distribution defined in $\mathbb{R}^n$ by analytic continuation from

$$\mathbf{Re}\alpha>\tfrac{n-1}{2},\qquad \Omega^\alpha(x) \;=\; \pi^{\alpha-\frac{n+1}{2}}\Gamma^{-1}\left(\alpha-\tfrac{n-1}{2}\right)\left(\frac{1}{1-|x|^2}\right)_+^{\frac{n+1}{2}-\alpha}.$$

Equivalently, it can be defined by

$$\begin{aligned}\widehat{\Omega}^\alpha(\xi) \;&=\; \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-\left[\frac{n+1}{2}-\alpha\right]}\mathbf{J}_{\frac{n}{2}-\left[\frac{n+1}{2}-\alpha\right]}\Big(2\pi|\xi|\Big)\\ &=\; \left(\frac{1}{|\xi|}\right)^{\alpha-\frac{1}{2}}\mathbf{J}_{\alpha-\frac{1}{2}}\Big(2\pi|\xi|\Big),\qquad \alpha\in\mathbb{C}.\end{aligned}\tag{3. 1}$$

See (A. 7)-(A. 8). By using the integral formula of Bessel functions in (A. 1), we find

$$\widehat{\Omega}^\alpha(\xi) \;=\; \pi^{\alpha-1}\Gamma(\alpha)\int_{-1}^{1}e^{2\pi\mathbf{i}|\xi|s}(1-s^2)^{\alpha-1}ds.\tag{3. 2}$$

On the other hand, Λ^α is a distribution defined in $\mathbb{R}^{n+1}$ by analytic continuation from

$$\mathbf{Re}\alpha>\tfrac{n-1}{2},\qquad \Lambda^\alpha(x,t) \;=\; \pi^{\alpha-\frac{n+1}{2}}\Gamma^{-1}\left(\alpha-\tfrac{n-1}{2}\right)\left(\frac{1}{t^2-|x|^2}\right)_+^{\frac{n+1}{2}-\alpha}.\tag{3. 3}$$

Let h be a Schwartz function defined in $\mathbb{R}^{n+1}$. We have

$$\begin{aligned}h*\Lambda^\alpha(x,t) \;&=\; \pi^{\alpha-\frac{n+1}{2}}\Gamma^{-1}\left(\alpha-\tfrac{n-1}{2}\right)\iint_{|u|<|r|}h(x-u,t-r)\left(\frac{1}{r^2-|u|^2}\right)^{\frac{n+1}{2}-\alpha}dudr\\ &=\; \iint_{\mathbb{R}^{n+1}}h(x-u,t-r)\Omega^\alpha\left(\frac{u}{r}\right)|r|^{2\alpha-1-n}dudr\\ &=\; \int_{\mathbb{R}}\left\{\int_{\mathbb{R}^n}e^{2\pi\mathbf{i}x\cdot\xi}\widehat{h}(\xi,t-r)\widehat{\Omega}^\alpha(r\xi)d\xi\right\}|r|^{2\alpha-1}dr\end{aligned}\tag{3. 4}$$

whenever $\mathbf{Re}\alpha>\frac{n-1}{2}$. Here, $\widehat{h}(\cdot,t-r)$ is the Fourier transform of $h(x,t-r)$ in $x\in\mathbb{R}^n$.

Remark 3.1 *By the principle of analytic continuation, we must have (3. 4) hold for every* $\mathbf{Re}\alpha>0$.

Let $0<\mathbf{Re}\alpha<1$. From (3. 4), we find

$$h*\Lambda^\alpha(x,t) \;=\; \lim_{N\longrightarrow\infty}\iint_{\mathbb{R}^{n+1}}e^{2\pi\mathbf{i}\left[x\cdot\xi+t\tau\right]}\widehat{h}(\xi,\tau)\left\{\int_{-N}^{N}e^{-2\pi\mathbf{i}\tau r}\widehat{\Omega}^\alpha(r\xi)|r|^{2\alpha-1}dr\right\}d\xi d\tau.$$

By using the asymptotic expansion of Bessel functions in (A. 2)-(A. 4), together with (3. 2), we write

$$
\begin{aligned}
\int_{\mathbb{R}} e^{-2\pi \mathbf{i}\tau r}\widehat{\Omega}^{\alpha}(r\xi)|r|^{2\alpha-1}dr &= \int_{\mathbb{R}} e^{-2\pi \mathbf{i}\tau r}\left(\frac{1}{|r\xi|}\right)^{\alpha-\frac{1}{2}} \mathbf{J}_{\alpha-\frac{1}{2}}\Big(2\pi|r\xi|\Big)|r|^{2\alpha-1}dr \qquad \text{by (3. 1)}\\
&= \frac{1}{\pi}\int_{\mathbb{R}} e^{-2\pi \mathbf{i}\tau r}\left(\frac{1}{|r\xi|}\right)^{\alpha}\cos\left[2\pi|r\xi|-\frac{\pi}{2}\alpha\right]|r|^{2\alpha-1}dr \\
&+ \int_{\mathbb{R}} e^{-2\pi \mathbf{i}\tau r}\mathcal{E}^{\alpha}\left(2\pi|r\xi|\right)|r|^{2\alpha-1}dr,
\end{aligned}
\tag{3. 5}
$$

$$
\left|\mathcal{E}^{\alpha}(\rho)\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\begin{cases} \rho^{-\mathbf{Re}\alpha}, & 0<\rho\leq 1,\\ \rho^{-\mathbf{Re}\alpha-1}, & \rho>1.\end{cases}
$$

Consider $\frac{1}{10}<|\xi|\leq 10$. The norm estimate for $\mathcal{E}^{\alpha}$ implies

$$
\left|\int_{\mathbb{R}} e^{-2\pi \mathbf{i}\tau r}\mathcal{E}^{\alpha}\left(2\pi|r\xi|\right)|r|^{2\alpha-1}dr\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}.
$$

Because of Euler's formulae, we replace the cosine function in (3. 5) with $e^{-\mathbf{i}\left(\frac{\pi}{2}\right)\alpha}e^{2\pi\mathbf{i}|r\xi|}$ or $e^{\mathbf{i}\left(\frac{\pi}{2}\right)\alpha}e^{-2\pi\mathbf{i}|r\xi|}$. By integration by parts *w.r.t* r, we find

$$
\begin{aligned}
&\left|\int_{2^{j-1}}^{2^j} e^{2\pi\mathbf{i}\left[|\xi|-\tau\right]r}\left(\frac{1}{|r\xi|}\right)^{\alpha}|r|^{2\alpha-1}dr\right| \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\begin{cases} 2^{\mathbf{Re}\alpha j}, & 2^j\leq \Big||\tau|-|\xi|\Big|^{-1},\\ \left|\dfrac{1}{|\tau|-|\xi|}\right| 2^{\left[\mathbf{Re}\alpha-1\right]j}, & 2^j> \Big||\tau|-|\xi|\Big|^{-1}.\end{cases}
\end{aligned}
\tag{3. 6}
$$

For symmetry reason, (3. 6) further implies

$$
\begin{aligned}
&\left|\int_{\mathbb{R}} e^{-2\pi \mathbf{i}\tau r}\left(\frac{1}{|r\xi|}\right)^{\alpha}\cos\left[2\pi|r\xi|-\frac{\pi}{2}\alpha\right]|r|^{2\alpha-1}dr\right| \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\sum_{2^j\leq ||\tau|-|\xi||^{-1}} 2^{\mathbf{Re}\alpha j} + \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\sum_{2^j> ||\tau|-|\xi||^{-1}}\left|\frac{1}{|\tau|-|\xi|}\right| 2^{\left[\mathbf{Re}\alpha-1\right]j} \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\left|\frac{1}{|\tau|-|\xi|}\right|^{\mathbf{Re}\alpha}.
\end{aligned}
\tag{3. 7}
$$

On the other hand, the Fourier transform of Λ^{α} defined by analytic continuation from (3. 3) agrees with the function

$$
\widehat{\Lambda}^{\alpha}(\xi,\tau) = \pi^{\frac{n-1}{2}-2\alpha}\Gamma(\alpha)\left\{\left(\frac{1}{\tau^2-|\xi|^2}\right)_{-}^{\alpha} - \sin\pi\left(\alpha-\tfrac{1}{2}\right)\left(\frac{1}{\tau^2-|\xi|^2}\right)_{+}^{\alpha}\right\}
$$

whenever $|\tau|\neq|\xi|$. See appendix B.

From (3. 4) to (3. 7), by applying Lebesgue's dominated convergence theorem, we have

$$
\begin{aligned}
h * \Lambda^\alpha(0,0) &= \lim_{N\longrightarrow\infty} \iint_{\mathbb{R}^{n+1}} \widehat{h}(\xi,\tau)\left\{\int_{-N}^{N} e^{-2\pi\mathbf{i}\tau r}\widehat{\Omega}^\alpha(r\xi)|r|^{2\alpha-1}dr\right\}d\xi d\tau \\
&= \iint_{\mathbb{R}^{n+1}} \widehat{h}(\xi,\tau)\left\{\int_{\mathbb{R}} e^{-2\pi\mathbf{i}\tau r}\widehat{\Omega}^\alpha(r\xi)|r|^{2\alpha-1}dr\right\}d\xi d\tau \qquad (3.\ 8) \\
&= \iint_{\mathbb{R}^{n+1}} \widehat{h}(\xi,\tau)\widehat{\Lambda}^\alpha(\xi,\tau)d\xi d\tau
\end{aligned}
$$

for every h defined in $\mathbb{R}^{n+1}$ whose Fourier transform is supported in the cylindrical region: $\left\{(\xi,\tau)\in\mathbb{R}^n\times\mathbb{R}\colon \frac{1}{10} < |\xi| \leq 10\right\}$.

Let $\varphi \in C_o^\infty(\mathbb{R})$ such that $\varphi(t) = 1$ if $|t| \leq 1$ and $\varphi(t) = 0$ if $|t| > 2$. Denote $\mathbf{c}_\varphi^{-1} = \int_{\mathbb{R}} \varphi(t)dt$. Given $(\eta, s) \in \mathbb{R}^n \times \mathbb{R}$ for $|s| \neq |\eta|$ and $\frac{1}{3} \leq |\eta| \leq 3$, we consider a family of *good kernels*: $\widehat{h_\varepsilon}(\xi,\tau) = \mathbf{c}_\varphi \varepsilon^{-(n+1)}\varphi\left[\varepsilon^{-1}\sqrt{|\xi-\eta|^2+(\tau-s)^2}\right]$, $0<\varepsilon<1/10$.

Replace $\widehat{h}(\xi,\tau)$ by $\widehat{h_\varepsilon}(\xi,\tau)$ in (3. 8). By taking $\varepsilon \longrightarrow 0$, we find

$$
\widehat{\Lambda}^\alpha(\xi,\tau) = \int_{\mathbb{R}} e^{-2\pi\mathbf{i}\tau r}\widehat{\Omega}^\alpha(r\xi)|r|^{2\alpha-1}dr, \qquad |\tau|\neq|\xi|, \qquad \tfrac{1}{3} < |\xi| \leq 3. \qquad (3.\ 9)
$$

By applying Lebesgue's dominated convergence theorem again, we have

$$
\widehat{\mathbf{I}^\alpha f}(\xi) = \widehat{f}(\xi)\widehat{\mathbf{P}}^\alpha(\xi),
$$

$$
\begin{aligned}
\widehat{\mathbf{P}}^\alpha(\xi) &= \widehat{\phi}(\xi)\int_0^1 \widehat{\Lambda}^\alpha(\xi,\tau)\tau d\tau \\
&= \widehat{\phi}(\xi)\int_{0<\tau<1,\ \tau\neq|\xi|}\left\{\int_{\mathbb{R}} e^{-2\pi\mathbf{i}\tau r}\widehat{\Omega}^\alpha(r\xi)|r|^{2\alpha-1}dr\right\}\tau d\tau \qquad \text{by (3. 9)} \\
&= \widehat{\phi}(\xi)\lim_{N\longrightarrow\infty}\int_{0<\tau<1,\ \tau\neq|\xi|}\left\{\int_{-N}^{N} e^{-2\pi\mathbf{i}\tau r}\widehat{\Omega}^\alpha(r\xi)|r|^{2\alpha-1}dr\right\}\tau d\tau \qquad (3.\ 10) \\
&= \widehat{\phi}(\xi)\int_{\mathbb{R}}\left\{\int_0^1 e^{-2\pi\mathbf{i}\tau r}\tau d\tau\right\}\widehat{\Omega}^\alpha(r\xi)|r|^{2\alpha-1}dr \\
&= \widehat{\phi}(\xi)\int_{\mathbb{R}} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1}dr,
\end{aligned}
$$

$$
\omega(r) = e^{2\pi\mathbf{i}r}\int_0^1 e^{-2\pi\mathbf{i}\tau r}\tau d\tau = \frac{-1}{2\pi\mathbf{i}}\frac{1}{r} - \frac{1}{4\pi^2r^2}\left[1-e^{-2\pi\mathbf{i}r}\right].
$$

Note that $\widehat{\phi}$ defined in (1. 9) has $\mathbf{supp}\widehat{\phi} \subset \left\{\xi\in\mathbb{R}^n\colon \frac{1}{3} < |\xi| \leq 3\right\}$.

Consider

$$\widehat{\mathbf{P}^\alpha}(\xi) = \widehat{\mathbf{P}^\alpha_<}(\xi) + \sum_{j>0} \widehat{\mathbf{P}^\alpha_j}(\xi),$$

$$\widehat{\mathbf{P}^\alpha_<}(\xi) = \widehat{\phi}(\xi) \int_{-1}^{1} e^{-2\pi \mathbf{i} r} \widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1} dr, \tag{3. 11}$$

$$\widehat{\mathbf{P}^\alpha_j}(\xi) = \widehat{\phi}(\xi) \int_{2^{j-1}\le|r|<2^j} e^{-2\pi \mathbf{i} r} \widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1} dr, \qquad j>0.$$

Furthermore, we assert

$$\widehat{\mathbf{P}^\alpha_j}(\xi) = \sum_{m=1}^{M} \widehat{\mathbf{P}^\alpha_{j\,m}}(\xi),$$

$$\widehat{\mathbf{P}^\alpha_{j\,m}}(\xi) = \widehat{\phi}(\xi) \int_{\lambda_{m-1}\le|r|<\lambda_m} e^{-2\pi \mathbf{i} r} \widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1} dr, \tag{3. 12}$$

$$\lambda_m \in [2^{j-1}, 2^j], \qquad \lambda_0 = 2^{j-1},\ \lambda_M = 2^j \ \text{ and } \ 2^{\sigma j-1} \le \lambda_m - \lambda_{m-1} < 2^{\sigma j}$$

where $0<\sigma=\sigma(\mathbf{Re}\alpha)<\frac{1}{2}$ can be chosen sufficiently small.

By using (3. 2), we find

$$\begin{aligned} \mathbf{P}^\alpha_<(x) &= \int_{\mathbb{R}^n} e^{2\pi \mathbf{i} x\cdot\xi} \widehat{\phi}(\xi) \left\{ \int_{-1}^{1} e^{-2\pi \mathbf{i} r} \widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1} dr \right\} d\xi \\ &= \pi^{\alpha-1}\Gamma(\alpha) \\ &\quad \int_{\mathbb{R}^n} e^{2\pi \mathbf{i} x\cdot\xi} \widehat{\phi}(\xi) \left\{ \int_{-1}^{1} e^{-2\pi \mathbf{i} r} \left\{ \int_{-1}^{1} e^{2\pi \mathbf{i} |r\xi| s} (1-s^2)^{\alpha-1} ds \right\} \omega(r)|r|^{2\alpha-1} dr \right\} d\xi. \end{aligned} \tag{3. 13}$$

An N-fold integration by parts *w.r.t* ξ inside (3. 13) shows $\left|\mathbf{P}^\alpha_<(x)\right| \le \mathfrak{B}_{N\ \mathbf{Re}\alpha} e^{\mathbf{c}|\mathbf{Im}\alpha|} \left(\frac{1}{1+|x|}\right)^N$. Hence that $\mathbf{P}^\alpha_<$ is an $\mathbf{L}^1$-function in $\mathbb{R}^n$.

Given $j>0$, $\left\{\xi^\nu_j\right\}_\nu$ is a collection of points almost equally distributed on $\mathbb{S}^{n-1}$ having a grid length between $2^{-j/2-1}$ and $2^{-j/2}$: **(1)** $|\xi^\mu_j - \xi^\nu_j| \ge 2^{-j/2-1}$ for every $\xi^\mu_j \ne \xi^\nu_j$. **(2)** For any $u\in\mathbb{S}^{n-1}$, there is a ξ^ν_j in the open set $\{\xi\in\mathbb{S}^{n-1}: |\xi-u|<2^{-j/2+1}\}$.

Let $\varphi\in C^\infty_o(\mathbb{R})$ be a smooth cut-off function such that $\varphi(t)=1$ if $|t|\le 1$ and $\varphi(t)=0$ if $|t|>2$. Recall (1. 22)-(1. 23). We have

$$\varphi^\nu_j(\xi) = \frac{\varphi\left[2^{j/2}\left|\frac{\xi}{|\xi|}-\xi^\nu_j\right|\right]}{\sum_{\nu:\xi^\nu_j\in\mathbb{S}^{n-1}} \varphi\left[2^{j/2}\left|\frac{\xi}{|\xi|}-\xi^\nu_j\right|\right]}, \tag{3. 14}$$

$$\mathbf{supp}\varphi^\nu_j \subset \Gamma^\nu_j = \left\{\xi\in\mathbb{R}^n: \left|\frac{\xi}{|\xi|}-\xi^\nu_j\right| < 2^{-j/2+1}\right\}.$$

Next, consider $\left\{\xi_j^\nu\right\}_\nu = \bigcup_{\ell=1}^L \mathcal{Z}_\ell$. Each $\mathcal{Z}_\ell$ is selected from $\left\{\xi_j^\nu\right\}_\nu \setminus \{\mathcal{Z}_1\cup\cdots\cup\mathcal{Z}_{\ell-1}\}$ as follows. **(a)** $|\xi_j^\mu-\xi_j^\nu|\geq \mathbf{c}2^{-j/2+\sigma j}$ whenever $\xi_j^\mu\neq\xi_j^\nu$ belonging $\mathcal{Z}_\ell$. **(b)** If ξ_j^μ satisfies $|\xi_j^\mu-\xi_j^\nu|\geq 2\mathbf{c}2^{-j/2+\sigma j}$ for every selected ξ_j^ν, then ξ_j^μ shall be selected.

Let

$$\widehat{\mathbf{P}}{}^{\alpha\ \nu}_{j\ m}(\xi) \ = \ \varphi_j^\nu(\xi)\widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1}dr. \tag{3. 15}$$

We write

$$\widehat{\mathbf{P}}{}^{\alpha}_{j\ m}(\xi) \ = \ \sum_{\ell=1}^L \widehat{\mathbf{P}}{}^{\alpha\ \ell}_{j\ m}(\xi), \qquad \widehat{\mathbf{P}}{}^{\alpha\ \ell}_{j\ m}(\xi) \ = \ \sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell} \widehat{\mathbf{P}}{}^{\alpha\ \nu}_{j\ m}(\xi) \tag{3. 16}$$

and

$$\mathbf{I}^\alpha_{j\ m}f(x) \ = \ \sum_{\ell=1}^L \mathbf{I}^{\alpha\ \ell}_{j\ m}f(x), \qquad \mathbf{I}^{\alpha\ \ell}_{j\ m}f(x) \ = \ \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\mathbf{P}}{}^{\alpha\ \ell}_{j\ m}(\xi)d\xi \tag{3. 17}$$

for every $j>0$ and $m=1,2,\ldots,M$.

Given $\xi_j^\mu\in\mathcal{Z}_\ell$ for some μ, $\mathbf{L}_\mu$ is an $n\times n$-orthogonal matrix with $\det\mathbf{L}_\mu=1$. Moreover, $\mathbf{L}_\mu^T\xi_j^\mu=(1,0)^T\in\mathbb{R}\times\mathbb{R}^{n-1}$. Recall (1. 25)-(1. 26). We have

$$\Psi^{\alpha\ \ell}_{j\ m}(x) \ = \ \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell} \Psi^{\alpha\ \mu}_{j\ m}(x),$$

$$\Psi^{\alpha\ \mu}_{j\ m}(x) \ = \ \varphi\left[2^{-\sigma j-1}\left|\lambda_m-\left(\mathbf{L}_\mu^T x\right)_1\right|\right]\prod_{i=2}^n\varphi\left[2^{-\left[\frac{1}{2}+\sigma\right]j}\left|\left(\mathbf{L}_\mu^T x\right)_i\right|\right],$$

$$\begin{aligned}\mathbf{supp}\Psi^{\alpha\ \mu}_{j\ m} \ &\subset \ \mathrm{R}^{\mu}_{j\ m}\\ &= \ \left\{x\in\mathbb{R}^n\colon \lambda_m-2^{\sigma j+2}<\left(\mathbf{L}_\mu^T x\right)_1<\lambda_m+2^{\sigma j+2},\ \left|\left(\mathbf{L}_\mu^T x\right)_i\right|<2^{\left[\frac{1}{2}+\sigma\right]j+1},\ i=2,\ldots,n\right\}.\end{aligned} \tag{3. 18}$$

Consider

$$\mathbf{I}^{\alpha\ \ell}_{j\ m}f(x) \ = \ \mathbf{II}^{\alpha\ \ell}_{j\ m}f(x)+\mathbf{III}^{\alpha\ \ell}_{j\ m}f(x) \tag{3. 19}$$

of which

$$\begin{aligned}\mathbf{II}^{\alpha\ \ell}_{j\ m}f(x) \ &= \ \int_{\mathbb{R}^n} f(x-u)\mathbf{U}^{\alpha\ \ell}_{j\ m}(u)du,\\ \mathbf{U}^{\alpha\ \ell}_{j\ m}(u) \ &= \ \mathbf{P}^{\alpha\ \ell}_{j\ m}(u)\left[1-\Psi^{\alpha\ \ell}_{j\ m}(u)\right]+\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(u)\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu\neq\xi_j^\mu}\mathbf{P}^{\alpha\ \nu}_{j\ m}(u)\end{aligned} \tag{3. 20}$$

and

$$\begin{aligned}\mathbf{III}^{\alpha\ \ell}_{j\ m}f(x) \ &= \ \int_{\mathbb{R}^n} f(x-u)\mathbf{V}^{\alpha\ \ell}_{j\ m}(u)du,\\ \mathbf{V}^{\alpha\ \ell}_{j\ m}(u) \ &= \ \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(u)\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}\mathbf{P}^{\alpha\ \nu}_{j\ m}(u).\end{aligned} \tag{3. 21}$$

3.1 $\,^\sharp\mathbf{U}^{\alpha\,\beta\,\ell}_{j\,m}$ and $^\sharp\mathbf{V}^{\alpha\,\beta\,\ell}_{j\,m}$

$^\sharp\Omega^\alpha$ is a distribution defined in $\mathbb{R}^n$ by analytic continuation from

$$\mathbf{Re}\alpha > \tfrac{n}{2}, \qquad {}^\sharp\Omega^\alpha(x) \;=\; \pi^{\alpha-\frac{n+2}{2}}\Gamma^{-1}\left(\alpha-\tfrac{n}{2}\right)\left(\frac{1}{1-|x|^2}\right)_+^{\frac{n+2}{2}-\alpha}. \tag{3. 22}$$

Equivalently, it can be defined by

$$\begin{aligned} {}^\sharp\widehat{\Omega}^\alpha(\xi) \;&=\; \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-\left[\frac{n+2}{2}-\alpha\right]}\mathbf{J}_{\frac{n}{2}-\left[\frac{n+2}{2}-\alpha\right]}\Big(2\pi|\xi|\Big) \\ &=\; \left(\frac{1}{|\xi|}\right)^{\alpha-1}\mathbf{J}_{\alpha-1}\Big(2\pi|\xi|\Big), \qquad \alpha\in\mathbb{C}. \end{aligned} \tag{3. 23}$$

Let $\mathbf{Re}\alpha \geq \left[\frac{2n}{2n-1}\right]\mathbf{Re}\beta$ and $\frac{2n-1}{4n} < \mathbf{Re}\beta < \frac{2n-1}{2n-2}$. Define

$$^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_j(\xi) \;=\; \widehat{\phi}(\xi)\int_{2^{j-1}\leq|r|<2^j} e^{-2\pi\mathbf{i}r}\,{}^\sharp\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr, \qquad j>0. \tag{3. 24}$$

Remark 3.2 *Let $T>0$. We have*

$$\left|\sum_{j>T}{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_j(\xi)\right| \;\leq\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}-\varepsilon} 2^{-T\varepsilon} \tag{3. 25}$$

for some $\varepsilon=\varepsilon(\mathbf{Re}\alpha,\mathbf{Re}\beta)>0$.

Consider

$$\begin{aligned} {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_j(\xi) \;&=\; \sum_{m=1}^M {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_{j\,m}(\xi), \\ {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_{j\,m}(\xi) \;&=\; \widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}\,{}^\sharp\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr, \end{aligned} \tag{3. 26}$$

$$\lambda_m\in[2^{j-1},2^j], \qquad \lambda_0=2^{j-1},\ \lambda_M=2^j\ \text{ and }\ 2^{\sigma j-1}\;\leq\;\lambda_m-\lambda_{m-1}<2^{\sigma j}$$

where $0<\sigma=\sigma(\mathbf{Re}\alpha,\mathbf{Re}\beta)<\frac{1}{2}$ can be chosen sufficiently small.

As described in **(a)-(b)** below (1. 24), $\mathcal{Z}_\ell$ consists of ξ^ν_j s almost equally distributed on $\mathbb{S}^{n-1}$ with a grid length between $\mathbf{c}2^{-j/2+\sigma j}$ and $2\mathbf{c}2^{-j/2+\sigma j}$.

Let φ^ν_j defined in (3. 14). Assert

$$\begin{aligned} &{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_{j\,m}(\xi) \;=\; \sum_{\ell=1}^L {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi), \qquad {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi) \;=\; \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell}{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\nu}_{j\,m}(\xi), \\ &{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\nu}_{j\,m}(\xi) \;=\; \varphi^\nu_j(\xi)\widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}\,{}^\sharp\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr. \end{aligned} \tag{3. 27}$$

For every $j>0$, $m=1,\ldots,M$ and $\ell=1,\ldots,L$, we define

$$
{}^\sharp\mathbf{U}^{\alpha~\beta~\ell}_{j~m}(x) ~=~ {}^\sharp\mathbf{P}^{\alpha~\beta~\ell}_{j~m}(x)\Big[1-\Psi^{\alpha~\ell}_{j~m}(x)\Big]+\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha~\mu}_{j~m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j\neq\xi^\mu_j}{}^\sharp\mathbf{P}^{\alpha~\beta~\nu}_{j~m}(x), \qquad (3.~28)
$$

$$
{}^\sharp\mathbf{V}^{\alpha~\beta~\ell}_{j~m}(x) ~=~ \sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha~\mu}_{j~m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j=\xi^\mu_j}{}^\sharp\mathbf{P}^{\alpha~\beta~\nu}_{j~m}(x). \qquad (3.~29)
$$

Proposition One *Let* $\mathbf{Re}\alpha\geq\left[\frac{2n}{2n-1}\right]\mathbf{Re}\beta$ *and* $\frac{2n-1}{4n}<\mathbf{Re}\beta<\frac{2n-1}{2n-2}$. *We have*

$$
\left|{}^\sharp\widehat{\mathbf{U}}^{\alpha~\beta~\ell}_{j~m}(\xi)\right| ~\leq~ \mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}2^{j/2}2^{-\varepsilon j} \qquad (3.~30)
$$

and

$$
\left|{}^\sharp\widehat{\mathbf{V}}^{\alpha~\beta~\ell}_{j~m}(\xi)\right| ~\leq~ \mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}2^{-\varepsilon j} \qquad (3.~31)
$$

for some $\varepsilon=\varepsilon(\mathbf{Re}\alpha,\mathbf{Re}\beta)>0$.

3.2 ${}^\flat\mathbf{U}^{\alpha~\beta~\ell}_{j~m}$ and ${}^\flat\mathbf{V}^{\alpha~\beta~\ell}_{j~m}$

${}^\flat\Omega^\alpha$ is a distribution defined in $\mathbb{R}^n$ by analytic continuation from

$$
\mathbf{Re}\alpha>0, \qquad {}^\flat\Omega^\alpha(x) ~=~ \pi^{\alpha-1}\Gamma^{-1}(\alpha)\left(\frac{1}{1-|x|^2}\right)_+^{1-\alpha}. \qquad (3.~32)
$$

Equivalently, it can be defined by

$$
\begin{aligned}
{}^\flat\widehat{\Omega}^\alpha(\xi) ~&=~ \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-(1-\alpha)}\mathbf{J}_{\frac{n}{2}-(1-\alpha)}\Big(2\pi|\xi|\Big)\\
~&=~ \left(\frac{1}{|\xi|}\right)^{\frac{n-1}{2}+\alpha-\frac{1}{2}}\mathbf{J}_{\frac{n-1}{2}+\alpha-\frac{1}{2}}\Big(2\pi|\xi|\Big), \qquad \alpha\in\mathbb{C}.
\end{aligned} \qquad (3.~33)
$$

Let $\mathbf{Re}\alpha>0$ and $0<\mathbf{Re}\beta<\frac{1}{2}$. Define

$$
{}^\flat\widehat{\mathbf{P}}^{\alpha~\beta}_j(\xi) ~=~ \widehat{\phi}(\xi)\int_{2^{j-1}\leq|r|<2^j} e^{-2\pi\mathbf{i}r}{}^\flat\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr, \qquad j>0. \qquad (3.~34)
$$

Consider

$$
\begin{aligned}
{}^\flat\widehat{\mathbf{P}}^{\alpha~\beta}_j(\xi) ~&=~ \sum_{m=1}^M {}^\flat\widehat{\mathbf{P}}^{\alpha~\beta}_{j~m}(\xi),\\
{}^\flat\widehat{\mathbf{P}}^{\alpha~\beta}_{j~m}(\xi) ~&=~ \widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}{}^\flat\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr,
\end{aligned} \qquad (3.~35)
$$

$$
\lambda_m\in[2^{j-1},2^j], \qquad \lambda_0=2^{j-1},\ \lambda_M=2^j \ \text{ and } \ 2^{\sigma j-1} ~\leq~ \lambda_m-\lambda_{m-1}<2^{\sigma j}
$$

where $0<\sigma=\sigma(\mathbf{Re}\alpha)<\frac{1}{2}$ can be chosen sufficiently small.

As described in **(a)-(b)** below (1. 24), $\mathcal{Z}_\ell$ consists of ξ_j^ν s almost equally distributed on $\mathbb{S}^{n-1}$ with a grid length between $\mathbf{c}2^{-j/2+\sigma j}$ and $2\mathbf{c}2^{-j/2+\sigma j}$.

Let φ_j^ν defined in (3. 14). Assert

$$
\begin{aligned}
{}^\flat\widehat{\mathbf{P}}^{\alpha\,\beta}_{j\,m}(\xi) &= \sum_{\ell=1}^{L} {}^\flat\widehat{\mathbf{P}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi), \qquad {}^\flat\widehat{\mathbf{P}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi) = \sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell} {}^\flat\widehat{\mathbf{P}}^{\alpha\,\beta\,\nu}_{j\,m}(\xi), \\
{}^\flat\widehat{\mathbf{P}}^{\alpha\,\beta\,\nu}_{j\,m}(\xi) &= \varphi_j^\nu(\xi)\widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}{}^\flat\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr.
\end{aligned} \tag{3. 36}
$$

For every $j>0$, $m=1,\ldots,M$ and $\ell=1,\ldots,L$, we define

$$
{}^\flat\mathbf{U}^{\alpha\,\beta\,\ell}_{j\,m}(x) = {}^\flat\mathbf{P}^{\alpha\,\beta\,\ell}_{j\,m}(x)\Big[1-\Psi^{\alpha\,\ell}_{j\,m}(x)\Big] + \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\Psi^{\alpha\,\mu}_{j\,m}(x)\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu\neq\xi_j^\mu}{}^\flat\mathbf{P}^{\alpha\,\beta\,\nu}_{j\,m}(x), \tag{3. 37}
$$

$$
{}^\flat\mathbf{V}^{\alpha\,\beta\,\ell}_{j\,m}(x) = \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\Psi^{\alpha\,\mu}_{j\,m}(x)\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}{}^\flat\mathbf{P}^{\alpha\,\beta\,\nu}_{j\,m}(x). \tag{3. 38}
$$

Proposition Two *Let* $\mathbf{Re}\alpha>0$ *and* $0<\mathbf{Re}\beta<\frac{1}{2}$. *We have*

$$
\int_{\mathbb{R}^n}\left|{}^\flat\mathbf{U}^{\alpha\,\beta\,\ell}_{j\,m}(x)\right|dx \leq \mathfrak{B}_{N\,\mathbf{Re}\alpha\,\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}\,2^{-(1-\sigma)j}2^{-Nj}2^{-\varepsilon j}, \qquad N\geq 0 \tag{3. 39}
$$

and

$$
\int_{\mathbb{R}^n}\left|{}^\flat\mathbf{V}^{\alpha\,\beta\,\ell}_{j\,m}(x)\right|dx \leq \mathfrak{B}_{\mathbf{Re}\alpha\,\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}\,2^{-(1-\sigma)j}2^{-\varepsilon j} \tag{3. 40}
$$

for some $\varepsilon=\varepsilon(\mathbf{Re}\alpha)>0$.

4 Proof of Theorem Two

Recall $\mathbf{I}^\alpha f$ defined in (3. 10) and $\widehat{\mathbf{P}}^\alpha_<$, $\widehat{\mathbf{P}}^\alpha_j$ and $\widehat{\mathbf{P}}^\alpha_{j\,m}$ defined in (3. 11) and (3. 12). We have

$$
\begin{aligned}
\mathbf{I}^\alpha f(x) &= \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\left\{\widehat{\mathbf{P}}^\alpha_<(\xi)+\sum_{0<j\leq T}\sum_{m=1}^{M}\widehat{\mathbf{P}}^\alpha_{j\,m}(\xi)+\sum_{j>T}\widehat{\mathbf{P}}^\alpha_j(\xi)\right\}d\xi \\
&= \mathbf{I}^\alpha_< f(x)+\sum_{0<j\leq T}\sum_{m=1}^{M}\mathbf{I}^\alpha_{j\,m}f(x)+\mathbf{R}^\alpha_T f(x);
\end{aligned} \tag{4. 1}
$$

$$
\mathbf{R}^\alpha_T f(x) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\phi}(\xi)\left\{\int_{|r|\geq 2^T} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\alpha-1}dr\right\}d\xi
$$

where $\widehat{\phi}$ is defined in (1. 9) and $\mathbf{supp}\widehat{\phi}\subset\left\{\xi\in\mathbb{R}^n\colon\frac{1}{3}<|\xi|\leq 3\right\}$.

The kernel of $\mathbf{I}^\alpha_<$ as shown in (3. 13) is an $\mathbf{L}^1$-function in $\mathbb{R}^n$. From now on, we focus on $\mathbf{I}^\alpha_{j\,m}f, 0<j\leq T, m=1,\ldots,M$ and $\mathbf{R}^\alpha_T f$.

Let $\frac{1}{2}<\mathbf{Re}\alpha<1$. We can find $\mathbf{a}_i=\mathbf{a}_i(\mathbf{Re}\alpha), \mathbf{b}_i=\mathbf{b}_i(\mathbf{Re}\alpha), i=1,2$ such that

$$\begin{array}{c}\mathbf{a}_1>0, \qquad 0<\mathbf{b}_1<\frac{1}{2}, \qquad \mathbf{a}_2\geq\left[\frac{2n}{2n-1}\right]\mathbf{b}_2, \qquad \frac{2n-1}{4n}<\mathbf{b}_2<\frac{2n-1}{2n-2}, \\ \mathbf{Re}\alpha \;=\; \frac{\mathbf{a}_1}{n}+\mathbf{a}_2\left(\frac{n-1}{n}\right) \;=\; \frac{\mathbf{b}_1}{n}+\mathbf{b}_2\left(\frac{n-1}{n}\right).\end{array} \tag{4. 2}$$

As $\mathbf{Re}\alpha\longrightarrow 1$, we necessarily have $\mathbf{b}_1\longrightarrow\frac{1}{2}$, $\mathbf{b}_2\longrightarrow\frac{2n-1}{2n-2}$ and $\mathbf{a}_1\longrightarrow 0$, $\mathbf{a}_2\longrightarrow\frac{n}{n-1}$.

Let $0\leq\mathbf{Re}z\leq 1$. We define

$$\begin{aligned}\widehat{\Omega}^{\alpha\,z}(\xi) \;&=\; \left(\frac{1}{|\xi|}\right)^{\mathbf{a}_1z+\mathbf{a}_2(1-z)-\frac{1}{2}+\left(\frac{n-1}{2}\right)z-\frac{1}{2}(1-z)+\mathbf{i}\mathbf{Im}\alpha} \mathbf{J}_{\mathbf{a}_1z+\mathbf{a}_2(1-z)-\frac{1}{2}+\left(\frac{n-1}{2}\right)z-\frac{1}{2}(1-z)+\mathbf{i}\mathbf{Im}\alpha}(2\pi|\xi|)\\ &=\; \pi^{\mathbf{a}_1z+\mathbf{a}_2(1-z)-1+\left(\frac{n-1}{2}\right)z-\frac{1}{2}(1-z)+\mathbf{i}\mathbf{Im}\alpha}\Gamma^{-1}\left(\mathbf{a}_1z+\mathbf{a}_2(1-z)+\left(\frac{n-1}{2}\right)z-\frac{1}{2}(1-z)+\mathbf{i}\mathbf{Im}\alpha\right)\\ &\int_{-1}^1 e^{2\pi\mathbf{i}|\xi|s}(1-s^2)^{\mathbf{a}_1z+\mathbf{a}_2(1-z)-1+\left(\frac{n-1}{2}\right)z-\frac{1}{2}(1-z)+\mathbf{i}\mathbf{Im}\alpha}ds \qquad \text{by (A. 1)}\end{aligned} \tag{4. 3}$$

and

$$\widehat{\mathbf{P}}^{\alpha\,z}_j(\xi) \;=\; e^{\left[z-\frac{1}{n}\right]^2}\widehat{\phi}(\xi)\int_{2^{j-1}\leq|r|<2^j} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^{\alpha\;\mathbf{a}}_z(r\xi)\omega(r)|r|^{2\left[\mathbf{b}_1z+\mathbf{b}_2(1-z)\right]-1+2\mathbf{i}\mathbf{Im}\alpha}dr, \qquad j>0. \tag{4. 4}$$

Let $\lambda_m\in[2^{j-1},2^j]$: $\lambda_0=2^{j-1}$, $\lambda_L=2^j$ and $2^{\sigma j-1}\leq\lambda_m-\lambda_{m-1}<2^{\sigma j}$ for which $0<\sigma<\frac{1}{2}$ can be chosen sufficiently small. We consider

$$\begin{aligned}\widehat{\mathbf{P}}^{\alpha\,z}_j(\xi) \;&=\; \sum_{m=1}^M\widehat{\mathbf{P}}^{\alpha\,z}_{j\,m}(\xi),\\ \widehat{\mathbf{P}}^{\alpha\,z}_{j\,m}(\xi) \;&=\; e^{\left[z-\frac{1}{n}\right]^2}\widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^{\alpha\;\mathbf{a}}_z(r\xi)\omega(r)|r|^{2\left[\mathbf{b}_1z+\mathbf{b}_2(1-z)\right]-1+2\mathbf{i}\mathbf{Im}\alpha}dr.\end{aligned} \tag{4. 5}$$

Observe that $\widehat{\mathbf{P}}^{\alpha\,z}_{j\,m}(\xi)$ is analytic for $0\leq\mathbf{Re}z\leq 1$. In particular, $\widehat{\mathbf{P}}^{\alpha\,\frac{1}{n}}_{j\,m}(\xi)=\widehat{\mathbf{P}}^{\alpha}_{j\,m}(\xi)$.

Define

$$\mathbf{I}^{\alpha\,z}_{j\,m}f(x) \;=\; \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\widehat{\mathbf{P}}^{\alpha\,z}_{j\,m}(\xi)d\xi \tag{4. 6}$$

for every $j>0$, $m=1,2,\ldots,M$ and

$$\mathbf{R}^{\alpha\,z}_T f(x) \;=\; \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\widehat{f}(\xi)\sum_{j>T}\widehat{\mathbf{P}}^{\alpha\,z}_j(\xi)d\xi, \qquad T>0. \tag{4. 7}$$

Given $f\in\mathbf{L}^p(\mathbb{R}^n)$, $g\in\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^n)$ to be simple functions and $p=\frac{2n}{n+1}$, we define

$$f_z(x) \;=\; \mathbf{sign}f(x)|f(x)|^{\left[\frac{1-z}{2}+z\right]p}, \qquad g_z(x) \;=\; \mathbf{sign}g(x)|g(x)|^{\left[1-\frac{1-z}{2}-z\right]\frac{p}{p-1}} \tag{4. 8}$$

for $0\leq\mathbf{Re}z\leq 1$.

Remark 4.1 *Note that* $f_{0+\mathbf{iIm}z} \in \mathbf{L}^2(\mathbb{R}^n)$, $g_{0+\mathbf{iIm}z} \in \mathbf{L}^2(\mathbb{R}^n)$ *and* $f_{1+\mathbf{iIm}z} \in \mathbf{L}^1(\mathbb{R}^n)$, $g_{1+\mathbf{iIm}z} \in \mathbf{L}^\infty(\mathbb{R}^n)$. *At* $z = \frac{1}{n}$, *we find* $f_z = f$ *and* $g_z = g$. *Without loss of generality, assume*

$$\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \;=\; \left\|g\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^n)} \;=\; 1.$$

We claim

$$\int_{\mathbb{R}^n} \mathbf{R}_T^{\alpha\ z} f_z(x) g_z(x) dx \;\longrightarrow\; 0, \qquad T \longrightarrow \infty \tag{4. 9}$$

for $0 \le \mathbf{Re}z \le 1$.

First, at $\mathbf{Re}z = 0$, we find $\widehat{\mathbf{P}}_j^{\alpha\ 0+\mathbf{iIm}z}(\xi) = e^{\left[\mathbf{iIm}z - \frac{1}{n}\right]^2} \sharp \widehat{\mathbf{P}}_j^{\mathbf{A}_2\ \mathbf{B}_2}(\xi)$ as defined in (3. 24) where $\mathbf{A}_2 = \mathbf{a}_2 + \mathbf{i}\left[\mathbf{a}_1 - \mathbf{a}_2 + \left(\frac{n-1}{2}\right) + \frac{1}{2}\right]\mathbf{Im}z + \mathbf{iIm}\alpha$ and $\mathbf{B}_2 = \mathbf{b}_2 + \mathbf{i}\left[\mathbf{b}_1 - \mathbf{b}_2\right]\mathbf{Im}z + \mathbf{iIm}\alpha$.

Note that $\mathbf{a}_2 \ge \left[\frac{2n}{2n-1}\right]\mathbf{b}_2$ and $\frac{2n-1}{4n} < \mathbf{b}_2 < \frac{2n-1}{2n-2}$ as shown in (4. 2). Recall **Remark 3.2**. We have

$$\begin{aligned} \left|\sum_{j>T} \widehat{\mathbf{P}}_j^{\alpha\ 0+\mathbf{iIm}z}(\xi)\right| \;&\le\; \mathfrak{B}_{\mathbf{a}_2\ \mathbf{b}_2}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, e^{\mathbf{c}|\mathbf{Im}z|}\, e^{-[\mathbf{Im}z]^2} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}-\varepsilon} 2^{-T\varepsilon} \\ &\le\; \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}-\varepsilon} 2^{-T\varepsilon}, \qquad 0 \le \mathbf{Re}z \le 1 \end{aligned} \tag{4. 10}$$

for some $\varepsilon = \varepsilon(\mathbf{a}_2, \mathbf{b}_2) = \varepsilon(\mathbf{Re}\alpha) > 0$.

As defined in (4. 4), $\mathbf{supp}\widehat{\mathbf{P}}_j^{\alpha\ z} = \mathbf{supp}\widehat{\phi} \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{3} < |\xi| \le 3\right\}$. From (4. 7), by using (4. 10) and Schwartz inequality, we find

$$\begin{aligned} \left|\mathbf{R}_T^{\alpha\ 0+\mathbf{iIm}z} f_{0+\mathbf{iIm}z}(x)\right| \;&\le\; \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \left\|\widehat{f}_{0+\mathbf{iIm}z}\right\|_{\mathbf{L}^2(\mathbb{R}^n)} \left\{\int_{|\xi|\le 3} \left|\frac{1}{1-|\xi|}\right|^{1-2\varepsilon} d\xi\right\}^{\frac{1}{2}} 2^{-T\varepsilon} \\ &\le\; \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-T\varepsilon} \left\|f_{0+\mathbf{iIm}z}\right\|_{\mathbf{L}^2(\mathbb{R}^n)} \qquad \text{by Plancherel theorem.} \end{aligned} \tag{4. 11}$$

Next, at $\mathbf{Re}z = 1$, we have

$$\begin{aligned} \left|\widehat{\Omega}^{\alpha\ 1+\mathbf{iIm}z}(r\xi)\right| \;&\le\; \left(\frac{1}{|r\xi|}\right)^{\mathbf{a}_1 - \frac{1}{2} + \frac{n-1}{2}} \left|\mathbf{J}_{\mathbf{a}_1(1+\mathbf{iIm}z) + \mathbf{ia}_2\mathbf{Im}z - \frac{1}{2} + \left(\frac{n-1}{2}\right)(1+\mathbf{iIm}z) - \frac{\mathbf{i}}{2}\mathbf{Im}z + \mathbf{iIm}\alpha}(2\pi|r\xi|)\right| \\ &\le\; \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, e^{\mathbf{c}|\mathbf{Im}z|} \left(\frac{1}{1+|r\xi|}\right)^{\mathbf{a}_1 + \frac{n-1}{2}} \qquad \text{by (A. 5).} \end{aligned} \tag{4. 12}$$

Recall $\omega(r)$ from (3. 10). We find

$$\omega(r) \;=\; e^{2\pi\mathbf{i}r} \int_0^1 e^{-2\pi\mathbf{i}\tau r} \tau d\tau \;=\; \frac{-1}{2\pi\mathbf{i}} \frac{1}{r} - \frac{1}{4\pi^2 r^2}\left[1 - e^{-2\pi\mathbf{i}r}\right]$$

implying

$$|\omega(r)| \;\le\; \mathfrak{B}\left[1 + |r|\right]^{-1}. \tag{4. 13}$$

Denote χ as an indicator function. From (4. 4), we have

$$
\begin{aligned}
\left|\sum_{j>T}\widehat{\mathbf{P}}_j^{\alpha~1+\mathbf{iIm}z}(\xi)\right| &= \left|e^{\left[z-\frac{1}{n}\right]^2}\widehat{\phi}(\xi)\int_{|r|\geq 2^T} e^{-2\pi\mathbf{i}r}\widehat{\Omega}^{\alpha~1+\mathbf{iIm}z}(r\xi)\omega(r)|r|^{2\left[\mathbf{b}_1 z+\mathbf{b}_2(1-z)\right]-1+2\mathbf{iIm}\alpha}dr\right| \\
&\leq \mathfrak{B}e^{-\left[\mathbf{Im}z\right]^2}\int_{|r|\geq 2^T}\left|\widehat{\Omega}^{\alpha~\mathbf{a}}_{1+\mathbf{iIm}z}(r\xi)\chi_{\left\{\frac{1}{3}<|\xi|\leq 3\right\}}(\xi)\right||r|^{2\mathbf{b}_1-2}dr \qquad \text{by (4. 13)} \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, e^{\mathbf{c}|\mathbf{Im}z|}\, e^{-\left[\mathbf{Im}z\right]^2}\int_{|r|\geq 2^T}\left(\frac{1}{1+|r|}\right)^{\mathbf{a}_1+\frac{n-1}{2}}|r|^{2\mathbf{b}_1-2}dr \qquad \text{by (4. 12)} \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\left(\frac{1}{1+2^T}\right)^{\mathbf{a}_1+\frac{n-1}{2}} 2^{T(2\mathbf{b}_1-1)} \qquad \mathbf{a}_1>0,\ 0<\mathbf{b}_1<\tfrac{1}{2} \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-T\left(\frac{n-1}{2}\right)}.
\end{aligned}
\tag{4. 14}
$$

Let $\mathbf{R}_T^{\alpha~z}$ defined (4. 7). By using (4. 14), we find

$$
\begin{aligned}
\left|\mathbf{R}_T^{\alpha~1+\mathbf{iIm}z}f_{1+\mathbf{iIm}z}(x)\right| &\leq \int_{|\xi|\leq 3}\left|\widehat{f}_{1+\mathbf{iIm}z}(\xi)\right|\left|\sum_{j>T}\widehat{\mathbf{P}}_j^{\alpha~1+\mathbf{iIm}z}(\xi)\right|d\xi \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-T\left(\frac{n-1}{2}\right)}\left\|\widehat{f}_{1+\mathbf{iIm}z}\right\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-T\left(\frac{n-1}{2}\right)}\left\|f_{1+\mathbf{iIm}z}\right\|_{\mathbf{L}^1(\mathbb{R}^n)}.
\end{aligned}
\tag{4. 15}
$$

Consider f_z, g_z given in (4. 8) and **Remark 4.1**. From (4. 11) and (4. 15), by applying the Three-Line lemma, we obtain

$$
\left|\mathbf{R}_T^{\alpha~z}f_z(x)\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-T\varepsilon}, \qquad 0\leq \mathbf{Re}z\leq 1. \tag{4. 16}
$$

This estimate further implies (4. 9). Consequently, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n}\sum_{j>0}\mathbf{I}_j^{\alpha~z}f_z(x)g_z(x)dx &= \lim_{T\longrightarrow\infty}\sum_{0<j\leq T}\int_{\mathbb{R}^n}\mathbf{I}_j^{\alpha~z}f_z(x)g_z(x)dx \\
&= \lim_{T\longrightarrow\infty}\sum_{0<j\leq T}\sum_{m=1}^{M}\int_{\mathbb{R}^n}\mathbf{I}_{j~m}^{\alpha~z}f_z(x)g_z(x)dx, \qquad 0\leq \mathbf{Re}z\leq 1.
\end{aligned}
\tag{4. 17}
$$

Given $j>0$, $\left\{\xi_j^\nu\right\}_\nu$ is a collection of points almost equally distributed on $\mathbb{S}^{n-1}$ having a grid length between $2^{-j/2-1}$ and $2^{-j/2}$. See **(1)-(2)** above (1. 22)-(1. 23). Furthermore, $\left\{\xi_j^\nu\right\}_\nu = \bigcup_{\ell=1}^{L}\mathcal{Z}_\ell$ where ξ_j^ν s belonging to each $\mathcal{Z}_\ell$ are almost equally distribution on $\mathbb{S}^{n-1}$ with a grid length between $\mathbf{c}2^{-j/2+\sigma j}$ and $2\mathbf{c}2^{-j/2+\sigma j}$. See **(a)-(b)** below (1. 24).

Let φ_j^ν defined in (3. 14) and $\Psi_{j\,m}^{\alpha\,\ell}, \Psi_{j\,m}^{\alpha\,\mu}$ defined in (3. 18). Consider

$$\widehat{\mathbf{P}}_{j\,m}^{\alpha\,z}(\xi) = \sum_{\ell=1}^{L} \widehat{\mathbf{P}}_{j\,m}^{\alpha\,\ell\,z}(\xi), \qquad \sum_{\ell=1}^{L} \widehat{\mathbf{P}}_{j\,m}^{\alpha\,\ell\,z}(\xi) = \sum_{\nu:\xi_j^\nu \in \mathcal{Z}_\ell} \widehat{\mathbf{P}}_{j\,m}^{\alpha\,\nu\,z}(\xi),$$

$$\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\nu\,z}(\xi) = e^{\left[z-\frac{1}{n}\right]^2} \widehat{\phi}(\xi)\varphi_j^\nu(\xi) \int_{\lambda_{m-1}\le|r|<\lambda_m} e^{-2\pi \mathbf{i} r} \widehat{\Omega}^{\alpha\,z}(r\xi)\omega(r)|r|^{2\left[\mathbf{b}_1 z+\mathbf{b}_2(1-z)\right]-1+2\mathbf{Im}\alpha} dr. \tag{4. 18}$$

For every $j > 0, m = 1, \ldots, M$ and $\ell = 1, \ldots, L$, we define

$$\mathbf{II}_{j\,m}^{\alpha\,\ell\,z} f(x) = \int_{\mathbb{R}^n} f(x-u)\mathbf{U}_{j\,m}^{\alpha\,\ell\,z}(u)du,$$

$$\mathbf{U}_{j\,m}^{\alpha\,\ell\,z}(u) = \mathbf{P}_{j\,m}^{\alpha\,\ell\,z}(u)\left[1-\Psi_{j\,m}^{\alpha\,\ell}(u)\right] + \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell} \Psi_{j\,m}^{\alpha\,\mu}(u) \sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell, \xi_j^\nu\neq\xi_j^\mu} \mathbf{P}_{j\,m}^{\alpha\,\nu\,z}(u) \tag{4. 19}$$

and

$$\mathbf{III}_{j\,m}^{\alpha\,\ell\,z} f(x) = \int_{\mathbb{R}^n} f(x-u)\mathbf{V}_{j\,m}^{\alpha\,\ell\,z}(u)du,$$

$$\mathbf{V}_{j\,m}^{\alpha\,\ell\,z}(u) = \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell} \Psi_{j\,m}^{\alpha\,\mu}(u) \sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell, \xi_j^\nu=\xi_j^\mu} \mathbf{P}_{j\,m}^{\alpha\,\nu\,z}(u). \tag{4. 20}$$

In particular, at $z=\frac{1}{n}$, we find $\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\nu\,\frac{1}{n}} = \widehat{\mathbf{P}}_{j\,m}^{\alpha\,\nu}$ and $\mathbf{II}_{j\,m}^{\alpha\,\ell\,\frac{1}{n}} f = \mathbf{II}_j^{\alpha\,\ell} f$, $\mathbf{III}_{j\,m}^{\alpha\,\ell\,\frac{1}{n}} f = \mathbf{III}_j^{\alpha\,\ell} f$ as defined in (3. 20)-(3. 21).

Let $\mathbf{Re}z = 0$. From (4. 19)-(4. 20), we have $\mathbf{II}_{j\,m}^{\alpha\,\ell\,0+\mathbf{iIm}z} f = e^{\left[\mathbf{iIm}z-\frac{1}{n}\right]^2} f * {}^\sharp\mathbf{U}_{j\,m}^{\mathbf{A}_2\,\mathbf{B}_2\,\ell}$ and $\mathbf{III}_{j\,m}^{\alpha\,\ell\,0+\mathbf{iIm}z} f = e^{\left[\mathbf{iIm}z-\frac{1}{n}\right]^2} f * {}^\sharp\mathbf{V}_j^{\mathbf{A}_2\,\mathbf{B}_2\,\ell} f$ for ${}^\sharp\mathbf{U}_{j\,m}^{\mathbf{A}_2\,\mathbf{B}_2\,\ell}$, ${}^\sharp\mathbf{V}_{j\,m}^{\mathbf{A}_2\,\mathbf{B}_2\,\ell}$ defined in (3. 28)-(3. 29) with $\mathbf{A}_2 = \mathbf{a}_2 + \mathbf{i}\left[\mathbf{a}_1 - \mathbf{a}_2 + \left(\frac{n-1}{2}\right) + \frac{1}{2}\right]\mathbf{Im}z + \mathbf{iIm}\alpha$ and $\mathbf{B}_2 = \mathbf{b}_2 + \mathbf{i}\left[\mathbf{b}_1 - \mathbf{b}_2\right]\mathbf{Im}z + \mathbf{iIm}\alpha$.

Recall $\mathbf{a}_2 \ge \left[\frac{2n}{2n-1}\right]\mathbf{b}_2$, $\frac{2n-1}{4n} < \mathbf{b}_2 < \frac{2n-1}{2n-2}$ in (4. 2). By applying (3. 30) in **Proposition One** together with Plancherel theorem and Schwartz inequality, we find

$$\begin{aligned}
&\int_{\mathbb{R}^n} \mathbf{II}_{j\,m}^{\alpha\,\ell\,0+\mathbf{iIm}z} f_{0+\mathbf{iIm}z}(x) g_{0+\mathbf{iIm}z}(x) dx \\
&\le \mathfrak{B}_{\mathbf{a}_2\,\mathbf{b}_2}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{j/2} 2^{-\varepsilon j} \left\|f_{0+\mathbf{iIm}z}\right\|_{\mathbf{L}^2(\mathbb{R}^n)} \left\|g_{0+\mathbf{iIm}z}\right\|_{\mathbf{L}^2(\mathbb{R}^n)} \\
&\le \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{j/2} 2^{-\varepsilon j}
\end{aligned} \tag{4. 21}$$

for some $\varepsilon = \varepsilon(\mathbf{a}_2, \mathbf{b}_2) = \varepsilon(\mathbf{Re}\alpha) > 0$.

On the other hand, by using (3. 31) in **Proposition One**, we have

$$\int_{\mathbb{R}^n} \mathbf{III}_{j\,m}^{\alpha\,\ell\,0+\mathbf{iIm}z} f_{0+\mathbf{iIm}z}(x) g_{0+\mathbf{iIm}z}(x) dx \le \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-\varepsilon j}. \tag{4. 22}$$

Let $\mathbf{Re}z = 1$. From (4. 19)-(4. 20), we find $\mathbf{II}^{\alpha\ \ell\ 1+\mathbf{iIm}z}_{j\ m} f = e^{\left[\mathbf{iIm}z+\frac{n-1}{n}\right]^2} f * {}^\flat\mathbf{U}^{\mathbf{A}_1\ \mathbf{B}_1\ \ell}_{j\ m}$ and $\mathbf{III}^{\alpha\ \ell\ 1+\mathbf{iIm}z}_{j\ m} f = e^{\left[\mathbf{iIm}z+\frac{n-1}{n}\right]^2} f * {}^\flat\mathbf{V}^{\mathbf{A}_1\ \mathbf{B}_1\ \ell}_{j\ m} f$ for ${}^\flat\mathbf{U}^{\mathbf{A}_1\ \mathbf{B}_1\ \ell}_{j\ m}, {}^\flat\mathbf{V}^{\mathbf{A}_1\ \mathbf{B}_1\ \ell}_{j\ m}$ defined in (3. 37)-(3. 38) with $\mathbf{A}_1 = \mathbf{a}_1 + \mathbf{i}\left[\mathbf{a}_1 - \mathbf{a}_2 + \left(\frac{n-1}{2}\right) + \frac{1}{2}\right]\mathbf{Im}z + \mathbf{iIm}\alpha$ and $\mathbf{B}_1 = \mathbf{b}_1 + \mathbf{i}\left[\mathbf{b}_1 - \mathbf{b}_2\right]\mathbf{Im}z + \mathbf{iIm}\alpha$.

Note that $\mathbf{a}_1 > 0, 0 < \mathbf{b}_1 < \frac{1}{2}$ in (4. 2). By applying (3. 39) in **Proposition Two** and using Hölder inequality, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \mathbf{II}^{\alpha\ \ell\ 1+\mathbf{iIm}z}_{j\ m} f_{1+\mathbf{iIm}z}(x) g_{1+\mathbf{iIm}z}(x) dx \\
&\leq\ \mathfrak{B}_{N\ \mathbf{a}_1\ \mathbf{b}_1}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-jN} 2^{-j\varepsilon}\ \left\|f_{1+\mathbf{iIm}z}\right\|_{\mathbf{L}^1(\mathbb{R}^n)} \left\|g_{1+\mathbf{iIm}z}\right\|_{\mathbf{L}^\infty(\mathbb{R}^n)} \\
&\leq\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, 2^{-(1-\sigma)j} 2^{-Nj} 2^{-\varepsilon j}, \qquad N \geq 0
\end{aligned}
\tag{4. 23}
$$

for some $\varepsilon = \varepsilon(\mathbf{a}_1, \mathbf{b}_1) = \varepsilon(\mathbf{Re}\alpha) > 0$.

On the other hand, by using (3. 40) in **Proposition Two**, we find

$$
\int_{\mathbb{R}^n} \mathbf{III}^{\alpha\ \ell\ 1+\mathbf{iIm}z}_{j\ m} f_{1+\mathbf{iIm}z}(x) g_{1+\mathbf{iIm}z}(x) dx\ \leq\ \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{-\varepsilon j}. \tag{4. 24}
$$

Recall **Remark 1.2** and **Remark 1.3**. There are at most a constant multiple of $2^{(1-\sigma)j}$ many $\lambda_m \in [2^{j-1}, 2^j]$ and $2^{(n-1)\sigma j}$ many $\mathcal{Z}_\ell \subset \left\{\xi^\nu_j\right\}_\nu$. As mentioned in **Remark 4.1**, at $z = \frac{1}{n}$, $f_z = f \in \mathbf{L}^p(\mathbb{R}^n)$ and $g_z = g \in \mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^n)$ which are simple functions and $p = \frac{2n}{n+1}$.

From (4. 21) and (4. 23) with $N \geq \frac{n-1}{2}$, the Three-Line lemma implies

$$
\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{II}^\alpha_j f(x) g(x) dx\ &=\ \sum_{\ell=1}^L \sum_{m=1}^M \int_{\mathbb{R}^n} \mathbf{II}^{\alpha\ \ell}_{j\ m} f(x) g(x) dx \\
&\leq\ \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{(n-1)\sigma j} 2^{-\varepsilon j}.
\end{aligned}
\tag{4. 25}
$$

From (4. 22) and (4. 24), by applying the Three-Line lemma, we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} \mathbf{III}^\alpha_j f(x) g(x) dx\ &=\ \sum_{\ell=1}^L \sum_{m=1}^M \int_{\mathbb{R}^n} \mathbf{III}^{\alpha\ \ell}_{j\ m} f(x) g(x) dx \\
&\leq\ \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{(n-1)\sigma j} 2^{-\varepsilon j}.
\end{aligned}
\tag{4. 26}
$$

A standard argument extends (4. 25)-(4. 26) to every $f \in \mathbf{L}^p(\mathbb{R}^n)$ and $g \in \mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^n)$ with $\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} = \left\|g\right\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^n)} = 1$.

Recall (4. 17) . Take into account $\mathbf{I}^\alpha_j = \mathbf{II}^\alpha_j + \mathbf{III}^\alpha_j, j > 0$. By summing over every $j > 0$ and using (4. 25)-(4. 26) with $0 < \sigma < \frac{1}{2}$ chosen sufficiently small, i.e: $n\sigma < \frac{1}{2}\varepsilon$, we conclude

$$
\left\|\mathbf{I}^\alpha f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}\ \leq\ \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad p = \frac{2n}{n+1}. \tag{4. 27}
$$

Lastly, the adjoint operator of $\mathbf{I}^\alpha$ defined in (1. 10) can be simply given by $\mathbf{I}^{\overline{\alpha}}$ where $\overline{\alpha}$ is the complex conjugate of α. Clearly, it satisfies all above estimates. By duality, we obtain

$$\left\|\mathbf{I}^\alpha f\right\|_{\mathbf{L}^p(\mathbb{R}^n)} \leq \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad p = \tfrac{2n}{n-1}. \tag{4. 28}$$

The $\mathbf{L}^p$-norm inequality in (1. 11) follows by applying Riesz-Thorin interpolation theorem.

5 Proof of Proposition One

Recall (3. 26). Let $\mathbf{Re}\alpha \geq \left[\frac{2n}{2n-1}\right]\mathbf{Re}\beta$ and $\frac{2n-1}{4n} < \mathbf{Re}\beta < \frac{2n-1}{2n-2}$. We have

$$^\sharp\widehat{\mathbf{P}}_j^{\alpha\,\beta}(\xi) = \sum_{m=1}^M \widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta}(\xi),$$

$$^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta}(\xi) = \widehat{\phi}(\xi)\int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r\,\sharp}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr$$

where $^\sharp\widehat{\Omega}^\alpha$ is defined in (3. 23).

Moreover, $\widehat{\phi}$ is defined in (1. 9). ω is given in (3. 10). $\lambda_m \in [2^{j-1}, 2^j]$: $\lambda_0 = 2^{j-1}$, $\lambda_M = 2^j$ and $2^{\sigma j-1} \leq \lambda_m - \lambda_{m-1} < 2^{\sigma j}$ for which $0 < \sigma = \sigma(\mathbf{Re}\alpha, \mathbf{Re}\beta) < \frac{1}{2}$ can be chosen sufficiently small.

Lemma One *Given $j > 0$ and $m = 1, 2, \ldots, M$, we have*

$$\left|^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta}(\xi)\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j}2^{j/2}2^{-\varepsilon j}, \qquad |1-|\xi|| \leq 2^{-j} \tag{5. 1}$$

and

$$\left|^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta}(\xi)\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}\left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}} 2^{-\varepsilon j}, \qquad |1-|\xi|| > 2^{-j} \tag{5. 2}$$

for some $\varepsilon = \varepsilon(\mathbf{Re}\alpha, \mathbf{Re}\beta) > 0$.

Remark 5.1 *As required for later estimates, we choose $\sigma = \sigma(\mathbf{Re}\alpha, \mathbf{Re}\beta)$ to satisfy $n\sigma \leq \frac{1}{2}\varepsilon$.*

Recall **Remark 1.2**. There are at most a constant multiple of $2^{(1-\sigma)j}$ many $\lambda_m \in [2^{j-1}, 2^j]$. Consequently, by applying (5. 1) and (5. 2), we find

$$\begin{aligned}\left|^\sharp\widehat{\mathbf{P}}_j^{\alpha\,\beta}(\xi)\right| &\leq \sum_{m=1}^M \left|^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta}(\xi)\right| \\ &\leq \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{j/2}2^{-\varepsilon j}, \qquad |1-|\xi|| \leq 2^{-j}\end{aligned} \tag{5. 3}$$

and

$$\begin{aligned}\left|^\sharp\widehat{\mathbf{P}}_j^{\alpha\,\beta}(\xi)\right| &\leq \sum_{m=1}^M \left|^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta}(\xi)\right| \\ &\leq \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}} 2^{-\varepsilon j}, \qquad |1-|\xi|| > 2^{-j}.\end{aligned} \tag{5. 4}$$

Recall **Remark 3.2**. Consider

$$\left|\sum_{j>T} {}^{\sharp}\widehat{\mathbf{P}}_j^{\alpha\ \beta}(\xi)\right| \ \leq \ \sum_{j>T:|1-|\xi||\leq 2^{-j}} \left|{}^{\sharp}\widehat{\mathbf{P}}_j^{\alpha\ \beta}(\xi)\right| + \sum_{j>T:|1-|\xi||>2^{-j}} \left|{}^{\sharp}\widehat{\mathbf{P}}_j^{\alpha\ \beta}(\xi)\right|.$$

We have

$$\begin{aligned}\sum_{j>T:|1-|\xi||\leq 2^{-j}} \left|{}^{\sharp}\widehat{\mathbf{P}}_j^{\alpha\ \beta}(\xi)\right| \ &\leq \ \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} \sum_{j>T:|1-|\xi||\leq 2^{-j}} 2^{j/2}2^{-\varepsilon j} \qquad \text{by (5. 3)} \\ &\leq \ \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}-\frac{\varepsilon}{2}} \sum_{j>T} 2^{-\left(\frac{\varepsilon}{2}\right)j} \qquad (5.\ 5) \\ &\leq \ \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}-\frac{\varepsilon}{2}} 2^{-T\varepsilon/2}\end{aligned}$$

and

$$\begin{aligned}\sum_{j>T:|1-|\xi||>2^{-j}} \left|{}^{\sharp}\widehat{\mathbf{P}}_j^{\alpha\ \beta}(\xi)\right| \ &\leq \ \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha}e^{\mathbf{cIm}\beta} \sum_{j>T:|1-|\xi||>2^{-j}} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}} 2^{-\varepsilon j} \qquad \text{by (5. 4)} \\ &\leq \ \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha}e^{\mathbf{cIm}\beta} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}-\frac{\varepsilon}{2}} \sum_{j>T} 2^{-\left(\frac{\varepsilon}{2}\right)j} \qquad (5.\ 6) \\ &\leq \ \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha}e^{\mathbf{cIm}\beta} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}-\frac{\varepsilon}{2}} 2^{-T\varepsilon/2}.\end{aligned}$$

From (5. 5)-(5. 6), we conclude (3. 25).

5.1 Proof of Lemma One

Recall ${}^{\sharp}\Omega^{\alpha}$ defined in (3. 22)-(3. 23). We have

$$\begin{aligned}{}^{\sharp}\widehat{\mathbf{P}}_{j\ m}^{\alpha\ \beta}(\xi) \ &= \ \widehat{\phi}(\xi) \int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r}{}^{\sharp}\Omega^{\alpha}(r\xi)\omega(r)|r|^{2\beta-1}dr \\ &= \ \widehat{\phi}(\xi) \int_{\lambda_{m-1}\leq|r|<\lambda_m} e^{-2\pi\mathbf{i}r} \left(\frac{1}{|r\xi|}\right)^{\alpha-1} \mathbf{J}_{\alpha-1}\left(2\pi|r\xi|\right)\omega(r)|r|^{2\beta-1}dr.\end{aligned}$$

Note that $\mathbf{supp}\widehat{\phi} \subset \left\{\xi\in\mathbb{R}^n\colon \frac{1}{3}<|\xi|\leq 3\right\}$ and

$$\omega(r) \ = \ \frac{-1}{2\pi\mathbf{i}}\frac{1}{r} - \frac{1}{4\pi^2r^2}\left[1-e^{-2\pi\mathbf{i}r}\right]$$

as shown in (3. 10).

For symmetry reason, we consider $r > 0$ only. Define

$$ {}^{\sharp}\widehat{\mathbf{Q}}_{j\ m}^{\alpha\ \beta}(\xi) \;=\; \widehat{\phi}(\xi) \int_{\lambda_{m-1}}^{\lambda_m} e^{-2\pi \mathbf{i} r} \left(\frac{1}{r|\xi|}\right)^{\alpha-1} \mathbf{J}_{\alpha-1}(2\pi r|\xi|)\, r^{2\beta-2} dr \tag{5. 7} $$

and

$$ {}^{\sharp}\widehat{\mathbf{R}}_{j\ m}^{\alpha\ \beta}(\xi) \;=\; \widehat{\phi}(\xi) \int_{\lambda_{m-1}}^{\lambda_m} e^{-2\pi \mathbf{i} r} \left[1 - e^{-2\pi \mathbf{i} r}\right] \left(\frac{1}{r|\xi|}\right)^{\alpha-1} \mathbf{J}_{\alpha-1}(2\pi r|\xi|)\, r^{2\beta-3} dr. \tag{5. 8} $$

Because $\mathbf{Re}\alpha \geq \left[\frac{2n}{2n-1}\right] \mathbf{Re}\beta$ and $\frac{2n-1}{4n} < \mathbf{Re}\beta < \frac{2n-1}{2n-2}$, we find

$$ \mathbf{Re}\alpha \;\geq\; \left[\frac{2n}{2n-1}\right] \mathbf{Re}\beta \;>\; \frac{1}{2}, \qquad 2\mathbf{Re}\beta - \mathbf{Re}\alpha \;\leq\; \left[\frac{2n-2}{2n-1}\right] \mathbf{Re}\beta \;<\; 1. \tag{5. 9} $$

By using the norm estimate of Bessel functions in (A. 5), we have

$$ \begin{aligned} \left|{}^{\sharp}\Omega^{\alpha}(r\xi)\right| \;&=\; \left(\frac{1}{r|\xi|}\right)^{\mathbf{Re}\alpha-1} |\mathbf{J}_{\alpha-1}(2\pi r|\xi|)| \\ &\leq\; \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \left\{\frac{1}{1+|r\xi|}\right\}^{\mathbf{Re}\alpha-\frac{1}{2}}. \end{aligned} $$

This further implies

$$ \begin{aligned} \left|{}^{\sharp}\widehat{\mathbf{R}}_{j}^{\alpha\ \beta}(\xi)\right| \;&\leq\; \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \int_{\lambda_{m-1}}^{\lambda_m} \left\{\frac{1}{1+r}\right\}^{\mathbf{Re}\alpha-\frac{1}{2}} r^{2\mathbf{Re}\beta-3} dr \qquad \left(\tfrac{1}{3} < |\xi| \leq 3\right) \\ &\leq\; \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} \int_{\lambda_{m-1}}^{\lambda_m} r^{2\mathbf{Re}\beta-\mathbf{Re}\alpha-2-\frac{1}{2}} dr \\ &\leq\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-2-\frac{1}{2}\right]j} |\lambda_m - \lambda_{m-1}| \\ &\leq\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-2-\frac{1}{2}\right]j} 2^{\sigma j} \\ &=\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-1\right]j} 2^{-j/2}, \qquad 2\mathbf{Re}\beta - \mathbf{Re}\alpha - 1 < 0. \end{aligned} \tag{5. 10} $$

Next, we aim to show

$$ \left|{}^{\sharp}\widehat{\mathbf{Q}}_{j\ m}^{\alpha\ \beta\ \ell}(\xi)\right| \;\leq\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{C}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j} 2^{j/2} 2^{-j\varepsilon}, \qquad |1 - |\xi|| \leq 2^{-j} \tag{5. 11} $$

and

$$ \left|{}^{\sharp}\widehat{\mathbf{Q}}_{j\ m}^{\alpha\ \beta\ \ell}(\xi)\right| \;\leq\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{C}|\mathbf{Im}\alpha|} e^{\mathbf{C}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}} 2^{-j\varepsilon}, \qquad |1 - |\xi|| > 2^{-j} \tag{5. 12} $$

for some $\varepsilon = \varepsilon(\mathbf{Re}\alpha, \mathbf{Re}\beta) > 0$.

Later, it should be clear that the same estimates regarding ${}^{\sharp}\widehat{\mathbf{Q}}_{j\ m}^{\alpha\ \beta}$ and ${}^{\sharp}\widehat{\mathbf{R}}_{j\ m}^{\alpha\ \beta}$ remain to be true for $r \in [-\lambda_m, -\lambda_{m-1}]$. Therefore, we conclude (5. 1) and (5. 2) from (5. 10) and (5. 11)-(5. 12).

By using the asymptotic expansion of Bessel functions in (A. 2), we write

$$ {}^{\sharp}\widehat{\mathbf{Q}}_{j\,m}^{\alpha\,\beta}(\xi) = \widehat{\phi}(\xi)\int_{\lambda_{m-1}}^{\lambda_m} e^{-2\pi\mathbf{i}r}\Big[{}^{\sharp}\mathfrak{S}^{\alpha}(r|\xi|) + {}^{\sharp}\mathcal{E}^{\alpha}(r|\xi|)\Big] r^{2\beta-2}dr \tag{5. 13} $$

where

$$ {}^{\sharp}\mathfrak{S}^{\alpha}(\rho) = \frac{1}{\pi}\left(\frac{1}{\rho}\right)^{\alpha-\frac{1}{2}}\cos\left[2\pi\rho - \frac{\pi}{2}\alpha + \frac{\pi}{4}\right], \qquad \rho > 0 \tag{5. 14} $$

and

$$ \left|{}^{\sharp}\mathcal{E}^{\alpha}(\rho)\right| \le \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\begin{cases} \rho^{-\mathbf{Re}\alpha+\frac{1}{2}}, & 0<\rho\le 1, \\ \rho^{-\mathbf{Re}\alpha-\frac{1}{2}}, & \rho > 1. \end{cases} \tag{5. 15} $$

By using (5. 15), we find

$$ \begin{aligned} \left|\widehat{\phi}(\xi)\int_{\lambda_{m-1}}^{\lambda_m} e^{-2\pi\mathbf{i}r\,\sharp}\mathcal{E}^{\alpha}(r|\xi|)\, r^{2\beta-2}dr\right| &\le \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\int_{\lambda_{m-1}}^{\lambda_m}\left(\frac{1}{r}\right)^{\mathbf{Re}\alpha+\frac{1}{2}} r^{2\mathbf{Re}\beta-2}dr \\ &\le \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-2-\frac{1}{2}\right]j}2^{\sigma j} \\ &= \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j}2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-1\right]j}2^{-j/2}. \end{aligned} \tag{5. 16} $$

Because of Euler's formulae, we replace the cosine function in (5. 14) with $e^{\pm 2\pi\mathbf{i}\rho}$ multiplied by $\mathfrak{B}_{\mathbf{Re}\alpha}e^{\mathbf{c}|\mathbf{Im}\alpha|}$. Assert

$$ \begin{aligned} {}^{\sharp}\widehat{\mathbf{A}}_{j\,m}^{\alpha\,\beta}(\xi) &= \widehat{\phi}(\xi)\int_{\lambda_{m-1}}^{\lambda_m} e^{2\pi\mathbf{i}\left[|\xi|-1\right]r}\left(\frac{1}{r|\xi|}\right)^{\alpha-\frac{1}{2}} r^{2\beta-2}dr \\ &= |\xi|^{\frac{1}{2}-\alpha}\widehat{\phi}(\xi)\int_{\lambda_{m-1}}^{\lambda_m} e^{2\pi\mathbf{i}\left[|\xi|-1\right]r} r^{2\beta-\alpha+\frac{1}{2}-2}dr. \end{aligned} \tag{5. 17} $$

Suppose $|1-|\xi|| \le 2^{-j}$. We have

$$ \begin{aligned} \left|{}^{\sharp}\widehat{\mathbf{A}}_{j\,m}^{\alpha\,\beta}(\xi)\right| &\le \mathfrak{B}_{\mathbf{Re}\alpha}\int_{\lambda_{m-1}}^{\lambda_m} r^{2\mathbf{Re}\beta-\mathbf{Re}\alpha+\frac{1}{2}-2}dr \\ &\le \mathfrak{B}_{\mathbf{Re}\alpha}\, 2^{j/2}2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-2\right]j}2^{\sigma j} \\ &= \mathfrak{B}_{\mathbf{Re}\alpha}\, 2^{-(1-\sigma)j}2^{j/2}2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-1\right]j}. \end{aligned} \tag{5. 18} $$

On the other hand, suppose $|1-|\xi|| > 2^{-j}$. By integration by parts *w.r.t* r in (5. 17), we find

$$ \begin{aligned} {}^{\sharp}\widehat{\mathbf{A}}_{j\,m}^{\alpha\,\beta}(\xi) &= \frac{1}{2\pi\mathbf{i}}|\xi|^{\frac{1}{2}-\alpha}\widehat{\phi}(\xi)\frac{1}{|\xi|-1}e^{2\pi\mathbf{i}\left[|\xi|-1\right]r}r^{2\beta-\alpha+\frac{1}{2}-2}\Big|_{\lambda_{m-1}}^{\lambda_m} \\ &-\frac{1}{2\pi\mathbf{i}}\left[2\beta-\alpha-\tfrac{3}{2}\right]|\xi|^{\frac{1}{2}-\alpha}\widehat{\phi}(\xi)\frac{1}{|\xi|-1}\int_{\lambda_{m-1}}^{\lambda_m} e^{2\pi\mathbf{i}\left[|\xi|-1\right]r}r^{2\beta-\alpha-\frac{1}{2}-2}dr. \end{aligned} \tag{5. 19} $$

From (5. 19), by using the mean-value theorem on the boundary term, we have

$$
\begin{aligned}
\left|{}^\sharp\widehat{\mathbf{A}}^{\alpha\,\beta}_{j\,m}(\xi)\right| \;&\le\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} e^{\mathbf{cIm}\beta} \left|\frac{1}{1-|\xi|}\right| 2^{j\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-\frac{1}{2}-2\right]} |\lambda_m-\lambda_{m-1}| \\
&+\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} e^{\mathbf{cIm}\beta} \left|\frac{1}{1-|\xi|}\right| \int_{\lambda_{m-1}}^{\lambda_m} r^{2\mathbf{Re}\beta-\mathbf{Re}\alpha-\frac{1}{2}-2} dr \\
&\le\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} e^{\mathbf{cIm}\beta} \left|\frac{1}{1-|\xi|}\right| 2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-\frac{1}{2}-2\right]j} 2^{\sigma j} \qquad (5.\ 20) \\
&\le\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} e^{\mathbf{cIm}\beta}\, 2^{-(1-\sigma)j} \left|\frac{1}{1-|\xi|}\right| 2^{-j/2} 2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-1\right]j} \\
&\le\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{cIm}\alpha} e^{\mathbf{cIm}\beta}\, 2^{-(1-\sigma)j} \left|\frac{1}{1-|\xi|}\right|^{\frac{1}{2}} 2^{\left[2\mathbf{Re}\beta-\mathbf{Re}\alpha-1\right]j}.
\end{aligned}
$$

Our estimates in (5. 18)-(5. 20) remain to be true if $e^{2\pi \mathbf{i} r|\xi|}$ in (5. 17) is replaced by $e^{-2\pi\mathbf{i}r|\xi|}$. Together with (5. 16), we conclude (5. 11)-(5. 12). This finishes the proof of **Lemma One**.

5.2 Size of ${}^\sharp\widehat{\mathbf{U}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi)$ and ${}^\sharp\widehat{\mathbf{V}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi)$

We begin to prove (3. 30) and (3. 31) in **Proposition One**. Recall (3. 27). We have

$$
\begin{aligned}
{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\nu}_{j\,m}(\xi) \;&=\; \varphi_j^\nu(\xi)\widehat{\phi}(\xi) \int_{\lambda_{m-1}\le|r|<\lambda_m} e^{-2\pi\mathbf{i}r\sharp}\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1} dr \\
&=\; \varphi_j^\nu(\xi)\, {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_{j\,m}(\xi)
\end{aligned}
$$

where φ_j^ν is defined in (3. 14). Moreover,

$$
{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi) \;=\; \sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell} {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\nu}_{j\,m}(\xi) \;=\; {}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_{j\,m}(\xi) \sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell} \varphi_j^\nu(\xi).
$$

$\mathcal{Z}_\ell$ is a subset of $\left\{\xi_j^\nu\right\}_\nu$: ξ_j^ν s belonging to $\mathcal{Z}_\ell$ are almost equally distributed on $\mathbb{S}^{n-1}$ with a grid length between $\mathbf{c}2^{-j/2}2^{\sigma j}$ and $2\mathbf{c}2^{-j/2}2^{\sigma j}$. See **(a)-(b)** below (1. 24).

Remark 5.2 *For φ_j^ν defined in (3. 14), it is easy to see*

$$
0\le\varphi_j^\nu(\xi)\le 1, \qquad \sum_{\nu:\xi_j^\nu\in\mathcal{U}_\ell}\varphi_j^\nu(\xi)\le 1.
$$

By applying (5. 1)-(5. 2) in **Lemma One**, we find

$$
\begin{aligned}
\left|{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta\,\ell}_{j\,m}(\xi)\right| \;&\le\; \left|{}^\sharp\widehat{\mathbf{P}}^{\alpha\,\beta}_{j\,m}(\xi)\right| \\
&\le\; \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|} e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j} 2^{j/2} 2^{-\varepsilon j}.
\end{aligned}
\qquad (5.\ 21)
$$

Recall $\Psi^{\alpha\ \ell}_{j\ m}, \Psi^{\alpha\ \mu}_{j\ m}$ defined in (3. 18) and ${}^\sharp\mathbf{U}^{\alpha\ \beta\ \ell}_{j\ m}, {}^\sharp\mathbf{V}^{\alpha\ \beta\ \ell}_{j\ m}$ defined in (3. 28)-(3. 29). We find

$$\begin{aligned}
{}^\sharp\mathbf{U}^{\alpha\ \beta\ \ell}_{j\ m}(x) + {}^\sharp\mathbf{V}^{\alpha\ \beta\ \ell}_{j\ m}(x) &= {}^\sharp\mathbf{P}^{\alpha\ \beta\ \ell}_{j\ m}(x)\left[1-\Psi^{\alpha\ \ell}_{j\ m}(x)\right] + \sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu\neq\xi^\mu_j}{}^\sharp\mathbf{P}^{\alpha\ \beta\ \nu}_{j\ m}(x)\\
&\quad+\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu=\xi^\mu_j}{}^\sharp\mathbf{P}^{\alpha\ \beta\ \nu}_{j\ m}(x)\\
&= {}^\sharp\mathbf{P}^{\alpha\ \beta\ \ell}_{j\ m}(x)\left[1-\Psi^{\alpha\ \ell}_{j\ m}(x)\right] + \Psi^{\alpha\ \ell}_{j\ m}(x){}^\sharp\mathbf{P}^{\alpha\ \beta\ \ell}_{j\ m}(x)\\
&= {}^\sharp\mathbf{P}^{\alpha\ \beta\ \ell}_{j\ m}(x).
\end{aligned}$$

Hence, (3. 30) can be obtained by putting together (3. 31) and (5. 21).

Our task is left to show

$$\left|{}^\sharp\widehat{\mathbf{V}}^{\alpha\ \beta\ \ell}_{j\ m}(\xi)\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}2^{-j\varepsilon}, \qquad \varepsilon=\varepsilon(\mathbf{Re}\alpha,\mathbf{Re}\beta)>0. \tag{5. 22}$$

Let ${}^\sharp\mathbf{V}^{\alpha\ \beta\ \ell}_{j\ m}$ defined in (3. 29). We have

$$\begin{aligned}
{}^\sharp\widehat{\mathbf{V}}^{\alpha\ \beta\ \ell}_{j\ m}(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi\mathbf{i}x\cdot\xi}\left\{\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j=\xi^\mu_j}{}^\sharp\mathbf{P}^{\alpha\ \beta\ \nu}_{j\ m}(x)\right\}dx\\
&= \sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\ \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j=\xi^\mu_j}\int_{\mathbb{R}^n}\widehat{\Psi}^{\alpha\ \mu}_{j\ m}(\xi-\zeta){}^\sharp\widehat{\mathbf{P}}^{\alpha\ \beta\ \nu}_{j\ m}(\zeta)d\zeta.
\end{aligned}\tag{5. 23}$$

On the other hand, for $\Psi^{\alpha\ \mu}_{j\ m}$ defined in (3. 18), we find

$$\widehat{\Psi}^{\alpha\ \mu}_{j\ m}(\xi-\zeta) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot(\xi-\zeta)}\varphi\left[2^{-\sigma j-1}\left|\lambda_m-\left(\mathbf{L}^T_\mu x\right)_1\right|\right]\prod_{i=2}^n\varphi\left[2^{-[\frac{1}{2}+\sigma]j}\left|\left(\mathbf{L}^T_\mu x\right)_i\right|\right]dx. \tag{5. 24}$$

$\mathbf{L}_\mu$ is an orthogonal matrix with $\det\mathbf{L}_\mu=1$. Moreover, $\mathbf{L}^T_\mu\xi^\mu_j=(1,0)^T\in\mathbb{R}\times\mathbb{R}^{n-1}$. Write $\zeta=\mathbf{L}_\mu\eta$ and $x=\mathbf{L}_\mu u$ inside (5. 23)-(5. 24). We have

$$\widehat{\Psi}^{\alpha\ \mu}_{j\ m}(\xi-\mathbf{L}_\mu\eta) = \int_{\mathbb{R}^n} e^{-2\pi\mathbf{i}\left[\mathbf{L}^T_\mu\xi-\eta\right]\cdot u}\varphi\left[2^{-\sigma j-1}\left|\lambda_m-u_1\right|\right]\prod_{i=2}^n\varphi\left[2^{-[\frac{1}{2}+\sigma]j}|u_i|\right]du \tag{5. 25}$$

and

$${}^\sharp\widehat{\mathbf{P}}^{\alpha\ \beta\ \nu}_{j\ m}(\mathbf{L}_\mu\eta) = \varphi^\nu_j(\mathbf{L}_\mu\eta)\widehat{\phi}(\mathbf{L}_\mu\eta)\int_{\lambda_{m-1}\leq|r|<\lambda_m}e^{-2\pi\mathbf{i}r}\,{}^\sharp\widehat{\Omega}^\alpha(r\mathbf{L}_\mu\eta)\omega(r)|r|^{2\beta-1}dr.$$

Let $\varphi\in C^\infty_o(\mathbb{R})$ be the smooth bump-function as before. We find

$$\mathbf{vol\ supp}\left\{\varphi\left[2^{-\sigma j-1}|\lambda_m-u_1|\right]\prod_{i=2}^n\varphi\left[2^{-[\frac{1}{2}+\sigma]j}|u_i|\right]\right\} \lesssim 2^{(n-1)[\frac{1}{2}+\sigma]j}2^{\sigma j} = 2^{(\frac{n-1}{2})j}2^{n\sigma j}. \tag{5. 26}$$

From (5. 25) and (5. 26), we have

$$\left|\widehat{\Psi}^{\alpha\ \mu}_{j\ m}(\xi-\mathbf{L}_\mu\eta)\right| \lesssim 2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}. \tag{5. 27}$$

Observe that

$$\begin{aligned}
&\partial_{u_1}\varphi\left[2^{-\sigma j-1}|\lambda_m\pm u_1|\right] = 0, \qquad \text{if} \qquad |\lambda_m\pm u_1|<2^{\sigma j+1},\\
&\partial_{u_i}\varphi\left[2^{-\left[\frac12+\sigma\right]j}|u_i|\right] = 0 \qquad \text{if} \qquad |u_i|<2^{\left[\frac12+\sigma\right]j},\ \ i=2,\dots,n.
\end{aligned}$$

A direct computation shows

$$\begin{aligned}
&\left|\partial^N_{u_1}\varphi\left[2^{-\sigma j-2}|\lambda_m\pm u_1|\right]\right| \leq \mathfrak{B}_N\,2^{-N\sigma j},\\
&\left|\partial^N_{u_i}\varphi\left[2^{-\left[\frac12+\sigma\right]j}|u_i|\right]\right| \leq \mathfrak{B}_N\,2^{-N\left[\frac12+\sigma\right]j}, \qquad i=2,\dots,n
\end{aligned} \tag{5. 28}$$

for every $N\geq 0$.

Case 1 Suppose $|\xi|\leq\frac{1}{10}$ or $|\xi|>10$. For $\frac13<|\eta|\leq 3$, we either have $\left|[\mathbf{L}^T_\mu\xi-\eta]_1\right|>\frac{1}{5\sqrt{n}}$ or $\left|[\mathbf{L}^T_\mu\xi-\eta]_i\right|>\frac{1}{5\sqrt{n}}$ for some $i=2,\dots,n$.

By integration by parts either *w.r.t* u_1 or u_i inside (5. 25) and using (5. 26)-(5. 28), we find

$$\left|\widehat{\Psi}^{\alpha\ \mu}_{j\ m}(\xi-\mathbf{L}_\mu\eta)\right| \leq \mathfrak{B}_N\,2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}\,2^{-N\sigma j}, \qquad N\geq 0.$$

By choosing N sufficiently large depending on $\sigma=\sigma(\mathbf{Re}\alpha,\mathbf{Re}\beta)>0$, we obtain

$$\left|\widehat{\Psi}^{\alpha\ \mu}_{j\ m}(\xi-\mathbf{L}_\mu\eta)\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}. \tag{5. 29}$$

Case 2 Let $\frac{1}{10}<|\xi|\leq 10$. We must have

$$\xi \in \Gamma^{\overline{\mu}}_j = \left\{\xi\in\mathbb{R}^n\colon\left|\frac{\xi}{|\xi|}-\xi^{\overline{\mu}}_j\right|<2^{-j/2+1}\right\} \qquad \text{for some } \overline{\mu}.$$

Lemma 5.1 *Let $\frac{1}{10}<|\xi|\leq 10$ and $\frac13<|\eta|\leq 3$. Suppose $\xi^\nu_j=\xi^\mu_j$ and $\xi^{\overline{\mu}}_j\neq\xi^\mu_j$ belonging to $\mathcal{Z}_\ell$. Either we have $\left|\left[\mathbf{L}^T_\mu\xi-\eta\right]_1\right|\approx 2$ or $\left|\left[\mathbf{L}^T_\mu\xi-\eta\right]_\imath\right|\gtrsim 2^{-j/2}2^{\sigma j}$ for some $\imath\in\{2,\dots,n\}$ whenever $\mathbf{L}_\mu\eta\in\Gamma^\nu_j$.*

Proof Note that $\mathbf{L}^T_\mu\xi^\mu_j=(1,0)^T\in\mathbb{R}\times\mathbb{R}^{n-1}$. Consider $\mathbf{L}_\mu\eta\in\Gamma^\nu_j$ for which $\xi^\nu_j=\xi^\mu_j$. We have $\left|\xi^\nu_j-\xi^\mu_j\right|=\left|\mathbf{L}^T_\mu\xi^\nu_j-(1,0)^T\right|=0$. By using the triangle inequality, we find

$$\begin{aligned}
\frac{|\eta_i|}{|\eta|} \leq \left|\frac{\eta}{|\eta|}-(1,0)^T\right| &\leq \left|\frac{\eta}{|\eta|}-\mathbf{L}^T_\mu\xi^\nu_j\right|+\left|\mathbf{L}^T_\mu\xi^\nu_j-(1,0)^T\right|\\
&\leq 2^{-j/2+1}, \qquad i=2,\dots,n.
\end{aligned} \tag{5. 30}$$

On the other hand, recall $\left|\xi_j^\mu - \xi_j^\nu\right| \geq \mathbf{c}2^{-j/2}2^{\sigma j}$ for every $\xi_j^\mu, \xi_j^\nu \in \mathcal{Z}_\ell$ and $\xi_j^\mu \neq \xi_j^\nu$. Therefore, we have $\left|\xi_j^{\overline{\mu}} - \xi_j^\mu\right| \geq \mathbf{c}2^{-j/2}2^{\sigma j}$ if $\xi_j^{\overline{\mu}} \neq \xi_j^\mu$.

Let $\xi \in \Gamma_j^{\overline{\mu}}$. By using the triangle inequality, we find

$$\begin{aligned}
\left|\frac{\mathbf{L}_\mu^T\xi}{|\xi|} - (1,0)^T\right| &= \left|\frac{\xi}{|\xi|} - \xi_j^\mu\right| \geq \left|\xi_j^{\overline{\mu}} - \xi_j^\mu\right| - \left|\frac{\xi}{|\xi|} - \xi_j^{\overline{\mu}}\right| \\
&\geq \mathbf{c}2^{-j/2}2^{\sigma j} - 2^{-j/2+1} \\
&> [\mathbf{c}-2]\, 2^{-j/2}2^{\sigma j}.
\end{aligned} \tag{5. 31}$$

Suppose $\left|\left(\frac{\mathbf{L}_\mu^T\xi}{|\xi|}\right)_1 - 1\right| \leq \left[\sum_{i=2}^n \left(\frac{\mathbf{L}_\mu^T\xi}{|\xi|}\right)_i^2\right]^{\frac{1}{2}}$. The estimate in (5. 31) further implies

$$\begin{aligned}
\left|\frac{\left(\mathbf{L}_\mu^T\xi\right)_\imath}{|\xi|}\right| &\geq \frac{1}{\sqrt{2}\sqrt{n-1}}\left|\frac{\mathbf{L}_\mu^T\xi}{|\xi|} - (1,0)^T\right| \\
&> \frac{1}{\sqrt{2}\sqrt{n-1}}[\mathbf{c}-2]\, 2^{-j/2}2^{\sigma j}, \qquad \text{for some} \qquad \imath \in \{2,\ldots,n\}.
\end{aligned} \tag{5. 32}$$

Suppose $\left|\left(\frac{\mathbf{L}_\mu^T\xi}{|\xi|}\right)_1 - 1\right| > \left[\sum_{i=2}^n \left(\frac{\mathbf{L}_\mu^T\xi}{|\xi|}\right)_i^2\right]^{\frac{1}{2}}$. As a geometric fact, $\frac{\mathbf{L}_\mu^T\xi}{|\xi|}$ belongs to the semi-sphere opposite to $(1,0)^T$. We either have $\left(\frac{\mathbf{L}_\mu^T\xi}{|\xi|}\right)_1 \approx -1$ or $\left|\left(\frac{\mathbf{L}_\mu^T\xi}{|\xi|}\right)_\imath\right| > \frac{1}{\sqrt{2}\sqrt{n-1}}[\mathbf{c}-2]\, 2^{-j/2}2^{\sigma j}$ for some $\imath \in \{2,\ldots,n\}$ as (5. 32).

By taking into account $\frac{1}{3} < |\eta| \leq 3$ and $\frac{1}{10} < |\xi| \leq 10$ in (5. 30) and (5. 31)-(5. 32), we find

$$|\eta_\imath| \leq 3\,2^{-j/2+1}, \qquad \left|\left(\mathbf{L}_\mu^T\xi\right)_\imath\right| \geq \frac{1}{10\sqrt{2}\sqrt{n-1}}[\mathbf{c}-2]\, 2^{-j/2}2^{\sigma j}.$$

Consequently, $\left|[\mathbf{L}_\mu^T\xi - \eta]_\imath\right| \gtrsim 2^{-j/2}2^{\sigma j}$ provided that $\mathbf{c}$ is large. $\square$

Let $\xi \in \Gamma_j^{\overline{\mu}}$ and $\xi_j^{\overline{\mu}} \neq \xi_j^\mu$ belonging to $\mathcal{Z}_\ell$. By applying **Lemma 5.1** and using the estimates in (5. 26)-(5. 28), an N-fold integration by parts either *w.r.t* u_1 or $u_\imath$ inside (5. 25) shows

$$\begin{aligned}
\left|\widehat{\Psi}^{\alpha\ \mu}_{j\ m}(\xi - \mathbf{L}_\mu\eta)\right| &\leq \mathfrak{B}_N\, 2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}\left[2^{-N\sigma j} + 2^{N\left[\frac{1}{2}-\sigma\right]j}2^{-N\left[\frac{1}{2}+\sigma\right]j}\right] \\
&\leq \mathfrak{B}_N\, 2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}\, 2^{-N\sigma j} \\
&\leq \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}, \quad N \text{ sufficiently large depending on } \sigma = \sigma(\mathbf{Re}\alpha, \mathbf{Re}\beta) > 0.
\end{aligned} \tag{5. 33}$$

Now, we assert

$$\int_{\mathbb{R}^n} \widehat{\Psi}_{j\,m}^{\alpha\,\mu}(\xi - \mathbf{L}_\mu\eta)^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta\,\nu}(\mathbf{L}_\mu\eta)d\eta = \sum_{k=0}^\infty \int_{\mathcal{S}_k} \widehat{\Psi}_{j\,m}^{\alpha\,\mu}(\xi - \mathbf{L}_\mu\eta)^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta\,\nu}(\mathbf{L}_\mu\eta)d\eta,$$
$$\mathcal{S}_k = \left\{\eta \in \mathbb{R}^n : 2^{-k} \le \big|1 - |\eta|\big| < 2^{-k+1}\right\}. \tag{5. 34}$$

Observe that

$$\mathbf{vol}\left\{\mathcal{S}_k \cap \mathbf{supp}^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta\,\nu}\right\} \lesssim 2^{-\left(\frac{n-1}{2}\right)j}2^{-k}. \tag{5. 35}$$

Recall ${}^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta\,\nu}(\mathbf{L}_\mu\eta) = \varphi_j^\nu(\mathbf{L}_\mu\eta)^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta}(\mathbf{L}_\mu\eta)$ where $0 \le \varphi_j^\nu(\mathbf{L}_\mu\eta) \le 1$. See **Remark 5.2**.

For $j \le k$, we have $\big|1 - |\eta|\big| \le 2^{-k+1} \le 2^{-j+1}$. By applying (5. 1) in **Lemma One**, we find

$$\begin{aligned}\left|{}^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta\,\nu}(\mathbf{L}_\mu\eta)\right| &\le \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j}2^{j/2}2^{-\varepsilon j} \\ &\le \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-(1-\sigma)j}2^{k/2}2^{-\varepsilon j}.\end{aligned} \tag{5. 36}$$

For $j > k$, we have $\big|1 - |\eta|\big| > 2^{-k} > 2^{-j}$. By applying (5. 2) in **Lemma One**, we find

$$\begin{aligned}\left|{}^\sharp\widehat{\mathbf{P}}_{j}^{\alpha\,\beta\,\nu}(\mathbf{L}_\mu\eta)\right| &\le \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}\left|\frac{1}{1-|\eta|}\right|^{\frac{1}{2}} 2^{-\varepsilon j} \\ &\le \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}2^{k/2}2^{-\varepsilon j}.\end{aligned} \tag{5. 37}$$

From (5. 35) and (5. 36)-(5. 37), we obtain

$$\begin{aligned}&\sum_{k=0}^\infty \int_{\mathcal{S}_k} \left|{}^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta\,\nu}(\mathbf{L}_\mu\eta)\right| d\eta \\ &\le \sum_{k=0}^\infty \left|{}^\sharp\widehat{\mathbf{P}}_{j}^{\alpha\,\beta\,\nu}(\mathbf{L}_\mu\eta)\right| \mathbf{vol}\left\{\mathcal{S}_k \cap \mathbf{supp}^\sharp\widehat{\mathbf{P}}_{j\,m}^{\alpha\,\beta\,\nu}\right\} \\ &\le \sum_{k=0}^\infty \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}2^{-(1-\sigma)j}2^{k/2}2^{-\varepsilon j}\, 2^{-\left(\frac{n-1}{2}\right)j}2^{-k} \\ &= \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j}\sum_{k=0}^\infty 2^{-k/2} \\ &\le \mathfrak{B}_{\mathbf{Re}\alpha\ \mathbf{Re}\beta}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\, 2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j}.\end{aligned} \tag{5. 38}$$

Recall **Remark 1.3**. For each $\mathcal{Z}_\ell \subset \left\{\xi_j^\nu\right\}_\nu$, there are at most a constant multiple of $2^{\left[\frac{n-1}{2}-(n-1)\sigma\right]j}$ many $\xi_j^\nu \in \mathcal{Z}_\ell$.

Consider $|\xi|\leq\frac{1}{10}$ or $|\xi|>10$. From (5. 23) and (5. 34), we have

$$\begin{array}{lr}
\left|{}^\sharp\widehat{\mathbf{V}}^{\alpha~\beta~\ell}_{j~m}(\xi)\right| ~\leq~ \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}\sum_{k=0}^\infty\left|\int_{\mathcal{S}_k}\widehat{\Psi}^{\alpha~\mu}_{j~m}(\xi-\mathbf{L}_\mu\eta){}^\sharp\widehat{\mathbf{P}}^{\alpha~\beta~\nu}_{j~m}(\mathbf{L}_\mu\eta)d\eta\right| & \\
\leq~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}\sum_{k=0}^\infty\int_{\mathcal{S}_k}\left|{}^\sharp\widehat{\mathbf{P}}^{\alpha~\beta~\nu}_{j~m}(\mathbf{L}_\mu\eta)\right|d\eta & \text{by (5. 29)}\\
\leq~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j} & \text{by (5. 38)}\\
\leq~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\,2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}\,2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j} & \text{by } \textbf{Remark 1.3}\\
=~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\,2^{-(n-1)\sigma j}2^{-(1-\sigma)j}2^{-\varepsilon j}. &
\end{array}\tag{5. 39}$$

Let $\frac{1}{10}<|\xi|\leq 10$. From (5. 23) and (5. 34), we have

$$\begin{array}{lr}
\left|{}^\sharp\widehat{\mathbf{V}}^{\alpha~\beta~\ell}_{j~m}(\xi)\right| ~\leq~ \sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}\sum_{k=0}^\infty\left|\int_{\mathcal{S}_k}\widehat{\Psi}^{\alpha~\mu}_{j~m}(\xi-\mathbf{L}_\mu\eta){}^\sharp\widehat{\mathbf{P}}^{\alpha~\beta~\nu}_{j~m}(\mathbf{L}_\mu\eta)d\eta\right| & \\
=~\left[\sum_{\mu:\xi_j^\mu\in\mathcal{U}_\ell,\xi_j^\mu\neq\xi_j^{\overline{\mu}}}+\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell,\xi_j^\mu=\xi_j^{\overline{\mu}}}\right]\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}\sum_{k=0}^\infty\left|\int_{\mathcal{S}_k}\widehat{\Psi}^{\alpha~\mu}_{j~m}(\xi-\mathbf{L}_\mu\eta){}^\sharp\widehat{\mathbf{P}}^{\alpha~\beta~\nu}_{j~m}(\mathbf{L}_\mu\eta)d\eta\right| & \\
\leq~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell,\xi_j^\mu\neq\xi_j^{\overline{\mu}}}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}\sum_{k=0}^\infty\int_{\mathcal{S}_k}\left|{}^\sharp\widehat{\mathbf{P}}^{\alpha~\beta~\nu}_{j~m}(\mathbf{L}_\mu\eta)\right|d\eta & \text{by (5. 33)}\\
+~\mathfrak{B}\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell,\xi_j^\mu=\xi_j^{\overline{\mu}}}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}\sum_{k=0}^\infty\int_{\mathcal{S}_k}\left|{}^\sharp\widehat{\mathbf{P}}^{\alpha~\beta~\nu}_{j~m}(\mathbf{L}_\mu\eta)\right|d\eta & \text{by (5. 27)}\\
\leq~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell,\xi_j^\mu\neq\xi_j^{\overline{\mu}}}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j} & \\
+~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\sum_{\mu:\xi_j^\mu\in\mathcal{Z}_\ell,\xi_j^\mu=\xi_j^{\overline{\mu}}}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell,\xi_j^\nu=\xi_j^\mu}2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}\,2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j} & \text{by (5. 38)}\\
\leq~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\,2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}\,2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j} & \\
+~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\,2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}\,2^{-(1-\sigma)j}2^{-\left(\frac{n-1}{2}\right)j}2^{-\varepsilon j} & \text{by } \textbf{Remark 1.3}\\
\leq~\mathfrak{B}_{\mathbf{Re}\alpha~\mathbf{Re}\beta}e^{\mathbf{c}|\mathbf{Im}\alpha|}e^{\mathbf{c}|\mathbf{Im}\beta|}\,2^{-(1-\sigma)j}2^{n\sigma j}2^{-\varepsilon j}. &
\end{array}\tag{5. 40}$$

We obtain (5. 22) provided that $n\sigma\leq\frac{1}{2}\varepsilon$ as discussed in **Remark 5.1**.

6 On the $\mathbf{L}^1$-norm of ${}^\flat\mathbf{U}^{\alpha\,\beta\,\ell}_{j\,m}$ and ${}^\flat\mathbf{V}^{\alpha\,\beta\,\ell}_{j\,m}$

Recall ${}^\flat\Omega^\alpha$ defined in (3. 32)-(3. 33) and ${}^\flat\widehat{\mathbf{P}}^{\alpha\,\beta\,\ell}_{j\,m}$ defined in (3. 36) for $\mathbf{Re}\alpha > 0$ and $0 < \mathbf{Re}\beta < \frac{1}{2}$. Given $j > 0$, $m = 1, 2, \ldots, M$ and $\ell = 1, \ldots, L$, we have

$$
{}^\flat\mathbf{P}^{\alpha\,\beta\,\ell}_{j\,m}(x) = \sum_{\nu:\xi^\nu_j \in \mathcal{Z}_\ell} {}^\flat\mathbf{P}^{\alpha\,\beta\,\nu}_{j\,m}(x),
$$

$$
{}^\flat\mathbf{P}^{\alpha\,\beta\,\nu}_{j\,m}(x) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\left\{\int_{\lambda_{m-1}\le|r|<\lambda_m} e^{-2\pi\mathbf{i}r}\varphi^\nu_j(\xi)\widehat{\phi}(\xi){}^\flat\widehat{\Omega}^\alpha(r\xi)\omega(r)|r|^{2\beta-1}dr\right\}d\xi,
$$

$$
{}^\flat\widehat{\Omega}^\alpha(\xi) = \left(\frac{1}{|\xi|}\right)^{\frac{n-1}{2}+\mathbf{Re}\alpha-\frac{1}{2}} \mathbf{J}_{\frac{n-1}{2}+\alpha-\frac{1}{2}}\Big(2\pi|\xi|\Big).
$$

$\lambda_m \in [2^{j-1}, 2^j]$: $\lambda_0 = 2^{j-1}$, $\lambda_L = 2^j$ and $2^{\sigma j-1} \le \lambda_m - \lambda_{m-1} < 2^{\sigma j}$ for some $0 < \sigma = \sigma(\mathbf{Re}\alpha) < \frac{1}{2}$ which can be chosen sufficiently small.

φ^ν_j is defined in (3. 14) whose support is contained in $\Gamma^\nu_j = \left\{\xi \in \mathbb{R}^n : \left|\frac{\xi}{|\xi|} - \xi^\nu_j\right| < 2^{-j/2+1}\right\}$.

$\mathcal{Z}_\ell$ is a subset of $\left\{\xi^\nu_j\right\}_\nu$: ξ^ν_j s belonging to each $\mathcal{Z}_\ell$ are almost equally distributed on $\mathbb{S}^{n-1}$ with a grid length between $\mathbf{c}2^{-j/2}2^{\sigma j}$ and $2\mathbf{c}2^{-j/2}2^{\sigma j}$. See **(a)-(b)** below (1. 24). Recall **Remark 1.3**. There are at most a constant multiple of $2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}$ many ξ^ν_j s inside $\mathcal{Z}_\ell$.

$\widehat{\phi}$ is defined in (1. 9) and $\mathbf{supp}\widehat{\phi} \subset \left\{\xi \in \mathbb{R}^n : \frac{1}{3} < |\xi| \le 3\right\}$. Moreover, $\omega(r)$ given in (3. 10) satisfies $|\omega(r)| \lesssim \left[1 + |r|\right]^{-1}$ as shown in (4. 13).

By using the norm estimate of Bessel functions in (A. 5), we find

$$
\begin{aligned}
\left|{}^\flat\widehat{\Omega}^\alpha(r\xi)\right| &= \left(\frac{1}{|r\xi|}\right)^{\frac{n-1}{2}+\mathbf{Re}\alpha-\frac{1}{2}} \left|\mathbf{J}_{\frac{n-1}{2}+\alpha-\frac{1}{2}}\Big(2\pi|r\xi|\Big)\right| \\
&\le \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\left(\frac{1}{1+|r\xi|}\right)^{\frac{n-1}{2}+\mathbf{Re}\alpha}.
\end{aligned} \tag{6. 1}
$$

Fubini's theorem allows us to write

$$
{}^\flat\mathbf{P}^{\alpha\,\beta\,\ell}_{j\,m}(x) = \int_{\lambda_{m-1}\le|r|<\lambda_m} e^{-2\pi\mathbf{i}r}\,{}^\flat\mathbf{P}^{\alpha\,\ell}_{r\,j}(x)\omega(r)|r|^{2\beta-1}dr \tag{6. 2}
$$

for which

$$
\begin{aligned}
{}^\flat\mathbf{P}^{\alpha\,\ell}_{r\,j}(x) &= \sum_{\nu:\xi^\nu_j\in\mathcal{U}_\ell} {}^\flat\mathbf{P}^{\alpha\,\nu}_{r\,j}(x) \\
{}^\flat\mathbf{P}^{\alpha\,\nu}_{r\,j}(x) &= \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}x\cdot\xi}\varphi^\nu_j(\xi)\widehat{\phi}(\xi){}^\flat\widehat{\Omega}^\alpha(r\xi)d\xi.
\end{aligned} \tag{6. 3}
$$

Recall $\Psi^{\alpha\,\ell}_{j\,m}, \Psi^{\alpha\,\mu}_{j\,m}$ defined in (3. 18) and ${}^\flat\mathbf{U}^{\alpha\,\beta\,\ell}_{j\,m}$, ${}^\flat\mathbf{V}^{\alpha\,\beta\,\ell}_{j\,m}$ defined in (3. 37)-(3. 38). We have

$$\begin{aligned} {}^\flat\mathbf{U}^{\alpha\,\beta\,\ell}_{j\,m}(x) &= {}^\flat\mathbf{P}^{\alpha\,\beta\,\ell}_{j\,m}(x)\Big[1-\Psi^{\alpha\,\ell}_{j\,m}(x)\Big] + \sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\,\mu}_{j\,m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu\neq\xi^\mu_j}{}^\flat\mathbf{P}^{\alpha\,\beta\,\nu}_{j\,m}(x),\\ {}^\flat\mathbf{V}^{\alpha\,\beta\,\ell}_{j\,m}(x) &= \sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\,\mu}_{j\,m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu=\xi^\mu_j}{}^\flat\mathbf{P}^{\alpha\,\beta\,\nu}_{j\,m}(x). \end{aligned}$$

In order to prove (3. 39)-(3. 40) in **Proposition Two**, it is suffice to show

$$\begin{aligned} &\int_{\mathbb{R}^n}\Big|{}^\flat\mathbf{P}^{\alpha\,\ell}_{r\,j}(x)\Big|\Big[1-\Psi^{\alpha\,\ell}_{j\,m}(x)\Big]dx+\int_{\mathbb{R}^n}\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\,\mu}_{j\,m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j\neq\xi^\mu_j}\Big|{}^\flat\mathbf{P}^{\alpha\,\nu}_{r\,j}(x)\Big|dx\\ &\le \mathfrak{B}_{N\,\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\,2^{-Nj}2^{-\varepsilon j},\qquad N\ge 0 \end{aligned}\tag{6. 4}$$

and

$$\int_{\mathbb{R}^n}\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\,\mu}_{j\,m}(x)\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j=\xi^\mu_j}\Big|{}^\flat\mathbf{P}^{\alpha\,\nu}_{r\,j}(x)\Big|dx \le \mathfrak{B}_{\mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\,2^{-\varepsilon j}\tag{6. 5}$$

for some $\varepsilon=\varepsilon(\mathbf{Re}\alpha)>0$ whenever $\lambda_{m-1}\le|r|<\lambda_m$.

This is an immediate consequence of using Minkowski integral inequality together with $|\omega(r)|\le\mathfrak{B}\Big[1+|r|\Big]^{-1}$, $0<\mathbf{Re}\beta<\frac{1}{2}$ and $|\lambda_m-\lambda_{m-1}|\le 2^{\sigma j}$.

Let ${}^\flat\mathbf{P}^{\alpha\,\nu}_{r\,j}$ defined in (6. 3). We write

$$\begin{aligned} {}^\flat\mathbf{P}^{\alpha\,\nu}_{r\,j}(x) &= \int_{\mathbb{R}^n}e^{2\pi\mathbf{i}x\cdot\xi}\varphi^\nu_j(\xi)\widehat{\phi}(\xi)\left(\frac{1}{|r\xi|}\right)^{\frac{n-1}{2}+\alpha-\frac{1}{2}}\mathbf{J}_{\frac{n-1}{2}+\alpha-\frac{1}{2}}\Big(2\pi|r\xi|\Big)d\xi\\ &= {}^\flat\mathbf{Q}^{\alpha\,\nu}_{r\,j}(x)+\sum_{k=1}^N{}^\flat\mathbf{Q}^{\alpha\,\nu}_{r\,j\,k}(x)+{}^\flat\mathbf{R}^{\alpha\,\nu}_{r\,j\,N}(x) \end{aligned}\tag{6. 6}$$

by using the asymptotic expansion of Bessel functions in (A. 2)-(A. 4). First,

$${}^\flat\mathbf{Q}^{\alpha\,\nu}_{r\,j}(x) = \frac{1}{\pi}\int_{\mathbb{R}^n}e^{2\pi\mathbf{i}x\cdot\xi}\varphi^\nu_j(\xi)\widehat{\phi}(\xi)\left(\frac{1}{|r\xi|}\right)^{\frac{n-1}{2}+\alpha}\cos\left[2\pi|r\xi|-\left(\frac{n}{2}+\alpha-1\right)\frac{\pi}{2}-\frac{\pi}{4}\right]d\xi.\tag{6. 7}$$

Next, there are finitely many

$$\begin{aligned} &{}^\flat\mathbf{Q}^{\alpha\,\nu}_{r\,j\,k}(x) =\\ &\quad(2\pi)^{-2k}\frac{\mathbf{a}_k}{\pi}\int_{\mathbb{R}^n}e^{2\pi\mathbf{i}x\cdot\xi}\varphi^\nu_j(\xi)\widehat{\phi}(\xi)\left(\frac{1}{|r\xi|}\right)^{\frac{n-1}{2}+\alpha+2k}\cos\left[2\pi|r\xi|-\left(\frac{n}{2}+\alpha-1\right)\frac{\pi}{2}-\frac{\pi}{4}\right]d\xi\\ &+(2\pi)^{-2k+1}\frac{\mathbf{b}_k}{\pi}\int_{\mathbb{R}^n}e^{2\pi\mathbf{i}x\cdot\xi}\varphi^\nu_j(\xi)\widehat{\phi}(\xi)\left(\frac{1}{|r\xi|}\right)^{\frac{n-1}{2}+\alpha+2k-1}\sin\left[2\pi|r\xi|-\left(\frac{n}{2}+\alpha-1\right)\frac{\pi}{2}-\frac{\pi}{4}\right]d\xi \end{aligned}\tag{6. 8}$$

where $\mathbf{a}_k$, $\mathbf{b}_k$ are constants given in (A. 2).

Finally, the remainder term

$$
\begin{aligned}
{}^{\flat}\mathbf{R}^{\alpha\ \nu}_{r\ j\ N}(x) &= \int_{\mathbb{R}^n} e^{2\pi \mathbf{i} x\cdot\xi}\varphi_j^{\nu}(\xi)\widehat{\phi}(\xi)\mathbf{H}^{\alpha}\Big(2\pi|r\xi|\Big)d\xi, \\
\partial_{\xi}^{\gamma}\mathbf{H}^{\alpha}(\rho) &= \mathbf{O}\Big(\rho^{-2N-\frac{1}{2}+|\gamma|}\Big), \qquad \rho\longrightarrow\infty
\end{aligned}
\tag{6. 9}
$$

with the implied constant bounded by $\mathfrak{B}_{\gamma\ \mathbf{Re}\alpha}e^{\mathbf{c}|\mathbf{Im}\alpha|}$.

Recall φ_j^{ν} defined in (3. 14). We find $\partial_{\xi}\varphi_j^{\nu}(\xi)=0$ whenever $\left|\frac{\xi}{|\xi|}-\xi_j^{\nu}\right|\leq 2^{-j/2}$. A direct computation shows

$$
\left|\partial_{\xi}^{\gamma}\left[\varphi_j^{\nu}(\xi)\right]\right| \leq \mathfrak{B}_{\gamma}\,2^{|\gamma|j/2}, \qquad \tfrac{1}{3}<|\xi|\leq 3
$$

for every multi-index γ.

We have $|r|\approx 2^j$ if $\lambda_{m-1}\leq|r|<\lambda_m$. An $(n+1)$-fold integration by parts *w.r.t* ξ inside (6. 9) gives

$$
\left|{}^{\flat}\mathbf{R}^{\alpha\ \nu}_{r\ j\ N}(x)\right| \leq \mathfrak{B}_{\mathbf{Re}\alpha}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}\left(\frac{1}{1+|x|}\right)^{n+1} 2^{-\left[2N+\frac{1}{2}-(n+1)\right]j}.
$$

Similarly to (6. 3), we define

$$
\begin{aligned}
&{}^{\flat}\mathbf{Q}^{\alpha\ \ell}_{r\ j}(x) = \sum_{\nu:\xi_j^{\nu}\in\mathcal{Z}_{\ell}}{}^{\flat}\mathbf{Q}^{\alpha\ \nu}_{r\ j}(x), \qquad {}^{\flat}\mathbf{Q}^{\alpha\ \ell}_{r\ j\ k}(x) = \sum_{\nu:\xi_j^{\nu}\in\mathcal{Z}_{\ell}}{}^{\flat}\mathbf{Q}^{\alpha\ \nu}_{r\ j\ k}(x), \qquad k=1,2,\ldots,N, \\
&{}^{\flat}\mathbf{R}^{\alpha\ \ell}_{r\ j\ N}(x) = \sum_{\nu:\xi_j^{\nu}\in\mathcal{Z}_{\ell}}{}^{\flat}\mathbf{R}^{\alpha\ \nu}_{r\ j\ N}(x).
\end{aligned}
\tag{6. 10}
$$

For N chosen sufficiently large, both ${}^{\flat}\mathbf{R}^{\alpha\ \nu}_{r\ j\ N}$ and ${}^{\flat}\mathbf{R}^{\alpha\ \ell}_{r\ j\ N}$ are integrable functions.

We aim to show

$$
\begin{aligned}
&\int_{\mathbb{R}^n}\left|{}^{\flat}\mathbf{Q}^{\alpha\ \ell}_{r\ j}(x)\right|\left[1-\Psi^{\alpha\ \ell}_{j\ m}(x)\right]dx+\int_{\mathbb{R}^n}\sum_{\mu:\xi_j^{\mu}\in\mathcal{Z}_{\ell}}\Psi^{\alpha\ \mu}_{j\ m}(x)\sum_{\nu:\xi_j^{\nu}\in\mathcal{Z}_{\ell},\xi_j^{\nu}\neq\xi_j^{\mu}}\left|{}^{\flat}\mathbf{Q}^{\alpha\ \nu}_{r\ j}(x)\right|dx \\
&\leq \mathfrak{B}_{N\ \mathbf{Re}\alpha}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}\,2^{-jN}2^{-j\varepsilon}, \qquad N\geq 0
\end{aligned}
\tag{6. 11}
$$

and

$$
\int_{\mathbb{R}^n}\sum_{\mu:\xi_j^{\mu}\in\mathcal{Z}_{\ell}}\Psi^{\alpha\ \mu}_{j\ m}(x)\sum_{\nu:\xi_j^{\nu}\in\mathcal{Z}_{\ell},\xi_j^{\nu}=\xi_j^{\mu}}\left|{}^{\flat}\mathbf{Q}^{\alpha\ \nu}_{r\ j}(x)\right|dx \leq \mathfrak{B}_{\mathbf{Re}\alpha}\,e^{\mathbf{c}|\mathbf{Im}\alpha|}\,2^{-j\varepsilon} \tag{6. 12}
$$

for some $\varepsilon=\varepsilon(\mathbf{Re}\alpha)>0$ whenever $\lambda_{m-1}\leq|r|<\lambda_m$.

Remark 6.1 *Later, it should be clear that (6. 11)-(6. 12) hold if* ${}^{\flat}\mathbf{Q}^{\alpha\ \ell}_{r\ j}$, ${}^{\flat}\mathbf{Q}^{\alpha\ \nu}_{r\ j}$ *are replaced by* ${}^{\flat}\mathbf{Q}^{\alpha\ \ell}_{r\ j\ k}$, ${}^{\flat}\mathbf{Q}^{\alpha\ \nu}_{r\ j\ k}$ *respectively. Furthermore, an extra term* $2^{-j[2k-1]}$ *appears on the right hand side of these inequalities.*

From (6. 11)-(6. 12) and **Remark 6.1**, we conclude (6. 4)-(6. 5).

7 Proof of Proposition Two

Let ${}^{\flat}\mathbf{Q}^{\alpha}_{r}{}^{\nu}_{j}(x)$ defined in (6. 7). Because of Euler's formulae, we replace the cosine function with $e^{2\pi\mathbf{i}|r\xi|}$ or $e^{-2\pi\mathbf{i}|r\xi|}$ multiplied by $\mathfrak{B}_{\mathbf{Re}\alpha}e^{\mathbf{c}|\mathbf{Im}\alpha|}$. This assertion leads us to study

$$\begin{aligned}
{}^{\flat}\mathbf{A}^{\alpha}_{r}{}^{\ell}_{j}(x) &= \sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell}} {}^{\flat}\mathbf{A}^{\alpha}_{r}{}^{\nu}_{j}(x), \qquad {}^{\flat}\mathbf{A}^{\alpha}_{r}{}^{\nu}_{j}(x) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}\left[x\cdot\xi-r|\xi|\right]}\varphi^{\nu}_{j}(\xi)\widehat{\phi}(\xi)\left(\frac{1}{r|\xi|}\right)^{\frac{n-1}{2}+\alpha} d\xi, \\
{}^{\flat}\mathbf{B}^{\alpha}_{r}{}^{\ell}_{j}(x) &= \sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell}} {}^{\flat}\mathbf{B}^{\alpha}_{r}{}^{\nu}_{j}(x), \qquad {}^{\flat}\mathbf{B}^{\alpha}_{r}{}^{\nu}_{j}(x) = \int_{\mathbb{R}^n} e^{2\pi\mathbf{i}\left[x\cdot\xi+r|\xi|\right]}\varphi^{\nu}_{j}(\xi)\widehat{\phi}(\xi)\left(\frac{1}{r|\xi|}\right)^{\frac{n-1}{2}+\alpha} d\xi
\end{aligned} \tag{7. 1}$$

where $\lambda_{m-1} \le r < \lambda_m$ for $\lambda_m \in [2^{j-1}, 2^j]$ as before.

We focus on ${}^{\flat}\mathbf{A}^{\alpha}_{r}{}^{\ell}_{j}(x)$ first. An analogue of our estimates also handle ${}^{\flat}\mathbf{B}^{\alpha}_{r}{}^{\ell}_{j}(x)$.

Note that $\mathbf{supp}\widehat{\phi} \subset \left\{\xi\in\mathbb{R}^n : \frac{1}{3} < |\xi| \le 3\right\}$ and $\mathbf{supp}\varphi^{\nu}_{j} \subset \Gamma^{\nu}_{j} = \left\{\xi\in\mathbb{R}^n : \left|\frac{\xi}{|\xi|}-\xi^{\nu}_{j}\right| < 2^{-j/2+1}\right\}$. We have

$$\begin{aligned}
\left|{}^{\flat}\mathbf{A}^{\alpha}_{r}{}^{\nu}_{j}(x)\right| &\le r^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]} \int_{\mathbf{supp}\widehat{\phi}\varphi^{\nu}_{j}} \left(\frac{1}{|\xi|}\right)^{\frac{n-1}{2}+\mathbf{Re}\alpha} d\xi \\
&\le \mathfrak{B}_{\mathbf{Re}\alpha}\, 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j} 2^{-\left(\frac{n-1}{2}\right)j}.
\end{aligned} \tag{7. 2}$$

Recall $\Psi^{\alpha}_{j}{}^{\ell}_{m}$ and $\Psi^{\alpha}_{j}{}^{\mu}_{m}$ defined in (3. 18). We find

$$\mathbf{vol}\ \mathbf{supp}\Psi^{\alpha}_{j}{}^{\mu}_{m} \lesssim 2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}. \tag{7. 3}$$

As indicated in **Remark 1.3**, there are at most a constant multiple of $2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}$ many ξ^{ν}_{j} s inside each $\mathcal{Z}_{\ell}$. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{\mu:\xi^{\mu}_{j}\in\mathcal{Z}_{\ell}} \Psi^{\alpha}_{j}{}^{\mu}_{m}(x) \sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell},\xi^{\nu}_{j}=\xi^{\mu}_{j}} \left|{}^{\flat}\mathbf{A}^{\alpha}_{r}{}^{\nu}_{j}(x)\right| dx \\
&\le \sum_{\mu:\xi^{\mu}_{j}\in\mathcal{Z}_{\ell}} \sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell},\xi^{\nu}_{j}=\xi^{\mu}_{j}} \int_{\mathbf{supp}\Psi^{\alpha}_{j}{}^{\mu}_{m}} \left|{}^{\flat}\mathbf{A}^{\alpha}_{r}{}^{\nu}_{j}(x)\right| dx \\
&\le \mathfrak{B}_{\mathbf{Re}\alpha} \sum_{\mu:\xi^{\mu}_{j}\in\mathcal{Z}_{\ell}} \sum_{\nu:\xi^{\nu}_{j}\in\mathcal{Z}_{\ell},\xi^{\nu}_{j}=\xi^{\mu}_{j}} 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-\left(\frac{n-1}{2}\right)j}\, 2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j} \qquad \text{by (7. 2)-(7. 3)} \\
&\le \mathfrak{B}_{\mathbf{Re}\alpha}\, 2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-\left(\frac{n-1}{2}\right)j}\, 2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j} \qquad \text{by } \textbf{Remark 1.3} \\
&= \mathfrak{B}_{\mathbf{Re}\alpha}\, 2^{-\left[\mathbf{Re}\alpha-\sigma\right]j}.
\end{aligned} \tag{7. 4}$$

Lemma Two *Given $j > 0$, $m = 1, \ldots, M$ and $\ell = 1, \ldots, L$, we have*

$$\begin{array}{l} \displaystyle\int_{\mathbb{R}^n} \left| {}^\flat\mathbf{A}^{\alpha\ \ell}_{r\ j}(x) \right| \left[1 - \Psi^{\alpha\ \ell}_{j\ m}(x) \right] dx + \sum_{\mu : \xi^\mu_j \in \mathcal{Z}_\ell} \int_{\mathbb{R}^n} \Psi^{\alpha\ \mu}_{j\ m}(x) \sum_{\nu : \xi^\nu_j \in \mathcal{Z}_\ell, \xi^\nu_j \neq \xi^\mu_j} \left| {}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x) \right| dx \\ \le\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-Nj} 2^{-\varepsilon j}, \qquad N \ge 0 \end{array} \tag{7. 5}$$

for some $\varepsilon = \varepsilon(\mathbf{Re}\alpha) > 0$ whenever $\lambda_{m-1} \le r < \lambda_m$.

7.1 Proof of Lemma Two

Let $\xi \longrightarrow r^{-1}\xi$. From (7. 1), we find

$$\begin{aligned} {}^\flat\mathbf{A}^{\alpha\ \ell}_{r\ j}(x) &= \sum_{\nu : \xi^\nu_j \in \mathcal{Z}_\ell} {}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x), \\ {}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x) &= r^{-n} \int_{\mathbb{R}^n} e^{2\pi \mathbf{i}\left[(r^{-1}x)\cdot\xi - |\xi|\right]} \varphi^\nu_j(\xi) \widehat{\phi}\left(r^{-1}\xi\right) \left(\frac{1}{|\xi|}\right)^{\frac{n-1}{2}+\alpha} d\xi. \end{aligned} \tag{7. 6}$$

Given ν, $\mathbf{L}_\nu$ is an $n \times n$-orthogonal matrix with $\det \mathbf{L}_\nu = 1$ and

$$\mathbf{L}^T_\nu \xi^\nu_j = (1, 0)^T \in \mathbb{R} \times \mathbb{R}^{n-1}.$$

Let $\xi = \mathbf{L}_\nu \eta$ and $x = \mathbf{L}_\nu u$ in (7. 6). We have

$${}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(\mathbf{L}_\nu u) = r^{-n} \int_{\mathbb{R}^n} e^{2\pi \mathbf{i}\left[(r^{-1}u)\cdot\eta - |\eta|\right]} \varphi^\nu_j(\mathbf{L}_\nu \eta) \widehat{\phi}\left(r^{-1}\eta\right) \left(\frac{1}{|\eta|}\right)^{\frac{n-1}{2}+\alpha} d\eta. \tag{7. 7}$$

Note that φ^ν_j is defined in (3. 14). $\widehat{\phi}$ defined in (1. 9) is radially symmetric.

Remark 7.1 *Write $\eta = (\eta_1, \eta') \in \mathbb{R} \times \mathbb{R}^{n-1}$. We have $\frac{1}{3}r < \eta_1 < 3r$ and $|\eta'| \lesssim 2^{j/2}$ whenever $\varphi^\nu_j(\mathbf{L}_\nu \eta) \widehat{\phi}\left(r^{-1}\eta\right) \neq 0$.*

Consider

$$\begin{aligned} {}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(\mathbf{L}_\nu u) &= r^{-n} \int_{\mathbb{R}^n} e^{2\pi \mathbf{i}\left[(r^{-1}u)\cdot\eta - \eta_1\right]} \Theta^{\alpha\ \nu}_{r\ j}(\eta) d\eta, \\ \Theta^{\alpha\ \nu}_{r\ j}(\eta) &= e^{-2\pi \mathbf{i}\left[|\eta| - \eta_1\right]} \varphi^\nu_j(\mathbf{L}_\nu \eta) \widehat{\phi}\left(r^{-1}\eta\right) \left(\frac{1}{|\eta|}\right)^{\frac{n-1}{2}+\alpha}. \end{aligned} \tag{7. 8}$$

Remark 7.1 implies

$$\mathbf{vol\ supp} \Theta^{\alpha\ \nu}_{r\ j} \lesssim 2^j 2^{\left(\frac{n-1}{2}\right)j}. \tag{7. 9}$$

Recall $\lambda_{m-1} \le r < \lambda_m$ where $\lambda_m \in [2^{j-1}, 2^j] : 2^{\sigma j - 1} \le \lambda_m - \lambda_{m-1} < 2^{\sigma j}$ where $0 < \sigma < \frac{1}{2}$ can be chosen sufficiently small. For brevity, we write

$$\lambda = \lambda_m.$$

We aim to show

$$\left|\left(\lambda\partial_{\eta_1}\right)^N\left(\lambda^{\frac{1}{2}}\partial^\gamma_{\eta'}\right)\Theta^{\alpha\ \nu}_{r\ j}(\eta)\right| \leq \mathfrak{B}_{N\ \gamma\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j} \tag{7. 10}$$

for every $N \geq 0$ and every multi-index γ.

1. Denote

$$\chi(\eta) = |\eta| - \eta_1, \qquad \eta_1 > 0.$$

We claim

$$\left|\partial^\gamma_\eta \chi(\eta)\right| \leq \mathfrak{B}_\gamma\, 2^{-\gamma_1 j} 2^{-\left[|\gamma|-\gamma_1\right]j/2} \tag{7. 11}$$

for every multi-index γ whenever $\mathbf{L}_\nu\eta \in \Gamma^\nu_j \cap \left\{\frac{1}{3}r < |\eta| \leq 3r\right\}$.

Note that $\chi(\eta)$ is homogeneous of degree 1 in η: $\chi(\rho\eta) = \rho\chi(\eta)$, $\rho > 0$. We have

$$\left|\partial^\gamma_\eta \chi(\eta)\right| \leq \mathfrak{B}_\gamma\, |\eta|^{1-|\gamma|} \tag{7. 12}$$

for every multi-index γ.

Indeed, $\partial^\gamma_\eta\chi(\rho\eta) = \rho^{|\gamma|}\partial^\gamma_\xi\chi(\xi)$ for $\xi = \rho\eta$. On the other hand, $\partial^\gamma_\eta\chi(\rho\eta) = \partial^\gamma_\eta\left[\rho\chi(\eta)\right] = \rho\partial^\gamma_\eta\chi(\eta)$. Choose $\rho = |\eta|^{-1}$. We find $\partial^\gamma_\eta\chi(\eta) = |\eta|^{1-|\gamma|}\partial^\gamma_\xi\chi(\xi)$ where $|\xi| = 1$.

Suppose $|\gamma| - \gamma_1 \geq 2$. From (7. 12), we find

$$\begin{aligned}\left|\partial^\gamma_\eta \chi(\eta)\right| &\leq \mathfrak{B}_\gamma\, 2^{-\left(|\gamma|-1\right)j} = \mathfrak{B}_\gamma\, 2^{-\gamma_1 j}2^{-\left[|\gamma|-\gamma_1-1\right]j} \\ &\leq \mathfrak{B}_\gamma\, 2^{-\gamma_1 j}2^{-\left[|\gamma|-\gamma_1\right]j/2}.\end{aligned} \tag{7. 13}$$

Suppose $|\gamma| - \gamma_1 = 0$ or 1. Denote $\eta^1 = (\eta_1, 0)^T \in \mathbb{R}\times\mathbb{R}^{n-1}$. A direct computation shows

$$\left(\nabla_\eta\chi\right)(\eta^1) = 0,$$

$$\left(\partial^N_{\eta_1}\nabla_{\eta'}\chi\right)(\eta^1) = \left(\nabla_{\eta'}\partial^N_{\eta_1}\chi\right)(\eta^1) = 0, \qquad N \geq 0.$$

By writing out the Taylor expansion of $\partial^N_{\eta_1}\chi(\eta)$ and $\nabla_{\eta'}\partial^N_{\eta_1}\chi(\eta)$ in the η'-subspace and using $\left(\nabla_{\eta'}\partial^N_{\eta_1}\chi\right)(\eta^1) = 0$, we have

$$\partial^N_{\eta_1}\chi(\eta) = \mathbf{O}\left(|\eta'|^2|\eta|^{-N-1}\right), \qquad \nabla_{\eta'}\partial^N_{\eta_1}\chi(x,\eta) = \mathbf{O}\left(|\eta'||\eta|^{-N-1}\right), \qquad N \geq 0. \tag{7. 14}$$

From (7. 14) and **Remark 7.1**, we conclude

$$\left|\partial^N_{\eta_1}\chi(\eta)\right| \leq \mathfrak{B}_N 2^{-Nj}, \qquad \left|\nabla_{\eta'}\partial^N_{\eta_1}\chi(\eta)\right| \leq \mathfrak{B}_N 2^{-Nj}2^{-j/2}, \qquad N \geq 0.$$

2. For $\widehat{\phi}$ defined in (1. 9), we easily find

$$\left|\partial^\gamma_\eta\left\{\widehat{\phi}\left(r^{-1}\eta\right)\left(\frac{1}{|\eta|}\right)^{\frac{n-1}{2}+\alpha}\right\}\right| \leq \mathfrak{B}_{\gamma\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-|\gamma|j} \tag{7. 15}$$

for every multi-index γ.

3. Recall φ_j^ν defined in (3. 14). We have

$$\varphi_j^\nu(\mathbf{L}_\nu\eta) = \frac{\varphi\left[2^{j/2}\left|\frac{\eta}{|\eta|}-\eta_j^\nu\right|\right]}{\sum_{\nu:\xi_j^\nu\in\mathbb{S}^{n-1}}\varphi\left[2^{j/2}\left|\frac{\eta}{|\eta|}-\eta_j^\nu\right|\right]}.$$

Observe that $\partial_\eta\varphi_j^\nu(\mathbf{L}_\nu\eta)=0$ whenever $\left|\frac{\eta}{|\eta|}-\eta_j^\nu\right|\le 2^{-j/2}$. A direct computation shows

$$\begin{aligned}\frac{\partial}{\partial\eta_1}\frac{\eta_1}{|\eta|} &= \frac{1}{|\eta|}-\frac{\eta_1^2}{|\eta|^3} = \frac{|\eta'|^2}{|\eta|^3},\\ \frac{\partial}{\partial\eta_1}\frac{\eta_i}{|\eta|} &= -\frac{\eta_1\eta_i}{|\eta|^3}, \qquad i = 2,\ldots,n.\end{aligned} \tag{7. 16}$$

Because $\eta_1\approx 2^j$ and $\eta'\lesssim 2^{j/2}$ as shown in **Remark 7.1**, (7. 16) implies

$$\left|\frac{\partial}{\partial\eta_1}\frac{\eta}{|\eta|}\right| \approx 2^{j/2}|\eta|^{-2} \approx 2^{-j/2}|\eta|^{-1}. \tag{7. 17}$$

By carrying out the differentiation *w.r.t* $\eta=(\eta_1,\eta')$ and using (7. 17), we find

$$\left|\partial_{\eta_1}^N\partial_{\eta'}^\gamma\left[\varphi_j^\nu(\mathbf{L}_\nu\eta)\right]\right| \le \mathfrak{B}_{N\,\gamma}\,|\eta|^{-N}2^{|\gamma|j/2}|\eta|^{-|\gamma|} \tag{7. 18}$$

for every multi-index γ and $N\ge 0$.

By putting together the above estimates in **1-3**, we obtain (7. 10).

Recall $\Psi_{j\,m}^{\alpha\,\mu}$ defined in (3. 18). Moreover, $\mathcal{Z}_\ell$ is a subset of $\left\{\xi_j^\nu\right\}_\nu$. See **(a)-(b)** below (1. 24).

Lemma 7.1 *Suppose $\xi_j^\mu,\xi_j^\nu\in\mathcal{Z}_\ell$ and $\xi_j^\mu\neq\xi_j^\nu$. Let $x=\mathbf{L}_\nu u\in\mathbf{supp}\Psi_{j\,m}^{\alpha\,\mu}$. We either have $u_1\approx -2^j$ or $|u_\imath|>2^{\left[\frac{1}{2}+\sigma\right]j+1}$ for some $\imath\in\{2,\ldots,n\}$.*

Remark 7.2 *Consequently, $\mathbf{supp}\Psi_{j\,m}^{\alpha\,\mu}\cap\mathbf{supp}\Psi_{j\,m}^{\alpha\,\nu}=\emptyset$ if $\xi_j^\mu,\xi_j^\nu\in\mathcal{U}_\ell$ and $\xi_j^\mu\neq\xi_j^\nu$.*

Proof Denote $\mathbb{S}_j^{n-1}$ to be a sphere centered at the origin with radius 2^j. Let $x_j^\nu\in\mathbb{S}_j^{n-1}$ in the same direction of ξ_j^ν. As a geometric fact, we have

$$\left|x_j^\mu-x_j^\nu\right| > \mathbf{c}\,2^{j/2}2^{\sigma j} \qquad \text{if} \qquad \xi_j^\mu,\xi_j^\nu\in\mathcal{U}_\ell \quad\text{and}\quad \xi_j^\mu\neq\xi_j^\nu.$$

Recall $\mathbf{L}_\nu^T\xi_j^\nu=(1,0)^T\in\mathbb{R}\times\mathbb{R}^{n-1}$. We find $\mathbf{L}_\nu^Tx_j^\nu=(2^j,0)^T\in\mathbb{R}\times\mathbb{R}^{n-1}$ and

$$\left|\mathbf{L}_\nu^Tx_j^\mu-\mathbf{L}_\nu^Tx_j^\nu\right| = \left|\mathbf{L}_\nu^Tx_j^\mu-(2^j,0)^T\right| > \mathbf{c}\,2^{j/2}2^{\sigma j}. \tag{7. 19}$$

Suppose $\left|\left(\mathbf{L}_\nu^T x_j^\mu\right)_1 - 2^j\right| \leq \left[\sup_{i=2}^n \left(\mathbf{L}_\nu^T x_j^\mu\right)_i^2\right]^{\frac{1}{2}}$. Then, (7. 19) implies

$$\left[\sum_{i=2}^n \left(\mathbf{L}_\nu^T x_j^\mu\right)_i^2\right]^{\frac{1}{2}} \;>\; \frac{1}{\sqrt{2}}\mathbf{c}\, 2^{j/2} 2^{\sigma j}.$$

Hence

$$\left|\left(\mathbf{L}_\nu^T x_j^\mu\right)_\imath\right| \;>\; \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-1}}\mathbf{c}\, 2^{j/2}2^{\sigma j} \qquad \text{for some} \qquad \imath \in \{2, \ldots, n\}. \tag{7. 20}$$

Suppose $\left|\left(\mathbf{L}_\nu^T x_j^\mu\right)_1 - 2^j\right| > \left[\sup_{i=2}^n \left(\mathbf{L}_\nu^T x_j^\mu\right)_i^2\right]^{\frac{1}{2}}$. Geometrically, $\mathbf{L}_\nu^T x_j^\mu$ belongs to the semi-sphere opposite to $(2^j, 0)^T$. We either have $\left(\mathbf{L}_\nu^T x_j^\mu\right)_1 \approx -2^j$ or $\left|\left(\mathbf{L}_\nu^T x_j^\mu\right)_\imath^2\right| > \frac{1}{\sqrt{2}}\frac{1}{\sqrt{n-1}}\mathbf{c}\; 2^{j/2}2^{\sigma j}$ for some $\imath \in \{2, \ldots, n\}$ as (7. 20).

Let $x \in \mathbf{supp}\Psi_{j\,m}^{\alpha\,\mu}$. By definition of $\Psi_{j\,m}^{\alpha\,\mu}$ in (3. 18), $u = \mathbf{L}_\nu^T x$ satisfies

$$\left|u - \mathbf{L}_\nu^T x_j^\mu\right| \;=\; \left|x - x_j^\mu\right| \;<\; \sqrt{n}2^{\left[\frac{1}{2}+\sigma\right]j}.$$

All together, we conclude $u_1 \approx -2^j$ or $|u_\imath| > 2^{\left[\frac{1}{2}+\sigma\right]j+1}$ for some $\imath \in \{2, \ldots, n\}$ provided that $\mathbf{c} > 0$ is large enough. □

Define a differential operator

$$\mathbf{D}_\lambda f \;=\; Id + \lambda^2 \partial_{\eta_1}^2 f + \lambda \Delta_{\eta'} f. \tag{7. 21}$$

Let ${}^\flat\mathbf{A}_{r\,j}^{\alpha\,\nu}(\mathbf{L}_\nu u)$ be given in (7. 8). An N-fold integration by parts *w.r.t* $\mathbf{D}_\lambda$ shows

$$\begin{aligned} {}^\flat\mathbf{A}_{r\,j}^{\alpha\,\nu}(\mathbf{L}_\nu u) \;&=\; (2\pi\mathbf{i})^{-2N}\left[1 + \left[\left(\lambda r^{-1}\right)u_1 - \lambda\right]^2 + \lambda\sum_{i=2}^n \left(r^{-1}u_i\right)^2\right]^{-N} \\ &\qquad r^{-n}\int_{\mathbb{R}^n} e^{2\pi\mathbf{i}\left[\left(r^{-1}u\right)\cdot\eta - \eta_1\right]}\mathbf{D}_\lambda^N \Theta_{r\,j}^{\alpha\,\nu}(\eta)d\eta. \end{aligned} \tag{7. 22}$$

By using (7. 9) and (7. 10), we have

$$\left|\int_{\mathbb{R}^n} e^{2\pi\mathbf{i}\left[\left(r^{-1}u\right)\cdot\eta - \eta_1\right]}\mathbf{D}_\lambda^N \Theta_{r\,j}^{\alpha\,\nu}(\eta)d\eta\right| \;\leq\; \mathfrak{B}_{N\;\mathbf{Re}\alpha}\; e^{\mathbf{c}|\mathbf{Im}\alpha|}\; 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^j 2^{\left(\frac{n-1}{2}\right)j} \tag{7. 23}$$

for every $N \geq 0$.

Recall $\Psi_{j\,m}^{\alpha\,\mu}$ defined in (3. 18). Define

$$\widetilde{\Psi}_{j\,m}^{\alpha\,\mu}(x) \;=\; \begin{cases} 1, & \Psi_{j\,m}^{\alpha\,\mu}(x) = 1, \\ 0, & \Psi_{j\,m}^{\alpha\,\mu}(x) \neq 1. \end{cases}$$

Note that $\widetilde{\Psi}^{\alpha\ \mu}_{j\ m}\Big(\mathbf{L}_\mu u\Big)$ is supported in the rectangle

$$\mathrm{R}^\sigma_{j\ \lambda} \ = \ \left\{ u\in\mathbb{R}^n \colon \lambda-2^{\sigma j+1}\le u_1\le \lambda+2^{\sigma j+1},\ \ |u_i|\le 2^{\left[\frac{1}{2}+\sigma\right]j},\ \ i=2,\ldots,n\right\}. \tag{7. 24}$$

By definition of $\Psi^{\alpha\ \mu}_{j\ m}$ in (3. 18) and **Remark 7.2**, we find

$$0 \ \le \ \Psi^{\alpha\ \mu}_{j\ m}(x) \ \le \ 1, \qquad 1-\sum_{\mu:\xi^\mu_j\in\mathcal{U}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(x) \ \ge \ 0.$$

Consider the first integral in (7. 5). We have

$$\begin{aligned}
\int_{\mathbb{R}^n}\left|{}^\flat\mathbf{A}^{\alpha\ \ell}_{r\ j}(x)\right|\Big[1-\Psi^{\alpha\ \ell}_{j\ m}(x)\Big]dx \ &= \ \int_{\mathbb{R}^n}\left|{}^\flat\mathbf{A}^{\alpha\ \ell}_{r\ j}(x)\right|\left[1-\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(x)\right]dx\\
&\le \ \int_{\mathbb{R}^n}\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell}\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x)\right|\left[1-\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(x)\right]dx\\
&= \ \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell}\int_{\mathbb{R}^n}\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x)\right|\left[1-\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\Psi^{\alpha\ \mu}_{j\ m}(x)\right]dx\\
&\le \ \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell}\int_{\mathbb{R}^n\setminus\bigcup_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\mathbf{supp}\widetilde{\Psi}^{\alpha\ \mu}_{j\ m}}\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x)\right|dx\\
&\le \ \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell}\int_{\mathbb{R}^n\setminus\mathbf{supp}\widetilde{\Psi}^{\alpha\ \nu}_{j\ m}}\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x)\right|dx\\
&= \ \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell}\int_{\mathbb{R}^n\setminus\mathrm{R}^\sigma_{j\ \lambda}}\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(\mathbf{L}_\nu u)\right|du.
\end{aligned}\tag{7. 25}$$

Recall $\lambda_{m-1}\le r<\lambda$ ($\lambda=\lambda_m$) of which $\lambda\in[2^{j-1},2^j]$ and $2^{\sigma j-1}\le\lambda-\lambda_{m-1}<2^{\sigma j}$. Moreover, $0<\sigma=\sigma(\mathbf{Re}\alpha)<\frac{1}{2}$ can be chosen sufficiently small.

Let $\mathrm{R}^\sigma_{j\ \lambda}$ defined in (7. 24). Suppose $u_1\notin[\lambda-2^{\sigma j+1},\lambda+2^{\sigma j+1}]$. We find

$$\begin{aligned}
\Big(\lambda r^{-1}\Big)u_1-\lambda \ &> \ \Big[\lambda+2^{\sigma j+1}\Big]-\lambda \ = \ 2^{\sigma j+1},\\
\Big(\lambda r^{-1}\Big)u_1-\lambda \ &< \ \frac{\lambda}{\lambda-2^{\sigma j}}\Big[\lambda-2^{\sigma j+1}\Big]-\lambda\\
&= \ \lambda\left\{\frac{\lambda-2^{\sigma j+1}}{\lambda-2^{\sigma j}}-1\right\} \ = \ \lambda\left[\frac{-2^{\sigma j+1}+2^{\sigma j}}{\lambda-2^{\sigma j}}\right]\\
&< \ -2^{\sigma j}.
\end{aligned}\tag{7. 26}$$

Recall **Remark 1.3**. There are at most a constant multiple of $2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}$ many ξ_j^ν s in $\mathcal{Z}_\ell$. By using (7. 22)-(7. 23), we have

$$\begin{array}{l}
\displaystyle\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell}\int_{\mathbb{R}^n\setminus\mathrm{R}^\sigma_{j\,\lambda}}\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(\mathbf{L}_\nu u)\right|du \ \le\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\sum_{\nu:\xi_j^\nu\in\mathcal{Z}_\ell} r^{-n}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}\,2^j2^{\left(\frac{n-1}{2}\right)j} \\
\qquad\qquad\displaystyle\int_{\mathbb{R}^n\setminus\mathrm{R}^\sigma_{j\,\lambda}}\left[1+\left[\left(\lambda r^{-1}\right)u_1-\lambda\right]^2+\lambda\sum_{i=2}^n\left(r^{-1}u_i\right)^2\right]^{-N}du \\
\le\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\ 2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}\ 2^{-nj}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}\,2^j2^{\left(\frac{n-1}{2}\right)j} \\
\qquad\qquad\displaystyle\int_{\mathbb{R}^n\setminus\mathrm{R}^\sigma_{j\,\lambda}}\left[1+\left[\left(\lambda r^{-1}\right)u_1-\lambda\right]^2+\lambda\sum_{i=2}^n\left(r^{-1}u_i\right)^2\right]^{-N}du \\
\le\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\ 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}\displaystyle\int_{\mathbb{R}^n\setminus\mathrm{R}^\sigma_{j\,\lambda}}\left[1+\left[\left(\lambda r^{-1}\right)u_1-\lambda\right]^2+\lambda\sum_{i=2}^n\left(r^{-1}u_i\right)^2\right]^{-N}du \\
=\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\ 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}r^{\frac{n-1}{2}} \\
\qquad\displaystyle\int_{\mathbb{R}^n\setminus\left\{\lambda-2^{\sigma j+1}\le u_1\le\lambda+2^{\sigma j+1},\ |u_i|\le(2^jr^{-1})^{\frac12}2^{\sigma j}\right\}}\left[1+\left[\left(\lambda r^{-1}\right)u_1-\lambda\right]^2+\left(\lambda r^{-1}\right)\sum_{i=2}^n u_i^2\right]^{-N}du \\
\qquad\qquad\qquad\qquad u_i\longrightarrow r^{\frac12}u_i,\ i=2,\dots,n \\
\le\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\ 2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}r^{\frac{n-1}{2}}2^{-2\sigma\left[N-n-1\right]j} \\
\qquad\displaystyle\int_{\mathbb{R}^n\setminus\left\{\lambda-2^{\sigma j+1}\le|u_1|\le\lambda+2^{\sigma j+1},\ |u_i|\le(2^jr^{-1})^{\frac12}2^{\sigma j}\right\}}\left[1+\left[\left(\lambda r^{-1}\right)u_1-\lambda\right]^2+\left(\lambda r^{-1}\right)\sum_{i=2}^n u_i^2\right]^{-n-1}du \\
\qquad\qquad\qquad\qquad \text{by (7. 26) and } \lambda\approx r\approx 2^j \\
=\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\ 2^{-\mathbf{Re}\alpha j}2^{-2\sigma\left[N-n-1\right]j}\displaystyle\int_{\mathbb{R}^n}\left[1+|u_1-r|^2+\sum_{i=2}^n u_i^2\right]^{-n-1}du.
\end{array}\tag{7. 27}$$

From (7. 25)-(7. 27), by taking N sufficiently large depending $\sigma=\sigma(\mathbf{Re}\alpha)>0$, we conclude

$$\int_{\mathbb{R}^n}\left|{}^\flat\mathbf{A}^{\alpha\ \ell}_{r\ j}(x)\right|\left[1-\Psi^{\alpha\ \ell}_{j\ m}(x)\right]dx\ \le\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\ 2^{-\mathbf{Re}\alpha j}2^{-Nj},\qquad N\ge 0.\tag{7. 28}$$

Next, we turn to the summation term in (7. 5). From (7. 8), we find

$${}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(\mathbf{L}_\nu u)\ =\ r^{-n}\int_{\mathbb{R}^n}e^{2\pi\mathbf{i}\left[\left(r^{-1}u\right)\cdot\eta-\eta_1\right]}\Theta^{\alpha\ \nu}_{r\ j}(\eta)d\eta.$$

Let $\mathbf{L}_\nu u\in\mathbf{supp}\Psi^{\alpha\ \mu}_{j\ m}$. Suppose $\xi_j^\mu,\xi_j^\nu\in\mathcal{Z}_\ell$ and $\xi_j^\nu\ne\xi_j^\mu$. **Lemma 7.1** shows $u_1\approx -2^j$ or $|u_\imath|>2^{\left[\frac12+\sigma\right]j+1}$ for some $\imath\in\{2,\dots,n\}$. Recall the differential inequality in (7. 10). We have

$$\left|\partial^N_{\eta_1}\partial^M_{\eta_\imath}\Theta^{\alpha\ \nu}_{r\ j}(\eta)\right|\ \le\ \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\ 2^{-j\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]}2^{-Nj}2^{-Mj/2},\qquad N,M\ge 0.\tag{7. 29}$$

As shown in (7. 9), the volume of $\mathbf{supp}\Theta^{\alpha\ \nu}_{r\ j}$ is bounded by a constant multiple of $2^j2^{\left(\frac{n-1}{2}\right)j}$.

Let $u_1 \approx -2^j$. An N-fold integration by parts $w.r.t$ η_1 shows

$$\begin{aligned}
\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(\mathbf{L}_\nu u)\right| &\leq \mathfrak{B}_N\, r^{-n}\left|r^{-1}u_1-1\right|^{-N}\int_{\mathbb{R}^n}\left|\partial^N_{\eta_\imath}\Theta^{\alpha\ \nu}_{r\ j}(\eta)\right|d\eta\\
&\leq \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, r^{-n}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-Nj}\, 2^j 2^{\left(\frac{n-1}{2}\right)j} \qquad (7.\ 30)\\
&\leq \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-Nj}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-\left(\frac{n-1}{2}\right)j}.
\end{aligned}$$

Let $|u_\imath| > 2^{\left[\frac{1}{2}+\sigma\right]j+1}$. An M-fold integration by parts $w.r.t$ $\eta_\imath$ shows

$$\begin{aligned}
\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(\mathbf{L}_\nu u)\right| &\leq \mathfrak{B}_M\, r^{-n}\left|r^{-1}u_\imath\right|^{-M}\int_{\mathbb{R}^n}\left|\partial^M_{\eta_\imath}\Theta^{\alpha\ \nu}_{r\ j}(\eta)\right|d\eta\\
&\leq \mathfrak{B}_{M\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, r^{-n}r^M 2^{-M\left[\frac{1}{2}+\sigma\right]j}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-Mj/2}\, 2^j 2^{\left(\frac{n-1}{2}\right)j} \qquad (7.\ 31)\\
&\leq \mathfrak{B}_{M\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-M\sigma j}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-\left(\frac{n-1}{2}\right)j}.
\end{aligned}$$

From (7. 3), we find $\mathbf{vol}\ \mathbf{supp}\Psi^{\alpha\ \mu}_{j\ m} \lesssim 2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}$. Moreover, there are at most a constant multiple of $2^{\left(\frac{n-1}{2}\right)j}2^{-(n-1)\sigma j}$ many ξ^ν_j s in $\mathcal{Z}_\ell$ as shown in **Remark 1.3**. We have

$$\begin{aligned}
&\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\ \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j\neq\xi^\mu_j}\int_{\mathbb{R}^n}\Psi^{\alpha\ \mu}_{j\ m}(x)\left|{}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}(x)\right|dx \qquad (x=\mathbf{L}_\nu u)\\
&\leq \mathfrak{B}_{M\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\sum_{\mu:\xi^\mu_j\in\mathcal{Z}_\ell}\ \sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell,\xi^\nu_j\neq\xi^\mu_j}2^{\left(\frac{n-1}{2}\right)j}2^{n\sigma j}\, 2^{-M\sigma j}2^{-\left[\frac{n-1}{2}+\mathbf{Re}\alpha\right]j}2^{-\left(\frac{n-1}{2}\right)j} \qquad (7.\ 32)\\
&\qquad\qquad \text{by (7. 30)-(7. 31)}\\
&\leq \mathfrak{B}_{N\ \mathbf{Re}\alpha}\, e^{\mathbf{c}|\mathbf{Im}\alpha|}\, 2^{-\mathbf{Re}\alpha j}2^{-Nj}, \qquad N\geq 0
\end{aligned}$$

provided that M is sufficiently large depending $\sigma=\sigma(\mathbf{Re}\alpha)>0$ and $N\geq 0$.

Lastly, recall ${}^\flat\mathbf{B}^{\alpha\ \ell}_{r\ j}(x)=\sum_{\nu:\xi^\nu_j\in\mathcal{Z}_\ell}{}^\flat\mathbf{B}^{\alpha\ \nu}_{r\ j}(x)$ from (7. 1). Write

$$\begin{aligned}
{}^\flat\mathbf{B}^{\alpha\ \nu}_{r\ j}(x) &= \int_{\mathbb{R}^n}e^{2\pi\mathbf{i}\left[x\cdot\xi+r|\xi|\right]}\varphi^\nu_j(\xi)\widehat{\phi}(\xi)\left(\frac{1}{r|\xi|}\right)^{\frac{n-1}{2}+\alpha}d\xi\\
&= (-1)^n\int_{\mathbb{R}^n}e^{-2\pi\mathbf{i}\left[x\cdot\xi-r|\xi|\right]}\varphi^\nu_j(-\xi)\widehat{\phi}(\xi)\left(\frac{1}{r|\xi|}\right)^{\frac{n-1}{2}+\alpha}d\xi, \qquad \xi\longrightarrow-\xi.
\end{aligned}$$

All regarding estimates from (7. 2) to (7. 32) remain valid if ${}^\flat\mathbf{A}^{\alpha\ \ell}_{r\ j}$ and ${}^\flat\mathbf{A}^{\alpha\ \nu}_{r\ j}$ are replaced by ${}^\flat\mathbf{B}^{\alpha\ \ell}_{r\ j}$ and ${}^\flat\mathbf{B}^{\alpha\ \nu}_{r\ j}$.

A Some estimates regarding Bessel functions

• For $a>-\frac{1}{2}, b\in\mathbb{R}$ and $\rho>0$, a Bessel function $\mathbf{J}_{a+\mathbf{i}b}$ has an integral formula

$$\mathbf{J}_{a+\mathbf{i}b}(\rho) = \frac{(\rho/2)^{a+\mathbf{i}b}}{\pi^{\frac{1}{2}}\Gamma\left(a+\frac{1}{2}+\mathbf{i}b\right)}\int_{-1}^1 e^{\mathbf{i}\rho s}(1-s^2)^{a-\frac{1}{2}+\mathbf{i}b}ds. \qquad (A.\ 1)$$

• For every $a > -\frac{1}{2}, b \in \mathbb{R}$ and $\rho > 0$, we have

$$\begin{aligned} \mathbf{J}_{a+\mathbf{i}b}(\rho) \sim{}& \left(\frac{\pi\rho}{2}\right)^{-\frac{1}{2}} \cos\left[\rho - (a+\mathbf{i}b)\frac{\pi}{2} - \frac{\pi}{4}\right] \\ &+ \left(\frac{\pi\rho}{2}\right)^{-\frac{1}{2}} \sum_{k=1}^{\infty} \cos\left[\rho - (a+\mathbf{i}b)\frac{\pi}{2} - \frac{\pi}{4}\right] \mathbf{a}_k \rho^{-2k} + \sin\left[\rho - (a+\mathbf{i}b)\frac{\pi}{2} - \frac{\pi}{4}\right] \mathbf{b}_k \rho^{-2k+1} \end{aligned} \tag{A. 2}$$

where

$$\mathbf{a}_k = (-1)^k [a+\mathbf{i}b, 2k] 2^{-2k}, \qquad \mathbf{b}_k = (-1)^{k+1} [a+\mathbf{i}b, 2k-1] 2^{-2k+1},$$

$$[a+\mathbf{i}b, m] = \frac{\Gamma\left(\frac{1}{2} + a + \mathbf{i}b + m\right)}{m!\Gamma\left(\frac{1}{2} + a + \mathbf{i}b - m\right)}, \qquad m = 0, 1, 2, \ldots \tag{A. 3}$$

in the sense of that

$$\begin{aligned} &\left(\frac{d}{d\rho}\right)^{\ell} \Bigg\{ \mathbf{J}_{a+\mathbf{i}b}(\rho) - \left(\frac{\pi\rho}{2}\right)^{-\frac{1}{2}} \cos\left[\rho - (a+\mathbf{i}b)\frac{\pi}{2} - \frac{\pi}{4}\right] \\ &\quad - \left(\frac{\pi\rho}{2}\right)^{-\frac{1}{2}} \sum_{k=1}^{N} \cos\left[\rho - (a+\mathbf{i}b)\frac{\pi}{2} - \frac{\pi}{4}\right] \mathbf{a}_k \rho^{-2k} + \sin\left[\rho - (a+\mathbf{i}b)\frac{\pi}{2} - \frac{\pi}{4}\right] \mathbf{b}_k \rho^{-2k+1} \Bigg\} \\ &= \mathbf{O}\left(\rho^{-2N-\frac{1}{2}}\right) \qquad N \geq 0, \qquad \ell \geq 0, \qquad \rho \longrightarrow \infty. \end{aligned} \tag{A. 4}$$

The implied constant is bounded by $\mathfrak{B}_a e^{\mathbf{c}|b|}$.

By putting together (A. 1) and (A. 2)-(A. 4), we find a norm estimate in below.

• For every $a > -\frac{1}{2}$, $b \in \mathbb{R}$ and $\rho > 0$,

$$\left| \frac{1}{\rho^{a+\mathbf{i}b}} \mathbf{J}_{a+\mathbf{i}b}(\rho) \right| \leq \mathfrak{B}_a \left(\frac{1}{1+\rho}\right)^{\frac{1}{2}+a} e^{\mathbf{c}|b|}. \tag{A. 5}$$

Remark A.1 *(A. 5) is in fact true for every $a, b \in \mathbb{R}$ and $\rho > 0$.*

This is a consequence of using the following identity.

• For every $a, b \in \mathbb{R}$ and $\rho > 0$, we have

$$\mathbf{J}_{a-1+\mathbf{i}b}(\rho) = 2\left[\frac{a+\mathbf{i}b}{\rho}\right] \mathbf{J}_{a+\mathbf{i}b}(\rho) - \mathbf{J}_{a+1+\mathbf{i}b}(\rho). \tag{A. 6}$$

More discussion of Bessel functions can be found in the book of Watson [24].

Let $z \in \mathbb{C}$. Ω^z is a distribution defined by analytic continuation from

$$\mathbf{Re} z < 1, \qquad \Omega^z(x) = \pi^{-z} \Gamma^{-1}(1-z) \left(\frac{1}{1-|x|^2}\right)_+^z. \tag{A. 7}$$

• Ω^z can be equivalently defined by

$$\widehat{\Omega^z}(\xi) = \left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-z} \mathbf{J}_{\frac{n}{2}-z}\left(2\pi|\xi|\right), \qquad z \in \mathbb{C}. \tag{A. 8}$$

Remark A.2 *From (A. 1), (A. 2)-(A. 4) and (A. 6), we conclude that $\widehat{\Omega^z}(\xi)$ for given $\xi \in \mathbb{R}^n$ is an analytic function of $z \in \mathbb{C}$.*

Denote $\omega_{n-2} = 2\pi^{\frac{n-1}{2}}\Gamma^{-1}\left(\frac{n-1}{2}\right)$ which is the area of $\mathbb{S}^{n-2}$. From (A. 7), we have

$$\begin{aligned}
\widehat{\Omega^z}(\xi) &= \pi^{-z}\Gamma^{-1}(1-z)\int_{|x|<1} e^{-2\pi\mathbf{i}x\cdot\xi}\left(\frac{1}{1-|x|^2}\right)^z dx \\
&= \pi^{-z}\Gamma^{-1}(1-z)\int_0^\pi \left\{\int_0^1 e^{-2\pi\mathbf{i}|\xi|r\cos\vartheta}(1-r^2)^{-z}r^{n-1}dr\right\}\omega_{n-2}\sin^{n-2}\vartheta d\vartheta \\
&= \omega_{n-2}\pi^{-z}\Gamma^{-1}(1-z)\int_{-1}^1 \left\{\int_0^1 e^{2\pi\mathbf{i}|\xi|rs}(1-r^2)^{-z}r^{n-1}dr\right\}(1-s^2)^{\frac{n-3}{2}}ds \quad (-s=\cos\vartheta) \\
&= \omega_{n-2}\pi^{-z}\Gamma^{-1}(1-z)\int_0^1 \left\{\int_{-1}^1 e^{2\pi\mathbf{i}|\xi|rs}(1-s^2)^{\frac{n-3}{2}}ds\right\}(1-r^2)^{-z}r^{n-1}dr \\
&= \omega_{n-2}\pi^{-z}\Gamma^{-1}(1-z)\int_0^1 \left\{\int_{-1}^1 \cos(2\pi|\xi|rs)(1-s^2)^{\frac{n-3}{2}}ds\right\}(1-r^2)^{-z}r^{n-1}dr.
\end{aligned} \tag{A. 9}$$

Recall the Beta function identity:

$$\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} = \int_0^1 r^{z-1}(1-r)^{w-1}dr \tag{A. 10}$$

for every $\mathbf{Re}z > 0$ and $\mathbf{Re}w > 0$.

By writing out the Taylor expansion of the cosine function inside (A. 9), we find

$$\begin{aligned}
&\omega_{n-2}\pi^{-z}\Gamma^{-1}(1-z)\int_0^1 \left\{\int_{-1}^1 \cos(2\pi|\xi|rs)(1-s^2)^{\frac{n-3}{2}}ds\right\}(1-r^2)^{-z}r^{n-1}dr \\
&= \omega_{n-2}\pi^{-z}\Gamma^{-1}(1-z) \\
&\quad \sum_{k=0}^\infty(-1)^k\frac{(2\pi|\xi|)^{2k}}{(2k)!}\left\{\int_{-1}^1 s^{2k}(1-s^2)^{\frac{n-3}{2}}ds\right\}\left\{\int_0^1 r^{2k+n-1}(1-r^2)^{-z}dr\right\} \\
&= \frac{1}{2}\omega_{n-2}\pi^{-z}\Gamma^{-1}(1-z) \\
&\quad \sum_{k=0}^\infty(-1)^k\frac{(2\pi|\xi|)^{2k}}{(2k)!}\left\{\int_0^1 t^{k+\frac{1}{2}-1}(1-t)^{\frac{n-1}{2}-1}dt\right\}\left\{\int_0^1 \rho^{k+\frac{n}{2}-1}(1-\rho)^{1-z-1}d\rho\right\} \\
&\qquad (t=s^2,\ \rho=r^2) \\
&= \frac{1}{2}\omega_{n-2}\pi^{-z}\Gamma^{-1}(1-z)\sum_{k=0}^\infty(-1)^k\frac{(2\pi|\xi|)^{2k}}{(2k)!}\,\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(k+\frac{n}{2}\right)}\,\frac{\Gamma\left(k+\frac{n}{2}\right)\Gamma(1-z)}{\Gamma\left(k+\frac{n}{2}+1-z\right)} \\
&\qquad \text{by (A. 10)} \\
&= \pi^{\frac{n-1}{2}-z}\sum_{k=0}^\infty(-1)^k\frac{(2\pi|\xi|)^{2k}}{(2k)!}\,\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{n}{2}+1-z\right)}.
\end{aligned} \tag{A. 11}$$

On the other hand, we have

$$
\begin{aligned}
\left(\frac{1}{|\xi|}\right)^{\frac{n}{2}-z} \mathbf{J}_{\frac{n}{2}-z}\big(2\pi|\xi|\big) &= \pi^{\frac{n-1}{2}-z}\Gamma^{-1}\left(\tfrac{n+1}{2}-z\right)\int_{-1}^{1} e^{2\pi\mathbf{i}|\xi|s}(1-s^2)^{\frac{n-1}{2}-z}ds \qquad \text{by (A. 1)} \\
&= \pi^{\frac{n-1}{2}-z}\Gamma^{-1}\left(\tfrac{n+1}{2}-z\right)\int_{-1}^{1}\cos\left(2\pi|\xi|s\right)(1-s^2)^{\frac{n-1}{2}-z}ds \\
&= \pi^{\frac{n-1}{2}-z}\Gamma^{-1}\left(\tfrac{n+1}{2}-z\right)\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi|\xi|)^{2k}}{(2k)!}\int_{-1}^{1}s^{2k}(1-s^2)^{\frac{n-1}{2}-z}ds \\
&= \pi^{\frac{n-1}{2}-z}\Gamma^{-1}\left(\tfrac{n+1}{2}-z\right)\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi|\xi|)^{2k}}{(2k)!}\int_{0}^{1}\rho^{k+\frac{1}{2}-1}(1-\rho)^{\frac{n+1}{2}-z-1}d\rho \qquad (\rho=s^2) \\
&= \pi^{\frac{n-1}{2}-z}\sum_{k=0}^{\infty}(-1)^k\frac{(2\pi|\xi|)^{2k}}{(2k)!}\,\frac{\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(k+\frac{n}{2}+1-z\right)} \qquad \text{by (A. 10).}
\end{aligned}
\tag{A. 12}
$$

B Fourier transform of Λ^α

We derive the formula of $\widehat{\Lambda}^\alpha(\xi,\tau)$ in (1. 8) by following the lines in p. 253 − 284, Chapter III of Gelfand and Shilov [10].

$\widehat{\mathrm{U}}^\alpha$, $\widehat{\mathrm{V}}^\alpha$ are distributions defined by analytic continuation from

$$
\widehat{\mathrm{U}}^\alpha(\xi,\tau) = \Gamma^{-1}(1-\alpha)\left(\frac{1}{\tau^2-|\xi|^2}\right)_-^{\alpha}, \qquad \widehat{\mathrm{V}}^\alpha(\xi,\tau) = \Gamma^{-1}(1-\alpha)\left(\frac{1}{\tau^2-|\xi|^2}\right)_+^{\alpha}, \qquad \mathbf{Re}\alpha<1.
\tag{B. 1}
$$

Denote $\lambda(\alpha)=\frac{n+1}{2}-\alpha,\ \alpha\in\mathbb{C}$. Π^α, Λ^α are distributions defined by analytic continuation from

$$
\Pi^\alpha(x,t) = \Gamma^{-1}(1-\lambda(\alpha))\left(\frac{1}{t^2-|x|^2}\right)_-^{\lambda(\alpha)}, \qquad \Lambda^\alpha(x,t) = \Gamma^{-1}(1-\lambda(\alpha))\left(\frac{1}{t^2-|x|^2}\right)_+^{\lambda(\alpha)}, \qquad \mathbf{Re}\lambda(\alpha)<1.
\tag{B. 2}
$$

Let $a>0, b>0$ and

$$
\rho(\xi,\tau,a,b) = \left\{\left[\tau^2-|\xi|^2\right]^2+\left[a\tau^2+b|\xi|^2\right]^2\right\}^{\frac{1}{2}}, \qquad \cos\theta(\xi,\tau,a,b) = \frac{\tau^2-|\xi|^2}{\rho(\xi,\tau,a,b)}.
\tag{B. 3}
$$

For $\mathbf{Re}\alpha<\frac{n+1}{2}$ and $|\xi|\neq|\tau|$, we assert

$$
\begin{aligned}
\widehat{\mathrm{P}}^{\alpha\,a\,b}(\xi,\tau) &= \left[\tau^2-|\xi|^2+\mathbf{i}a\tau^2+\mathbf{i}b|\xi|^2\right]^{-\alpha} & \widehat{\mathrm{R}}^{\alpha\,a\,b}(\xi,\tau) &= \left[\tau^2-|\xi|^2-\mathbf{i}a\tau^2-\mathbf{i}b|\xi|^2\right]^{-\alpha} \\
&= \rho(\xi,\tau,a,b)^{-\alpha}e^{-\mathbf{i}\alpha\theta(\xi,\tau,a,b)}, & &= \rho(\xi,\tau,a,b)^{-\alpha}e^{\mathbf{i}\alpha\theta(\xi,\tau,a,b)}.
\end{aligned}
\tag{B. 4}
$$

$\widehat{\mathrm{P}}^{\alpha\ a\ b}$ and $\widehat{\mathrm{R}}^{\alpha\ a\ b}$ are distributions defined in $\mathbb{R}^{n+1}$ agree with (B. 4) whenever $|\xi| \neq |\tau|$. Define

$$\widehat{\mathrm{P}}^{\alpha} = \lim_{a\longrightarrow 0,\ b\longrightarrow 0} \widehat{\mathrm{P}}^{\alpha\ a\ b}, \qquad \widehat{\mathrm{R}}^{\alpha} = \lim_{a\longrightarrow 0,\ b\longrightarrow 0} \widehat{\mathrm{R}}^{\alpha\ a\ b}, \qquad \mathbf{Re}\alpha < \tfrac{n+1}{2}. \tag{B. 5}$$

For $\mathbf{Re}\lambda(\alpha) < \frac{n+1}{2}$ and $|x| \neq |t|$, we consider

$$\begin{aligned} \Phi^{\alpha\ a\ b}(x,t) &= \left[t^2 - |x|^2 + \mathbf{i}at^2 + \mathbf{i}b|x|^2\right]^{-\lambda(\alpha)} & \Psi^{\alpha\ a\ b}(x,t) &= \left[t^2 - |x|^2 - \mathbf{i}at^2 - \mathbf{i}b|x|^2\right]^{-\lambda(\alpha)} \\ &= \rho(x,t,a,b)^{-\lambda(\alpha)} e^{-\mathbf{i}\lambda(\alpha)\theta(x,t,a,b)}, & &= \rho(x,t,a,b)^{-\lambda(\alpha)} e^{\mathbf{i}\lambda(\alpha)\theta(x,t,a,b)}. \end{aligned} \tag{B. 6}$$

$\Phi^{\alpha\ a\ b}$ and $\Psi^{\alpha\ a\ b}$ are distributions defined in $\mathbb{R}^{n+1}$ agree with (B. 6) whenever $|x| \neq |t|$. Define

$$\Phi^{\alpha} = \lim_{a\longrightarrow 0,\ b\longrightarrow 0} \Phi^{\alpha\ a\ b}, \qquad \Psi^{\alpha} = \lim_{a\longrightarrow 0,\ b\longrightarrow 0} \Psi^{\alpha\ a\ b}, \qquad \mathbf{Re}\lambda(\alpha) < \tfrac{n+1}{2}. \tag{B. 7}$$

• $\widehat{\mathrm{P}}^{\alpha}, \widehat{\mathrm{R}}^{\alpha}$ are analytic for $\mathbf{Re}\alpha < \frac{n+1}{2}$. • $\Phi^{\alpha}, \Psi^{\alpha}$ are analytic for $\mathbf{Re}\lambda(\alpha) < \frac{n+1}{2}$.

Regarding details can be found in 2.2, Chapter III of Gelfand and Shilov [10].

Define $\widehat{\mathrm{P}}^{\alpha}$ and $\widehat{\mathrm{R}}^{\alpha}$ by analytic continuation from (B. 5). Recall (B. 1) and (B. 3)-(B. 5). We find

$$\widehat{\mathrm{P}}^{\alpha} = \Gamma(1-\alpha)\left[e^{-\mathbf{i}\pi\alpha}\widehat{\mathrm{U}}^{\alpha} + \widehat{\mathrm{V}}^{\alpha}\right], \qquad \widehat{\mathrm{R}}^{\alpha} = \Gamma(1-\alpha)\left[e^{\mathbf{i}\pi\alpha}\widehat{\mathrm{U}}^{\alpha} + \widehat{\mathrm{V}}^{\alpha}\right]. \tag{B. 8}$$

Define Φ^{α} and Ψ^{α} by analytic continuation from (B. 7). Recall (B. 2), (B. 3) and (B. 6)-(B. 7). We have

$$\Phi^{\alpha} = \Gamma(1-\lambda(\alpha))\left[e^{-\mathbf{i}\pi\lambda(\alpha)}\Pi^{\alpha} + \Lambda^{\alpha}\right], \qquad \Psi^{\alpha} = \Gamma(1-\lambda(\alpha))\left[e^{\mathbf{i}\pi\lambda(\alpha)}\Pi^{\alpha} + \Lambda^{\alpha}\right]. \tag{B. 9}$$

For $\mathbf{Im}z \neq 0, \mathbf{Im}w \neq 0$ and $0 < \mathbf{Re}\lambda(\alpha) < \frac{n+1}{2}$, we assert

$$Q^{\alpha\ z\ w}(x,t) = \left\{\frac{1}{z|x|^2 + wt^2}\right\}^{\lambda(\alpha)}, \qquad (x,t) \neq (0,0).$$

Let $a > 0, b > 0$. From direct computation, we find

$$\begin{aligned} \widehat{Q}^{\alpha\ \mathbf{i}a\ \mathbf{i}b}(\xi,\tau) &= (-\mathbf{i})^{\lambda(\alpha)} \iint_{\mathbb{R}^{n+1}} e^{-2\pi\mathbf{i}\left[x\cdot\xi + t\tau\right]} \left\{\frac{1}{a|x|^2 + bt^2}\right\}^{\lambda(\alpha)} dxdt \\ &= (-\mathbf{i})^{\lambda(\alpha)} \frac{1}{(\sqrt{a})^n\sqrt{b}} \iint_{\mathbb{R}^{n+1}} e^{-2\pi\mathbf{i}\left[x\cdot\xi/\sqrt{a} + t\tau/\sqrt{b}\right]} \left(\frac{1}{|x|^2 + t^2}\right)^{\lambda(\alpha)} dxdt \qquad x \longrightarrow x/\sqrt{a},\ t \longrightarrow t/\sqrt{b} \\ &= (-\mathbf{i})^{\lambda(\alpha)} \frac{\pi^{-\frac{n+1}{2}+2\lambda(\alpha)}}{(\sqrt{a})^n\sqrt{b}} \frac{\Gamma\left(\frac{n+1}{2} - \lambda(\alpha)\right)}{\Gamma(\lambda(\alpha))} \left\{\frac{1}{|\xi|^2/a + \tau^2/b}\right\}^{\frac{n+1}{2}-\lambda(\alpha)}, \qquad 0 < \mathbf{Re}\lambda(\alpha) < \tfrac{n+1}{2} \end{aligned} \tag{B. 10}$$

whenever $(\xi,\tau) \neq (0,0)$. The last equality in (B. 10) is derived by using

$$\int_{\mathbb{R}^{\mathbf{N}}} e^{-2\pi\mathbf{i}x\cdot\xi}|x|^{\gamma-\mathbf{N}}dx = \frac{\pi^{\frac{\mathbf{N}-\gamma}{2}}\Gamma\left(\frac{\gamma}{2}\right)}{\pi^{\frac{\gamma}{2}}\Gamma\left(\frac{\mathbf{N}-\gamma}{2}\right)}|\xi|^{-\gamma}, \qquad \xi \neq 0, \qquad 0 < \mathbf{Re}\gamma < \mathbf{N}.$$

Replace $a = -\mathbf{i}z$ and $b = -\mathbf{i}w$ inside (B. 10). We have

$$\begin{aligned}\widehat{Q}^{\alpha\ z\ w}(\xi,\tau) &= \iint_{\mathbb{R}^{n+1}} e^{-2\pi\mathbf{i}(x\cdot\xi+t\tau)}\left\{\frac{1}{z|x|^2+wt^2}\right\}^{\lambda(\alpha)} dxdt \\ &= \frac{\pi^{\lambda(\alpha)-\alpha}}{(\sqrt{z})^n\sqrt{w}}\frac{\Gamma(\alpha)}{\Gamma(\lambda(\alpha))}\left\{\frac{1}{|\xi|^2/z+\tau^2/w}\right\}^{\alpha}, \qquad (\xi,\tau)\neq(0,0).\end{aligned} \tag{B. 11}$$

Remark B.1 *(B. 11) is true for every* $\mathbf{Im}z > 0$ *and* $\mathbf{Im}w > 0$.

Let $w = \mathbf{i}b$ fixed. Both sides of (B. 11) are analytic whenever $\mathbf{Im}z > 0$. Moreover, they are equal at $z = \mathbf{i}a$ for every $a > 0$. By analytic continuation, this equality holds for $\mathbf{Im}z > 0$. Given z, $\mathbf{Im}z > 0$, a vice versa argument shows that (B. 11) is true for $\mathbf{Im}w > 0$.

On the other hand, we find

$$\widehat{Q}^{\alpha\ -\mathbf{i}a\ -\mathbf{i}b}(\xi,\tau) = (-\mathbf{i})^{-\lambda(\alpha)}\frac{\pi^{-\frac{n+1}{2}+2\lambda(\alpha)}}{(\sqrt{a})^n\sqrt{b}}\frac{\Gamma\left(\frac{n+1}{2}-\lambda(\alpha)\right)}{\Gamma(\lambda(\alpha))}\left\{\frac{1}{|\xi|^2/a+\tau^2/b}\right\}^{\frac{n+1}{2}-\lambda(\alpha)}, \qquad (\xi,\tau)\neq(0,0), \quad 0<\mathbf{Re}\lambda(\alpha)<\frac{n+1}{2} \tag{B. 12}$$

by carrying out the same estimate in (B. 10).

Replace $a = \mathbf{i}z$ and $b = \mathbf{i}w$ inside (B. 12), we obtain (B. 11) again. Furthermore, an analogue argument below **Remark B.1** shows (B. 11) hold for $\mathbf{Im}z < 0, \mathbf{Im}w < 0$.

Let

$$\rho(a) = \sqrt{1+a^2}, \qquad \cos\theta(a) = -1/\rho(a), \qquad \sin\theta(a) = a/\rho(a). \tag{B. 13}$$

Note that

$$(-1\pm\mathbf{i}a)^{\frac{n}{2}} = \rho(a)^{\frac{n}{2}}e^{\pm\mathbf{i}\theta\left(\frac{n}{2}\right)} \longrightarrow e^{\pm\mathbf{i}\pi\left(\frac{n}{2}\right)} \qquad \text{as} \qquad a\longrightarrow 0. \tag{B. 14}$$

Consider $z = -1\pm\mathbf{i}a$ and $w = 1\pm\mathbf{i}b$ inside (B. 11). We have

$$\begin{aligned}\widehat{Q}^{\alpha\ z\ w}(\xi,\tau) &\doteq \widehat{Q}^{\alpha\ a\ b}(\xi,\tau) \\ &= \iint_{\mathbb{R}^{n+1}} e^{-2\pi\mathbf{i}(x\cdot\xi+t\tau)}\left\{\frac{1}{(-1\pm\mathbf{i}a)|x|^2+(1\pm\mathbf{i}b)t^2}\right\}^{\lambda(\alpha)} dxdt \\ &= \frac{\pi^{\lambda(\alpha)-\alpha}}{(-1\pm\mathbf{i}a)^{\frac{n}{2}}(1\pm\mathbf{i}b)^{\frac{1}{2}}}\frac{\Gamma(\alpha)}{\Gamma(\lambda(\alpha))}\left\{\frac{(-1\pm\mathbf{i}a)(1\pm\mathbf{i}b)}{(1\pm\mathbf{i}b)|\xi|^2+(-1\pm\mathbf{i}a)\tau^2}\right\}^{\alpha}\end{aligned} \tag{B. 15}$$

for $(\xi,\tau)\neq(0,0)$ and $0<\mathbf{Re}\lambda(\alpha)<\frac{n+1}{2}$.

Define $\widehat{Q}^{\alpha\ a\ b}$, $a>0, b>0, 0<\mathbf{Re}\lambda(\alpha)<\frac{n+1}{2}$ as a distribution in $\mathbb{R}^{n+1}$ agree with (B. 15) whenever $|\xi|\neq|\tau|$. Recall (B. 4)-(B. 7). By using (B. 14) and taking $a\longrightarrow 0, b\longrightarrow 0$, we find

$$\widehat{\Phi}^{\alpha} = \pi^{\lambda(\alpha)-\alpha}e^{-\mathbf{i}\pi\left(\frac{n}{2}\right)}\frac{\Gamma(\alpha)}{\Gamma(\lambda(\alpha))}\widehat{\mathrm{R}}^{\alpha}, \qquad \widehat{\Psi}^{\alpha} = \pi^{\lambda(\alpha)-\alpha}e^{\mathbf{i}\pi\left(\frac{n}{2}\right)}\frac{\Gamma(\alpha)}{\Gamma(\lambda(\alpha))}\widehat{\mathrm{P}}^{\alpha} \tag{B. 16}$$

simultaneously for $0<\mathbf{Re}\lambda(\alpha)<\frac{n+1}{2}$.

Moreover, $\widehat{\mathrm{P}}^{\alpha}, \widehat{\mathrm{R}}^{\alpha}$ and $\Phi^{\alpha}, \Psi^{\alpha}$ can be given in terms of $\Pi^{\alpha}, \Lambda^{\alpha}$ and $\mathrm{U}^{\alpha}, \mathrm{V}^{\alpha}$ as (B. 8) and (B. 9). A direct computation shows

$$\begin{aligned} \widehat{\Lambda}^{\alpha}\left[e^{i\pi\lambda(\alpha)} - e^{-i\pi\lambda(\alpha)}\right] &= \Gamma^{-1}(1-\lambda(\alpha))\left[e^{\mathbf{i}\pi\lambda(\alpha)}\widehat{\Phi}^{\alpha} - e^{-\mathbf{i}\pi\lambda(\alpha)}\widehat{\Psi}^{\alpha}\right] \\ &= \Gamma^{-1}(1-\lambda(\alpha))\Gamma^{-1}(\lambda(\alpha))\pi^{\lambda(\alpha)-\alpha}\Gamma(\alpha) \\ &\quad \left[e^{-\mathbf{i}\pi\left[\frac{n}{2}-\lambda(\alpha)\right]}\widehat{\mathrm{R}}^{\alpha} - e^{\mathbf{i}\pi\left[\frac{n}{2}-\lambda(\alpha)\right]}\widehat{\mathrm{P}}^{\alpha}\right], \qquad 0 < \mathbf{Re}\lambda(\alpha) < \tfrac{n+1}{2}. \end{aligned} \tag{B. 17}$$

From (B. 17), by using the identity $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$, $z \in \mathbb{C}$, we obtain

$$\begin{aligned} \widehat{\Lambda}^{\alpha} &= \pi^{\lambda(\alpha)-\alpha-1}\Gamma(\alpha)\frac{1}{2\mathbf{i}}\left\{-e^{\mathbf{i}\pi\left[\frac{n}{2}-\lambda(\alpha)\right]}\widehat{\mathrm{P}}^{\alpha} + e^{-\mathbf{i}\pi\left[\frac{n}{2}-\lambda(\alpha)\right]}\widehat{\mathrm{R}}^{\alpha}\right\} \\ &= \pi^{\lambda(\alpha)-\alpha-1}\Gamma(\alpha)\Gamma(1-\alpha)\left\{\widehat{\mathrm{U}}^{\alpha} - \sin\pi\left(\alpha - \tfrac{1}{2}\right)\widehat{\mathrm{V}}^{\alpha}\right\}, \qquad 0 < \mathbf{Re}\lambda(\alpha) < \tfrac{n+1}{2}. \end{aligned} \tag{B. 18}$$

Lastly, for $0 < \mathbf{Re}\alpha < 1$, $\widehat{\mathrm{U}}^{\alpha}$ and $\widehat{\mathrm{V}}^{\alpha}$ agree with $\widehat{\mathrm{U}}^{\alpha}(\xi,\tau)$ and $\widehat{\mathrm{V}}^{\alpha}(\xi,\tau)$ in (B. 1) whenever $|\xi| \neq |\tau|$. As a consequence of (B. 18), we have

$$\widehat{\Lambda}^{\alpha}(\xi,\tau) = \pi^{\lambda(\alpha)-\alpha-1}\Gamma(\alpha)\left\{\left(\frac{1}{\tau^2 - |\xi|^2}\right)_-^{\alpha} - \sin\pi\left(\alpha - \tfrac{1}{2}\right)\left(\frac{1}{\tau^2 - |\xi|^2}\right)_+^{\alpha}\right\}, \qquad |\xi| \neq |\tau|.$$

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References

[1] J. Bourgain, *Some new estimates for oscillatory integrals*, Essays on Fourier analysis in honor of Elias M. Stein, Princeton University Press, 1994.

[2] J. Bourgain, **L**^{p}*-estimates for oscillatory integrals in several variables*, Geometric and Functional Analysis **1**: no.4, 321-374,1991.

[3] J. Bourgain and L. Guth, *Bounds on oscillatory integral operators based on multilinear estimates*, Geometric and Functional Analysis **21**: no.6,1239-1295,2011.

[4] L. Carlson and P. Sjölin, *Oscillatory integrals and a multiplier problem for the disc*, Studia Mathematica **44**: 87-99, 1972.

[5] A. Carbery, *Restriction implies Bochner-Riesz for paraboloids*, Mathematical Proceedings of the Cambridge Philosophical Society **111**: no.3, 525-529, 1992.

[6] C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math **15**: 44-52, 1973.

[7] C. Fefferman, *The multiplier problem for the ball*, Annals of Mathematics, **94**: 330-336, 1971.

[8] L. Guth, *A restriction estimate using polynomial partitioning*, J. Amer. Math. Soc **29**: 371-413, 2016.

[9] L. Guth, *Restriction estimate using polynomial partitioning II*, Acta Mathematica **221**:no.1, 81-142, 2018.

[10] I. M. Gelfand and G. E. Shilov, *Generalized Functions*, vol. I, Academic Press, New York, 1964.

[11] S. Guo, C. Oh, H. Wang, S. Wu and R. Zhang, *The Bochner-Riesz problem, an old approach revisited*, Peking Mathematical Journal **8**: 201-270, 2025.

[12] S. Guo, H. Wang and R. Zhang, *A dichotomy for Hormander-type oscillatory integral operators*, Invention Mathematicae **238**: 503-584, 2024.

[13] P. Gressman, *On the Oberlin affine curvature condition*, Duke Mathematics Journal **168**: no.11, 2075-2126, 2019.

[14] P. Gressman, *A Local curvature characterization of maximally-nondegenerate Radon-like transforms*, arXiv:2303.03325.

[15] C. S. Herz, *On the mean inversion of Fourier and Hankel transforms*, Proc. Nat. Acad. Sci. U.S.A. **40**: 996-999, 1954.

[16] S. Lee, *Improved bounds for Bochner-Riesz and maximal Bochner-Riesz operators*, Duke Mathematics Journal, **122**: no.1, 205-232,2004.

[17] A. Seeger, C. D. Sogge and E. M. Stein, *Regularity Properties of Fourier Integral Operators*, Annals of Mathematics, **134**: 231-251, 1991.

[18] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton University Press, 1993.

[19] E. M. Stein, *Interpolation of Linear Operators*, Transactions of the A. M. S **83**: no.2, 482-492, 1956.

[20] R. Strichartz, *Convolutions with kernels having singularity on a sphere*, Transaction of the American Mathematical Society **148**: 461-471, 1970.

[21] T. Tao, *The Bochner-Riesz conjecture implies the Restriction conjecture*, Duke Mathematics Journal **96**: 363-376, 1999.

[22] T. Tao and A. Vargas, *A bilinear approach to the restriction and Kakeya conjectures*, J. Amer. Math. Soc **11**: 967-1000, 1998.

[23] P. Tomas, *A restriction theorem for the Fourier transform*, Bulletin of A.M.S **81**: 477-478, 1975.

[24] G. N. Watson, *Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1944.

[25] T. Wolff, *A sharp bilinear cone restriction estimate*, Annals of Mathematics **153**: 661-698, 2001.

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