$\mathbf{H}^1 \longrightarrow \mathbf{L}^q$-boundedness of fractional integral operators having a flag kernel

Jiashu Zhang and Zipeng Wang

Abstract

We study a family of fractional integral operators

$$\mathbf{I}^{\rho}_{\alpha\beta}f(x,y) \;=\; \iint_{\mathbb{R}^n\times\mathbb{R}^m} f(x-u,y-v)\left(\frac{1}{|u|}\right)^{n-\alpha}\left[\frac{1}{|u|^{\rho}+|v|}\right]^{m-\beta} dudv$$

for $0<\alpha<n, 0<\beta<m$ and $\rho\geq 1$. First, we show $\mathbf{I}^{\rho}_{\alpha\beta}: \mathbf{L}^p(\mathbb{R}^{n+m}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{n+m}), 1<p<q<\infty$ if and only if $\frac{\alpha}{n}\geq\frac{\beta}{m}$ and $\frac{\alpha+\rho\beta}{n+\rho m}=\frac{1}{p}-\frac{1}{q}$. Second, we prove that $\mathbf{I}^{\rho}_{\alpha\beta}$ is bounded from the classical, atom decomposable $\mathbf{H}^1$-Hardy space to $\mathbf{L}^q(\mathbb{R}^{n+m})$ if and only if $\frac{\alpha}{n}>\frac{\beta}{m}$ and $\frac{\alpha+\rho\beta}{n+\rho m}=1-\frac{1}{q}$.

1 Introduction

Let $0<\alpha<n$. A fractional integral operator $\mathbf{I}_{\alpha}$ is initially defined as

$$\mathbf{I}_{\alpha}f(x) \;=\; \int_{\mathbb{R}^n} f(u)\left[\frac{1}{|x-u|}\right]^{n-\alpha} du. \tag{1. 1}$$

In 1928, Hardy and Littlewood [1] have obtained an regularity theorem for $\mathbf{I}_{\alpha}$ when $\mathbf{N}=1$. Ten years later, Sobolev [2] made extensions on every higher dimensional space.

$\diamond$ Throughout, $\mathfrak{B}>0$ is regarded as a generic constant depending on its sub-indices.

Hardy-Littlewood-Sobolev theorem *Let $\mathbf{I}_{\alpha}$ defined in (1. 1) for $0<\alpha<n$. We have*

$$\begin{gathered}\left\|\mathbf{I}_{\alpha}f\right\|_{\mathbf{L}^q(\mathbb{R}^n)} \;\leq\; \mathfrak{B}_{p\,q}\,\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^n)}, \qquad 1<p<q<\infty \\ \textit{if and only if} \qquad \frac{\alpha}{n} \;=\; \frac{1}{p}-\frac{1}{q}.\end{gathered} \tag{1. 2}$$

This classical result has been re-investigated by Krantz [7] on Hardy spaces.

Krantz theorem *Let $\mathbf{I}_{\alpha}$ defined in (1. 1) for $0<\alpha<n$. We have*

$$\begin{gathered}\left\|\mathbf{I}_{\alpha}f\right\|_{\mathbf{H}^q(\mathbb{R}^n)} \;\leq\; \mathfrak{B}_{p\,q}\,\left\|f\right\|_{\mathbf{H}^p(\mathbb{R}^n)}, \qquad 0<p<q<\infty \\ \textit{if and only if} \qquad \frac{\alpha}{n} \;=\; \frac{1}{p}-\frac{1}{q}.\end{gathered} \tag{1. 3}$$

Remark 1.1. *$\mathbf{H}^p(\mathbb{R}^n), 0<p\leq 1$ is the classical $\mathbf{H}^p$-Hardy space investigated by Fefferman and Stein [3]. Moreover, it has a characterization of atomic decomposition established by Coifman [4].*

Recently, **Krantz theorem** is extended to the multi-parameter setting by Tang [16] whereas $\mathbf{I}_\alpha$ in (1. 1) is replaced with strong fractional integral operators whose kernels have singularity on every coordinate subspace. To better illustrate the difference between this new result and ours, we focus on its bi-parameter version. Let $0 < \alpha < n$, $0 < \beta < m$. Define

$$\mathbf{I}_{\alpha\beta} f(x,y) \; = \; \iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left[\frac{1}{|x-u|}\right]^{n-\alpha}\left[\frac{1}{|y-v|}\right]^{m-\beta} dudv. \tag{1. 4}$$

Tang theorem Let $\mathbf{I}_{\alpha\beta}$ defined in (1. 4) for $0 < \alpha < n$, $0 < \beta < m$. We have

$$\begin{aligned} \left\|\mathbf{I}_{\alpha\beta} f\right\|_{\mathbf{H}^q\times\mathbf{H}^q(\mathbb{R}^n\times\mathbb{R}^m)} \; &\leq \; \mathfrak{B}_{p\;q} \left\|f\right\|_{\mathbf{H}^p\times\mathbf{H}^p(\mathbb{R}^n\times\mathbb{R}^m)}, \qquad 0<p<q<\infty \\ \text{if and only if} \qquad & \frac{\alpha}{n} \; = \; \frac{\beta}{m} \; = \; \frac{1}{p}-\frac{1}{q}. \end{aligned} \tag{1. 5}$$

Remark 1.2. $\mathbf{H}^p \times \mathbf{H}^p(\mathbb{R}^n \times \mathbb{R}^m)$, $0 < p \leq 1$ *is the product Hardy space introduced by Gundy and Stein [6]. Furthermore, it cannot be characterized in terms of "rectangle atoms". See the counter-example of Carleson [5].*

In this paper, we study a family of fractional integral operators whose kernels satisfying non-isotropic dilations have singularity on a coordinate subspace, commonly known as flag kernels. An initial motivation to assert these operators comes from certain sub-elliptic boundary value problems. The solution turns out to be a composition of two singular integral operators. One of them is elliptic. The other is parabolic associated with an non-isotropic dilation. Singular integrals of this type have been systematically studied. For instance, see the paper by Phong and Stein [8], Muller, Ricci and Stein [9], Nagel, Ricci and Stein [10], Nagel, Ricci, Stein and Wainger [11], Han-et-al [12], Han, Lu and Sawyer [13] and Han, Lee, Li and Wick [14]-[15]. This direction remains largely open for fractional integrals.

Let $0 < \alpha < n$, $0 < \beta < m$ and $\rho \geq 1$. We define

$$\mathbf{I}^\rho_{\alpha\beta} f(x,y) \; = \; \iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\Omega^{\alpha\beta}_\rho(x-u,y-v)dudv \tag{1. 6}$$

where $\Omega^{\alpha\beta}_\rho$ is a distribution on $\mathbb{R}^{n+m}$ agree with

$$\Omega^{\alpha\beta}_\rho(x,y) \; = \; \left(\frac{1}{|x|}\right)^{n-\alpha}\left[\frac{1}{|x|^\rho+|y|}\right]^{m-\beta}, \qquad x\neq 0. \tag{1. 7}$$

A concrete example is given by Muller, Ricci and Stein [9]. Let $x = (z,w) \in \mathbb{R}^d \times \mathbb{R}^d$. Consider $\mathcal{L}^{-\mathbf{a}}T^{-\mathbf{b}}$ for $0 < \mathbf{a} < d$, $0 < \mathbf{b} < 1$: $T = \partial_t$ and $\mathcal{L}$ is the sub-Laplacian: $\mathcal{L} = -\sum_{i=1}^d \mathbf{Z}_i^2 + \mathbf{W}_i^2$, $\mathbf{Z}_i = \partial_{z_i} + 2w_i\partial_t$, $\mathbf{W}_i = \partial_{w_i} - 2z_i\partial_t$. The inverse of $\mathcal{L}^{\mathbf{a}}$ is given as the Riesz potential:

$$\mathcal{L}^{-\mathbf{a}} \; = \; \frac{1}{\Gamma(\mathbf{a})}\int_0^\infty s^{\mathbf{a}-1}e^{-s\mathcal{L}}ds$$

where Γ denotes Gamma function. The kernel of $\mathcal{L}^{-\mathbf{a}}T^{-\mathbf{b}}$ is a distribution in $\mathbb{R}^{2d+1}$ agree with a function similar to $\Gamma\left(\frac{1-\mathbf{b}}{2}\right)\left(\frac{1}{|x|}\right)^{2d-2\mathbf{a}}\left[\frac{1}{|x|^2+|y|}\right]^{1-\mathbf{b}}$ for $x \neq 0$.[1] In compare to (1. 7), we find $n = 2d$, $m = 1$, $\alpha = 2\mathbf{a}$, $\beta = \mathbf{b}$ and $\rho = 2$.

[1]We say A similar to B if $\mathbf{c}^{-1}B \leq A \leq \mathbf{c}B$ for some $\mathbf{c} > 0$.

For $1 < p < q < \infty$, we have $\mathcal{L}^{-\mathbf{a}} T^{-\mathbf{b}} \colon \mathbf{L}^p(\mathbb{R}^{2d+1}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{2d+1})$ if and only if $\mathbf{a} \geq d\mathbf{b}$ and $\frac{\mathbf{a}+\mathbf{b}}{d+1} = \frac{1}{p} - \frac{1}{q}$. This is proved in section 6 of [9] by using complex interpolation. One of the two end-point estimates relies on the $\mathbf{L}^p$-theorem developed there. First, we show that every convolution operator with a kernel similar to (1. 7) satisfies the desired $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-regularity.

Theorem One *Let* $\mathbf{I}^{\rho}_{\alpha\beta}$ *defined in (1. 6)-(1. 7) for* $0 < \alpha < n$, $0 < \beta < m$ *and* $\rho \geq 1$. *We have*

$$\left\| \mathbf{I}^{\rho}_{\alpha\beta} f \right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})} \leq \mathfrak{B}_{\alpha\,\beta\,\rho\,p\,q} \left\| f \right\|_{\mathbf{L}^p(\mathbb{R}^{n+m})}, \qquad 1 < p < q < \infty \tag{1. 8}$$

if and only if

$$\frac{\alpha}{n} \geq \frac{\beta}{m}, \qquad \frac{\alpha + \rho\beta}{n + \rho m} = \frac{1}{p} - \frac{1}{q}. \tag{1. 9}$$

Our main result is to give a characterization for the $\mathbf{H}^1 \longrightarrow \mathbf{L}^q$-boundedness of $\mathbf{I}^{\rho}_{\alpha\beta}$.

Theorem Two *Let* $\mathbf{I}^{\rho}_{\alpha\beta}$ *defined in (1. 6)-(1. 7) for* $0 < \alpha < n$, $0 < \beta < m$ *and* $\rho \geq 1$. *We have*

$$\left\| \mathbf{I}^{\rho}_{\alpha\beta} f \right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})} \leq \mathfrak{B}_{\alpha\,\beta\,\rho\,q} \left\| f \right\|_{\mathbf{H}^1(\mathbb{R}^{n+m})}, \qquad 1 < q < \infty \tag{1. 10}$$

if and only if

$$\frac{\alpha}{n} > \frac{\beta}{m}, \qquad \frac{\alpha + \rho\beta}{n + \rho m} = 1 - \frac{1}{q}. \tag{1. 11}$$

Note that $\mathbf{H}^1(\mathbb{R}^{n+m})$ is the same $\mathbf{H}^1$-Hardy space introduced by Fefferman and Stein [3] which has an atomic decomposition due to Coifman [4]. On the other hand, the product Hardy space $\mathbf{H}^1 \times \mathbf{H}^1(\mathbb{R}^n \times \mathbb{R}^m)$ defined by Gundy and Stein [6] is a subspace of $\mathbf{H}^1(\mathbb{R}^{n+m})$.

The fractional integral operator $\mathbf{I}^{\rho}_{\alpha\beta}$ whose kernel $\Omega^{\alpha\beta}_{\rho}$ carries certain multi-parameter structure as defined in (1. 7) is still bounded from the classical, atom decomposable $\mathbf{H}^1$-Hardy space to $\mathbf{L}^q(\mathbb{R}^{n+m})$.

The remaining paper is organized as follows. In the next section, we show $\mathbf{I}^{\rho}_{\alpha\beta} \colon \mathbf{L}^p(\mathbb{R}^{n+m}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{n+m})$ implying $\frac{\alpha}{n} \geq \frac{\beta}{m}$ and $\frac{\alpha+\rho\beta}{n+\rho m} = \frac{1}{p} - \frac{1}{q}$ for $1 \leq p < q < \infty$. Moreover, we give a counter example for $\mathbf{I}^{\rho}_{\alpha\beta} \colon \mathbf{H}^1(\mathbb{R}^{n+m}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{n+m})$ when $\frac{\alpha}{n} = \frac{\beta}{m}$ and $\frac{\alpha+\rho\beta}{n+\rho m} = 1 - \frac{1}{q}$. In section 3, we show that (1. 9) implies (1. 8). Section 4 is devoted to the proof of (1. 11) $\Longrightarrow$ (1. 10). By the characterization of atomic decomposition for $\mathbf{H}^1(\mathbb{R}^{n+m})$, it is suffice to show $\left\| \mathbf{I}^{\rho}_{\alpha\beta} a \right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})} \leq \mathfrak{B}_{\alpha\,\beta\,\rho\,q}$ of which a is an $\mathbf{H}^1$-atom:

$$\mathbf{supp} a \subset \mathbf{Q}, \qquad |a(x,y)| \leq \frac{1}{\mathbf{vol}\{\mathbf{Q}\}}, \qquad \iint_{\mathbf{Q}} a(x,y) dx dy = 0 \tag{1. 12}$$

where $\mathbf{Q} \subset \mathbb{R}^{n+m}$ is some cube parallel to the coordinates.

2 Proof of necessary conditions

Let $\mathbf{I}^{\rho}_{\alpha\beta}$ defined in (1. 6)-(1. 7) for $0 < \alpha < n$, $0 < \beta < m$ and $\rho \geq 1$. We have

$$\mathbf{I}^{\rho}_{\alpha\beta} f(x,y) = \iint_{\mathbb{R}^n \times \mathbb{R}^m} f(u,v) \left(\frac{1}{|x-u|} \right)^{n-\alpha} \left[\frac{1}{|x-u|^{\rho} + |y-v|} \right]^{m-\beta} du dv. \tag{2. 1}$$

By changing dilations $(x,y)\longrightarrow(\delta x,\delta^\rho\lambda y)$ and $(u,v)\longrightarrow(\delta u,\delta^\rho\lambda v)$ for $\delta>0,\lambda>1$, we find

$$
\begin{array}{l}
\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f\left[\delta^{-1}u,\delta^{-\rho}\lambda^{-1}v\right]\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho+|y-v|}\right]^{m-\beta}dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\
= \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{\delta|x-u|}\right)^{n-\alpha}\left[\frac{1}{\delta^\rho|x-u|^\rho+\delta^\rho\lambda|y-v|}\right]^{m-\beta}\delta^{n+\rho m}\lambda^m dudv\right\}^q \delta^{n+\rho m}\lambda^m dxdy\right\}^{\frac{1}{q}}\\
= \delta^{\alpha+\rho\beta}\delta^{\frac{n+\rho m}{q}}\lambda^{\frac{m}{q}}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho+\lambda|y-v|}\right]^{m-\beta}\lambda^m dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\
\geq \delta^{\alpha+\rho\beta}\delta^{\frac{n+\rho m}{q}}\lambda^{\frac{m}{q}}\\
\quad \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho+|y-v|}\right]^{m-\beta}\lambda^{\beta-m}\lambda^m dudv\right\}^q dxdy\right\}^{\frac{1}{q}} \qquad (\lambda>1)\\
= \delta^{\alpha+\rho\beta}\delta^{\frac{n+\rho m}{q}}\lambda^{\beta}\lambda^{\frac{m}{q}}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho+|y-v|}\right]^{m-\beta}dudv\right\}^q dxdy\right\}^{\frac{1}{q}}.
\end{array}
\tag{2. 2}
$$

Consider

$$
\left\|\mathbf{I}^\rho_{\alpha\beta}f\right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})} \lesssim \left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{n+m})}, \qquad 1\leq p<q<\infty \tag{2. 3}
$$

which implies that the last line of (2. 2) is bounded by a constant multiple of

$$
\begin{array}{l}
\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f\left[\delta^{-1}u,\delta^{-\rho}\lambda^{-1}v\right]\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho+|y-v|}\right]^{m-\beta}dudv\right\}^q dxdy\right\}^{\frac{1}{q}}\\
\lesssim \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left[f\left(\delta^{-1}x,\delta^{-\rho}\lambda^{-1}y\right)\right]^p dxdy\right\}^{\frac{1}{p}}\\
= \delta^{\frac{n+\rho m}{p}}\lambda^{\frac{m}{p}}\left\|f\right\|_{\mathbf{L}^p(\mathbb{R}^{n+m})}.
\end{array}
\tag{2. 4}
$$

This must be true for every $\delta>0$ and $\lambda>1$. We necessarily have

$$
\frac{\alpha+\rho\beta}{n+\rho m} = \frac{1}{p}-\frac{1}{q} \tag{2. 5}
$$

and

$$
\beta \leq \frac{m}{p}-\frac{m}{q}. \tag{2. 6}
$$

By putting together (2. 5) and (2. 6), we find

$$
\alpha m \geq \beta n. \tag{2. 7}
$$

2.1 A counter example for $\mathbf{H}^1 \longrightarrow \mathbf{L}^q$-estimate

Now, we give a counterexample for $\mathbf{I}^\rho_{\alpha\beta}\colon \mathbf{H}^1(\mathbb{R}^{n+m}) \longrightarrow \mathbf{L}^q(\mathbb{R}^{n+m})$ when $\alpha m = n\beta$.

Let $\mathbf{Q}_o = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \colon |x_i| \leq 1, i = 1,2,\ldots,n;\ \ |y_i| \leq 1, i = 1,2,\ldots,m\}$. Consider $a(x,y) = \mathrm{sgn}(x_1)\chi_{\mathbf{Q}_o}(x,y)$ which is an $\mathbf{H}^1$-atom in $\mathbb{R}^{n+m}$. Define $\mathbf{U} \subset \mathbb{R}^n$ by $\mathbf{U} = \Big\{x \in \mathbb{R}^n \colon 2 \leq x_i \leq 4, i = 1,2,\ldots,n\Big\}$. We aim to show

$$\iint_{\mathbf{U}\times\mathbb{R}^m} \Big|\mathbf{I}^\rho_{\alpha\beta} a(x,y)\Big|^q dxdy \ = \ \infty. \tag{2. 8}$$

Denote $\mathbf{e}_1$ to be the unit vector in x_1-axis and $\mathbf{Q}_o^+ = \mathbf{Q}_o \cap \{x_1 > 0\}$. For $(x,y) \in \mathbf{U}\times\mathbb{R}^m$, we find

$$\begin{aligned}
\mathbf{I}^\rho_{\alpha\beta} a(x,y) \ &= \ \iint_{\mathbf{Q}_o} a(u,v)\Omega^{\alpha\beta}_\rho(x-u,y-v)dudv \\
&= \ \iint_{\mathbf{Q}_o^+} \left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho + |y-v|}\right]^{m-\beta} - \left(\frac{1}{|x-u+\mathbf{e}_1|}\right)^{n-\alpha}\left[\frac{1}{|x-u+\mathbf{e}_1|^\rho + |y-v|}\right]^{m-\beta} dudv \\
&\gtrsim \ \iint_{\mathbf{Q}_o^+} \left[\frac{1}{1+|y-v|}\right]^{m-\beta} dudv \\
&\gtrsim \ \left(\frac{1}{1+|y|}\right)^{m-\beta}.
\end{aligned} \tag{2. 9}$$

Note that $\frac{\alpha}{n} = \frac{\beta}{m}$ and $\frac{\alpha+\rho\beta}{n+\rho m} = 1 - \frac{1}{q}$ together imply $\beta = m - \frac{m}{q}$. We have

$$\begin{aligned}
\iint_{\mathbf{U}\times\mathbb{R}^m} \Big|\mathbf{I}^\rho_{\alpha\beta} a(x,y)\Big|^q dxdy \ &\gtrsim \ \iint_{\mathbf{U}\times\mathbb{R}^m} \left(\frac{1}{1+|y|}\right)^{q(m-\beta)} dxdy \qquad \text{by (2. 9)} \\
&\approx \ \int_{\mathbb{R}^m} \left(\frac{1}{1+|y|}\right)^m dy \ = \ \infty.
\end{aligned} \tag{2. 10}$$

3 Proof of Theorem One

Recall (1. 9). We have $\frac{\alpha}{n} \geq \frac{\beta}{m}$ and $\frac{\alpha+\rho\beta}{n+\rho m} = \frac{1}{p} - \frac{1}{q}$ for $1 < p < q < \infty$. Define $0 < \mathbf{a} \leq \alpha < n$ and $0 < \beta \leq \mathbf{b} < m$ implicitly by requiring

$$\frac{\mathbf{a}}{n} \ = \ \frac{\mathbf{b}}{m}, \qquad \mathbf{a} + \rho\mathbf{b} \ = \ \alpha + \rho\beta. \tag{3. 1}$$

By solving the two equations in (3. 1), we find

$$\mathbf{a} = \frac{\alpha+\rho\beta}{1+\rho(\frac{m}{n})}, \qquad \mathbf{b} = \frac{\alpha+\rho\beta}{\frac{n}{m}+\rho}. \tag{3. 2}$$

Furthermore, (3. 1) together with the homogeneity condition $\frac{\alpha+\rho\beta}{n+\rho m} = \frac{1}{p} - \frac{1}{q}$ imply

$$\frac{\mathbf{a}}{n} \ = \ \frac{\mathbf{b}}{m} \ = \ \frac{\mathbf{a}+\rho\mathbf{b}}{n+\rho m} \ = \ \frac{\alpha+\rho\beta}{n+\rho m} \ = \ \frac{1}{p} - \frac{1}{q}. \tag{3. 3}$$

Let $\mathbf{I}^{\rho}_{\alpha\beta} f = f * \Omega^{\alpha\beta}_{\rho}$ defined in (1. 6)-(1. 7). Observe that

$$\begin{aligned}
\Omega^{\alpha\beta}_{\rho}(x,y) &= \left(\frac{1}{|x|}\right)^{n-\alpha}\left[\frac{1}{|x|^{\rho}+|y|}\right]^{m-\beta} \\
&= \left(\frac{1}{|x|}\right)^{n-\alpha}\left[\frac{1}{|x|^{\rho}+|y|}\right]^{m-\mathbf{b}}\left[\frac{1}{|x|^{\rho}+|y|}\right]^{\mathbf{b}-\beta} \\
&\le \left(\frac{1}{|x|}\right)^{n-\alpha}\left[\frac{1}{|x|^{\rho}+|y|}\right]^{m-\mathbf{b}}\left(\frac{1}{|x|}\right)^{\rho\mathbf{b}-\rho\beta} \qquad (3.\ 4) \\
&= \left(\frac{1}{|x|}\right)^{n-\mathbf{a}}\left[\frac{1}{|x|^{\rho}+|y|}\right]^{m-\mathbf{b}} \\
&\le \left(\frac{1}{|x|}\right)^{n-\mathbf{a}}\left(\frac{1}{|y|}\right)^{m-\mathbf{b}} \qquad x \neq 0.
\end{aligned}$$

Recall **Hardy-Littlewood-Sobolev theorem** stated in the beginning of this paper. We have

$$\begin{aligned}
\left\|\mathbf{I}^{\rho}_{\alpha\beta} f\right\|_{\mathbf{L}^{q}(\mathbb{R}^{n+m})} &= \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\Omega^{\alpha\beta}_{\rho}(x-u,y-v)dudv\right\}^{q} dxdy\right\}^{\frac{1}{q}} \\
&\le \left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m}\left\{\iint_{\mathbb{R}^n\times\mathbb{R}^m} f(u,v)\left(\frac{1}{|x-u|}\right)^{n-\mathbf{a}}\left(\frac{1}{|y-v|}\right)^{m-\mathbf{b}} dudv\right\}^{q} dxdy\right\}^{\frac{1}{q}} \qquad \text{by (3. 4)} \\
&\le \left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m} f(x,v)\left(\frac{1}{|y-v|}\right)^{m-\mathbf{b}} dv\right\}^{p} dx\right\}^{\frac{q}{p}} dy\right\}^{\frac{1}{q}} \qquad \text{by (3. 3) and (1. 2)} \\
&\le \left\{\int_{\mathbb{R}^n}\left\{\int_{\mathbb{R}^m}\left\{\int_{\mathbb{R}^m} f(x,v)\left(\frac{1}{|y-v|}\right)^{m-\mathbf{b}} dv\right\}^{q} dy\right\}^{\frac{p}{q}} dx\right\}^{\frac{1}{p}} \qquad \text{by Minkowski intergal inequality} \\
&\le \left\|f\right\|_{\mathbf{L}^{p}(\mathbb{R}^{n+m})} \qquad \text{by (3. 3) and (1. 2).}
\end{aligned} \qquad (3.\ 5)$$

4 Proof of Theorem Two

Let $0 < \alpha < n$, $0 < \beta < m$ and $\rho \ge 1$. From (1. 7), we have

$$\Omega^{\alpha\beta}_{\rho}(x,y) = \left(\frac{1}{|x|}\right)^{n-\alpha}\left[\frac{1}{|x|^{\rho}+|y|}\right]^{m-\beta}, \qquad x \neq 0.$$

A direct computation gives

$$
\begin{aligned}
\left|\nabla_x \Omega_\rho^{\alpha\beta}(x,y)\right| &\le \mathfrak{B}_{\alpha\,\beta\,\rho}\, \Omega_\rho^{\alpha\beta}(x,y)\ \max\left\{\frac{1}{|x|}, \frac{|x|^{\rho-1}}{|x|^\rho+|y|}\right\},\\
\left|\nabla_y \Omega_\rho^{\alpha\beta}(x,y)\right| &\le \mathfrak{B}_{\alpha\,\beta\,\rho}\, \Omega_\rho^{\alpha\beta}(x,y)\ \frac{1}{|x|^\rho+|y|}.
\end{aligned}
\tag{4. 1}
$$

From (4. 1), we conclude

$$
\left|\nabla \Omega_\rho^{\alpha\beta}(x,y)\right| \le \mathfrak{B}_{\alpha\,\beta\,\rho}\, \Omega_\rho^{\alpha\beta}(x,y) \max\left\{\frac{1}{|x|}, \frac{1}{|x|^\rho+|y|}\right\}. \tag{4. 2}
$$

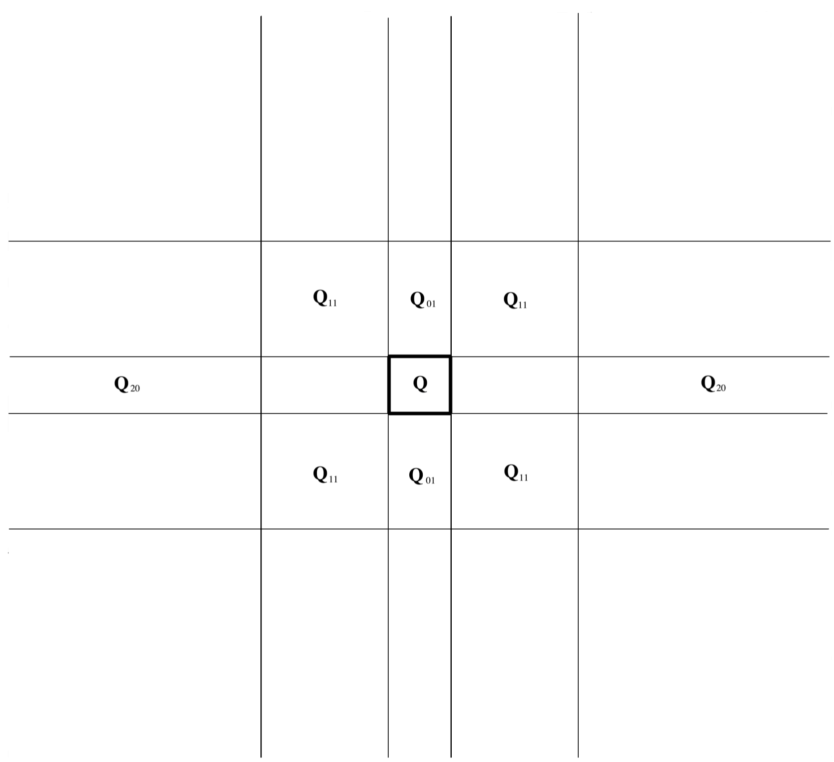
Let $\mathbf{I}_{\alpha\beta}^\rho f = f * \Omega_\rho^{\alpha\beta}$ defined in (1. 6)-(1. 7). We aim to prove

$$
\left\|\mathbf{I}_{\alpha\beta}^\rho a\right\|_{\mathbf{L}^q(\mathbb{R}^{n+m})} \le \mathfrak{B}_{\alpha\,\beta\,\rho\,q} \qquad \text{for} \qquad \frac{\alpha}{n} > \frac{\beta}{m}, \quad \frac{\alpha+\rho\beta}{n+\rho m} = 1-\frac{1}{q} \tag{4. 3}
$$

where a is an $\mathbf{H}^1$-atom in $\mathbb{R}^{n+m}$ satisfying (1. 12).

Denote $\mathbf{Q} \subset \mathbb{R}^{n+m}$ to be a cube centered on the origin with a side length equal to 2^L. Without lose of the generality, we assume a supported in the cube $\frac{1}{2}\mathbf{Q}$: co-center with $\mathbf{Q}$ having a side length 2^{L-1}. For $k, \ell \ge 0$, we define

$$
\mathbf{Q} = \mathbf{Q}_{00}, \qquad \mathbf{Q}_{k\ell} = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : 2^{L+k-1} \le |x| < 2^{L+k},\ \ 2^{L+\ell-1} \le |y| < 2^{L+\ell}\right\}. \tag{4. 4}
$$



We obtain (4. 5) by estimating

$$\iint_{\mathbf{Q}_{k\ell}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^q dxdy, \qquad \frac{\alpha}{n} > \frac{\beta}{m}, \quad \frac{\alpha+\rho\beta}{n+\rho m} = 1-\frac{1}{q} \tag{4. 5}$$

w.r.t **Case 1**: $k=\ell=0$, **Case 2**: $k>0, \ell>0$, **Case 3**: $k>0, \ell=0$ and **Case 4**: $k=0, \ell>0$.

Before moving forward, we note that $\frac{\alpha}{n} > \frac{\beta}{m}$ and $\frac{\alpha+\rho\beta}{n+\rho m} = 1-\frac{1}{q}$ together imply

$$\frac{\alpha}{n} > 1-\frac{1}{q} \tag{4. 6}$$

and

$$\frac{\beta}{m} < 1-\frac{1}{q}. \tag{4. 7}$$

These two strict inequalities will be used later.

4.1 Case 1: $k=\ell=0$

Recall that a is supported in $\frac{1}{2}\mathbf{Q}$ and $|a(x,y)| \leq 2^{n+m}\mathbf{vol}\{\mathbf{Q}\}^{-1}$. Let $(x,y)\in\mathbf{Q}$. We have

$$\begin{aligned}
\left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right| &\leq \iint_{\frac{1}{2}\mathbf{Q}} |a(u,v)| \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left[\frac{1}{|x-u|^{\rho}+|y-v|}\right]^{m-\beta} dudv \\
&\lesssim \sum_{s\leq L}\sum_{r\leq L} \iint_{2^{s-1}<|x-u|\leq 2^s, 2^{r-1}<|y-v|\leq 2^r} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} 2^{s(\alpha-n)} \left[2^r+2^{\rho s}\right]^{\beta-m} dudv \qquad (4.\ 8)\\
&= 2^{-(n+m)L} \sum_{s\leq L}\sum_{r\leq L} \iint_{2^{s-1}<|x-u|\leq 2^s, 2^{r-1}<|y-v|\leq 2^r} 2^{s(\alpha-n)} \left[2^r+2^{\rho s}\right]^{\beta-m} dudv.
\end{aligned}$$

Split the above summation into two: $\mathbf{S}_1+\mathbf{S}_2$ *w.r.t* $r\geq\rho s$ and $r<\rho s$. We find

$$\mathbf{S}_1 = 2^{-(n+m)L} \sum_{s\leq\min\{L/\rho, L\}} \sum_{\rho s\leq r\leq L} \iint_{2^{s-1}<|x-u|\leq 2^s,\ 2^{r-1}<|y-v|\leq 2^r} 2^{s(\alpha-n)} \left[2^r+2^{\rho s}\right]^{\beta-m} dudv \tag{4. 9}$$

and

$$\mathbf{S}_2 = 2^{-(n+m)L} \sum_{r\leq\min\{L, \rho L\}} \sum_{\frac{r}{\rho}<s\leq L} \iint_{2^{s-1}<|x-u|\leq 2^s,\ 2^{r-1}<|y-v|\leq 2^r} 2^{s(\alpha-n)} \left[2^r+2^{\rho s}\right]^{\beta-m} dudv. \tag{4. 10}$$

Consider $\mathbf{S}_1$ in (4. 9). We have

$$\begin{aligned}
\mathbf{S}_1 &\lesssim 2^{-(n+m)L} \sum_{s\leq\min\{L/\rho, L\}} \sum_{\rho s\leq r\leq L} 2^{s(\alpha-n)} \left[2^r+2^{\rho s}\right]^{\beta-m} 2^{sn} 2^{rm} \\
&\leq 2^{-(n+m)L} \sum_{s\leq\min\{L/\rho, L\}} \sum_{\rho s\leq r\leq L} 2^{s\alpha} 2^{r\beta} \qquad (0<\beta<m) \qquad (4.\ 11)\\
&\leq \mathfrak{B}_{\beta}\, 2^{-(n+m-\beta)L} \sum_{s\leq\min\{L/\rho, L\}} 2^{s\alpha}.
\end{aligned}$$

For $L > 0$, $\min\{L/\rho, L\} = L/\rho$. $\mathbf{S}_1$ in (4. 11) is further bounded by $\mathfrak{B}_{\alpha\,\beta} 2^{\left[(\alpha/\rho)+\beta-n-m\right]L}$. We find

$$\begin{aligned}
\iint_{\mathbf{Q}} \mathbf{S}_1^q dxdy \;&\le\; \mathfrak{B}_{\alpha\,\beta}\, 2^{q\left[(\alpha/\rho)+\beta-n-m\right]L} 2^{(n+m)L} \\
&=\; \mathfrak{B}_{\alpha\,\beta}\, 2^{q\left[(\alpha/\rho)+\beta\right]L} 2^{-(q-1)(n+m)L} \;=\; \mathfrak{B}_{\alpha\,\beta}\, 2^{(q-1)\left[\frac{n}{\rho}+m\right]L} 2^{-(q-1)(n+m)L} \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m} = 1-\tfrac{1}{q}\right) \\
&=\; \mathfrak{B}_{\alpha\,\beta}\, 2^{(q-1)\left[\frac{n}{\rho}-n\right]L} \;\le\; \mathfrak{B}_{\alpha\,\beta\,\rho\,q}.
\end{aligned} \tag{4. 12}$$

On the other hand, (4. 6) implies $q\alpha > (q-1)n$. For $L \le 0$, $\min\{L/\rho, L\} = L$. $\mathbf{S}_1$ in (4. 11) is further bounded by $\mathfrak{B}_{\alpha\,\beta} 2^{\left[\alpha+\beta-n-m\right]L}$. We have

$$\begin{aligned}
\iint_{\mathbf{Q}} \mathbf{S}_1^q dxdy \;&\le\; \mathfrak{B}_{\alpha\,\beta}\, 2^{q\left[\alpha+\beta-n-m\right]L} 2^{(n+m)L} \\
&=\; \mathfrak{B}_{\alpha\,\beta}\, 2^{q\alpha\left[1-\frac{1}{\rho}\right]L} 2^{q\left[(\alpha/\rho)+\beta\right]L} 2^{-(q-1)(n+m)L} \\
&=\; \mathfrak{B}_{\alpha\,\beta}\, 2^{q\alpha\left[1-\frac{1}{\rho}\right]L} 2^{(q-1)\left[\frac{n}{\rho}+m\right]L} 2^{-(q-1)(n+m)L} \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m} = 1-\tfrac{1}{q}\right) \\
&=\; \mathfrak{B}_{\alpha\,\beta}\, 2^{q\alpha\left[1-\frac{1}{\rho}\right]L} 2^{(q-1)\left[\frac{n}{\rho}-n\right]L} \\
&=\; \mathfrak{B}_{\alpha\,\beta}\, 2^{\left[q\alpha-(q-1)n\right]\left[1-\frac{1}{\rho}\right]L} \\
&\le\; \mathfrak{B}_{\alpha\,\beta\,\rho\,q}.
\end{aligned} \tag{4. 13}$$

Consider $\mathbf{S}_2$ in (4. 10). We have

$$\begin{aligned}
\mathbf{S}_2 \;&\lesssim\; 2^{-(n+m)L} \sum_{r\le\min\{L,\rho L\}} \sum_{\frac{r}{\rho}<s\le L} 2^{s(\alpha-n)}\left[2^r + 2^{\rho s}\right]^{\beta-m} 2^{sn} 2^{rm} \\
&\le\; 2^{-(n+m)L} \sum_{r\le\min\{L,\rho L\}} \sum_{\frac{r}{\rho}<s\le L} 2^{s\left[\alpha+\rho\beta-\rho m\right]} 2^{rm} \qquad (0<\beta<m) \\
&\le\; \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-(n+m)L} \sum_{r\le\min\{L,\rho L\}} \max\left\{2^{(r/\rho)(\alpha+\rho\beta)}, 2^{L\left[\alpha+\rho\beta-\rho m\right]} 2^{rm}\right\}.
\end{aligned} \tag{4. 14}$$

For $L > 0$, $\min\{L, \rho L\} = L$. $\mathbf{S}_2$ in (4. 14) is further bounded by $\mathfrak{B}_{\alpha\,\beta\,\rho} 2^{-(n+m)L} \max\left\{2^{\left[(\alpha/\rho)+\beta\right]L}, 2^{\left[\alpha+\rho\beta-\rho m\right]L} 2^{mL}\right\}$.

We find

$$\iint_{\mathbf{Q}} \mathbf{S}_2^q dxdy \leq \mathfrak{B}_{\alpha\,\beta\,\rho} \max\left\{2^{q\left[(\alpha/\rho)+\beta\right]L}, 2^{q\left[\alpha+\rho\beta-\rho m\right]L} 2^{qmL}\right\} 2^{-(q-1)(n+m)L}$$

$$= \mathfrak{B}_{\alpha\,\beta\,\rho} \max\left\{2^{(q-1)\left[\frac{n}{\rho}+m\right]L}, 2^{\left[(q-1)n-\rho m\right]L} 2^{qmL}\right\} 2^{-(q-1)(n+m)L} \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m} = 1-\tfrac{1}{q}\right) \tag{4. 15}$$

$$= \mathfrak{B}_{\alpha\,\beta\,\rho} \max\left\{2^{(q-1)\left[\frac{n}{\rho}-n\right]L}, 2^{(1-\rho)mL}\right\} \leq \mathfrak{B}_{\alpha\,\beta\,\rho\,q}.$$

For $L \leq 0$, $\min\{L, \rho L\} = \rho L$. $\mathbf{S}_2$ in (4. 14) is further bounded by $\mathfrak{B}_{\alpha\,\beta\,\rho} 2^{-(n+m)L} 2^{(\alpha+\rho\beta)L}$. We have

$$\iint_{\mathbf{Q}} \mathbf{S}_2^q dxdy \leq \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{q(\alpha+\rho\beta)L} 2^{-(q-1)(n+m)L}$$

$$= \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{(q-1)(n+\rho m)L} 2^{-(q-1)(n+m)L} \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m} = 1-\tfrac{1}{q}\right) \tag{4. 16}$$

$$= \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{(q-1)(\rho-1)mL} \leq \mathfrak{B}_{\alpha\,\beta\,\rho\,q}.$$

4.2 Case 2: $k > 0$, $\ell > 0$

Let $(x, y) \in \mathbf{Q}_{k\ell}$. We have $|x-u| \sim 2^{k+L}$, $|y-v| \sim 2^{\ell+L}$. Consider $k+L \geq 0$ or $\ell \geq k$. We find

$$|x-u|^\rho + |y-v| \gtrsim |x-u|. \tag{4. 17}$$

By using the cancellation property of a: $\iint_{\frac{1}{2}\mathbf{Q}} a(x, y)dxdy = 0$, we have

$$\mathbf{I}^\rho_{\alpha\beta} a(x, y) = \iint_{\frac{1}{2}\mathbf{Q}} a(u, v)\Big[\Omega^{\alpha\beta}_\rho(x-u, y-v) - \Omega^{\alpha\beta}_\rho(x, y)\Big]dudv. \tag{4. 18}$$

Recall the estimate in (4. 2). From (4. 18), we find

$$\begin{aligned}
\left|\mathbf{I}^\rho_{\alpha\beta} a(x, y)\right| &\leq \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left|\nabla\Omega^{\alpha\beta}_\rho(x-u, y-v)\right| dudv \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left[\frac{1}{|x-u|^\rho + |y-v|}\right]^{m-\beta} \\
&\qquad \max\left\{\frac{1}{|x-u|}, \frac{1}{|x-u|^\rho + |y-v|}\right\} dudv \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left[\frac{1}{|x-u|^\rho + |y-v|}\right]^{m-\beta} \frac{1}{|x-u|} dudv \qquad \text{by (4. 17)} \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\frac{1}{2}\mathbf{Q}} 2^{-(n+m)L} 2^L 2^{(k+L)(\alpha-n)} \Big[2^{\rho(k+L)} + 2^{\ell+L}\Big]^{\beta-m} 2^{-(k+L)} dudv \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-k} 2^{(k+L)(\alpha-n)} \Big[2^{\rho(k+L)} + 2^{\ell+L}\Big]^{\beta-m}.
\end{aligned} \tag{4. 19}$$

Suppose $\rho(k+L) \geq \ell + L$. The last line of (4. 19) can be further bounded by $\mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-k} 2^{(k+L)(\alpha-n)} 2^{\rho(k+L)(\beta-m)}$. We have

$$
\begin{aligned}
&\iint_{\mathbf{Q}_{k\ell}} \left| \mathbf{I}^{\rho}_{\alpha\beta} a(x,y) \right|^q dxdy \\
&\leq\ \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}_{k\ell}} 2^{-qk} 2^{q(k+L)(\alpha-n)} 2^{q\rho(k+L)(\beta-m)} dxdy \\
&=\ \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}_{k\ell}} 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} dxdy \\
&\leq\ \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} 2^{(k+L)n} 2^{(\ell+L)m}.
\end{aligned}
\tag{4. 20}
$$

By using (4. 33) and taking the summation over every $k, \ell \geq 0$: $\rho(k+L) \geq \ell + L$, we obtain

$$
\begin{aligned}
&\sum_{k,\ell\geq 0:\ \rho(k+L)\geq \ell+L} \iint_{\mathbf{Q}_{k\ell}} \left| \mathbf{I}^{\rho}_{\alpha\beta} a(x,y) \right|^q dxdy \\
&\leq\ \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0} \sum_{\ell\leq \rho(k+L)-L} 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} 2^{(k+L)n} 2^{(\ell+L)m} \\
&\leq\ \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0} 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} 2^{(k+L)n} 2^{\rho(k+L)m} \\
&=\ \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0} 2^{-qk} 2^{(k+L)\left[q(\alpha+\rho\beta)-(q-1)(n+\rho m)\right]} \\
&=\ \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0} 2^{-qk} \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m} = 1 - \tfrac{1}{q}\right) \\
&\leq\ \mathfrak{B}_{\alpha\,\beta\,\rho\,q}.
\end{aligned}
\tag{4. 21}
$$

Suppose $\rho(k+L) \leq \ell + L$. The last line of (4. 19) is further bounded by $\mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-k} 2^{(k+L)(\alpha-n)} 2^{(\ell+L)(\beta-m)}$. We have

$$
\begin{aligned}
&\iint_{\mathbf{Q}_{k\ell}} \left| \mathbf{I}^{\rho}_{\alpha\beta} a(x,y) \right|^q dxdy \\
&\leq\ \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}_{k\ell}} 2^{-qk} 2^{q(k+L)(\alpha-n)} 2^{q(\ell+L)(\beta-m)} dxdy \\
&\leq\ \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-qk} 2^{q(k+L)(\alpha-n)} 2^{q(\ell+L)(\beta-m)} 2^{(k+L)n} 2^{(\ell+L)m} \\
&=\ \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-qk} 2^{q(k+L)(\alpha-n)} 2^{(\ell+L)\left[q(\beta-m)+m\right]} 2^{(k+L)n}.
\end{aligned}
\tag{4. 22}
$$

Recall (4. 7). Note that $\frac{\beta}{m} < 1 - \frac{1}{q}$ implies $q(\beta - m) < -m$.

By using (4. 22) and taking the summation over every $k, \ell \geq 0$: $\rho(k+L) \leq \ell + L$, we obtain

$$\begin{aligned}
&\sum_{k,\ell\geq 0:\ \rho(k+L)\leq \ell+L} \iint_{\mathbf{Q}_{k\ell}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^q dxdy \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\geq 0} \sum_{\ell\geq \rho(k+L)-L} 2^{-qk} 2^{q(k+L)(\alpha-n)} 2^{(\ell+L)\left[q(\beta-m)+m\right]} 2^{(k+L)n} \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\geq 0} 2^{-qk} 2^{q(k+L)(\alpha-n)} 2^{\rho(k+L)\left[q(\beta-m)+m\right]} 2^{(k+L)n} \qquad (q(\beta-m) < -m) \qquad (4.\ 23) \\
&= \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\geq 0} 2^{-qk} 2^{(k+L)\left[q(\alpha+\rho\beta)-(q-1)(n+\rho m)\right]} \\
&= \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\geq 0} 2^{-qk} \leq \mathfrak{B}_{\alpha\ \beta\ \rho\ q}. \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m} = 1-\tfrac{1}{q}\right)
\end{aligned}$$

Note that $|x-u| \sim 2^{k+L}$, $|y-v| \sim 2^{\ell+L}$ for $(x,y) \in \mathbf{Q}_{k\ell}$. Consider $k+L<0$ and $\ell<k$. We find

$$|x-u|^{\rho} + |y-v| \lesssim |x-u|. \qquad (4.\ 24)$$

Recall (4. 2) and (4. 18). We have

$$\begin{aligned}
\left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right| &\leq \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left|\nabla\Omega^{\alpha\beta}_{\rho}(x-u,y-v)\right| dudv \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho} \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left[\frac{1}{|x-u|^{\rho}+|y-v|}\right]^{m-\beta} \\
&\qquad\qquad \max\left\{\frac{1}{|x-u|}, \frac{1}{|x-u|^{\rho}+|y-v|}\right\} dudv \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho} \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left[\frac{1}{|x-u|^{\rho}+|y-v|}\right]^{m-\beta} \frac{1}{|x-u|^{\rho}+|y-v|} dudv \\
&\qquad\qquad \text{by } (4.\ 24) \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho} \iint_{\frac{1}{2}\mathbf{Q}} 2^{-(n+m)L} 2^{L} 2^{(k+L)(\alpha-n)} \left[2^{\rho(k+L)} + 2^{\ell+L}\right]^{\beta-m} \left[2^{\rho(k+L)} + 2^{\ell+L}\right]^{-1} dudv \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\, 2^{L} 2^{(k+L)(\alpha-n)} \left[2^{\rho(k+L)} + 2^{\ell+L}\right]^{\beta-m} \left[2^{\rho(k+L)} + 2^{\ell+L}\right]^{-1}.
\end{aligned} \qquad (4.\ 25)$$

Suppose $\rho(k+L) \geq \ell+L$. The last line in (4. 25) is further bounded by $\mathfrak{B}_{\alpha\ \beta\ \rho} 2^{L} 2^{(k+L)(\alpha-n)} 2^{\rho(k+L)(\beta-m)} 2^{-\rho(k+L)}$. We have

$$\begin{aligned}
\iint_{\mathbf{Q}_{k\ell}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^q dxdy &\leq \mathfrak{B}_{\alpha\ \beta\ \rho} \iint_{\mathbf{Q}_{k\ell}} 2^{qL} 2^{q(k+L)(\alpha-n)} 2^{q\rho(k+L)(\beta-m)} 2^{-q\rho(k+L)} dxdy \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\, 2^{qL} 2^{q(k+L)(\alpha-n)} 2^{q\rho(k+L)(\beta-m)} 2^{-q\rho(k+L)} 2^{(k+L)n} 2^{(\ell+L)m} \qquad (4.\ 26) \\
&= \mathfrak{B}_{\alpha\ \beta\ \rho}\, 2^{q\left[L-\rho(k+L)\right]} 2^{(k+L)\left[q(\alpha+\rho\beta-n-\rho m)+n\right]} 2^{(\ell+L)m}.
\end{aligned}$$

By using (4. 26) and summing over every $k,\ell\ge 0$: $\rho(k+L)\ge \ell+L$, we obtain

$$
\begin{aligned}
&\sum_{k,\ell\ge 0:\ \rho(k+L)\ge \ell+L} \iint_{\mathbf{Q}_{k\ell}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^q dxdy \\
&\le\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\ge -(\rho-1)L/\rho}\ \sum_{\ell\le \rho(k+L)-L} 2^{q\left[L-\rho(k+L)\right]} 2^{(k+L)\left[q(\alpha+\rho\beta-n-\rho m)+n\right]} 2^{(\ell+L)m} \\
&\le\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\ge -(\rho-1)L/\rho} 2^{q\left[L-\rho(k+L)\right]} 2^{(k+L)\left[q(\alpha+\rho\beta-n-\rho m)+n\right]} 2^{\rho(k+L)m} \qquad (4.\ 27)\\
&=\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\ge -(\rho-1)L/\rho} 2^{q\left[L-\rho(k+L)\right]} 2^{(k+L)\left[q(\alpha+\rho m)-(q-1)(n+\rho m)\right]} \\
&=\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{k\ge -(\rho-1)L/\rho} 2^{q\left[L-\rho(k+L)\right]}\ \le\ \mathfrak{B}_{\alpha\ \beta\ \rho\ q}. \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m}=1-\tfrac{1}{q}\right)
\end{aligned}
$$

Suppose $\rho(k+L)<\ell+L$. The last line in (4. 25) is further bounded by $\mathfrak{B}_{\alpha\ \beta\ \rho}\ 2^L 2^{(k+L)(\alpha-n)} 2^{(\ell+L)(\beta-m)} 2^{-(\ell+L)}$.

$$
\begin{aligned}
\iint_{\mathbf{Q}_{k\ell}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^q dxdy\ &\le\ \mathfrak{B}_{\alpha\ \beta\ \rho} \iint_{\mathbf{Q}_{k\ell}} 2^{qL} 2^{q(k+L)(\alpha-n)} 2^{q(\ell+L)(\beta-m)} 2^{-q(\ell+L)} dxdy \\
&\le\ \mathfrak{B}_{\alpha\ \beta\ \rho} 2^{qL} 2^{q(k+L)(\alpha-n)} 2^{q(\ell+L)(\beta-m)} 2^{-q(\ell+L)} 2^{(k+L)n} 2^{(\ell+L)m} \qquad (4.\ 28)\\
&=\ \mathfrak{B}_{\alpha\ \beta\ \rho}\ 2^{-q\ell} 2^{(k+L)\left[q(\alpha-n)+n\right]} 2^{(\ell+L)\left[q(\beta-m)+m\right]}.
\end{aligned}
$$

Recall (4. 6). Note that $\frac{\alpha}{n}>1-\frac{1}{q}$ implies $q(\alpha-n)>-n$. By using (4. 28) and summing over every $k,\ell\ge 0$: $\rho(k+L)<\ell+L$, we obtain

$$
\begin{aligned}
&\sum_{k,\ell\ge 0:\ \rho(k+L)<\ell+L} \iint_{\mathbf{Q}_{k\ell}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^q dxdy \\
&\le\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{\ell\ge 0}\ \sum_{k<\ell/\rho+L/\rho-L} 2^{-q\ell} 2^{(k+L)\left[q(\alpha-n)+n\right]} 2^{(\ell+L)\left[q(\beta-m)+m\right]} \\
&\le\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{\ell\ge 0} 2^{-q\ell} 2^{\left[\ell/\rho+L/\rho\right]\left[q(\alpha-n)+n\right]} 2^{(\ell+L)\left[q(\beta-m)+m\right]} \qquad (q(\alpha-n)>-n) \qquad (4.\ 29)\\
&=\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{\ell\ge 0} 2^{-q\ell} 2^{(\ell+L)\left[q(\alpha+\rho\beta)-(q-1)(n+\rho m)\right]/\rho} \\
&=\ \mathfrak{B}_{\alpha\ \beta\ \rho} \sum_{\ell\ge 0} 2^{-q\ell}\ \le\ \mathfrak{B}_{\alpha\ \beta\ \rho\ q}. \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m}=1-\tfrac{1}{q}\right)
\end{aligned}
$$

4.3 Case 3: $k > 0, \ell = 0$ or $k = 0, \ell > 0$

Let $(x, y) \in \mathbf{Q}_{k0}$. We have $|x-u| \sim 2^{k+L}$. Consider $\rho(k+L) \geq L$. If $k+L \geq 0$, we find

$$|x-u|^{\rho} \gtrsim |x-u|. \tag{4. 30}$$

Recall (4. 2) and (4. 18). We have

$$\begin{aligned}
\left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right| &\leq \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left|\nabla\Omega^{\alpha\beta}_{\rho}(x-u, y-v)\right| dudv \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}} \frac{1}{\mathbf{vol}\{\frac{1}{2}\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left[\frac{1}{|x-u|^{\rho}+|y-v|}\right]^{m-\beta} \max\left\{\frac{1}{|x-u|}, \frac{1}{|x-u|^{\rho}+|y-v|}\right\} dudv \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}} \sqrt{|u|^2+|v|^2} \left(\frac{1}{|x-u|}\right)^{n-\alpha} \left[\frac{1}{|x-u|^{\rho}+|y-v|}\right]^{m-\beta} \frac{1}{|x-u|} dudv \qquad \text{by (4. 30)} \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\frac{1}{2}\mathbf{Q}} 2^{-(n+m)L} 2^{L} 2^{(k+L)(\alpha-n)} \left[2^{\rho(k+L)} + 2^{L}\right]^{\beta-m} 2^{-(k+L)} dudv \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-k} 2^{(k+L)(\alpha-n)} \left[2^{\rho(k+L)} + 2^{L}\right]^{\beta-m}.
\end{aligned} \tag{4. 31}$$

The last line of (4. 31) is further bounded by $\mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-k} 2^{(k+L)(\alpha-n)} 2^{\rho(k+L)(\beta-m)}$. We have

$$\begin{aligned}
\iint_{\mathbf{Q}_{k0}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^{q} dxdy &\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}_{k0}} 2^{-qk} 2^{q(k+L)(\alpha-n)} 2^{q\rho(k+L)(\beta-m)} dxdy \\
&= \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}_{k0}} 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} dxdy \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} 2^{(k+L)n} 2^{Lm}.
\end{aligned} \tag{4. 32}$$

By using (4. 32) and taking the summation over every $k \geq 0$: $\rho(k+L) \geq L$, we obtain

$$\begin{aligned}
&\sum_{k\geq 0:\ \rho(k+L)\geq L} \iint_{\mathbf{Q}_{k0}} \left|\mathbf{I}^{\rho}_{\alpha\beta} a(x,y)\right|^{q} dxdy \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0:\ \rho(k+L)\geq L} 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} 2^{(k+L)n} 2^{Lm} \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0} 2^{-qk} 2^{q(k+L)\left[\alpha+\rho\beta-(n+\rho m)\right]} 2^{(k+L)n} 2^{\rho(k+L)m} \\
&= \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0} 2^{-qk} 2^{(k+L)\left[q(\alpha+\rho\beta)-(q-1)(n+\rho m)\right]} \\
&= \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{k\geq 0} 2^{-qk} \leq \mathfrak{B}_{\alpha\,\beta\,\rho\,q}. \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m} = 1-\tfrac{1}{q}\right)
\end{aligned} \tag{4. 33}$$

On the other hand, if $k+L<0$, we find

$$|x-u|^{\rho}+|y-v| \lesssim 2^{\rho(k+L)}+2^{L} \lesssim 2^{k+L} \approx |x-u|. \tag{4. 34}$$

Recall (4. 2) and (4. 18). We have

$$\begin{aligned}
\left|\mathbf{I}^{\rho}_{\alpha\beta}a(x,y)\right| &\leq \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\sqrt{|u|^2+|v|^2}\left|\nabla\Omega^{\alpha\beta}_{\rho}(x-u,y-v)\right| dudv \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\sqrt{|u|^2+|v|^2}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^{\rho}+|y-v|}\right]^{m-\beta} \\
&\qquad\qquad \max\left\{\frac{1}{|x-u|},\frac{1}{|x-u|^{\rho}+|y-v|}\right\}dudv \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\sqrt{|u|^2+|v|^2}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^{\rho}+|y-v|}\right]^{m-\beta}\frac{1}{|x-u|^{\rho}+|y-v|}dudv \\
&\qquad\qquad \text{by (4. 34)} \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\iint_{\frac{1}{2}\mathbf{Q}} 2^{-(n+m)L}2^{L}2^{(k+L)(\alpha-n)}\left[2^{\rho(k+L)}+2^{L}\right]^{\beta-m}\left[2^{\rho(k+L)}+2^{L}\right]^{-1}dudv \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\, 2^{L}2^{(k+L)(\alpha-n)}\left[2^{\rho(k+L)}+2^{L}\right]^{\beta-m}\left[2^{\rho(k+L)}+2^{L}\right]^{-1}.
\end{aligned} \tag{4. 35}$$

Because $\rho(k+L)\geq L$, the last line in (4. 35) can be further bounded by $\mathfrak{B}_{\alpha\ \beta\ \rho}2^{L}2^{(k+L)(\alpha-n)}2^{\rho(k+L)(\beta-m)}2^{-\rho(k+L)}$. We have

$$\begin{aligned}
\iint_{\mathbf{Q}_{k\ell}}\left|\mathbf{I}^{\rho}_{\alpha\beta}a(x,y)\right|^{q}dxdy &\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\iint_{\mathbf{Q}_{k\ell}} 2^{qL}2^{q(k+L)(\alpha-n)}2^{q\rho(k+L)(\beta-m)}2^{-q\rho(k+L)}dxdy \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}2^{qL}2^{q(k+L)(\alpha-n)}2^{q\rho(k+L)(\beta-m)}2^{-q\rho(k+L)}2^{(k+L)n}2^{Lm} \\
&= \mathfrak{B}_{\alpha\ \beta\ \rho}\, 2^{q\left[L-\rho(k+L)\right]}2^{(k+L)\left[q(\alpha+\rho\beta-n-\rho m)+n\right]}2^{Lm}.
\end{aligned} \tag{4. 36}$$

By using (4. 36) and summing over every $k\geq 0\colon \rho(k+L)\geq L$, we obtain

$$\begin{aligned}
&\sum_{k\geq 0\colon\ \rho(k+L)\geq L}\iint_{\mathbf{Q}_{k0}}\left|\mathbf{I}^{\rho}_{\alpha\beta}a(x,y)\right|^{q}dxdy \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\sum_{k\geq 0\colon\ \rho(k+L)\geq L} 2^{q\left[L-\rho(k+L)\right]}2^{(k+L)\left[q(\alpha+\rho\beta-n-\rho m)+n\right]}2^{Lm} \\
&\leq \mathfrak{B}_{\alpha\ \beta\ \rho}\sum_{k\geq -(\rho-1)L/\rho} 2^{q\left[L-\rho(k+L)\right]}2^{(k+L)\left[q(\alpha+\rho\beta-n-\rho m)+n\right]}2^{\rho(k+L)m} \\
&= \mathfrak{B}_{\alpha\ \beta\ \rho}\sum_{k\geq -(\rho-1)L/\rho} 2^{q\left[L-\rho(k+L)\right]}2^{(k+L)\left[q(\alpha+\rho m)-(q-1)(n+\rho m)\right]} \\
&= \mathfrak{B}_{\alpha\ \beta\ \rho}\sum_{k\geq -(\rho-1)L/\rho} 2^{q\left[L-\rho(k+L)\right]} \leq \mathfrak{B}_{\alpha\ \beta\ \rho\ q}. \qquad \left(\tfrac{\alpha+\rho\beta}{n+\rho m}=1-\tfrac{1}{q}\right)
\end{aligned} \tag{4. 37}$$

Consider $\rho(k+L)<L$. Recall $\mathbf{I}^{\rho}_{\alpha\beta}f=f*\Omega^{\alpha\beta}_{\rho}$ defined in (1. 6)-(1. 7). We have

$$
\begin{aligned}
\left|\mathbf{I}^{\rho}_{\alpha\beta}a(x,y)\right| &\leq \iint_{\frac{1}{2}\mathbf{Q}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\Omega^{\alpha\beta}_{\rho}(x-u,y-v)dudv\\
&\leq \sum_{s\leq L}\iint_{\frac{1}{2}\mathbf{Q}\cap\{\sqrt{m}2^{s-1}\leq|y-v|<\sqrt{m}2^{s}\}} \frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\Omega^{\alpha\beta}_{\rho}(x-u,y-v)dudv\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{s\leq L}2^{-(n+m)L}2^{(k+L)(\alpha-n)}\Big[2^{\rho(k+L)}+2^{s}\Big]^{\beta-m}2^{Ln}2^{sm}\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{s\leq L}2^{-Lm}2^{(k+L)(\alpha-n)}\Big[2^{\rho(k+L)}+2^{s}\Big]^{\beta-m}2^{sm}\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{s\leq L}2^{-Lm}2^{(k+L)(\alpha-n)}2^{s\beta}\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\,2^{(k+L)(\alpha-n)}2^{L(\beta-m)}.
\end{aligned}
\tag{4. 38}
$$

By using (4. 38), we find

$$
\begin{aligned}
\iint_{\mathbf{Q}_{k0}}\left|\mathbf{I}^{\rho}_{\alpha\beta}a(x,y)\right|^{q}dxdy &\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\iint_{\mathbf{Q}_{k0}}2^{q(k+L)(\alpha-n)}2^{qL(\beta-m)}dxdy\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\,2^{q(k+L)(\alpha-n)}2^{qL(\beta-m)}2^{(k+L)n}2^{Lm}\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho}\,2^{(k+L)\big[q(\alpha-n)+n\big]}2^{L\big[q(\beta-m)+m\big]}.
\end{aligned}
\tag{4. 39}
$$

Recall (4. 6). Note that $\frac{\alpha}{n}>1-\frac{1}{q}$ implies $q(\alpha-n)>-n$. By using (4. 39) and summation over $k\geq 0\colon\ \rho(k+L)<L$, we obtain

$$
\begin{aligned}
\sum_{k\geq 0\colon\ \rho(k+L)<L}\iint_{\mathbf{Q}_{k0}}\left|\mathbf{I}^{\rho}_{\alpha\beta}a(x,y)\right|^{q}dxdy &\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{k\geq 0\colon\ \rho(k+L)<L}2^{(k+L)\big[q(\alpha-n)+n\big]}2^{L\big[q(\beta-m)+m\big]}\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{0\leq k<L/\rho-L}2^{(k+L)\big[q(\alpha-n)+n\big]}2^{L\big[q(\beta-m)+m\big]}\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho\,q}\,2^{L\big[q(\alpha-n)+n\big]/\rho}2^{L\big[q(\beta-m)+m\big]} \qquad (q(\alpha-n)+n>0)\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho\,q}\,2^{L\big[q(\alpha+\rho\beta)-(q-1)(n+\rho m)\big]/\rho}\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho\,q}. \qquad \left(\frac{\alpha+\rho\beta}{n+\rho m}=1-\frac{1}{q}\right)
\end{aligned}
\tag{4. 40}
$$

4.4 Case 4: $k=0$ and $\ell>0$

Let $(x,y)\in\mathbf{Q}_{0\ell}$. We have $|y-v|\sim 2^{\ell+L}$. Recall $\Omega_\rho^{\alpha\beta}(x,y)$ defined in (1. 7). Moreover, $\mathbf{supp}a\subset\frac{1}{2}\mathbf{Q}$ and $|a(x,y)|\leq 2^{n+m}\mathbf{vol}\{\mathbf{Q}\}^{-1}$. We have

$$\begin{aligned}
\left|\mathbf{I}_{\alpha\beta}^{\rho}a(x,y)\right| &\leq \iint_{\frac{1}{2}\mathbf{Q}}|a(u,v)|\Omega_\rho^{\alpha\beta}(x-u,y-v)dudv\\
&\lesssim \iint_{\frac{1}{2}\mathbf{Q}}\frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho+|y-v|}\right]^{m-\beta}dudv\\
&\leq \sum_{s\leq L}\iint_{\frac{1}{2}\mathbf{Q}\cap\{\sqrt{n}2^{s-1}\leq|x-u|<\sqrt{n}2^s\}}\frac{1}{\mathbf{vol}\{\mathbf{Q}\}}\left(\frac{1}{|x-u|}\right)^{n-\alpha}\left[\frac{1}{|x-u|^\rho+|y-v|}\right]^{m-\beta}dudv \qquad (4.\ 41)\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{s\leq L}2^{-(n+m)L}2^{s(\alpha-n)}\left[2^{\rho s}+2^{\ell+L}\right]^{\beta-m}2^{sn}2^{Lm}\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{s\leq L}2^{-Ln}2^{s\alpha}\left[2^{\rho s}+2^{\ell+L}\right]^{\beta-m}.
\end{aligned}$$

Suppose $\ell\geq(\rho-1)L$. We find $\rho s\leq\rho L\leq\ell+L$. The last line of (4. 41) can be further bounded by

$$\mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{s\leq L}2^{-Ln}2^{s\alpha}2^{(\ell+L)(\beta-m)} \leq \mathfrak{B}_{\alpha\,\beta\,\rho}\,2^{L(\alpha-n)}2^{(\ell+L)(\beta-m)}. \qquad (4.\ 42)$$

From (4. 41)-(4. 42), we have

$$\begin{aligned}
\iint_{\mathbf{Q}_{0\ell}}\left|\mathbf{I}_{\alpha\beta}^{\rho}a(x,y)\right|^q dxdy &\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\iint_{\mathbf{Q}_{0\ell}}2^{qL(\alpha-n)}2^{q(\ell+L)(\beta-m)}dxdy\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\,2^{qL(\alpha-n)}2^{q(\ell+L)(\beta-m)}2^{Ln}2^{(\ell+L)m} \qquad (4.\ 43)\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho}\,2^{L\left[q(\alpha-n)+n\right]}2^{(\ell+L)\left[q(\beta-m)+m\right]}.
\end{aligned}$$

Recall (4. 7). We have $\frac{\beta}{m}<1-\frac{1}{q}\Longrightarrow q(\beta-m)+m<0$. By using (4. 43) and summing over $\ell\geq0\colon\ \ell\geq(\rho-1)L$, we obtain

$$\begin{aligned}
\sum_{\ell\geq0:\ \ell\geq(\rho-1)L}\iint_{\mathbf{Q}_{0\ell}}\left|\mathbf{I}_{\alpha\beta}^{\rho}a(x,y)\right|^q dxdy &\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\sum_{\ell\geq0:\ \ell\geq(\rho-1)L}2^{L\left[q(\alpha-n)+n\right]}2^{(\ell+L)\left[q(\beta-m)+m\right]}\\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho\,q}\,2^{L\left[q(\alpha-n)+n\right]}2^{\rho L\left[q(\beta-m)+m\right]} \qquad (4.\ 44)\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho\,q}\,2^{L\left[q(\alpha+\rho\beta)-(q-1)(n+\rho m)\right]}\\
&= \mathfrak{B}_{\alpha\,\beta\,\rho\,q}. \qquad \left(\frac{\alpha+\rho\beta}{n+\rho m}=1-\frac{1}{q}\right)
\end{aligned}$$

Suppose $\ell < (\rho-1)L$. The last line of (4. 41) can be further bounded by

$$
\begin{aligned}
&\mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{s\leq L} 2^{-Ln} 2^{s\alpha} \Big[2^{\rho s} + 2^{\ell+L} \Big]^{\beta-m} \leq \\
&\mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{s\leq(\ell+L)/\rho} 2^{-Ln} 2^{s\alpha} 2^{(\ell+L)(\beta-m)} + \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{(\ell+L)/\rho<s\leq L} 2^{-Ln} 2^{s\alpha} 2^{\rho s(\beta-m)} \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-Ln} 2^{(\ell+L)\big[\alpha+\rho\beta-\rho m\big]/\rho} + \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{L\big[\alpha+\rho\beta-(n+\rho m)\big]}.
\end{aligned}
\tag{4. 45}
$$

From (4. 41) and (4. 45), we have

$$
\begin{aligned}
&\iint_{\mathbf{Q}_{0\ell}} \Big| \mathbf{I}^{\rho}_{\alpha\beta} a(x,y) \Big|^q dxdy \leq \\
&\mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}_{0\ell}} 2^{-qLn} 2^{q(\ell+L)\big[\alpha+\rho\beta-\rho m\big]/\rho} dxdy + \mathfrak{B}_{\alpha\,\beta\,\rho} \iint_{\mathbf{Q}_{0\ell}} 2^{qL\big[\alpha+\rho\beta-(n+\rho m)\big]} dxdy \\
&\leq \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-qLn} 2^{q(\ell+L)\big[\alpha+\rho\beta-\rho m\big]/\rho} 2^{Ln} 2^{(\ell+L)m} + \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{qL\big[\alpha+\rho\beta-(n+\rho m)\big]} 2^{Ln} 2^{(\ell+L)m} \\
&= \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{-L(q-1)n} 2^{(\ell+L)\big[q(\alpha+\rho\beta)-(q-1)\rho m\big]/\rho} + \mathfrak{B}_{\alpha\,\beta\,\rho}\, 2^{L\big[q(\alpha+\rho\beta)-(q-1)n-q\rho m\big]} 2^{(\ell+L)m}.
\end{aligned}
\tag{4. 46}
$$

Note that $\frac{\alpha+\rho\beta}{n+\rho m} = 1-\frac{1}{q} \Longrightarrow q(\alpha+\rho\beta) = (q-1)(n+\rho m)$ which further shows $q(\alpha+\rho\beta) > (q-1)\rho m$. By using (4. 46) and summing over $\ell \geq 0\colon \ell < (\rho-1)L$, we obtain

$$
\begin{aligned}
&\sum_{\ell\geq 0\colon\, \ell<(\rho-1)L} \iint_{\mathbf{Q}_{0\ell}} \Big| \mathbf{I}^{\rho}_{\alpha\beta} a(x,y) \Big|^q dxdy \leq \\
&\mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{\ell<(\rho-1)L} 2^{-L(q-1)n} 2^{(\ell+L)\big[q(\alpha+\rho\beta)-(q-1)\rho m\big]/\rho} + \mathfrak{B}_{\alpha\,\beta\,\rho} \sum_{\ell<(\rho-1)L} 2^{L\big[q(\alpha+\rho\beta)-(q-1)n-q\rho m\big]} 2^{(\ell+L)m} \\
&= \mathfrak{B}_{\alpha\,\beta\,\rho\,q}\, 2^{L\big[q(\alpha+\rho\beta)-(q-1)(n+\rho m)\big]} = \mathfrak{B}_{\alpha\,\beta\,\rho\,q}.
\end{aligned}
\tag{4. 47}
$$

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School of Mathematical Sciences, Fudan University, Shanghai, 200433, China
email: 20114010001@fudan.edu.cn

Westlake University, Hangzhou, 310010, China
email: wangzipeng@westlake.edu.cn