Stein-Weiss inequality on product spaces

Zipeng Wang

In communication with Charles Fefferman

Abstract

We give the characterization between weighted norm inequalities of strong fractional integral operators and their associated multi parameter Muckenhoupt characteristics, by considering the weights to be power functions. As a result, we extend the classical Stein-Weiss theorem to product spaces.

1 Introduction

Let $0 < \alpha < \mathbf{N}$. A fractional integral operator I_α is defined by

$$I_\alpha f(x) \;=\; \int_{\mathbb{R}^\mathbf{N}} f(y)\left[\frac{1}{|x-y|}\right]^{\mathbf{N}-\alpha} dy. \tag{1. 1}$$

In 1928, Hardy and Littlewood [1] first established a two-weight $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality for I_α in the one dimensional space, by considering the *weights* as suitable powers. This result has been extended to higher dimensions by Stein and Weiss [3]. Today, it bears the name of Stein-Weiss inequality.

$\diamond$ Throughout, $\mathfrak{B} > 0$ is regarded as a generic constant depending on its subindices.

Theorem A: Stein and Weiss (1958) *Let I_α defined in (1. 1) for $0 < \alpha < \mathbf{N}$. Suppose that $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}$ for $x \neq 0$ and $\gamma, \delta \in \mathbb{R}$. We have*

$$\left\|\omega I_\alpha f\right\|_{\mathbf{L}^q(\mathbb{R}^\mathbf{N})} \;\leq\; \mathfrak{B}_{p\ q\ \alpha\ \gamma\ \delta}\ \left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})} \tag{1. 2}$$

for $1 < p \leq q < \infty$ if

$$\gamma \;<\; \frac{\mathbf{N}}{q}, \qquad \delta \;<\; \mathbf{N}\left(\frac{p-1}{p}\right), \qquad \gamma+\delta \;\geq\; 0 \tag{1. 3}$$

and

$$\frac{\alpha}{\mathbf{N}} \;=\; \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}}. \tag{1. 4}$$

In the case of $\gamma = \delta = 0$, **Theorem A** was proved in $\mathbb{R}^\mathbf{N}$ by Sobolev [2]. This is also known as Hardy-Littlewood-Sobolev inequality.

Fractional integration in weighted norms have been systematically studied over the several past decades. For instance, see Muckenhoupt and Wheeden [5], Coifman and Fefferman [9], Fefferman and Muckenhoupt [8], Pérez [10] and Sawyer and Wheeden [6].

Denote Q to be a cube in $\mathbb{R}^{\mathbf{N}}$ parallel to coordinates. As a well known estimate, (1. 2) implies

$$\sup_{Q\subset\mathbb{R}^{\mathbf{N}}} |Q|^{\frac{\alpha}{\mathbf{N}}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|Q|}\int_Q \omega^q(x)dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|Q|}\int_Q \left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)dx\right\}^{\frac{p-1}{p}} < \infty. \tag{1. 5}$$

The supremum in (1. 5) is called Muckenhoupt characteristic, as was first introduced by Muckenhoupt for which ω^q and $\sigma^{-\frac{p}{p-1}}$ are nonnegative and locally integrable.

Take into account $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}$ for $x \neq 0$ and $\gamma, \delta \in \mathbb{R}$. We find the two-weight $\mathbf{L}^p \longrightarrow \mathbf{L}^q$-norm inequality in (1. 5) implying the constraints in (1. 3)-(1. 4). Hence that (1. 2), (1. 3)-(1. 4) and (1. 5) are in fact equivalent conditions.

Now, consider $\mathbb{R}^{\mathbf{N}} = \mathbb{R}^{\mathbf{N}_1} \times \mathbb{R}^{\mathbf{N}_2} \times \cdots \times \mathbb{R}^{\mathbf{N}_n}$ as a product space. Let

$$0 < \alpha_i < \mathbf{N}_i, \qquad i = 1, 2, \dots, n \qquad \text{and} \qquad \alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n. \tag{1. 6}$$

In this paper, we give an extension of **Theorem A** to multi-parameter settings by asserting the *strong* fractional integral operator $\mathbf{I}_\alpha$ defined by

$$\mathbf{I}_\alpha f(x) = \int_{\mathbb{R}^{\mathbf{N}}} f(y) \prod_{i=1}^{n} \left[\frac{1}{|x_i - y_i|}\right]^{\mathbf{N}_i - \alpha_i} dy. \tag{1. 7}$$

Observe that the kernel of $\mathbf{I}_\alpha$ has singularity appeared on every coordinate subspace.

The study of certain operators that commute with a multi-parameter family of dilations, dates back to the time of Jessen, Marcinkiewicz and Zygmund. A number of pioneering results have been accomplished, for example by Cordoba and Fefferman [11], Fefferman [12], Fefferman and Stein [13], Müller, Ricci and Stein [14], Journé [15] and Pipher [16]. This area remains largely open for fractional integration.

2 Statement of the main result

Denote $\mathbf{Q} = \mathbf{Q}_1 \times \mathbf{Q}_2 \times \cdots \times \mathbf{Q}_n \subset \mathbb{R}^{\mathbf{N}_1} \times \mathbb{R}^{\mathbf{N}_2} \times \cdots \times \mathbb{R}^{\mathbf{N}_n} = \mathbb{R}^{\mathbf{N}}$ where $\mathbf{Q}_i$ is a cube in $\mathbb{R}^{\mathbf{N}_i}$ parallel to coordinates for every $i = 1, 2, \dots, n$.

Theorem A*: *Let $\mathbf{I}_\alpha$ defined in (1. 6)-(1. 7). Suppose $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}$, $\gamma, \delta \in \mathbb{R}$ at $x \neq 0$. For $1 < p \leq q < \infty$, the following conditions are equivalent:*

1.

$$\sup_{\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}} \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^q(x)\, dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)\, dx\right\}^{\frac{p-1}{p}} < \infty. \tag{2. 1}$$

2.

$$\gamma < \frac{\mathbf{N}}{q}, \qquad \delta < \mathbf{N}\left(\frac{p-1}{p}\right), \qquad \gamma + \delta \geq 0 \tag{2. 2}$$

and

$$\frac{\alpha}{\mathbf{N}} = \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}}. \tag{2. 3}$$

For $\gamma \geq 0, \delta \leq 0$,

$$\alpha_i - \frac{\mathbf{N}_i}{p} \ < \ \delta, \qquad i = 1, 2, \ldots, n. \tag{2. 4}$$

For $\gamma \leq 0, \delta \geq 0$,

$$\alpha_i - \mathbf{N}_i \left(\frac{q-1}{q}\right) \ < \ \gamma, \qquad i = 1, 2, \ldots, n. \tag{2. 5}$$

For $\gamma > 0, \delta > 0$,

$$\begin{aligned} \sum_{i \in \mathcal{U}} \alpha_i - \frac{\mathbf{N}_i}{p} \ < \ \delta, \qquad \mathcal{U} \ = \ \left\{ i \in \{1, 2, \ldots, n\} : \alpha_i - \frac{\mathbf{N}_i}{p} \ \geq \ 0 \right\}, \\ \sum_{i \in \mathcal{V}} \alpha_i - \left(\frac{q-1}{q}\right) \mathbf{N}_i \ < \ \gamma, \qquad \mathcal{V} \ = \ \left\{ i \in \{1, 2, \ldots, n\} : \alpha_i - \mathbf{N}_i \left(\frac{q-1}{q}\right) \ \geq \ 0 \right\}. \end{aligned} \tag{2. 6}$$

3.

$$\left\| \omega \mathbf{I}_\alpha f \right\|_{\mathbf{L}^q(\mathbb{R}^\mathbf{N})} \ \leq \ \mathfrak{B}_{p\, q\, \alpha\, \gamma\, \delta} \ \left\| f \sigma \right\|_{\mathbf{L}^p(\mathbb{R}^\mathbf{N})}, \qquad 1 < p \leq q < \infty. \tag{2. 7}$$

Remark 2.1 *In the 2-parameter setting ($n = 2$),* **Theorem A*** *is first proved in the joint work of Sawyer and Wang [7]. For $\gamma \geq 0, \delta \leq 0$ or $\gamma \leq 0, \delta \geq 0$, the "sandwiching" idea introduced there applies to the general multi-parameter situation. However, the difficult case occurs at $\gamma > 0, \delta > 0$. The method used in [7] relies on solving a system of algebraic equations, which is no longer solvable for $n > 2$.*

The remaining paper is organized as follows. In section 3, we introduce a new framework, where the product space is decomposed into an infinitely many of dyadic cones. Every partial sum operator defined on a dyadic cone is essentially an one-parameter fractional integral operator, satisfying the desired regularity.

In section 4, we prove that the multi-parameter Muckenhoupt characteristic in (2. 1) implies the constraints in (2. 2)-(2. 6).

In section 5, by using (2. 2)-(2. 6), we show that

$$\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \frac{1}{p} + \frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_\mathbf{Q} \omega^{qr}(x)\, dx \right\}^{\frac{1}{qr}} \left\{ \frac{1}{|\mathbf{Q}|} \int_\mathbf{Q} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}} (x)\, dx \right\}^{\frac{p-1}{pr}}, \qquad r > 1 \tag{2. 8}$$

decays exponentially as the eccentricity of $\mathbf{Q}$ getting large, whenever

$$\alpha_i \ > \ \mathbf{N}_i \left(\tfrac{1}{p} - \tfrac{1}{q}\right), \qquad i = 1, 2, \ldots, n.$$

On the other hand, we handle the case $\alpha_i = \mathbf{N}_i \left(\frac{1}{p} - \frac{1}{q}\right),\ i = 1, 2, \ldots, n$ in section 6.

We prove **Theorem A*** in the last section, by decomposing $\mathbf{I}_\alpha$ so that the resulting estimates can be reduced to either of the above two cases.

Because $\mathbf{I}_\alpha$ is positive definite, we assume $f \geq 0$ from now on.

3 Cone decomposition on product spaces

Denote $\mathbf{t}$ as an n-tuple $(2^{-t_1}, 2^{-t_2}, \ldots, 2^{-t_n})$ where $t_i,\ i = 1, 2, \ldots, n$ are non-negative integers. We require $t_\nu \doteq \min\{t_i\colon\ i = 1, 2, \ldots, n\} = 0$. Define

$$\Delta_{\mathbf{t}} \mathbf{I}_\alpha f(x) \;=\; \int_{\Gamma_{\mathbf{t}}(x)} f(y) \prod_{i=1}^n \left[\frac{1}{|x_i - y_i|}\right]^{\mathbf{N}_i - \alpha_i} dy \tag{3. 1}$$

where

$$\Gamma_{\mathbf{t}}(x) \;\doteq\; \bigotimes_{i=1}^n \left\{ y_i \in \mathbb{R}^{\mathbf{N}_i}\colon\ 2^{-t_i} \leq \frac{|x_i - y_i|}{|x_\nu - y_\nu|} < 2^{-t_i+1} \right\}. \tag{3. 2}$$

Observe that $\Gamma_{\mathbf{t}}(x)$ in (3. 2) is a dyadic cone with vertex on x whose eccentricity depends on $\mathbf{t}$. In particular, we write

$$\Gamma_o(x) \;\doteq\; \Gamma_{\mathbf{t}}(x), \qquad t_1 = t_2 = \cdots = t_n = 0. \tag{3. 3}$$

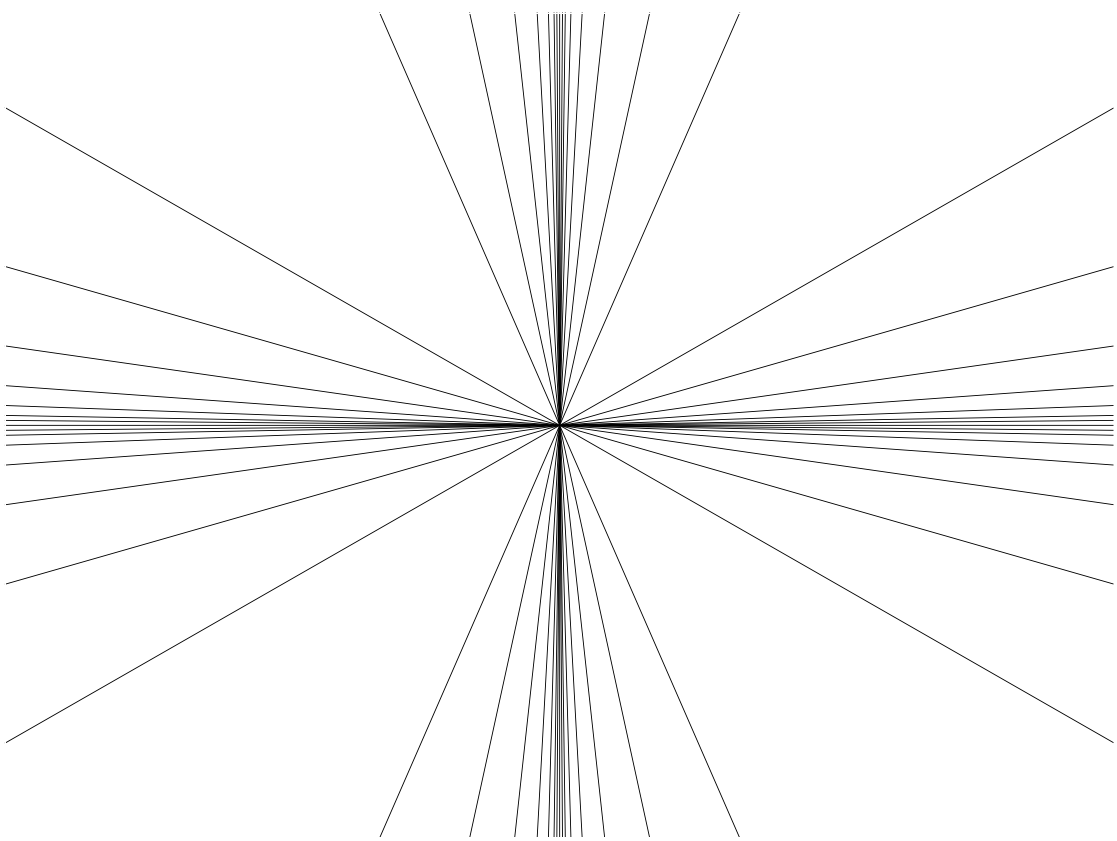

Figure 1: dyadic cones in a 2-parameter setting.

Let $\mathbf{Q}^{\mathbf{t}}$ be a dilated of $\mathbf{Q}$ such that $|\mathbf{Q}_i^{\mathbf{t}}|^{\frac{1}{\mathbf{N}_i}} = 2^{-t_i} |\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}, i = 1, 2, \ldots, n$.

Write $\mathbf{t}x = \left(2^{-t_1}x_1, 2^{-t_2}x_2, \ldots, 2^{-t_n}x_n\right)$. We have

$$\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(\mathbf{t}x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(\mathbf{t}x)\,dx\right\}^{\frac{p-1}{pr}}$$

$$= \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \omega^{qr}(x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)\,dx\right\}^{\frac{p-1}{pr}} \tag{3. 4}$$

$$= \prod_{i=1}^{n} 2^{t_i\left[\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right]} \prod_{i=1}^{n} |\mathbf{Q}_i^{\mathbf{t}}|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \omega^{qr}(x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)\,dx\right\}^{\frac{p-1}{pr}}$$

for every $\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}}$.

Given $\mathbf{t}$, consider

$$\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}} : |\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}/|\mathbf{Q}_\nu|^{\frac{1}{\mathbf{N}_\nu}} = 2^{-t_i}, \quad i = 1, 2, \ldots, n. \tag{3. 5}$$

For $r > 1$, we define

$$\mathbf{A}^{\alpha}_{pqr}(\mathbf{t} : \omega, \sigma) = \sup_{\mathbf{Q} \text{ in } (3.\,5)} \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)\,dx\right\}^{\frac{p-1}{pr}}. \tag{3. 6}$$

Suppose that $\mathbf{Q}$ is a cube satisfying

$$|\mathbf{Q}_1|^{\frac{1}{\mathbf{N}_1}} = |\mathbf{Q}_2|^{\frac{1}{\mathbf{N}_2}} = \cdots = |\mathbf{Q}_n|^{\frac{1}{\mathbf{N}_n}}. \tag{3. 7}$$

We find

$$|\mathbf{Q}|^{\frac{\alpha}{\mathbf{N}}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(\mathbf{t}x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(\mathbf{t}x)\,dx\right\}^{\frac{p-1}{pr}}$$

$$= \prod_{i=1}^{n} 2^{t_i\left[\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right]} \prod_{i=1}^{n} |\mathbf{Q}_i^{\mathbf{t}}|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \omega^{qr}(x)\,dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}^{\mathbf{t}}|}\int_{\mathbf{Q}^{\mathbf{t}}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)\,dx\right\}^{\frac{p-1}{pr}} \quad \text{by (3. 4)}$$

$$\le \prod_{i=1}^{n} 2^{t_i\left[\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right]} \mathbf{A}^{\alpha}_{pqr}(\mathbf{t} : \omega, \sigma) \qquad \text{by (3. 5)-(3. 6).} \tag{3. 8}$$

Now, we recall Sawyer-Wheeden theorem for one-parameter fractional integral operators, stated as Theorem 1 in [6]:

$$\left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)\left[\frac{1}{|x-y|}\right]^{\mathbf{N}-\alpha} dy\right\}^{q} \omega^{q}(x)dx\right\}^{\frac{1}{q}} \le \mathfrak{B}_{p\,q\,r\,\alpha}\, A^{\alpha}_{pqr}(\omega, \sigma) \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^{p}(x)dx\right\}^{\frac{1}{p}} \tag{3. 9}$$

for $1 < p \le q < \infty$, if

$$A^{\alpha}_{pqr}(\omega,\sigma) \ = \sup_{\mathbf{Q}\,:\,|\mathbf{Q}_1|^{\frac{1}{\mathbf{N}_1}} = \cdots = |\mathbf{Q}_n|^{\frac{1}{\mathbf{N}_n}}} |\mathbf{Q}|^{\frac{\alpha}{\mathbf{N}}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \omega^{qr}(x)dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{\sigma}\right)^{\frac{pr}{p-1}}(x)dx\right\}^{\frac{p-1}{pr}} < \infty \qquad (3.\ 10)$$

for some $r > 1$.

Remark 3.1 *The constant* $\mathfrak{B}_{p\ q\ r\ \alpha}\, A^{\alpha}_{pqr}(\omega,\sigma)$ *in (3. 9) is not written explicitly in the statement of Theorem 1 by Sawyer and Wheeden [6]. But it can be computed directly by carrying out the proof given in section 2 of [6].*

By applying (3. 9)-(3. 10) and using the estimate in (3. 8), we have

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f(\mathbf{t}y)\left[\frac{1}{|x-y|}\right]^{\mathbf{N}-\alpha} dy\right\}^{q} \omega^{q}(\mathbf{t}x)dx\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{B}_{p\ q\ r\ \alpha} \prod_{i=1}^{n} 2^{t_i\left[\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right]} \mathbf{A}^{\alpha}_{pqr}(\mathbf{t}:\omega,\sigma) \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^{p}(\mathbf{t}x)dx\right\}^{\frac{1}{p}} .
\end{aligned} \qquad (3.\ 11)$$

Recall from (3. 1)-(3. 2). By changing dilations $x \longrightarrow \mathbf{t}x$, $y \longrightarrow \mathbf{t}y$, we have

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(\Delta_{\mathbf{t}}\mathbf{I}_{\alpha} f\right)^{q}(x)\omega^{q}(x)dx\right\}^{\frac{1}{q}} \\
&= \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\Gamma_{\mathbf{t}}(x)} f(y)\prod_{i=1}^{n}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i} dy\right\}^{q} \omega^{q}(x)dx\right\}^{\frac{1}{q}} \\
&= \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\Gamma_{o}(x)} f\left(\mathbf{t}y\right)\left\{\prod_{i=1}^{n} 2^{-t_i\mathbf{N}_i}\left[\frac{1}{2^{-t_i}|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}\right\}dy\right\}^{q} \omega^{q}\left(\mathbf{t}x\right)\prod_{i=1}^{n}2^{-t_i\mathbf{N}_i}dx\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{B}_{\alpha} \prod_{i=1}^{n} 2^{-t_i\left(\alpha_i+\frac{\mathbf{N}_i}{q}\right)} \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left\{\int_{\mathbb{R}^{\mathbf{N}}} f\left(\mathbf{t}y\right)\left[\frac{1}{|x-y|}\right]^{\mathbf{N}-\alpha} dy\right\}^{q} \omega^{q}\left(\mathbf{t}x\right)dx\right\}^{\frac{1}{q}} \\
&\leq \mathfrak{B}_{p\ q\ r\ \alpha} \prod_{i=1}^{n} 2^{-t_i\left(\alpha_i+\frac{\mathbf{N}_i}{q}\right)} 2^{t_i\left[\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right]} \mathbf{A}^{\alpha}_{pqr}\left(\mathbf{t}:\omega,\sigma\right) \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^{p}\left(\mathbf{t}x\right)dx\right\}^{\frac{1}{p}} \qquad \text{by (3. 11)} \\
&= \mathfrak{B}_{p\ q\ r\ \alpha}\, \mathbf{A}^{\alpha}_{pqr}\left(\mathbf{t}:\omega,\sigma\right) \prod_{i=1}^{n} 2^{-t_i\left(\alpha_i+\frac{\mathbf{N}_i}{q}\right)} 2^{t_i\left[\alpha_i-\frac{\mathbf{N}_i}{p}+\frac{\mathbf{N}_i}{q}\right]} \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^{p}(x)\prod_{i=1}^{n}2^{t_i\mathbf{N}_i}dx\right\}^{\frac{1}{p}} \\
&= \mathfrak{B}_{p\ q\ r\ \alpha}\, \mathbf{A}^{\alpha}_{pqr}\left(\mathbf{t}:\omega,\sigma\right) \left\{\int_{\mathbb{R}^{\mathbf{N}}} \left(f\sigma\right)^{p}(x)\,dx\right\}^{\frac{1}{p}} .
\end{aligned} \qquad (3.\ 12)$$

Observe that $\Delta_{\mathbf{t}}\mathbf{I}_\alpha$ is essentially an one-parameter fractional integral operator, satisfying

$$\left\|\left(\Delta_{\mathbf{t}}\mathbf{I}_\alpha f\right)\omega\right\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \ \leq\ \mathfrak{B}_{p\ q\ r\ \alpha}\ \mathbf{A}^\alpha_{pqr}\left(\mathbf{t}:\omega,\sigma\right)\left\|f\sigma\right\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})}, \qquad 1<p\leq q<\infty. \tag{3. 13}$$

By applying Minkowski inequality, we obtain (2. 7) provided that

$$\sum_{\mathbf{t}}\mathbf{A}^\alpha_{pqr}\left(\mathbf{t}:\omega,\sigma\right)\ <\ \infty. \tag{3. 14}$$

4 Necessary constraints

First, as investigated by Muckenhoupt, we choose $f=\sigma^{-\frac{p}{p-1}}\chi_{\mathbf{Q}}$ in (2. 7). This two-weight $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality implies

$$\mathbf{A}^\alpha_{pq}(\omega,\sigma)\ =\ \sup_{\mathbf{Q}\subset\mathbb{R}^{\mathbf{N}}}\prod_{i=1}^n|\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\omega^q(x)dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{\sigma}\right)^{\frac{p}{p-1}}(x)dx\right\}^{\frac{p-1}{p}}\ <\ \infty. \tag{4. 1}$$

Let $\omega(x)=|x|^{-\gamma},\sigma(x)=|x|^{\delta}$ for $x\neq0$ and $\gamma,\delta\in\mathbb{R}$. We aim to show (4. 1) implying (2. 2)-(2. 6).

For the remaining section, assume $\mathbf{Q}$ centered on the origin of $\mathbb{R}^{\mathbf{N}}$. Denote $\mathbf{Q}^\lambda$ to be a dilated variant of $\mathbf{Q}$ for $\lambda>0$ such that $\mathbf{Q}^\lambda=\mathbf{Q}^\lambda_1\times\mathbf{Q}^\lambda_2\times\cdots\times\mathbf{Q}^\lambda_n$ and $|\mathbf{Q}^\lambda_i|^{\frac{1}{\mathbf{N}_i}}=\lambda|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}},\ i=1,2,\ldots,n$. From (4. 1) we find

$$\begin{aligned}
&\prod_{i=1}^n|\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\\
&=\ \lambda^{\gamma+\delta-\alpha+\mathbf{N}\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^n|\mathbf{Q}^\lambda_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}^\lambda|}\int_{\mathbf{Q}^\lambda}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}^\lambda|}\int_{\mathbf{Q}^\lambda}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\\
&\leq\ \lambda^{\gamma+\delta-\alpha+\mathbf{N}\left(\frac{1}{p}-\frac{1}{q}\right)}\ \mathbf{A}^\alpha_{pq}\left(|x|^{-\gamma},|x|^{\delta}\right).
\end{aligned} \tag{4. 2}$$

Consider $|\mathbf{Q}_1|^{\frac{1}{\mathbf{N}_1}}=|\mathbf{Q}_2|^{\frac{1}{\mathbf{N}_2}}=\cdots=|\mathbf{Q}_n|^{\frac{1}{\mathbf{N}_n}}=1$. The first line of (4. 2) is bounded from below. Suppose $\gamma+\delta-\alpha+\mathbf{N}\left(\frac{1}{p}-\frac{1}{q}\right)\neq0$. By either taking $\lambda\longrightarrow0$ or $\lambda\longrightarrow\infty$, the last line of (4. 2) is vanished. We must have $\gamma+\delta-\alpha+\mathbf{N}\left(\frac{1}{p}-\frac{1}{q}\right)=0$ which is (2. 3).

Write $x=(x_i,x_i^\dagger)\in\mathbb{R}^{\mathbf{N}_i}\times\mathbb{R}^{\mathbf{N}-\mathbf{N}_i},\ i=1,2,\ldots,n$ and $\mathbf{Q}^\dagger_i=\bigotimes_{j\neq i}\mathbf{Q}_j$. Suppose $\mathbf{Q}_i$ centered on the origin of $\mathbb{R}^{\mathbf{N}_i}$. Let $\mathbf{Q}_i$ shrink to 0 and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}}=1,\ j\neq i$ in (4. 1). By applying Lebesgue Differentiation Theorem, we have

$$\begin{aligned}
&\lim_{|\mathbf{Q}_i|\longrightarrow0}|\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}^\dagger_i|}\int_{\mathbf{Q}^\dagger_i}\left(\frac{1}{|x^\dagger_i|}\right)^{\gamma q}dx^\dagger_i\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}^\dagger_i|}\int_{\mathbf{Q}^\dagger_i}\left(\frac{1}{|x^\dagger_i|}\right)^{\frac{\delta p}{p-1}}dx^\dagger_i\right\}^{\frac{p-1}{p}}\\
&\leq\ \mathbf{A}^\alpha_{pq}\left(|x|^{-\gamma},|x|^{\delta}\right),\qquad i\ =\ 1,2,\ldots,n.
\end{aligned} \tag{4. 3}$$

The boundedness of $\mathbf{A}^{\alpha}_{pq}\left(|x|^{-\gamma},|x|^{\delta}\right)$ requires

$$\frac{\alpha_i}{\mathbf{N}_i} \ \geq\ \frac{1}{p}-\frac{1}{q}, \qquad i \ =\ 1,2,\ldots,n. \tag{4. 4}$$

By putting together (2. 3) and (4. 4), we find $\gamma+\delta\geq 0$. On the other and, it is essential to require $\gamma q<\mathbf{N}$ and $\delta\left(\frac{p}{p-1}\right)<\mathbf{N}$ for the local integrability of $|x|^{-\gamma q}$ and $|x|^{-\delta\left(\frac{p}{p-1}\right)}$ respectively. These are the constraints in (2. 2).

Let $\mathcal{S}$ to be a proper subset of $\{1,2,\ldots,n\}$. We define ${}^{\mathbf{t}}\mathbf{Q}_i=\mathbf{Q}_i\cap\{2^{-t_i}\leq|x_i|\leq 2^{-t_i+1}\}$ for every $t_i>0$ and $i\in\mathcal{S}$. Denote ${}^{\mathbf{t}}\mathbf{Q}=\bigotimes_{i\in\mathcal{S}}{}^{\mathbf{t}}\mathbf{Q}_i\times\bigotimes_{i\in\mathcal{S}^c}\mathbf{Q}_i$ and $\mathbf{Q}_{\mathcal{S}}=\bigotimes_{i\in\mathcal{S}}\mathbf{Q}_i$, $\mathbf{Q}_{\mathcal{S}^c}=\bigotimes_{i\in\mathcal{S}^c}\mathbf{Q}_i$. Moreover, we write $x=(x_{\mathcal{S}},x_{\mathcal{S}^c})\in\mathbb{R}^{\mathbf{N}_{\mathcal{S}}}\times\mathbb{R}^{\mathbf{N}-\mathbf{N}_{\mathcal{S}}}$ for which $\mathbf{N}_{\mathcal{S}}=\sum_{i\in\mathcal{S}}\mathbf{N}_i$

Suppose that there exists at least one $i\in\mathcal{S}^c$ such that $\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)>0$. Let $0<\lambda<1$. Consider $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}=1$ for $i\in\mathcal{S}$ and $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}=\lambda$ for $i\in\mathcal{S}^c$. We have

$$\begin{aligned}
&\prod_{i=1}^n|\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\\
&=\ \lambda^{\sum_{i\in\mathcal{S}^c}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\sum_{\mathbf{t}}\frac{1}{|\mathbf{Q}|}\int_{{}^{\mathbf{t}}\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\left\{\sum_{\mathbf{t}}\frac{1}{|\mathbf{Q}|}\int_{{}^{\mathbf{t}}\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\\
&=\ \left\{\lambda^{\left[\sum_{i\in\mathcal{S}^c}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)\right]\frac{q}{2}}\sum_{\mathbf{t}}\frac{1}{|\mathbf{Q}|}\int_{{}^{\mathbf{t}}\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\right\}^{\frac{1}{q}}\\
&\qquad\left\{\lambda^{\left[\sum_{i\in\mathcal{S}^c}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)\right]\left(\frac{p}{p-1}\right)\frac{1}{2}}\sum_{\mathbf{t}}\frac{1}{|\mathbf{Q}|}\int_{{}^{\mathbf{t}}\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\right\}^{\frac{p-1}{p}}\\
&\doteq\ \left\{\sum_{\mathbf{t}}\mathbf{A}_{\mathbf{t}}(\lambda)\right\}^{\frac{1}{q}}\left\{\sum_{\mathbf{t}}\mathbf{B}_{\mathbf{t}}(\lambda)\right\}^{\frac{p-1}{p}}.
\end{aligned}\tag{4. 5}$$

By applying Lebesgue differentiation theorem, we find

$$\begin{aligned}
\lim_{\lambda\longrightarrow 0}\frac{1}{|\mathbf{Q}|}\int_{{}^{\mathbf{t}}\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q}dx\ &=\ \int\cdots\int_{\bigotimes_{i\in\mathcal{S}}{}^{\mathbf{t}}\mathbf{Q}_i}\left[\frac{1}{\sum_{i\in\mathcal{S}}|x_i|^2}\right]^{\frac{\gamma q}{2}}\prod_{i\in\mathcal{S}}dx_i,\\
\lim_{\lambda\longrightarrow 0}\frac{1}{|\mathbf{Q}|}\int_{{}^{\mathbf{t}}\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}}dx\ &=\ \int\cdots\int_{\bigotimes_{i\in\mathcal{S}}{}^{\mathbf{t}}\mathbf{Q}_i}\left[\frac{1}{\sum_{i\in\mathcal{S}}|x_i|^2}\right]^{\frac{1}{2}\frac{\delta p}{p-1}}\prod_{i\in\mathcal{S}}dx_i.
\end{aligned}\tag{4. 6}$$

Because $\sum_{i\in\mathcal{S}^c}\alpha_i-\mathbf{N}_i(\frac{1}{p}-\frac{1}{q})>0$, we have $\mathbf{A}_{\mathbf{t}}(0)=\mathbf{B}_{\mathbf{t}}(0)=0$ for every $\mathbf{t}$. This remains to be true if $\sum_{i\in\mathcal{S}^c}\alpha_i-\mathbf{N}_i(\frac{1}{p}-\frac{1}{q})$ is replaced by any smaller positive number. Therefore, each $\mathbf{A}_{\mathbf{t}}(\lambda)$ and $\mathbf{B}_{\mathbf{t}}(\lambda)$ is Hölder continuous for $\lambda\geq 0$ whose exponent is strictly positive. Furthermore, $\sum_{\mathbf{t}}\mathbf{A}_{\mathbf{t}}(\lambda)$ and $\sum_{\mathbf{t}}\mathbf{B}_{\mathbf{t}}(\lambda)$ are uniformly bounded for every $\lambda>0$. Consequently, both summations are continuous at $\lambda=0$ and

$$\lim_{\lambda\longrightarrow 0}\sum_{\mathbf{t}}\mathbf{A}_{\mathbf{t}}(\lambda)\ =\ 0,\qquad \lim_{\lambda\longrightarrow 0}\sum_{\mathbf{t}}\mathbf{B}_{\mathbf{t}}(\lambda)\ =\ 0. \tag{4. 7}$$

Suppose $\alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) = 0$ for every $i \in \mathcal{S}^c$. Shrink $\mathbf{Q}_i$ to the origin of $\mathbb{R}^{\mathbf{N}_i}$ for $i \in \mathcal{S}^c$ in (4. 1). By applying Lebesgue differentiation theorem, we have

$$\mathbf{A}^{\alpha}_{pq}\left(|x|^{-\gamma}, |x|^{\delta}\right) \;\geq\; \prod_{i\in\mathcal{S}} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}_{\mathcal{S}}|}\int_{\mathbf{Q}_{\mathcal{S}}}\left(\frac{1}{|x_{\mathcal{S}}|}\right)^{\gamma q} dx_{\mathcal{S}}\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}_{\mathcal{S}}|}\int_{\mathbf{Q}_{\mathcal{S}}}\left(\frac{1}{|x_{\mathcal{S}}|}\right)^{\frac{\delta p}{p-1}} dx_{\mathcal{S}}\right\}^{\frac{p-1}{p}} \tag{4. 8}$$

where $\gamma q < \mathbf{N}_{\mathcal{S}}$ and $\delta\left(\frac{p}{p-1}\right) < \mathbf{N}_{\mathcal{S}}$ become necessities.

Case One: Consider $\gamma \geq 0, \delta \leq 0$. Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for $i \in \{1, 2, \dots, n\}$ and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}} = \lambda$ for all $j \neq i$. Suppose $\alpha_j - \mathbf{N}_j\left(\frac{1}{p}-\frac{1}{q}\right) = 0$ for every $j \neq i$. We have

$$\begin{aligned}
&\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}} dx\right\}^{\frac{p-1}{p}} \\
&\geq\; \mathfrak{B}_{q\,\gamma} \left\{\int_{\mathbf{Q}_i}\left(\frac{1}{\lambda+|x_i|}\right)^{\gamma q} dx_i\right\}^{\frac{1}{q}} \left\{\int_{\mathbf{Q}_i}\left(\frac{1}{|x_i|}\right)^{\frac{\delta p}{p-1}} dx_i\right\}^{\frac{p-1}{p}} \qquad (\delta \leq 0) \\
&\geq\; \mathfrak{B}_{p\,q\,\gamma\,\delta} \left\{\int_{\lambda<|x_i|\leq 1}\left(\frac{1}{\lambda+|x_i|}\right)^{\gamma q} dx_i\right\}^{\frac{1}{q}}
\end{aligned} \tag{4. 9}$$

where

$$\begin{aligned}
\int_{\lambda<|x_i|\leq 1}\left(\frac{1}{\lambda+|x_i|}\right)^{\gamma q} dx_i \;&=\; \mathfrak{B}\,\ln\left(\frac{1+\lambda}{2\lambda}\right) \quad \text{if } \gamma = \frac{\mathbf{N}_i}{q}, \\
\int_{\lambda<|x_i|\leq 1}\left(\frac{1}{\lambda+|x_i|}\right)^{\gamma q} dx_i \;&=\; \mathfrak{B}\,\frac{1}{\gamma q - \mathbf{N}_i}\left[\left(\frac{1}{2\lambda}\right)^{\gamma q-\mathbf{N}_i} - \left(\frac{1}{\lambda+1}\right)^{\gamma q-\mathbf{N}_i}\right] \quad \text{if} \quad \text{if } \gamma > \frac{\mathbf{N}_i}{q}.
\end{aligned} \tag{4. 10}$$

From (4. 9)-(4. 10), to satisfy the inequality in (4. 2) as $\lambda \longrightarrow 0$, we need

$$\gamma \;<\; \frac{\mathbf{N}_i}{q}, \qquad i = 1, 2, \dots, n. \tag{4. 11}$$

Suppose that there exists $j \neq i$ such that $\alpha_j - \mathbf{N}_j\left(\frac{1}{p}-\frac{1}{q}\right) > 0$. We have

$$\begin{aligned}
&\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}} dx\right\}^{\frac{p-1}{p}} \\
&\geq\; \mathfrak{B}_{q\,\gamma} \prod_{j\neq i} \lambda^{\alpha_j-\mathbf{N}_j\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\int_{\mathbf{Q}_i}\left(\frac{1}{\lambda+|x_i|}\right)^{\gamma q} dx_i\right\}^{\frac{1}{q}} \left\{\int_{\mathbf{Q}_i}\left(\frac{1}{|x_i|}\right)^{\frac{\delta p}{p-1}} dx_i\right\}^{\frac{p-1}{p}} \quad (\delta \leq 0) \\
&\geq\; \mathfrak{B}_{p\,q\,\gamma\,\delta} \prod_{j\neq i} \lambda^{\alpha_j-\mathbf{N}_j\left(\frac{1}{p}-\frac{1}{q}\right)} \left\{\int_{0<|x_i|\leq\lambda}\left(\frac{1}{\lambda}\right)^{\gamma q} dx_i\right\}^{\frac{1}{q}} \;=\; \mathfrak{B}_{p\,q\,\gamma\,\delta}\, \lambda^{\frac{\mathbf{N}_i}{q}-\gamma+\sum_{j\neq i}\alpha_j-\mathbf{N}_j\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned} \tag{4. 12}$$

Recall the estimates in (4. 5)-(4. 7). Take $\mathcal{S} = \{i\}$. The first line of (4. 12) equals to zero as $\lambda \longrightarrow 0$. Together with (4. 11), we find

$$\gamma \ < \ \frac{\mathbf{N}_i}{q} + \sum_{j\neq i} \alpha_j - \mathbf{N}_j \left(\frac{1}{p} - \frac{1}{q}\right), \qquad i \ = \ 1, 2, \ldots, n. \tag{4. 13}$$

Case Two: Consider $\gamma \leq 0, \delta \geq 0$. Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for $i \in \{1, 2, \ldots, n\}$ and $|\mathbf{Q}_j|^{\frac{1}{\mathbf{N}_j}} = \lambda$ for all $j \neq i$. Suppose $\alpha_j - \mathbf{N}_j \left(\frac{1}{p} - \frac{1}{q}\right) = 0$ for every $j \neq i$. We have

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \frac{1}{p} + \frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}} dx \right\}^{\frac{p-1}{p}} \\
&\geq \ \mathfrak{B}_{p\ \delta} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{|x_i|}\right)^{\gamma q} dx_i \right\}^{\frac{1}{q}} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{\lambda + |x_i|}\right)^{\frac{\delta p}{p-1}} dx_i \right\}^{\frac{p-1}{p}} \qquad (\gamma \leq 0) \\
&\geq \ \mathfrak{B}_{p\ q\ \gamma\ \delta} \left\{ \int_{\lambda < |x_i| \leq 1} \left(\frac{1}{\lambda + |x_i|}\right)^{\frac{\delta p}{p-1}} dx_i \right\}^{\frac{p-1}{p}}
\end{aligned} \tag{4. 14}$$

where

$$\begin{aligned}
&\int_{\lambda<|x_i|\leq 1} \left(\frac{1}{\lambda + |x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \ = \ \mathfrak{B}_{\mathbf{N}} \ \ln\left(\frac{1+\lambda}{2\lambda}\right) \quad \text{if } \delta = \mathbf{N}_i \left(\tfrac{p-1}{p}\right), \\
&\int_{\lambda<|x_i|\leq 1} \left(\frac{1}{\lambda + |x_i|}\right)^{\delta\left(\frac{p}{p-1}\right)} dx_i \ = \ \mathfrak{B}_{\mathbf{N}} \ \frac{1}{\delta\left(\frac{p}{p-1}\right) - \mathbf{N}_i} \left[\left(\frac{1}{2\lambda}\right)^{\delta\left(\frac{p}{p-1}\right) - \mathbf{N}_i} - \left(\frac{1}{\lambda+1}\right)^{\delta\left(\frac{p}{p-1}\right) - \mathbf{N}_i} \right] \\
&\hspace{20em} \text{if } \ \delta > \mathbf{N}_i \left(\tfrac{p-1}{p}\right).
\end{aligned} \tag{4. 15}$$

From (4. 14)-(4. 15), to satisfy the inequality in (4. 2) as $\lambda \longrightarrow 0$, we need

$$\delta \ < \ \mathbf{N}_i \left(\frac{p-1}{p}\right), \qquad i \ = 1, 2, \ldots, n. \tag{4. 16}$$

Suppose that there exists $j \neq i$ such that $\alpha_j - \mathbf{N}_j \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. We have

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \frac{1}{p} + \frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}} dx \right\}^{\frac{p-1}{p}} \\
&\geq \ \mathfrak{B}_{p\ \delta\ n} \prod_{j\neq i} \lambda^{\alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{|x_i|}\right)^{\gamma q} dx_i \right\}^{\frac{1}{q}} \left\{ \int_{\mathbf{Q}_i} \left(\frac{1}{\lambda + |x_i|}\right)^{\frac{\delta p}{p-1}} dx_i \right\}^{\frac{p-1}{p}} \quad (\gamma \leq 0) \\
&\geq \ \mathfrak{B}_{p\ q\ \gamma\ \delta} \prod_{j\neq i} \lambda^{\alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \int_{0<|x_i|\leq\lambda} \left(\frac{1}{\lambda}\right)^{\frac{\delta p}{p-1}} dx_i \right\}^{\frac{p-1}{p}} \\
&= \ \mathfrak{B}_{p\ q\ \gamma\ \delta} \ \lambda^{\left(\frac{p-1}{p}\right)\mathbf{N}_i - \delta + \sum_{j\neq i} \alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right)}.
\end{aligned} \tag{4. 17}$$

Recall the estimates in (4. 5)-(4. 7). Take $\mathcal{S} = \{i\}$. The first line of (4. 17) equals to zero as $\lambda \longrightarrow 0$. Together with (4. 16), we find

$$\delta \ < \ \mathbf{N}_i \left(\frac{p-1}{p} \right) + \sum_{j \neq i} \alpha_j - \mathbf{N}_j \left(\frac{1}{p} - \frac{1}{q} \right), \qquad i = 1, 2, \ldots, n. \tag{4. 18}$$

Case Three: Consider $\gamma > 0, \delta > 0$. Note that (4. 1) is invariant by changing dilation in one-parameter as shown in (4. 2) because of (2. 3).

Recall the definition of $\mathcal{U}$ and $\mathcal{V}$ from (2. 6). We write $x_{\mathcal{U}} \in \mathbb{R}^{\mathbf{N}_{\mathcal{U}}}$ and $x_{\mathcal{V}} \in \mathbb{R}^{\mathbf{N}_{\mathcal{V}}}$ where $\mathbb{R}^{\mathbf{N}_{\mathcal{U}}} = \bigotimes_{i \in \mathcal{U}} \mathbb{R}^{\mathbf{N}_i}$ and $\mathbb{R}^{\mathbf{N}_{\mathcal{V}}} = \bigotimes_{i \in \mathcal{V}} \mathbb{R}^{\mathbf{N}_i}$.

Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = \lambda^{-1}$ for every $i \in \mathcal{U}$ and $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for all other $i \notin \mathcal{U}$. We have

$$\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \frac{1}{p} + \frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\gamma q} dx \right\}^{\frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\frac{\delta p}{p-1}} dx \right\}^{\frac{p-1}{p}} \\
&\geq \ \mathfrak{B}_{p\,q\,\gamma\,\delta} \prod_{i \in \mathcal{U}} \left(\frac{1}{\lambda} \right)^{\alpha_i - \frac{\mathbf{N}_i}{p}} \left\{ \int \cdots \int_{\bigotimes_{i \in \mathcal{U}} \mathbf{Q}_i} \left[\frac{1}{1 + \sum_{i \in \mathcal{U}} |x_i|} \right]^{\gamma q} \prod_{i \in \mathcal{U}} dx_i \right\}^{\frac{1}{q}} \\
&\qquad \left\{ \prod_{i \in \mathcal{U}} \lambda^{\mathbf{N}_i} \int \cdots \int_{\bigotimes_{i \in \mathcal{U}} \mathbf{Q}_i} \lambda^{\frac{\delta p}{p-1}} \prod_{i \in \mathcal{U}} dx_i \right\}^{\frac{p-1}{p}} \qquad (0 < \lambda < 1) \\
&\geq \ \mathfrak{B}_{p\,q\,\gamma\,\delta} \prod_{i \in \mathcal{U}} \left(\frac{1}{\lambda} \right)^{\alpha_i - \frac{\mathbf{N}_i}{p}} \left\{ \int \cdots \int_{\bigotimes_{i \in \mathcal{U}} 0 < |x_i| \leq 1} \prod_{i \in \mathcal{U}} dx_i \right\}^{\frac{1}{q}} \\
&\qquad \left\{ \prod_{i \in \mathcal{U}} \lambda^{\mathbf{N}_i} \int \cdots \int_{\bigotimes_{i \in \mathcal{U}} \mathbf{Q}_i} \lambda^{\frac{\delta p}{p-1}} \prod_{i \in \mathcal{U}} dx_i \right\}^{\frac{p-1}{p}} \\
&\geq \ \mathfrak{B}_{p\,q\,\gamma\,\delta} \left(\frac{1}{\lambda} \right)^{\sum_{i \in \mathcal{U}} \alpha_i - \frac{\mathbf{N}_i}{p} - \delta}.
\end{aligned} \tag{4. 19}$$

In the case of $\mathcal{U} = \{1, 2, \ldots, n\}$, as $\gamma < \frac{\mathbf{N}}{q}$, we find

$$\begin{aligned}
\delta \ &= \ \frac{\mathbf{N}}{q} - \gamma + \sum_{i=1}^n \alpha_i - \frac{\mathbf{N}_i}{p} \qquad \text{by (2. 3)} \\
&> \ \sum_{i=1}^n \alpha_i - \frac{\mathbf{N}_i}{p} \ = \ \sum_{i \in \mathcal{U}} \alpha_i - \frac{\mathbf{N}_i}{p}.
\end{aligned} \tag{4. 20}$$

Suppose that $\mathcal{U}$ is a proper subset of $\{1, 2, \ldots, n\}$. There exists at least one $i \in \mathcal{U}^c$ such that $\alpha_i - \mathbf{N}_i \left(\frac{1}{p} - \frac{1}{q} \right) > 0$. Recall the estimates in (4. 5)-(4. 7). Take $\mathcal{S} = \mathcal{U}$. The first line of (4. 19) equals to zero as $\lambda \longrightarrow 0$. We must have

$$\sum_{i \in \mathcal{U}} \alpha_i - \frac{\mathbf{N}_i}{p} \ < \ \delta. \tag{4. 21}$$

Suppose that $\mathcal{U}$ is a proper subset of $\{1,2,\ldots,n\}$ where $\alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) = 0$ for every $i \in \mathcal{U}^c$.

Let $\mathcal{S} = \mathcal{U}$. We have $|x_{\mathcal{U}}|^{-\gamma}$ and $|x_{\mathcal{U}}|^{\delta}$ satisfying the Muckenhoupt characteristic in (4. 8) on $\mathbb{R}^{\mathbf{N}_{\mathcal{U}}} \doteq \bigotimes_{i\in\mathcal{U}} \mathbb{R}^{\mathbf{N}_i}$. Denote

$$\alpha_{\mathcal{U}} \;=\; \sum_{i\in\mathcal{U}} \alpha_i.$$

By carrying out the same estimate in (4. 2), we find

$$\gamma \;<\; \frac{\mathbf{N}_{\mathcal{U}}}{q}, \qquad \delta \;<\; \mathbf{N}_{\mathcal{U}}\left(\frac{p-1}{p}\right), \qquad \frac{\alpha_{\mathcal{U}}}{\mathbf{N}_{\mathcal{U}}} \;=\; \frac{1}{p}-\frac{1}{q}+\frac{\gamma+\delta}{\mathbf{N}_{\mathcal{U}}}. \tag{4. 22}$$

This further implies

$$\begin{aligned} \delta \;&=\; \frac{\mathbf{N}_{\mathcal{U}}}{q} - \gamma + \sum_{i\in\mathcal{U}} \alpha_i - \frac{\mathbf{N}_i}{p} \\ &>\; \sum_{i\in\mathcal{U}} \alpha_i - \frac{\mathbf{N}_i}{p}. \end{aligned} \tag{4. 23}$$

Let $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = \lambda^{-1}$ for every $i \in \mathcal{V}$ and $|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 1$ for all other $i \notin \mathcal{V}$. We have

$$\begin{aligned} &\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q} dx\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\frac{\delta p}{p-1}} dx\right\}^{\frac{p-1}{p}} \\ &\geq\; \mathfrak{B}_{p\,q\,\gamma\,\delta} \left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathcal{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i} \left\{\prod_{i\in\mathcal{V}} \lambda^{\mathbf{N}_i} \int\cdots\int_{\bigotimes_{i\in\mathcal{V}} \mathbf{Q}_i} \lambda^{\gamma q} \prod_{i\in\mathcal{V}} dx_i\right\}^{\frac{1}{q}} \\ &\qquad \left\{\int\cdots\int_{\bigotimes_{i\in\mathcal{V}} \mathbf{Q}_i} \left[\frac{1}{1+\sum_{i\in\mathcal{V}}|x_i|}\right]^{\frac{\delta p}{p-1}} \prod_{i\in\mathcal{V}} dx_i\right\}^{\frac{p-1}{p}} \\ &\geq\; \mathfrak{B}_{p\,q\,\gamma\,\delta} \left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathcal{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i} \left\{\prod_{i\in\mathcal{V}} \lambda^{\mathbf{N}_i} \int\cdots\int_{\bigotimes_{i\in\mathcal{V}} \mathbf{Q}_i} \lambda^{\gamma q} \prod_{i\in\mathcal{V}} dx_i\right\}^{\frac{1}{q}} \\ &\qquad \left\{\int\cdots\int_{\bigotimes_{i\in\mathcal{V}} 0<|x_i|\leq 1} \prod_{i\in\mathcal{V}} dx_i\right\}^{\frac{p-1}{p}} \\ &\geq\; \mathfrak{B}_{p\,q\,\gamma\,\delta} \left(\frac{1}{\lambda}\right)^{\sum_{i\in\mathcal{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i - \gamma}. \end{aligned} \tag{4. 24}$$

In the case of $\mathcal{V} = \{1,2,\ldots,n\}$, as $\delta < \mathbf{N}\left(\frac{p-1}{p}\right)$, we find

$$\begin{aligned} \gamma \;&=\; \left(\frac{p-1}{p}\right)\mathbf{N} - \delta + \sum_{i=1}^n \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) \qquad \text{by (2. 3)} \\ &>\; \sum_{i=1}^n \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) \;=\; \sum_{i\in\mathcal{V}} \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right). \end{aligned} \tag{4. 25}$$

Suppose that $\mathcal{V}$ is a proper subset of $\{1, 2, \ldots, n\}$. There exists at least one $i \in \mathcal{V}^c$ such that $\alpha_i - \mathbf{N}_i \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. Recall the estimates in (4. 5)-(4. 7). Take $\mathcal{S} = \mathcal{V}$. The first line of (4. 24) equals to zero at $\lambda \longrightarrow 0$. We must have

$$\sum_{i\in\mathcal{V}} \alpha_i - \left(\frac{q-1}{q}\right)\mathbf{N}_i \ < \ \gamma. \tag{4. 26}$$

Suppose that $\mathcal{V}$ is a proper subset of $\{1, 2, \ldots, n\}$ where $\alpha_i - \mathbf{N}_i \left(\frac{1}{p} - \frac{1}{q}\right) = 0$ for every $i \in \mathcal{V}^c$. Let $\mathcal{S} = \mathcal{V}$. We have $|x_{\mathcal{V}}|^{-\gamma}$ and $|x_{\mathcal{V}}|^{\delta}$ satisfying the Muckenhoupt characteristic in (4. 8) on $\mathbb{R}^{\mathbf{N}_{\mathcal{V}}} \doteq \bigotimes_{i\in\mathcal{V}} \mathbb{R}^{\mathbf{N}_i}$. Denote

$$\alpha_{\mathcal{V}} \ = \ \sum_{i\in\mathcal{V}} \alpha_i.$$

By carrying out the same estimate in (4. 2), we find

$$\gamma \ < \ \frac{\mathbf{N}_{\mathcal{V}}}{q}, \qquad \delta \ < \ \mathbf{N}_{\mathcal{V}}\left(\frac{p-1}{p}\right), \qquad \frac{\alpha_{\mathcal{V}}}{\mathbf{N}_{\mathcal{V}}} \ = \ \frac{1}{p} - \frac{1}{q} + \frac{\gamma+\delta}{\mathbf{N}_{\mathcal{V}}}. \tag{4. 27}$$

This further implies

$$\begin{aligned} \gamma \ &= \ \left(\frac{p-1}{p}\right)\mathbf{N}_{\mathcal{V}} - \delta + \sum_{i\in\mathcal{V}} \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) \\ &> \ \sum_{i\in\mathcal{V}} \alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right). \end{aligned} \tag{4. 28}$$

Remark 4.1 *By using the formula in (2. 3), we can verify that the constraints in (4. 13) and (4. 18) are equivalent to (2. 4) and (2. 5) respectively: For $\gamma \geq 0, \delta \leq 0$, we have*

$$\gamma \ < \ \frac{\mathbf{N}_i}{q} + \sum_{j\neq i} \alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right) \qquad \Longleftrightarrow \qquad \alpha_i - \frac{\mathbf{N}_i}{p} \ < \ \delta, \quad i \ = \ 1, 2, \ldots, n. \tag{4. 29}$$

For $\gamma \leq 0, \delta \geq 0$, we find

$$\begin{aligned} &\delta \ < \ \mathbf{N}_i\left(\frac{p-1}{p}\right) + \sum_{j\neq i} \alpha_j - \mathbf{N}_j\left(\frac{1}{p} - \frac{1}{q}\right) \qquad \Longleftrightarrow \\ &\alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) \ < \ \gamma, \quad i \ = \ 1, 2, \ldots, n. \end{aligned} \tag{4. 30}$$

5 Decay estimate on varying eccentricities

Recall $\mathbf{A}^{\alpha}_{pqr}(\mathbf{t} : \omega, \sigma)$ defined in (3. 6). We aim to show

$$\mathbf{A}^{\alpha}_{pqr}(\mathbf{t} : \omega, \sigma) \ \leq \ \mathfrak{B}_{p\ q\ r\ \alpha\ \gamma\ \delta} \prod_{i=1}^{n} 2^{-\varepsilon|t_i|}$$

for some $\varepsilon = \varepsilon(p, q, r, \alpha, \gamma, \delta) > 0$. This further implies the summability in (3. 14).

Principal Lemma: *Let γ, δ satisfying (2. 2)-(2. 6). Suppose*

$$\frac{\alpha_i}{\mathbf{N}_i} > \frac{1}{p} - \frac{1}{q}, \qquad i = 1, 2, \ldots, n. \tag{5. 1}$$

For $0 < \lambda_i \leq 1, i = 1, 2, \ldots, n$, define

$$^{\lambda}\mathbf{Q} \subset \mathbb{R}^{\mathbf{N}} : |\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} / |\mathbf{Q}_\nu|^{\frac{1}{\mathbf{N}_\nu}} = \lambda_i. \tag{5. 2}$$

There exists an $\varepsilon > 0$ such that

$$\begin{aligned} \sup_{^{\lambda}\mathbf{Q}} \prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \frac{1}{p} + \frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\gamma q r} dx \right\}^{\frac{1}{qr}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|} \right)^{\frac{\delta p r}{p-1}} dx \right\}^{\frac{p-1}{pr}} & \\ \leq \mathfrak{B}_{p\, q\, r\, \alpha\, \gamma\, \delta} \prod_{i=1}^{n} (\lambda_i)^{\varepsilon} & \end{aligned} \tag{5. 3}$$

for some $r > 1$. The values of ε and r depend only on $p, q, \alpha, \gamma, \delta$.

Proof: By carrying out the same estimate in (4. 2) and using the formula (2. 3), we find that the *r-bump* characteristic in (5. 3) is invariant by changing one-parameter dilation. Therefore, it is suffice to consider $|\mathbf{Q}_\nu|^{\frac{1}{\mathbf{N}_\nu}} = 1$.

Let $\mathbf{Q}_i^o$ and $\mathbf{Q}_i^* \subset \mathbb{R}^{\mathbf{N}_i}$ to be centered on the origin of $\mathbb{R}^{\mathbf{N}_i}$ and

$$\begin{aligned} |\mathbf{Q}_i^o|^{\frac{1}{\mathbf{N}_i}} &= |\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}}, \qquad i = 1, 2, \ldots, n; \\ |\mathbf{Q}_i^*|^{\frac{1}{\mathbf{N}_i}} &= 3|\mathbf{Q}_i|^{\frac{1}{\mathbf{N}_i}} = 3\lambda_i, \qquad i = 1, 2, \ldots, n. \end{aligned} \tag{5. 4}$$

Remark 5.1 *Suppose $\mathbf{Q}_i \cap \mathbf{Q}_i^o = \emptyset$. We must have $|x_i| \geq |x_i^o| / \sqrt{n}$ for every $x_i \in \mathbf{Q}_i$ and every $x_i^o \in \mathbf{Q}_i^o$. Otherwise, if $\mathbf{Q}_i$ intersects $\mathbf{Q}_i^o$, then $\mathbf{Q}_i \subset \mathbf{Q}_i^*$.*

After a permutation on indices $i = 1, 2, \ldots, n$, we can assume $\nu = 1$ and

$$1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n. \tag{5. 5}$$

Case One: Let $\gamma \geq 0, \delta \leq 0$ satisfying (2. 2)-(2. 4). By adjusting the value of r, we assume

$$\sum_{i=1}^{m-1} \mathbf{N}_i < \gamma q r < \sum_{i=1}^{m} \mathbf{N}_i, \qquad 1 \leq m \leq n. \tag{5. 6}$$

Suppose that $\mathbf{Q}$ is centered on some $z\in\mathbb{R}^{\mathbf{N}}$ with $|z|\le 3$. We have

$$
\begin{array}{l}
\displaystyle\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma qr} dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}} dx\right\}^{\frac{p-1}{pr}} \\[2ex]
\displaystyle\le\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\left\{\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i}\int\cdots\int_{\bigotimes_{i=1}^n\mathbf{Q}_i}\left[\frac{1}{|x_1|+\cdots+|x_n|}\right]^{\gamma qr}dx_1\cdots dx_n\right\}^{\frac{1}{qr}} \\
\hfill (\delta\le 0) \\[2ex]
\displaystyle\le\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}} \\[2ex]
\displaystyle\qquad\left\{\int\cdots\int_{\bigotimes_{i=m}^n\mathbf{Q}_i}\left\{\int\cdots\int_{\bigotimes_{i=1}^{m-1}\mathbb{R}^{\mathbf{N}_i}}\left[\frac{1}{|x_1|+\cdots+|x_n|}\right]^{\gamma qr}dx_1\cdots dx_{m-1}\right\}dx_m\cdots dx_n\right\}^{\frac{1}{qr}} \\[2ex]
\displaystyle\le\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\left\{\int\cdots\int_{\bigotimes_{i=m}^n\mathbf{Q}_i}\left[\frac{1}{|x_m|+\cdots+|x_n|}\right]^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i}dx_m\cdots dx_n\right\}^{\frac{1}{qr}} \\[2ex]
\displaystyle\le\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^n\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\left\{\int\cdots\int_{\bigotimes_{i=m}^n\mathbf{Q}_i}\left(\frac{1}{|x_m|}\right)^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i}dx_m\cdots dx_n\right\}^{\frac{1}{qr}} \\[2ex]
\displaystyle\le\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^m\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\left\{\int_{\mathbf{Q}_m^*}\left(\frac{1}{|x_m|}\right)^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i}dx_m\right\}^{\frac{1}{qr}} \qquad \text{by } \textbf{Remark 5.1} \\[2ex]
\displaystyle\le\ \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\,(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^m\mathbf{N}_i-\gamma}\prod_{i=1}^m\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{array}
\tag{5. 7}
$$

From direct computation, the formula in the last line of (5. 7) can be rewritten as

$$
\begin{array}{l}
\displaystyle(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^m\mathbf{N}_i-\gamma}\prod_{i=1}^m(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mathbf{N}_i}{qr}}\prod_{i=m+1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \\[2ex]
\displaystyle=\ (\lambda_m)^{\frac{1}{qr}\sum_{i=1}^m\mathbf{N}_i-\gamma}\prod_{i=2}^m(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{\mathbf{N}_i}{qr}}\prod_{i=m+1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \qquad (\lambda_1=1) \\[2ex]
\displaystyle=\ (\lambda_m)^{\frac{\mathbf{N}_1}{qr}+\sum_{i=2}^n\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\gamma}\prod_{i=2}^m\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\frac{\mathbf{N}_i}{p}+\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\prod_{i=m+1}^n\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{array}
\tag{5. 8}
$$

Recall **Remark 4.1**. $\gamma\ge 0,\delta\le 0$ satisfy the two equivalent strict inequalities in (4. 29)-(4. 30). Define $0\le\vartheta\le 1$ implicitly by requiring $\lambda_m=(\lambda_n)^\vartheta$. For r sufficiently close to 1, we find

$$
\vartheta\left[\frac{\mathbf{N}_1}{qr}+\sum_{i=2}^n\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\gamma\right]+(1-\vartheta)\left[\alpha_n-\mathbf{N}_n\left(\frac{1}{p}-\frac{1}{q}\right)\right]\ >\ 0 \tag{5. 9}
$$

and

$$\alpha_i - \frac{\mathbf{N}_i}{p} + \left(1 - \frac{1}{r}\right)\frac{\mathbf{N}_i}{q} \ < \ 0, \qquad i = 1, 2, \ldots n. \tag{5. 10}$$

Note that $\alpha_n - \mathbf{N}_n \left(\frac{1}{p} - \frac{1}{q}\right) > 0$. By using (5. 9)-(5. 10), we conclude that (5. 8) is bounded by $\mathfrak{B}_{p\ q\ r\ \gamma\ \delta}(\lambda_n)^{\varepsilon}$ for some $\varepsilon = \varepsilon(p, q, r, \alpha, \gamma, \delta) > 0$.

Case Two: Let $\gamma \leq 0, \delta \geq 0$ satisfying (2. 2)-(2. 3) and (2. 5). By adjusting the value of r, assume

$$\sum_{i=1}^{m-1} \mathbf{N}_i \ < \ \delta\left(\frac{pr}{p-1}\right) \ < \ \sum_{i=1}^{m} \mathbf{N}_i, \qquad 1 \ \leq \ m \ \leq \ n. \tag{5. 11}$$

Suppose that $\mathbf{Q}$ is centered on some $z \in \mathbb{R}^{\mathbf{N}}$ with $|z| \leq 3$. We have

$$\begin{aligned}
&\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i} - \frac{1}{p} + \frac{1}{q}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma q r} dx \right\}^{\frac{1}{qr}} \left\{ \frac{1}{|\mathbf{Q}|} \int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta \frac{pr}{p-1}} dx \right\}^{\frac{p-1}{pr}} \\
&\leq \ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right)} \left\{ \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i} \int \cdots \int_{\bigotimes_{i=1}^{n} \mathbf{Q}_i} \left[\frac{1}{|x_1| + \cdots + |x_n|}\right]^{\delta \frac{pr}{p-1}} dx_1 \cdots dx_n \right\}^{\frac{p-1}{pr}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\gamma \leq 0) \\
&\leq \ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right)} \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i} \\
&\quad \left\{ \int \cdots \int_{\bigotimes_{i=m}^{n} \mathbf{Q}_i} \left\{ \int \cdots \int_{\bigotimes_{i=1}^{m-1} \mathbb{R}^{\mathbf{N}_i}} \left[\frac{1}{|x_1| + \cdots + |x_n|}\right]^{\delta \frac{pr}{p-1}} dx_1 \cdots dx_{m-1} \right\} dx_m \cdots dx_n \right\}^{\frac{p-1}{pr}} \\
&\leq \ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right)} \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i} \\
&\quad \left\{ \int \cdots \int_{\bigotimes_{i=m}^{n} \mathbf{Q}_i} \left[\frac{1}{|x_m| + \cdots + |x_n|}\right]^{\delta \frac{pr}{p-1} - \sum_{i=1}^{m-1} \mathbf{N}_i} dx_m \cdots dx_n \right\}^{\frac{p-1}{pr}} \\
&\leq \ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right)} \prod_{i=1}^{n} \left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i} \left\{ \int \cdots \int_{\bigotimes_{i=m}^{n} \mathbf{Q}_i} \left(\frac{1}{|x_m|}\right)^{\delta \frac{pr}{p-1} - \sum_{i=1}^{m-1} \mathbf{N}_i} dx_m \cdots dx_n \right\}^{\frac{p-1}{pr}} \\
&\leq \ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right)} \prod_{i=1}^{m} \left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i} \left\{ \int_{\mathbf{Q}_m^*} \left(\frac{1}{|x_m|}\right)^{\delta \frac{pr}{p-1} - \sum_{i=1}^{m-1} \mathbf{N}_i} dx_m \right\}^{\frac{p-1}{pr}} \quad \text{by } \textbf{Remark 5.1} \\
&\leq \ \mathfrak{B}_{p\ q\ r\ \gamma\ \delta} \ (\lambda_m)^{\frac{p-1}{pr} \sum_{i=1}^{m} \mathbf{N}_i - \delta} \prod_{i=1}^{m} \left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i - \mathbf{N}_i\left(\frac{1}{p} - \frac{1}{q}\right)}.
\end{aligned} \tag{5. 12}$$

From direct computation, the formula in the last line of (5. 12) can be rewritten as

$$
\begin{aligned}
&(\lambda_m)^{\frac{p-1}{pr}\sum_{i=1}^m \mathbf{N}_i-\delta}\prod_{i=1}^m (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{p-1}{pr}\mathbf{N}_i}\prod_{i=m+1}^n (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\\
=\;&(\lambda_m)^{\frac{p-1}{pr}\sum_{i=1}^m \mathbf{N}_i-\delta}\prod_{i=2}^m (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{p-1}{pr}\mathbf{N}_i}\prod_{i=m+1}^n (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\qquad(\lambda_1=1) \qquad (5.\ 13)\\
=\;&(\lambda_m)^{\frac{p-1}{pr}\mathbf{N}_1+\sum_{i=2}^n \alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\delta}\prod_{i=2}^m \left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\frac{q-1}{q}+\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}\prod_{i=m+1}^n \left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned}
$$

Recall **Remark 4.1**. $\gamma\le 0,\delta\ge 0$ satisfy the two equivalent strict inequalities in (4. 30). Define $0\le\vartheta\le 1$ implicitly by requiring $\lambda_m=(\lambda_n)^{\vartheta}$. For r sufficiently close to 1, we find

$$
\vartheta\left[\mathbf{N}_1\left(\frac{p-1}{pr}\right)+\sum_{i=2}^n \alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-\delta\right]+(1-\vartheta)\left[\alpha_n-\mathbf{N}_n\left(\frac{1}{p}-\frac{1}{q}\right)\right]\;>\;0 \qquad (5.\ 14)
$$

and

$$
\alpha_i-\mathbf{N}_i\left(\frac{q-1}{q}\right)+\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i\;<\;0,\qquad i=1,2,\ldots n. \qquad (5.\ 15)
$$

Note that $\alpha_n-\mathbf{N}_n\left(\frac{1}{p}-\frac{1}{q}\right)>0$. By using (5. 14)-(5. 15), we conclude that (5. 13) is bounded by $\mathfrak{B}_{p\ q\ r\ \gamma\ \delta}(\lambda_n)^{\varepsilon}$ for some $\varepsilon=\varepsilon(p,q,r,\alpha,\gamma,\delta)>0$.

Suppose that $\mathbf{Q}$ is centered on some $z\in\mathbb{R}^{\mathbf{N}}$ with $|z|>3$. Because $\mathbf{Q}$ has a diameter 1, we have

$$
\frac{1}{2}|z|\;\le\;|x|\;\le\;2|z| \qquad (5.\ 16)
$$

whenever $x\in\mathbf{Q}$. By using (5. 16), we obtain

$$
\begin{aligned}
&\prod_{i=1}^n |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\gamma qr}dx\right\}^{\frac{1}{qr}}\left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}}\left(\frac{1}{|x|}\right)^{\delta\left(\frac{pr}{p-1}\right)}dx\right\}^{\frac{p-1}{pr}}\\
\le\;&\mathfrak{B}_{\gamma\ \delta}\left(\frac{1}{|z|}\right)^{\gamma+\delta}\prod_{i=1}^n (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\;\le\;\mathfrak{B}_{\gamma\ \delta}\prod_{i=1}^n (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.\qquad(\gamma+\delta\ge 0)
\end{aligned}
\qquad (5.\ 17)
$$

Case Three: Let $\gamma>0,\delta>0$ satisfying (2. 2)-(2. 3) and (2. 6). By adjusting the value of r, assume

$$
\sum_{i=1}^{m-1}\mathbf{N}_i\;<\;\gamma qr\;<\;\sum_{i=1}^m \mathbf{N}_i,\qquad 1\;\le\;m\;\le\;n, \qquad (5.\ 18)
$$

$$
\sum_{i=1}^{\ell-1}\mathbf{N}_i\;<\;\delta\left(\frac{pr}{p-1}\right)\;<\;\sum_{i=1}^{\ell}\mathbf{N}_i,\qquad 1\;\le\;\ell\;\le\;n. \qquad (5.\ 19)
$$

We have

$$
\begin{aligned}
&\prod_{i=1}^{n} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\gamma qr} dx\right\}^{\frac{1}{qr}} \left\{\frac{1}{|\mathbf{Q}|}\int_{\mathbf{Q}} \left(\frac{1}{|x|}\right)^{\delta\frac{pr}{p-1}} dx\right\}^{\frac{p-1}{pr}}\\
&\le \mathfrak{B}_{p\,q\,r\,\gamma\,\delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}} \prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i\frac{p-1}{pr}}\\
&\quad\left\{\int\cdots\int_{\bigotimes_{i=m}^{n}\mathbf{Q}_i}\left\{\int\cdots\int_{\bigotimes_{i=1}^{m-1}\mathbb{R}^{\mathbf{N}_i}}\left[\frac{1}{|x_1|+\cdots+|x_n|}\right]^{\gamma qr} dx_1\cdots dx_{m-1}\right\} dx_m\cdots dx_n\right\}^{\frac{1}{qr}}\\
&\quad\left\{\int\cdots\int_{\bigotimes_{i=\ell}^{n}\mathbf{Q}_i}\left\{\int\cdots\int_{\bigotimes_{i=1}^{\ell-1}\mathbb{R}^{\mathbf{N}_i}}\left[\frac{1}{|x_1|+\cdots+|x_n|}\right]^{\delta\frac{pr}{p-1}} dx_1\cdots dx_{\ell-1}\right\} dx_\ell\cdots dx_n\right\}^{\frac{p-1}{pr}}\\
&\le \mathfrak{B}_{p\,q\,r\,\gamma\,\delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}} \prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i\frac{p-1}{pr}}\\
&\quad\left\{\int\cdots\int_{\bigotimes_{i=m}^{n}\mathbf{Q}_i}\left[\frac{1}{|x_m|+\cdots+|x_n|}\right]^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i} dx_m\cdots dx_n\right\}^{\frac{1}{qr}}\\
&\quad\left\{\int\cdots\int_{\bigotimes_{i=\ell}^{n}\mathbf{Q}_i}\left[\frac{1}{|x_\ell|+\cdots+|x_n|}\right]^{\delta\frac{pr}{p-1}-\sum_{i=1}^{\ell-1}\mathbf{N}_i} dx_\ell\cdots dx_n\right\}^{\frac{p-1}{pr}}\\
&\le \mathfrak{B}_{p\,q\,r\,\gamma\,\delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}} \prod_{i=1}^{n}\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i\frac{p-1}{pr}}\\
&\quad\left\{\int\cdots\int_{\bigotimes_{i=m}^{n}\mathbf{Q}_i}\left(\frac{1}{|x_m|}\right)^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i} dx_m\cdots dx_n\right\}^{\frac{1}{qr}}\\
&\quad\left\{\int\cdots\int_{\bigotimes_{i=\ell}^{n}\mathbf{Q}_i}\left(\frac{1}{|x_\ell|}\right)^{\delta\frac{pr}{p-1}-\sum_{i=1}^{\ell-1}\mathbf{N}_i} dx_\ell\cdots dx_n\right\}^{\frac{p-1}{pr}}\\
&\le \mathfrak{B}_{p\,q\,r\,\gamma\,\delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{m}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}} \prod_{i=1}^{\ell}\left(\frac{1}{\lambda_i}\right)^{\mathbf{N}_i\frac{p-1}{pr}}\\
&\quad\left\{\int_{\mathbf{Q}_m^*}\left(\frac{1}{|x_m|}\right)^{\gamma qr-\sum_{i=1}^{m-1}\mathbf{N}_i} dx_m\right\}^{\frac{1}{qr}}\left\{\int_{\mathbf{Q}_\ell^*}\left(\frac{1}{|x_\ell|}\right)^{\delta\frac{pr}{p-1}-\sum_{i=1}^{\ell-1}\mathbf{N}_i} dx_\ell\right\}^{\frac{p-1}{pr}} \qquad \text{by } \textbf{Remark 5.1}\\
&\le \mathfrak{B}_{p\,q\,r\,\gamma\,\delta}\ (\lambda_m)^{\frac{1}{qr}\sum_{i=1}^{m}\mathbf{N}_i-\gamma}\ (\lambda_\ell)^{\left(\frac{p-1}{pr}\right)\sum_{i=1}^{\ell}\mathbf{N}_i-\delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{m}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}} \prod_{i=1}^{\ell}\left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i}.
\end{aligned}
\tag{5. 20}
$$

Let $0 \leq k \leq n-1$. From direct computation, we find

$$\begin{aligned}
&\frac{1}{r}\left[\frac{1}{q}+\frac{p-1}{p}\right]\sum_{i=1}^{k}\mathbf{N}_i-(\gamma+\delta)+\sum_{i=k+1}^{n}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)\\
&= \frac{\mathbf{N}}{r}-\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}-(\gamma+\delta)+\sum_{i=k+1}^{n}\alpha_i-\frac{\mathbf{N}_i}{r}-\mathbf{N}_i\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\\
&= \frac{\mathbf{N}}{r}-\frac{1}{r}\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}-\alpha+\mathbf{N}\left(\frac{1}{p}-\frac{1}{q}\right)+\sum_{i=k+1}^{n}\alpha_i-\frac{\mathbf{N}_i}{r}-\mathbf{N}_i\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right) \qquad \text{by (2. 3)}\\
&= \left(\frac{\mathbf{N}}{r}-\alpha\right)+\mathbf{N}\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)+\sum_{i=k+1}^{n}\alpha_i-\frac{\mathbf{N}_i}{r}-\mathbf{N}_i\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\\
&= \sum_{i=1}^{k}\frac{\mathbf{N}_i}{r}-\alpha_i+\mathbf{N}_i\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right).
\end{aligned} \tag{5. 21}$$

Suppose $\ell \leq m$. The formula in the last line of (5. 20) can be rewritten as

$$\begin{aligned}
&(\lambda_m)^{\frac{1}{r}\left[\frac{1}{q}+\frac{p-1}{p}\right]\sum_{i=1}^{l-1}\mathbf{N}_i-(\gamma+\delta)}\left(\frac{\lambda_\ell}{\lambda_m}\right)^{\frac{p-1}{pr}\sum_{i=1}^{l-1}\mathbf{N}_i-\delta}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=l}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{\ell-1}\left(\frac{1}{\lambda_i}\right)^{\frac{1}{r}\left[\frac{1}{q}+\frac{p-1}{p}\right]\mathbf{N}_i}\\
&= (\lambda_m)^{\frac{1}{r}\left[\frac{1}{q}+\frac{p-1}{p}\right]\sum_{i=1}^{\ell-1}\mathbf{N}_i+\sum_{i=l}^{n}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-(\gamma+\delta)}\left(\frac{\lambda_\ell}{\lambda_m}\right)^{\frac{p-1}{pr}\sum_{i=1}^{\ell-1}\mathbf{N}_i-\delta}\\
&\quad\prod_{i=\ell}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=\ell}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{\ell-1}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&= \left(\frac{\lambda_\ell}{\lambda_m}\right)^{\frac{p-1}{pr}\sum_{i=1}^{l-1}\mathbf{N}_i-\delta}\prod_{i=\ell}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=\ell}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{\ell-1}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i} \qquad \text{by (5. 21)}\\
&= \left(\frac{\lambda_\ell}{\lambda_m}\right)^{\frac{p-1}{pr}\sum_{i=1}^{\ell-1}\mathbf{N}_i-\delta}\prod_{i=m+1}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=\ell}^{m}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\frac{\mathbf{N}_i}{p}+\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}\prod_{i=1}^{\ell-1}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&= \prod_{i=m+1}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^{\ell-1}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&\quad\left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i=1}^{l-1}\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta}\prod_{i=\ell}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}}.
\end{aligned} \tag{5. 22}$$

Recall the subset $\mathcal{U}$ defined in (2. 6) where $\alpha_i-\mathbf{N}_i/p<0$ for every $i \notin \mathcal{U}$.

Note that $\lambda_m \le \lambda_\ell$ when $\ell \le m$. For r sufficiently close to 1, we have

$$
\begin{array}{ll}
& \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i=1}^{\ell-1} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta} \prod_{i=\ell}^{m} \left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}} \\[2ex]
\le & \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i\in\mathcal{U}\cap\{1,\dots,\ell-1\}} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta} \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i\in\mathcal{U}\cap\{\ell,\dots,m\}} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}} \\
& \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i\in\mathcal{U}^c\cap\{1,\dots,\ell-1\}} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}} \prod_{i\in\mathcal{U}^c\cap\{\ell,\dots,m\}} \left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}} \\[2ex]
\le & \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i\in\mathcal{U}\cap\{1,\dots,m\}} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta}.
\end{array}
\tag{5. 23}
$$

By bringing the estimates in (5. 22)-(5. 23) back to (5. 20), we find

$$
\begin{array}{ll}
& (\lambda_m)^{\frac{1}{qr}\sum_{i=1}^{m}\mathbf{N}_i-\gamma} (\lambda_\ell)^{\frac{p-1}{pr}\sum_{i=1}^{\ell}\mathbf{N}_i-\delta} \prod_{i=1}^{n} (\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{m}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}} \prod_{i=1}^{\ell}\left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i} \\[2ex]
= & \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i=1}^{\ell-1} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta} \prod_{i=l}^{m} \left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}} \\
& \prod_{i=m+1}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)} \prod_{i=1}^{\ell-1}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i} \\[2ex]
\le & \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\sum_{i\in\mathcal{U}\cap\{1,2,\dots,m\}} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta} \prod_{i=1}^{\ell}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i} \prod_{i=m+1}^{n}\left(\frac{\lambda_i}{\lambda_m}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{array}
\tag{5. 24}
$$

Recall that $\delta > 0$ satisfies the first strict inequality in (2. 6). From (5. 1) and (1. 6), we also have $\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i < \alpha_i < \mathbf{N}_i$ for every $i = 1, 2, \dots, n$. Define implicitly $0 \le \vartheta_1 \le \vartheta_2 \le 1$ by requiring $\lambda_\ell = (\lambda_n)^{\vartheta_1}$ and $\lambda_m = (\lambda_n)^{\vartheta_2}$.

For r sufficiently close to 1, we have

$$
\begin{array}{l}
\vartheta_1\left[\frac{\mathbf{N}_1}{r}-\alpha_1+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_1\right] + (1-\vartheta_2)\left(\alpha_n-\mathbf{N}_n\left(\frac{1}{p}-\frac{1}{q}\right)\right) \\[2ex]
+ (\vartheta_2-\vartheta_1)\left[\sum_{i\in\mathcal{U}\cup\{1,2,\dots,m\}} \frac{\mathbf{N}_i}{p}-\alpha_i-\left(1-\frac{1}{r}\right)\frac{\mathbf{N}_i}{q}+\delta\right] > 0.
\end{array}
\tag{5. 25}
$$

The estimate in (5. 25) implies that (5. 24) is bounded by $\mathfrak{B}_{p\,q\,r\,\alpha\,\gamma\,\delta}(\lambda_n)^{\varepsilon}$ for some $\varepsilon = \varepsilon(p\,q\,r\,\alpha\,\gamma\,\delta) > 0$.

On the other hand, suppose $m \le \ell$. The last line of (5. 20) can be rewritten as

$$
\begin{aligned}
&(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}(\lambda_\ell)^{\frac{p-1}{pr}\sum_{i=1}^{\ell}\mathbf{N}_i-\delta}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^{m-1}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^{\ell}\left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i}\\
&= (\lambda_\ell)^{\frac{1}{r}\left[\frac{1}{q}+\frac{p-1}{p}\right]\sum_{i=1}^{m-1}\mathbf{N}_i-(\gamma+\delta)}\left(\frac{\lambda_m}{\lambda_\ell}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\prod_{i=1}^{n}(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{\ell}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{1}{\lambda_i}\right)^{\frac{1}{r}\left[\frac{1}{q}+\frac{p-1}{p}\right]\mathbf{N}_i}\\
&= (\lambda_\ell)^{\frac{1}{r}\left[\frac{1}{q}+\frac{p-1}{p}\right]\sum_{i=1}^{m-1}\mathbf{N}_i+\sum_{i=m}^{n}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)-(\gamma+\delta)}\left(\frac{\lambda_m}{\lambda_\ell}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\\
&\quad\prod_{i=m}^{n}\left(\frac{\lambda_i}{\lambda_\ell}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{\ell}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&= \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\prod_{i=m}^{n}\left(\frac{\lambda_i}{\lambda_\ell}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{\ell}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i} \qquad \text{by (5. 21)}\\
&= \left(\frac{\lambda_m}{\lambda_\ell}\right)^{\frac{1}{qr}\sum_{i=1}^{m-1}\mathbf{N}_i-\gamma}\prod_{i=\ell+1}^{n}\left(\frac{\lambda_i}{\lambda_\ell}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=m}^{\ell}\left(\frac{\lambda_i}{\lambda_\ell}\right)^{\alpha_i-\frac{q-1}{q}\mathbf{N}_i+\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}\prod_{i=1}^{m-1}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&= \prod_{i=\ell+1}^{n}\left(\frac{\lambda_i}{\lambda_\ell}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^{m}\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&\quad\left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i=1}^{m-1}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i+\gamma}\prod_{i=m}^{\ell}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}.
\end{aligned}
\tag{5. 26}
$$

Recall the subset $\mathcal{V}$ defined in (2. 6) where $\alpha_i - \mathbf{N}_i\left(\frac{q-1}{q}\right) < 0$ for every $i \notin \mathcal{V}$.

Note that $\lambda_\ell \le \lambda_m$ when $m \le \ell$. For r sufficiently close to 1, we find

$$
\begin{aligned}
&\left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i=1}^{m-1}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i+\gamma}\prod_{i=m}^{\ell}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}\\
&\le \left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i\in\mathcal{V}\cap\{1,\dots,m-1\}}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i+\gamma}\left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i\in\mathcal{V}\cap\{m,\dots,\ell\}}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}\\
&\quad\left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i\in\mathcal{V}^c\cap\{1,\dots,m-1\}}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}\prod_{i\in\mathcal{V}^c\cap\{m,\dots,\ell\}}\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}\\
&\le \left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i\in\mathcal{V}\cap\{1,\dots,\ell\}}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i+\gamma}.
\end{aligned}
\tag{5. 27}
$$

By bringing the estimates in (5. 26)-(5. 27) back to (5. 20), we find

$$
\begin{aligned}
&(\lambda_m)^{\frac{1}{qr}\sum_{i=1}^m \mathbf{N}_i-\gamma}(\lambda_\ell)^{\frac{p-1}{pr}\sum_{i=1}^\ell \mathbf{N}_i-\delta}\prod_{i=1}^n(\lambda_i)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^m\left(\frac{1}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{qr}}\prod_{i=1}^\ell\left(\frac{1}{\lambda_i}\right)^{\frac{p-1}{pr}\mathbf{N}_i}\\
&= \prod_{i=\ell+1}^n\left(\frac{\lambda_i}{\lambda_\ell}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{i=1}^m\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\\
&\quad\left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i=1}^{m-1}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i+\gamma}\prod_{i=m}^\ell\left(\frac{\lambda_\ell}{\lambda_i}\right)^{\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i}\\
&\le \left(\frac{\lambda_\ell}{\lambda_m}\right)^{\sum_{i\in\mathcal{V}\cap\{1,\dots,\ell\}}\frac{q-1}{q}\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\frac{p-1}{p}\mathbf{N}_i+\gamma}\prod_{i=1}^m\left(\frac{\lambda_m}{\lambda_i}\right)^{\frac{\mathbf{N}_i}{r}-\alpha_i+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i}\prod_{i=\ell+1}^n\left(\frac{\lambda_i}{\lambda_\ell}\right)^{\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)}.
\end{aligned}
\tag{5. 28}
$$

Recall that $\gamma > 0$ satisfies the second strict inequality in (2. 6). From (5. 1) and (1. 6), we also have $\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_i < \alpha_i < \mathbf{N}_i$ for every $i = 1, 2, \dots, n$. Define implicitly $0 \le \vartheta_1 \le \vartheta_2 \le 1$ by requiring $\lambda_m = (\lambda_n)^{\vartheta_1}$ and $\lambda_\ell = (\lambda_n)^{\vartheta_2}$. For r sufficiently close to 1, we have

$$
\begin{aligned}
&\vartheta_1\left[\frac{\mathbf{N}_1}{r}-\alpha_1+\left(1-\frac{1}{r}\right)\left(\frac{1}{p}-\frac{1}{q}\right)\mathbf{N}_1\right] + (1-\vartheta_2)\left(\alpha_n-\mathbf{N}_n\left(\frac{1}{p}-\frac{1}{q}\right)\right)\\
&+ (\vartheta_2-\vartheta_1)\left[\sum_{i\in\mathcal{V}\cap\{1,\dots,\ell\}}\left(\frac{q-1}{q}\right)\mathbf{N}_i-\alpha_i-\left(1-\frac{1}{r}\right)\left(\frac{p-1}{p}\right)\mathbf{N}_i+\gamma\right] > 0.
\end{aligned}
\tag{5. 29}
$$

The estimate in (5. 29) implies that (5. 28) is bounded by $\mathfrak{B}_{p\,q\,r\,\alpha\,\gamma\,\delta}(\lambda_n)^{\varepsilon}$ for some $\varepsilon = \varepsilon(p, q, r, \alpha, \gamma, \delta) > 0$. □

6 One-weight inequality on product spaces

In this section, we consider the special case of balance indices:

$$
\frac{\alpha_i}{\mathbf{N}_i} = \frac{1}{p}-\frac{1}{q}, \qquad i = 1, 2, \dots, n. \tag{6. 1}
$$

Consequently, we must have $\frac{\alpha_i}{\mathbf{N}_i} = \frac{\alpha}{\mathbf{N}}$ for every $i = 1, 2, \dots, n$. Recall (2. 3). $\frac{\alpha}{\mathbf{N}} = \frac{1}{p}-\frac{1}{q}$ implies $\gamma + \delta = 0$. Consider $\omega(x) = |x|^{-\gamma}, \sigma(x) = |x|^{\delta}$ for $x \neq 0$ and $\gamma, \delta \in \mathbb{R}$. Our assertion reduces to the one-weight situation: $\omega = \sigma$.

Write $x = (x_i, x_i^\dagger) \in \mathbb{R}^{\mathbf{N}_i} \times \mathbb{R}^{\mathbf{N}-\mathbf{N}_i}$ and $\mathbf{Q}_i^\dagger = \bigotimes_{i\neq j}\mathbf{Q}_i$ for $i = 1, 2, \dots, n$. Shrink $\mathbf{Q}_i^\dagger$ to $x_i^\dagger$ in (2. 1). By applying the Lebesgue Differentiation Theorem, we find

$$
\left\{\frac{1}{|\mathbf{Q}_i|}\int_{\mathbf{Q}_i}\omega^q(x_i, x_i^\dagger)dx_i\right\}^{\frac{1}{q}}\left\{\frac{1}{|\mathbf{Q}_i|}\int_{\mathbf{Q}_i}\left(\frac{1}{\omega}\right)^{\frac{p}{p-1}}(x_i, x_i^\dagger)dx_i\right\}^{\frac{p-1}{p}} < \infty \tag{6. 2}
$$

for every $\mathbf{Q}_i \subset \mathbb{R}^{\mathbf{N}_i}$ and $x_i^\dagger \in \mathbb{R}^{\mathbf{N}-\mathbf{N}_i}$, $i = 1, 2, \dots, n$.

Observe that (6. 1)-(6. 2) are sufficient conditions of **Muckenhoupt-Wheeden theorem** [5] which implies

$$\begin{aligned}&\left\{\int_{\mathbb{R}^{\mathbf{N}_i}}\left\{\int_{\mathbb{R}^{\mathbf{N}_i}} f(y_i,x_i^\dagger)\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i} dy_i\right\}^q \omega^q(x_i,x_i^\dagger)dx_i\right\}^{\frac{1}{q}}\\ &\qquad\leq \mathfrak{B}_{p\,q\,\omega}\left\{\int_{\mathbb{R}^{\mathbf{N}_i}}\Big(f\omega\Big)^p(x_i,x_i^\dagger)dx_i\right\}^{\frac{1}{p}}\end{aligned} \tag{6. 3}$$

for $1<p<q<\infty$ and $x_i^\dagger\in\mathbb{R}^{\mathbf{N}-\mathbf{N}_i},\ i=1,2,\ldots,n$.

By using (6. 3), we have

$$\begin{aligned}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(\omega\mathbf{I}_\alpha f\Big)^q(x)dx\right\}^{\frac{1}{q}} &= \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)\prod_{i=1}^n\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy\right\}^q\omega^q(x)dx\right\}^{\frac{1}{q}}\\ &\leq \mathfrak{B}_{p\,q\,\omega}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_i}}\left\{\int_{\mathbb{R}^{\mathbf{N}_i}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_i}} f(x_i,y_i^\dagger)\prod_{j\neq i}\left[\frac{1}{|x_j-y_j|}\right]^{\mathbf{N}_j-\alpha_j}dy_i^\dagger\right\}^p\omega^p(x_i,x_i^\dagger)dx_i\right\}^{\frac{q}{p}}dx_i^\dagger\right\}^{\frac{1}{q}}\\ &\leq \mathfrak{B}_{p\,q\,\omega}\left\{\int_{\mathbb{R}^{\mathbf{N}_i}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_i}}\left\{\int_{\mathbb{R}^{\mathbf{N}-\mathbf{N}_i}} f(x_i,y_i^\dagger)\prod_{j\neq i}\left[\frac{1}{|x_j-y_j|}\right]^{\mathbf{N}_j-\alpha_j}dy_i^\dagger\right\}^q\omega^q(x_i,x_i^\dagger)dx_i^\dagger\right\}^{\frac{p}{q}}dx_i\right\}^{\frac{1}{p}}\\ &\qquad\vdots \qquad\qquad\qquad\qquad \text{by Minkowski integral inequality}\\ &\leq \mathfrak{B}_{p\,q\,\omega}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(f\omega\Big)^p(x)dx\right\}^{\frac{1}{p}},\qquad 1<p<q<\infty.\end{aligned} \tag{6. 4}$$

7 Proof of Theorem A*

Suppose that $\omega(x)=|x|^{-\gamma},\sigma(x)=|x|^{\delta}$ for $x\neq 0$ and $\gamma,\delta\in\mathbb{R}$ satisfy the multi-parameter Muckenhoupt characteristic in (2. 1). Let $\{1,2,\ldots n\}=\mathbf{U}\cup\mathbf{V}$ where

$$\mathbf{U} = \left\{i\in\{1,2,\ldots,n\}:\ \frac{\alpha_i}{\mathbf{N}_i}=\frac{1}{p}-\frac{1}{q}\right\},\qquad \mathbf{V} = \left\{i\in\{1,2,\ldots,n\}:\ \frac{\alpha_i}{\mathbf{N}_i}>\frac{1}{p}-\frac{1}{q}\right\}. \tag{7. 1}$$

Define

$$\begin{aligned}\alpha_{\mathbf{U}} &= \sum_{i\in\mathbf{U}}\alpha_i,\quad \mathbf{Q}_{\mathbf{U}}=\bigotimes_{i\in\mathbf{U}}\mathbf{Q}_i,\quad \mathbb{R}^{\mathbf{N}_{\mathbf{U}}}=\bigotimes_{i\in\mathbf{U}}\mathbb{R}^{\mathbf{N}_i},\\ \alpha_{\mathbf{V}} &= \sum_{i\in\mathbf{V}}\alpha_i,\quad \mathbf{Q}_{\mathbf{V}}=\bigotimes_{i\in\mathbf{V}}\mathbf{Q}_i,\quad \mathbb{R}^{\mathbf{N}_{\mathbf{V}}}=\bigotimes_{i\in\mathbf{V}}\mathbb{R}^{\mathbf{N}_i}.\end{aligned} \tag{7. 2}$$

Write $x=(x_{\mathbf{U}},x_{\mathbf{V}})\in\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}\times\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}$. Denote the cardinality of $\mathbf{U}$ and $\mathbf{V}$ by $|\mathbf{U}|$ and $|\mathbf{V}|$.

Consider $\mathbf{Q}_i$ centered on the origin of $\mathbb{R}^{\mathbf{N}_i}$ for $i \in \mathbf{U}$. Shrink $\mathbf{Q}_i, i \in \mathbf{U}$ to 0 in (2. 1). By applying Lebesgue Differentiation Theorem, we have

$$\begin{aligned}&\sup_{\mathbf{Q_V}\subset\mathbb{R}^{\mathbf{N_V}}} \prod_{i\in\mathbf{V}} |\mathbf{Q}_i|^{\frac{\alpha_i}{\mathbf{N}_i}-\frac{1}{p}+\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q_V}|}\int_{\mathbf{Q_V}} \left(\frac{1}{|x_\mathbf{V}|}\right)^{q\gamma} dx_\mathbf{V}\right\}^{\frac{1}{q}} \left\{\frac{1}{|\mathbf{Q_V}|}\int_{\mathbf{Q_V}} \left(\frac{1}{|x_\mathbf{V}|}\right)^{\frac{\delta p}{p-1}} dx_\mathbf{V}\right\}^{\frac{p-1}{p}} \\ &\leq \mathbf{A}^{\alpha}_{pq}\left(|x|^{-\gamma},|x|^{\delta}\right) < \infty.\end{aligned} \tag{7. 3}$$

Remark 7.1 *By carrying out the same estimates in section 4, we find (7. 3) implying (2. 2)-(2. 6) with $\alpha, n, \mathbf{N}$ replaced by $\alpha_\mathbf{V}$, $|\mathbf{V}|$, $\mathbf{N_V}$ respectively.*

Proposition 7.1 *Suppose $\omega(x_\mathbf{V}) = |x_\mathbf{V}|^{-\gamma}, \sigma(x_\mathbf{V}) = |x_\mathbf{V}|^{\delta}$ for $x_\mathbf{V} \neq 0$ and $\gamma, \delta \in \mathbb{R}$ satisfying (7. 3). For a.e $x_\mathbf{U} \in \mathbb{R}^{\mathbf{N_U}}$, we have*

$$\begin{aligned}&\left\{\int_{\mathbb{R}^{\mathbf{N_V}}}\left\{\int_{\mathbb{R}^{\mathbf{N_V}}} f(x_\mathbf{U},y_\mathbf{V}) \prod_{i\in\mathbf{V}}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i} dy_\mathbf{V}\right\}^q \omega^q(x_\mathbf{V})dx_\mathbf{V}\right\}^{\frac{1}{q}} \\ &\leq \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N_V}}}\left(f(x_\mathbf{U},x_\mathbf{V})\right)^p \sigma^p(x_\mathbf{V})dx_\mathbf{V}\right\}^{\frac{1}{p}}, \qquad 1 < p \leq q < \infty.\end{aligned} \tag{7. 4}$$

Proof: Let $|\mathbf{V}| = 1$. **Theorem A** by Stein and Weiss [3] together with **Remark 7.1** imply (7. 4).

Consider $|\mathbf{V}| \geq 2$. By applying **Principal Lemma** in the beginning of section 5, we have $\omega(x_\mathbf{V}) = |x_\mathbf{V}|^{-\gamma}, \sigma(x_\mathbf{V}) = |x_\mathbf{V}|^{\delta}$ satisfying the estimate in (5. 2)-(5. 3) for every $\mathbf{Q_V} \subset \mathbb{R}^{\mathbf{N_V}}$ whereas $\alpha_i > \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)$, $i \in \mathbf{V}$.

Denote $\mathbf{t}$ as a $|\mathbf{V}|$-tuple $\left(2^{-t_1}, 2^{-t_2}, \ldots, 2^{-t_{|\mathbf{V}|}}\right)$. Recall $\mathbf{A}^{\alpha}_{pqr}(\mathbf{t}\colon \omega,\sigma)$ defined in (3. 6). We find

$$\sum_{\mathbf{t}} \mathbf{A}^{\alpha_\mathbf{V}}_{pqr}\left(\mathbf{t} : |x_\mathbf{V}|^{-\gamma}, |x_\mathbf{V}|^{\delta}\right) < \infty \tag{7. 5}$$

for some $r = r(p,q,\alpha,\gamma,\delta) > 1$. This is the desired summability in (3. 14). □

Remark 7.2 *Note that $\mathbf{V} = \{1,2,\ldots,n\}$ when $1 < p = q < \infty$.* **Proposition 7.1** *already implies* **Theorem A***.

For the remaining section, we assert $1 < p < q < \infty$ only.

In the case of $\gamma > 0, \delta > 0$, we observe

$$\omega(x) = |x|^{-\gamma} \leq |x_\mathbf{V}|^{-\gamma} = \omega(x_\mathbf{V}), \qquad \sigma(x_\mathbf{V}) = |x_\mathbf{V}|^{\delta} \leq |x|^{\delta} = \sigma(x). \tag{7. 6}$$

From (7. 6), we have

$$\left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(\omega\mathbf{I}_\alpha f\Big)^q(x)dx\right\}^{\frac{1}{q}} \leq \left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{\mathbb{R}^{\mathbf{N}}} f(y)\prod_{i=1}^n\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy\right\}^q\omega^q\left(x_{\mathbf{V}}\right)dx\right\}^{\frac{1}{q}}$$

$$\leq \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}} f\left(y_{\mathbf{U}},x_{\mathbf{V}}\right)\prod_{i\in\mathbf{U}}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy_{\mathbf{U}}\right\}^p\sigma^p\left(x_{\mathbf{V}}\right)dx_{\mathbf{V}}\right\}^{\frac{q}{p}}dx_{\mathbf{U}}\right\}^{\frac{1}{q}}$$

by **Proposition 7.1**

$$\leq \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}} f\left(y_{\mathbf{U}},x_{\mathbf{V}}\right)\prod_{i\in\mathbf{U}}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy_{\mathbf{U}}\right\}^q dx_{\mathbf{U}}\right\}^{\frac{p}{q}}\sigma^p\left(x_{\mathbf{V}}\right)dx_{\mathbf{V}}\right\}^{\frac{1}{p}}$$

by Minkowski integral inequality

$$\leq \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\iint_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}\times\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}}\Big(f\left(x_{\mathbf{U}},x_{\mathbf{V}}\right)\Big)^p\sigma^p\left(x_{\mathbf{V}}\right)dx_{\mathbf{U}}dx_{\mathbf{V}}\right\}^{\frac{1}{p}} \qquad \text{by (6. 2)-(6. 4) with } \omega=1$$

$$\leq \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(f\sigma\Big)^p(x)dx\right\}^{\frac{1}{p}}, \qquad 1<p<q<\infty. \tag{7. 7}$$

Consider $\gamma\geq 0,\delta\leq 0$ or $\gamma\leq 0,\delta\geq 0$. Note that it is suffice to study one of these two cases because $\mathbf{I}_\alpha$ is self-adjoint and

$$\Big\|\omega\mathbf{I}_\alpha f\Big\|_{\mathbf{L}^q(\mathbb{R}^{\mathbf{N}})} \lesssim \Big\|f\sigma\Big\|_{\mathbf{L}^p(\mathbb{R}^{\mathbf{N}})} \qquad \text{if and only if} \qquad \Big\|\sigma^{-1}\mathbf{I}_\alpha g\Big\|_{\mathbf{L}^{\frac{p}{p-1}}(\mathbb{R}^{\mathbf{N}})} \lesssim \Big\|g\omega^{-1}\Big\|_{\mathbf{L}^{\frac{q}{q-1}}(\mathbb{R}^{\mathbf{N}})}. \tag{7. 8}$$

Let $\gamma\geq 0,\delta\leq 0$. Suppose f supported in the region where $|x_{\mathbf{U}}|\leq|x_{\mathbf{V}}|$. By using (7. 6) and a repeat estimate in (7. 7), we obtain

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^{\mathbf{N}}}\left\{\int_{|y_{\mathbf{U}}|\leq|y_{\mathbf{V}}|} f(y)\prod_{i=1}^n\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy\right\}^q\omega^q(x)dx\right\}^{\frac{1}{q}}\\
&\leq \left\{\iint_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}\times\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}}\left\{\iint_{|y_{\mathbf{U}}|\leq|y_{\mathbf{V}}|} f\left(y_{\mathbf{U}},y_{\mathbf{V}}\right)\prod_{i\in\mathbf{U}\cup\mathbf{V}}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy_{\mathbf{U}}dy_{\mathbf{V}}\right\}^q\omega^q(x_{\mathbf{V}})dx\right\}^{\frac{1}{q}}\\
&\leq \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\iint_{|x_{\mathbf{U}}|\leq|x_{\mathbf{V}}|}\Big(f(x_{\mathbf{U}},x_{\mathbf{V}})\Big)^p\sigma^p(x_{\mathbf{V}})dx_{\mathbf{U}}dx_{\mathbf{V}}\right\}^{\frac{1}{p}}\\
&\leq \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}}}\Big(f\sigma\Big)^p(x)dx\right\}^{\frac{1}{p}}.
\end{aligned}\tag{7. 9}$$

The last inequality holds in (7. 9) because $\sigma(x_{\mathbf{V}})=|x_{\mathbf{V}}|^\delta\approx|x|^\delta=\sigma(x)$ for $|x_{\mathbf{U}}|\leq|x_{\mathbf{V}}|$.

On the other hand, suppose f supported in the region $|x_{\mathbf{U}}| > |x_{\mathbf{V}}|$. Recall that $\gamma \geq 0, \delta \leq 0$ satisfy (2. 2)-(2. 4). In particular,

$$\gamma + \delta \;=\; \sum_{i=1}^{n} \alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) \;=\; \sum_{i\in\mathbf{V}} \alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) \qquad \text{by (7. 1)} \tag{7. 10}$$

and

$$\alpha_i - \frac{\mathbf{N}_i}{p} \;<\; \delta \;\leq\; 0 \quad \text{for every} \quad i \in \{1,2,\ldots,n\} = \mathbf{U}\cup\mathbf{V}. \tag{7. 11}$$

By putting together (7. 10) and (7. 11), we find

$$0 \;\leq\; \gamma+\delta \;=\; \sum_{i\in\mathbf{V}} \alpha_i - \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right) \;<\; \frac{\mathbf{N}_{\mathbf{V}}}{q}, \qquad 0 \;<\; \mathbf{N}_{\mathbf{V}}\left(\frac{p-1}{p}\right). \tag{7. 12}$$

Proposition 7.2 *Let* $\rho\,(x_{\mathbf{V}}) = |x_{\mathbf{V}}|^{-(\gamma+\delta)}$, $\eta\,(x_{\mathbf{V}}) \equiv 1$. *For a.e* $x_{\mathbf{U}} \in \mathbb{R}^{\mathbf{N}_{\mathbf{U}}}$, *we have*

$$\begin{aligned} &\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}} \left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}} f(x_{\mathbf{U}},y_{\mathbf{V}}) \prod_{i\in\mathbf{V}} \left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i} dy_{\mathbf{V}}\right\}^q \rho^q(x_{\mathbf{V}})dx_{\mathbf{V}}\right\}^{\frac{1}{q}} \\ &\leq\; \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{V}}}} \Big(f(x_{\mathbf{U}},x_{\mathbf{V}})\Big)^p \eta^p(x_{\mathbf{V}})dx_{\mathbf{V}}\right\}^{\frac{1}{p}}, \qquad 1 \;<\; p \;\leq\; q \;<\; \infty. \end{aligned} \tag{7. 13}$$

Proof: Observe that (7. 10)-(7. 12) imply the constraints in (2. 2)-(2. 4) with $\gamma,\delta,\alpha,n,\mathbf{N}$ replaced by $\gamma+\delta, 0, \alpha_{\mathbf{V}}, |\mathbf{V}|, \mathbf{N}_{\mathbf{V}}$ respectively.

Let $|\mathbf{V}| = 1$. **Theorem A** by Stein and Weiss [3] shows that these constraints are sufficient conditions to imply (7. 13).

Consider $|\mathbf{V}| \geq 2$. By applying **Principal Lemma**, $\rho(x_{\mathbf{V}}) = |x_{\mathbf{V}}|^{-(\gamma+\delta)}, \eta(x_{\mathbf{V}}) \equiv 1$ satisfy the estimate in (5. 2)-(5. 3) for every $\mathbf{Q}_{\mathbf{V}} \subset \mathbb{R}^{\mathbf{N}_{\mathbf{V}}}$ whereas $\alpha_i > \mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right)$, $i \in \mathbf{V}$.

Denote $\mathbf{t}$ as a $|\mathbf{V}|$-tuple $\left(2^{-t_1}, 2^{-t_2}, \ldots, 2^{-t_{|\mathbf{V}|}}\right)$. We find

$$\sum_{\mathbf{t}} \mathbf{A}^{\alpha_{\mathbf{V}}}_{pqr}\left(\mathbf{t} : |x_{\mathbf{V}}|^{-(\gamma+\delta)}, 1\right) \;<\; \infty \tag{7. 14}$$

for some $r = r(p,q,\alpha,\gamma,\delta) > 1$. Again, this is the desired summability in (3. 14). $\square$

Proposition 7.3 *Let* $\omega(x_{\mathbf{U}}) = \sigma\,(x_{\mathbf{U}}) = |x_{\mathbf{U}}|^{\delta}$. *For a.e* $x_{\mathbf{V}} \in \mathbb{R}^{\mathbf{N}_{\mathbf{V}}}$, *we have*

$$\begin{aligned} &\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}} \left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}} f(y_{\mathbf{U}},x_{\mathbf{V}}) \prod_{i\in\mathbf{U}} \left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i} dy_{\mathbf{U}}\right\}^q \omega^q(x_{\mathbf{U}})dx_{\mathbf{U}}\right\}^{\frac{1}{q}} \\ &\leq\; \mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}_{\mathbf{U}}}} \Big(f(x_{\mathbf{U}},x_{\mathbf{V}})\Big)^p \omega^p(x_{\mathbf{U}})dx_{\mathbf{U}}\right\}^{\frac{1}{p}}, \qquad 1 \;<\; p \;<\; q \;<\; \infty. \end{aligned} \tag{7. 15}$$

Proof: From (7. 1) and (7. 11), we find

$$-\delta+\delta \;=\; 0 \;=\; \sum_{i\in\mathbf{U}}\alpha_i-\mathbf{N}_i\left(\frac{1}{p}-\frac{1}{q}\right), \qquad -\delta \;<\; \frac{\mathbf{N}_i}{p}-\alpha \;=\; \frac{\mathbf{N}_i}{q} \quad \text{for} \quad i\in\mathbf{U}. \tag{7. 16}$$

These constraints are sufficient conditions for **Theorem A** on every subspace $\mathbb{R}^{\mathbf{N}_i}$ for $i\in\mathbf{U}$. The one-weight $\mathbf{L}^p\longrightarrow\mathbf{L}^q$-norm inequality in (7. 15) follows the iteration argument given in section 6. □

Let $\rho(x_\mathbf{V})=|x_\mathbf{V}|^{-(\gamma+\delta)}$ and $\sigma(x_\mathbf{U})=|x_\mathbf{U}|^\delta$ where $\gamma+\delta\geq 0$ and $\delta\leq 0$. Observe that

$$\omega(x) \;=\; |x|^{-\gamma} \;\leq\; \rho\,(x_\mathbf{V})\,\sigma\,(x_\mathbf{U})\,. \tag{7. 17}$$

We have

$$\begin{aligned}
&\left\{\int_{\mathbb{R}^\mathbf{N}}\left\{\int_{|y_\mathbf{U}|>|y_\mathbf{V}|}f(y)\prod_{i=1}^n\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy\right\}^q\omega^q(x)dx\right\}^{\frac{1}{q}}\\
&\leq\;\left\{\int_{\mathbb{R}^\mathbf{N}}\left\{\int_{|y_\mathbf{U}|>|y_\mathbf{V}|}f(y)\prod_{i=1}^n\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy\right\}^q\rho^q\,(x_\mathbf{V})\,\sigma^q\,(x_\mathbf{U})\,dx\right\}^{\frac{1}{q}} \qquad \text{by (7. 17)}\\
&=\;\left\{\iint_{\mathbb{R}^{\mathbf{N}_\mathbf{U}}\times\mathbb{R}^{\mathbf{N}_\mathbf{V}}}\left\{\iint_{|y_\mathbf{U}|>|y_\mathbf{V}|}f(y_\mathbf{U},y_\mathbf{V})\prod_{i\in\mathbf{U}\cup\mathbf{V}}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy_\mathbf{U}dy_\mathbf{V}\right\}^q\rho^q\,(x_\mathbf{V})\,\sigma^q\,(x_\mathbf{U})\,dx_\mathbf{U}dx_\mathbf{V}\right\}^{\frac{1}{q}}\\
&\leq\;\mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}_\mathbf{U}}}\left\{\int_{\mathbb{R}^{\mathbf{N}_\mathbf{V}}}\left\{\int_{|y_\mathbf{U}|>|x_\mathbf{V}|}f(y_\mathbf{U},x_\mathbf{V})\prod_{i\in\mathbf{U}}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy_\mathbf{U}\right\}^p dx_\mathbf{V}\right\}^{\frac{q}{p}}\sigma^q\,(x_\mathbf{U})\,dx_\mathbf{U}\right\}^{\frac{1}{q}}\\
&\qquad\qquad\text{by } \textbf{Proposition 7.2}\\
&\leq\;\mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^{\mathbf{N}_\mathbf{V}}}\left\{\int_{\mathbb{R}^{\mathbf{N}_\mathbf{U}}}\left\{\int_{|y_\mathbf{U}|>|x_\mathbf{V}|}f(y_\mathbf{U},x_\mathbf{V})\prod_{i\in\mathbf{U}}\left[\frac{1}{|x_i-y_i|}\right]^{\mathbf{N}_i-\alpha_i}dy_\mathbf{U}\right\}^q\sigma^q\,(x_\mathbf{U})\,dx_\mathbf{U}\right\}^{\frac{p}{q}}dx_\mathbf{V}\right\}^{\frac{1}{p}}\\
&\qquad\qquad\text{by Minkowski integral inequality}\\
&\leq\;\mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\iint_{|x_\mathbf{U}|>|x_\mathbf{V}|}\Big(f(x_\mathbf{U},x_\mathbf{V})\Big)^p\sigma^p\,(x_\mathbf{U})\,dx_\mathbf{U}dx_\mathbf{V}\right\}^{\frac{1}{p}} \qquad \text{by } \textbf{Proposition 7.3}\\
&\leq\;\mathfrak{B}_{p\,q\,\alpha\,\gamma\,\delta}\left\{\int_{\mathbb{R}^\mathbf{N}}\Big(f\sigma\Big)^p(x)dx\right\}^{\frac{1}{p}}, \qquad 1<p<q<\infty.
\end{aligned} \tag{7. 18}$$

The last inequality holds because $\sigma(x_\mathbf{U})\approx\sigma(x)$ for $|x_\mathbf{U}|>|x_\mathbf{V}|$.

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Westlake University, Hangzhou, 310010, China
email: wangzipeng@westlake.edu.cn