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Saturated sliding mode control with limited magnitude and rate

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Abstract: This study is concerned with the sliding mode control design subject to actuator saturation. The actuator saturation is formulated with the rate limitation and the standard magnitude limitation. Both the saturated sliding mode control approaches against the known and the unknown bounded uncertainties are discussed. The stability analysis for the resulting overall closed-loop systems is conducted. Moreover, several optimisation conditions are established to estimate the attraction domain of the saturated system under the designed controllers, by determining the invariant set of the system. Finally, simulation examples are shown to illustrate the proposed control design scheme.

1 Introduction

Actuator saturation is a common phenomenon in real control systems. When actuator force is reaching its physical limitation, it is said to be saturated. As a result, the corresponding output is no longer linear to the input. Therefore, it inevitably brings difficulties to the control design and stability analysis of the control systems. Generally, actuator saturation in each control channel includes magnitude limitation and rate limitation. The former refers to the position displacement between input and output signal magnitudes, while the latter refers to the velocity change. When focusing on the treatment of the non-linearity characteristics of the saturation, plenty works [1–10] have been developed to tackle with it. For example, Chen and Wang [11] employed quadratical and Lure's type Lyapunov functions to estimate the attraction domain based on circle and Popov criterions. The work in [2, 4] adopted the absolute stability analysis method to analyse the stability of the saturated system. Since circle and Popov criterions were used to deal with the general memoryless section bounded non-linearity, the specific characteristics of saturated non-linearity were ignored. As a result, it somehow leads to an over conservative estimation of the attraction domain of closed-loop system. To deal with the saturation segment, the work in [3, 12] proposed a new method by employing an auxiliary matrix to obtain a less conservative estimation of the attraction domain, and the stability conditions can be expressed as some linear matrix inequalities, which simplifies the synthesis of the controllers. Moreover, to deal with the uncertainties in the control systems, the $\mathscr{L}_2$ performance against the disturbances were considered in [13, 14] based on the type-2 fuzzy framework. Zhou *et al.* [7] generalised the method in [3, 15], which represented the saturation non-linearity in a politopic expression. Recently, a polynomial control law was designed in [6] for the resulting saturated system. The saturated non-linearity was dealt with in [1] by a convex combination of state feedback and auxiliary time-delay feedback.

As a flexible control strategy with strong robustness to the matched uncertainties and parameter variations, sliding mode control (SMC) is widely used in most practical systems with uncertainty-inevitable factor [16–23]. To sum up, the above works concerned about the closed-loop stability under linear feedback control mainly. Since the sliding mode controller is designed to force the state trajectories of the closed-loop system to reach a predesigned sliding surface in a finite time, the reaching law of the sliding surface is usually required to promise the rapidness. Thus, some large control forces always lead to actuator saturation. There are some explorations about saturated systems under SMC strategy, for example, the method based on the switched second-order SMC [24], the observer-based SMC [25], the adaptive SMC based on fuzzy rules [26], the method based on static anti-windup compensator [27] and others [28]. It is noteworthy that the work in [29] refined the SMC limits respected in a region of the state space by designing a specific sliding mode surface, based on algebraic Ricotta equation approach and parametric Lyapunov equation. Torchani *et al.* developed a new SMC design methodology in [30] for a class of linear system subjected to saturation and norm bounded uncertainty, however without the consideration of the norm unbounded uncertainty. Furthermore, as discussed in [22], the merit of adaptive SMC is that it counteracts the uncertainties and perturbations with a dynamic adaptation of control gain, which can be as small as possible. Zhu *et al.* employed some adaptive laws in [31] to estimate the disturbances and the attitude stabilisation of a spacecraft system was achieved, without the requirement of prior knowledge of inertia moment. Wu *et al.* in [32] concerned with event-triggered SMC for uncertain stochastic systems subject to limited communication capacity. However, the saturated properties as well as corresponding attraction domain analyses of the closed-loop system were ignored, and the specific SMC design method when considering actuator rate saturation was out of consideration.

In this paper, the aim is to propose a new saturated SMC design and stability analysis method for linear systems with uncertainties. In terms of the SMC design under actuator saturation against the uncertain disturbances, two cases of the uncertainty are discussed: the saturated SMC when the norm boundedness of the uncertainties is known, and the saturated adaptive SMC when the norm boundedness of the uncertainties is unknown. Furthermore, the attraction domain of the resulting closed-loop system is estimated under the both saturated control cases. Motivated by the work in [3], we set the matrix P which is contained in both the Lyapunov functions and the expressions of ellipsoids as an unknown variable. It would not influence the selection of Lyapunov functions, and a less conservative estimation of the attraction domain of the system could also be obtained. Based on the results in [33, 34], a set of solvable conditions used for the estimation are expressed in terms of a series of linear matrix inequalities. In this study, the main contributions are listed as follows:

(1) for a class of uncertain linear systems subjected to actuator magnitude and rate saturation, the structures of sliding mode controllers are designed based on a linear sliding mode surface which facilitates the implementation of the sliding mode controllers,

(2) the attraction domains subject to some designed conditions of the closed-loop SMC system are estimated under the actuator saturation limitation,
(3) a set of solvable conditions with less conservativeness are derived for the controller and the sliding surface design.

The rest sections of this paper are as follows. Section 2 describes the control problem of the linear system under actuator saturation. Section 3 addresses the analysis and design for the uncertain system with the norm boundedness of the uncertainties known, while Section 4 discusses the adaptive SMC synthesis for the uncertain system with the norm boundedness of the uncertainties unknown. Section 5 focuses on the SMC design subject to the actuator rate saturation. Section 6 exemplifies two numerical examples to validate the effectiveness of the proposed theory and approaches. Section 7 gives the concluding remarks.

Notations: Throughout the paper, the symbol $\| \cdot \|$ is used to denote the 2-norm of a vector or a matrix. $\mathrm{sgn}(s)$ denotes $[\mathrm{sgn}(s_1) \quad \mathrm{sgn}(s_2) \quad \dots \quad \mathrm{sgn}(s_n)]^{\mathrm{T}}$ in which for any $s_i, i = 1, 2, \dots n$,

$$\mathrm{sgn}(s_i) = \begin{cases} -1, \text{ if } s_i < 0, \\ [-1, 1], \text{ if } s_i = 0, \\ 1, \text{ if } s_i > 0\,. \end{cases}$$

$\bar{\sigma}(\Gamma)$ denotes the maximum eigenvalue of Γ. $\mathscr{E}(P, \rho)$ denotes an ellipsoid which could be expressed as $\{x(t) \in R: x^{\mathrm{T}}(t)Px(t) \le \rho\}$ with P a positive definite matrix and ρ the measurement of the volume of the ellipsoid. $\mathscr{V}$ represents a set satisfying $\{\theta \in \mathbb{R}^m: \theta_i = 1 \text{ or } 0\}$. There are 2^m elements in $\mathscr{V}$, and $\theta_i \in \mathscr{V}$ will be used to choose ith row of H and $\hat{H}$ to form a new matrix $M(\theta, H, \hat{H})$. $\sigma(\cdot)$ denotes the standard saturation function.

2 Problem description

Consider the following linear system with a series of uncertainties:

$$\dot{x}(t) = Ax(t) + B\sigma(u(t)) + Ew(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the plant state vector and $w(t) \in \mathbb{R}^l$ is the external disturbance or the system parametric perturbation. $\sigma(u(t)) \in \mathbb{R}^m$ denotes the actual control force under saturation, which will be designed subject to the limited magnitude and rate in each channel (detailed formulations can be seen in [3, 7]). In each control channel, we define

$$\begin{aligned} u &= [u_1 \quad u_2 \quad \cdots \quad u_m]^{\mathrm{T}}, \\ \sigma(u) &= [\sigma_1(u_1) \quad \sigma_2(u_2) \quad \cdots \quad \sigma_m(u_m)]^{\mathrm{T}}, \\ \sigma_i(u_i) &= \mathrm{sgn}(u_i) \min\{u_{i\max}, |u_i|\}, \end{aligned}$$

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$ and $E \in \mathbb{R}^{n \times l}$ are the system constant matrices.

Different from the analysis and synthesis of the saturated linear system discussed in [3], we will explore the stability analysis of the saturated system under the SMC strategy in this study, including:

(1) the stability analysis of the SMC-based system with actuator saturation (1),
(2) the finite-time SMC under the limited magnitude and rate of the actuator,
(3) the attraction domain analysis of the resulting closed-loop system against the uncertainty.

We introduce the following definitions and lemmas which are useful to derive the main results of this work.

Assumption 1: Assume $\| w(t) \| \le v$ with v being a constant.

Definition 1: Let the original state of system (1) be $x(t_0) = x_0 \in \mathbb{R}^n$, and the state trajectory is denoted as $\psi(t, x_0)$, then the attraction domain of the origin is $\mathscr{F} := \{x_0 \in \mathbb{R}^n: \lim_{t \to \infty} \psi(t, x_0) = 0\}$ [3].

Definition 2: For any positive-definite matrix $P \in \mathbb{R}^{n \times n}$, define a corresponding ellipsoid: $\mathscr{E}(P, \rho) = \{x(t) \in \mathbb{R}^n: x^{\mathrm{T}}(t)Px(t) \le \rho\}$. Consider the Lyapunov function $V(t) = x^{\mathrm{T}}(t)Px(t)$ for system (1), then the ellipsoid $\mathscr{E}(P, \rho)$ is said to be contractively invariant if $\dot{V}(t) < 0$ establishes for all $x(t) \in \mathscr{E}(P, \rho) \setminus \{0\}$ [3].

Definition 3: Let $X_R = \mathscr{E}(R, 1) = \{x \in \mathbb{R}^n: x^{\mathrm{T}}Rx \le 1\}$ with matrix $R > 0$ be an ellipsoid type set, which is a prescribed bounded convex set. Define $\kappa_R(\mathscr{S}) := \sup\{\kappa > 0: \kappa \cdot X_R \subset \mathscr{S}\}$ for a set $\mathscr{S} \subset \mathbb{R}^n$. If $\kappa_R(\mathscr{S}) \ge 1$, then $\kappa \cdot X_R \subset \mathscr{S}$ [3].

Lemma 1: Consider a Lyapunov functional $V(t) = x(t)^{\mathrm{T}}Q^{-1}x(t)$ with $Q = Q^{\mathrm{T}} > 0$, and a hyperplane $\mathscr{H} := \{x(t) | Cx(t) = r\}$ with $C \in \mathbb{R}^n$ and r being a non-zero scalar. Then the minimum of $V(t)$ along the hyperplane $\mathscr{H}$ is $\alpha_r = r^2/(CQC^{\mathrm{T}})$. Define the α-sublevel set of $V(t)$ as $\mathrm{lev}_\alpha V = \mathscr{E}(Q^{-1}, \alpha)$, and the region $R_r := \{x(t) | \, |C_{q_i}x(t)| \le \quad r_i, i = 1, \dots, n_q\}$. The necessary and sufficient condition of the $\mathrm{lev}_\alpha V$ being inside of the R_r is $\alpha \le \alpha_r$ with $\alpha_r = \min_{i = 1, \dots, n_q}(r_i^2/(C_{q,i}QC_{q,i}^{\mathrm{T}}))$ and $C_{q,i}$ denoting the ith column of $C_q \in \mathbb{R}^{n_q}$ [2].

Lemma 2: Assume that the matrix $J \in \mathbb{R}^{n \times m}$ be of full column rank ($n \geqslant m$), and there exists $Jx = b$ from some vectors b, then the linear equation concerned with the state vector x has unique solution $x = J_L^{-}b$ with $J_L^{-1} = (J^{\mathrm{T}}J)^{-1}J^{\mathrm{T}}$.

Remark 1: Since the system states which satisfy conditions described in Definitions 2 and 3 will be designed to remain inside of the contractively invariant set in subsequent time, Lemma 1 also provides a solution to estimate the attraction domain. Consequently, the estimation of the attraction domain is given after the controller synthesis in subsequent sections.

3 SMC synthesis under magnitude saturation

3.1 Saturated SMC scheme

We use the following well-studied sliding surface for the linear system (1):

$$s(t) = Gx(t), \tag{2}$$

where the matrix $G \in \mathbb{R}^{m \times n}$ is a given matrix which satisfies that the matrix GB be of non-singular. According to $\dot{s}(t) = 0$, the equivalent control can be computed as

$$u_e(t) = -(GB)^{-1}(GAx(t) - GEw(t))\,.$$

In this paper, we employ the exponential reaching law as follows [35]:

$$u_s(t) = -\mu\,\mathrm{sgn}(s(t)) - \Gamma s(t),$$

where μ is a given positive scalar, and $\Gamma \in \mathbb{R}^{m \times m}$ is a properly chosen matrix. If the uncertainty $w(t)$ is measurable in practice, the control law can be $u(t) = u_e(t) + u_s(t)$. However, the uncertainty is hard to maintain in most practical systems. Since in Assumption 1 $\| w(t) \| \le v$, the desired control strategies under actuator saturation will be processed considering the information of the bound v in the following context. First, we design the following sliding mode controller if the bound v is prior known:

$$\begin{aligned} u(t) = &-(GB)^{-1}GAx - (GB)^{-1}(\mu + \| GE \| v)\mathrm{sgn}(s(t)) \\ &-(GB)^{-1}\Gamma s(t)\,. \end{aligned} \tag{3}$$

Based on the controller (3), the analysis of the overall SMC-based system is of following two subsections.

3.2 Reachability analysis

Before presenting the reachability condition of the sliding surface (2), we introduce the following lemma.

Lemma 3: Consider the Lyapunov functional $V(s)$ of the sliding surface dynamics $\dot{s}(t)$. If the following inequality holds

$$\dot{V}(s) + \alpha V(s) + \beta V^{\gamma}(s) \le 0,$$

where $\alpha > 0$, $\beta > 0$ and $0 < \gamma < 1$ [36]. Then the sliding surface will converge to $s = 0$ in a finite time, and the settling time is determined by

$$T \le \frac{1}{\alpha(1-\gamma)} \ln \frac{\alpha V^{1-\gamma}(s_0) + \beta}{\beta}.$$

Under the SMC law in (3), the reachability of the surface is analysed in the following theorem.

Theorem 1: Consider the open-loop system (1) and the sliding surface (2). With the force of sliding mode controller (3), the state trajectories of the system (1) can reach the sliding mode surface $s(t) = 0$ in the finite time t_{ss}

$$t_{ss} \le \frac{2}{\bar{\sigma}(\Gamma)} \ln \frac{\bar{\sigma}(\Gamma) V^{\frac{1}{2}}(s_0) + \mu}{\mu}, \tag{4}$$

and maintain on it subsequently.

Proof: Select the Lyapunov functional $V_1(t) = \frac{1}{2} s^{\mathrm{T}}(t) s(t)$ for the sliding surface dynamics. One can easily obtain $\dot{V}_1(t) \le -\mu \parallel s(t) \parallel - \bar{\sigma}(\Gamma) \parallel s(t) \parallel^2$ under the SMC law (3). Thus, according to Lemma 3, the finite time t_{ss} can be derived like the expression in (4). Detailed proof procedure is omitted for space limitation. □

3.3 Stability analysis of the closed-loop system

According to the structure of the controller $u(t)$ in (3), we define the state set contained in the region

$$\begin{aligned} \mathscr{L}(G, \mu) &:= \{x \in \mathbb{R}^n : |h_i x(t) + \delta k_i \mathrm{sgn}(s(t))| \le u_{i\max}\}, \\ & i = 1, 2, \ldots m, \end{aligned} \tag{5}$$

where $H \triangleq (GB)^{-1}(GA + \Gamma G)$, $\delta \triangleq \mu + \parallel GE \parallel v$, $K \triangleq (GB)^{-1}$, h_i and k_i denote the ith vector of the matrices H and K, respectively, for simplicity. To facilitate the subsequent analysis, we define a new set $\mathscr{L}'(G, \mu)$

$$\begin{aligned} \mathscr{L}'(G, \mu) &:= \{x \in \mathbb{R}^n : |h_i x(t)| + |\delta k_i \mathrm{sgn}(s(t))| \le u_{i\max}\}, \\ & i = 1, 2, \ldots m, \end{aligned} \tag{6}$$

which fulfills $\mathscr{L}'(G, \mu) \subseteq \mathscr{L}(G, \mu)$.

Moreover, motivated by Hu *et al.* [3], we adopt an auxiliary parameter matrix $\hat{G} \in \mathbb{R}^{m \times n}$ such that the state set contained in the region $\mathscr{L}'(\hat{G}, \mu)$, which is helpful to obtain a less conservative invariant set (specific unitisation can be seen in Theorem 2). Accordingly, the definition of the parameters are of the following form: $\hat{H} \triangleq (\hat{G}B)^{-1}(\hat{G}A + \Gamma \hat{G})$, $\hat{\delta} \triangleq \mu + \parallel \hat{G}E \parallel v$, $\hat{K} \triangleq (\hat{G}B)^{-1}$, $\hat{h}_i$ and $\hat{k}_i$ denote the ith vector of the $\hat{H}$ and $\hat{K}$, respectively. The following theorem gives a stability criterion of the SMC-based system (1).

Theorem 2: Consider the closed-loop system (1) with the sliding mode controller (3). For a given set $\mathscr{E}(P, \rho)$, matrix $G \in \mathbb{R}^{m \times n}$, diagonal matrix $\Gamma \in \mathbb{R}^{m \times m}$ and positive scalar μ, if there exist matrix $\hat{G} \in \mathbb{R}^{m \times n}$, positive scalars λ and η such that

$$\mathscr{E}(P, \rho) \subset \mathscr{L}'(\hat{G}, \mu), \tag{7}$$

$$QX_1^{\mathrm{T}} + X_1 Q + \Pi_1 \Lambda \Pi_1^{\mathrm{T}} < 0, \quad \forall \theta \in \mathscr{V}, \tag{8}$$

where

$$\begin{aligned} X_1 &\triangleq A + BM_1(\theta, G, \hat{G}), \\ \Pi_1 &\triangleq [B \quad E \quad \beta_1 I \quad vI], \\ \Lambda &\triangleq \mathrm{diag}\{\lambda I, \quad \eta I, \quad \rho(2I - \lambda Q), \quad \rho(2I - \eta Q)\}^{-1}, \\ \beta_1 &\triangleq \sum_{i=1}^{m} \Big(\theta_i \delta \sum_{j=1}^{m} k_{ij} + (1 - \theta_i)\hat{\delta} \sum_{j=1}^{m} \hat{k}_{ij}\Big), \end{aligned}$$

then $\mathscr{E}(P, \rho)$ is the strictly invariant set of the closed-loop system (1).

Proof: According to the system (1) and the sliding mode controller (3), we consider Lyapunov function as $V_2(t) = x^{\mathrm{T}}(t) P x(t)$ with $P = Q^{-1}$, then

$$\begin{aligned} \dot{V}_2(t) &= x^{\mathrm{T}}(t) P \dot{x}(t) + \dot{x}^{\mathrm{T}}(t) P x(t) \\ &= 2x^{\mathrm{T}}(t) P [Ax(t) + Ew(t) \\ &\quad + B\sigma(-Hx(t) - \delta K \mathrm{sgn}(s(t)))] \\ &= 2x^{\mathrm{T}}(t) PAx(t) + 2x^{\mathrm{T}} PEw(t) \\ &\quad + \sum_{i=1}^{m} 2x^{\mathrm{T}}(t) Pb_i \sigma(-h_i x(t) - \delta k_i \mathrm{sgn}(s(t))). \end{aligned} \tag{9}$$

In term of the term $x^{\mathrm{T}}(t) Pb_i \sigma(-h_i x(t) - \delta k_i \mathrm{sgn}(s(t)))$, we consider $-\delta k_i \mathrm{sgn}(s(t)) \le \delta \sum_{j=1}^{m} |k_{ij}| = p_{i\max}$. Similarly, $-\hat{\delta}\hat{k}_i \mathrm{sgn}(s(t)) \le \hat{\delta} \sum_{j=1}^{m} |\hat{k}_{ij}| = \hat{p}_{i\max}$. Then there left four cases:

(1) If $2x^{\mathrm{T}}(t) Pb_i \ge 0$ and $\sigma(-h_i x(t) - \delta k_i \mathrm{sgn}(s(t))) \ge u_{i\max}$, then we note that

$$\begin{aligned} 2x^{\mathrm{T}}(t) Pb_i \sigma(u_i(t)) &= 2x^{\mathrm{T}}(t) Pb_i u_{i\max} \\ &\le 2x^{\mathrm{T}}(t) Pb_i (-h_i x(t) + p_{i\max}). \end{aligned}$$

(2) If $2x^{\mathrm{T}}(t) Pb_i \ge 0$ and $\sigma(-h_i x(t) - \delta k_i \mathrm{sgn}(s(t))) \le -u_{\max}$, then $\forall x \in \mathscr{E}(P, \rho) \setminus \{0\}$ and $-u_{i\max} \le -\hat{h}_i x(t) - \hat{\delta}\hat{k}_i \mathrm{sgn}(s(t))$, it follows that

$$\begin{aligned} 2x^{\mathrm{T}}(t) Pb_i \sigma(u_i(t)) &= 2x^{\mathrm{T}}(t) Pb_i (-u_{i\max}) \\ &\le 2x^{\mathrm{T}} Pb_i (-\hat{h}_i x(t) + \hat{p}_{i\max}). \end{aligned}$$

(3) If $2x^{\mathrm{T}}(t) Pb_i \le 0$ and $\sigma(-h_i x(t) - \delta k_i \mathrm{sgn}(s(t))) \ge u_{i\max}$, then $\forall x \in \mathscr{E}(P, \rho) \setminus \{0\}$ and $u_{i\max} \ge -\hat{h}_i x(t) - \hat{\delta}\hat{k}_i \mathrm{sgn}(s(t))$, it follows that

$$\begin{aligned} 2x^{\mathrm{T}}(t) Pb_i \sigma(u_i(t)) &= 2x^{\mathrm{T}}(t) Pb_i u_{i\max} \\ &\le 2x^{\mathrm{T}}(t) Pb_i (-\hat{h}_i x(t) + \hat{p}_{i\max}). \end{aligned}$$

(4) If $2x^{\mathrm{T}}(t) Pb_i \le 0$ and $\sigma(-h_i x(t) - \delta k_i \mathrm{sgn}(s(t))) \le -u_{i\max}$, then we note that

$$\begin{aligned} 2x^{\mathrm{T}}(t) Pb_i \sigma(u_i(t)) &= 2x^{\mathrm{T}}(t) Pb_i (-u_{i\max}) \\ &\le 2x^{\mathrm{T}}(t) Pb_i (-h_i x(t) + p_{i\max}). \end{aligned}$$

Summarising from the above discussion, we can obtain

$$\begin{aligned} 2x^{\mathrm{T}}(t) Pb_i \sigma(u_i(t)) \le \max \big\{ & 2x^{\mathrm{T}}(t) Pb_i (-h_i x(t) + p_{i\max}), \\ & 2x^{\mathrm{T}}(t) Pb_i (-\hat{h}_i x(t) + \hat{p}_{i\max}) \big\}. \end{aligned}$$

If there exists

$$x^{\mathrm{T}}(t)Pb_i(-h_ix(t)+p_{i\max}) \le x^{\mathrm{T}}(t)Pb_i(-\hat{h}_ix(t)+\hat{p}_{i\max}),$$

then we set $\theta_i = 0$, otherwise, $\theta_i = 1$. Then we define

$$\begin{aligned} M(\theta, G, \hat{G}) &= \begin{bmatrix} \theta_1(-h_1x(t)+p_{1\max}) \\ +(1-\theta_1)(-\hat{h}_1x(t)+\hat{p}_{1\max}) \\ \vdots \\ \theta_m(-h_mx(t)+p_{m\max}) \\ +(1-\theta_m)(-\hat{h}_mx(t)+\hat{p}_{m\max}) \end{bmatrix} \\ &= M_1(\theta, G, \hat{G})x(t) + M_2(\theta, G, \hat{G}), \end{aligned} \tag{10}$$

with

$$M_1(\theta, G, \hat{G}) = \begin{bmatrix} \theta_1(-h_1)+(1-\theta_1)(-\hat{h}_1) \\ \vdots \\ \theta_m(-h_m)+(1-\theta_m)(-\hat{h}_m) \end{bmatrix},$$
$$M_2(\theta, G, \hat{G}) = \begin{bmatrix} \theta_1p_{1\max}+(1-\theta_1)\hat{p}_{1\max} \\ \vdots \\ \theta_mp_{m\max}+(1-\theta_m)\hat{p}_{m\max} \end{bmatrix}.$$

Taking use of the inequality $a^{\mathrm{T}}b + b^{\mathrm{T}}a < (1/\eta)a^{\mathrm{T}}a + \eta b^{\mathrm{T}}b$ for positive scalar η and vectors a and b, we have

$$\begin{aligned} 2x^{\mathrm{T}}(t)PBM_2 &\le \frac{1}{\lambda}x^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) + \lambda M_2^{\mathrm{T}}M_2 \\ &\le \frac{1}{\lambda}x^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) + \lambda\beta_1^2, \end{aligned} \tag{11}$$

and

$$\begin{aligned} 2x^{\mathrm{T}}(t)PEw(t) &\le \frac{1}{\eta}x^{\mathrm{T}}(t)PEE^{\mathrm{T}}Px(t) + \eta w(t)^{\mathrm{T}}w(t) \\ &\le \frac{1}{\eta}x^{\mathrm{T}}(t)PEE^{\mathrm{T}}Px(t) + \eta v^2. \end{aligned} \tag{12}$$

Combining (9) and (10)–(12), one can obtain

$$\begin{aligned} \dot{V}_2(t) \le x^{\mathrm{T}}(t)\Big[&2P(A+BM_1(\theta, G, \hat{G}))^{\mathrm{T}} + \frac{1}{\lambda}PBB^{\mathrm{T}}P \\ &+\frac{1}{\eta}PEE^{\mathrm{T}}P\Big]x(t) + \lambda\beta_1^2 + \eta v^2. \end{aligned} \tag{13}$$

Further, considering $(\lambda Q)^{-1} > 2I - \lambda Q$, then from the inequality (8), we have

$$QX_1^{\mathrm{T}} + X_1Q + \Pi_1\Lambda'\Pi_1^{\mathrm{T}} < 0, \quad \forall\theta \in \mathscr{V},$$

with $\Lambda' = \mathrm{diag}\{(1/\lambda)I,\ (1/\eta)I,\ (\lambda/\rho)Q,\ (\eta/\rho)Q\}$, to which applying Schur complement, and being multiplied by Q^{-1} on both sides, we have

$$\begin{aligned} &(A+BM_1(\theta, G, \hat{G}))^{\mathrm{T}}P + P(A+BM_1(\theta, G, \hat{G})) \\ &+\frac{1}{\lambda}PBB^{\mathrm{T}}P + \frac{1}{\eta}PEE^{\mathrm{T}}P + (\lambda\beta_1^2+\eta v^2)P < 0. \end{aligned} \tag{14}$$

Combining (13) with (14), we can note that for $\forall x \in \mathscr{E}(P, \rho)\setminus\{0\}$, there exists

$$\dot{V}_2(t) < -\frac{\lambda\beta_1^2+\eta v^2}{\rho}x^{\mathrm{T}}(t)Px(t) + (\lambda\beta_1^2+\eta v^2) < 0.$$

This completes the proof. □

Based on the discussion in Remark 2, the following corollary can be developed which follows on Theorem 2.

Remark 2: From condition (8), it can be noticed that matrix G is coupled with Q, which leaves the obstacle to solve for a solution of G via standard calculation tools. To cope with the problem, we select the sliding mode surface parameter matrix as $G = \alpha(B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}$ with α being a scalar. As a result, $H = FA + \Gamma F$, $\delta = \mu + \alpha \parallel FE \parallel v$, $K = \alpha^{-1}I_m$, $\hat{H} = FA + \hat{\Gamma}F$, $\hat{\delta} = \mu + \hat{\alpha} \parallel FE \parallel v$, $\hat{K} = \hat{\alpha}^{-1}I_m$ with $F \triangleq (B^{\mathrm{T}}B)^{-1}B^{\mathrm{T}}$. Thus, the problem of solving matrix $\hat{G}$ can be transferred to the problem of solving $\hat{\Gamma}$ and $\hat{\alpha}$.

Corollary 1: Consider the closed-loop system (1) under the sliding mode controller (3), for a given set $\mathscr{E}(P, \rho)$, diagonal matrix $\Gamma \in \mathbb{R}^{m\times m}$ and positive scalar α. If there exist matrix $\hat{\Gamma} \in \mathbb{R}^{m\times m}$, positive scalars $\hat{\alpha}$, λ and η such that condition (7) holds and

$$QX_2^{\mathrm{T}} + X_2Q + \Pi_2\Lambda\Pi_2^{\mathrm{T}} < 0, \quad \forall\theta \in \mathscr{V} \tag{15}$$

where

$$\begin{aligned} X_2 &\triangleq A - B(FA - M_1'(\theta, \Gamma, \hat{\Gamma})), \\ M_1'(\theta, \Gamma, \hat{\Gamma}) &\triangleq \begin{bmatrix} \theta_1(-\Gamma F)_1 + (1-\theta_1)(-\hat{\Gamma}F)_1 \\ \vdots \\ \theta_m(-\Gamma F)_m + (1-\theta_m)(-\hat{\Gamma}F)_m \end{bmatrix}, \\ \Pi_2 &\triangleq \begin{bmatrix} B & E & \beta_2I & vI \end{bmatrix}, \\ \beta_2 &\triangleq \max\Big\{\frac{\mu}{\alpha}, \frac{\mu}{\hat{\alpha}}\Big\} + v \parallel FE \parallel, \end{aligned}$$

and $\hat{\alpha}$ is the auxiliary scalar corresponding α, then $\mathscr{E}(P, \rho)$ is a strictly invariant set of the system.

Proof: The proof is same as that of Theorem 2. □

Remark 3: Referring to Definitions 2 and 3, the states of the system which satisfies the conditions (7) and (8) are constrained in the contractively invariant set $\mathscr{E}(P, \rho)$ defined by the solution of the above two conditions, and they will remain inside of it in subsequent time. Corollary 1 is the expansion of Theorem 2, which simplifies the selection of the auxiliary parameters. It also provides a solution to estimate the attraction domain in virtue of the invariant set $\mathscr{E}(P, \rho)$.

3.4 Estimation of the attraction domain

In this research, we estimate the attraction domain of the plant state based on an ellipsoid type set. According to Definition 3, given that Corollary 1 provides the condition of the contractively invariant set, we can search for one with κ maximised. Therefore, above discussion can be represented as the following result, which is a convex optimisation problem:

$$\begin{aligned} &\sup_{P>0,\, \rho,\, \hat{\alpha},\, \hat{\Gamma}} \quad \kappa \\ &\text{s.t.(a) } \kappa X_R \subset \mathscr{E}(P, \rho), \\ &\text{(b) } X_2^{\mathrm{T}}P + PX_2 + \frac{1}{\lambda}PBB^{\mathrm{T}}P + \frac{1}{\eta}PEE^{\mathrm{T}}P \\ &\quad +(\lambda\beta_2^2+\eta v^2)P < 0, \forall\theta \in \mathscr{V}, \\ &\text{(c) } \mathscr{E}(P, \rho) \subset \mathscr{L}'(\hat{G}, \mu). \end{aligned} \tag{16}$$

According to Lemma 1, considering the limited control input $u_{i\max}$, the following condition can be derived based on the representation (c) in (16).

Lemma 4: Consider the parametric matrices presented in Theorem 2, the condition of the actuator saturation for the sliding mode controller (3) is of the following matrix inequalities for all $i \in [1, \cdots, m]$:

$$\begin{bmatrix} (u_{i\max} - \hat{p}_{i\max})^2 & (\hat{H}Q)_i \\ (\hat{H}Q)_i^{\mathrm{T}} & Q \end{bmatrix} \geq 0. \tag{17}$$

Proof: First, according to the definitions in (5) and (6), we have

$$\hat{h}_i x(t) + \hat{p}_{i\max} \leq |\hat{h}_i x(t)| + |\hat{\delta}\hat{k}_i \mathrm{sgn}(s(t))| \leq u_{i\max}$$
$$\Rightarrow \hat{h}_i x(t) \leq u_{i\max} - \hat{p}_{i\max}.$$

Considering the result given by Lemma 1 as boundary condition, then we have

$$x^* = \frac{(u_{i\max} - \hat{p}_{i\max})Q\hat{h}_i^{\mathrm{T}}}{\hat{h}_i Q \hat{h}_i^{\mathrm{T}}},$$
$$x^{*\mathrm{T}} Q^{-1} x^* = \frac{(u_{i\max} - \hat{p}_{i\max})^2}{\hat{h}_i Q \hat{h}_i^{\mathrm{T}}} = 1.$$

Thus, the inequality (17) can be derived. This completes the proof.□

To facilitate the calculation, we set $\gamma = 1/\kappa^2$, $W = \hat{H}Q$, and w_i denotes the ith row of matrix W. The optimisation problem can be transferred to searching the invariant set with γ minimised, that is

$$\begin{aligned} &\inf_{P>0,\,\rho,\,\hat{\alpha},\,\hat{\Gamma}} \gamma \\ &\text{s.t. (a)} \begin{bmatrix} \gamma & x^{\mathrm{T}} \\ x & Q \end{bmatrix} \geq 0, \\ &\text{(b) (15)}, \\ &\text{(c)} \begin{bmatrix} (u_{i\max} - \hat{p}_{i\max})^2 & w_i \\ w_i^{\mathrm{T}} & Q \end{bmatrix} \geq 0, \\ &i = 1, 2, \ldots m. \end{aligned} \tag{18}$$

Above all, the attraction domain of the plant states has been obtained by searching for the maximum estimate of the invariant set as far as possible. Above discussion has been represented in the form of optimisation problem (16) and corresponding specific linear matrix inequalities (18).

Remark 4: It is obvious that there are two unknown variables Q and $\hat{\Gamma}$ in $Q(B\hat{\Gamma}F)^{\mathrm{T}} = QF^{\mathrm{T}}\hat{\Gamma}^{\mathrm{T}}B^{\mathrm{T}}$, respectively, which lead to the unexpected bilinearlity. To treat the unexpected problem, let us define $\hat{Z} = QF^{\mathrm{T}}\hat{\Gamma}^{\mathrm{T}}$ with $QF^{\mathrm{T}}, \hat{Z} \in \mathbb{R}^{n\times m}$ and $\hat{\Gamma}^{\mathrm{T}} \in \mathbb{R}^{m\times m}$. Assume that QF^{T} be of full column rank, thus, according to Lemma 2, a solvable matrix form, i.e. $\hat{\Gamma}^{\mathrm{T}} = (QF^{\mathrm{T}})_L^{-1}\hat{Z}$, can be obtained for further calculation using standard calculation tools.

4 Adaptive SMC synthesis under magnitude saturation

In this section, adaptive control method is adopted to deal with the upper bound v when v is unknown in practice. The estimate of v is defined as $\hat{v}(t)$, and the estimation error is $\tilde{v}(t) = v - \hat{v}(t)$. The adaptive sliding mode controller is designed as follows:

$$\begin{aligned} u(t) = &-(GB)^{-1}GAx(t) - (GB)^{-1}(\mu + \| GE \| \hat{v}(t))\mathrm{sgn}(s(t)) \\ &-(GB)^{-1}\Gamma s(t), \end{aligned} \tag{19}$$

with the corresponding adaptive law

$$\dot{\hat{v}}(t) = \begin{cases} \bar{\hat{v}} \| s(t) \| \mathrm{sgn}(\| s(t) \| - \chi), & \text{if } \hat{v}(t) > \phi, \\ \varepsilon, \quad \text{if } \hat{v}(t) \leq \phi, \end{cases}$$

where $\bar{\hat{v}} > 0$, $\phi > 0$, $\chi > 0$ and $\varepsilon > 0$ are given adaptive parameters.

Remark 5: Once the sliding mode of $s(t)$ is developed, the adaptive law (20) will allow $\hat{v}(t)$ decline if $\| s(t) \| < \chi$, which promises the stability of the sliding mode while keeping $\hat{v}(t)$ at a minimum level. It also enables the adaptive law acquire enough gain to compensate the uncertainty, which promises the reachability of the sliding mode surface.

Remark 6: In terms of the selection of χ, it will be hard for $s(t)$ to reach the sliding surface or get into the layer boundary if χ is selected too small, it will lead to the persistent increasing of the $\hat{v}(t)$ and larger chattering. If χ is selected too large, it will lead to the poor ideality of the accuracy of the controller. Though the accuracy is not ideal enough, the system still could be ultimately uniformly bounded. So we prefer to select it larger rather than smaller than the real value. Meanwhile, it should satisfy that $\| s(t) \| < \chi$ when $\hat{v}(t) > \phi$.

4.1 Reachability analysis of the sliding surface

The following theorem states the reachability of the sliding surface (2) under the designed adaptive sliding mode controller (19).

Theorem 3: Consider the system (1) and the sliding surface (2), with the force of sliding mode controller (19). The state trajectories of the controlled system (1) could reach the sliding mode surface $s(t) = 0$ in a finite time

$$t_{sa} \leq \frac{2}{\underline{\sigma}(\Gamma)} \ln \frac{\underline{\sigma}(\Gamma)V^{\frac{1}{2}}(s_0) + \Psi_{\min}}{\Psi_{\min}}, \tag{20}$$

and maintain on it subsequently.

Proof: Assume that $0 < \hat{v}_m \leq \hat{v}(t) \leq \hat{v}_M$, with $\hat{v}_m$ and $\hat{v}_M$ being the unknown lower and upper bounds of $\hat{v}(t)$, respectively. As discussed in [37], the exact information of $\hat{v}_M$ and $\hat{v}_m$ is not required, and the corresponding Lyapunov function is selected as

$$V_3(t) = \frac{1}{2}s^{\mathrm{T}}(t)s(t) + \frac{1}{2\gamma}(\hat{v}(t) - \hat{v}_m)^2.$$

Then it follows that

$$\begin{aligned} \dot{V}_3(t) &= s^{\mathrm{T}}(t)\dot{s}(t) + \frac{1}{\gamma}(\hat{v}(t) - \hat{v}_m)\dot{\hat{v}}(t) \\ &= s^{\mathrm{T}}(t)(-(\mu + \| GE \| \hat{v}(t))\mathrm{sgn}(s(t)) + GEw(t) - \Gamma s(t)) \\ &\quad + \frac{1}{\gamma}(\hat{v}(t) - \hat{v}_m)\bar{\hat{v}} \| s(t) \| \mathrm{sgn}(\| s(t) \| - \chi) \\ &\leq -\mu \| s(t) \| + (v - \hat{v}(t)) \| GE \| \| s(t) \| - \underline{\sigma}(\Gamma) \| s(t) \|^2 \\ &\quad - \hat{v}_m \| GE \| \| s(t) \| + \hat{v}_m \| GE \| \| s(t) \| \\ &\quad + \frac{1}{\gamma}(\hat{v}(t) - \hat{v}_m)\bar{\hat{v}} \| s(t) \| \mathrm{sgn}(\| s(t) \| - \chi) \\ &= -(\mu - (v - \hat{v}_m) \| GE \|) \| s(t) \| - \underline{\sigma}(\Gamma) \| s(t) \|^2 \\ &\quad + \frac{1}{\gamma}\bar{\hat{v}} \| s(t) \| \mathrm{sgn}(\| s(t) \| - \chi)(\hat{v}(t) - \hat{v}_m) \\ &\quad - \| GE \| \| s(t) \| (\hat{v}(t) - \hat{v}_m) \\ &\leq -\min\{\sqrt{2}\Psi_1, \sqrt{2\gamma}\Psi_2\}\left(\frac{\| s(t) \|}{\sqrt{2}} + \frac{|\hat{v}(t) - \hat{v}_m|}{\sqrt{2\gamma}}\right) \\ &\quad - \underline{\sigma}(\Gamma) \| s(t) \|^2 \\ &\leq -\Psi_{\min}V_3^{1/2} - \underline{\sigma}(\Gamma) \| s(t) \|^2 < 0, \end{aligned}$$

with

$$\Psi_{\min} = \min\{\sqrt{2}\Psi_1, \sqrt{2\gamma}\Psi_2\}, \quad \Psi_1 = \mu - (v - \hat{v}_m) \| GE \|,$$
$$\Psi_2 = -\frac{1}{\gamma}\bar{\hat{v}} \| s(t) \| \mathrm{sgn}(\| s(t) \| - \chi) + \| GE \| \| s(t) \|.$$

According to Lemma 3, the finite time is

$$t_{sa} \le \frac{2}{\underline{\sigma}(\Gamma)} \ln \frac{\underline{\sigma}(\Gamma) V^{\frac{1}{2}}(s_0) + \Psi_{\min}}{\Psi_{\min}}.$$

Moreover, to make sure that both Ψ_1 and Ψ_2 are positive scalars, we should choose $\mu > (v - \hat{v}_m) \parallel GE \parallel$, $\gamma > \bar{\hat{v}} / \parallel GE \parallel$ such that $\Psi_1 > 0, \Psi_2 > 0$. This ends the proof. □

4.2 Stability analysis of the closed-loop system

According to (5) and (6), the definitions of the state sets are updated as follows:

$$\begin{aligned}\mathscr{L}(G,\tau) &:= \{x \in \mathbb{R}^n : |h_i x(t) + (\mu + \tau\hat{v}(t))k_i \mathrm{sgn}(s(t))| \\ &\le u_{i\max}\}, \quad i = 1, 2, \dots m,\\ \mathscr{L}'(G,\tau) &:= \{x \in \mathbb{R}^n : |h_i x(t)| + |(\mu + \tau\hat{v}(t))k_i \mathrm{sgn}(s(t))| \\ &\le u_{i\max}\}, \quad i = 1, 2, \dots m,\end{aligned}$$

with $\tau \triangleq \parallel GE \parallel$. The following theorem gives a stability criterion of the adaptive SMC-based system (1).

Theorem 4: Consider the closed-loop system in (1) under the sliding mode controller (19). Then for a given set $\mathscr{E}(P,\rho)$, matrix $G \in \mathbb{R}^{n\times m}$ and diagonal matrix $\Gamma \in \mathbb{R}^{m\times m}$, if there exist matrix $\hat{G} \in \mathbb{R}^{n\times m}$, positive scalars λ and η such that

$$\mathscr{E}(P,\rho) \subset \mathscr{L}'(\hat{G},\hat{\tau}), \tag{21}$$

$$QX_3^{\mathrm{T}} + X_3 Q + \Pi_3 \Lambda \Pi_3^{\mathrm{T}} < 0, \quad \forall \theta \in \mathscr{V}, \tag{22}$$

where

$$\begin{aligned}X_3 &\triangleq A + BN_1(\theta, G, \hat{G}),\\ \Pi_3 &\triangleq [B \quad E \quad \beta_3 I \quad \varpi I],\\ \beta_3 &\triangleq 2\mu \sum_{i=1}^{m} \left(\theta_i \sum_{j=1}^{m} k_{ij} + (1-\theta_i) \sum_{j=1}^{m} \hat{k}_{ij}\right),\\ \varpi &\triangleq \frac{\mu}{\parallel GE \parallel},\end{aligned}$$

then $\mathscr{E}(P,\rho)$ is a strictly invariant set of the system (1).

Proof: Based on the closed-loop system composed by the system (1) and the sliding mode controller (19), considering Lyapunov function as $V_4(t) = x^{\mathrm{T}}(t)Px(t)$, then we have

$$\begin{aligned}\dot{V}_4(t) &= x^{\mathrm{T}}(t)P\dot{x}(t) + \dot{x}^{\mathrm{T}}(t)Px(t)\\ &= 2x^{\mathrm{T}}(t)P[Ax(t) + B\sigma(-Hx(t)\\ &\quad -(\mu + \tau\hat{v}(t))K\mathrm{sgn}(s(t))) + Ew(t)]\\ &= 2x^{\mathrm{T}}(t)PAx(t) + 2x^{\mathrm{T}}(t)PEw(t)\\ &\quad \sum_{i=1}^{m} 2x^{\mathrm{T}}(t)Pb_i\sigma(-h_i x(t) - (\mu + \tau\hat{v}(t))k_i \mathrm{sgn}(s(t))).\end{aligned} \tag{23}$$

Since the upper bound of $\hat{v}(t)$ is selected as $\mu / \parallel GE \parallel$ to promise the reachability of the sliding mode, the upper bound of $\hat{v}(t)$ could be selected as $\mu / \parallel \hat{G}E \parallel$ correspondingly when the control law is $u(t) = -\hat{G}x(t) - (\mu + \hat{\tau}\hat{v}(t))\hat{K}\mathrm{sgn}(s(t))$. Meanwhile, $-(\mu + \tau\hat{v}(t))K\mathrm{sgn}(s(t))$ and $-(\mu + \hat{\tau}\hat{v}(t))\hat{K}\mathrm{sgn}(s(t))$ are replaced by their maximums which both are $2\mu\sum_{j=1}^{m} |k_{ij}| = q_{i\max}$. Similar to (24), we come to the conclusion

$$\begin{aligned}2x^{\mathrm{T}}(t)Pb_i\sigma(u_i(t)) \le \max\{&2x^{\mathrm{T}}(t)Pb_i(-h_i x(t) + q_{i\max}),\\ &2x^{\mathrm{T}}(t)Pb_i(-\hat{h}_i x(t) + q_{i\max})\}.\end{aligned} \tag{24}$$

If there exists

$$x^{\mathrm{T}}(t)Pb_i(-h_i x(t) + q_{i\max}) < x^{\mathrm{T}}(t)Pb_i(-\hat{h}_i x(t) + q_{i\max}),$$

we set $\theta_i = 0$, otherwise, $\theta_i = 1$. Defining

$$\begin{aligned}N(\theta, G, \hat{G}) &= \begin{bmatrix}\theta_1(-h_1 x(t) + q_{1\max})\\ +(1-\theta_1)(-\hat{h}_1 x(t) + q_{1\max})\\ \vdots\\ \theta_m(-h_m x(t) + q_{m\max})\\ +(1-\theta_m)(-\hat{h}_m x(t) + q_{m\max})\end{bmatrix}\\ &= N_1(\theta, G, \hat{G})x(t) + N_2(\theta, G, \hat{G}),\end{aligned} \tag{25}$$

with

$$N_1(\theta, G, \hat{G}) = \begin{bmatrix}\theta_1(-h_1) + (1-\theta_1)(-\hat{h}_1)\\ \vdots\\ \theta_m(-h_m) + (1-\theta_m)(-\hat{h}_m)\end{bmatrix},$$

$$N_2(\theta, G, \hat{G}) = [q_{1\max}, \dots, q_{m\max}]^{\mathrm{T}}.$$

Similar to (11) and (12), we have

$$\begin{aligned}2x^{\mathrm{T}}(t)PBN_2 &\le \frac{1}{\lambda}x^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) + \lambda N_2^{\mathrm{T}}N_2\\ &\le \frac{1}{\lambda}x^{\mathrm{T}}(t)PBB^{\mathrm{T}}Px(t) + \lambda\beta_3^2,\end{aligned} \tag{26}$$

and for $2x(t)^{\mathrm{T}}PEw(t)$,

$$\begin{aligned}2x(t)^{\mathrm{T}}PEw(t) &\le \frac{1}{\eta}x(t)^{\mathrm{T}}PEE^{\mathrm{T}}Px(t) + \eta w(t)^{\mathrm{T}}w(t)\\ &\le \frac{1}{\eta}x(t)^{\mathrm{T}}PEE^{\mathrm{T}}Px(t) + \eta\varpi^2,\end{aligned} \tag{27}$$

with $\varpi = \frac{\mu}{\parallel GE \parallel}$. Combining with (23), (25)–(27), we have

$$\begin{aligned}\dot{V}_4(t) \le x^{\mathrm{T}}(t)[&2P(A + BN_1(\theta, G, \hat{G}))^{\mathrm{T}} + \frac{1}{\lambda}PBB^{\mathrm{T}}P\\ &+\frac{1}{\eta}PEE^{\mathrm{T}}P]x(t) + \lambda\beta_3^2 + \eta\varpi^2.\end{aligned}$$

Referring to the proof of last section, with (22), we have

$$\begin{aligned}&(A + BN_1(\theta, G, \hat{G}))^{\mathrm{T}}P + P(A + BN_1(\theta, G, \hat{G}))\\ &+\frac{1}{\lambda}PBB^{\mathrm{T}}P + \frac{1}{\eta}PEE^{\mathrm{T}}P + (\lambda\beta_3^2 + \eta\varpi^2)P < 0.\end{aligned}$$

Referring to the proof of Theorem 2, then for $\forall x \in \mathscr{E}(P,\rho)\setminus\{0\}$, there exists

$$\dot{V}_4(t) < -\frac{\lambda\beta_3^2 + \eta\varpi^2}{\rho}x^{\mathrm{T}}(t)Px(t) + (\lambda\beta_3^2 + \eta\varpi^2) < 0. \tag{28}$$

This ends the proof. □

Recalling the statement in Remark 2, we have the following corollary for the solution of the sliding mode surface parametric matrix G.

Corollary 2: Consider the closed-loop system in (1) with sliding mode controller (19). For a given set $\mathscr{E}(P,\rho)$, diagonal matrix $\Gamma \in \mathbb{R}^{m\times m}$ and positive scalar α, if there exist matrix $\hat{\Gamma} \in \mathbb{R}^{m\times m}$, positive scalars $\hat{\alpha}$, λ and η such that condition (21) holds and

$$QX_4^{\mathrm{T}} + X_4 Q + \Pi_4 \Lambda \Pi_4^{\mathrm{T}} < 0, \quad \forall \theta \in \mathscr{V} \tag{29}$$

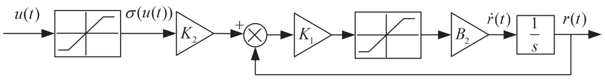


Fig. 1 *Actuator unit with magnitude and rate limitations*

where

$$
\begin{aligned}
X_4 &\triangleq A - B(FA - N_1'(\theta, \Gamma, \hat{\Gamma})),\\
\Pi_4 &\triangleq \begin{bmatrix} B & E & \beta_4 I & \varpi I \end{bmatrix},\\
\beta_4 &\triangleq 2\mu \cdot \max\left\{\frac{1}{\alpha}, \frac{1}{\hat{\alpha}}\right\},
\end{aligned}
$$

then $\mathscr{E}(P, \rho)$ is a strictly invariant set of the system.

Proof: The proof is same as that of Theorem 4. □

4.3 Estimation of the attraction domain

For simplicity, we present the approach to the estimation of the attraction domain as follows. Specific explanation and proof can be referred to Section 3.4. The following optimisation problem gives the conditions of the contractively invariant set of the closed-loop system (1) with the controller (19)

$$
\begin{aligned}
&\sup_{P>0,\,\rho,\,\hat{\alpha},\,\hat{\Gamma}} \kappa\\
&\text{s.t.(a) } \kappa X_R \subset \mathscr{E}(P, \rho),\\
&\text{(b) } X_4^{\mathrm{T}} P + P X_4 + \frac{1}{\lambda} P B B^{\mathrm{T}} P + \frac{1}{\eta} P E E^{\mathrm{T}} P\\
&+(\lambda \beta_4^2 + \eta \varpi^2) P < 0, \forall \theta \in \mathscr{V},\\
&\text{(c) } \mathscr{E}(P, \rho) \subset \mathscr{L}'(\hat{G}, \hat{\tau}).
\end{aligned}
\tag{30}
$$

Further, a solvable minimisation scheme of α can be transferred as follows:

$$
\begin{aligned}
&\inf_{P>0,\,\rho,\,\hat{\alpha},\,\hat{\Gamma}} \gamma\\
&\text{s.t. (29) and (a)} \begin{bmatrix} \gamma & x^{\mathrm{T}} \\ x & Q \end{bmatrix} \geq 0,\\
&\text{(b)} \begin{bmatrix} (u_{\max} - \hat{q}_{i\max})^2 & w_i \\ w_i^{\mathrm{T}} & Q \end{bmatrix} \geq 0,\\
&i = 1, 2, \ldots, m.
\end{aligned}
\tag{31}
$$

5 Control synthesis under rate saturation

5.1 Two-nested saturation system description

When considering the actuator subjected to magnitude and rate saturation simultaneously, we adopt the two-nested saturation system given by Bateman and Lin [38]. Fig. 1 represents the first-order actuator model with magnitude and rate saturation.

The control structure of Fig. 1 can be represented as

$$
\dot{r}(t) = B_2 \sigma\big(K_1(K_2 \sigma(u(t)) - r(t))\big),
$$

where $r(t) \in \mathbb{R}^m$ represents actuator position, $K_1 \in \mathbb{R}^{m \times m}$ is the control gain for the actuators, and $K_2 \in \mathbb{R}^{m \times m}$ is a matrix of control employed to allow different position saturation levels to be represented. Then the resulting closed-loop system can be reformulated as

$$
\begin{cases}
\dot{x}(t) = Ax(t) + Br(t) + Ew(t),\\
u(t) = -Hx(t) - \delta K \operatorname{sgn}(s(t)),\\
\dot{r}(t) = B_2 \sigma[K_1(K_2 \sigma(u(t)) - r(t))].
\end{cases}
$$

With the aim to represent it as the form of the two-nested saturation system [38], we obtain

$$
\begin{aligned}
\dot{\tilde{x}}(t) = {} & \tilde{A} \tilde{x}(t) + \tilde{B} \sigma[-H_1 \tilde{x}(t) - \delta \tilde{K} \operatorname{sgn}(s(t))\\
& + D_2 \sigma(-H_2 \tilde{x}(t) - \delta \tilde{K} \operatorname{sgn}(s(t)))] + \tilde{E} \tilde{w}(t),
\end{aligned}
\tag{32}
$$

where

$$
\begin{aligned}
&\tilde{A} \triangleq \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \tilde{B} \triangleq \begin{bmatrix} 0 \\ B_2 \end{bmatrix}, \tilde{E} \triangleq \begin{bmatrix} E & 0 \\ 0 & \delta K \end{bmatrix},\\
&H_1 \triangleq [0 \quad K_1], H_2 \triangleq [H \quad 0], \tilde{K} \triangleq \alpha^{-1} I,\\
&\tilde{x}(t) \triangleq \begin{bmatrix} x(t) \\ r(t) \end{bmatrix}, \tilde{w}(t) \triangleq \begin{bmatrix} w(t) \\ \operatorname{sgn}(s(t)) \end{bmatrix}, D_2 \triangleq K_1 K_2.
\end{aligned}
$$

5.2 Stability analysis of the closed-loop system

Under the designed control law (3) or the control law (19), the stability of the control system with the considered rate saturation is analysed in the following theorem.

Theorem 5: Consider the closed-loop system (32) with the controller (3) or (19). For a given set $\mathscr{E}(P, \rho)$, matrix $\Gamma \in \mathbb{R}^{m \times m}$, positive scalar α, if there exist matrix $\hat{\Gamma} \in \mathbb{R}^{m \times m}$, positive scalars $\hat{\alpha}$, λ and η such that

$$
\mathscr{E}(P, \rho) \subset \mathscr{L}'(\hat{G}_1, \mu),
\tag{33}
$$

$$
\mathscr{E}(P, \rho) \subset \mathscr{L}'(\hat{G}_2, \mu),
\tag{34}
$$

$$
Q X_5^{\mathrm{T}} + X_5 Q + \Pi_5 \Lambda \Pi_5^{\mathrm{T}} < 0,
\tag{35}
$$

where

$$
\begin{aligned}
X_5 &\triangleq \tilde{A} + \tilde{B} M_3(\theta, \hat{G}_1, Z, \hat{Z}),\\
\hat{G}_1 &\triangleq [0 \quad \hat{K}_1], \quad \hat{G}_2 \triangleq [\hat{G} \quad 0],\\
Z &\triangleq [D_2(FA + \Gamma F) \quad K_1],\\
\hat{Z} &\triangleq [D_2(FA + \hat{\Gamma} F) \quad K_1],\\
\Pi_5 &\triangleq \begin{bmatrix} \tilde{B} & E & \beta_5 I & v I \end{bmatrix},\\
\beta_5 &\triangleq \vartheta\left(\max\left\{\frac{\mu}{\alpha}, \frac{\mu}{\hat{\alpha}}\right\} + v \, \| FE \| \right),\\
\vartheta &\triangleq \sqrt{\sum_{i=1}^{m} \Big(\sum_{j=1}^{m} |B_2(K_1 K_2 + I)|_{ij} \Big)^2}
\end{aligned}
$$

then $\mathscr{E}(P, \rho)$ is a strictly invariant set of the system (1).

Proof: Consider the Lyapunov functional $\tilde{V}(t) = \tilde{x}^{\mathrm{T}}(t) P \tilde{x}(t)$, then

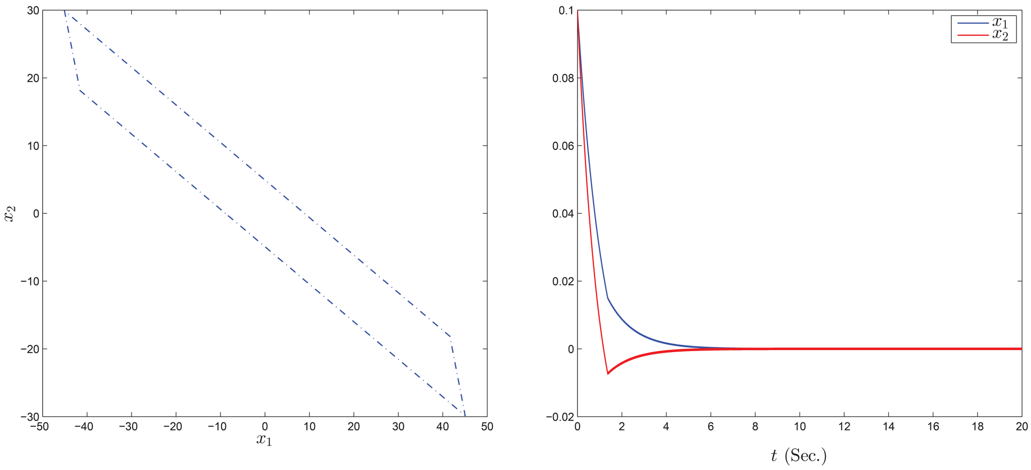
$$
\begin{aligned}
\dot{\tilde{V}}(t) = {} & 2\tilde{x}^{\mathrm{T}}(t) P \big[\tilde{A} \tilde{x}(t) + \tilde{B} \sigma\big(-H_1 \tilde{x} + \delta \tilde{K} \operatorname{sgn}(s)\\
& + D_2 \sigma(-H_2 \tilde{x} + \delta \tilde{K} \operatorname{sgn}(s))\big) + \tilde{E} \tilde{w}\big].
\end{aligned}
\tag{36}
$$

It can be observed that the switched terms can be replaced by its maximum exactly during the former proof. Thus, these switched terms in (36) could be replaced by $B_2(K_1 K_2 + I)\delta \sum_{j=1}^{m} |k_{ij}|$. As a result, for the rest terms in (36), similarly, we demand for

$$
\begin{aligned}
&(\tilde{A} + \tilde{B} M_3(\theta, \hat{G}_1, H_1 + D_2 \hat{G}_2, H_1 + D_2 H_2))^{\mathrm{T}} P\\
&+ P(\tilde{A} + \tilde{B} M_3(\theta, \hat{G}_1, H_1 + D_2 \hat{G}_2, H_1 + D_2 H_2))\\
&+ \frac{1}{\lambda} P \tilde{B} \tilde{B}^{\mathrm{T}} P + \frac{1}{\eta} P E E^{\mathrm{T}} P + (\lambda \beta_5^2 + \eta v^2) P < 0, \quad \forall \theta \in \mathscr{V},
\end{aligned}
\tag{37}
$$

Since the sliding mode surface has been selected as (2), we have

$$
\begin{aligned}
H_1 + D_2 \hat{G}_2 &\triangleq [D_2 \hat{G} \quad K_1] \triangleq [D_2(FA + \hat{\Gamma} F) \quad K_1],\\
H_1 + D_2 H_2 &\triangleq [D_2 H \quad K_1] \triangleq [D_2(FA + \Gamma F) \quad K_1],
\end{aligned}
$$



a *b*

Fig. 2 *Simulation of the numerical system subjected to saturation*
(a) Invariant set of the closed-loop system subjected to magnitude saturation under the sliding mode controller (40), ***(b)*** State trajectories of the closed-loop system subjected to two-nested saturation under the sliding mode controller (41)

Since then, (37) is equivalent to (35). According to (34)–(35), we can see that for $\forall\, x \in \mathscr{E}(P, \rho)\setminus\{0\}$ there exists

$$\dot{\tilde{V}}(t) < -\frac{\lambda\beta_5^2 + \eta v^2}{\rho} x^{\mathrm{T}}(t) P x(t) + (\lambda\beta_5^2 + \eta v^2) < 0.$$

This completes the proof. □

5.3 Estimation of the attraction domain

Based on the discussion in the above sections, we summarise the results on the estimation of the attraction domain of the overall control system (32) with magnitude and rate limitations in the following statements:

$$\begin{aligned} &\sup_{P>0,\, \rho,\, \hat{\alpha},\, \hat{\Gamma}} \quad \kappa \\ &\quad \text{s.t.(a) } \kappa X_R \subset \mathscr{E}(P, \rho), \\ &\text{(b) } X_5^{\mathrm{T}} P + P X_5 + \frac{1}{\lambda} P \tilde{B} \tilde{B}^{\mathrm{T}} P + \frac{1}{\eta} P E E^{\mathrm{T}} P \\ &+ (\lambda\beta_5^2 + \eta v^2) P < 0, \forall\, \theta \in \mathscr{V}, \\ &\text{(c) } \mathscr{E}(P, \rho) \subset \mathscr{L}'(\hat{G}_1, \mu), \\ &\text{(d) } \mathscr{E}(P, \rho) \subset \mathscr{L}'(\hat{G}_2, \mu). \end{aligned} \tag{38}$$

Let $W_1 = \hat{H}_1 Q$, $W_2 = \hat{H}_2 Q$, and w_{1i} as well as w_{2i} denote the ith row of W_1 and W_2, respectively. The solvable optimisation scheme corresponding to the above optimisation problem is presented as follows:

$$\begin{aligned} &\inf_{P>0,\, \rho,\, \hat{\alpha},\, \hat{\Gamma}} \quad \gamma \\ &\text{s.t. (35) and (a)} \begin{bmatrix} \gamma & x^{\mathrm{T}} \\ x & Q \end{bmatrix} \ge 0, \\ &\text{(b)} \begin{bmatrix} (u_{i\max} - \hat{p}_{i\max})^2 & w_{1i} \\ w_{1i}^{\mathrm{T}} & Q \end{bmatrix} \ge 0, \\ &i = 1, 2, \ldots m, \\ &\text{(c)} \begin{bmatrix} (u_{i\max} - \hat{p}_{i\max})^2 & w_{2i} \\ w_{2i}^{\mathrm{T}} & Q \end{bmatrix} \ge 0, \\ &i = 1, 2, \ldots m. \end{aligned} \tag{39}$$

6 Simulation examples

Example 1: In this example, we will validate the effectiveness of the designed control results subject to saturation. In terms of the plant (1), the system parameters are given as follows:

$$A = \begin{bmatrix} -0.4 & -0.8 \\ 0.8 & 0.6 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, E = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}.$$

We set the initial state as $x_0(t) = [0.1 \quad 0.1]^{\mathrm{T}}$, and the uncertainty is $w(t) = 0.1\sin(x_1(t))$.

(i) *Simulation of the numerical system subjected to magnitude saturation*

Since ρ as a measurement of the volume of ellipsoids, which could also be contained in α, here we can set $\rho = 1$ without loss of generality, and

$$\alpha = 1, \quad \mu = 0.01, \quad v = 0.01, \quad \Gamma = 1, \quad u_{i\max} = 1.0.$$

By adopting the SMC design method, according to Theorem 2, we have

$$\begin{cases} s(t) = Gx(t), \\ u(t) = -FAx(t) - \delta K \mathrm{sgn}(s(t)) - K\Gamma(s(t)), \end{cases} \tag{40}$$

where $G = [0.1000 \quad 0.2000]$. Under the designed controller (40), the responses of the closed-loop system are shown in Figs. 2*a* and *b*.

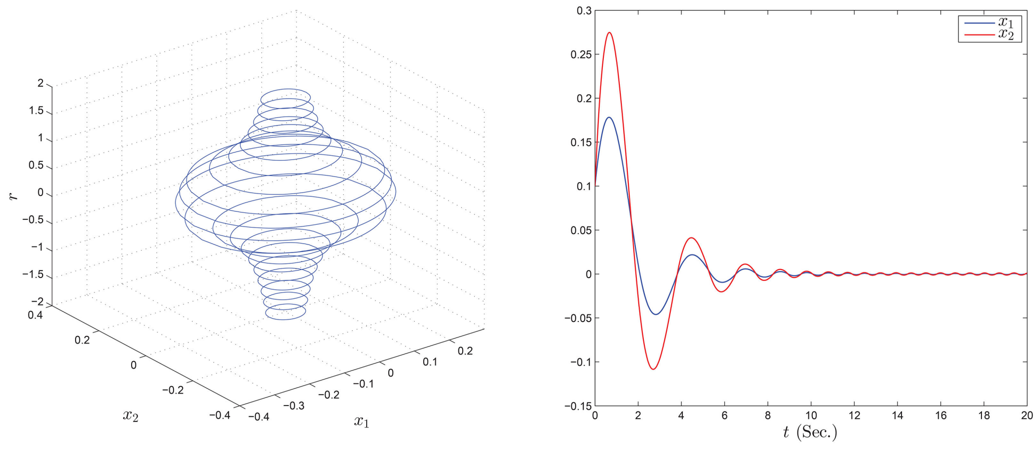
Fig. 2*a* shows that invariant set is obtained by using the SMC design method. The boundary of the circle was formed by $\kappa^*(x)x$, with $\kappa^*(x)$ means the optimal solution of κ. Fig. 2*b* is given to prove the performance of the saturated system under the SMC method, and it depicts the states trajectories of the closed-loop system.

(ii) *Simulation of the system subjected to magnitude and rate saturation*

On the basis of the magnitude saturation, we set the same values for some common parameters, and let $K_1 = 2$, $K_2 = 1$, $B_2 = 1$ and $u_{i\max} = 15.0$, $r_{i\max} = 15.0$. Considering the third variable of the augmented system represents actuator position, we pick up the two former state variables to illustrate the range of the estimated invariant set. According to Theorem 5, we have

$$\begin{aligned} u_r(t) = &-H_1 x(t) - \delta K \mathrm{sgn}(s(t)) \\ &+ D_2(-H_2 x(t) - \delta K \mathrm{sgn}(s(t))) \end{aligned} \tag{41}$$

where $H_1 = [0 \quad 0 \quad 2.000]$, $H_2 = [0.2600 \quad 0.4000 \quad 0]$ and $D_2 = 2$.



a b

Fig. 3 *Simulation of the numerical system subjected to magnitude and rate saturation*
(a) Invariant sets of the closed-loop system, ***(b)*** State trajectories of the closed-loop system

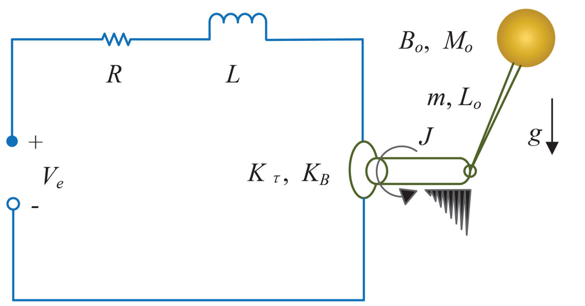


Fig. 4 *Schematic diagram of the electromechanical system*

Table 1 Description of system parameters

Symbol	Description	Value (units)
J	rotor inertia	$1.625 \times 10^3\,\mathrm{kg \cdot m^2}$
L	armature inductance	$1.0\,\mathrm{H}$
R	armature resistance	$5.0\,\Omega$
m	link mass	$0.506\,\mathrm{kg}$
M_o	load mass	$0.434\,\mathrm{kg}$
B_o	coefficient of viscous friction	$16.25 \times 10^3\,\mathrm{N \cdot m \cdot s/rad}$
R_o	radius of the load	$0.023\,\mathrm{m}$
K_B	back-emf coefficient	$0.90\,\mathrm{N \cdot m/A}$
K_τ	coefficient of conversion	$0.90\,\mathrm{N \cdot m/A}$
g	gravity coefficient	$9.81\,\mathrm{m \cdot s^2}$

Under the designed controller in (41), the responses of the closed-loop system are shown in Figs. 3*a* and *b*. Fig. 3*a* depicts the estimated attraction domain of the closed-loop system subjected to magnitude and rate saturation, with the limit of $r(t)$ varies from -2.0 to 2.0 with 0.2 increment. Fig. 3*b* illustrates that the augmented system under the sliding mode controller (41) is stabilised.

Example 2: This example provides the electromechanical system [39] to validate the effectiveness of the proposed control results. The schematic diagram of the electromechanical system is shown in Fig. 4.
The dynamics of the electromechanical system are formulated as

$$I = \left(M + \frac{J}{K_\tau}\right)\ddot{q} + \frac{B_o}{K_\tau}\dot{q} + N\sin(q),$$
$$L\dot{I} = V_e - RI - K_B\dot{q},$$

with

$$M \triangleq \frac{5(m + 3M_o)L_o^2 + 6M_oR_o^2}{15K_\tau}, \quad N \triangleq \frac{(m + 2M_o)L_og}{2K_\tau}.$$

Besides, the parameter descriptions are given in Table 1.
Let $x_1(t) = q(t)$, $x_2(t) = \dot{q}(t)$ and $x_3(t) = I(t)$ be the angular motor position, the position of the load, the motor armature current, respectively. The state-space representation of the two dynamic subsystems is as follows:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -\dfrac{NK_\tau}{MK_\tau + J}\sin(x_1(t)) - \dfrac{BK_\tau}{MK_\tau + J}x_2(t), \\ \dot{x}_3(t) = -\dfrac{K_B}{L}x_2(t) - \dfrac{R}{L}x_3(t) + \dfrac{1}{L}u(t). \end{cases}$$

Consequently, the system matrices in terms of the plant form in (1) for the employed electromechanical system are given as follows:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\dfrac{BK_\tau}{MK_\tau + J} & 0 \\ 0 & -\dfrac{K_B}{L} & -\dfrac{R}{L} \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{L} \end{bmatrix},$$
$$E = \begin{bmatrix} 0 \\ -\dfrac{NK_\tau}{MK_\tau + J} \\ 0 \end{bmatrix}.$$

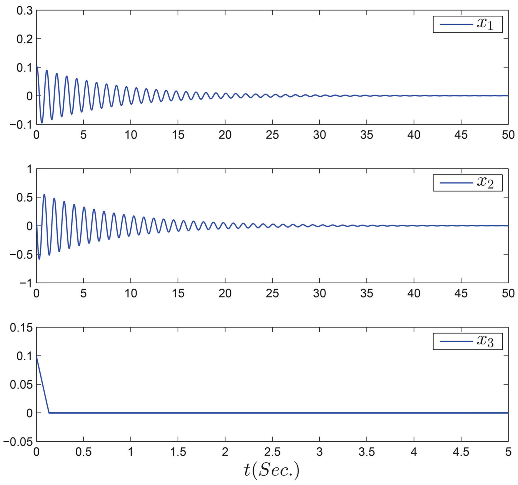
(i) *Simulation of the electromechanical system subjected to magnitude saturation*:

First, the real control force, according to the statement in Theorem 4, the detailed parameters of the sliding surface and adaptive SMC are as follows: $\Gamma = 1.4$, $\alpha = 0.8$, $\mu = 0.5$, $u_{i\max} = 1.0$, then

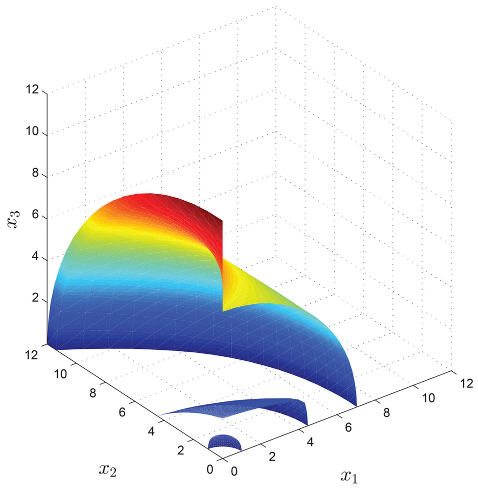
$$\begin{cases} s(t) = Gx(t), \\ u(t) = -FAx(t) - (\mu + \| GE \| \hat{v}(t))K\mathrm{sgn}(s(t)) - K\Gamma(s(t)), \end{cases} \tag{42}$$

where $G = [0 \quad 0 \quad 0.8000]$. The corresponding adaptive law (20), with $\bar{\hat{v}} = 0.1$, $\chi = 0.1$, $\phi = 0.1$, $\varepsilon = 0.1$, $\gamma = 1$ is set to satisfy $\gamma > \bar{\hat{v}} / \| GE \|$.

Under the designed controller (42), the responses of the closed-loop system are shown in Fig. 5*a*. It illustrates the state trajectories to approve the reachability of the sliding surface and the stability of the closed-loop system.
(ii) *Simulation of the electromechanical system subjected to magnitude and rate saturation*



a



b

Fig. 5 *Simulation results of the electromechanical system subjected to saturation*

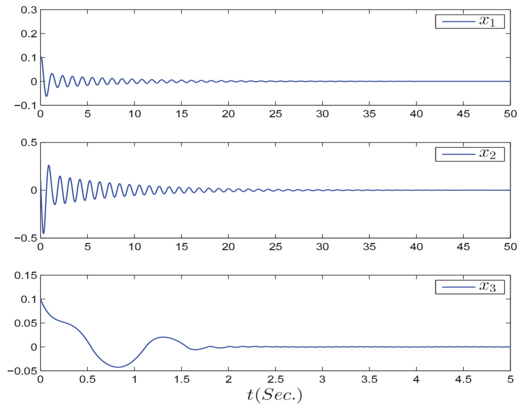
(a) Trajectories of the saturated system under adaptive-sliding mode controller (42), ***(b)*** Invariant sets of closed-system under adaptive sliding mode controllers (42) and (43)

According to Theorem 5, the control parameters are given as follows: $\Gamma = 10$, $\alpha = 8$, $\mu = 0.1$, $B_2 = 1$, $K_1 = 2$, $K_2 = 1$, $u_{i\max} = 100$, $r_{i\max} = 10$. Then the adaptive sliding mode controller

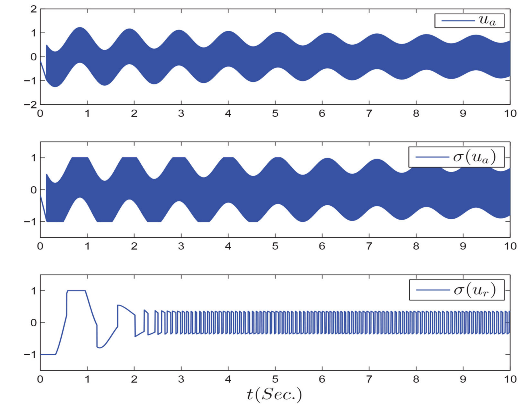
$$\begin{aligned} u(t) = & -H_1x(t) - (\mu + \parallel GE \parallel \hat{v}(t))K\mathrm{sgn}(s(t)) \\ & + D[-H_2x(t) - (\mu + \parallel GE \parallel \hat{v}(t))K\mathrm{sgn}(s(t))], \end{aligned} \tag{43}$$

with $H_1 = [0 \quad 0 \quad 0 \quad 2.0]$, $H_2 = [0 \quad -0.9 \quad -4.6 \quad 0]$ and $D = 2$. Besides, the adaptive law is (20), which is the same as Example 2-i).

Under the designed controller (43), the responses of the closed-loop system are shown in Fig. 5*b*. It depicts the estimate of the attraction domain of the closed-loop system subject to magnitude and rate saturation. The surface is formed of $\kappa^*(x)x$, which means that in every direction, the ratio of the point on the surface to the unit circle represents the optimal κ we searched for. The largest ellipsoid represents the invariant set of the system subject to magnitude saturation, the medium one represents the invariant set of system subject to two-nested saturation, and the smallest one is a unit circle as a reference. Fig. 6*a* illustrates that the state responses of the augmented system under the controller (43) is stable. In Fig. 6*b*, u_a, $\sigma(u_a)$ and $\sigma(u_r)$ represent the adaptive sliding mode controller output, and the real control force subjected to magnitude and rate saturation, respectively.



a



b

Fig. 6 *Simulation results of the electromechanical system subjected to magnitude and rate saturation*

(a) Trajectories of saturated system under adaptive sliding mode controller (43), ***(b)*** Real control forces

7 Conclusion

This paper has developed a new saturated SMC design method, and the reachability of the sliding mode as well as the stability of the saturated system under the SMC is analysed. Apart from the actuator magnitude saturation, the rate saturation of actuator has been also considered. Besides, convergence conditions in terms of a set of solvable matrix inequalities have been provided to estimate the attraction domain. Finally, numerical examples have been presented to demonstrate the applicability of the established saturated SMC methods.

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9 References

[1] Chen, Y., Fei, S., Li, Y.: 'Stabilization of neutral time-delay systems with actuator saturation via auxiliary time-delay feedback', *Automatica*, 2015, **52**, pp. 242–247

[2] Hindi, H., Boyd, S.: 'Analysis of linear systems with saturation using convex optimization'. Proc. the 37th IEEE Conf. Decision and Control, 1998, vol. 1, pp. 903–908

[3] Hu, T., Lin, Z., Chen, B.M.: 'An analysis and design method for linear systems subject to actuator saturation and disturbance', *Automatica*, 2002, **38**, (2), pp. 351–359

[4] Pittet, C., Tarbouriech, S., Burgat, C.: 'Stability regions for linear systems with saturating controls via circle and popov criteria'. Proc. the 36th IEEE Conf. Decision and Control, vol. 5, 1997, pp. 4518–4523

[5] Prieur, C., Tarbouriech, S., da Silva, J.M.G.: 'Wave equation with cone-bounded control laws', *IEEE Trans. Autom. Control*, 2016, **61**, (11), pp. 3452–3463

[6] Valmorbida, G., Tarbouriech, S., Garcia, G.: 'Design of polynomial control laws for polynomial systems subject to actuator saturation', *IEEE Trans. Autom. Control*, 2013, **58**, (7), pp. 1758–1770

[7] Zhou, B., Zheng, W.X., Duan, G.R.: 'An improved treatment of saturation nonlinearity with its application to control of systems subject to nested saturation', *Automatica*, 2011, **47**, (2), pp. 306–315

[8] Shen, C., Li, Y., Chen, Y., *et al.*: 'Performance Analysis of Multi-Motion Sensor Behavior for Active Smartphone Authentication', *IEEE Trans. Inf. Forensics Sec.*, 2018, **13**, (1), pp. 48–62

[9] Zhao, Y., Shen, Y., Bernard, A., *et al.*: 'Evaluation and comparison of current biopsy needle localization and tracking methods using 3D ultrasound', *Ultrasonics*, 2017, **73**, pp. 206–220

[10] Zhao, D., Xu, L., Huangfu, Y., *et al.*: 'Semi-physical modeling and control of a centrifugal compressor for the air feeding of a PEM fuel cell', *Energy Convers. Manag.*, 2017, **154**, (15), pp. 380–386

[11] Chen, B.S., Wang, S.S.: 'The stability of feedback control with nonlinear saturating actuator: time domain approach', *IEEE Trans. Autom. Control*, 1988, **33**, (5), pp. 483–487

[12] Hu, T., Lin, Z., Chen, B.M.: 'Analysis and design for discrete-time linear systems subject to actuator saturation', *Syst. Control Lett.*, 2002, **45**, (2), pp. 97–112

[13] Fang, H., Lin, Z., Hu, T.: 'Analysis of linear systems in the presence of actuator saturation and L2-disturbances', *Automatica*, 2004, **40**, (7), pp. 1229–1238

[14] Liu, J., Wu, C., Wang, Z.: 'Reliable filter design for sensor networks in the type-2 fuzzy framework', *IEEE Trans. Ind. Inf.*, 2017, **13**, (4), pp. 1742–1752

[15] Hu, T., Lin, Z.: '*Control systems with actuator saturation: analysis and design*' (Springer Science Business Media, Birkhäuser Boston, USA, 2001)

[16] Bartolini, G., Ferrara, A., Usai, E.: 'Chattering avoidance by second-order sliding mode control', *IEEE Trans. Autom. Control*, 1998, **43**, (2), pp. 241–246

[17] Basin, M., Panathula, C.B., Shtessel, Y.: 'Adaptive uniform finite-/fixed-time convergent second-order sliding-mode control', *Int. J. Control*, 2016, **89**, (9), pp. 1777–1787

[18] Feng, Y., Yu, X., Han, F.: 'nonsingular terminal sliding-mode control of nonlinear systems', *Automatica*, 2013, **49**, (6), pp. 1715–1722

[19] Su, X., Liu, X., Shi, P., *et al.*: 'Sliding mode control of discrete-time switched systems with repeated scalar nonlinearities', *IEEE Trans. Autom. Control*, 2017, **62**, (9), pp. 4606–4610

[20] Utkin, V.: 'Sliding mode control of dc/dc converters', *J. Franklin. Inst.*, 2013, **350**, (8), pp. 2146–2165

[21] Utkin, V., Shi, J.: 'Integral sliding mode in systems operating under uncertainty conditions'. Proc. the 35th IEEE Conf. Decision and Control, vol. 4, 1996, pp. 4591–4596

[22] Utkin, V.I., Poznyak, A.S.: 'Adaptive sliding mode control with application to super-twist algorithm: Equivalent control method', *Automatica*, 2013, **49**, (1), pp. 39–47

[23] Wu, L., Zheng, W.X., Gao, H.: 'Dissipativity-based sliding mode control of switched stochastic systems', *IEEE Trans. Autom. Control*, 2013, **58**, (3), pp. 785–791

[24] Tanelli, M., Punta, E., Ferrara, A.: 'A switched second-order sliding mode control algorithm for non-affine systems with saturations'. Proc. the 51st IEEE Conf. Decision and Control, 2012, pp. 5488–5493

[25] Golkani, M.A., Fridman, L., Koch, S., *et al.*: 'Observer-based saturated output feedback control using twisting algorithm'. IEEE Proc. of the 14th Int. Workshop on Variable Structure Systems, 2016, pp. 246–250

[26] Saghafinia, A., Ping, H., Uddin, M., *et al.*: 'Adaptive fuzzy sliding-mode control into chattering-free', *IEEE Trans. Ind. Appl.*, 2015, **51**, (1), pp. 692–701

[27] Septanto, H., Syaichu-Rohman, A., Mahayana, D.: 'Static anti-windup compensator design of linear sliding mode control for input saturated systems'. IEEE Proc. Int. Conf. Electrical Engineering and Informatics, 2011, pp. 1–4

[28] Lebedev, A.: 'The analysis of sliding mode control features in variable structure system with saturation'. Proc. IEEE Conf. Information and Automation, 2015, pp. 2203–2208

[29] Bandyopadhyay, B., Janardhanan, S., Spurgeon, S.K.: '*Advances in sliding mode control*' (Springer, Verlag Berlin Heidelberg, 2013)

[30] Torchani, B., Sellami, A., Garcia, G.: 'Sliding mode control of saturated systems with norm bounded uncertainty'. 16th IEEE Mediterranean Electrotechnical Conf. (MELECON), 2012, pp. 15–18

[31] Zhu, Z., Xia, Y., Fu, M.: 'Adaptive sliding mode control for attitude stabilization with actuator saturation', *IEEE Trans. Ind. Electron.*, 2011, **58**, (10), pp. 4898–4907

[32] Wu, L., Gao, Y., Liu, J., *et al.*: 'Event-triggered sliding mode control of stochastic systems via output feedback', *Automatica*, 2017, **82**, pp. 79–92

[33] Choi, H.H.: 'A new method for variable structure control system design: A linear matrix inequality approach', *Automatica*, 1997, **33**, (11), pp. 2089–2092

[34] Choi, H.H.: 'An explicit formula of linear sliding surfaces for a class of uncertain dynamic systems with mismatched uncertainties', *Automatica*, 1998, **34**, (8), pp. 1015–1020

[35] Ioannou, P.A., Sun, J.: '*Robust adaptive control*', vol. 1 (PTR Prentice-Hall, Upper Saddle River, NJ, 1996)

[36] Yu, S., Yu, X., Shirinzadeh, B., *et al.*: 'Continuous finite-time control for robotic manipulators with terminal sliding mode', *Automatica*, 2005, **41**, (11), pp. 1957–1964

[37] Shtessel, Y., Taleb, M., Plestan, F.: 'A novel adaptive-gain supertwisting sliding mode controller: Methodology and application', *Automatica*, 2012, **48**, (5), pp. 759–769

[38] Bateman, A., Lin, Z.: 'An analysis and design method for linear systems under nested saturation', *Syst. Control Lett.*, 2003, **48**, (1), pp. 41–52

[39] Dawson, D.M., Carroll, J.J., Schneider, M.: 'Integrator backstepping control of a brush dc motor turning a robotic load', *IEEE Trans. Control. Syst. Technol.*, 1994, **2**, (3), pp. 233–244