Poisson Point Process

A point process Φ on some underlying space S is a Poisson point process with intensity measure Λ if it has the following two properties:

1. The number of points in a bounded Borel set $B \subset S$ is a Poisson random variable with mean $\lambda(B)$.
2. The number of points in n disjoint Borel sets forms n independent random variables.

Generally, we have:

$$\underbrace{\Lambda(B)}_{\text{Intensity Measure}} = \int_B \underbrace{\lambda(x)}_{\text{Intensity}} \, \mathrm{d}x \tag{1}$$

For homogeneous PPP, $\lambda(x) = \lambda$, and accordingly, $\Lambda(B) = \lambda|B|$.

1 Point process transformations

1.1 Mapping Theorem

Let Φ be a PPP on $\mathbb{R}^d$ with intensity function λ. Let $f : \mathbb{R}^d \to \mathbb{R}^s$ be measurable and $\Lambda(f^{-1}\{y\}) = 0$, $\forall y \in \mathbb{R}^s$, i.e., f does not shrink a compact set to a point. Let,

$$\mu(B) \triangleq \Lambda(f^{-1}(B)) = \int_{f^{-1}(B)} \lambda(x)dx, \tag{2}$$

where it satisfies that $\mu(B) < \infty$ $\forall$ compact B. Then,

$$f(\Phi) = \bigcup_{x \in \Phi} f(x) \tag{3}$$

is a PPP with intensity measure μ.

Corollary 1. For a PPP Φ with intensity λ let the mapping be $A : \mathbb{R}^d \to \mathbb{R}^d$. The,

$$A(\Phi) = \{Ax : x \in \Phi\} \tag{4}$$

is a stationary PPP with intensity $\lambda \det(A^{-1})$.

Example 2.1: Let $\Phi = \{x_i\}$ be a homogeneous PPP with intensity λ on $\mathbb{R}^2$. What is the intensity function $\mu(x)$ of the one dimensional PPPs:

1. $\Psi = \{||x_i||\}$?
2. $\Phi' = \{||x_i||^2\}$?

1.2 Independent Thinning

Let $g : \mathbb{R}^d \to [0, 1]$ be a thinning function. Consider each point of a stationary PPP Φ and delete each point with probability $1 - g(x)$, independently. Let us call the resulting point process as $\tilde{\Phi}$. What can we say about $\tilde{\Phi}$? Is it a PPP as well?

Let us consider the probability that there are k points of $\tilde{\Phi}$ in a set B:

$$\mathbb{P}\left(\tilde{\Phi}(B) = k\right) = \sum_{i=k}^{\infty} \mathbb{P}\left(\Phi(B) = i\right) \mathbb{P}\left(\tilde{\Phi}(B) = k | \Phi(B) = i\right). \tag{5}$$

Now, for one point, the probability to remain (i.e., not be deleted) is evaluated as:

$$\mathbb{P}\left(\tilde{\Phi}(B) = 1 | \Phi(B) = 1\right) = \frac{1}{|B|} \int_B g(\mathbf{x}) \mathrm{d}x. \tag{6}$$

This implies,

$$\mathbb{P}\left(\tilde{\Phi}(B) = k | \Phi(B) = i\right) = \binom{i}{k} \left(\frac{1}{|B|} \int_B g(\mathbf{x}) \mathrm{d}x\right)^k \left(1 - \frac{1}{|B|} \int_B g(\mathbf{x}) \mathrm{d}x\right)^{i-k}. \tag{7}$$

Substituting this in (5), results in

$$\begin{aligned}\mathbb{P}\left(\tilde{\Phi}(B) = k\right) &= \sum_{i=k}^{\infty} \frac{\exp(-\lambda|B|)(\lambda|B|)^i}{i!} \binom{i}{k} \left(\frac{1}{|B|} \int_B g(\mathbf{x}) \mathrm{d}x\right)^k \left(1 - \frac{1}{|B|} \int_B g(\mathbf{x}) \mathrm{d}x\right)^{i-k} \\ &= \frac{\exp\left(-\lambda \left(\frac{1}{|B|} \int_B g(\mathbf{x}) \mathrm{d}x\right) |B|\right) \left(\lambda \left(\frac{1}{|B|} \int_B g(\mathbf{x}) \mathrm{d}x\right) |B|\right)^k}{k!} \end{aligned} \tag{8}$$

1.3 Superposition

Let us consider two independent homogeneous PPPs Φ_1 and Φ_2 with intensities λ_1 and λ_2, respectively. Then, $\Phi = \Phi_1 \cup \Phi_2$ is also a homogeneous PPP with intensity $\lambda_1 + \lambda_2$.

Proof. Left to the reader. $\square$