Binomial Point Process

1 Single Point in a Compact Set

Let us consider the most elementary of the point processes - one containing a single point in a compact subset W of $\mathbb{R}^d$. For simplicity, let us assume $d = 2$. Consider a subset $A \subseteq W$. If a point with location $X \in W$ is uniformly located in W, the probability that it falls inside A is

$$p_A = \mathbb{P}(X \in A) = \frac{\nu(A)}{\nu(W)}. \tag{1}$$

Here $\nu(\cdot)$ represents the volume (area in case of $\mathbb{R}^2$) of the region. This resembles a single Bernoulli trial with success probability p_A, e.g., tossing a coin with p_A being the probability of getting "Heads."

In case $W = \mathcal{B}(\mathbf{0}, R)$, i.e., a ball of radius R centered at the origin and $A = \mathcal{B}(\mathbf{x_0}, r)$ with $||\mathbf{x_0}|| + r \leq R$, then, we have

$$p_A = \frac{\pi r^2}{\pi R^2} = \frac{r^2}{R^2} \tag{2}$$

1.1 PDF of the distance from the center

First let us re-purpose the above observation to derive the CDF:

$$F_x(r) = \mathbb{P}(x \leq r) = \mathbb{P}(X \in \mathcal{B}(\mathbf{0}, r)) = \frac{r^2}{R^2}$$

Naturally the pdf can be expressed as

$$f_x(r) = \frac{\mathrm{d}}{\mathrm{d}r} F_x(r) = \frac{2r}{R^2} \tag{3}$$

Example 1: Let us consider a simple path-loss model excluding fading where the received power at the center of the room from an AP at a distance x is $PKx^{-\alpha}$. Here K is the path-loss constant and α is the path-loss exponent. Let the channel noise be N_0. A metric of interest is called the signal to noise ratio (SNR) coverage probability where the SNR is defined as

$$\xi = \frac{PKx^{-\alpha}}{N_0} \tag{4}$$

Then, the SNR coverage probability is evaluated as:

$$\begin{aligned}\mathcal{P}_C(\gamma_0) &= \mathbb{P}\left(\xi \geq \gamma_0\right) = \mathbb{P}\left(\frac{PKx^{-\alpha}}{N_0} \geq \gamma_0\right) \\ &= \mathbb{P}\left(x \leq \left(\frac{\gamma N_0}{PK}\right)^{\frac{-1}{\alpha}}\right) \\ &= F_x\left(\left(\frac{\gamma N_0}{PK}\right)^{\frac{-1}{\alpha}}\right) \\ &= \frac{1}{R^2}\left(\frac{\gamma N_0}{PK}\right)^{\frac{-2}{\alpha}}\end{aligned} \tag{5}$$

1.2 Simulation of Points

1.2.1 Inverse Transform Sampling

Let us recall the inverse transform sampling technique to generate instances of a random variable with a given distribution. Let $F_X(\cdot)$ be the continuous cumulative density function (CDF) and $F^{-1}(\cdot)$ be its inverse function, i.e.,

$$F^{-1}(u) = \mathtt{inf}\{x|F(x) \geq u\}, \quad 0 < u < 1. \tag{6}$$

Here we have considered `inf` since CDFs are weakly monotonic and right continuous.

Proposition 1. The inverse transform sampling method claims that if U is a continuous random variable uniformly distributed on $(0, 1)$, then the random variable $F^{-1}(U)$ has F as its CDF.

Proof. Consider the CDF of $F^{-1}(U)$:

$$\mathbb{P}\left(F^{-1}(U) \leq x\right) = \mathbb{P}\left(U \leq F(x)\right) = F(x). \tag{7}$$

The last equality is due to the fact that $\mathbb{P}(U \leq t) = t$, when t is uniform in (0,1). □

Example 2: In this example, we will use the inverse transform sampling method to generate uniformly distributed points in a circle. Recall that the CDF of the distance of such a point is $F_x(r) = \frac{r^2}{R^2}$. Thus,

$$F_X^{-1}(u) = R\sqrt{u}. \tag{8}$$

Accordingly, in order to generate a uniformly distributed point in a circle, we generate a uniformly distributed random variable U and employ the transformation $R\sqrt{U}$, which in turn creates instances of a random variable representing its distance from the origin. It can be noted that angle for the polar coordinate of the point is uniformly distributed between 0 and 2π.

1.3 MATLAB Code

```
clc; clear all;
gammavec = logspace(-8,-2,30);
R = 100;
P = 0.1;
Cov = zeros(1,length(gammavec));
experiments = 2000;
for i = 1:length(gammavec)
gamma0 = gammavec(i);
for iter = 1:experiments
d = R*(rand);
Recpow = P*d^-2;
if Recpow > gamma0
Cov(i) = Cov(i) + 1;
end
end
Cov(i) = Cov(i)/experiments;
end
plot(pow2db(gammavec),Cov);
hold on
plot(pow2db(gammavec), min(1,(1/R^2) * (P./gammavec)),'*');
```

2 Multiple points

If the same process as the single uniformly located point in W is repeated for n_B points, followed by a superposition of all the points, the resulting process is called a binomial point process (BPP) of n_B points. For such a point process, with locations represented by random variables $X_1, X_2, \ldots, X_{n_B}$, we have

$$\begin{aligned} \mathbb{P}\left(X_1 \in A_1, X_2 \in A_2, \ldots, X_{n_B} \in A_{n_B}\right) &= \mathbb{P}\left(X_1 \in A_1\right) \mathbb{P}\left(X_2 \in A_2\right) \ldots \mathbb{P}\left(X_{n_B} \in A_{n_B}\right) \\ &= \frac{\nu(A_1)\nu(A_2)\ldots\nu(A_{n_B})}{(\nu(W))^{n_B}} \end{aligned} \tag{9}$$

The number of randomly located points in $A \subset W$, say $N_W(A)$ follows

$$\mathbb{P}(N_W(A) = k) = \binom{n}{k} p_A^k (1 - p_A)^{n-k}, \tag{10}$$

where $p_A = \frac{\nu(A)}{\nu(W)}$. The resemblance of the above with the Binomial distribution is the reason for the nomenclature BPP.

2.1 Distribution of the distance to the nearest point

If the nearest point of a BPP with n_B points defined on W is at $\mathbf{x}$ from the origin, the CDF of its distance is evaluated as:

$$
\begin{aligned}
F_{d1}(r) = \mathbb{P}\left(||\mathbf{x}|| \leq r\right) &= 1 - \mathbb{P}\left(||\mathbf{x}|| > r\right) \\
&= \left(\frac{r^2}{R^2}\right)^0 \left(\frac{R^2 - r^2}{R^2}\right)^{n_B} = \left(\frac{R^2 - r^2}{R^2}\right)^{n_B}
\end{aligned}
$$

3 Poisson Limit

Let the number of points n_B go to infinity by expanding W, while keeping $p = \frac{|A|}{|W|}$ fixed. Then, what will be the distribution of the number of points in A? Specifically, we are interested in

$$
\mathbb{P}\left(\Phi_B(A) = k\right) = \lim_{n_B \to \infty} \binom{n_B}{k} p^k (1-p)^{n-k} \tag{11}
$$

Let $\lambda = \frac{n_B}{|W|}$. Therefore, $p = \frac{\lambda|A|}{n_B}$. Accordingly,

$$
\begin{aligned}
\mathbb{P}\left(\Phi_B(A) = k\right) &= \lim_{n_B \to \infty} \binom{n_B}{k} \left(\frac{\lambda|A|}{n_B}\right)^k \left(1 - \frac{\lambda|A|}{n_B}\right)^{n_B - k} \\
&= \lim_{n_B \to \infty} \frac{1 \cdot \left(1 - \frac{1}{n_B}\right) \cdot \left(1 - \frac{2}{n_B}\right) \cdots \left(1 - \frac{n_B - k + 1}{n_B}\right)}{k!} \left(\lambda|A|\right)^k \left(1 - \frac{\lambda|A|}{n_B}\right)^{n_B - k} \\
&= \frac{\left(\lambda|A|\right)^k}{k!} \exp\left(-\lambda|A|\right)
\end{aligned} \tag{12}
$$

The above is the well-known Poisson distribution.

4 Exercises

1. Consider a point $P = (x_t, 0)$ in $\mathbb{R}^2$. Find out the distance distribution of the nearest point of a BPP Φ_B of n_B points defined on $\mathbb{B}((0,0), R)$ from P. Assume that $x_t < R$.
2. For a BPP Φ_B of n_B points defined on $\mathbb{B}((0,0), R)$, find out the distance distribution of the farthest point from the origin.
3. Plot the distance distributions (CDF and pdf) of the nearest and the farthest points of a BPP Φ_B of n_B points defined on $\mathbb{B}((0,0), R)$.
4. Assume a forest in the shape of a disc of radius R. Let in the forest area n sensors are deployed uniformly to detect forest fires. Each sensor has a range $t < R$, i.e., it can detect a fire within t distance from it. Let a fire originate at a uniformly located point the forest. What is the probability that the fire will go undetected?