APPENDIX A

This section provides the proofs of Theorem 1, Theorem 2, and Corollary 1, respectively.

Proof of Theorem 1. We prove this theorem by showing that the fixed point $\tilde{\boldsymbol{s}}$ presented in Theorem 1 satisfies the condition $\frac{\mathrm{d}\tilde{s}_i(t)}{\mathrm{d}t}=0$ for any $i\geq 0$.

Based on the expected drift $\boldsymbol{F}(\boldsymbol{s})$, the fixed point condition $\dot{s}_i(t)=0$ implies that

$$\begin{aligned}\bar{\lambda}^{[\mathrm{M}]}s_{i-1}^d-\theta x s_i+\eta y\left(1-s_i\right)^b=&\\ \bar{\lambda}^{[\mathrm{M}]}s_i^d-\theta x s_{i+1}+\eta y\left(1-s_{i+1}\right)^b,&\ \forall i\in\mathcal{B}.\end{aligned}\tag{41}$$

Accordingly, we define the parameter γ as follows

$$\gamma\triangleq\bar{\lambda}^{[\mathrm{M}]}s_i^d-\theta x s_{i+1}+\eta y\left(1-s_{i+1}\right)^b,\quad\forall i\in\mathcal{B}.\tag{42}$$

Note that γ is a constant to be determined later. Rearranging the terms yields

$$\begin{aligned}s_i=&\left[\frac{\theta x s_{i+1}-\eta y(1-s_{i+1})^b+\gamma}{\bar{\lambda}^{[\mathrm{M}]}}\right]^{\frac{1}{d}}\\ =&g(s_{i+1};\gamma),\end{aligned}\tag{43}$$

where $s_{\mathrm{B}+1}=0$ and function $g(\cdot;\gamma)$ is defined in (20). Furthermore, the boundary condition in Definition 1 leads to

$$1=g^{\mathrm{B}+1}(0;\gamma).\tag{44}$$

This completes the proof of this theorem. □

Proof of Theorem 2. We will prove this theorem based on Lemma 3 and Lemma 4. Specifically, Proposition 1 indicates

$$\begin{aligned}\lim_{\delta\to\infty}L^{[\mathrm{N}][\delta]}(\boldsymbol{z},\boldsymbol{\Phi})=&\lim_{\delta\to\infty}\frac{1}{\delta}\int_0^\delta L^{[\mathrm{N}]}(\tau,\boldsymbol{z},\boldsymbol{\Phi})\mathrm{d}\tau\\ \overset{(a)}{=}&\lim_{\delta\to\infty}\frac{1}{\delta}\int_0^\delta\sum_{i=1}^{\infty}S_i^{[\mathrm{N}]}(\tau)\mathrm{d}\tau\\ \overset{(b)}{=}&\sum_{i=1}^{\infty}\lim_{\tau\to\infty}S_i^{[\mathrm{N}]}(\tau),\end{aligned}\tag{45}$$

where the equality (a) follows from Proposition 1, and the equality (b) follows from the limiting probability of the state $S_i^{[\mathrm{N}]}(\tau)$.

Based on Lemma 3 and Lemma 4, we have

$$\lim_{N\to\infty}\lim_{\tau\to\infty}\boldsymbol{S}^{[\mathrm{N}]}(\tau)=\tilde{\boldsymbol{s}}.\tag{46}$$

Combining (45) and (46) leads to

$$\lim_{\substack{M\to\infty\\ N=M\theta}}\lim_{\delta\to\infty}\left\|L^{[\mathrm{N}][\delta]}(\boldsymbol{z},\boldsymbol{\Phi})-l\big(\boldsymbol{z},\bar{\lambda}^{[\mathrm{M}]}\big)\right\|=0,\tag{47}$$

which completes the proof of this theorem. □

Lemma 3. *For any initial condition $\boldsymbol{s}^{\mathbf{0}}$, the mean field model converges exponentially to the fixed point $\tilde{\boldsymbol{s}}$, i.e.,*

$$\lim_{\tau\to\infty}\boldsymbol{s}(\tau)=\tilde{\boldsymbol{s}}.\tag{48}$$

Proof of Lemma 3. Given the mean field model $\boldsymbol{s}(\tau)$ in Definition 1 and the fixed point $\tilde{\boldsymbol{s}}$ in (14), we define the following Lyapunov function

$$\psi(\tau)\triangleq\sum_{i=1}^{\infty}\omega_i\big|s_i(\tau)-\tilde{s}_i\big|,\tag{49}$$

where $w_i\geq 1$ for any $i\geq 0$.

Next we prove this lemma based on Lemma 5 and Lemma 6, which are some intermediate results presented in Appendix C. We start with defining two bounds (i.e., $\boldsymbol{s}^{High}$ and $\boldsymbol{s}^{Low}$) for the original initial point $\boldsymbol{s}^{\mathbf{0}}$ as follows:

$$s_i^{High}\triangleq\max\left(s_i^0,\tilde{s}_i\right),\quad\forall i\geq 0,\tag{50a}$$

$$s_i^{Low}\triangleq\min\left(s_i^0,\tilde{s}_i\right),\quad\forall i\geq 0.\tag{50b}$$

Accordingly, we let $\boldsymbol{s}^{Low}(t)$ and $\boldsymbol{s}^{High}(t)$ denote the solution of the mean field model with the initial condition $\boldsymbol{s}^{Low}$ and $\boldsymbol{s}^{High}$, respectively. Lemma 6 indicates that

$$s_i^{Low}(\tau)\leq s_i(\boldsymbol{s}^{\mathbf{0}},\tau)\leq s_i^{High}(\tau),\quad\forall i\geq 1,\tau>0.\tag{51}$$

Lemma 5 indicates that

$$\lim_{\tau\to\infty}\boldsymbol{s}^{Low}(\tau)=\tilde{\boldsymbol{s}},\tag{52a}$$

$$\lim_{\tau\to\infty}\boldsymbol{s}^{High}(\tau)=\tilde{\boldsymbol{s}}.\tag{52b}$$

Based on the above discussions, Squeeze Theorem leads to

$$\lim_{\tau\to\infty}\boldsymbol{s}(\boldsymbol{s}^{\mathbf{0}},\tau)=\tilde{\boldsymbol{s}},\tag{53}$$

which completes the proof. □

Lemma 4. *Given the initial condition $\boldsymbol{s}(0)=\lim_{N\to\infty}\boldsymbol{S}^{[\mathrm{N}]}(0)$ for the mean field model, the following is true*

$$\lim_{N\to\infty}\sup_{\tau\in[0,\delta]}\left\|\boldsymbol{S}^{[\mathrm{N}]}(\tau)-\boldsymbol{s}(\tau)\right\|=0.\tag{54}$$

Proof of Lemma 4. We prove this lemma based on Kurtz's Theorem (e.g., [18], [30], [47], [48]), which consists of three conditions.

First, $\{\boldsymbol{S}^{[\mathrm{N}]}(\tau):\forall\tau\}$ is a density dependent Markov chain.

Second, we show that the state transition rate is bounded. Specifically, $\mathcal{E}\triangleq\{\pm\boldsymbol{e}^a:\forall a\geq 1\}$ denotes the state transition set. When the system stays in state $\boldsymbol{S}=(S_i\in[0,1]:\forall i\geq 0)$. The rates of the transitions $\boldsymbol{e}^a$ and $-\boldsymbol{e}^a$ are given by

$$\begin{aligned}\beta_{\boldsymbol{e}^a}(\boldsymbol{S})=&\frac{\lambda}{\theta}\left(S_{a-1}^d-S_a^d\right),\\ \beta_{-\boldsymbol{e}^a}(\boldsymbol{S})=&x\left(S_a-S_{a+1}\right)+\frac{\eta}{\theta}y\left[(1-S_{a+1})^b-(1-S_a)^b\right].\end{aligned}\tag{55}$$

Note that $S_a\in[0,1]$ for any $a\geq 1$. Hence the rate at which an transition occurs is bounded by $\frac{\lambda}{\theta}+x+\frac{\eta}{\theta}y$.

Third, the expected drift is Lipschitz continuous with the parameter $2\left(x+\frac{\lambda}{\theta}d+\frac{\eta}{\theta}yb\right)$ according to Lemma 7.

The three aspects above show that the three conditions of Kurtz's Theorem hold, which completes this proof. □

Proof of Corollary 1. When $(d,b)=(1,1)$, the fixed point condition $\dot{s}_i(t)=0$ implies that

$$\begin{aligned}\bar{\lambda}^{[\mathrm{M}]}s_{i-1}-\theta x s_i-\eta y s_i=&\\ \bar{\lambda}^{[\mathrm{M}]}s_i-\theta x s_{i+1}-\eta y s_{i+1},&\ \forall i\in\mathcal{B}.\end{aligned}\tag{56}$$

We suppose that the above equalities equals to v. That is,

$$v=\bar{\lambda}^{[\mathrm{M}]}s_i-\theta x s_{i+1}-\eta y s_{i+1},\ \forall i\in\mathcal{B}.\tag{57}$$

Accordingly, we let $\rho=\frac{\bar{\lambda}^{[\mathrm{M}]}}{\theta x+\eta y}$ and obtain

$$s_{i+1}=\rho s_i-\frac{v}{\theta x+\eta y}.\tag{58}$$

Based on the definition of the density-based state, we have

$$\begin{cases} s_0 = 1, \\ s_i = \rho^i - \dfrac{v}{\theta x + \eta y} \displaystyle\sum_{k=0}^{i-1} \rho^k, \quad \forall i \in \{1, 2, ..., \mathrm{B}\}, \\ s_{\mathrm{B}+1} = 0. \end{cases} \tag{59}$$

Therefore, the parameter v satisfies the following equality

$$\frac{v}{\theta x + \eta y} \sum_{k=0}^{\mathrm{B}} \rho^k = \rho^{\mathrm{B}+1}. \tag{60}$$

Now we calculate $l(x, y, \bar{\lambda}^{[\mathrm{M}]})$ as follows:

$$\begin{aligned} l(x, y, \bar{\lambda}^{[\mathrm{M}]}) &= \sum_{i=1}^{\mathrm{B}} \rho^i - \frac{v}{\theta x + \eta y} \sum_{i=1}^{\mathrm{B}} \sum_{k=0}^{i-1} \rho^k \\ &\overset{(a)}{=} \sum_{i=1}^{\mathrm{B}} \rho^i - \frac{\rho^{\mathrm{B}+1}}{\sum_{k=0}^{\mathrm{B}} \rho^k} \sum_{i=1}^{\mathrm{B}} \sum_{k=0}^{i-1} \rho^k \\ &= \frac{1}{1-\rho} + \mathrm{B} - \frac{1+\mathrm{B}}{1-\rho^{1+\mathrm{B}}} \end{aligned} \tag{61}$$

where the equality (a) follows from (60). This completes the proof of this corollary. □

Appendix B

This section provides the proofs of Lemma 1, Lemma 2, Theorem 3, Corollary 2, and Corollary 3.

Proof of Lemma 2. We define Φ_t as follows

$$\Phi_t \triangleq \sum_{(i,j)\in\mathcal{A}} w_t(i, j), \tag{62}$$

and prove this lemma in the following three steps.

Part I: We derive a lower bound for Φ_{T+1}. Specifically, for any tuple $(i, j) \in \mathcal{A}$, we have

$$\begin{aligned} \Phi_{T+1} &\overset{(a)}{\geq} w_{T+1}(i, j) \\ &\overset{(b)}{=} w_1(i, j) \prod_{t=1}^{T} (1-\epsilon)^{c_t(i,j)} \\ &\overset{(c)}{=} (1-\epsilon)^{\sum_{t=1}^{T} c_t(i,j)}, \end{aligned} \tag{63}$$

where the inequality (a) follows that $w_{T+1}(i, j) \geq 0$ for any $(i, j) \in \mathcal{A}$, the equality (b) follows the updating policy (9), and the equality (c) follows that $w_1(i, j) = 1$ for any $(i, j) \in \mathcal{A}$.

Part II: Based on the definition (62), we have

$$\begin{aligned} \frac{\Phi_{t+1}}{\Phi_t} &\overset{(a)}{=} \sum_{(i,j)\in\mathcal{A}} \frac{w_{t+1}(i, j)}{\Phi_t} \\ &\overset{(b)}{=} \sum_{(i,j)\in\mathcal{A}} \frac{w_t(i, j)}{\Phi_t} \cdot (1-\epsilon)^{c_t(i,j)} \\ &\overset{(c)}{=} \sum_{(i,j)\in\mathcal{A}} p_t(i, j) \cdot (1-\epsilon)^{c_t(i,j)} \\ &\overset{(d)}{\leq} \sum_{(i,j)\in\mathcal{A}} p_t(i, j) \cdot [1 - \epsilon \cdot c_t(i, j)] \\ &= 1 - \epsilon \cdot \sum_{(i,j)\in\mathcal{A}} c_t(i, j) p_t(i, j), \end{aligned} \tag{64}$$

where the equality (a) follows the definition of Φ_{t+1} in (62), the equality (b) is due to (30), the inequality (c) is due to (28), and the inequality (d) follows that $(1-\epsilon)^x \leq 1 - \epsilon x$ for any $x \in [0, 1]$.

Taking the logarithm on both sides yields

$$\begin{aligned} \ln\left(\frac{\Phi_{t+1}}{\Phi_t}\right) &\leq \ln\left(1 - \epsilon \cdot \sum_{(i,j)\in\mathcal{A}} c_t(i, j) p_t(i, j)\right) \\ &\overset{(a)}{\leq} -\epsilon \cdot \sum_{(i,j)\in\mathcal{A}} c_t(i, j) p_t(i, j) \\ &\overset{(b)}{=} -\epsilon \cdot \sum_{(i,j)\in\mathcal{A}} \frac{C_t(x_{[i]}, y_{[j]}, \bar{\lambda}_t^{[\mathrm{M}]})}{\bar{C}} \cdot p_t(i, j) \\ &\overset{(c)}{=} -\frac{\epsilon}{\bar{C}} \cdot \mathbb{E}\left[C_t\big(x_{[i]}, y_{[j]}, \bar{\lambda}_t^{[\mathrm{M}]}\big) \Big| \boldsymbol{p_t}\right], \end{aligned} \tag{65}$$

where the inequality (a) follows that $\ln(1-x) + x \leq 0$ for any $x \in [0, 1]$, the equality (b) follows the definition of $c_t(i, j)$, and the equality (c) follows $\sum_{(i,j)\in\mathcal{A}} p_t(i, j) = 1$. Rearranging the terms leads to

$$\mathbb{E}\left[C_t\big(x_{[i]}, y_{[j]}, \bar{\lambda}_t^{[\mathrm{M}]}\big) \Big| \boldsymbol{p_t}\right] \leq \frac{\bar{C}}{\epsilon} \left(\ln \Phi_t - \ln \Phi_{t+1}\right). \tag{66}$$

Summing over the above inequality over t, we obtain

$$\begin{aligned} \sum_{t=1}^{T} \mathbb{E}\left[C_t\big(x_{[i]}, y_{[j]}, \bar{\lambda}_t^{[\mathrm{M}]}\big) \Big| \boldsymbol{p_t}\right] &\leq \frac{\bar{C}}{\epsilon} \sum_{t=1}^{T} (\ln \Phi_t - \ln \Phi_{t+1}) \\ &\overset{(a)}{=} \frac{\bar{C}}{\epsilon} [\ln \Phi_1 - \ln \Phi_{T+1}] \\ &\overset{(b)}{=} \frac{\bar{C}}{\epsilon} [\ln(A_x A_y) - \ln \Phi_{T+1}], \end{aligned} \tag{67}$$

where the equality (a) uses the telescoping technique, and the equality (b) is due to $\Phi_1 = A_x A_y$.

Part III: Combining Part I and Part II leads to

$$\begin{aligned} C_T^{\mathfrak{A}} &= \sum_{t=1}^{T} \mathbb{E}\left[C_t\big(x_{[i]}, y_{[j]}, \bar{\lambda}_t^{[\mathrm{M}]}\big) \Big| \boldsymbol{p_t}\right] \\ &\leq \frac{\bar{C}}{\epsilon} [\ln(A_x A_y) - \ln \Phi_{T+1}] \\ &\overset{(a)}{\leq} \frac{\bar{C}}{\epsilon} \left[\ln(A_x A_y) + \ln\left(\frac{1}{1-\epsilon}\right) \sum_{t=1}^{T} c_t(i, j)\right] \\ &\overset{(b)}{=} \frac{\bar{C}}{\epsilon} \left[\ln(A_x A_y) + \ln\left(\frac{1}{1-\epsilon}\right) \sum_{t=1}^{T} \frac{C_t(x_{[i]}, y_{[j]}, \bar{\lambda}_t^{[\mathrm{M}]})}{\bar{C}}\right] \\ &= \frac{\bar{C} \ln(A_x A_y)}{\epsilon} + \frac{1}{\epsilon} \ln\left(\frac{1}{1-\epsilon}\right) \sum_{t=1}^{T} C_t(x_{[i]}, y_{[j]}, \bar{\lambda}_t^{[\mathrm{M}]}), \end{aligned} \tag{68}$$

where the inequality (a) substitutes (63), the equality (b) follows the definition (29). Therefore, for any tuple $(i, j) \in \mathcal{A}$, we have

$$C_T^{\mathfrak{A}} \leq \frac{\bar{C} \ln(A_x A_y)}{\epsilon} + \frac{1}{\epsilon} \ln\left(\frac{1}{1-\epsilon}\right) \sum_{t=1}^{T} C_t(x_{[i]}, y_{[j]}, \lambda_t). \tag{69}$$

Next we derive an upper bound for $\ln(A_x A_y)$

$$\begin{aligned}
&\ln(A_x A_y) \\
=&\ln\left[\left(1+\left\lfloor \log_{1+\beta}\left(\tfrac{x_H}{x_L}\right)\right\rfloor\right)\left(1+\left\lfloor \log_{1+\beta}\left(\tfrac{y_H}{y_L}\right)\right\rfloor\right)\right] \\
\le&\ln\left[\left(1+\log_{1+\beta}\left(\tfrac{x_H}{x_L}\right)\right)\left(1+\log_{1+\beta}\left(\tfrac{y_H}{y_L}\right)\right)\right] \\
=&\ln\left(1+\tfrac{\ln\left(\frac{x_H}{x_L}\right)}{\ln(1+\beta)}\right)+\ln\left(1+\tfrac{\ln\left(\frac{y_H}{y_L}\right)}{\ln(1+\beta)}\right) \\
=&\frac{\epsilon}{\bar{C}}\Psi(\epsilon,\beta).
\end{aligned} \tag{70}$$

Therefore, for any $(i,j)\in\mathcal{A}$ we obtain

$$C_T^{\mathfrak{A}} \le \Psi(\epsilon,\beta)+\frac{1}{\epsilon}\ln\left(\frac{1}{1-\epsilon}\right)\sum_{t=1}^{T} C_t(x_{[i]},y_{[j]},\bar{\lambda}_t^{[\mathrm{M}]}). \tag{71}$$

This completes the proof of this lemma. □

Proof of Lemma 1. We prove this lemma by deriving an upper bound for $C_t(x_{[i^*]},y_{[j^*]},\bar{\lambda}_t^{[\mathrm{M}]})$ in two steps.

$$\begin{aligned}
&C_t(x_{[i^*]},y_{[j^*]},\bar{\lambda}_t^{[\mathrm{M}]}) \\
=&G(l(x_{[i^*]},y_{[j^*]},\bar{\lambda}_t^{[\mathrm{M}]}))+\xi_t^{\mathsf{P}}\theta x_{[i^*]}^{\sigma}+\xi_t^{\mathsf{A}}\eta y_{[j^*]}^{\sigma} \\
\overset{(a)}{=}&G(l(x_{[i^*]},y_{[j^*]},\bar{\lambda}_t^{[\mathrm{M}]}))+\xi_t^{\mathsf{P}}\theta\left[x_{[i^*-1]}(1+\beta)\right]^{\sigma} \\
&+\xi_t^{\mathsf{A}}\eta\left[y_{[j^*-1]}(1+\beta)\right]^{\sigma} \\
\overset{(b)}{\le}&G(l(x_{[i^*]},y_{[j^*]},\bar{\lambda}_t^{[\mathrm{M}]}))+\xi_t^{\mathsf{P}}\theta(x^*)^{\sigma}(1+\beta)^{\sigma} \\
&+\xi_t^{\mathsf{A}}\eta(y^*)^{\sigma}(1+\beta)^{\sigma} \\
\overset{(c)}{\le}&G(l(x^*,y^*,\bar{\lambda}_t^{[\mathrm{M}]}))+\xi_t^{\mathsf{P}}\theta(x^*)^{\sigma}(1+\beta)^{\sigma} \\
&+\xi_t^{\mathsf{A}}\eta(y^*)^{\sigma}(1+\beta)^{\sigma} \\
\le&(1+\beta)^{\sigma}C_t(x^*,y^*,\bar{\lambda}_t^{[\mathrm{M}]}),
\end{aligned} \tag{72}$$

where the equality (a) follows from the definitions in (26) and (27). The inequality (b) follows from the condition in (33). The inequality (c) follows that $G(l(x,y,\bar{\lambda}^{[\mathrm{M}]}))$ is decreasing in (x,y). This completes the proof. □

Proof of Theorem 3. Combining Lemma 2 and Lemma 1 completes the proof of this theorem. □

Proof of Corollary 2. Note that the parameter setting in this corollary implies that $\beta=\frac{\alpha}{2^{1+\sigma}}\in(0,1]$, since $\alpha\in(0,2^{\sigma+1}]$. Hence we obtain

$$\begin{aligned}
\Psi(\epsilon,\beta)=&\tfrac{\bar{C}}{\epsilon}\left[\ln\left(1+\tfrac{\ln\left(\frac{x_H}{x_L}\right)}{\ln(1+\beta)}\right)+\ln\left(1+\tfrac{\ln\left(\frac{y_H}{y_L}\right)}{\ln(1+\beta)}\right)\right] \\
\overset{(a)}{\le}&\tfrac{\bar{C}}{\epsilon}\left[\ln\left(\tfrac{\ln\left(\frac{2x_H}{x_L}\right)}{\ln(1+\beta)}\right)+\ln\left(\tfrac{\ln\left(\frac{2y_H}{y_L}\right)}{\ln(1+\beta)}\right)\right] \\
\overset{(b)}{=}&\tfrac{\bar{C}}{\epsilon}\left[2\ln\left(\tfrac{1}{\ln(1+\beta)}\right)+J-2\ln(2)\right] \\
\overset{(c)}{\le}&\tfrac{\bar{C}}{\epsilon}\left[2\ln\left(\tfrac{1}{2}+\tfrac{1}{\beta}\right)+J-2\ln(2)\right] \\
\overset{(d)}{\le}&\frac{\bar{C}}{\epsilon}\left[2\ln\left(\frac{1}{\beta}\right)+J\right]
\end{aligned} \tag{73}$$

where the last inequality follows that $1\le\frac{\ln(2)}{\ln(1+\beta)}$ for any $\beta\in(0,1)$. The equality (b) follows from $J=\ln\left(\ln\frac{2x_H}{x_L}\right)+\ln\left(\ln\frac{2y_H}{y_L}\right)+2\ln(2)$. The inequality (c) follows that $\frac{1}{\ln(1+\beta)}\le\frac{1}{2}+\frac{1}{\beta}$ for any $\beta>0$. The inequality (d) follows from $\frac{1}{2}\le\frac{1}{\beta}$. When $(\epsilon,\beta)=\left(\frac{\alpha}{2^{1+\sigma}},\frac{\alpha}{2^{1+\sigma}}\right)$, we obtain

$$\Psi\left(\frac{\alpha}{2^{1+\sigma}},\frac{\alpha}{2^{1+\sigma}}\right)\le\frac{2^{1+\sigma}\bar{C}}{\alpha}\left[2\ln\left(\frac{2^{2+\sigma}}{\alpha}\right)+J\right]\le\frac{\alpha C_T^*}{2^{1+\sigma}}, \tag{74}$$

where the last inequality is due to the condition (38).

Furthermore, we have

$$\begin{aligned}
\frac{1}{\epsilon}\ln\left(\frac{1}{1-\epsilon}\right)(1+\beta)^{\sigma}\overset{(a)}{\le}&(1+\epsilon)(1+\beta)^{\sigma} \\
\le&[1+\max(\epsilon,\beta)]^{1+\sigma} \\
\overset{(b)}{\le}&1+(2^{1+\sigma}-1)\cdot\max(\epsilon,\beta) \\
=&1+\frac{2^{1+\sigma}-1}{2^{1+\sigma}}\cdot\alpha,
\end{aligned} \tag{75}$$

where the inequality (a) follows that $\frac{1}{\epsilon}\ln\left(\frac{1}{1-\epsilon}\right)\le 1+\epsilon$ for any $\epsilon\in[0,\frac{2}{3}]$, the inequality (b) follows that $\max(\epsilon,\beta)\in[0,1]$ and $(1+v)^a\le 1+(2^a-1)v$ for any $v\in[0,1]$.

Combining the above two steps leads to

$$C_T^{\mathfrak{A}}\le\frac{\alpha C_T^*}{2^{1+\sigma}}+\left(1+\frac{2^{1+\sigma}-1}{2^{1+\sigma}}\cdot\alpha\right)C_T^*=(1+\alpha)C_T^*, \tag{76}$$

which completes the proof of this corollary. □

Proof of Corollary 3. With the parameters $\epsilon=\sqrt{\frac{\ln(T)}{T}}$ and $\beta=\frac{1}{\sqrt{T}}$, we have $\epsilon>\beta$ and $\beta\in(0,1)$.

First, based on the inequality (75) and $\epsilon>\beta$, we obtain

$$\frac{1}{\epsilon}\ln\left(\frac{1}{1-\epsilon}\right)(1+\beta)^{\sigma}\le 1+\left(2^{1+\sigma}-1\right)\epsilon. \tag{77}$$

Second, based on (73) and $\beta\in(0,1)$, we obtain

$$\Psi(\epsilon,\beta)\le\frac{\bar{C}}{\epsilon}\left[2\ln\left(\frac{1}{\beta}\right)+J\right]. \tag{78}$$

Combining the above two steps yields

$$C_T^{\mathfrak{A}}\le\frac{\bar{C}}{\epsilon}\left[2\ln\left(\frac{1}{\beta}\right)+J\right]+\left[1+\left(2^{1+\sigma}-1\right)\epsilon\right]C_T^*. \tag{79}$$

Rearranging the terms leads to

$$\begin{aligned}
&C_T^{\mathfrak{A}}-C_T^* \\
\le&\frac{\bar{C}}{\epsilon}\left[2\ln\left(\frac{1}{\beta}\right)+J\right]+\left(2^{1+\sigma}-1\right)\epsilon C_T^* \\
\overset{(a)}{=}&\bar{C}\sqrt{\frac{T}{\ln(T)}}\left[\ln(T)+J\right]+\left(2^{1+\sigma}-1\right)C_T^*\sqrt{\frac{\ln(T)}{T}} \\
\overset{(b)}{\le}&\bar{C}\sqrt{\frac{T}{\ln(T)}}\left[\ln(T)+J\right]+\left(2^{1+\sigma}-1\right)\bar{C}\sqrt{T\ln(T)} \\
=&\bar{C}\sqrt{T\ln(T)}\left[2^{1+\sigma}+\frac{J}{\ln(T)}\right]
\end{aligned} \tag{80}$$

where the equality (a) substitutes the parameters $\epsilon=\sqrt{\frac{\ln(T)}{T}}$ and $\beta=\frac{1}{\sqrt{T}}$. The inequality (b) follows that $C_T^*\le T\bar{C}$. This completes the proof. □

Appendix C

This section introduces three lemmas used in Appendix A.

Lemma 5. *If $s_i(0) \le \tilde{s}_i$ (or $s_i(0) \ge \tilde{s}_i$) for any $i \ge 1$, then $\psi(\tau)$ converges exponentially to zero, i.e.,*

$$\lim_{\tau\to\infty} \psi(\tau) = 0. \tag{81}$$

Proof of Lemma 5. We prove this lemma by showing that there exist a constant $\delta > 0$ such that we have

$$\frac{\mathrm{d}\psi(t)}{\mathrm{d}t} \le \delta\psi(t), \tag{82}$$

which indicates $\psi(t) \le \psi(0)\exp(-\delta t)$.

For notation simplicity, we define $\varepsilon_i(t)$ as follows:

$$\varepsilon_i(t) \triangleq s_i(t) - \tilde{s}_i, \quad \forall i \ge 0. \tag{83}$$

The Lyapunov function defined in (49) can be expressed as $\psi(t) = \sum_{i\ge1}^{\infty} w_i|\varepsilon_i(t)|$. Next we show how to prove (82) for the case $\varepsilon_i(t) \le 0$ for any $i \ge 1$. In this case, we have $\psi(t) = -\sum_{i\ge1}^{\infty} w_i\varepsilon_i(t)$. For notation simplicity, we define function $H(\cdot)$ as follows:

$$H(\varepsilon_i(t), \tilde{s}_i, d) \triangleq [\varepsilon_i(t) + \tilde{s}_i]^d - \tilde{s}_i^d. \tag{84}$$

Based on the mean field model, we have

$$\begin{aligned}
\frac{\mathrm{d}\varepsilon(t)}{\mathrm{d}t} =& \frac{\mathrm{d}s_i(t)}{\mathrm{d}t} \\
=& \frac{\lambda}{\theta}\left[s_{i-1}^d(t) - s_i^d(t)\right] - x\left[s_i(t) - s_{i+1}(t)\right] \\
& - \frac{\eta}{\theta}\cdot y\left(\left[1 - s_{i+1}(t)\right]^b - \left[1 - s_i(t)\right]^b\right) \\
\overset{(a)}{=}& \frac{\lambda}{\theta}\left(\left[\varepsilon_{i-1}(t) + \tilde{s}_{i-1}\right]^d - \left[\varepsilon_i(t) + \tilde{s}_i\right]^d\right) \\
& - x\left[\varepsilon_i(t) + \tilde{s}_i - \varepsilon_{i+1}(t) - \tilde{s}_{i+1}\right] \\
& - \frac{\eta}{\theta}\cdot y\left(\left[1 - \varepsilon_{i+1}(t) - \tilde{s}_{i+1}\right]^b - \left[1 - \varepsilon_i(t) - \tilde{s}_i\right]^b\right) \\
\overset{(b)}{=}& \frac{\lambda}{\theta}\left[H(\varepsilon_{i-1}(t), \tilde{s}_{i-1}, d) + \tilde{s}_{i-1}^d - H(\varepsilon_i(t), \tilde{s}_i, d) - \tilde{s}_i^d\right] \\
& - x\left[\varepsilon_i(t) - \varepsilon_{i+1}(t) + \tilde{s}_i - \tilde{s}_{i+1}\right] \\
& - \frac{\eta}{\theta}\cdot y\Big[H(-\varepsilon_{i+1}(t), 1 - \tilde{s}_{i+1}, b) + (1 - \tilde{s}_{i+1})^b \\
& - H(-\varepsilon_i(t), 1 - \tilde{s}_i, b) - (1 - \tilde{s}_i)^b\Big] \\
\overset{(c)}{=}& \frac{\lambda}{\theta}\left[H(\varepsilon_{i-1}(t), \tilde{s}_{i-1}, d) - H(\varepsilon_i(t), \tilde{s}_i, d)\right] \\
& - x\left[\varepsilon_i(t) - \varepsilon_{i+1}(t)\right] \\
& - \frac{\eta}{\theta}\cdot y\Big[H(-\varepsilon_{i+1}(t), 1 - \tilde{s}_{i+1}, b) \\
& - H(-\varepsilon_i(t), 1 - \tilde{s}_i, b)\Big],
\end{aligned} \tag{85}$$

where the equality (a) follows from (83), the equality (b) follows from the definition in (84), and the equality (c) follows that $\tilde{\boldsymbol{s}}$ is the fixed point. Therefore, the terms of $\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}$ involving $\varepsilon_i(t)$ is given by (86). We want to choose w_{i-1}, w_i, and w_{i+1} such that (87) holds. Note that (87) is equivalent to (88). Moreover, $\frac{H(\varepsilon_i(t), \tilde{s}_i, d)}{-\varepsilon_i(t)}$ and $\frac{H(-\varepsilon_i(t), 1-\tilde{s}_i, b)}{-\varepsilon_i(t)}$ is decreasing in $\varepsilon_i(t) \in [-1, 1]$. Therefore, the condition (88) holds if we ensure

$$w_{i-1} \le w_i \le w_{i+1}, \tag{89a}$$

$$\delta w_i \le \tfrac{w_{i+1} - w_i}{\theta}\lambda\tfrac{H(1, \tilde{s}_i, d)}{-1} \tag{89b}$$

$$+ \tfrac{w_i - w_{i-1}}{\theta}\left(\theta x + \eta y\tfrac{H(-1, 1-\tilde{s}_i, b)}{-1}\right). \tag{89c}$$

Moreover, (89c) is equivalent to

$$\begin{aligned}
& w_{i+1} \\
\le\;& w_i + \frac{(w_i - w_{i-1})\left[x - \frac{\eta}{\theta}yH(-1, 1 - \tilde{s}_i, b)\right] - \delta w_i}{\frac{\lambda}{\theta}H(1, \tilde{s}_i, d)} \\
=\;& w_i + \frac{(w_i - w_{i-1})\left(x + \frac{\eta}{\theta}y\left[(1 - \tilde{s}_i)^d - (-\tilde{s}_i)^d\right]\right) - \delta w_i}{\frac{\lambda}{\theta}\left[(1 + \tilde{s}_i)^d - \tilde{s}_i^d\right]}.
\end{aligned} \tag{90}$$

We are able to choose an increasing sequence $\{w_i\}$ (starting with $w_0 = 0$ and $w_1 = 1$) and δ that satisfy (90). □

Lemma 6. *Let $\boldsymbol{s}^\star(\tau)$ and $\boldsymbol{s}(\tau)$ denote the solutions of the mean field model with initial conditions $\boldsymbol{s}^\star(0)$ and $\boldsymbol{s}(0)$, respectively. If $s_i^\star(0) \le s_i(0)$ for any $i \ge 1$, then we have*

$$s_i^\star(\tau) \le s_i(\tau), \quad \forall i \ge 1 \text{ and } \tau > 0. \tag{91}$$

Proof of Lemma 6. We prove this lemma by contradiction. Given the initial conditions satisfying $s_i^\star(0) < s_i(0)$ for any $i \ge 1$, we suppose that the inequality is first violated at t'. That is, there exits $k \ge 1$ such that $s_k^\star(t') = s_k(t')$. Then we consider the following cases.
1) Case $s_{k-1}^\star(t') < s_{k-1}(t')$ and $s_{k+1}^\star(t') < s_{k+1}(t')$. In this case, we have

$$\begin{aligned}
& \frac{\mathrm{d}s_k^\star(t')}{\mathrm{d}t} - \frac{\mathrm{d}s_k(t')}{\mathrm{d}t} \\
=& \frac{\lambda}{\theta}\left(\left[s_{k-1}^\star(t')\right]^d - \left[s_k^\star(t')\right]^d\right) - x\big(s_k^\star(t') - s_{k+1}^\star(t')\big) \\
& - \frac{\eta}{\theta}y\left(\left[1 - s_{k+1}^\star(t')\right]^b - \left[1 - s_k^\star(t')\right]^b\right) \\
& - \frac{\lambda}{\theta}\left(\left[s_{k-1}(t')\right]^d - \left[s_k(t')\right]^d\right) + x\big(s_k(t') - s_{k+1}(t')\big) \\
& + \frac{\eta}{\theta}y\left(\left[1 - s_{k+1}(t')\right]^b - \left[1 - s_k(t')\right]^b\right) \\
=& \frac{\lambda}{\theta}\left(\left[s_{k-1}^\star(t')\right]^d - \left[s_{k-1}(t')\right]^d\right) - x\big(s_{k+1}(t') - s_{k+1}^\star(t')\big) \\
& - \frac{\eta}{\theta}y\left(\left[1 - s_{k+1}^\star(t')\right]^b - \left[1 - s_{k+1}(t')\right]^b\right) \\
\le& 0.
\end{aligned} \tag{92}$$

Therefore, there exists $t'' < t'$ such that

$$\begin{cases} s_k^\star(t'') - s_k(t'') < 0, \\ \dfrac{\mathrm{d}s_k^\star(t)}{\mathrm{d}t} - \dfrac{\mathrm{d}s_k(t)}{\mathrm{d}t} < 0, \quad \forall t \in [t'', t'], \end{cases} \tag{93}$$

which leads to

$$\begin{aligned}
& s_k^\star(t') - s_k(t') \\
=& s_k^\star(t'') - s_k(t'') + \int_{t''}^{t'}\left[\frac{\mathrm{d}s_k^\star(t)}{\mathrm{d}t} - \frac{\mathrm{d}s_k(t)}{\mathrm{d}t}\right]\mathrm{d}t < 0.
\end{aligned} \tag{94}$$

This contradicts with $s_k^\star(t') = s_k(t')$.

$$
\begin{aligned}
&-\frac{w_i}{\theta}\Big(-\lambda H(\varepsilon_i(t),\tilde{s}_i,d)-\theta x\varepsilon_i(t)+\eta y H(-\varepsilon_i(t),1-\tilde{s}_i,b)\Big)-\frac{w_{i+1}}{\theta}\lambda H(\varepsilon_i(t),\tilde{s}_i,d)\\
&\quad-\frac{w_{i-1}}{\theta}\Big(\theta x\varepsilon_i(t)-\eta y H(-\varepsilon_i(t),1-\tilde{s}_i,b)\Big)\\
=&\frac{w_i-w_{i+1}}{\theta}\lambda H(\varepsilon_i(t),\tilde{s}_i,d)+\frac{w_i-w_{i-1}}{\theta}\Big(\theta x\varepsilon_i(t)-\eta y H(-\varepsilon_i(t),1-\tilde{s}_i,b)\Big).
\end{aligned}
\tag{86}
$$

$$
\frac{w_i-w_{i+1}}{\theta}\lambda H(\varepsilon_i(t),\tilde{s}_i,d)+\frac{w_i-w_{i-1}}{\theta}\Big(\theta x\varepsilon_i(t)-\eta y H(-\varepsilon_i(t),1-\tilde{s}_i,b)\Big)\le\delta w_i\epsilon_i(t).
\tag{87}
$$

$$
\frac{w_{i+1}-w_i}{\theta}\lambda\frac{H(\varepsilon_i(t),\tilde{s}_i,d)}{-\varepsilon_i(t)}+\frac{w_i-w_{i-1}}{\theta}\left(\theta x+\eta y\frac{H(-\varepsilon_i(t),1-\tilde{s}_i,b)}{-\varepsilon_i(t)}\right)\ge\delta w_i.
\tag{88}
$$

2) Case $s^{\star}_{k-1}(t')=s_{k-1}(t')$ and $s^{\star}_{k+1}(t')<s_{k+1}(t')$. Similar contradiction still exists.
3) Case $s^{\star}_{k-1}(t')<s_{k-1}(t')$ and $s^{\star}_{k+1}(t')=s_{k+1}(t')$. Similar contradiction still exists.
4) Case $s^{\star}_{k-1}(t')=s_{k-1}(t')$ and $s^{\star}_{k+1}(t')=s_{k+1}(t')$. If $s^{\star}_{k-2}(t')<s_{k-2}(t')$ or $s^{\star}_{k+2}(t')<s_{k+2}(t')$, then similar contradiction still exists.

The discussions complete this proof. □

Lemma 7. *The expected drift $\boldsymbol{F}(\cdot)$ defined in Proposition 2 satisfies*

$$
\big\|\boldsymbol{F}(\boldsymbol{s})-\boldsymbol{F}(\boldsymbol{r})\big\|\le 2\left(x+\frac{\lambda}{\theta}d+\frac{\eta}{\theta}yb\right)\big\|\boldsymbol{s}-\boldsymbol{r}\big\|.
\tag{95}
$$

Proof of Lemma 7. Based on the definition (14), we obtain

$$
\begin{aligned}
\|\boldsymbol{F}(\boldsymbol{s})-\boldsymbol{F}(\boldsymbol{r})\|=&\sum_{i=0}^{\infty}\Big\|\frac{\lambda}{\theta}\left(s_{i-1}^d-s_i^d\right)-x\big(s_i-s_{i+1}\big)\\
&-\frac{\eta}{\theta}y\left[(1-s_{i+1})^b-(1-s_i)^b\right]\\
&-\frac{\lambda}{\theta}\left(r_{i-1}^d-r_i^d\right)+x\big(r_i-r_{i+1}\big)\\
&+\frac{\eta}{\theta}y\left[(1-r_{i+1})^b-(1-r_i)^b\right]\Big\|\\
=&\sum_{i=0}^{\infty}\Big\|\frac{\lambda}{\theta}\left(s_{i-1}^d-r_{i-1}^d\right)-\frac{\lambda}{\theta}\left(s_i^d-r_i^d\right)\\
&-x\left(s_i-r_i\right)+x\left(s_{i+1}-r_{i+1}\right)\\
&-\frac{\eta}{\theta}y\left[(1-s_{i+1})^b-(1-r_{i+1})^b\right]\\
&+\frac{\eta}{\theta}y\left[(1-s_i)^b-(1-r_i)^b\right]\Big\|\\
\le&\sum_{i=0}^{\infty}2x\|s_i-r_i\|+\frac{2\lambda}{\theta}\Big\|s_i^d-r_i^d\Big\|\\
&+\frac{\eta}{\theta}2y\Big\|(1-s_i)^b-(1-r_i)^b\Big\|,
\end{aligned}
\tag{96}
$$

where the last inequality follows from the triangle inequality. Furthermore, we have

$$
\Big\|s_i^d-r_i^d\Big\|=\left\|(s_i-r_i)\sum_{k=0}^{d-1}s_i^k\cdot r_i^{d-1-k}\right\|\le\big\|s_i-r_i\big\|d,
\tag{97}
$$

and

$$
\begin{aligned}
&\Big\|(1-s_i)^b-(1-r_i)^b\Big\|\\
=&\left\|(r_i-s_i)\sum_{k=0}^{b-1}(1-s_i)^k\cdot(1-r_i)^{b-1-k}\right\|\\
\le&\big\|r_i-s_i\big\|b,
\end{aligned}
\tag{98}
$$

where the inequalities follows that $0\le s_i,r_i\le 1$.

Therefore, we have

$$
\begin{aligned}
\|\boldsymbol{F}(\boldsymbol{s})-\boldsymbol{F}(\boldsymbol{r})\|\le&2\left(x+\frac{\lambda}{\theta}d+\frac{\eta}{\theta}yb\right)\sum_{i=0}^{\infty}\big\|s_i-r_i\big\|\\
=&2\left(x+\frac{\lambda}{\theta}d+\frac{\eta}{\theta}yb\right)\|\boldsymbol{s}-\boldsymbol{r}\|.
\end{aligned}
\tag{99}
$$

This completes the proof. □