Gram-Schmidt orthogonalization

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- Least squares and Gram-Schmidt method

Suppose you have linearly independent vectors $v_1, v_2, \ldots, v_n$. How to find an orthonormal basis for span$\{v_1, v_2, \ldots, v_n\}$?
Step 1. Pick any vector, say v_1, and make it of lenth 1:

$$e_1 = v_1/|v_1|.$$

Step 2. Pick any of the remaining vectors, say v_2, make an orhtogonal to v_1:

$$v_2' = v_2 - P_{e_1}(v_2),$$

where P_{v_1} is the projection operator on vector e_1. Hence

$$v_2' = v_2 - \langle v_2, e_1\rangle e_1 = v_2 - \langle v_2, v_1\rangle v_1/\langle v_1, v_1\rangle$$

Step 3. make v_2' of lenth 1:

$$e_2 = v_2'/|v_2'|$$

Inductive step:

$$v_k' = v_k - \sum_{i<k} \langle v_k, v_i\rangle v_i/\langle v_i, v_i\rangle$$

$$e_k = v_k'/|v_k'|.$$

Question: Is it possible to do the same when you have $v_1, v_2, \ldots, v_n, \ldots$?
Example 1. Legendre polynomials.
Consider polynomials of degree $n-1$ on $[-1,1]$ with the L_2 inner product. Then polynomials with the leading coefficient 1

$$1, x, x^2 - 1/3, \ldots$$

is an orthogonal basis. Note, that Legendre polynomials are not an orthonormal basis.
Question: Is is possible to make Legendre polynomials orthonormal?

Example 2. Chebyshev Polynomials. Consider the vector space of polynomials on the interval $[-1,1]$ with the inner product

$$\langle g,h\rangle = \int_{-1}^{1} g(x)h(x)/(1-x^2)^{1/2}dx$$

We want to construct an orthogonal basis $T_k, k = 0,1,2,3,...$ for such polynomials, such that for every k T_k is a polynomial of degree at most k, $\langle T_0, T_0\rangle = \pi$ $\langle T_k T_k\rangle = \pi/2$, $k > 0$:

$$T_0 = 1, T_1 = x, T_2 = 2x^2 - 1, T_{n+1} = 2xT_n - T_{n-1}.$$

Definition A square matrix is called orthogonal if its columns are orthonormal.
Lemma If Q is orthogonal, then $Q^{-1} = Q^t$
Question: Recall the matrix of the Fourier transform, is it orthogonal?
Example 1: Rotation matrix in $\mathbb{R}^2$
Question: Is a permutation matrix orthogonal?
Question: Is a projection matrix orthogonal?
Lemma Multiplication by an orthogonal matrix preserves inner product.
Question: What does it tell us about angles and lengths? Why?
Lemma The Gram-Schmidt method implies that

$$\begin{pmatrix} v_1 & v_2 & \dots & v_n \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & \dots & e_n \end{pmatrix} \begin{pmatrix} \langle v_1, e_1\rangle & \langle v_2, e_1\rangle & \dots & \langle v_n, e_1\rangle \\ 0 & \langle v_2, e_2\rangle & \dots & \langle v_n, e_2\rangle \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \langle v_n, e_n\rangle \end{pmatrix}$$

for linearly independent vectors $v_1, v_2, \dots, v_n$.
Corollary $A = QR$, if columns of A are linearly independent.
Question: Suppose $A = QR$, how does it simplify the least squares method?