CMPS 102 — Fall 2018 — Homework 2

"I have read and agree to the collaboration policy." - ***Kevin Wang***

Solution to Problem 1: Hadamard Matrix

Claim 1. $H_n = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$ *where* $H_n[i,j] = \frac{1}{\sqrt{2^n}}(-1)^{i \cdot j}$

Proof. Given that $H_n[i,j] = \frac{1}{\sqrt{2^n}}(-1)^{i \cdot j}$, let matrix $H_n = \frac{1}{\sqrt{2^n}}H'_n$ where $H'_n[i,j] = (-1)^{i \cdot j}$.
Let A_n, B_n, C_n, and D_n be the four $2^{n-1} \times 2^{n-1}$ sized submatrices formed when dividing matrix H_n into quarters as shown below:

$$\begin{array}{cc} & \begin{array}{cc} i & i' \end{array} \\ \begin{array}{c} j \\ j' \end{array} & \begin{bmatrix} A & B \\ C & D \end{bmatrix} \end{array}$$

where $i, j \in [0, 2^{n-1})$
where $i', j' \in [2^{n-1}, 2^n)$

Note that $i', j' = i, j + 2^{n-1}$, where 2^{n-1} is the n-bit binary number consisting of the hi-order bit 1 followed by $(n-1)$ 0 bits.

By definition of a Walsh-Hadamard matrix, the submatrix A_n is constructed such that the (i,j)-th entry of $A_n[i,j] = \frac{1}{\sqrt{2^n}}(-1)^{i \cdot j}$ where $i, j \in [0, 2^{n-1})$. As such,

$$\begin{aligned} A_n[i,j] &= \frac{1}{\sqrt{2^n}}H'_{n-1}[i,j] \\ &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2^{n-1}}}H'_{n-1}[i,j] \\ \implies A_n &= \frac{1}{\sqrt{2}}H_{n-1} \end{aligned}$$

By definition of a Walsh-Hadamard matrix, the submatrix B_n is constructed such that the (i',j)-th entry of $B_n[i',j] = \frac{1}{\sqrt{2^n}}(-1)^{i' \cdot j}$ where $i' = i + 2^{n-1}$. As such,

$$\begin{aligned} i' \cdot j &= (i + 2^{n-1}) \cdot j \\ &= (i \cdot j) + (2^{n-1} \cdot j) \\ &= (i \cdot j) + [(1)(0) + (0)(j_{bit}) + (0)...] \\ &= (i \cdot j) + 0 \\ B_n[i',j] &= \frac{1}{\sqrt{2^n}}H'_{n-1}[i',j] \\ &= \frac{1}{\sqrt{2^n}}H'_{n-1}[i,j] \\ &= \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2^{n-1}}}H'_{n-1}[i,j] \\ \implies B_n &= \frac{1}{\sqrt{2}}H_{n-1} \end{aligned}$$

We note that the vice versa i, j' is an equivalent case. As such, $C_n = \frac{1}{\sqrt{2}} H_{n-1}$.

Finally, by definition of a Walsh-Hadamard matrix, the submatrix D_n is constructed such that the (i', j')-th entry of $D_n[i', j'] = \frac{1}{\sqrt{2^n}}(-1)^{i' \cdot j'}$ where $i', j' = i, j + 2^{n-1}$. As such,

$$\begin{aligned}
i' \cdot j' &= (i + 2^{n-1}) \cdot (j + 2^{n-1}) \\
&= (i \cdot (j + 2^{n-1})) + (2^{n-1} \cdot (j + 2^{n-1})) \\
&= (i \cdot j) + (i \cdot 2^{n-1}) + (j \cdot 2^{n-1}) + (2^{n-1} \cdot 2^{n-1}) \\
&= (i \cdot j) + 2[(1)(0) + (0)(i, j_{bit}) + (0)...] + [(1)(1) + (0)(0) + ...] \\
&= (i \cdot j) + (0) + (1) \\
&= (i \cdot j) + 1 \\
D_n[i', j'] &= \frac{1}{\sqrt{2^n}} H'_{n-1}[i', j'] \\
&= \frac{1}{\sqrt{2^n}} [H'_{n-1}[i, j] \times (-1^1)] \\
&= -\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2^{n-1}}} H'_{n-1}[i, j] \\
\implies D_n &= -\frac{1}{\sqrt{2}} H_{n-1}
\end{aligned}$$

Lastly, we piece the matrix together and get the recursive representation:

$$\begin{aligned}
H_n &= \begin{bmatrix} A_n & B_n \\ C_n & D_n \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} H_{n-1} & \frac{1}{\sqrt{2}} H_{n-1} \\ \frac{1}{\sqrt{2}} H_{n-1} & -\frac{1}{\sqrt{2}} H_{n-1} \end{bmatrix} \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}
\end{aligned}$$

$\square$

Claim 2. *The Euclidian norm of every column and every row is 1.*

Proof. Let matrix $H_n = \frac{1}{\sqrt{2^n}} H'_n$ where $H'_n[i, j] = (-1)^{i \cdot j}$.

We define the Euclidian norm as the square root of the absolute squares of its elements. Because each element of H_n is equal to $\frac{1}{\sqrt{2^n}} \times \pm 1$, the Euclidian norm of every column and row in the $2^n \times 2^n$ Walsh-Hadamard matrix H_n is equal to:

$$\begin{aligned} \|H_n\| &= \sqrt{\left(\pm \frac{1}{\sqrt{2^n}} \right)^2 \times 2^n} \\ &= \sqrt{\frac{2^n}{2^n}} \\ &= 1 \end{aligned}$$

$\square$

Lemma 1. Let M be a matrix with size $2^n \times 2^n$ and let M^T be its transpose. If M is an orthonormal matrix, then $M^T M = I_{2^n \times 2^n}$.

Proof. The result of $M^T M$ is a matrix where $[i, j]$ is equal to the dot product of columns i and j. As the dot product of every column must be 0 and each column has a Euclidian norm of 1, the result is the Identity matrix. $\square$

Claim 3. *The columns of H_n form an orthonormal basis.*

Proof. Let $H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$ where $H_n[i, j] = \frac{1}{\sqrt{2^n}}(-1)^{i \cdot j}$.

Let $H_n^T = H_n$ due to the commutative property of the bitwise dot product $i \cdot j$. By Lemma 1, the base case H_1 is an orthonormal matrix:

$$\begin{aligned}
H_1^T H_1 &= H_1 H_1 \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 1+1 & 1-1 \\ 1-1 & 1+1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= I_{2^1 \times 2^1}
\end{aligned}$$

Let us assume that H_{n-1} is orthonormal:

$$H_{n-1}^T H_{n-1} = H_{n-1} H_{n-1} = I_{2^{n-1} \times 2^{n-1}}$$

Then:

$$\begin{aligned}
H_n^T H_n &= H_n H_n \\
&= \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} (H_{n-1}H_{n-1}) + (H_{n-1}H_{n-1}) & (H_{n-1}H_{n-1}) - (H_{n-1}H_{n-1}) \\ (H_{n-1}H_{n-1}) - (H_{n-1}H_{n-1}) & (H_{n-1}H_{n-1}) + (H_{n-1}H_{n-1}) \end{bmatrix} \\
&= \begin{bmatrix} H_{n-1}H_{n-1} & 0 \\ 0 & H_{n-1}H_{n-1} \end{bmatrix} \\
&= \begin{bmatrix} I_{2^{n-1} \times 2^{n-1}} & 0 \\ 0 & I_{2^{n-1} \times 2^{n-1}} \end{bmatrix} \\
&= I_{2^n \times 2^n}
\end{aligned}$$

$\square$

Let a vector $v \in \mathbb{R}^{2^n}$ and let $H_n = \frac{1}{\sqrt{2}} \begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$

Algorithm 1 Uses divide-and-conquer to compute $H_n \cdot v$

Hadamard-Vector-Product (H_n , v , n):

Let $v_1 = \begin{bmatrix} v_0 \\ \vdots \\ v_{(m/2)-1} \end{bmatrix}$ and $v_2 = \begin{bmatrix} v_{m/2} \\ \vdots \\ v_{m-1} \end{bmatrix}$, where $m = 2^n$

Let $H_n \cdot v = \begin{bmatrix} (H_n \cdot v)_A \\ (H_n \cdot v)_B \end{bmatrix}$, where

$$(H_n \cdot v)_A = H_{n-1} \cdot v_A + H_{n-1} \cdot v_B = H_{n-1} \cdot (v_A + v_B)$$
$$(H_n \cdot v)_B = H_{n-1} \cdot v_A - H_{n-1} \cdot v_B = H_{n-1} \cdot (v_A - v_B)$$

if $n = 1$ **then**
  Return $\frac{1}{\sqrt{2}} \begin{bmatrix} H_n \cdot v_A \\ H_n \cdot v_B \end{bmatrix}$
else $\{n > 1\}$
  Return $\begin{bmatrix} \text{Hadamard-Vector-Product } (H_{n-1}, (v_A + v_B), n-1) \\ \text{Hadamard-Vector-Product } (H_{n-1}, (v_A - v_B), n-1) \end{bmatrix}$
end if

Claim 4. *The time complexity of the algorithm is $O(m \log m)$, where $m = 2^n$.*

Proof. Each half of $H_n \cdot v$ takes time $T\left(\frac{m}{2}\right)$. Each vector/matrix addition or subtraction takes time $O(m)$. Therefore, the recurrence relation for the algorithm is defined as:

$$T(m) = 2T\left(\frac{m}{2}\right) + O(m)$$

Thus, by Case 2 of the Master Theorem, the algorithm's time complexity is $O(m \log m)$. □