CMPS 102 — Fall 2018 — Homework 1

*"I have read and agree to the collaboration policy." - **Kevin Wang***

Solution to Problem 4: Perfectly Imperfect Masterpieces

Given Bob's specifications, we want to write an algorithm that helps him design the alignment of the L-shaped blocks for any square with length 2^n units, where $n \in \mathbb{N}$. The black 1 unit square shown in Figure 0 is a blank spot on the masterpiece.

Figure 0

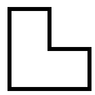


Figure 1

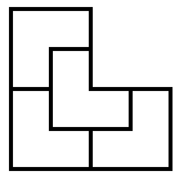
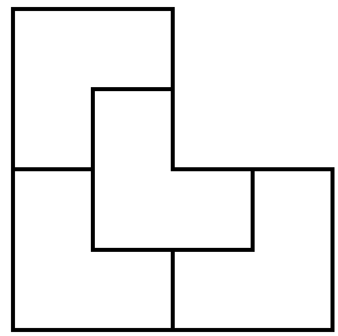


Figure 2



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Algorithm 1 Recursively creates a perfect imperfect masterpiece for any n.

PERFECT-IMPERFECT (n):
Initialize a blank masterpiece M
Initialize L as the L-shaped block shown in Figure 1.
FILL-MP (M, L, n)

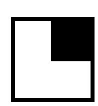
FILL-MP (M, L', n):
if $n = 0$ **then**
  Add the imperfection shown in Figure 0 to the top right corner of M
else $\{n > 0\}$
  Add the segment L' to the bottom left of M
  Let L' be the augmented segment created with 4 of itself as shown in Figure 2.
  FILL-MP $(M, L', n-1)$
end if

Claim 1. *The algorithm is correct for all $n \in \mathbb{N}$.*

Proof. Let $P(n)$ be the statement: "The algorithm designs a perfect imperfect masterpiece for a square of size 2^n".

The base cases of $P(0)$ and $P(1)$ are true.

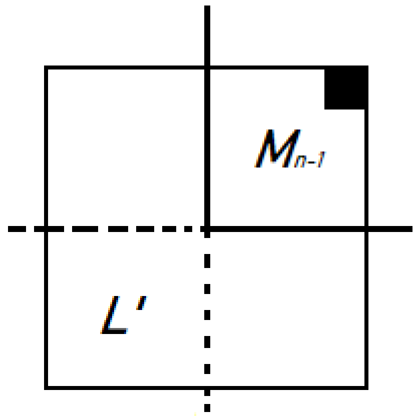
P(0)



P(1)

Assume for a square with side 2^n, where $n > n_0$ and $n_0 = 1$, that $P(n)$ is true. We need to prove that $P(n+1)$ is true.

Let a square with side of 2^{n+1} be equivalent to a square with side 2^n in quadrant I, in addition to the L' segment that fills quadrants II, III, and IV.



Using the induction hypothesis, we assume that the square in quadrant I is a correctly designed masterpiece. The additional L' segment is formed solely of nested L' segments of smaller size. Thus, the final square maintains the properties that define a perfect imperfect masterpiece. Thus $P(n) \implies P(n+1)$ is true, proving that the algorithm will design a correct masterpiece. $\square$

The run time of adding a block L to a masterpiece takes $O(1)$. When creating the augmented L' segment, it takes four individual L' segment placements. The recursion is run a total of n times. Thus, a total of 4^n L-shaped blocks are placed. The time complexity of this algorithm is: $O(2^{2n})$.