CMPS 102 — Fall 2018 — Homework 1

"I have read and agree to the collaboration policy." - ***Kevin Wang***

Solution to Problem 3: Tokens in the Bag

Given a bag of red, green, and/or blue tokens, we want to prove that performing the following algorithm will terminate:

Algorithm 1 Repeats until the bag of tokens is empty

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while there are more than 2 tokens in the bag do
    Take 2 random tokens out of the bag
    if one of the tokens is red then
        Do nothing
    else if both tokens are green then
        Put 1 green token and 2 blue tokens back into the bag
    else {At least 1 blue token and the other token is not red}
        Put 3 red tokens back into the bag
    end if
end while [Exactly 2 or fewer tokens in the bag]
Empty the bag
```

We assume that there is always enough tokens to put back into the bag.

Observation 1. A selection with a red token decreases the token count of the bag.

Lemma 1. A selection with a blue token results in a decrease of the bag's token count.

Proof.

Case 1: If the blue token is selected with a red one, by Observation 1, the token count of the bag is decreased.

Case 2: If the blue token is selected with a green or another blue token, they are replaced by 3 red tokens. By Observation 1, this subset of red tokens will result in a decrease of the bag's token count.

□

Lemma 2. A selection with a green token results in a decrease of the bag's token count.

Proof.

Case 1: If the green token is selected with a red one, by Observation 1, the token count of the bag is decreased.

Case 2: If the green token is selected with a blue token, by Lemma 1, the selection results in a decrease of the bag's token count.

Case 3: If the green token is selected with another green token, they are replaced by 1 green and 2 blue tokens. By Lemma 1, all possible combinations of 2 tokens selected from this subset result in a decrease of the bag's token count.

□

Lemma 3. All possible color combinations will lead to a decrease of a bag's total token count.

Proof. By Observation 1, Lemma 1, and Lemma 2, a selection with any color will lead to the eventual removal of the affected subset of tokens. □

Claim 1. *This process will always terminate.*

Proof. Let $P(n)$ be the statement: "A bag of n tokens will always terminate", where n is the number of tokens in the bag and $n \in \mathbb{N}$.

It is stated that the bag is emptied – and thus the program terminated – when there are exactly 2 or fewer tokens left in the bag prior to a loop. Therefore, the base cases of $P(1)$ and $P(2)$ are true.

Assume for a bag of n tokens, where $n > n_0$ and $n_0 = 2$, that $P(n)$ is true. We need to prove that $P(n+1)$ is true.

Let a bag of $n + 1$ tokens be equivalent to a bag of n tokens plus a single token. Using the induction hypothesis, we assume that the process terminates on the subset of n tokens. By Lemma 3, the additional token – whether it be red, blue, or green, respectively – will always result in a net decrease of tokens. We also know that 1 single token left-over will still result in termination. Thus $P(n) \implies P(n+1)$ is true, proving that the process will always terminate. □