1 Boundary-free shear flows

1.1 Almost parallel, two-dimensional flows

Plane flows and axisymmetric flows: the mean velocity field is entirely confied to planes.
Plane flows: mean flow in planes parallel to a given plane is identical;
Axisymmetric flows: mean flow in planes through the axis of symmetry is identical.
Plane flows
principal mean-velocity component is in x direction, confied to the x,y plane, evolve slowly in the x direction.

$$U_i = \{U, V, 0\}$$

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

Wake, jet and shear layer.

$$U_s = max\,|U_0 - U|$$

For far wakes, $U = \mathscr{O}(U_0)$;
For jets and shear layers, $U = \mathscr{O}(U_s)$.
Generally, $U = \mathscr{O}(\tilde{U})$, $\tilde{U} = U_0$ for wakes and $\tilde{U} = U_s$ for jets and shear layers.
Cross-stream scale ℓ,

$$\frac{\partial U}{\partial y} = \mathscr{O}\left(\frac{U_s}{\ell}\right)$$

Length scale in the x direction, L

$$\frac{\partial U}{\partial x} = \mathscr{O}\left(\frac{U_s}{L}\right)$$

Velocity scale for turbulence, $\mathcal{u}$,

$$-\overline{uv} = \overline{u^2} = \overline{v^2} = \mathscr{O}(\mathcal{u}^2)$$

From the equation for continuity, $\partial U/\partial x + \partial V/\partial y = 0$, the cross-stream component V of the mean velocity,

$$V = \mathcal{O}\left(\frac{U_s \ell}{L}\right)$$

The cross-stream momentum equation

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \frac{\partial}{\partial x}\left(\overline{uv}\right) + \frac{\partial}{\partial y}\left(\overline{v^2}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

We assume

$$\frac{\tilde{U}}{u}\frac{U_s}{u}\left(\frac{\ell}{L}\right)^2 \to 0, \quad \frac{U_s}{u}\frac{1}{R_\ell}\left(\frac{\ell}{L}\right) \to 0$$

Identify the orders of magnitudes, the pressure term should be of the same order as $\partial\overline{v^2}/\partial y$. Thus,

$$\frac{\partial\overline{v^2}}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

Integrate,

$$\frac{P}{\rho} + \overline{v^2} = \frac{P_0}{\rho}$$

P_0: the pressure outside the turbulent part of the field ($y \to \pm\infty$). We assume $\partial P_0/\partial x = 0$.
Take derivative w.r.t x,

$$\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial\overline{v^2}}{\partial x} = 0$$

The streamwise momentum equation

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial x}\left(\overline{u^2} - \overline{v^2}\right) + \frac{\partial}{\partial y}\left(\overline{uv}\right) = \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$$

Similarly, the following relation should keep:

$$\frac{\tilde{U}}{u}\frac{U_s}{u}\frac{\ell}{L} = \mathcal{O}(1)$$

Turbulent wakes

If $u/U_s = \mathcal{O}(1)$, then

$$\frac{u}{\tilde{U}} = \mathcal{O}\left(\frac{\ell}{L}\right)$$

This situation occurs in far wakes. Thus, we have

$$U\frac{\partial U}{\partial x} + \frac{\partial(\overline{uv})}{\partial y} = 0$$

For wakes, we have $\tilde{U} = U_0$, $u \sim U_s$,

$$\frac{U - U_0}{U_0} = \mathcal{O}\left(\frac{U_s}{U_0}\right) = \mathcal{O}\left(\frac{\ell}{L}\right)$$

Which imples we can replace U by U_0,

$$U_0\frac{\partial U}{\partial x} + \frac{\partial(\overline{uv})}{\partial y} = 0$$

Turbulent jets and mixing layers

If $\tilde{U} = U_s$, then

$$\frac{u}{U_s} = \mathcal{O}\left(\frac{\ell}{L}\right)^{1/2}$$

Hence, the momentum equation

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial(\overline{uv})}{\partial y} = 0$$

The momentum integral

U_0 is not a function of position,

$$U\frac{\partial(U - U_0)}{\partial x} + V\frac{\partial(U - U_0)}{\partial y} + \frac{\partial(\overline{uv})}{\partial y} = 0$$

Use the continuity equation,

$$U\frac{\partial[U(U - U_0)]}{\partial x} + V\frac{\partial[V(U - U_0)]}{\partial y} + \frac{\partial(\overline{uv})}{\partial y} = 0$$

For large values of y, $U - U_0$ and $\overline{uv}$ vanishes,

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} U(U - U_0)\,\mathrm{d}y = 0$$

$$\rho\int_{-\infty}^{\infty} U(U - U_0)\,\mathrm{d}y = M$$

Momentum thickness

Momentum thickness θ: width of the completely separated stagnant region.
Net momentum defect per unit time and depth

$$-\rho U_0^2 \theta = M$$

$$\theta = \int_{-\infty}^{\infty} \frac{U}{U_0}\left(1 - \frac{U}{U_0}\right) \mathrm{d}y$$

It is independent of x.
Drag coefficient c_d,

$$D = c_d \frac{1}{2} \rho U_0^2 d$$

D is the drag per unit depth and d is the frontal height of the obstacle.
Clearly, $D = -M$,

$$c_d = 2\frac{2\theta}{d}$$

1.2 Turbulent wakes

Self-preservation

In wakes, we expect

$$\frac{U_0 - U}{U_s} = f(\frac{y}{\ell}, \frac{\ell}{L}, \frac{\ell U_s}{\nu}, \frac{U_s}{U_0})$$

Assume $\ell/L \to 0, \ell U_s/\nu \to \infty, U_s/U_0 \to 0$, thus

$$\frac{U_0 - U}{U_s} = f\left(\frac{y}{\ell}\right)$$

The turbulence intensity $\boldsymbol{u}$ is of oerder U_s,

$$-\overline{uv} = U_s^2 g\left(\frac{y}{\ell}\right)$$

Self-preservation hypothesis: the velocity defect and the Reynolds stress become invariant w.r.t x if they are expressed in terms of the local length and velocity scales ℓ and U_s.
Define $\xi = y/\ell$,

$$\frac{\partial U}{\partial x} = -\frac{\mathrm{d}U_s}{\mathrm{d}x} f + \frac{U_s}{\ell}\frac{\mathrm{d}l}{\mathrm{d}x}\xi f'$$
$$\frac{\partial \overline{uv}}{\partial y} = -\frac{U_s^2}{\ell} g'$$

(primes denote diff w.r.t ξ)

Thus we have,

$$-\frac{U_0\ell}{U_s^2}\frac{\mathrm{d}U_s}{\mathrm{d}x} f + \frac{U_0}{U_s}\frac{\mathrm{d}l}{\mathrm{d}x}\xi f' = g'$$

The coefficients of f and $\xi f'$ must be constant, thus

$$\frac{\ell}{U_s^2}\frac{\mathrm{d}U_s}{\mathrm{d}x} = const, \quad \frac{1}{U_s}\frac{\mathrm{d}\ell}{\mathrm{d}x} = const$$

The general solution could be $\ell \sim x^n$, $U_s \sim x^{n-1}$.

The momentum integral can be written as

$$U_0 U_s \ell \int_{-\infty}^{\infty} f(\xi)\,\mathrm{d}\xi - U_s^2 \ell \int_{-\infty}^{\infty} f^2(\xi)\,\mathrm{d}\xi = -\frac{M}{\rho}$$

The second term should be neglected, thus we have

$$U_s \ell \int_{-\infty}^{\infty} f(\xi)\,\mathrm{d}\xi = U_0\theta$$

$U_s\ell$ should be independent of x, thus $n = 1/2$. Hereby,

$$U_s = Ax^{-1/2}, \quad \ell = Bx^{1/2}$$

A self-preserving solution is possible only if the velocity and length scales behave as the equation above.

The mean-velocity profile

Substitute back into the differential equation,

$$\frac{1}{2}\left(\frac{U_0 B}{A}\right)(\xi f' + f) = g'$$

We need a relation between f and g.

Define eddy viscosity ν_T:

$$-\overline{uv} = \nu_T \frac{\partial U}{\partial y}$$

then

$$\nu_T = -\frac{U_s \ell g}{f'}$$

Assume ν_T is a constant,

$$\frac{\nu_T}{U_s \ell} = \frac{1}{R_T} = -\frac{g}{f'}$$

R_T: Turbulent Reynolds number, determined by experimental data. The equation is likely to be valid only near the center line of the wake.

$$\alpha(\xi f' + f) + f'' = 0, \quad \alpha = \frac{1}{2} R_T U_0 \frac{B}{A}$$

The solution is

$$f = \exp\left(-\frac{1}{2}\alpha\xi^2\right)$$

Since $f(0) = 1$, the convention is to define $\alpha = 1$, so that for $\xi = 1, y = \ell$, $f \approx 0.6$.

$$\int_{-\infty}^{\infty} f(\xi)\,\mathrm{d}\xi = (2\pi)^{1/2}$$

Sub back, we have

$$\begin{aligned} \frac{U_s}{U_0} &= 1.58 \left(\frac{\theta}{x}\right)^{1/2} \\ \frac{\ell}{\theta} &= 0.252 \left(\frac{x}{\theta}\right)^{1/2} \end{aligned}$$

This equation is sufficiently accurate.

Axisymmetric wakes

For axisymmetric wakes, $U_s \sim x^{-2/3}$, $\ell \sim x^{1/3}$, thus $R_\ell = U_s \ell/\nu \sim x^{-1/3}$, the structure of the axisymmetric wake is similar from the structure of the plane wake.

When R_ℓ is reduced to a value of the order unity, the wake ceases to be turbulent.

$$R_\ell \sim \left(\frac{U_0\theta}{\nu}\right)\left(\frac{\theta}{x}\right)^{1/3}$$

For moderate Reynolds numbers, R_ℓ reaches unity when x/θ is of order $(U_0\theta/\nu)^3$, which is a large distance.

Scale relations

The Reynolds stress

$$-\overline{uv} = -\frac{U_s^2 f}{R_T}$$

and its maximum

$$\left(-\frac{\overline{uv}}{U_s^2}\right)_{max} = (R_T^2 e)^{1/2} = 0.05$$

Take the correlation coefficient between u and v as 0.4, and we can estimate for the rms velocity fluctuation

$$u = \left(\frac{0.05U_s^2}{0.4}^{1/2}\right) = 0.35U_s$$

And the wake propagate rate is

$$\frac{\mathrm{d}\ell}{\mathrm{d}t} = U_0\frac{\mathrm{d}\ell}{\mathrm{d}x} \approx 0.08U_s$$

The ratio of two time scales $t_d/t_p \approx 2$, indicates that turbulence can never be in equilibrium because it never has time to adjust to its changing environment.

The turbulent energy budget

The equation for the kinetic energy:

$$0 = -U_0\frac{\partial}{\partial x}\left(\overline{\frac{1}{2}q^2}\right) - \overline{uv}\frac{\partial U}{\partial y} - \frac{\partial}{\partial y}\overline{\nu\left(\frac{1}{2}q^2 + \frac{p}{\rho}\right)} - \epsilon$$

Here, $\overline{q^2} = \overline{u_i u_i}$, is twice the kinetic energy per unit mass.

The first term is called advection, which is the convection by the mean flow, **A**.

The second term is production, **P**.

The third term is transport by turbulent motion, **T**.

The last term is dissipation, **D**.

TBD - ???

1.3 The wake of a self-propelled body