1 Boundary-free shear flows

1.1 Almost parallel, two-dimensional flows

Plane flows and axisymmetric flows: the mean velocity field is entirely confied to planes.
Plane flows: mean flow in planes parallel to a given plane is identical;
Axisymmetric flows: mean flow in planes through the axis of symmetry is identical.
Plane flows
principal mean-velocity component is in x direction, confied to the x,y plane, evolve slowly in the x direction.

$$U_i = \{U, V, 0\}$$

$$\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

Wake, jet and shear layer.

$$U_s = max\,|U_0 - U|$$

For far wakes, $U = \mathscr{O}(U_0)$;
For jets and shear layers, $U = \mathscr{O}(U_s)$.
Generally, $U = \mathscr{O}(\tilde{U})$, $\tilde{U} = U_0$ for wakes and $\tilde{U} = U_s$ for jets and shear layers.
Cross-stream scale ℓ,

$$\frac{\partial U}{\partial y} = \mathscr{O}\left(\frac{U_s}{\ell}\right)$$

Length scale in the x direction, L

$$\frac{\partial U}{\partial x} = \mathscr{O}\left(\frac{U_s}{L}\right)$$

Velocity scale for turbulence, $\mathcal{u}$,

$$-\overline{uv} = \overline{u^2} = \overline{v^2} = \mathscr{O}(\mathcal{u}^2)$$

From the equation for continuity, $\partial U/\partial x + \partial V/\partial y = 0$, the cross-stream component V of the mean velocity,

$$V = \mathcal{O}\left(\frac{U_s \ell}{L}\right)$$

The cross-stream momentum equation

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} + \frac{\partial}{\partial x}\left(\overline{uv}\right) + \frac{\partial}{\partial y}\left(\overline{v^2}\right) = -\frac{1}{\rho}\frac{\partial P}{\partial y} + \nu\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

We assume

$$\frac{\tilde{U}}{u}\frac{U_s}{u}\left(\frac{\ell}{L}\right)^2 \to 0, \quad \frac{U_s}{u}\frac{1}{R_\ell}\left(\frac{\ell}{L}\right) \to 0$$

Identify the orders of magnitudes, the pressure term should be of the same order as $\partial\overline{v^2}/\partial y$. Thus,

$$\frac{\partial\overline{v^2}}{\partial y} = -\frac{1}{\rho}\frac{\partial P}{\partial y}$$

Integrate,

$$\frac{P}{\rho} + \overline{v^2} = \frac{P_0}{\rho}$$

P_0: the pressure outside the turbulent part of the field ($y \to \pm\infty$). We assume $\partial P_0/\partial x = 0$.

Take derivative w.r.t x,

$$\frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\partial\overline{v^2}}{\partial x} = 0$$

The streamwise momentum equation

$$U\frac{\partial U}{\partial x} + V\frac{\partial U}{\partial y} + \frac{\partial}{\partial x}\left(\overline{u^2} - \overline{v^2}\right) + \frac{\partial}{\partial y}\left(\overline{uv}\right) = \nu\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right)$$

Similarly, we need

$$\frac{\tilde{U}}{u}\frac{U_s}{u}\frac{\ell}{L} = \mathcal{O}(1)$$