1 Introduction

Most flows in reality are turbulent.

1.1 The nature of turbulence

Irregularity
Irregularity (randomness) makes it impossible to analyze all turbulent flows with a deterministic approach.
Diffusivity
Diffusivity leads to rapid mixing and high rates of transfer in momentuum, heat and mass. A flow without diffusivity is not turbulent. Diffusivity prevents boundary-layer separation on airfoils at large angles of attack.
Large Reynolds numbers
In the equation of motion, the viscous terms and the nonlinear terms interact, which causes instablity, and when Reynolds number becomes large, turbulence originates.
3D vorticity fluctuations
Turbulence is rotational and three-dimensional, and is characterized by high levels of fluctuating vorticity. 2D velocity fluctuations could not maintain themselves, thus two dimensional flow such as the cyclones are not turbulence themselves.
Dissipation
Turbulent fows are always dissipative. Deformation work is performed by viscous shear stresses, thus the internal energy is increased while the kinetic energy is decreased. Hence, without a continuous supply of energy, the turbulence will decay very soon. Random waves are random, but not dissipative, thus they are not turbulence.
Continuum
Turbulence is governed by equations of fluid mechanics, and the scales are larger than molecular length scale.
Turbulent flow are flows
Turbulece is a feature of fluid flows. There is not a general solution to the N-S equations, thus there is no general soluions for turbulent flow. Every turbulent flow is different, however, they share some characteristics in common.

1.2 Methods of analysis

The equations of motion are too complicated, thus we cannot make quantitative predictions without empirical data. Statistical studies could always introduce more unknowns.
Hence, we need to make some ad hoc assumptions to make the number of equations equal to the number of unknowns – **Closure problem of turbulence theory**.

One approach: we assume that the relation between stress and rate of strain involves a turbulence-generated "viscosity", which plays a similar role to molecular viscosity in laminar flows. Here we introduce "eddy viscosity" and "mixing length".

Dimensional analysis
Dimensional analysis is powerful in the study of turbulence when some aspects of the structure of turbulence depends only on a few independent variables or parameters.

Asymptotic invariance
High Reynolds number appears in turbulent flows, thus it reasonable to propose that the turbulence should behave in the limit as the Reynolds number approaches infinity. The theory of turbulent boundary layers is an application of this method.

Local invariance
The characteristics of the turbulent motion at some point in time and space appear to be controlled mainly by the immediate environment (local).
As long as time scales are small enough, and the environment is changing gradually, we can assume that the turbulence is dynamically similar everywhere **if nondimensionalized with local length and time scales**.

1.3   The origin of turbulence

Laminar flow will turn into turbulence from instabilities (large Reynolds numbers).
Turbulence requires its environment to obtain energy so that it could maintain itself (due to dissipation). The source of energy could be shear in mean flow, or buoyancy. If there is no source of energy, the turbulence decays, and it again turns back into laminar flow.
If we wish to maintain a laminar flow, or make a turbulent flow laminar, we shall seek for a mechanism that consumes turbulent kinetic energy.

1.4   Diffusivity of turbulence

Turbulent motion has an ability to transport or mix **momentum, kinetic energy**, and contaminants such as **heat, particles and moisture.**
Diffusion in a problem with an imposed length scale
Suppose a room with characteristic linear dimension L where a heating element is installed.
θ is the temperature; γ is the thermal diffusivity (constant).
If there is no air motion, heat has to be distributed by molecular diffusion.

$$\frac{\partial \theta}{\partial t} = \gamma \frac{\partial^2 \theta}{\partial x_i \partial x_i}$$

Dimensionaly interpret it,

$$\frac{\Delta\theta}{T_m} \sim \gamma\frac{\Delta\theta}{L^2}$$

Thus,

$$T_m \sim \frac{L^2}{\gamma}$$

Let $L = 5\ m$, $\gamma = 0.20\ cm^2/sec$, we have $T_m \sim 10^6\ sec$, which is rather ineffective.

However, if there is motions, which is represented by characteristic velocity u, we shall have

$$T_t \sim \frac{L}{u}$$

To determine T_t, we need to find u. Suppose the difference in temperature is $\Delta\theta$, the buoyant acceleration could be $g\Delta\theta/\theta$, and if the height $h = 0.1\ m$, we have $u^2 \sim gh\Delta\theta/\theta\ 0.03(m/sec)^2$, which means that $u \sim 17\ cm/sec$.

If we assume $u = 5\ cm/s$ as an average throught the room and $L = 5\ m$, $T_t \sim 100\ sec$.

Compared to molecular diffusion, diffusion by random motion is much more rapid,

$$\frac{T_t}{T_m} \sim \frac{L}{u}\frac{\gamma}{L^2} = \frac{\gamma}{uL}$$

Which is the inverse of **Péclet number.**

As we know, Prandtl number $Pr = \nu/\gamma$, and for gases, Prandtl number is close to 1, we can substitute ν to γ,

$$\frac{T_t}{T_m} \sim \frac{\nu}{uL} = \frac{1}{Re}$$

In the example above, the Reynolds number Re is about 15000.

Eddy diffusivity

Due to the complication of turbulent flow, we can use a method effective diffusivity.

If we write for the diffusion of heat by turbulent motions,

$$\frac{\partial\theta}{\partial t} = K\frac{\partial^2\theta}{\partial x_i \partial x_i}$$

where K is the **representative diffusivity (or eddy diffusivity, exchange coefficient for heat)**.

From the equation above, we can estimate a time scale,

$$T \sim \frac{L^2}{K}$$

3

Since we have the atcual time scale T_t before, equating these two time scales, we have

$$K \sim uL$$

And finally arrive at

$$\frac{K}{\gamma} \sim \frac{K}{\nu} \sim \frac{uL}{\nu} = Re$$

This indicates that Reynolds number can also be interpreted as a ratio of turbulent viscosity to molecular viscosity.

N.B. eddy diffusivity K is an artifice, which might not actually exist.

Diffusion in a problem with an imposed time scale

An example of diffusivity of turbulence, the boundary layer of the atmosphere, ommited here.

1.5 Length scales in turbulent flows

In turbulent flows, a wide range of length scales exists. That is why spectral analysis of turbulent motion is useful.

Laminar boundary layers

For steady flow of an incompressible fluid with constant viscosity, the Navier-Stokes equations are

$$u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

If we have U as the charateristic velocity and L as the charateristic length, we can estimate the inertia term as U^2/L, and the viscous terms are $\nu U/L^2$. The ratio of inertia terms and viscous terms is $UL/\nu = Re$, which means that viscous terms should be negligible at large Reynolds numbers. However, since boundary conditions and initial conditions exist, we shall not neglect viscous terms everywhere in the flow field.

As viscous terms can survive at high Reynolds numbers, we shall choose a new length scale l, so that the viscous terms are of the same order of magnitude as the inertia terms.

$$U^2/L \sim \nu U/l^2$$

thus

$$\frac{l}{L} \sim \left(\frac{\nu}{UL}\right)^{1/2} = Re^{-1/2}$$

Hence, we know that the viscous length l is a transverse length scale, which represents the width or thickness of the boundary layer.

Diffusive and convective length scales

Diffusive length scale: across the flow;

Convective length scale: along the flow.

In most shear flows, their width is much smaller than their length. This is useful to some simplifying approximations. $l/L \to 0$ is a good example.

Turbulent boundary layers

We can compare turbulent shear flows to laminar flows in a rudimentary way.

Characteristic velocity fluctuations: u

Boundary-layer thickness: l

We have $dl/dt \sim u$. And

Time interval: L/U

thus $l \sim ut \sim uL/U$.

In effect, we equate turbulent "diffusion" time scale l/u to the convective time scale L/U. As a result, we can write the scale relations as

$$l/L \sim u/U$$
$$l/u \sim L/U$$

These relations lead to an assumption that the time scale of the external flow must be commensurate with the time scale of the flow. However, this assumption conflicts with eddy-viscosity concepts.

Laminar and turbulent friction

Experimental data suggest that u/U is of the order of 10^{-2} over a wide range of Reynolds numbers, we could know that the growth of turbulent shear flows is relatively rapid.

For laminar boundary layer in two-dimensional flow, drag (D)

$$D \sim \rho U^2 l$$

Thus the drag coefficient C_d is defined by

$$C_d = \frac{D}{\frac{1}{2}\rho U^2 L}$$

Hence,

$$C_d \sim \frac{2l}{L} = 2Re^{-1/2}$$

For a turbulent boundary layer, the mass flow deficit at the end of the plate is proportional to $\rho u l$, so that the rate of loss of momentum is proportional to $(\rho u l)U$. Thus,

$$D \sim \rho u U l$$

Then the drag coefficient becomes

$$C_d \sim 2\frac{u}{U}\frac{l}{L} \sim 2\left(\frac{u}{U}\right)^2$$

5

Which shows that the drag coefficient of a turbulent boundary layer should be much greater than the drag coefficient of a laminar boundary layer.

Small scales in turbulence

So far, we have only considered the largest eddy sizes in turbulent flows, since they do most of the transport of momentum and contaminants, and we also assumed that the large eddies are as big as the width of the flow, which is the relevant length scale in the analysis.

However, we shall attempt to find the smallest length scales in turbulent flows. At very small length scales, viscosity canbe effective in smoothing out velocity fluctuations.

Parameters governing small-scale motion: ϵ the dissipation rate per unit mass, ν kinematic viscosity. With these parameters, we can form length, time and velocity scales:

$$\eta = (\nu^3/\epsilon)^{1/4} \tau = (\nu/\epsilon)^{1/2} \qquad\qquad v = (\nu\epsilon)^{1/4}$$

Which are referred to as Kolmogorov microscales of length, time and velocity, a.k.a "inner" scales.

An inviscid estimate for the dissipation rate

Assumption: the rate at which large eddies supply energy to small eddies **is proportional to** the reciprocal of the time scale of the large eddies.

For per unit mass, the amount of kinetic energy is proportional to u^2; the rate of transfer of energy is proportional to u/l, where l represents the size of the largest eddies or the width of the flow, "integral scale", or "outer scale".

The rate of energy supply to the small-scale eddies is of order $u^2 \cdot u/l$. The dissipation rate ϵ,

$$\epsilon \sim u^3/l$$

From the equation above, we know that viscous dissipation of energy can be estimated from the large-scale dynamics, and viscosity is not involved.

It also indicates that large eddies lose a significant fraction of their kinetic energy $1/2u^2$ within one "turnover" time l/u.

Scale relations

$$\eta/l \sim (ul/\nu)^{-3/4} = Re^{-3/4}$$
$$\tau u/l \sim \tau/t = (ul/\nu)^{-1/2} = Re^{-1/2}$$
$$\nu/u \sim (ul/\nu)^{-1/4} = Re^{-1/4}$$

These relations show that the length, time, and velocity scales of the smallest eddies are very much smaller than those of the largest eddies.

Vorticity has the dimensions of sec^{-1}, which is reciprocal of τ. Thus, we know hat vorticity of the small-scale eddies is very much larger than that of the large-scale motion.

Molecular and turbulent scales

Kolmogorov length and time scales are the smallest scales in the turbulent motion, and thus we can say that most turbulent flows are continuum phenomena. If the dissipation rate ϵ is high, then the Kolmogorov scales would decrease, and this is usually associated with large values of u.

The relevant molecular length scale is the mean free path ξ, and the velocity scale of molecular motion in a gas is proportional to the speed of sound a in the gas.

Kinetic theory of gases shows,

$$\nu \sim a\xi$$

The ratio of ξ to η is called microstructure Knudsen number,

$$\xi/\eta \sim Ma/R^{1/4}$$

With a high Mach number and a low Reynolds number, turbulence might interfere with molecular motion, but this situation is not likely to occur.

The ratio of the time scale τ to the collision time scale ξ/a of molecular motion is

$$\tau a/\xi \sim Re^{1/2}Ma^{-2}$$

The smallest time scales in turbulent motion tend to be much larger than molecular time scales, we can assume that the motion of the gas molecules is in statistical equilibrium.