1 Turbulent transport of momentum and heat

Turbulent velocity fluctuations can generate large momentum fluxes between different parts of a flow. A momentum flux can be thought of as a stress, turbulent momentum fluxes are commonly called Reynolds stresses.

1.1 The Reynolds equations

The equations of motion of an incompressible fluid are

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \tilde{\sigma}_{ij}$$
$$\frac{\partial \tilde{u}_i}{\partial x_i} = 0$$

$\tilde{\sigma}_{ij}$: the sress tensor.
Repeated indices: summation over all three values of the index.
Tilde ($\tilde{}$): instantaneous value at (x_i, t) of a variable without Reynolds decomposition.
For Newtonian fluid,

$$\tilde{\sigma}_{ij} = -\tilde{p}\delta_{ij} + 2\mu \tilde{s}_{ij}$$

$\tilde{p}$: hydrodynamic pressure.
μ: dynamic viscosity (constant).
$\tilde{s}_{ij}$: rate of strain,

$$\tilde{s}_{ij} = \frac{1}{2}\left(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i}\right)$$

Sub into former N-S equation,

$$\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{p}}{\partial x_i} + \nu \frac{\partial^2 \tilde{u}_i}{\partial x_j \partial x_j}$$

The Reynolds decomposition
For a steady flow,

$$\frac{\partial U_i}{\partial t} = 0$$

We can have a time average of the flow. We decompose velocity into a mean flow U_i and velocity fluctuations u_i,

$$\tilde{u}_i = U_i + u_i$$

U_i should be a time average, thus

$$U_i = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} \tilde{u}_i dt$$

and then the average of fluctuation term

$$\bar{u}_i = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} (tildeu_i - U_i)dt \equiv 0$$

The mean value of a spatial derivative of a variable is equal to the corresponding spatial derivative of the mean value of that variable, in other words, we are able to change the order of averaging and taking derivatives.

$$\begin{aligned}
\frac{\partial \bar{\tilde{u}}_i}{\partial x_j} &= \frac{\partial U_i}{\partial x_j} \\
\frac{\partial \bar{u}_i}{\partial x_j} &= \frac{\partial}{\partial x_j}\bar{u}_i = 0
\end{aligned}$$

For the pressure $\tilde{p}$ and the stress $\tilde{\sigma}_{ij}$, we can also decompose them in a similar way.

$$\begin{aligned}
\tilde{p} &= P + p \\
\bar{p} &\equiv 0 \\
\tilde{\sigma}_{ij} &= \Sigma_{ij} + \sigma_{ij} \\
\bar{\sigma}_{ij} &\equiv 0
\end{aligned}$$

Mean stress tensor Σ_{ij} is given by

$$\Sigma_{ij} = -P\delta_{ij} + 2\mu S_{ij}$$

And stress fluctuations σ_{ij} is given by

$$\sigma_{ij} = -p\delta_{ij} + 2\mu s_{ij}$$

Whre S_{ij} and s_{ij} are the mean strain rate and strain-rate fluctuations,

$$\begin{aligned}
S_{ij} &= \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right) \\
s_{ij} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)
\end{aligned}$$

Correlated variables
An example (the proof is simple):

$$\begin{aligned}
\overline{\tilde{u}_i\tilde{u}_j} &= \overline{(U_i + u_i)(U_j + u_j)} \\
&= U_i U_J + \overline{u_i u_j}
\end{aligned}$$

If $\overline{u_i u_j} \neq 0$, then they are said to be correlated; if $\overline{u_i u_j} = 0$, then they are uncorrelated.
Correlation coefficient:

$$c_{ij} = \frac{\overline{u_i u_j}}{\sqrt{\overline{u_i^2}\,\overline{u_j^2}}}$$

Standard deviation or root-mean-square (rms) amplitude (denoted by $'$):

$$u_i' = \sqrt{\overline{u_i^2}}$$

Equations for the mean flow
Decompose the equation of continuity $\frac{\partial \tilde{u}_i}{\partial x_i} = 0$, we have

$$\frac{\partial U_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0$$

Take average of all terms, $\partial u_i/\partial x_i$ vanishes, thus $\partial U_i/\partial x_i = 0$. Sub back into the former equation, then $\partial u_i/\partial x_i = 0$.
The mean flow is incompressible, and the turbulent fluctuations are also incompressible.
Go back to the equations of motion, we take average

$$U_j \frac{\partial U_i}{\partial x_i} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\Sigma_{ij}$$

Where

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial}{\partial x_j}\overline{u_i u_j}$$

represents the mean transport of fluctuating momentum.
Rearrange, we have the Reynolds momentum equation:

$$U_j \frac{\partial U_i}{\partial x_j} = \frac{1}{\rho}\frac{\partial}{\partial x_j}\left(\Sigma_{ij} - \rho\overline{u_i u_j}\right)$$

Thus, we have the total mean stress T_{ij} defined as

$$T_{ij} = \Sigma_{ij} - \rho\overline{u_i u_j} = -P\delta_{ij} + 2\mu S_{ij} - \rho\overline{u_i u_j}$$

The Reynolds stress
The Reynolds stress tensor, representing the contribution of the turbulent motion to the mean stress tensor,

$$\tau_{ij} = -\rho\overline{u_i u_j}$$

The diagonal components are normal stresses, they contribute little to the transport of mean momentum; the off-diagonal components are shear stresses, $\tau_{ij} = \tau_{ji}$.

Turbulent transport of heat
We start from the diffusion equation for heat:

$$\frac{\partial \tilde{\theta}}{\partial t} + \tilde{u_j}\frac{\partial \tilde{\theta}}{\partial x_j} = \gamma \frac{\partial^2 \tilde{\theta}}{\partial x_j \partial x_j}$$

In the same way, we decompose the temperature $\tilde{\theta}$:

$$\tilde{\theta} = \Theta + \theta, \bar{\theta} = 0, \partial\Theta/\partial t = 0$$

Then we have

$$U_j \frac{\partial \Theta}{\partial x_j} = \frac{\partial}{\partial x_j}\left(-\theta \bar{u}_j + \gamma \frac{\partial \Theta}{\partial x_j}\right)$$
$$Q_j = c_p \rho \left(\theta u_j - \bar{\gamma} \partial \Theta / \partial x_j\right)$$

The transport of heat and momentum are analogical, thus we believe that in turbulence, they are transported in the same way.

1.2 Elements of the kinetic theory of gases

Pure shear flow
A steady pure shear flow, where $U_2 = U_3 = 0$, $U_1 = U_1(x_2)$. The viscous shear stress

$$\sigma_{12} = \sigma_{21} = \mu \partial U_1 / \partial x_2$$

It must result from molecular transport of momentum in the x_2 direction.
Here, v_1, v_2 are the velocity of a molecule relative to the mean flow.

$$\sigma_{12} = -\rho \bar{v_1} v_2$$

Molecular collisions
Mean free path: ξ.