1 The dynamics of turbulence

Two major questions:

- How is the kinetic energy of the turbulence maintained?
- Why are vorticity and vortex stretching so important to the study of turbulence?

1.1 Kinetic energy of the mean flow

Mean flow energy: $\frac{1}{2}U_iU_i$
We already have

$$U_j\frac{\partial U_i}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\frac{T_{ij}}{\rho}\right)$$

Multiply U_i on both sides,

$$\rho U_j\frac{\partial}{\partial x_j}\left(\frac{1}{2}U_iU_i\right)=\frac{\partial}{\partial x_j}(T_{ij}U_i)-T_{ij}\frac{\partial U_i}{\partial x_j}$$

Note that

$$T_{ij}\frac{\partial U_i}{\partial x_j}=T_{ij}S_{ij}$$

by the definition of S_{ij}.
This term,

$$\frac{\partial}{\partial x_j}(T_{ij}U_i)$$

we can use the Gauss' theorem,

$$\int_V\frac{\partial}{\partial x_j}(T_{ij}U_i)dV=\int_S\frac{\partial}{\partial x_j}(T_{ij}U_i)ds$$

If we use a control volume, and on its surface T_{ij} or U_i is zero, then we know that the total amount of kinetic energy is changed by $T_{ij}S_{ij}$.
$T_{ij}S_{ij}$ is called the deformation work.
Pure shear flow (omitted)
The effects of viscosity

Substitute T_{ij},

$$U_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}U_iU_i\right) = \frac{\partial}{\partial x_j}\left(-\frac{P}{\rho}U_j + 2\nu U_i S_{ij} - \overline{u_iu_j}U_i\right) + 2\nu S_{ij}S_{ij} + \overline{u_iu_j}S_{ij}$$

- $-\frac{\partial}{\partial x_j}\frac{P}{\rho}U_j$: pressure work
- $2\frac{\partial}{\partial x_j}\nu U_i S_{ij}$: transport of mean-flow energy by viscous stresses
- $-\frac{\partial}{\partial x_j}\overline{u_iu_j}U_i$: transport of mean-flow energy by Reynolds stresses

And the last two terms are negligible in most flows.
We define the representative velocity $\mathcal{u}$

$$\mathcal{u}^2 = \frac{1}{3}\overline{u_iu_i}$$

1.2 Kinetic energy of the turbulence

The turbulent energy budget

$$U_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_iu_i}\right) = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_jP} + \frac{1}{2}\overline{u_iu_iu_j} - 2\nu\overline{u_is_{ij}}\right) - \overline{u_iu_j}S_{ij} - 2\nu\overline{s_{ij}s_{ij}}$$

The fluctuating rate of strain

$$s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

The deformation work terms $\overline{u_iu_j}S_{ij}$ occurs in both equations with opposite signs. That is the exchange of energy between the mean flow and the turbulence.
Viscous dissipation: $-2\nu\overline{s_{ij}s_{ij}}$, it is a drain of energy and could not be neglected.
Production equals dissipation
In a steady, homogeneous, pure shear flow, we have

$$-\overline{u_iu_j}S_{ij} = 2\nu\overline{s_{ij}s_{ij}}$$

The rate of production of turbulent energy by Reynolds stresses euqals the rate of viscous dissipation. In most flows, this is not always true, but they are always of the same order of magnitude.
With scale relation, we can find

$$\overline{s_{ij}s_{ij}} \gg S_{ij}S_{ij}$$

When the Reynolds number is large, the fluctuating strain rate is much larger than the mean rate of strain.
Taylor microscale
In isotropic turbulence, the dissipation rate

$$\epsilon = 2\nu\overline{s_{ij}}s_{ij} = 15\nu\overline{(\partial u_1/\partial x_1)^2}$$

We define a new length scale λ (Taylor microscale) by

$$\overline{(\partial u_1/\partial x_1)^2} = \frac{\overline{u-1}}{\lambda^2}$$

Then

$$\epsilon = 15\nu u^2/\lambda^2$$

If S_{ij} is of order u/ℓ and $-\overline{u_iu_j}$ is of order u^2, then

$$\mathscr{A}u^3/\ell = 15\nu u^2/\lambda^2$$

Thus

$$\frac{\lambda}{\ell} = \left(\frac{15}{\mathscr{A}}\right)^1/2R_\ell^{-1/2}$$

Where $\mathscr{A}$ is an undetermined constant, presumably of order one. We know that the Taylor microscale is always much smaller than the integral scale.
Scale relations
The smallest scale is the Kolmogorov microscale η

$$\eta = (\nu^3/\epsilon)^1/4$$

The relation between ℓ, λ and η

$$\frac{\lambda}{\ell} = \frac{15}{\mathscr{A}} R_\lambda^{-1}$$
$$\frac{\lambda}{\eta} = 15^{1/4} R_\lambda^1/2$$

Where $R_\lambda = u\lambda/\nu$ is called the microscale Reynolds number.

Sectral energy transfer

If there is only one charateristic length ℓ, the dissipation rate can be estimated as

$$\epsilon = \mathscr{A} u^3/\ell$$

Considering the production $\mathscr{P}$ and dissipation ϵ

$$-\overline{u_i u_j} S_{ij} \sim \mathscr{A} u^3/\ell$$

Further estimates

The pressure-work term

$$-\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\overline{u_j p}\right) \sim \frac{u^3}{\ell}$$

Mean transport of turbulent energy by turbulent motion

$$-\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{u_i u_i u_j}\right) \sim \frac{u^3}{\ell}$$

Transport by viscous stresses

$$2\nu \frac{\partial}{\partial x_j} \sim \frac{u^3}{\ell} R_l^{-1}$$

Wind-tunnel turbulence

Nearly homogeneous turbulence in a low-speed wind tunnel.

Choose U_1 as the only non-zero component of the mean velocity.

$$U_1 \frac{\partial}{\partial x_1}\left(\frac{1}{2}\overline{u_i u_i}\right) = -\frac{\partial}{\partial x_1}\left(\frac{1}{\rho}\overline{u_1 p} + \frac{1}{2}\overline{u_i u_i u_1}\right) - \epsilon$$

We assume that R_ℓ is large, the viscous transport term can be neglected.

$$U_1 \frac{\partial}{\partial x_1}\left(\frac{1}{2}\overline{u_i u_i}\right) = \mathcal{O}\left(\frac{U_1}{x_1}u^2\right)$$

$$-\frac{\partial}{\partial x_1}\left(\frac{1}{\rho}\overline{u_1 p} + \frac{1}{2}\overline{u_i u_i u_1}\right) - \epsilon = \mathcal{O}\left(\frac{u^3}{x_1}\right)$$

$$\epsilon = \mathcal{O}\left(\frac{u^3}{\ell}\right)$$

In grid turbulence, velocity fluctuations are small: $u \ll U$. Thus,

$$U_1 \frac{\partial}{\partial x_1}\left(\frac{1}{2}\overline{u_i u_i}\right) = -\epsilon$$

And dimensionally,

$$\frac{U_1}{x_1} = C\frac{u}{\ell}$$

Which suggest that the time scale of the flow is of the same order of the time scale of the turbulence.

The change of ℓ and u downstream is omitted here.

Pure shear flow

Omitted here.

1.3 Vorticity dynamics

What do we mean by saying that turbulence is rotational and dissipative

Vorticity vector and rotation tensor

The vorticity is the curl of the velocity vector:

$$\tilde{\omega_i} = \epsilon_{ijk}\frac{\partial \tilde{u_k}}{\partial x_j}$$

ϵ_{ijk}: Levi-Civita notation, $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$, $\epsilon_{321} = \epsilon_{132} = \epsilon_{213} = -1$.

The deformation rate can be split into two parts: symmetric and skew-symmetric part.

$$\frac{\partial \tilde{u_i}}{\partial x_j} = \tilde{s_{ij}} + \tilde{r_{ij}}$$

Symmetric: strain rate

$$\tilde{s_{ij}} = \frac{1}{2}\left(\frac{\partial \tilde{u_i}}{\partial x_j} + \frac{\partial \tilde{u_j}}{\partial x_i}\right)$$

Skew-symmetric: rotation tensor

$$\tilde{r_{ij}} = \frac{1}{2}\left(\frac{\partial \tilde{u_i}}{\partial x_j} - \frac{\partial \tilde{u_j}}{\partial x_i}\right)$$

The vorticity vector is related only to the skew-symmetric part

$$\tilde{\omega_i} = \epsilon_{ijk}\tilde{r_{kj}}$$

And

$$\tilde{r_{ij}} = -\frac{1}{2}\epsilon_{ijk}\tilde{\omega_k}$$

Vortex terms in the equations of motion

Omitted here.

Reynolds stress and vorticity

Decompose the vorticity $\tilde{\omega_i}$

$$\tilde{\omega_i} = \Omega_i + \omega_i, \quad \overline{\omega_i} = 0$$

The equation for the mean velocity U_i

$$0 = -\frac{\partial}{\partial x_i}\left(\frac{P}{\rho} + \frac{1}{2}U_jU_j + \frac{1}{2}\overline{u_ju_j}\right) + \epsilon_{ijk}\left(U_j\Omega_k + \overline{u_j\omega_k}\right) + \nu\frac{\partial^2 U_i}{\partial x_j\,\partial x_j}$$

Since $\overline{u_ju_j} \ll U_jU_j$, and we consider such a two-dimensional flow that $U_1 \gg U_2$, $U_3 = 0$, $\partial/\partial x_1 \ll \partial/\partial x_2$, also neglect the viscous term and the contribution of the turbulence,

$$U_1\frac{\partial U_1}{\partial x_1} + U_2\frac{\partial U_1}{\partial x_2} = -\frac{1}{\rho}\frac{\partial P}{\partial x_1} + \overline{u_2\omega_3} - \overline{u_3\omega_2}$$

For the Reynolds shear stress $-\overline{u_1u_2}$,

$$\frac{\partial}{\partial x_2}\left(-\overline{u_1u_2}\right) = \overline{u_2\omega_3} - \overline{u_3\omega_2}$$

The vorticity equation

Taking curl operator to the Navier-Stokes equation ($\epsilon_{pqi}\,\partial/\,\partial x_q$), and some analysis,

$$\frac{\partial \tilde{\omega}_p}{\partial t} = \tilde{\omega}_k \frac{\partial \tilde{u}_p}{\partial x_k} - \tilde{u}_k \frac{\partial \tilde{\omega}_k}{\partial x_k} - \nu \frac{\partial}{\partial x_p}\left(\frac{\partial \tilde{\omega}_k}{\partial x_k}\right) + \nu \frac{\partial^2 \tilde{\omega}_p}{\partial x_k\,\partial x_k}$$

The first viscous term

$$\frac{\partial \tilde{\omega}_k}{\partial x_k} = \epsilon_{ijk} \frac{\partial^2 \tilde{u}_j}{\partial x_i\,\partial x_k} = 0$$

Thus, finally

$$\frac{\partial \tilde{\omega}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial x_j} = \tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial x_j} + \nu \frac{\partial^2 \tilde{\omega}_i}{\partial x_j\,\partial x_j}$$

Note that the skew-symmetric part $\tilde{r}_{ij}$ does not contribute to the equation, the equation could be further simplified as

$$\frac{\partial \tilde{\omega}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{\omega}_i}{\partial x_j} = \tilde{\omega}_j \tilde{s}_{ij} + \nu \frac{\partial^2 \tilde{\omega}_i}{\partial x_j\,\partial x_j}$$

Vorticity in turbulent flows

$$U_j \frac{\partial \Omega_i}{\partial x_j} = -\overline{u_j \frac{\partial \omega_i}{\partial x_j}} + \overline{\omega_j s_{ij}} + \Omega_j S_{ij} + \nu \frac{\partial^2 \Omega_i}{\partial x_j\,\partial x_j}$$

The mean vorticity and fluctuating vorticity are divergenceless

$$\frac{\partial \Omega_i}{\partial x_i} = 0, \qquad \frac{\partial \omega_i}{\partial x_i} = 0$$

Continuity:

$$\frac{\partial u_i}{\partial x_i} = 0$$

We have

$$\overline{u_j \frac{\partial \omega_i}{\partial x_j}} = \frac{\partial}{\partial x_j}\left(\overline{u_j \omega_i}\right)$$

$$\overline{\omega_j s_{ij}} = \frac{\partial}{\partial x_j}\left(\overline{\omega_j u_i}\right)$$

The first term is a transport "divergence", while the second term is the gain of mean vorticity caused by the stretching and rotation of fluctuating voriticity components by fluctuating strain rates.

Two-dimensional mean flow

The dynamics of $\Omega_i\Omega_i$

The equation for $\overline{\omega_i\omega_i}$

Turbulence is rotational

An approximate vorticity budget

Multiple length scales

Stretching of magnetic field lines

1.4 The dynamics of temperature fluctuations

$$U_j \frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{\theta^2}\right) = -\frac{\partial}{\partial x_j}\left[\frac{1}{2}\overline{\theta^2 u_j} - \gamma\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{\theta^2}\right)\right] - \overline{\theta u_j}\frac{\partial \Theta}{\partial x_j} - \gamma\overline{\frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial x_j}}$$

For steady homogeneous shear flow

$$-\overline{\theta u_j}\frac{\partial \Theta}{\partial x_j} = \gamma\overline{\frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial x_j}}$$

Microscales in the temperature field

Let us define

$$\overline{\left(\frac{\partial \theta}{\partial x_1}\right)^2} = 2\frac{\overline{\theta^2}}{\lambda_\theta^2}$$

Then the isotropic temperature field has

$$\gamma\overline{\frac{\partial \theta}{\partial x_j}\frac{\partial \theta}{\partial x_j}} = 6\frac{\overline{\theta^2}}{\lambda_\theta^2}$$

With order analysis we obtain

$$\frac{\lambda_\theta}{\lambda} = C\left(\frac{\gamma}{\nu}\right)^{1/2}$$

C is of order one.

...

Buoyant convection

Richardson numbers

Monin-Oboukhov length

Convection in the atmospheric boundary layer